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# FANO MANIFOLDS WITH NEF TANGENT BUNDLES ARE WEAKLY ALMOST KÄHLER-EINSTEIN

JEAN-PIERRE DEMAILLY

ABSTRACT. The goal of this short note is to point out that every Fano manifold with a nef tangent bundle possesses an almost Kähler-Einstein metric, in a weak sense. The technique relies on a regularization theorem for closed positive  $(1, 1)$ -currents. We also discuss related semistability questions and Chern inequalities.

*dedicated to Professor Ngaiming Mok on the occasion of his sixtieth birthday*

## 1. INTRODUCTION

Recall that a holomorphic vector bundle  $E$  on a projective manifold  $X$  is said to be numerically effective (nef) if the line bundle  $\mathcal{O}_{\mathbb{P}(E)}(1)$  is nef on  $Y = \mathbb{P}(E)$ . Clearly, every homogeneous projective manifold  $X$  has its tangent bundle  $T_X$  generated by sections, so  $T_X$  is nef; moreover, by [DPS94], every compact Kähler manifold with  $T_X$  nef admits a finite étale cover  $\tilde{X}$  that is a locally trivial fibration over its Albanese torus, and the fibers are themselves Fano manifolds  $F$  with  $T_F$  nef. Hence, the classification problem is essentially reduced to the case when  $X$  is a Fano manifold with  $T_X$  nef. In this direction, Campana and Peternell [CP91] conjectured in 1991 that Fano manifolds with nef tangent bundles are rational homogeneous manifolds  $G/P$ , namely quotients of linear algebraic groups by parabolic subgroups. Although this can be checked up to dimension 3 by inspecting the classification of Fano 3-folds by Manin-Iskovskii and Mukai, very little is known in higher dimensions. It would be tempting to use the theory of VMRT's developed by J.M. Hwang and N. Mok ([Hwa01], [Mok08]), since the expected homogeneity property should be reflected in the geometry of rational curves. Even then, the difficulties to be solved remain formidable; see e.g. [Mok02] and also [MOSWW] for a recent account of the problem.

On the other hand, every rational homogeneous manifold  $X = G/P$  carries a Kähler metric that is invariant by a compact real form  $G^{\mathbb{R}}$  of  $G$  (cf. [AP86]), and the corresponding Ricci curvature form (i.e. the curvature of  $-K_{G/P}$ ) is then a Kähler-Einstein metric. A stronger condition than nefness of  $T_X$  is the existence of a Kähler metric on  $X$  whose holomorphic bisectional curvature is nonnegative. N. Mok [Mok88] characterized those manifolds, they are exactly the products of hermitian symmetric spaces of compact type by flat compact complex tori  $\mathbb{C}^q/\Lambda$  and projective spaces  $\mathbb{P}^{n_j}$  with a Kähler metric of nonnegative holomorphic bisectional curvature. However, hermitian symmetric spaces of compact type are a smaller class than rational homogeneous manifolds (for instance, they do not include complete flag

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manifolds), thus one cannot expect these Kähler-Einstein metrics to have a nonnegative bisectional curvature in general.

It is nevertheless a natural related question to investigate whether every Fano manifold  $X$  with nef tangent bundle actually possesses a Kähler-Einstein metric. Our main observation is the following (much) weaker statement, whose proof is based on regularization techniques for closed positive currents [Dem92], [Dem99]. We denote here by  $\text{PC}^{1,1}(X)$  the cone of positive currents of bidegree  $(1, 1)$  on  $X$  and put  $n = \dim_{\mathbb{C}} X$ . Though it is a convex set of infinite dimension, the intersection  $\text{PC}^{1,1}(X) \cap c_1(X)$  is compact and metrizable for the weak topology (see § 2), and therefore it carries a unique uniform structure that can be defined by any compatible metric.

