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Jean-Pierre Demailly

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FANO MANIFOLDS WITH NEF TANGENT BUNDLES ARE WEAKLY ALMOST KÄHLER-EINSTEIN

JEAN-PIERRE DEMAILLY

ABSTRACT. The goal of this short note is to point out that every Fano manifold with a nef tangent bundle possesses an almost Kähler-Einstein metric, in a weak sense. The technique relies on a regularization theorem for closed positive $(1, 1)$ -currents. We also discuss related semistability questions and Chern inequalities.

dedicated to Professor Ngaiming Mok on the occasion of his sixtieth birthday

1. INTRODUCTION

Recall that a holomorphic vector bundle E on a projective manifold X is said to be numerically effective (nef) if the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef on $Y = \mathbb{P}(E)$. Clearly, every homogeneous projective manifold X has its tangent bundle T_X generated by sections, so T_X is nef; moreover, by [DPS94], every compact Kähler manifold with T_X nef admits a finite étale cover \tilde{X} that is a locally trivial fibration over its Albanese torus, and the fibers are themselves Fano manifolds F with T_F nef. Hence, the classification problem is essentially reduced to the case when X is a Fano manifold with T_X nef. In this direction, Campana and Peternell [CP91] have conjectured in 1991 that Fano manifolds with nef tangent bundles are rational homogeneous manifolds G/P , namely quotients of linear algebraic groups by parabolic subgroups. Although this can be checked up to dimension 3 by inspecting the classification of Fano 3-folds by Manin-Iskovskii and Mukai, very little is known in higher dimensions. It would be tempting to use the theory of VMRT's developed by J.M. Hwang and N. Mok ([Hwa01], [Mok08]), since the expected homogeneity property should be reflected in the geometry of rational curves. Even then, the difficulties to be solved remain formidable; see e.g. [Mok02] and also [MOSWW] for a recent account of the problem.

On the other hand, every rational homogeneous manifold $X = G/P$ carries a Kähler metric that is invariant by a compact real form $G^{\mathbb{R}}$ of G (cf. [AP86]), and the bisectional curvature of such a metric is easily seen to be nonnegative. Another path to proving the theorem could thus be to produce a Kähler metric with nonnegative bisectional curvature, since compact complex manifolds carrying such a metric have been characterized by N. Mok [Mok88]. The Ricci curvature form (i.e. the curvature of $-K_{G/P}$) is then itself a Kähler-Einstein metric with nonnegative bisectional curvature, hence a natural related question is whether every Fano manifold X with nef tangent bundle actually possesses a Kähler-Einstein metric, and whether such a metric will have nonnegative bisectional curvature. Our main observation is the following (much) weaker statement, whose proof is based on regularization techniques for

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closed positive currents [Dem92], [Dem99]. We denote here by $\text{PC}^{1,1}(X)$ the cone of positive currents of bidegree $(1, 1)$ on X and put $n = \dim_{\mathbb{C}} X$.

Theorem 1.1. *Let X be a Fano manifold such that the tangent bundle T_X is nef. Then*

- (i) *there exists a family of smoothing operators $(J_\varepsilon)_{\varepsilon \in]0,1]}$ that map every closed positive current $T \in \text{PC}^{1,1}(X) \cap c_1(X)$ to a smooth closed positive definite $(1, 1)$ -form*

$$\alpha_\varepsilon = J_\varepsilon(T) \in c_1(X),$$

in such a way that $J_\varepsilon(T)$ converges weakly to T as $\varepsilon \rightarrow 0$ (uniformly for all T), and $T \mapsto J_\varepsilon(T)$ is continuous with respect to the weak topology of currents and the strong topology of C^∞ convergence on smooth $(1, 1)$ -forms;

- (ii) *for every $\varepsilon \in]0, 1]$, there exists a Kähler metric ω_ε on X such that $\text{Ricci}(\omega_\varepsilon) = J_\varepsilon(\omega_\varepsilon)$, in other words ω_ε is “weakly almost Kähler-Einstein”.*

The construction of J_ε that we have is unfortunately not very explicit, and we do not even know if J_ε can be constructed as a natural linear (say convolution-like) operator. The proof of (ii) is based on the use of the Schauder fixed point theorem. Let us denote by

$$0 < \rho_{1,\varepsilon}(x) \leq \dots \leq \rho_{n,\varepsilon}(x)$$

the eigenvalues of $\text{Ricci}(\omega_\varepsilon)$ with respect to ω_ε at each point $x \in X$, and $\zeta_{1,\varepsilon}, \dots, \zeta_{n,\varepsilon} \in T_{X,x}$ a corresponding orthonormal family of eigenvectors with respect to ω_ε . We know that

