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# Dedekind's and Frege's views on logic

Hourya Benis Sinaceur

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**Abstract** A contextual and comparative analysis shows that Dedekind and Frege do not understand the terms “logic” and “arithmetic” in the same way. More specifically the meaning and the scope of the corresponding concepts are essentially different for them. Consequently Dedekind and Frege have different conceptions of the relationship between arithmetic and logic.

**Keywords** Arithmetic · Logic · Number · Structure · Concept · Function · Abbildung

**Zusammenfassung** Eine vergleichende und Kontextgebundene Analyse zeigt, dass Dedekind und Frege die Ausdrücke „Logik“ und „Arithmetik“ sehr unterschiedlich verstehen. Sinn und Umfang der entsprechenden Begriffe sind jeweils wesentlich verschieden. Folglich Dedekind und Frege nicht dieselbe Auslegung der Beziehung zwischen Logik und Arithmetik geben.

## 1 Introduction

Logicism is, broadly, the philosophical theory that arithmetic, and therefore all of mathematics, can be deduced from logic alone or can be reduced to logic.

The meaning of this thesis depends strongly on the meaning attributed to the term “logic”. Moreover should we understand “deduced from logic alone” as meaning “deduced by logical means alone” or “deduced from logical primitives alone by

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logical means alone”? Of course, the latter interpretation is much stronger than the first.

Naturally there are many ways to understand “logic” and “reduced to logic”, and one can therefore rightly argue that there are many logicisms. My aim here is to show that Dedekind and Frege understood “logic” and “arithmetic” in a very different way; more precisely, that Dedekind aimed at achieving arithmetical deductions by logical means alone, but he did not aim at reducing numbers to purely logical notions [3].<sup>1</sup>

## 2 Arithmetic

First of all, the term “arithmetic” does not have the same meaning for Dedekind and for Frege. Frege deals only with cardinal integers, and he claims that

with the definition of fractions, complex numbers and the rest, everything will in the end come down to the search for a *judgeable content* which can be transformed into an identity whose sides precisely are the new numbers. In other words, what we must do is fix the sense of a *recognition-judgement* for the case of these numbers [...] then the new numbers are given to us as *extensions of concepts*.<sup>2</sup>

\*<sup>3</sup> *Extensions of concepts* are objects (the reverse being false); they amount to sets, naively conceived. But Frege’s interest into extensions of concepts stems from concerns of *general logic* not from any idea of sets. Let me quote two comments extracted from “On Schönflies, The Logical Paradoxes of Set Theory”:

– [...] an extension of a concept is at bottom very different from an aggregate. The aggregate is composed of its parts. *Whereas the extension of a concept is not composed of the objects that belong to it.* For the case is conceivable that no objects belong to it. The extension of a concept simply has its being in the concept, not in the objects which belong to it; these are not its parts.<sup>4</sup>

<sup>1</sup> For a much more detailed account one might read chapter I of *Functions and generality of Logic. Reflections on Dedekind’s and Frege’s Logicisms*, Springer, 2015. I developed there the ideas that I already put forward in the introductions and comments of my French translation of *Stetigkeit und irrationale Zahlen* and *Was sind und was sollen die Zahlen?* in *Richard Dedekind : La création des nombres*, Paris, Vrin, 2008 [8].

<sup>2</sup> *Die Grundlagen der Arithmetik: eine logisch mathematische Untersuchung über den Begriff der Zahl* (Gl), Breslau, Koebner, 1884 [11], §104: “Es wird zuletzt auch bei der Definition der Brüche, complexen Zahlen u.s.w. Alles darauf ankommen, einen beurtheilbaren Inhalt aufzusuchen, der in eine Gleichung verwandelt werden kann, deren Seiten dann eben die neuen Zahlen sind. Mit anderen Worten: wir müssen den Sinn eines Wiedererkennungsurtheils für solche Zahlen festsetzen. [...], so werden uns die neue Zahlen als Umfänge von Begriffen gegeben” (emphasis added).

