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# Evolution Problem Associated with a Moving Convex Set in a Hilbert Space

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## I. INTRODUCTION

### 1a. Strong Formulation

The following problem arises from the theory of *elastoplastic* mechanical systems [14. 17].

Let  $H$  be a real Hilbert space; let  $I$  be an interval of  $\mathbf{R}$  containing its origin  $t_0$  but not necessarily bounded nor closed on the right. One gives a *multifunction* (i.e., a set-valued mapping)  $t \mapsto C(t)$  from  $I$  into  $H$ , such that the sets  $C(t)$  are *nonempty closed and convex*. When the language of kinematics is used,  $t$  is interpreted as the time and  $C$  is called a *moving set*. Denote by  $x \mapsto \psi(t, x)$  the *indicator function* of  $C(t)$  (i.e.,  $\psi(t, x) = 0$  if  $x \in C(t)$  and  $+\infty$  otherwise).

*Given  $a \in C(t_0)$ , the problem is that of finding a (single-valued) mapping  $u: I \rightarrow H$ , absolutely continuous on every compact subinterval of  $I$ , such that*

$$u(t_0) = a \tag{1.1}$$

*and that, for almost every  $t$  in  $I$ ,*

$$-du/dt \in \partial\psi(t, u(t)). \tag{1.2}$$

Recall that the set  $\partial\psi(t, x)$ , the *subdifferential* of the convex function  $x \mapsto \psi(t, x)$  at the point  $x$ , equals the closed convex cone formed by the elements of  $H$  which are, in a classical sense, *outward normal* to the convex set  $C(t)$  at the point  $x$ . This cone is empty if and only if  $x \notin C(t)$ ; when  $x \in C(t)$  it contains at least the zero of  $H$  and may reduce to that single element (for instance, if  $x$  is internal to  $C(t)$ ).

The evolution process defined by condition (1.2) may be depicted in a mechanical language, especially clear if  $C(t)$  possesses a nonempty interior: The moving point  $t \mapsto u(t)$  remains at rest as long as it happens to lie in this interior; when caught up with by the boundary of the moving set, it can only proceed in an inward normal direction, as if pushed by this boundary, so as to go on belonging to  $C(t)$ .

Because of this image, we shall refer to a function  $u$  satisfying (1.1) and (1.2) as a *solution of the sweeping process by the moving set  $C$ , for the initial value  $a$* .

Another vivid mechanical interpretation holds when the space  $H$  has the dimension 2. In that case let us picture the product  $H \times \mathbf{R}$  as the physical three-dimensional space, with  $\mathbf{R}$  corresponding to a vertical axis oriented downward. Imagine the set  $G = \{(x, t) \in H \times \mathbf{R} : x \in C(t)\}$  as a solid cavity, supposed to have a smooth boundary, and the curve  $\gamma = \{(x, t) \in H \times \mathbf{R} : x = u(t)\}$ , i.e., the graph of the unknown function  $u$ , as a tiny stationary waterstream falling down the cavity. Condition (1.2) is equivalent to the statements: (i) any arc of this stream which happens to be loose from the cavity wall is rectilinear and vertical; (ii) when water is running over the wall, it describes a line orthogonal to the level curves of the wall surface, i.e., a *line of steepest descent*; this agrees with hydrodynamics under the simplifying assumption that inertia may be neglected comparatively to friction and gravity; (iii) the minus sign in (1.2) precisely expresses that such a flow in contact with the wall may take place only on a part of this wall exposed upward; if it crosses the rim of a possible overhang, water will get loose from the surface and fall vertically down as formulated in (i).

The preceding are two examples of mechanical problems with *unilateral* requirements. As a matter of fact the convexity of the set  $C(t)$  appears in these two examples as a mere mathematical convenience. On the contrary, in the more sophisticated problems arising from the dynamics of mechanical systems with perfect unilateral constraints [12, 13] and whose mathematical treatment is closely related to the study of (1.2), convexity is involved in an essential way; such is also the case in the application to plasticity [17] which constitutes our main motivation.

Condition (1.2) is a special case of

$$-du/dt \in \partial\varphi(t, u(t)), \quad (1.3)$$

where  $\varphi(t, \cdot) : H \rightarrow ]-\infty, +\infty]$  denotes, for each  $t$ , a convex, l.s.c. function; this itself is a special case of

$$-du/dt \in A(t, u(t)), \quad (1.4)$$

where  $A(t, \cdot)$  is, for each  $t$ , a multifunction  $H \rightarrow H$ , *maximal monotone* in the sense of Minty. By elementary calculation, the monotonicity of  $A$  implies that for every two locally absolutely continuous solutions  $u_1$  and  $u_2$  of (1.4), the real function  $t \mapsto \|u_1(t) - u_2(t)\|$  is nonincreasing (in all the paper we shall denote by  $\|\cdot\|$  the norm in  $H$ ); thus at most one of such solutions can agree with some initial condition. At the time the author delivered his first multigraph seminar report on the subject [15, 16], evolution "equations" (1.3) or (1.4) were only studied under stringent assumptions, involving that the "domain"  $\text{dom } A(t, \cdot) = \{x \in H : A(t, x) \neq \emptyset\}$  was independent of  $t$ . The case where  $A$  does not depend on  $t$  amounts to the generation of semigroups of nonlinear contractions in  $H$

[5, 8, 11]. Concerning (1.2), on the contrary, nothing remains to study if the set  $\text{dom } \partial\psi(t, \cdot) = C(t)$  is constant. Thus one of the motives for a thorough investigation of (1.2) is that it displays, so to speak in a pure state, the difficulties raised by an evolutive domain.

Another specific feature of (1.2) is that the right member is a *conical* subset of  $H$ , i.e., it is invariant under the multiplication by positive real numbers; one easily checks [21] that every monotone multifunction possessing this property is actually contained in some  $\partial\psi$ , thus equal to  $\partial\psi$  if it is maximal monotone. The main consequence of the right member being a cone is the invariance of (1.2) under absolutely continuous increasing changes of variable. In the language of mechanics this expresses that the chain of the successive positions of the moving point  $u$  are associated with the chain of the successive configurations of the moving set  $C$  in a way which does not depend on the timing. This more generally is a typical feature of the *quasi-static* evolution of elastoplastic systems. in the study of which (1.2) plays a key part. Mathematically, performing such changes of timing amounts to *remove the privilege of the Lebesgue measure*. In fact the last section of this paper shows how the replacement of the Lebesgue measure by nonnegative real measures, possibly possessing some atoms, provides the natural extension of the formulation (1.2) to *discontinuous* motions, a case of mechanical importance.

Most of the work devoted to Eqs. (1.3) or (1.4) rests on the technique of *Yosida regularization*. In the special case of (1.2) this technique amounts to replacing  $\psi$  by some *penalty function* of the set  $C(t)$ ; when (1.2) figures as an intermediate step in the solution of some elastoplastic problem, Yosida technique finally results in the consideration of some *viscosity* which may be physically meaningful; concerning this aspect, the reader will refer to [17], where an unfamiliar extension of Yosida regularization is also developed: By making the Yosida coefficient vary abruptly with time, one may establish in the same line the convergence of an algorithm of time discretization.

No use of regularization is made in the present paper; some evolved technique of time discretization is developed instead which provides deeper results. The efficiency of this method may be ascribed to the fact that it relies on the ordering of the successive configurations of  $C(t)$  rather than on the proper timing and also that the essentially *unilateral* character of the process is taken into account.

Concerning the evolution problems of the general form (1.3) or (1.4) the reader may refer to [1, 2, 9, 10].

On the other hand a *stochastic* version of (1.2) is studied in [6, 7].

Some nonclassical concepts have to be summarized before we state the results of the present paper.

### 1b *Retraction of a Multifunction*

The existence of a locally absolutely continuous solution of (1.1), (1.2) obviously requires a certain regularity of the given multifunction  $t \mapsto C(t)$ .

In the first approach [15], the author assumed that this multifunction was locally absolutely continuous in the sense of the *Hausdorff distance* between subsets of  $H$ . This is actually a requirement of useless strength: if, on some interval  $[t_1, t_1 + \epsilon]$ , the multifunction  $C$  is nondecreasing, or even if  $t \mapsto V \cap C(t)$  is so, where  $V$  denotes some neighborhood of a point  $x_1 \in C(t_1)$ , visibly the constant function  $t \mapsto u(t) = x_1$  is a solution of (1.2) on this interval. Some regularity for the motion of the set  $C$  is needed only when this set *retracts*, effectively sweeping the point  $u$ .

This induces to some *unilateral* rating for the displacement of sets. For two subsets  $A, B$  of an arbitrary metric space  $(E, \delta)$  let us consider the dissymmetric *ecart*

$$e(A, B) = \sup_{a \in A} \inf_{b \in B} \delta(a, b), \quad (1.5)$$

where the supremum and the infimum are relative to the ordering of  $[0, +\infty]$ ; this entails, for instance,  $e(\emptyset, B) = 0$ . Clearly  $e(A, B) = 0$  if and only if  $A$  is contained in  $\text{cl } B$ , the closure of  $B$ . The Hausdorff “distance”  $h(A, B)$  equals the maximum of  $e(A, B)$  and  $e(B, A)$ .

For any three subsets of  $(E, \delta)$ , one proves that the corresponding values of  $e$  verify the “triangle inequality”. Let  $t \mapsto C(t)$  denote a multifunction defined on a real interval  $I$ , whose values are subsets of  $(E, \delta)$ . Let  $[s, t] \subset I$ ; by considering all the finite sequences  $(\tau_i)$  of the form  $s = \tau_0 \leq \dots \leq \tau_n = t$  and taking the supremum of  $\sum_{i=1}^n e(C(\tau_{i-1}), C(\tau_i))$  one defines the *retraction* of the multifunction  $C$  over the interval  $[s, t]$ ; notation  $\text{ret}(C; s, t)$ . This is zero if and only if the multifunction  $\tau \mapsto \text{cl } C(\tau)$  is nondecreasing everywhere in  $[s, t]$ . Obviously the *variation*, similarly defined by means of the Hausdorff distance, majorizes  $\text{ret}(C; s, t)$ .