$$(*) \quad \int_X \sum_{j=1}^n \rho_{j,\varepsilon} \omega_\varepsilon^n = n \int_X \text{Ricci}(\omega_\varepsilon) \wedge \omega_\varepsilon^{n-1} = n c_1(X)^n,$$

in particular the left-hand side is bounded. Also, as $\text{Ricci}(\omega_\varepsilon) - \omega_\varepsilon = J_\varepsilon(\omega_\varepsilon) - \omega_\varepsilon$ converges weakly to 0, we know that

$$\int_X \sum_{j=1}^n (\rho_{j,\varepsilon} - 1) \zeta_{1,\varepsilon}^* \wedge \overline{\zeta_{1,\varepsilon}^*} \wedge u$$

converges to 0 for every smooth $(n-1, n-1)$ -form u on X , hence the eigenvalues $\rho_{j,\varepsilon}$ “converge weakly to 1” in the sense that $\sum_{j=1}^n (\rho_{j,\varepsilon} - 1) \zeta_{1,\varepsilon}^* \wedge \overline{\zeta_{1,\varepsilon}^*}$ converges weakly to 0 in the space of smooth $(1, 1)$ -forms. Therefore, it does not seem too unrealistic to expect that the family (ω_ε) is well behaved in the following sense.

Definition 1.2. *We say the family (ω_ε) of weak almost Kähler-Einstein metrics is well behaved in L^p norm if*

$$(**)^p \quad \int_X (\rho_{n,\varepsilon} - \rho_{1,\varepsilon})^p \omega_\varepsilon^n$$

converges to 0 as ε tends to 0.

Standard curvature inequalities then yield the following result.

Theorem 1.3. *Assume that X is Fano with T_X nef, and possesses a family of weak almost Kähler-Einstein metrics that is well behaved in L^p norm. Then*

- (i) *if $p \geq 1$, T_X is $c_1(X)$ -semistable;*
(ii) *if $p \geq 2$, the Guggenheimer-Yau-Bogomolov-Miyaoka Chern class inequality*

$$[nc_1(X)^2 - (2n+2)c_2(X)] \cdot c_1(X)^{n-2} \leq 0$$

is satisfied.

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2. PROOFS

EXISTENCE OF REGULARIZATION OPERATORS IN 1.1 (i). Let us fix once for all a Kähler metric $\beta \in c_1(X)$. It is well known that $K = \text{PC}^{1,1}(X) \cap c_1(X)$ is compact for the weak topology of currents; this comes from the fact that currents $T \in K$ have a uniformly bounded mass

$$\int_X T \wedge \beta^{n-1} = c_1(X)^n.$$

We also know from [Dem92], [Dem99] that every such current T admits a family of regularizations $T_\varepsilon \in C^\infty(X) \cap c_1(X)$ converging weakly to T , such that $T_\varepsilon \geq -\varepsilon\beta$. Here, the fact that we can achieve an arbitrary small lower bound uses in an essential way the assumption that T_X is nef. After replacing T_ε by $T'_\varepsilon = (1 - \varepsilon)T_\varepsilon + \varepsilon\beta \geq \varepsilon^2\beta$, we can assume that T_ε is a Kähler form. An important issue is that we want to produce a continuous operator $J_\varepsilon(T) = T_\varepsilon$ defined on the weakly compact set K . This is clearly the case when the regularization is produced by convolution, as is done [Dem94], but in that case one needs a more demanding condition on T_X , e.g. that T_X possesses a hermitian metric with nonnegative bisectional curvature (i.e. that T_X is Griffiths semipositive). However, the existence of a continuous global operator J_ε is an easy consequence of the convexity of K . In fact, the weak topology of K is induced by the L^2 topology on the space of potentials φ when writing $T = \beta + dd^c\varphi$ (and taking the quotient by constants) – it is also induced by the L^p topology for any $p \in [1, \infty[$, but L^2 has the advantage of being a Hilbert space topology. By compactness, for every $\delta \in]0, 1]$ we can then find finitely many currents $(T_j)_{1 \leq j \leq N(\delta)}$ such that the Hilbert balls $B(T_j, \delta)$ cover K . Let K_δ be the convex hull of the family $(T_j)_{1 \leq j \leq N(\delta)}$ and let $p_\delta : K \rightarrow K_\delta$ be induced by the (nonlinear) Hilbert projection $L^2/\mathbb{R} \rightarrow K_\delta$. Since K_δ is finite dimensional and is a finite union of simplices, we can take regularizations $T_{j,\varepsilon}$ of the vertices T_j and construct a piecewise linear operator $\tilde{J}_{\delta,\varepsilon} : K_\delta \rightarrow K \cap C^\infty(X)^+$ to Kähler forms, simply by taking linear combinations of the $T_{j,\varepsilon}$'s. Then $\tilde{J}_{\delta,\varepsilon} \circ p_\delta(T)$ is the continuous regularization operator we need. The uniform weak convergence to T is guaranteed if we take $J_\varepsilon := \tilde{J}_{\delta(\varepsilon),\varepsilon}$ with $\varepsilon \ll \delta(\varepsilon) \rightarrow 0$, e.g. with a step function $\varepsilon \mapsto \delta(\varepsilon)$ that converges slowly to 0 compared to ε . These operators have the drawback of being non explicit and a priori non linear.