<sup>3</sup> The sign \* introduces a brief reminder of the meaning of technical concepts.

<sup>4</sup> *Posthumous Writings*, Basil Blackwell, Oxford, 1979, [15, p. 183] (emphasis added). Original in *Nachgelassene Schriften*, Hermes H., Kambartel F. & Kaulbach F. eds., Felix Meiner Verlag Hamburg, 1969, [13, p. 199]: “Schon hieraus geht hervor, daß im Grunde ein Begriffsumfang von einem Aggregate ganz verschieden ist. Das Aggregat besteht aus seinen Teilen. Dagegen besteht der Begriffsumfang nicht aus den

The relation of a part to the aggregate must still always be distinguished from that of an object to the extension of the concept to which it belongs. The extension of the concept is not determined by the aggregate even in [the] case, where they apparently coincide.<sup>5</sup>

\* *Concepts* are predicates of judgements, like in Kant's *Critique of Pure Reason* (Transcendental Analytic I, Chap. 1). But Frege sees predicates as mathematical functions, i. e. *unsaturated* expressions.

\* The number must be considered "in the context of a judgement which brings out its basic use" (*Gl*, § 46).

By contrast, Dedekind does not consider the notion of judgement or judgeable content, because he does not tackle the philosophical question of Truth. Additionally, a "concept" in Dedekind's sense is what can be defined by a few characteristic conditions (paradigmatically the four conditions  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$  for the abstract concept of number in *Was sind und was sollen die Zahlen?* [6], the conditions defining a field, an ideal, etc.). "Concept" is then what we now would call a "structure". I have highlighted a long time ago that Dedekind, Hilbert, Emmy Nøther, Emil Artin, Helmut Hasse, van der Waerden and others used the term "concept" [Begriff] with the meaning of our term "structure", as was later on popularized by Bourbaki.<sup>6</sup>

Also according to Frege, fractions, real, and complex numbers are not successively obtained as *numerical* extensions out of natural numbers, but as extensions of concepts. In *Grundgesetze der Arithmetik* (*Gg*) II § 57, Frege rejects explicitly "*das stückweise Definieren*", which was Dedekind's way of defining any domain of numbers out of natural integers.<sup>7</sup> Frege bases the definition of real numbers on the concepts of magnitude, measure and ratio. According to Frege, real numbers and natural numbers are two *different* sorts of numbers belonging to two "completely separate domains": the natural ones are those which answer the question "how many?", the real ones, which he calls "measuring numbers [Maßzahlen]", are those that are used in measuring continuous magnitudes.

Dedekind's understanding of "arithmetic" is quite different. By "arithmetic" Dedekind means two things.

First he means the whole body of numbers, be they natural numbers, negative, rational, irrational or complex numbers. Dedekind's goal is to achieve in a uniform way the gradual [*stückweise*] *numerical* extension of natural numbers without any help of any non-numerical notion<sup>8</sup>, in particular without summoning geometrical notions or appealing to the notion of measurable magnitude. "Arithmetic must develop

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Gegenständen, die ihm angehören. Der Fall ist nämlich denkbar, daß ihm keine Gegenstände angehören. Der Begriffsumfang hat eben seinen Bestand im Begriffe, nicht in den Gegenständen, die ihm angehören; diese sind nicht seine Teile."

<sup>5</sup> Ibidem : "So muß doch immer die Beziehung eines Teils zum Aggregate unterschieden werden von der des Gegenstandes zum Begriffsumfang, dem er angehört. Durch das Aggregat ist der Begriffsumfang auch in diesem Falle nicht bestimmt, wo sie scheinbar zusammenfallen."

<sup>6</sup> See *Corps et modèles*, Paris, Vrin [1], deuxième partie, chapitre II, where I described the "begriffliche Mathematik", as B. van der Waerden and Pavel Alexandroff called it.