**Theorem 1.1.** *Let  $X$  be a Fano manifold such that the tangent bundle  $T_X$  is nef. Then*

- (i) *there exists a family of smoothing operators  $(J_\varepsilon)_{\varepsilon \in ]0,1]}$  that map every closed positive current  $T \in \text{PC}^{1,1}(X) \cap c_1(X)$  to a smooth closed positive definite  $(1, 1)$ -form*

$$\alpha_\varepsilon = J_\varepsilon(T) \in c_1(X),$$

*in such a way that  $J_\varepsilon(T)$  converges weakly to  $T$  as  $\varepsilon \rightarrow 0$  (uniformly for all  $T$ ), and  $T \mapsto J_\varepsilon(T)$  is continuous with respect to the weak topology of currents and the strong topology of  $C^\infty$  convergence on smooth  $(1, 1)$ -forms;*

- (ii) *for every  $\varepsilon \in ]0, 1]$ , there exists a Kähler metric  $\omega_\varepsilon$  on  $X$  such that  $\text{Ricci}(\omega_\varepsilon) = J_\varepsilon(\omega_\varepsilon)$ , in other words  $\omega_\varepsilon$  is “weakly almost Kähler-Einstein”.*

The construction of  $J_\varepsilon$  that we have is unfortunately not very explicit, and we do not even know if  $J_\varepsilon$  can be constructed as a natural linear (say convolution-like) operator. The proof of (ii) is based on the use of the Schauder fixed point theorem. Let us denote by

$$0 < \rho_{1,\varepsilon}(x) \leq \dots \leq \rho_{n,\varepsilon}(x)$$

the eigenvalues of  $\text{Ricci}(\omega_\varepsilon)$  with respect to  $\omega_\varepsilon$  at each point  $x \in X$ , and  $\zeta_{1,\varepsilon}, \dots, \zeta_{n,\varepsilon} \in T_{X,x}$  a corresponding orthonormal family of eigenvectors with respect to  $\omega_\varepsilon$ . We know that

$$(*) \quad \int_X \sum_{j=1}^n \rho_{j,\varepsilon} \omega_\varepsilon^n = n \int_X \text{Ricci}(\omega_\varepsilon) \wedge \omega_\varepsilon^{n-1} = n c_1(X)^n,$$

in particular the left-hand side is bounded. Also, as  $\text{Ricci}(\omega_\varepsilon) - \omega_\varepsilon = J_\varepsilon(\omega_\varepsilon) - \omega_\varepsilon$  converges weakly to 0, we know that

$$\int_X \sum_{j=1}^n (\rho_{j,\varepsilon} - 1) \zeta_{1,\varepsilon}^* \wedge \overline{\zeta_{1,\varepsilon}^*} \wedge u$$

converges to 0 for every smooth  $(n-1, n-1)$ -form  $u$  on  $X$ , hence the eigenvalues  $\rho_{j,\varepsilon}$  “converge weakly to 1” in the sense that  $\sum_{j=1}^n (\rho_{j,\varepsilon} - 1) \zeta_{1,\varepsilon}^* \wedge \overline{\zeta_{1,\varepsilon}^*}$  converges weakly to 0 in the space of smooth  $(1, 1)$ -forms. Therefore, it does not seem too unrealistic to expect that the family  $(\omega_\varepsilon)$  is well behaved in the following sense.

**Definition 1.2.** *We say the family  $(\omega_\varepsilon)$  of weak almost Kähler-Einstein metrics is well behaved in  $L^p$  norm if*

$$(**)^p \quad \int_X (\rho_{n,\varepsilon} - \rho_{1,\varepsilon})^p \omega_\varepsilon^n$$

*converges to 0 as  $\varepsilon$  tends to 0.*

Standard curvature inequalities then yield the following result.

**Theorem 1.3.** *Assume that  $X$  is Fano with  $T_X$  nef, and possesses a family of weak almost Kähler-Einstein metrics that is well behaved in  $L^p$  norm. Then*

- (i) *if  $p \geq 1$ ,  $T_X$  is  $c_1(X)$ -semistable;*
- (ii) *if  $p \geq 2$ , the Guggenheimer-Yau-Bogomolov-Miyaoka Chern class inequality*

$$[nc_1(X)^2 - (2n + 2)c_2(X)] \cdot c_1(X)^{n-2} \leq 0$$

*is satisfied.*

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## 2. PROOFS

EXISTENCE OF REGULARIZATION OPERATORS IN 1.1 (i). Let us fix once for all a Kähler metric  $\beta \in c_1(X)$ . It is well known that  $K = \text{PC}^{1,1}(X) \cap c_1(X)$  is compact for the weak topology of currents; this comes from the fact that currents  $T \in K$  have a uniformly bounded mass