If  $\text{ret}(C; s, t)$  is finite for every  $[s, t] \subset I$  the multifunction is said to have a *finite retraction* on  $I$ ; let us make this assumption for all the following. Equivalently there exists a nondecreasing real function  $r$  defined on  $I$  up to an additive constant, such that for every  $s \leq t$

$$\text{ret}(C; s, t) = r(t) - r(s).$$

In all the paper,  $I$  is supposed to possess an origin  $t_0$  contained in it and we shall specify  $r$ , the *retraction function* of  $C$ , as  $r(t) = \text{ret}(C; t_0, t)$ .

The reader will refer to [19] for the general study of this concept; we only quote here the fact to be used in the sequel.

For every  $s \leq t$  in  $I$  one has

$$e(C(s), C(t)) \leq r(t) - r(s). \quad (1.6)$$

As this is finite, we conclude by making  $s = t_0$  that the assumption on  $C(t_0) \neq \emptyset$  implies  $C(t) \neq \emptyset$  for every  $t$  in  $I$ .

From  $r$  being finite it also ensues [19, Proposition 4.a] that when  $\tau$  tends to  $t$  from the right, the multifunction  $\tau \mapsto C(\tau)$  possesses a *limit* in the classical sense that the two sets

$$\liminf_{\tau \downarrow t} C(\tau) = \{x \in E: \lim_{\tau \downarrow t} \delta(x, C(\tau)) = 0\},$$

$$\limsup_{\tau \downarrow t} C(\tau) = \{x \in E: 0 \text{ is a cluster value of the function } \tau \mapsto \delta(x, C(\tau)) \text{ for } \tau \downarrow t\}$$

are *equal* to a closed, possibly empty, subset of  $E$  which will be denoted by  $C^+(t)$ .

Actually the assumptions made in this paper, i.e.,  $(E, \delta)$  is a Hilbert space and the sets  $C(t)$  are closed convex and nonempty, entail [19, Propositions 4.b and 4.d] that  $C^+(t)$  is nonempty convex, and that

$$e(C(t), C^+(t)) = \lim_{\tau \downarrow t} e(C(t), C(\tau)) = r^+(t) - r(t) \quad (1.7)$$

where  $r^+(t)$  stands for  $\lim_{\tau \downarrow t} r(\tau)$ .

### 1c. *Local-Step Functions and Local-Step Multifunctions*

We still suppose that the interval  $I$  possesses an origin  $t_0$  and contains it. Let us denote by  $\text{lfp}(I)$  the set of the elements  $P$  described as follows:  $P$  is a *partition* of  $I$  into a family of subintervals of any sort (some of them possibly reduced to single points) such that every compact subset of  $I$  is covered by finitely many members of this partition; such a partition may be called *locally finite*. Then  $P$  is a countable or finite family of intervals which will be indexed in concordance with their succession in  $I$ , namely,

$$P: I_0, I_1, \dots, I_i, \dots \quad (1.8)$$

The origin of  $I_0$  is  $t_0$ , the origin of  $I$ , necessarily contained in it; call  $t_i$  the origin of  $I_i$ , nonnecessarily belonging to this interval.

A function (resp. multifunction) defined on  $I$  is said to be *local-step* if it is constant on each member of some  $P \in \text{lfp}(I)$ . If  $F$  is such a multifunction of  $I$  into a metric space, the retraction of  $F$  over a compact subinterval  $[s, t]$  of  $I$  is easily constructed: call  $J_0, J_1, \dots, J_m$ , indexed according to their succession, those of the intersections  $[s, t] \cap I_i$  which are nonempty: for each  $j = 0, 1, \dots, m$ , choose  $\sigma_j \in J_j$ ; then

$$\text{ret}(F; s, t) = \sum_{j=1}^m e(F(\sigma_{j-1}), F(\sigma_j)). \quad (1.9)$$

Starting now from the multifunction  $C$  into the Hilbert space  $H$ , considered in the preceding section, and from  $P \in \text{lfp}(I)$  as described in (1.8), we shall

denote by  $C_P$  the local-step multifunction taking on each member  $I_i$  of  $P$  the constant value determined as follows:

$$C_P(I_i) = \begin{cases} C(t_i) & \text{if } t_i \in I_i, \\ C^+(t_i) & \text{if } t_i \notin I_i. \end{cases} \quad (1.10)$$

Using (1.7) and (1.9) some *approximation property* may be established [19, Proposition 5.c]: *Suppose the oscillation of the function  $r$  over every member of  $P$  is  $\leq \epsilon$ ; then*

$$\forall t \in I: \quad e(C_P(t), C(t)) \leq \epsilon, \quad (1.11)$$

$$\forall [s, t] \subset I: \quad \text{ret}(C_P; s, t) \leq \text{ret}(C; s, t) + \epsilon \quad (1.12)$$

(if  $s = t_0$  the term  $\epsilon$  in (1.12) may be dropped).

Clearly, since the function  $r$  is nondecreasing, a  $P \in \text{lfp}(I)$  satisfying the above condition may be obtained by taking the inverse image under  $r$  of a locally finite partition of  $\mathbf{R}$  into subintervals with length  $\leq \epsilon$ .

#### 1d. Results of Part 2

The preceding approximation property leads, in Part 2 of this paper, to a definition and study of the concept of a *weak solution* of the sweeping process by  $C$ , when  $C$  is a multifunction with closed convex values in  $H$  and finite retraction. With every  $P \in \text{lfp}(I)$  is associated as above the local-step multifunction  $C_P$ ; quite naturally, in view of the mechanical image presented in Section 1a, some local-step function  $u_P: I \rightarrow H$  is considered as the (weak) solution of the sweeping process by  $C_P$  for the initial value  $a \in C(t_0)$ . By definition,  $u_P$  takes on each member  $I_i$  of  $P$  the constant value  $u_i$  inductively determined by

$$u_0 = a, \quad (1.13)$$

$$u_{i+1} = \text{proj}(u_i, C_{i+1}) \quad (1.14)$$

(i.e.,  $u_{i+1}$  is the nearest point to  $u_i$  in the nonempty closed convex set  $C_{i+1} = C_P(I_{i+1})$ ). Now observe that the set  $\text{lfp}(I)$  is *directed* by the following partial order. One writes  $P < P'$  if the partition  $P'$  is a *refinement* of  $P$  (i.e., every member of  $P'$  is contained in a member of  $P$ ). This makes of the family  $(u_P)$ ,  $P \in \text{lfp}(I)$ , of mappings  $I \rightarrow H$  a net. Proposition 2a states that, under the assumptions made above, this net converges uniformly on  $I$ . The limit  $u$  constitutes by definition the weak solution of the sweeping process by  $C$  for the initial value  $a \in C(t_0)$ . The subsequent Sections of Part 2 are devoted to studying the properties of such weak solutions. Proposition 2b states that the flow of the weak solutions of the sweeping process by  $C$  is *nonexpansive*; as we have recalled before, the same is a familiar feature when (locally absolutely continuous) solutions of (1.4) are

considered. Proposition 3b states that  $u(t) \in C(t)$  for every  $t \in I$  and that the *variation*  $\text{var}(u; s, t)$  of  $u$  over every compact subinterval  $[s, t]$  of  $I$  is majorized by  $\text{ret}(C; s, t)$ ; hence, if the real function  $r$  is continuous, resp. locally absolutely continuous, resp. Lipschitz with ratio  $k$ , so is  $u$  with regard to the norm of  $H$ . Local absolute continuity, for a function  $u$  with values in a reflexive Banach space is known [11] to imply the existence of the strong derivative  $du/dt$  for almost every  $t \in I$ ; in fact it will be established at the end of the paper that in this case of a locally absolutely continuous  $r$ , the weak solution is actually *strong*, i.e., it satisfies (1.2).

At the discontinuity points of  $r$ , the function  $u$  may itself possess discontinuities; precise jump relations, from the left and from the right, are established in Sections 2d and 2e.

The differential condition (1.2) is of *local* character; the object of Section 2f, devoted to restrictions and piecings, is to exhibit the same character for weak solutions.

The physical applicability of the theory requires the *stability* of the solution with regard to the possible alterations of the data. Proposition 2g makes precise this stability by formulating a majoration of  $|u - u'|$ , where  $u$  and  $u'$  are weak solutions corresponding to two different multifunctions  $C$  and  $C'$  and different initial values. This inequality shall play an essential part when the sweeping process is inserted as a component in some more complicated mathematical problems arising from mechanics.

As stressed in Section 1a, the sweeping process, in its strong formulation, is invariant under nondecreasing change of variables. Section 2i establishes that the same holds for weak solutions.

From the computational standpoint, a natural way of approximately solving (1.1), (1.2) would consist in *time discretizations*. One chooses a sequence  $t_0 < t_1 < \dots < t_i < \dots$  of points of  $I$ ; the quotient  $(u(t_{i+1}) - u(t_i))(t_{i+1} - t_i)^{-1}$  is taken as an approximant of the derivative  $du/dt$  at the point  $t_{i+1}$ . Substituting it in (1.2) yields an inductive relation of the *implicit type*

$$\frac{u(t_i) - u(t_{i+1})}{t_{i+1} - t_i} \in \partial\psi(t_{i+1}, u(t_{i+1})).$$

By the elementary convex analysis, this is equivalent to

$$u(t_{i+1}) = \text{proj}(u(t_i), C(t_{i+1})).$$

Starting from  $u(t_0) = a$  the sequence  $u(t_i)$  is determined inductively. It may be said that, instead of being continuously swept by the moving set  $C$ , the moving point  $u$  is left at rest except that, at the chosen instants  $t_i$ , it catches up with this set instantly, by the shortest way, hence the name the *catching-up algorithm* we propose to give to this computation procedure. Section 2j makes precise the

connection between this algorithm and the construction of the local-step functions  $u_p$ . A majorization of  $|u_p - u|$  has been established in Section 2h; it applies in particular to the catching-up algorithm and proves its convergence.