<sup>7</sup> See also "Logic in mathematics", in *Posthumous Writings*, Basil Blackwell, Oxford, 1979, [15, p. 242 ff].

<sup>8</sup> To a certain extent this requirement is similar to Aristotle's refusal of *μετάβασις εἰς ἄλλο γένος*.

itself out of itself<sup>9</sup>: the radical difference is between number and magnitude, not between natural and real numbers. Dedekind rejects experience and intuition not only for the definition of the natural numbers, but also in the “stückweise” definition of the whole realm of numbers, and he wants to derive arithmetical (i. e. also analytical and algebraic) theorems from a little number of conditions through purely logical deduction. So when he defines in *Stetigkeit und irrationale Zahlen* [5] real numbers (up to isomorphism) out of rational numbers alone, Dedekind is in fact extending the totally ordered structure of the field  $\mathbf{Q}$  to the field  $\mathbf{R}$ . In modern terms, he is *embedding*  $\mathbf{Q}$  in  $\mathbf{R}$ , he is then identifying  $\mathbf{Q}$  with its *image* [Bild] in  $\mathbf{R}$ .<sup>10</sup> One must note that this identification constitutes a step which cannot be conceived in Frege’s setting, because in Frege’s view the rational number 2 and the real number 2 are different individuals, which are definable on the basis of different primitive notions.

We find another conception of arithmetic in Dedekind’s *Was sind und was sollen die Zahlen?* [6]. There, “arithmetic” points to a more general framework, to a kind of general arithmetic dealing not only with numbers but also with unspecified elements. For that purpose Dedekind generalizes the concept of Abbildung and he shapes the concept of System (set in modern terms). Thus when Dedekind asserts that arithmetic is “a part of logic”, he does not aim, in my view, at *reducing* arithmetic to logic<sup>11</sup>, i. e. reducing arithmetical notions to logical primitives, but at *developing*, on the basis of natural numbers and with logical means alone, a *general arithmetical framework*, which would appear as a structure of the mind. Dedekind’s assertion that the concept of number “flows *immediately* from the pure laws of thought”<sup>12</sup> means that the concept of number belongs to the structure of the knowing mind.

Naturally, one can argue that “the structure of the mind” is “logic” more or less. But one should then stress that it is “logic” in a definitely pre-Fregean sense, a sense more or less close to the Kantian fundamental view of the structural creativity of the mind. I think that Dedekind had an *epistemological* perspective, focused on the creative power and the constitutive traits of the rational mind<sup>13</sup> – without making the Kantian distinction between reason [Vernunft] and understanding [Verstand]. Although for the sake of precision we must note that Dedekind did not endorse Kant’s transcendentalism. Indeed, he did neither accept Kant’s transcendental Aesthetik<sup>14</sup>

<sup>9</sup> *Stetigkeit und irrationale Zahlen*, § 3: “[Ich] fordere, daß die Arithmetik sich aus sich selbst heraus entwickeln soll”.

<sup>10</sup> Such identifications are very usual in mathematical practice, but the philosophical question about how to conceive of, e. g., the identity of 2-rational and 2-real gives rise to subtle discussions.

<sup>11</sup> As I mentioned in the first chapter of *Functions and generality of Logic* [3], it’s not uncommon to argue for Dedekind’s logicism. See for instance José Ferreirós, *The labyrinth of thought*, Preface to the second revised edition, Birkhäuser Verlag, [9], and Michael Detlefsen, “Dedekind Against Intuition: Rigor, Scope and the Motives of his Logicism”, in *Logic and Knowledge*, C. Cellucci, E. Grosholz & E. Ippoliti (eds.), Cambridge Scholars Publishing, [4], 205–217.

<sup>12</sup> Vorwort zur ersten Auflage of *Zahlen* [6]: “Indem ich die Arithmetik (Algebra, Analysis) nur einen Teil der Logik nenne, spreche ich schon aus, daß ich den Zahlbegriff für gänzlich unabhängig von den Vorstellungen oder Anschauungen des Raumes und der Zeit, daß ich ihn vielmehr für einen unmittelbaren Ausfluß der reinen Denkgesetze halte”.