$$\int_X T \wedge \beta^{n-1} = c_1(X)^n.$$

We also know from [Dem92], [Dem99] that every such current  $T$  admits a family of regularizations  $T_\varepsilon \in C^\infty(X) \cap c_1(X)$  converging weakly to  $T$ , such that  $T_\varepsilon \geq -\varepsilon\beta$ . Here, the fact that we can achieve an arbitrary small lower bound uses in an essential way the assumption that  $T_X$  is nef. After replacing  $T_\varepsilon$  by  $T'_\varepsilon = (1 - \varepsilon)T_\varepsilon + \varepsilon\beta \geq \varepsilon^2\beta$ , we can assume that  $T_\varepsilon$  is a Kähler form. An important issue is that we want to produce a continuous operator  $J_\varepsilon(T) = T_\varepsilon$  defined on the weakly compact set  $K$ . This is clearly the case when the regularization is produced by convolution, as is done [Dem94], but in that case one needs a more demanding condition on  $T_X$ , e.g. that  $T_X$  possesses a hermitian metric with nonnegative bisectional curvature (i.e. that  $T_X$  is Griffiths semipositive). However, the existence of a continuous global operator  $J_\varepsilon$  is an easy consequence of the convexity of  $K$ . In fact, the weak topology of  $K$  is induced by the  $L^2$  topology on the space of potentials  $\varphi$  when writing  $T = \beta + dd^c\varphi$  (and taking the quotient by constants) – it is also induced by the  $L^p$  topology for any  $p \in [1, \infty[$ , but  $L^2$  has the advantage of being a Hilbert space topology. By compactness, for every  $\delta \in ]0, 1]$  we can then find finitely many currents  $(T_j)_{1 \leq j \leq N(\delta)}$  such that the Hilbert balls  $B(T_j, \delta)$  cover  $K$ . Let  $K_\delta$  be the convex hull of the family  $(T_j)_{1 \leq j \leq N(\delta)}$  and let  $p_\delta : K \rightarrow K_\delta$  be induced by the (nonlinear) Hilbert projection  $L^2/\mathbb{R} \rightarrow K_\delta$ . Since  $K_\delta$  is finite dimensional and is a finite union of simplices, we can take regularizations  $T_{j,\varepsilon}$  of the vertices  $T_j$  and construct a piecewise linear operator  $\tilde{J}_{\delta,\varepsilon} : K_\delta \rightarrow K \cap C^\infty(X)^+$  to Kähler forms, simply by taking linear combinations of the  $T_{j,\varepsilon}$ 's. Then  $\tilde{J}_{\delta,\varepsilon} \circ p_\delta(T)$  is the continuous regularization operator we need. The uniform weak convergence to  $T$  is guaranteed if we take  $J_\varepsilon := \tilde{J}_{\delta(\varepsilon),\varepsilon}$  with  $\varepsilon \ll \delta(\varepsilon) \rightarrow 0$ , e.g. with a step function  $\varepsilon \mapsto \delta(\varepsilon)$  that converges slowly to 0 compared to  $\varepsilon$ . These operators have the drawback of being non explicit and a priori non linear.

**Question 2.1.** *Is it possible to construct a linear regularization operator  $J_\varepsilon$  with the same properties as above, e.g. by means of a convolution process ?*

In fact, such a “linear” regularization operator  $J_\varepsilon$  is constructed in [Dem94] by putting  $T = \beta + dd^c\varphi$  and  $J_\varepsilon(T) = \beta + dd^c\varphi_\varepsilon$  where

$$\varphi_\varepsilon(z) = \int_{\zeta \in T_{X,z}} \varphi(\text{exph}_z(\zeta)) \chi(|\zeta|_h^2/\varepsilon^2) dV_h(\zeta)$$

for a suitable hermitian metrics  $h$  on  $T_X$ , taking  $\text{exph}$  to be the holomorphic part of the exponential map associated with the Chern connection  $\nabla_h$ . However, the positivity of  $J_\varepsilon(T)$  is in general not preserved, unless one assumes that  $(T_X, h)$  is (say) Griffiths semipositive. As the relation between nefness and Griffiths semipositivity is not yet elucidated, one would perhaps need to extend the above formula to the case of *Finsler metrics*. This is eventually possible by applying some of the techniques of [Dem99] to produce suitable dual Finsler metrics on  $T_X$  (notice that any assumption that  $T_X \otimes L$  is ample for some  $\mathbb{Q}$ -line bundle  $L$  translates into the existence of a strictly plurisubharmonic Finsler metric on the total space of  $(T_X \otimes L)^* \setminus \{0\}$ , so one needs to dualize).