Actually the catching-up algorithm works properly in the case where the function  $r$  is *right continuous*, a situation which receives a special emphasis in Part 3 of this paper.

### 1e. *Results of Part 3*

In view of Proposition 2c the weak solutions of the sweeping process are mappings from  $I$  into  $H$  with *bounded variation* over every compact subinterval of  $I$ . To such a mapping  $u$  classically corresponds an  $H$ -valued vector measure  $du$ , which we call its *differential measure*. The purpose of this last part of the paper is to investigate whether this concept of differential yields a characterization of the considered solution by a condition of the same form as (1.2). In fact, in the locally absolutely continuous case, the derivative  $du/dt$  is nothing but the density of the vector measure  $du$  relative to the Lebesgue measure  $dt$  of  $I$ , and we stressed in Section 1a that the considered changes of variables amounted to remove the privilege of the Lebesgue measure. Unfortunately, to the difference with the conventional continuous case, the differential measure of a discontinuous function  $u$  with bounded variation does not determine the function up to an additive constant. It turns out that the special case where the retraction function  $r$  is *right continuous*, implying right continuity for all the weak solutions of the sweeping process, provides the correct framework. In that case, Propositions 3a and 3b state that the weak solution  $u$  agreeing with the initial condition  $u(t_0) = a \in C(t_0)$  is the only right-continuous function, with locally bounded variation, agreeing with this initial condition, and possessing the following property: There exists (nonuniquely) a nonnegative real measure  $d\mu$  on  $I$  and a function  $u' \in \mathcal{L}_{loc}^1(d\mu, H)$  such that  $du = u'd\mu$  and that

$$-u'(t) \in \partial\psi(t, u(t))$$

for every  $t$  in  $I$ .

Actually  $dr$  may be taken as  $d\mu$  and, finally, Proposition 3c states as a corollary that, if  $r$  is locally absolutely continuous, the weak solutions of the sweeping process are effectively *strong*, i.e., they satisfy (1.2).

## 2. WEAK SOLUTIONS

### 2a. *Definition and Existence*

Recall that the real interval  $I$  is always supposed to contain its origin  $t_0$ ; using the notations defined in Sections 1c and 1d, one has:

PROPOSITION (2a). *Let  $C$  be a multifunction defined on  $I$  with closed convex values in the Hilbert space  $H$  and finite retraction; let  $a \in C(t_0)$  (this implies  $C(t_0) \neq \emptyset$ , thus  $C(t) \neq \emptyset$  for every  $t \in I$  since the retraction is finite). With every  $P \in \text{lfp}(I)$  described as in (1.8), let us associate the local-step multifunction  $C_P$  defined by (1.10). To  $C_P$  corresponds the local-step function  $u_P: I \rightarrow H$  inductively constructed by means of (1.13) and (1.14). Then the net of the functions  $(u_P)$ ,  $P \in \text{lfp}(I)$ , converges uniformly on  $I$ .*

DEFINITION. *The limit  $u$  in the above Proposition is called the weak solution of the sweeping process for the multifunction (or moving set)  $C$  and the initial value  $a$ .*

We shall base the proof on two lemmas.

LEMMA 1(2a). *Let  $\Gamma$  denote a closed convex nonempty subset of  $H$ ; denoting by  $\delta(x, \Gamma)$  and  $\delta(y, \Gamma)$  the distances from two arbitrary points  $x, y$  to this set, one has*

$$|x - \text{proj}(y, \Gamma)|^2 - |x - y|^2 \leq 2\delta(x, \Gamma)\delta(y, \Gamma). \quad (2.1)$$

*Proof.* This might be deduced from Lemma 2g below; more directly, by putting  $x' = \text{proj}(x, \Gamma)$  and  $y' = \text{proj}(y, \Gamma)$ , the left member of this inequality becomes

$$(y - y' | 2x - y - y') = 2(y - y' | x - x') + 2(y - y' | x' - y') - |y - y'|^2,$$

where  $( | )$  represents the scalar product in  $H$ . The first term on the right is less than or equal to  $2\delta(x, \Gamma)\delta(y, \Gamma)$  and, as  $x' \in \Gamma$ , the second term is  $\leq 0$  by virtue of a classical characterization of  $\text{proj}(y, \Gamma)$ .

LEMMA 2(2a). *Let  $I_i$  be one of the intervals composing  $P$ , with origin  $t_i$  (non-necessarily belonging to it). Let  $P' \in \text{lfp}(I)$  be a refinement of  $P$ ; denote by  $u_P$  and  $u_{P'}$  the local-step functions respectively corresponding in the same way as above to  $C_P$  and  $C_{P'}$  and to arbitrary initial values. Denote by  $J_1, J_2, \dots$ , indexed according to their succession in  $\mathbf{R}$ , the intervals of  $P'$  which are contained in  $I_i$ . In the event of  $I_i$  not being the last interval of  $P$ , these intervals necessarily form a finite sequence  $J_1, J_2, \dots, J_m$ ; in that case denote by  $J_{m+1}$  the interval of  $P'$  following  $J_m$ . With these notations, whichever are the real numbers  $\sigma_i \in J_1$  and  $\tau_i \in I_i \cup J_{m+1}$  such that  $\sigma_i \leq \tau_i$ , one has*

$$|u_P(\tau_i) - u_{P'}(\tau_i)|^2 - |u_P(\sigma_i) - u_{P'}(\sigma_i)|^2 \leq 2\rho_i^2 \quad (2.2)$$

with

$$\rho_i = \sup\{\text{ret}(C_{P'}; \theta, \theta'): [\theta, \theta'] \subset I_i\}.$$

*Proof.* First let us establish (2.2) in the event  $\tau_i \in J_{m+1}$ . Denote by  $x_i$  and

$C_i$ , respectively, the constant values of  $u_p$  and  $C_p$  on the interval  $I_i$ ; denote by  $y_j$  and  $D_j$  the constant values of  $u_{p'}$  and  $C_{p'}$  on the interval  $J_j$ . The definition of  $u_{p'}$  implies

$$y_{j+1} = \text{proj}(y_j, D_{j+1});$$

hence, by inequality (2.1),

$$|x_i - y_{j+1}|^2 - |x_i - y_j|^2 \leq 2\delta(x_i, D_{j+1})\delta(y_j, D_{j+1}). \quad (2.3)$$

By making  $j = 1, 2, \dots, m - 1$  and adding, one obtains

$$|x_i - y_m|^2 - |x_i - y_1|^2 \leq 2 \sum_{j=1}^{m-1} \delta(x_i, D_{j+1})\delta(y_j, D_{j+1}). \quad (2.4)$$

In the event  $m = 1$  interpret the right member as zero. The definitions of  $C_p$  and  $C_{p'}$  imply that  $C_{i+1}$  and  $D_{m+1}$  are the same set (namely,  $C(t_{i+1})$  if  $t_{i+1} \in I_{i+1}$  and  $C^+(t_{i+1})$  if  $t_{i+1} \notin I_{i+1}$ ). As the projection on this convex set is a nonexpanding mapping, one has  $|x_{i+1} - y_{m+1}|^2 \leq |x_i - y_m|^2$ . It follows that the left member of (2.2) is less than or equal to the right member of (2.4). Recall the expression given in (1.9) for the retraction of a local-step multifunction; since  $x_i$  belongs to  $C_i = D_1$ , one has, for every  $j = 1, 2, \dots, m - 1$ ,

$$\delta(x_i, D_{j+1}) \leq e(D_1, D_{j+1}) \leq \sup\{\text{ret}(C_{p'}; \theta, \theta') : [\theta, \theta'] \subset I_i\}.$$

Therefore the right member of (2.4) is less than or equal to the expressions

$$2\rho_i \sum_{j=1}^{m-1} e(D_j, D_{j+1}) \leq 2\rho_i^2. \quad (2.5)$$

This proves (2.2) in the considered case.

It remains to adapt this proof to the case  $\tau_i \in I_i$ ; then call  $\mu$  the value of  $j$  such that  $\tau_i \in J_j$  (no matter if  $I_i$  happens to be the last interval of  $P$  and the sequence  $(J_j)$  infinite). Add member to member the inequalities of the form (2.3), with  $j = 1, 2, \dots, \mu - 1$ . On the left this yields the first member of (2.2) and on the right the same result as in (2.4), except for replacing  $m$  by  $\mu$ ; again this right member is majorized by the right member of (2.5).

*Proof of the Proposition.* Due to  $H$  being complete, we are to establish that, for every  $\epsilon > 0$ , there exists  $P \in \text{lfp}(I)$  such that, for every  $P'$  and  $P''$  greater than  $P$  in the directed set  $\text{lfp}(I)$  (i.e., refinements of  $P$ ), the corresponding functions  $u_{p'}$  and  $u_{p''}$ , agreeing with the initial value  $a$ , verify for every  $t \in I$

$$|u_{p'}(t) - u_{p''}(t)| \leq \epsilon.$$

Let  $\beta$  be a strictly positive number and let  $\gamma_0, \gamma_1, \dots, \gamma_n, \dots$  be an infinite sequence of strictly positive numbers such that  $\sum_{n=0}^{\infty} \gamma_n = +\infty$  and  $\sum_{n=0}^{\infty} (\gamma_n)^2 \leq \beta$ . The semiopen intervals

$$[0, \gamma_0[, [\gamma_0, \gamma_0 + \gamma_1[, \dots, \left[ \sum_{n=0}^k \gamma_n, \sum_{n=0}^{k+1} \gamma_n \right], \dots$$

form a partition of  $\mathbf{R}_+$ . As the function  $r: t \mapsto \text{ret}(C; t_0, t)$  is nondecreasing the inverse images of these intervals by  $r$  are subintervals of  $I$  (of any sort; some of them possibly empty or reduced to singletons), and among these inverse images, those which are nonempty constitute a locally finite partition of  $I$ ; such we specify from now on the partition  $P: I_0, I_1, I_2, \dots$  considered in the foregoing.