<sup>13</sup> This is exactly the perspective that Frege rejected in favour of a logical-ontological perspective.

<sup>14</sup> Indeed, numbers are prior to space and time (see footnote 12).

nor discuss specific Kantian questions, such as the question of the synthetic a priori or the unifying function of consciousness.

However and overall the global Kantian view remains the one that permeates most of Dedekind's work.

### 3 Logical vs mathematical project

Frege's explicit goal was a "renewal of logic"<sup>15</sup>, which consists of

1. replacing the traditional Aristotelian splitting subject-predicate by the function-argument analysis, and
2. inventing a formal language appropriate for expressing the very logic of "pure thought" and the relations [*Beziehungen*] between concepts.

The *Begriffsschrift* (*Bg*), [10], establishes a "formula language" [*Formelsprache*] of logic (which contains the essentials of first- and second-order quantification with identity) with specific symbols and definite rules according to which derivations are carried out exclusively by virtue of the "logical form"<sup>16</sup> of expressions. The relations of concepts are analysed in terms of function-argument<sup>17</sup>: "It is easy to see", writes Frege in the Preface of *Bg*, "how taking a content as a function of an argument gives rise to concept formation"<sup>18</sup>. Let me summarize Frege's conception.

\* According to Frege a function is what a functional expression stands for. For instance " $3 + x = 7$ " is a functional expression. If " $x$ " is replaced by "4" then the expression becomes a true sentence or formula of arithmetic; if it is replaced by any other numeral the resulting sentence or formula will be false.

A potential argument for a function is an entity denoted by an expression that serves to supplement the functional expression so as to yield a complete sentence.

A concept is a function whose value always is a truth-value for any suitable argument<sup>19</sup>, and a relation is a function that has more than one argument place.

In part III of *Bg*, Frege presents a logical reconstruction of "a general theory of sequences". Mathematical induction is presented as a special case of a logical relation. And in the introduction of the *Grundlagen*, Frege claims again that he will "make clear that even an inference like that from  $n$  to  $n + 1$ , which on the face

<sup>15</sup> *Grundgesetze der Arithmetik* (*Gg*) I, Jena 1893 [12], Preface, last sentence: "May my book, then, even if belatedly, contribute to a renewal of logic." ("Und so möge denn dies Buch, wenn auch spät, zu einer Erneuerung der Logik beitragen.")

<sup>16</sup> This locution is absent from Dedekind's work.

<sup>17</sup> Later, in his essay "Function and concept", Frege will define a function as an "unsaturated" [ungesättigt] expression and introduce the notion of a values range as referring to a set of pairings of arguments with values.

<sup>18</sup> *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*, Verlag von Louis Nebert, Halle, 1879, [10], Vorwort, p. VII: "Insbesondere glaube ich, daß die Ersetzung der Begriffe *Subject* und *Praedical* durch *Argument* und *Function* sich auf die Dauer bewähren wird. Man erkennt leicht, wie die Auffassung eines Inhalts als Function eines Argumentes begriffbildend wirkt." (Frege's emphasis).

<sup>19</sup> This specifically Fregean understanding of the term "concept" is totally outside the scope of Dedekind's thinking.

of it is peculiar to mathematics, is based on the general laws of logic”.<sup>20</sup> In the conclusion (§ 87) Frege writes that, in the framework of his analysis, “calculation becomes deduction” (“Rechnen wäre Schlussfolgern”).

Dedekind clearly did not aim at building a formal language. He did not pay attention to the logical structure of mathematical formulas (or of linguistic expressions in general) but to their mathematical content. The scope of his conception of content is limited to mathematical concepts and structures, be they concrete or abstract. Considering for instance that the general concept of function constitutes a mathematical content is not the same thing as Frege’s “taking a content [of an expression] as a function of an argument”.