Question 2.1. *Is it possible to construct a linear regularization operator J_ε with the same properties as above, e.g. by means of a convolution process ?*

In fact, such a “linear” regularization operator J_ε is constructed in [Dem94] by putting $T = \beta + dd^c\varphi$ and $J_\varepsilon(T) = \beta + dd^c\varphi_\varepsilon$ where

$$\varphi_\varepsilon(z) = \int_{\zeta \in T_{X,z}} \varphi(\text{exph}_z(\zeta)) \chi(|\zeta|_h^2/\varepsilon^2) dV_h(\zeta)$$

for a suitable hermitian metrics h on T_X , taking exph to be the holomorphic part of the exponential map associated with the Chern connection ∇_h . However, the positivity of $J_\varepsilon(T)$ is in general not preserved, unless one assumes that (T_X, h) is (say) Griffiths semipositive. As the relation between nefness and Griffiths semipositivity is not yet elucidated, one would

perhaps need to extend the above formula to the case of *Finsler metrics*. This is eventually possible by applying some of the techniques of [Dem99] to produce suitable dual Finsler metrics on T_X (notice that any assumption that $T_X \otimes L$ is ample for some \mathbb{Q} -line bundle L translates into the existence of a strictly plurisubharmonic Finsler metric on the total space of $(T_X \otimes L)^* \setminus \{0\}$, so one needs to dualize).

USE OF THE SCHAUDER FIXED POINT THEOREM FOR 1.1 (ii). By Yau's theorem [Yau78], for every closed $(1, 1)$ -form $\rho \in c_1(X)$, there exists a unique Kähler metric $\gamma(\rho) \in c_1(X)$ such that $\text{Ricci}(\gamma(\rho)) = \rho$. Moreover, by the regularity theory of nonlinear elliptic operators, the map $\rho \mapsto \gamma(\rho)$ is continuous in C^∞ topology. We consider the composition

$$\gamma \circ J_\varepsilon : K \mapsto K \cap C^\infty(X)^+ \rightarrow K \cap C^\infty(X)^+ \subset K, \quad T \mapsto \rho = J_\varepsilon(T) \mapsto \gamma(\rho).$$

Since J_ε is continuous from the weak topology to the strong C^∞ topology, we infer that $\gamma \circ J_\varepsilon$ is continuous on K in the weak topology. Now, K is convex and weakly compact, therefore $\gamma \circ J_\varepsilon$ must have a fixed point $T = \omega_\varepsilon$ by the theorem of Schauder. By construction ω_ε must be a Kähler metric in $c_1(X)$, since $\gamma \circ J_\varepsilon$ maps K into the space of Kähler metrics contained in $c_1(X)$. This proves Theorem 1.1.

SEMISTABILITY OF T_X (1.3 (i)). Let $\mathcal{F} \subset \mathcal{O}(T_X)$ be a coherent subsheaf such that $\mathcal{O}(T_X)/\mathcal{F}$ is torsion free. We can view \mathcal{F} as a holomorphic subbundle of T_X outside of an algebraic subset of codimension 2 in X . The Chern curvature tensor $\Theta_{T_X, \omega_\varepsilon} = \frac{i}{2\pi} \nabla^2$ satisfies the Hermite-Einstein condition

$$\Theta_{T_X, \omega_\varepsilon} \wedge \omega_\varepsilon^{n-1} = \frac{1}{n} \rho_\varepsilon \omega_\varepsilon^n,$$

where $\rho_\varepsilon \in C^\infty(X, \text{End}(T_X))$ is the Ricci operator (with the same eigenvalues $\rho_{j, \varepsilon}$ as $\text{Ricci}(\omega_\varepsilon)$). It is well known that the curvature of a subbundle is always bounded above by the restriction of the full curvature tensor, i.e. $\Theta_{\mathcal{F}, \omega_\varepsilon} \leq \Theta_{T_X, \omega_\varepsilon}|_{\mathcal{F}}$ (say, in the sense of Griffiths positivity, viewing the curvature tensors as hermitian forms on $T_X \otimes \mathcal{F}$). By taking the trace with respect to ω_ε , we get

$$\Theta_{\mathcal{F}, \omega_\varepsilon} \wedge \omega_\varepsilon^{n-1} \leq \frac{1}{n} \rho_{\varepsilon|_{\mathcal{F}}} \omega_\varepsilon^n.$$