By this construction of  $P$ , the oscillations of the real function  $r$  over the intervals  $I_0, I_1, I_2, \dots$  are, respectively, majorized by some numbers  $\eta_0, \eta_1, \eta_2, \dots$  forming a (possibly finite) subsequence of the sequence  $(\gamma_n)$ ; thus

$$\sum (\eta_i)^2 \leq \beta. \quad (2.6)$$

The retraction of  $C$  over any subinterval of  $I_i$  is majorized by  $\eta_i$ ; in view of (1.12) this implies that the retraction of  $C_{P'}$  over such a subinterval of  $I_i$  is majorized by  $2\eta_i$ ; thus inequality (2.2) holds, with  $\rho_i$  replaced by  $2\eta_i$ .

Let  $t \in I$ ; there exists  $\nu$  such that  $t \in I_\nu$ . Fix the  $\sigma_i$  and  $\tau_i$  involved in Lemma 2 according to the rules

$$\begin{aligned} \text{for } i = 0: & \quad \sigma_0 = t_0, \\ \text{for } 0 < i < \nu: & \quad \tau_i = \sigma_{i+1}; \\ \text{for } i = \nu: & \quad \tau_\nu = t. \end{aligned}$$

Then, adding member to member the inequalities of the form (2.2) for  $i = 0, 1, \dots, \nu$  yields

$$|u_P(t) - u_{P'}(t)|^2 - |u_P(t_0) - u_{P'}(t_0)|^2 \leq 8 \sum_{i=0}^{\nu} (\eta_i)^2. \quad (2.7)$$

Since  $u_P(t_0) = u_{P'}(t_0) = a$  and in view of (2.6) this reduces to  $|u_P(t) - u_{P'}(t)| \leq (8\beta)^{1/2}$ . As the same is true for any other  $P'' \in \text{lfp}(I)$  greater than  $P$ , one finally has  $|u_{P'}(t) - u_{P''}(t)| \leq (32\beta)^{1/2}$  for every  $t \in I$ . Since  $\beta$  may be chosen arbitrarily small this establishes the Cauchy property.

From the above proof one might easily derive an upper bound of the error committed when  $u_P$  is considered as an approximant of the weak solution  $u$ ; a better majoration will be found in Section 2h.

## 2b. Nonexpansion

**PROPOSITION 2b.** *If  $u$  and  $u'$  are two weak solutions of the sweeping process by the same moving convex set  $C$ , the real function  $t \mapsto |u(t) - u'(t)|$  is nonincreasing.*

*Proof.* If  $u$  and  $u'$  correspond to initial values  $a$  and  $a'$ , these functions are, respectively, the limits, for  $P \in \text{lfp}(I)$ , of the local-step functions  $u_P$  and  $u'_P$  constructed from these initial values. As the successive values of  $u_P$  and  $u'_P$  are obtained by performing a sequence of projections on the closed convex sets  $C_1, C_2, \dots$ , the nonexpansion property of such projections implies that, for  $s \leq t$ , one has  $|u_P(s) - u'_P(s)| \geq |u_P(t) - u'_P(t)|$ . Take the limits of both members of this inequality for  $P \in \text{lfp}(I)$ .

## 2c. Variation

PROPOSITION 2c. *Let  $u$  be a weak solution of the sweeping process by  $C$  on the interval  $I$ ; then*

$$\forall t \in I: \quad u(t) \in C(t), \quad (2.8)$$

$$\forall [s, t] \subset I: \quad \text{var}(u; s, t) \leq \text{ret}(C; s, t) = r(t) - r(s). \quad (2.9)$$

*Proof.* The construction of  $u_P$  makes that  $u_P(t) \in C_P(t)$ , thus

$$\delta(u_P(t), C(t)) \leq e(C_P(t), C(t)).$$

The approximation property expressed by (1.11) implies that the second member has the limit zero for  $P$  ranging in the directed set  $\text{lfp}(I)$ ; hence (2.8) holds, since  $C(t)$  is closed.

On the other hand, if  $\sigma$  and  $\tau$ , with  $\sigma < \tau$ , respectively, belong to two adjacent intervals of  $P$ , the constructive law (1.14) clearly implies

$$|u_P(\sigma) - u_P(\tau)| \leq \text{ret}(C_P; \sigma, \tau). \quad (2.10)$$

For arbitrary  $\sigma$  and  $\tau$ , with  $\sigma \leq \tau$ , the same is found true by adding the similar inequalities corresponding to intermediate points. Taking the limits for  $P \in \text{lfp}(I)$  of both members of (2.10) yields, in view of (1.12),

$$|u(\sigma) - u(\tau)| \leq \text{ret}(C; \sigma, \tau).$$

Then (2.9) follows from the definition of  $\text{var}(u; s, t)$ .

As an immediate consequence of (2.9) one has:

COROLLARY 2c. *If the real function  $t \mapsto r(t) = \text{ret}(C; t_0, t)$  is continuous, resp. locally absolutely continuous, resp. Lipschitz with ratio  $k$ , such is, relative to the norm of  $H$ , every weak solution  $u$  of the sweeping process by  $C$ .*

Recall that the local absolute continuity of  $u$ , a function with values in a

reflexive Banach space, implies that the strong derivative  $du/dt$  exists for almost every  $t \in I$ . In that situation, (2.9) entails, for almost every  $t \in I$ ,

$$| du/dt | \leq | dr/dt |. \quad (2.11)$$

In the language of kinematics it may be said that the *speed of the moving point*  $u$  is majorized by the *retraction speed* of the moving set  $C$ .

Observe on the other hand that the local absolute continuity of  $r$  is equivalent to the following. For every compact subinterval  $[s, t]$  of  $I$  and every  $\epsilon > 0$ , there exists  $\eta > 0$  such that if  $]\sigma_i, \tau_i[$  is a finite family of nonoverlapping sub-intervals of  $[s, t]$ , one has the implication

$$\sum (\tau_i - \sigma_i) \leq \eta \Rightarrow \sum e(C(\sigma_i), C(\tau_i)) \leq \epsilon.$$

## 2d. Limit from the Left

Let  $t_1 \in I$ , strictly greater than  $t_0$ . As the real function  $r$  is nondecreasing, it possesses for  $t \uparrow t_1$  a finite limit denoted by  $r^-(t_1)$ . Then, from inequality (2.9) one concludes that every weak solution  $t \mapsto u(t)$  of the sweeping process presents the Cauchy property for  $t \uparrow t_1$ , implying the existence of

$$\lim_{t \uparrow t_1} u(t) \text{ denoted by } u^-(t_1).$$

PROPOSITION 2d. *With the above notations*

$$| u(t_1) - u^-(t_1) | \leq r(t_1) - r^-(t_1), \quad (2.12)$$

$$u(t_1) = \text{proj}(u^-(t_1), C(t_1)). \quad (2.13)$$

*Proof.* Inequality (2.12) immediately results from (2.9). For the proof of (2.13), observe that the subset  $\mathcal{P}$  of  $\text{lfp}(I)$ , consisting of partitions in which figures an interval reduced to the singleton  $\{t_1\}$ , is terminal in this directed set. Thus  $u$  may be considered as the limit of the net  $(u_P)$ ,  $P \in \mathcal{P}$ . If  $P \in \mathcal{P}$ , the definition of  $C_P$  makes that  $C_P(t_1) = C(t_1)$  and the construction of the local-step function  $u_P$  implies

$$u_P(t_1) = \text{proj}(u_P^-(t_1), C(t_1)).$$

As the convergence of the net  $(u_P)$ ,  $P \in \mathcal{P}$ , to  $u$  is uniform with regard to  $t$ , the following commutation of limits holds

$$u^-(t_1) = \lim_{P \in \mathcal{P}} u_P^-(t_1).$$

Then (2.13) results from the mapping  $x \mapsto \text{proj}(x, C(t_1))$  being continuous.

*Remark.* Inequality (2.12) shows that the continuity of  $r$  from the left at  $t_1$  implies the same for  $u$ . On the other hand it is easily found [19] that the left continuity of  $r$  is equivalent to

$$\lim_{t \uparrow t_1} e(C(t), C(t_1)) = 0,$$

which in turn amounts to the following: For every  $\omega$ , a neighborhood of zero in  $H$ , there exists  $\eta > 0$  such that

$$t_1 - \eta < t < t_1 \Rightarrow C(t) \subset C(t_1) + \omega.$$

Clearly this property holds in particular when the multifunction  $C$  is *upper semicontinuous from the left* at  $t_1$ , in the classical sense that, for every open set  $\Omega$  containing  $C(t_1)$ , there exists  $\eta > 0$  such that

$$t_1 - \eta < t < t_2 \Rightarrow C(t) \subset \Omega.$$

Statements involving in the same connection the limit set  $C^-(t_1)$  may be found in [19, Sect. 4e].

## 2e. *Limit from the Right*

Let  $t_1 \in I$ , suppose it is not the greatest element of  $I$ ; as before, the limit of  $r$  from the right at the point  $t_1$  is denoted by  $r^+(t_1)$ . Using again inequality (2.9) and the completeness of  $H$ , one concludes to the existence of  $u^+(t_1)$ , the limit of  $u$  from the right at the point  $t_1$ .