USE OF THE SCHAUDER FIXED POINT THEOREM FOR 1.1 (ii). By Yau’s theorem [Yau78], for every closed  $(1, 1)$ -form  $\rho \in c_1(X)$ , there exists a unique Kähler metric  $\gamma(\rho) \in c_1(X)$  such that  $\text{Ricci}(\gamma(\rho)) = \rho$ . Moreover, by the regularity theory of nonlinear elliptic operators, the map  $\rho \mapsto \gamma(\rho)$  is continuous in  $C^\infty$  topology. We consider the composition

$$\gamma \circ J_\varepsilon : K \mapsto K \cap C^\infty(X)^+ \rightarrow K \cap C^\infty(X)^+ \subset K, \quad T \mapsto \rho = J_\varepsilon(T) \mapsto \gamma(\rho).$$

Since  $J_\varepsilon$  is continuous from the weak topology to the strong  $C^\infty$  topology, we infer that  $\gamma \circ J_\varepsilon$  is continuous on  $K$  in the weak topology. Now,  $K$  is convex and weakly compact, therefore  $\gamma \circ J_\varepsilon$  must have a fixed point  $T = \omega_\varepsilon$  by the theorem of Schauder. By construction  $\omega_\varepsilon$  must be a Kähler metric in  $c_1(X)$ , since  $\gamma \circ J_\varepsilon$  maps  $K$  into the space of Kähler metrics contained in  $c_1(X)$ . This proves Theorem 1.1.

SEMISTABILITY OF  $T_X$  (1.3 (i)). Let  $\mathcal{F} \subset \mathcal{O}(T_X)$  be a coherent subsheaf such that  $\mathcal{O}(T_X)/\mathcal{F}$  is torsion free. We can view  $\mathcal{F}$  as a holomorphic subbundle of  $T_X$  outside of an algebraic subset of codimension 2 in  $X$ . The Chern curvature tensor  $\Theta_{T_X, \omega_\varepsilon} = \frac{i}{2\pi} \nabla^2$  satisfies the Hermite-Einstein condition

$$\Theta_{T_X, \omega_\varepsilon} \wedge \omega_\varepsilon^{n-1} = \frac{1}{n} \rho_\varepsilon \omega_\varepsilon^n,$$

where  $\rho_\varepsilon \in C^\infty(X, \text{End}(T_X))$  is the Ricci operator (with the same eigenvalues  $\rho_{j,\varepsilon}$  as  $\text{Ricci}(\omega_\varepsilon)$ ). It is well known that the curvature of a subbundle is always bounded above by the restriction of the full curvature tensor, i.e.  $\Theta_{\mathcal{F}, \omega_\varepsilon} \leq \Theta_{T_X, \omega_\varepsilon|_{\mathcal{F}}}$  (say, in the sense of Griffiths positivity, viewing the curvature tensors as hermitian forms on  $T_X \otimes \mathcal{F}$ ). By taking the trace with respect to  $\omega_\varepsilon$ , we get

$$\Theta_{\mathcal{F}, \omega_\varepsilon} \wedge \omega_\varepsilon^{n-1} \leq \frac{1}{n} \rho_{\varepsilon|_{\mathcal{F}}} \omega_\varepsilon^n.$$

By the minimax principle, if  $r = \text{rank } \mathcal{F}$ , the eigenvalues of  $\rho_{\varepsilon|_{\mathcal{F}}}$  are bounded above by

$$\rho_{n-r+1, \varepsilon} \leq \dots \leq \rho_{n, \varepsilon},$$

hence

$$\int_X c_1(\mathcal{F}) \wedge \omega_\varepsilon^{n-1} = \int_X \operatorname{tr}_{\mathcal{F}} \Theta_{\mathcal{F}, \omega_\varepsilon} \wedge \omega_\varepsilon^{n-1} \leq \frac{1}{n} \int_X \sum_{j=1}^r \rho_{n-r+j, \varepsilon} \omega_\varepsilon^n.$$

The left hand side is unchanged if we replace  $\omega_\varepsilon$  by  $\beta \in c_1(X)$ , and our assumption  $(**)^p$  for  $p = 1$  (cf. Definition 1.2) implies

$$\int_X c_1(\mathcal{F}) \wedge \beta^{n-1} \leq \lim_{\varepsilon \rightarrow 0} \frac{1}{n} \int_X \sum_{j=1}^r \rho_{n-r+j, \varepsilon} \omega_\varepsilon^n = \lim_{\varepsilon \rightarrow 0} \frac{r}{n^2} \int_X \sum_{j=1}^n \rho_{j, \varepsilon} \omega_\varepsilon^n = \frac{r}{n} \int_X c_1(X) \wedge \beta^{n-1}$$

by (\*). This means that  $T_X$  is  $c_1(X)$ -semistable.