One would say that Dedekind was continuing or completing Euclid’s mathematics whereas Frege was transforming totally Aristotle’s logic and ontology.

As is well known, Dedekind’s project was to arithmetize important mathematical concepts and theories such as the concept of real number and the theory of algebraic functions. That means that he tried to apply arithmetical laws to different domains than that of natural numbers. For that purpose he extracted a more general arithmetical structure from the arithmetic of the natural numbers. Commenting very briefly on Frege’s *Grundlagen* in the preface of the second edition of *Zahlen*, Dedekind stressed that mathematical induction is the *arithmetical* method per excellence. Actually, he reduced the inference from  $n$  to  $n + 1$  to an order-setting *Abbildung*, injective but not surjective, defined on an infinite *System* whose elements are not necessarily numbers. The concept of chain<sup>21</sup> allows us to obtain the general theorem of complete induction (Satz 59), and provides the arithmetical induction (Satz 80) and the theorem of definition by induction – finite recursion – (Satz 126). What matters is the *structural generalization*, not the *logical reduction*, of the *arithmetical relation of order*.<sup>22</sup>

Moreover, it is striking that after Cantor pointed out to him several times that a logical paradox was involved in his construction of Systems, Dedekind did not even try to amend anything. In the Vorwort of the third and last edition of *Zahlen* (1911), he affirms that “his trust in the internal harmony of our logic is not shaken” and he expressed his conviction that some means will certainly be found out in order to ground “the creative force thanks to which our mind creates out of some determinate elements a new element which is the System of them” rigorously.<sup>23</sup> The

<sup>20</sup> *Die Grundlagen der Arithmetik. Eine logisch mathematische Untersuchung über den Begriff der Zahl* (G), [11], Einleitung, p. IV: “Man wird aus dieser Schrift ersehen können, daß auch ein scheinbar eigentümlich mathematischer Schluss wie der von  $n$  auf  $n + 1$  auf den allgemeinen logischen Gesetzen beruht, daß es besonderer Gesetze des aggregativen Denkens nicht bedarf.”

<sup>21</sup> In contemporary terminology, given a function  $\varphi: S \rightarrow S$ , a  $\varphi$ -chain is a subset  $A$  of  $S$  such that  $\varphi(A) \subseteq A$ ; given a subset  $A$  of  $S$ , the  $\varphi$ -chain  $A_0$  generated by  $A$  is the intersection of all  $\varphi$ -chains containing  $A$ .

<sup>22</sup> For details see E. Reck paper “Dedekind’s structuralism: an interpretation and partial defense”, *Synthese* 137, [18], especially §§ 4–5; Wilfred Sieg and Dirk Schlimm, “Dedekind’s Analysis of Number: Systems and Axioms”, *Synthese* 147, [19], 121–170; W. Sieg & B. Morris, “Dedekind’s Structuralism: creating concepts and deriving theorems”, October 2015 [20], preprint of a paper forthcoming in *Logic, Philosophy of Mathematics, and their History: Essays in Honor of W.W. Tait*.

<sup>23</sup> “Die Bedeutung und teilweise Berechtigung dieser Zweifel [an der Sicherheit wichtiger Grundlagen meiner Auffassung] verkenne ich auch heute nicht. Aber mein Vertrauen in die innere Harmonie unserer

“internal<sup>24</sup> harmony of our logic” is rather puzzling, but “the creative force” of the mind obviously refers to a Kantian epistemological conception.

The mathematical concept of *function* plays a very important role in Frege's and in Dedekind's definitions of the concept of natural number. But the ways Frege and Dedekind go from mathematical function to the concept of natural number differ and reveal two obviously dissimilar lines of generalization. Frege substitutes mathematics for grammar in a logical analysis of language. He does so in order to modify the traditional *logic* and *ontology*, taking the “linguistic turn” and building quantification theory. Frege's question and answer are of the following kind.