PROPOSITION 2e. *With these notations*

$$|u^+(t_1) - u(t_1)| \leq r^+(t_1) - r(t_1), \quad (2.14)$$

$$u^+(t_1) = \text{proj}(u(t_1), C^+(t_1)). \quad (2.15)$$

*Proof.* Inequality (2.14) results immediately from (2.9). As in the preceding section, call  $\mathcal{P}$  the terminal subset of  $\text{lpf}(I)$  consisting of partitions which include the singleton  $\{t_1\}$ . For  $P \in \mathcal{P}$  one has

$$u_P^+(t_1) = \text{proj}(u_P(t_1), C^+(t_1))$$

since the interval of  $P$  following  $\{t_1\}$  is necessarily open at the left. As the convergence of the net  $(u_P)$ ,  $P \in \mathcal{P}$ , is uniform relative to  $t$ , the left member converges to  $u^+(t_1)$ ; then (2.15) results from the continuity of  $x \mapsto \text{proj}(x, C^+(t_1))$ .

*Remark 1.* Using [19, Proposition 4d] and elementary inequalities of Hilbert

space geometry, it may be proved that the right member of (2.15) equals  $\lim_{t \downarrow t_1} \text{proj}(u(t_1), C(t))$ .

*Remark 2.* Inequality (2.14) shows that the continuity from the right for  $r$  implies the same for  $u$ . On the other hand [19, Proposition 4c] this right continuity of  $r$  at the point  $t_1$  is equivalent to  $t_0$

$$\lim_{t \downarrow t_1} e(C(t_1), C(t)) = 0,$$

which in turn is equivalent to  $C(t_1) \subset C^+(t_1)$ .

## 2f. Restriction and Piecing

The purpose of this section is to display the *local* character of the concept of a weak solution.

**PROPOSITION 2f.** *Let  $u: I \rightarrow H$  be a weak solution of the sweeping process by  $C$ . Let  $I'$ , containing its origin  $t_0'$ , be a subinterval of  $I$ . Then the restriction  $u|_{I'}$  is a weak solution of the sweeping process by  $C|_{I'}$ .*

*Proof.* Take a partition  $P_0 \in \text{lfp}(I)$  whose  $I'$  is a member; in  $\text{lfp}(I)$  the set  $\mathcal{P}$  of the refinements of  $P_0$  is terminal; hence  $u$  is the limit of the net  $(u_P)$ ,  $P \in \mathcal{P}$ . Every  $P' \in \text{lfp}(I')$  is the *trace* on  $I'$  of at least one  $P \in \mathcal{P}$ ; we shall write  $P' = \text{tr } P$  defining thereby an order preserving surjective mapping of the directed set  $\mathcal{P}$  onto the directed set  $\text{lfp}(I')$ . According to Section 2a, the definition of  $u': I' \rightarrow H$ , the solution of the sweeping process by  $C|_{I'}$ , for an initial value  $a' \in C(t_0')$  equal to  $u(t_0')$ , rests on the construction of local-step functions taking this value at the origin  $t_0'$  of  $I'$ , each of them corresponding to an element  $P'$  of  $\text{lfp}(I')$ . If  $P' = \text{tr } P$ , let us denote by  $u'_P$  this local-step function; clearly  $u'$  may as well be defined as the limit of the net  $(u'_P)$ ,  $P \in \mathcal{P}$ . The restriction  $u_P|_{I'}$  is a local-step function on  $I'$ , constant on the same intervals as  $u'_P$ : both local-step functions are constructed inductively by successive projections on the same closed convex sets. As such projections are nonexpanding one has

$$\forall t \in I': \quad |u'_P(t) - u_P(t)| \leq |u'_P(t_0') - u_P(t_0')|,$$

with  $u'_P(t_0') = u(t_0')$  by hypothesis. The right member has the limit zero by the definition of  $u$ ; this proves  $u' = u|_{I'}$ .

**COROLLARY.** *Let  $I_0, I_1, \dots, I_i, \dots$  be a locally finite partition of  $I$  into intervals containing their respective origins  $t_0, t_1, \dots, t_i, \dots$  and let  $u: I \rightarrow H$  satisfy the conditions:*

(i) For every  $i$  the restriction of  $u$  to  $I_i$  is a solution of the sweeping process by the restriction of  $C$  to this interval (for  $i > 0$  this implies the existence of  $u^-(t_i)$ ).

(ii) For  $i > 0$

$$u(t_i) = \text{proj}(u^-(t_i), C(t_i)).$$

Then  $u$  is a weak solution of the sweeping process for  $C$  on  $I$ .

In fact there is at most one  $u: I \rightarrow H$  satisfying these conditions. Now, by virtue of the above proposition and of Proposition 2d, the solution of the sweeping process by  $C$  on  $I$ , with the initial value  $u(t_0)$ , actually satisfies them.

To this corollary may be reduced immediately the case of a cover of  $I$  by a sequence of overlapping intervals: A function  $I \rightarrow H$  whose restriction to each of these intervals is a weak solution of the corresponding sweeping process is also a weak solution of the sweeping process with regard to the whole of  $I$ .

## 2g. Alteration of the Data

The following proposition enables us to estimate to what extent some uncertainty about the multifunction  $C$  and the initial value  $a$  affects the corresponding weak solution of the sweeping process.

**PROPOSITION 2g.** *Let  $C$  and  $C'$  be two multifunctions from  $I$  into  $H$ , with closed convex values and finite retractions; let  $u$  and  $u'$  be some respective weak solutions of the sweeping process; let  $\mu(t)$  be a majorant of the Hausdorff distance  $h(C(\theta), C'(\theta))$  for  $\theta \in [t_0, t]$ . Then, for every  $t$  in  $I$ ,*

$$\begin{aligned} & |u(t) - u'(t)|^2 - |u(t_0) - u'(t_0)|^2 \\ & \leq 2\mu(t)(\text{ret}(C; t_0, t) + \text{ret}(C'; t_0, t)). \end{aligned} \quad (2.16)$$

We establish first:

**LEMMA.** *Let  $C_+$  and  $C'_+$  be two nonempty closed convex subsets of  $H$ ; let  $u_-$  and  $u'_-$  be two points of  $H$ ; put  $u_+ = \text{proj}(u_-, C_+)$  and  $u'_+ = \text{proj}(u'_-, C'_+)$ . Then*

$$\begin{aligned} & |u_+ - u'_+|^2 - |u_- - u'_-|^2 \\ & \leq 2|u_+ - u_-|e(C'_+, C_+) + 2|u'_+ - u'_-|e(C_+, C'_+). \end{aligned} \quad (2.17)$$

In fact, if  $\psi_+$  and  $\gamma_+$ , respectively, denote the indicator function and the support function of  $C_+$ , the above definition of  $u_+$  is classically equivalent to  $u_- - u_+ \in \partial\psi_+(u_+)$ , itself equivalent to  $u_+ \in \partial\gamma_+(u_- - u_+)$ , i.e.,

$$\forall y \in H: \quad (u_+ | y - u_- + u_+) + \gamma_+(u_- - u_+) \leq \gamma(y).$$

Apply this with  $y = u_- - u_+$ , then write the same after exchanging primed and nonprimed letters; adding these two inequalities member to member yields

$$\begin{aligned} & (u_+ - u_+' \mid u_+ - u_+' - u_- + u_-') \\ & \leq \gamma_+(u_- - u_+') - \gamma_+'(u_- - u_+') + \gamma_+'(u_- - u_+) - \gamma_+(u_- - u_+). \end{aligned}$$

Twice the left member is easily found to majorize the left member of (2.17). On the other hand the elementary techniques of convex analysis (see, e.g., [19, Sect. 2c]) yield for every  $y \in H$  such that  $\gamma_+(y)$  and  $\gamma_+'(y)$  are finite,

$$\gamma_+(y) - \gamma_+'(y) \leq |y| e(C_+, C_+'),$$

which finally entails inequality (2.17).

*Proof of the Proposition.* The functions  $u$  and  $u'$  are the respective limits of nets  $(u_P)$ ,  $P \in \text{lfp}(I)$ , and  $(u'_P)$ ,  $P \in \text{lfp}(I)$ . Let  $I_i$  and  $I_{i+1}$  denote two consecutive intervals of  $P$ ; by the definition of  $u_P$  one has

$$u_P(I_{i+1}) = \text{proj}(u_P(I_i), C_P(I_{i+1}))$$

and similarly for primed letters; then the lemma yields an inequality of the form (2.17). Let  $m$  be the value of  $i$  such that  $t \in I_i$ ; by adding member to member the inequalities obtained in this way for  $i = 0, 1, \dots, m - 1$ , one gets

$$|u_P(t) - u'_P(t)|^2 - |u_P(t_0) - u'_P(t_0)|^2 \leq 2\mu_P(t)(\text{var}(u_P; t_0, t) + \text{var}(u'_P; t_0, t)).$$

Here  $\mu_P(t)$  denotes a majorant of  $e(C_P(\theta), C'_P(\theta))$  and  $e(C'_P(\theta), C_P(\theta))$  for  $\theta \in [t_0, t]$ . In view of  $C_P$  and  $C'_P$  being defined by (1.10), elementary properties of limit sets show that the number  $\mu(t)$  of the proposition may stand for  $\mu_P(t)$ . As  $P$  ranges over  $\text{lfp}(I)$ , it remains to take the limits of the respective nets and observe that

$$\text{var}(u_P; t_0, t) \leq \text{ret}(C_P; t_0, t) \leq \text{ret}(C; t_0, t)$$

by virtue of (2.9) and (1.12), with similar inequalities for primed letters.

## 2h. Upper Bound of $|u_P - u|$

The following proposition provides a majoration of the error committed when the local-step function  $u_P$  in Proposition 2a is used as an approximant of the weak solution  $u$ . The proof will display a typical application of Proposition 2g.