CHERN CLASS INEQUALITY (1.3 (ii)). Let us write

$$\Theta_{T_X, \omega_\varepsilon} = (\theta_{\alpha\beta})_{1 \leq \alpha, \beta \leq n} = \left( \sum_{j,k} \theta_{\alpha\beta jk} dz_j \wedge d\bar{z}_k \right)_{1 \leq \alpha, \beta \leq n}$$

as an  $n \times n$  matrix of  $(1, 1)$ -forms with respect to an orthonormal frame of  $T_X$  that diagonalizes the Ricci operator  $\rho_\varepsilon$  (with respect to the metric  $\omega_\varepsilon$ ). A standard calculation yields

$$\begin{aligned} & 2 \left[ n c_1(X)_h^2 - (2n+2) c_2(X) \right] \wedge \frac{\omega_\varepsilon^{n-2}}{(n-2)!} \\ &= \left( - \sum_{\alpha \neq \beta, j \neq k} |\theta_{\alpha\alpha jk} - \theta_{\beta\beta jk}|^2 - (n+1) \sum_{\alpha, \beta, j, k, \text{pairwise } \neq} |\theta_{\alpha\beta jk}|^2 \right. \\ &\quad - (4n+8) \sum_{\alpha \neq j < k \neq \alpha} |\theta_{\alpha\alpha jk}|^2 - 4 \sum_{\alpha \neq j} |\theta_{\alpha\alpha\alpha j}|^2 \\ &\quad - \sum_{\alpha \neq \beta \neq j \neq \alpha} |\theta_{\alpha\alpha j j} - \theta_{\beta\beta j j}|^2 - \sum_{\alpha \neq \beta} |\theta_{\alpha\alpha\alpha\alpha} - 2\theta_{\alpha\alpha\beta\beta}|^2 \\ &\quad \left. + n \sum_{\alpha} \rho_{\alpha, \varepsilon}^2 - \left( \sum_{\alpha} \rho_{\alpha, \varepsilon} \right)^2 \right) \omega_\varepsilon^n \end{aligned}$$

where all terms in the summation are nonpositive except the ones involving the Ricci eigenvalues  $\rho_{\alpha, \varepsilon}$ . However, the assumption  $(**)^p$  for  $p \geq 2$  implies that the integral of the difference appearing in the last line converges to 0 as  $\varepsilon \rightarrow 0$ . Theorem 1.3 is proved.

Our results lead to the following interesting question.

**Question 2.2.** *Assuming that a “sufficiently good” family of regularization operators  $J_\varepsilon$  is used, can one infer that the resulting family  $(\omega_\varepsilon)$  of fixed points such that  $\operatorname{Ricci}(\omega_\varepsilon) = J_\varepsilon(\omega_\varepsilon)$  (or some subsequence) is well behaved in  $L^1$ , resp.  $L^2$  norm?*

**Remark 2.3.** A strategy to attack the Campana-Peternell conjecture could be as follows. The first step would be to prove that  $H^0(X, T_X) \neq 0$  (if  $\dim X > 0$ ). Assume therefore  $H^0(X, T_X) = 0$  and try to reach a contradiction. It is well known in that case that there are much fewer obstructions to the existence of Kähler-Einstein metrics, for instance one has the Bando-Mabuchi uniqueness theorem [BM85] and (obviously) the vanishing of the classical Futaki invariant [Fut83]; hence we could expect our family of weakly almost Kähler-Einstein metrics to converge to a genuine Kähler-Einstein metric. But then the resulting known Chern class inequalities (the ones of Theorem 1.3 (ii) combined with the Fulton-Lazarsfeld

inequalities [FL83], [DPS94]) might help to contradict  $H^0(X, T_X) = 0$ , e.g. by the Riemann-Roch formula or by ad hoc nonvanishing theorems such as the generalized hard Lefschetz theorem [DPS01]. Now, the existence of holomorphic vector fields on  $X$  implies that  $X$  has a non trivial group of automorphisms  $H = \text{Aut}(X)$ , and one could then try to apply induction on dimension on a suitable desingularization  $Y$  of  $X // H$ , if  $Y$  is not reduced to a point and one can achieve  $Y$  to have  $T_Y$  nef.

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