By the minimax principle, if $r = \text{rank } \mathcal{F}$, the eigenvalues of $\rho_{\varepsilon|_{\mathcal{F}}}$ are bounded above by

$$\rho_{n-r+1, \varepsilon} \leq \dots \leq \rho_{n, \varepsilon},$$

hence

$$\int_X c_1(\mathcal{F}) \wedge \omega_\varepsilon^{n-1} = \int_X \text{tr}_{\mathcal{F}} \Theta_{\mathcal{F}, \omega_\varepsilon} \wedge \omega_\varepsilon^{n-1} \omega_\varepsilon^{n-1} \leq \frac{1}{n} \int_X \sum_{j=1}^r \rho_{n-r+j, \varepsilon} \omega_\varepsilon^n.$$

The left hand side is unchanged if we replace ω_ε by $\beta \in c_1(X)$, and our assumption $(**)^p$ for $p = 1$ (cf. Definition 1.2) implies

$$\int_X c_1(\mathcal{F}) \wedge \beta^{n-1} \leq \lim_{\varepsilon \rightarrow 0} \frac{1}{n} \int_X \sum_{j=1}^r \rho_{n-r+j, \varepsilon} \omega_\varepsilon^n \leq \lim_{\varepsilon \rightarrow 0} \frac{r}{n^2} \int_X \sum_{j=1}^n \rho_{j, \varepsilon} \omega_\varepsilon^n = \frac{r}{n} \int_X c_1(X) \wedge \beta^{n-1}$$

by (*). This means that T_X is $c_1(X)$ -semistable.

CHERN CLASS INEQUALITY (1.3 (ii)). Let us write

$$\Theta_{T_X, \omega_\varepsilon} = (\theta_{\alpha\beta})_{1 \leq \alpha, \beta \leq n} = \left(\sum_{j, k} \theta_{\alpha\beta j k} dz_j \wedge d\bar{z}_k \right)_{1 \leq \alpha, \beta \leq n}$$

as an $n \times n$ matrix of $(1, 1)$ -forms with respect to an orthonormal frame of T_X that diagonalizes the Ricci operator ρ_ε (with respect to the metric ω_ε). A standard calculation yields

$$\begin{aligned} & 2 \left[n c_1(X)_h^2 - (2n + 2) c_2(X) \right] \wedge \frac{\omega_\varepsilon^{n-2}}{(n-2)!} \\ &= \left(- \sum_{\alpha \neq \beta, j \neq k} |\theta_{\alpha\alpha jk} - \theta_{\beta\beta jk}|^2 - (n+1) \sum_{\alpha, \beta, j, k, \text{ pairwise } \neq} |\theta_{\alpha\beta jk}|^2 \right. \\ &\quad - (4n+8) \sum_{\alpha \neq j < k \neq \alpha} |\theta_{\alpha\alpha jk}|^2 - 4 \sum_{\alpha \neq j} |\theta_{\alpha\alpha\alpha j}|^2 \\ &\quad - \sum_{\alpha \neq \beta \neq j \neq \alpha} |\theta_{\alpha\alpha jj} - \theta_{\beta\beta jj}|^2 - \sum_{\alpha \neq \beta} |\theta_{\alpha\alpha\alpha\alpha} - 2\theta_{\alpha\alpha\beta\beta}|^2 \\ &\quad \left. + n \sum_{\alpha} \rho_{\alpha, \varepsilon}^2 - \left(\sum_{\alpha} \rho_{\alpha, \varepsilon} \right)^2 \right) \omega_\varepsilon^n \end{aligned}$$

where all terms in the summation are nonpositive except the ones involving the Ricci eigenvalues $\rho_{\alpha, \varepsilon}$. However, the assumption $(**)^p$ for $p \geq 2$ implies that the integral of the difference appearing in the last line converges to 0 as $\varepsilon \rightarrow 0$. Theorem 1.3 is proved.

Our results lead to the following interesting question.

Question 2.2. *Assuming that a “sufficiently good” family of regularization operators J_ε is used, can one infer that the resulting family (ω_ε) of fixed points such that $\text{Ricci}(\omega_\varepsilon) = J_\varepsilon(\omega_\varepsilon)$ (or some subsequence) is well behaved in L^1 , resp. L^2 norm?*

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Jean-Pierre Demailly

Université Grenoble Alpes, Institut Fourier, F-38000 Grenoble, France

CNRS, Institut Fourier, F-38000 Grenoble, France

e-mail: jean-pierre.demailly@univ-grenoble-alpes.fr