**PROPOSITION 2h.** *Let  $t$  be fixed in  $I$ ; let  $I_0, I_1, \dots, I_\nu$  be the members of the partition  $P$  which cover the interval  $[t_0, t]$ . Let  $p$  denote a majorant of the oscillation*

of the function  $r$  over every one of these intervals (i.e., a majorant of the retraction of  $C$  over any subinterval of any one of them). Then

$$|u_p(t) - u(t)| \leq 2(p \operatorname{ret}(C; t_0, t))^{1/2}. \quad (2.18)$$

*Proof.* Denote by  $B(\rho)$  the closed ball in  $H$  centered at the origin, with radius  $\rho$ . Let us define a multifunction  $\tau \mapsto D(\tau)$  with closed convex values as follows. Call  $I_i$  the member of  $P$  containing  $\tau$ ; denote by  $r_i$  the “inceptive value” of the function  $r$  in  $I_i$ , i.e.,  $r_i = r(t_i)$  if the origin  $t_i$  of  $I_i$  belongs to this interval and  $r_i = r^+(t_i)$  if not; then put

$$D(\tau) = C(\tau) + B(r(\tau) - r_i).$$

Clearly, if  $t_i \in I_i$  one has  $D(t_i) = C(t_i)$ ; if  $t_i \notin I_i$ , by writing, for every  $x \in H$ ,

$$\delta(x, C(\tau)) \leq \delta(x, D(\tau)) + e(D(\tau), C(\tau))$$

and observing that  $e(D(\tau), C(\tau)) = r(\tau) - r^+(t_i)$  one concludes  $D^+(t_i) = C^+(t_i)$ .

On the other hand, the multifunction  $D$  is nondecreasing over each of the intervals  $I_i$ . In fact let  $\sigma$  and  $\tau$  be in  $I_i$  with  $\sigma \leq \tau$  and let  $x \in D(\sigma)$ ; then

$$\begin{aligned} \delta(x, C(\tau)) &\leq \delta(x, C(\sigma)) + e(C(\sigma), C(\tau)) \\ &\leq r(\sigma) - r_i + r(\tau) - r(\sigma) = r(\tau) - r_i, \end{aligned}$$

which entails  $x \in D(\tau)$ .

From  $D$  being nondecreasing over each of the intervals  $I_i$  it first results that its retraction over any subinterval of them is zero; thus  $\operatorname{ret}(D; t_0, t)$  equals the sum of jumps corresponding to the values  $1, 2, \dots, \nu$  of  $i$ , equal respectively, to  $e(D^-(t_i), D(t_i))$  if  $t_i \in I_i$  and to  $e(D(t_i), D^+(t_i))$  if  $t_i \notin I_i$  [19, Sect. 4d and 4e]. Now, if  $t_i \in I_i$  one has  $D(t_i) = C(t_i)$  and by the “triangle inequality” for the ecart  $e$ ,

$$e(D^-(t_i), D(t_i)) \leq e(D^-(t_i), C^-(t_i)) + e(C^-(t_i), C(t_i));$$

the definition of  $D$  entails that the first term on the right is majorized by  $r^-(t_i) - r_{i-1}$ ; the second term equals  $r(t_i) - r^-(t_i)$  [19, Sect. 4e]; so the considered jump is majorized by  $r(t_i) - r_{i-1}$ . In the case  $t_i \notin I_i$  one has  $D^+(t_i) = C^+(t_i)$ , therefore

$$e(D(t_i), D^+(t_i)) \leq e(D(t_i), C(t_i)) + e(C(t_i), C^+(t_i))$$

where the right member equals  $r^+(t_i) - r_{i-1}$ .

Adding all the jumps, one finally obtains

$$\operatorname{ret}(D; t_0, t) \leq \operatorname{ret}(C; t_0, t).$$

Thus  $D$  has a finite retraction; denote by  $v: I \rightarrow H$  the weak solution of the sweeping process by  $D$ , with initial value  $v(t_0) = u(t_0)$ . As  $D$  is nondecreasing over each of the intervals  $I_i$ , the function  $v$  is constant over each of them. In view of the jump conditions found in Sections 2d and 2e, one concludes that  $v$  is nothing else than  $u_p$ . It remains to apply proposition 2g with  $C' = D$ ; as the Hausdorff distance  $h(C(\tau), D(\tau))$  for  $\tau \in I_i$  equals  $r(\tau) - r_i$ , this distance is majorized by  $p$ .

## 2i. Change of Variable

**PROPOSITION 2i.** *Let  $\pi$  denote a nondecreasing surjective (thus continuous) mapping from an interval  $I'$ , containing its origin  $t'_0$ , onto an interval  $I$  (thus, containing its origin  $t_0$ ). Let  $C: I \rightarrow H$  be a multifunction with closed convex values and finite retraction. Then the multifunction  $C' = C \circ \pi: I' \rightarrow H$  (i.e.,  $C'(t') = C(\pi(t'))$  for every  $t' \in I'$ ) has a finite retraction. A function  $u': I' \rightarrow H$  is a weak solution of the sweeping process by  $C'$  if and only if it admits a factorization  $u' = u \circ \pi$ , where  $u: I \rightarrow H$  is a weak solution of the sweeping process by  $C$ .*

*Proof.* The definition of the retraction immediately yields that, if  $s' \leq t'$  in  $I'$ , if  $s = \pi(s')$  and  $t = \pi(t')$ , one has  $\text{ret}(C'; s', t') = \text{ret}(C; s, t)$ . With the notations of Section 2a, the solution  $u: I \rightarrow H$  of the sweeping process by  $C$  corresponding to some initial value  $a \in C(t_0) = C'(t'_0)$  is the limit of the net  $(u_p)$ ,  $P \in \text{lfp}(I)$ . The inverse images under  $\pi$  of the intervals constituting  $P$  are subintervals of  $I'$  constituting a locally finite partition  $P'$  of  $I'$ . Since  $\pi$  is continuous, both an interval  $I_i \in P$  and its inverse image  $I'_i \in P'$  contain or do not contain their respective origins  $t_i$  and  $t'_i$ ; furthermore  $C'^+(t'_i) = C^+(\pi(t'_i))$ . Thus the local-step multifunction  $C'_{P'}$ , associated in our usual way with the multifunction  $C'$  and with the partition  $P'$  equals  $C_P \circ \pi$ . The local-step function  $u'_{P'}: I' \rightarrow H$  corresponding to  $C'_{P'}$  and the initial value  $a$  is defined inductively by

$$u'_{P'}(I'_{i+1}) = \text{proj}(u'_{P'}(I'_i), C'_{P'}(I'_{i+1}));$$

then  $u'_{P'} = u_p \circ \pi$ . Consequently, the limit of  $u'_{P'}$ , for  $P$  ranging over the directed set  $\text{lfp}(I)$  equals  $u \circ \pi$ . Observe that, if  $\pi$  is not strictly increasing,  $P'$  does not range over the whole of  $\text{lfp}(I')$ ; however, Proposition 2h entails that the limit of  $u_{P'}$  is the solution  $u'$  of the sweeping process by  $C'$ . In fact if  $p$  is a majorant of the retraction of  $C$  over any subinterval of  $I_i \in P$ , it is also a majorant of the retraction of  $C'$  over any subinterval of  $I'_i \in P'$ , the inverse image under  $\pi$ .

*Remark.* For the study in the same connection of some *discontinuous* change of variable, see [18].

## 2j. The Catching-up Algorithm

The definition of  $u$  as the limit of a net has proved convenient for the subsequent theoretical study. But, since the considered functions take their values

in a metric space, one could as well construct  $u$  as the (uniform) limit of a sequence of local-step functions.

From the standpoint of computation, the situation would be the following. The interval  $I$  shall be replaced by some compact subinterval  $[t_0, t_0 + T]$ . If the retraction function  $r$  is known (or any real function more rapidly increasing than it), the inverse image of a finite partition of the interval  $[0, r(t_0 + T)]$  into subintervals of length  $\leq p$  yields a finite partition  $P$  of  $[t_0, t_0 + T]$  into subintervals satisfying the conditions of Proposition 2h. Thus the corresponding step-function  $u_p$  approximates  $u$  with a controlled error.

A drawback is the possible presence in  $P$  of some interval  $I_i$  which would not contain its origin  $t_i$ ; then the value  $u_p(I_i)$  is defined by projection on the limit set  $C^+(t_i)$  which is not directly given. This difficulty is partially overcome by observing (cf. Section 2c, Remark) that for every  $x \in H$

$$\text{proj}(x, C^+(t_i)) = \lim_{t \downarrow t_i} \text{proj}(x, C(t)).$$

This difficulty is avoided when *the function  $r$  is continuous from the right*, a case which will be specially emphasized in Part 3. Then, if one starts from a partition of  $[0, r(t_0 + T)]$  into intervals containing their origins, such are also their inverse images under  $r$ .

The simplest situation is that of a *continuous* function  $r$ , thus uniformly continuous on the compact interval  $[t_0, t_0 + T]$ . This yields that, for every  $\epsilon > 0$ , there exists  $\eta > 0$  such that, taking as  $P$  a finite partition of  $[t_0, t_0 + T]$  into intervals of the form  $[t_i, t_{i+1}[$ , the error  $|u_p - u|$  is uniformly majorized by  $\epsilon$  as soon as all the  $t_{i+1} - t_i$  are majorized by  $\eta$ . The construction of such a  $u_p$  properly constitutes the *catching up algorithm* introduced in Section 1d.

### 3. RIGHT-CONTINUOUS AND STRONG SOLUTIONS

#### 3a. Solutions in the Sense of Differential Measures

We shall say that a mapping  $u$  of the real interval  $I$  into the Hilbert (or Banach) space  $H$  has a *locally bounded variation* if it has a bounded variation over every compact subinterval of  $I$ ; notation  $u \in \text{lbv}(I, H)$ .

To every  $u \in \text{lbv}(I, H)$  classically corresponds an  $H$ -valued measure on  $I$  denoted by  $du$ ; we shall call it the *differential measure* of  $u$ . With the notations previously used for the left and right limits, a characteristic property of this vector measure is that, for every  $[s, t] \subset I$ , one has  $\int_{[s, t]} du = u^+(t) - u^-(s)$ . Consequently, the measure  $du$  determines, up to an additive constant, the two functions  $t \mapsto u^+(t)$  and  $t \mapsto u^-(t)$ , but, in general, the very value of  $u$  at a possible point of discontinuity remains independent of  $du$ .

In all the following, the considered functions will be supposed *right continuous*; if such is  $u$

$$\forall t \in I: \quad u(t) = u(t_0) + \int_{]t_0, t[} du. \quad (3.1)$$

DEFINITION 3a. *The function  $u \in \text{lbv}(I, H)$  is called a solution of the sweeping process in the sense of differential measures if there exists (nonuniquely) a nonnegative real measure  $d\mu$  on  $I$  and a function  $u' \in \mathcal{L}_{\text{loc}}^1(d\mu, H)$  such that  $du = u' d\mu$  and that*

$$-u'(t) \in \partial\psi(t, u(t)) \quad (3.2)$$

for every  $t$  in  $I$ .

An equivalent definition would consist in requiring  $u'$  to satisfy (3.2), except possibly on some  $d\mu$ -negligible set, but adding the specification that  $u(t) \in C(t)$  for every  $t$ . In fact the set  $\partial\psi(t, x)$  is nonempty if and only if  $x \in C(t)$ .

The following proposition entails the uniqueness of such a solution  $u$ , *right continuous* and agreeing with some initial condition  $u(t_0) = a$ .

PROPOSITION 3a. *If  $u_1$  and  $u_2$  are right continuous and solutions of the sweeping process in the above sense, the real function  $t \mapsto |u_1(t) - u_2(t)|$  is nonincreasing on  $I$ .*

*Proof.* The definition involves the existence of nonnegative real measures  $d\mu_m$ ,  $m = 1, 2$ , and functions  $u_m' \in \mathcal{L}_{\text{loc}}^1(d\mu_m, H)$  satisfying (3.2). Relative to  $d\mu = d\mu_1 + d\mu_2$ , the measure  $d\mu_m$  possesses a density  $\mu_m' \in \mathcal{L}^\infty(d\mu, \mathbf{R})$  with nonnegative values; thus  $du_m = u_m' \mu_m' d\mu$ . As, for every  $t \in I$ , the values of the multifunction  $\partial\psi(t, \cdot)$  are conical subsets of  $H$ , the monotonicity of this multifunction yields

$$(u_1 - u_2 | u_1' \mu_1' - u_2' \mu_2') \leq 0. \quad (3.3)$$

The function  $t \mapsto \frac{1}{2} |u_1(t) - u_2(t)|^2$  belongs to  $\text{lbv}(I, \mathbf{R})$  and, as it is right continuous, its increment from the point  $s \in I$  to the point  $t > s$  equals the integral of its differential measure on the interval  $]s, t[$ . As a result of [21, 22] this real measure is *majorized* by the measure  $(u_1 - u_2 | u_1' \mu_1' - u_2' \mu_2') d\mu$  which, in view of (3.3), is nonpositive.

Elementary counterexamples show that this conclusion does not hold in the absence of right continuity.

### 3b. Existence Theorem

PROPOSITION 3b. *If the retraction function  $r$  is right continuous, every weak solution  $u$  of the sweeping process (right continuous in view of Proposition 2e) is also a solution in the sense of differential measures; the measures  $dr$  may be taken as the measure  $d\mu$  of Definition 3a.*

*Proof.* Using a covering argument one may reduce the proof to the case where  $I$  is a compact interval  $[t_0, t_0 + T]$ .

Let  $(\epsilon_n)$  be an infinite sequence of strictly positive real numbers, converging to zero. As the real function  $r$  is nondecreasing, the hypothesis equivalently means that it is u.s.c.; thus the inverse image under  $r$  of any interval of the form  $[p, +\infty[$  is a closed (possibly empty) subinterval of  $I$ . By taking the inverse images of intervals of the form  $[p_i^n, p_{i+1}^n[$ , with  $p_{i+1}^n - p_i^n \leq \epsilon_n$ , and proceeding if necessary to subsequent refinements one constructs an infinite sequence  $(P_n)$  of finite partitions of  $I$  into subintervals of the form

$$P_n: [t_0, t_1^n[, [t_1^n, t_2^n[, \dots, [t_{\nu(n)-1}^n, t_0 + T[, \{t_0 + T\}$$

(we shall write  $t_i^n = t_0$  for  $i = 0$  and  $t_i^n = t_0 + T$  for  $i = \nu(n)$ ) possessing the properties:

- (i) the oscillation of  $r$  on each of the intervals constituting  $P_n$  is  $\leq \epsilon_n$ ;
- (ii)  $\lim_{n \rightarrow \infty} \max\{t_i^n - t_{i-1}^n : i = 1, 2, \dots, \nu(n)\} = 0$ .

For each  $n \in \mathbf{N}$ , let us define inductively a finite sequence  $(x_i^n)$ ,  $i = 0, 1, \dots, \nu(n)$ , of points of  $H$  by

$$x_0^n = a, \tag{3.4}$$

$$x_i^n = \text{proj}(x_{i-1}^n, C(t_i^n)). \tag{3.5}$$

This implies

$$|x_i^n - x_{i-1}^n| \leq e(C(t_{i-1}^n), C(t_i^n)) \leq r(t_i^n) - r(t_{i-1}^n). \tag{3.6}$$

Observe that, because of the refinements used in constructing the sequence of partitions  $P_n$ , it may happen that  $r(t_{i-1}^n) = r(t_i^n)$ . For every  $n \in \mathbf{N}$ , let us construct a mapping  $u^n: I \rightarrow H$  by defining its restriction to each interval  $[t_{i-1}^n, t_i^n[$  as follows:

- (1) If  $r(t_{i-1}^n) = r(t_i^n)$ ,  $u^n$  is given a constant value on the interval, namely,

$$\forall t \in [t_{i-1}^n, t_i^n[ : u^n(t) = x_{i-1}^n = x_i^n. \tag{3.7}$$

- (2) If  $r(t_i^n) > r(t_{i-1}^n)$ , one takes

$$\forall t \in [t_{i-1}^n, t_i^n[ : u^n(t) = \frac{[r(t_i^n) - r(t)] x_{i-1}^n + [r(t) - r(t_{i-1}^n)] x_i^n}{r(t_i^n) - r(t_{i-1}^n)}. \tag{3.8}$$

- (3) Finally, for  $t = t_{\nu(n)}^n = t_0 + T$ ,

$$u(t_0 + T) = x_{\nu(n)}^n.$$

Comparing adjacent intervals, one observes that the expressions of  $u(t)$  written in (3.7) or (3.8) are also valid for  $t = t_i^n$ ; hence, for every  $t$  and  $t'$  in  $[t_{i-1}^n, t_i^n]$ ,

$$u^n(t') - u^n(t) = \frac{r(t') - r(t)}{r(t_i^n) - r(t_{i-1}^n)} (x_i^n - x_{i-1}^n).$$

Thus, in view of (3.6), if  $t \leq t'$ ,

$$|u^n(t') - u^n(t)| \leq r(t') - r(t) \quad (3.9)$$

and, by addition, this also holds for every  $t \leq t'$  in  $I$ . This inequality entails that  $u^n$ , like the real function  $r$ , is right continuous, with bounded variation.

LEMMA 1. *The vector measure  $du^n$  equals  $u'^n dr$ , where  $u'^n$  denotes the step-function, with values in the unit ball of  $H$ , constructed as follows, on each interval of the form  $]t_{i-1}^n, t_i^n]$ :*

- (1) *if  $r(t_i^n) = r(t_{i-1}^n)$  one takes  $u'^n = 0$  on the interval;*
- (2) *otherwise  $u'^n$  has the constant value*

$$\forall t \in ]t_{i-1}^n, t_i^n]: u'^n(t) = \frac{x_i^n - x_{i-1}^n}{r(t_i^n) - r(t_{i-1}^n)}; \quad (3.10)$$

- (3) *in addition  $u'^n(t_0) = 0$ .*

Let us prove this lemma by checking that the measures  $du^n$  and  $u'^n dr$  yield the same integral over any subinterval of  $I$ . By unions or differences, one is reduced to the two cases:

*Case of an open subinterval  $]\sigma, \tau[$  of  $]t_{i-1}^n, t_i^n]$ .* Denoting as before by  $v^-$  and  $v^+$  the left and right limits of a function  $v$ , one has  $\int_{] \sigma, \tau[} dv = v^-(\tau) - v^+(\sigma)$ . Thus the expected equality is trivial if  $r(t_i^n) = r(t_{i-1}^n)$ ; otherwise

$$\int_{] \sigma, \tau[} du^n = \frac{r^-(\tau) - r^+(\sigma)}{r(t_i^n) - r(t_{i-1}^n)} (x_i^n - x_{i-1}^n),$$

which equals  $\int_{] \sigma, \tau[} u'^n dr$  in view of the constant value of  $u'^n$  over the interval  $]t_{i-1}^n, t_i^n]$ .

*Case of an interval consisting in the singleton  $t_i^n$ .* For  $0 < i \leq \nu(n)$  one has

$$\int_{\{t_i^n\}} du^n = u^{n+}(t_i^n) - u^{n-}(t_i^n)$$

which trivially vanishes if  $r(t_i^n) = r(t_{i-1}^n)$ ; otherwise, in view of (3.8), one finds

$$\int_{\{t_i^n\}} du^n = \frac{r(t_i^n) - r^-(t_i^n)}{r(t_i^n) - r(t_{i-1}^n)} (x_i^n - x_{i-1}^n)$$

while

$$\int_{\{t_i^n\}} u'^n dr = u'^n(t_i^n)[r(t_i^n) - r^-(t_i^n)];$$

equality follows from the definition of  $u'^n$ .

Finally, for  $i = 0$ , the right-continuity of  $r$  and  $u^n$  yields

$$\int_{\{t_0\}} du^n = \int_{\{t_0\}} u'^n dr = 0.$$

Let us observe now that the functions  $u'^n$ , taking their values in the unit ball of  $H$ , define elements of the closed ball with radius  $(r(t_0 + T))^{1/2}$  in the Hilbert space  $L^2(dr, H)$ . This balls is weakly sequentially compact, so that a subsequence may be extracted which converges in the weak topology of  $L^2(dr, H)$ . Suppose a change of indices such that this subsequence is identified with  $u'^n$  itself and denote by  $u'$  its weak limit. Define  $u: I \rightarrow H$  by

$$u(t) = u_0 + \int_{]t_0, t]} u' dr. \quad (3.11)$$

*This is a right-continuous function, with bounded variation, and  $u' dr$  is its differential measure.* The product of any element  $h$  of  $H$  by the characteristic function  $\chi_{]t_0, t]}$  of the interval  $]t_0, t]$  yields an element of  $L^2(dr, H)$ . Denoting by  $\langle \cdot, \cdot \rangle$  the scalar product in this space we have

$$(u(t) | h) = (u_0 | h) + \langle u', \chi_{]t_0, t]} h \rangle.$$

A similar expression holds for  $(u^n(t) | h)$ , proving that for every  $t \in I$ , the sequence  $u^n(t)$  converges to  $u(t)$  weakly in  $H$ .

Coming back to the partitions  $P_n$ , one observes that Proposition 2h applies to the step-functions constructed on the sequences  $x_i^n$ : this sequence of step-functions converges strongly and uniformly to the weak solution of the sweeping process admitting  $a$  as initial value. And by inequality (3.9) and the property (i) of  $P_n$  such is also the case for the sequence  $u^n$ ; thus the function  $u$ , as constructed in (3.11), is identical with the considered weak solution of the sweeping process and *the convergence of the sequence  $u^n$  is actually strong and uniform on  $I$ .*

Let us put now a notation: For every  $n \in \mathbf{N}$  and every  $t \in ]t_0, t_0 + T]$ , there exists  $i$  such that  $t \in ]t_{i-1}^n, t_i^n]$ ; the corresponding value of  $t_i^n$  will be denoted by  $\tau^n(t)$ ; in addition let us put  $\tau^n(t_0) = t_0$ . Then:

LEMMA 2. *For every  $t \in I$ ,*

$$-u'^n(t) \in \partial\psi(\tau^n(t), u^n(\tau^n(t))) \quad (3.12)$$

or equivalently, denoting by  $\gamma(t, \cdot)$  the support function of  $C(t)$ ,

$$u^n(\tau^n(t)) \in C(\tau^n(t)), \quad (3.13)$$

$$\gamma(\tau^n(t), -u^n(t)) + (u^n(\tau^n(t)) | u^n(t)) = 0. \quad (3.14)$$

This is trivial if  $t = t_0$ ; if  $\tau^n(t) = t_i^n$ ,  $i > 0$ , one has

$$u^n(\tau^n(t)) = u^n(t_i^n) = x_i^n \in C(t_i^n) = C(\tau^n(t));$$

on the other hand, (3.5) is equivalent to

$$x_{i-1}^n - x_i^n \in \partial\psi(t_i^n, x_i^n);$$

thus the expression (3.10) of  $u^n(t)$  entails (3.12) if  $r(t_{i-1}^n) < r(t_i^n)$ ; otherwise (3.12) trivially holds as  $u^n(t)$  is zero.

We are now to complete the proof of the proposition, by establishing that  $u$  complies with Definition 3a, with  $d\mu = dr$ . Actually  $u'$ , the density of the vector measure  $du$  relative to the real measure  $dr$ , has just been obtained as an element of  $L^2(dr, H)$ ; in the following we shall precisely understand  $u'$  as some representation of this element, i.e., a function everywhere defined on  $I$ . By passing to the limit in inequality (3.9), one gets a majoration of the vector measure  $du$  allowing for  $u'$  to take all its values in the unit ball of  $H$ . Then the elementary expression of the ecart  $e(C(t), C(\tau^n(t)))$  in terms of the respective support functions of the considered sets [19, Sect. 2c], already used in Section 2g of this paper, yields

$$\gamma(t, -u^n(t)) \leq \gamma(\tau^n(t), -u^n(t)) + r(\tau^n(t)) - r(t). \quad (3.15)$$

On the other hand, (3.9) implies

$$|u^n(\tau^n(t)) - u^n(t)| \leq r(\tau^n(t)) - r(t),$$

thus

$$(u^n(t) | u^n(t)) \leq (u^n(\tau^n(t)) | u^n(t)) + r(\tau^n(t)) - r(t). \quad (3.16)$$

By the strong convergence of  $u^n(t)$ , one has

$$\lim_{n \rightarrow \infty} (u(t) - u^n(t) | u^n(t)) = 0. \quad (3.17)$$

As  $r$  is right continuous and in view of the requirement (ii) observed in constructing the partitions  $P_n$

$$\lim_{n \rightarrow \infty} (r(\tau^n(t)) - r(t)) = 0. \quad (3.18)$$

Putting together (3.14), (3.15), (3.16), (3.17) one obtains, for every  $t$  in  $I$ ,

$$\limsup_{n \rightarrow \infty} [\gamma(t, -u'^n(t)) + (u(t) | u'^n(t))] \leq 0. \quad (3.19)$$

Let us make use now of Mazur's trick in the Hilbert space  $L^2(dr, H)$ ; there exists in this space a sequence  $(v'^m)$ ,  $m \in \mathbf{N}$ , which strongly converges to  $u'$  and such that each of the  $v'^m$  has the form

$$v'^m = \sum_{j \in J(m)} \alpha_j^m u'^j, \quad (3.20)$$

where  $J(m)$  is a finite set of integers  $\geq m$  and the  $\alpha_j^m$  are nonnegative real numbers such that

$$\sum_{j \in J(m)} \alpha_j^m = 1.$$

By (3.20) the  $v'^m$  are defined as functions on  $I$  and the strong convergence in  $L^2(dr, H)$  implies that, possibly after replacing them by some subsequence, one has the strong convergence of  $v'^m(t)$  to  $u'(t)$  for every  $t$  outside of some  $dr$ -negligible subset  $\omega$  of  $I$ . The convexity of the function  $x \mapsto \gamma(t, -x) + (u(t) | x)$  establishes that

$$\gamma(t, -v'^m(t)) + (u(t) | v'^m(t)) \leq \sum_{j \in J(m)} \alpha_j^m [\gamma(t, -u'^j(t)) + (u(t) | u'^j(t))];$$

therefore, in view of (3.19)

$$\limsup_{m \rightarrow \infty} [\gamma(t, -v'^m(t)) + (u(t) | v'^m(t))] \leq 0.$$

Now the lower semicontinuity of  $\gamma$  yields, for every  $t \notin \omega$ ,

$$\gamma(t, -u'(t)) + (u(t) | u'(t)) \leq \liminf_{m \rightarrow \infty} [\gamma(t, -v'^m(t)) + (u(t) | v'^m(t))].$$

By changing for zero the value that  $u'$  takes at every point of  $\omega$ , one finally concludes, for every  $t \in I$

$$\gamma(t, -u'(t)) + (u(t) | u'(t)) \leq 0.$$

As  $u(t) \in C(t)$ , this is equivalent to  $u'$  verifying (3.2).

### 3c. Strong Solutions

We conclude this paper by coming back to the strong formulation of Section 1a.

It will be supposed that the retraction function  $r$  is locally absolutely continuous on  $I$  (i.e., absolutely continuous on every compact subinterval of  $I$ ). Elementary this is found equivalent to the following: For every compact subinterval  $K$  of  $I$

and every  $\epsilon > 0$ , there exists  $\eta > 0$  such that, if  $]\sigma_i, \tau_i[$  denotes a finite family of nonoverlapping subintervals of  $K$ , one has the implication

$$\sum (\tau_i - \sigma_i) \leq \eta \Rightarrow \sum e(C(\sigma_i), C(\tau_i)) \leq \epsilon.$$

We shall express this by saying that the multifunction  $C$  has a *locally absolutely continuous retraction* on  $I$ . Such is a fortiori the case if  $C$  has a locally absolutely continuous *variation* in the sense of Hausdorff distance. Usual examples of multifunctions satisfying this condition and obtained as *intersections* of two others can be found in [17, 20].

Rapproaching Propositions 2a, 2c, 3b, one obtains:

PROPOSITION 3c. *If the multifunction  $C: I \rightarrow H$  with closed convex values has a locally absolutely continuous retraction, for every  $a \in C(t_0)$  there exists a unique mapping  $u: I \rightarrow H$ , locally absolutely continuous, such that  $u(t_0) = a$  and that*

$$-(du/dt) \in \partial\psi(t, u(t)) \quad (3.21)$$

for almost every  $t$  in  $I$ .

In fact, if  $dt$  denotes the Lebesgue measure of  $I$ , there exists  $\dot{r} \in \mathcal{L}_{loc}^1(dt, \mathbf{R})$  with nonnegative values such that  $\dot{d}r = \dot{r} dt$ . By Proposition 3b,  $du = u' \dot{d}r$ , with  $u' \in \mathcal{L}^\infty(\dot{d}r, H)$  satisfying (3.2); now  $u' \dot{r} \in \mathcal{L}_{loc}^1(dt, H)$  is the density of the vector measure  $du$  with regard to the Lebesgue measure  $dt$ , i.e., this function equals the strong derivative  $du/dt$  for almost every  $t$  in  $I$ . As  $\partial\psi$  is a cone (3.2) implies (3.21).

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