What is the number 1 itself or what means the sign 1<sup>25</sup>

What is the number itself? [...] We may try to know something about the number itself from the way of using the numerals and number-words. We use the number-words as names of objects, as proper names.<sup>26</sup>

For his part, Dedekind uses a mode of thinking presented as “logical” inasmuch as it is a fundamental way of *knowledge*. Nowhere does Dedekind try to make an analysis formally powered with specific logical tools; nor does he attempt to characterize numbers by using number-words or making a judgement involving numbers. He definitely turns away from a philosophy of judgement and faces towards a more objective perspective. On this point he escapes Kant's influence and most likely follows a trend developed by Hermann Lotze, without producing, as Frege

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Logik ist dadurch nicht erschüttert; ich glaube, daß eine strenge Untersuchung der Schöpferkraft des Geistes, aus bestimmten Elementen eine neues Bestimmtes, ihr System zu erschaffen, das notwendig von jeden dieser Elemente verschieden ist, gewiß dazu führen wird, die Grundlagen meiner Schrift einwandfrei zu gestalten.”.

<sup>24</sup> The term “internal” is a leitmotiv of Dedekind's thinking. Already in “Über die Einführung neuer Funktionen in die Mathematik” (1854), Dedekind speaks of “the internal nature of science and of “the internal necessity” of the retroaction on the whole scientific organism of some local progress” [*Gesammelte mathematische Werke* III, p. 430]. In the correspondence with Keferstein (1890) Dedekind tells that he did not understand the “internal connection” [den inneren Zusammenhang] of Keferstein's arguments, *Revue d'histoire des sciences*, 27/3, 1974, p. 261. The term “internal” comes again under Dedekind's pen in a comment about Gauss' preference for concepts as opposed to notation: as example Dedekind gives Riemann's definition of a function by its “characteristic and internal properties” [innerliche charakteristische Eigenschaften], “Über die Begründung der Idealtheorie” (1895), *Gesammelte mathematische Werke* II, 54–55. All those passages suggest that here Dedekind means that the internal deductive structure of *Zahlen* is safe despite of the local flaw in the way of stating the creation of a system out of some given elements.

<sup>25</sup> *Gl*, first sentence of the introduction : “Auf die Frage, was die Zahl Eins sei, oder was das Zeichen 1 bedeute, wird man meistens die Antwort erhalten: nun, ein Ding. Und wenn man dann darauf aufmerksam macht, daß der Satz “die Zahl Eins ist ein Ding” keine Definition ist, weil auf der einen Seite der bestimmte Artikel, auf der anderen der unbestimmte steht, daß es nur besagt, die Zahl Eins gehöre zu den Dingen, aber nicht, welches Ding sie sei, so wird man vielleicht aufgefordert, sich irgendein Ding zu wählen, das man Eins nennen wolle. Wenn aber Jeder das Recht hätte, unter diesem Namen zu verstehe, was er will, so würde derselbe Satz von der Eins für Verschiedene Verschiedenes bedeuten; es gäbe keinen gemeinsamen Inhalt solcher Sätze.”.

<sup>26</sup> “Number”, in *Posthumous Writings*, p. 265 – *Nachgelassene Schriften*, p. 284: “Was ist denn nun die Zahl selbst? [...] Aus der Gebrauchsweise der Zahlzeichen und Zahlwörter kann man etwas über die Zahl selbst zu erkennen suchen. Man gebraucht die Zahlzeichen und Zahlwörter wie Namen von Gegenständen als Eigennamen.” See also “Notes for Ludwig Darmstaedter”, *Posthumous Writings*, p. 256.

did, a logico-philosophical analysis of the judgement-content [*Urteilsinhalt*] based on the distinction between Denken and Erkennen.

Dedekind viewed the ascent from arithmetic proper to a general arithmetical setting, i. e. to general arithmetical structures, as a structural abstraction “freeing the elements from every other content” (*Zahlen*, Definition 73).

If in the consideration of a simply infinite system  $N$  set in order by a transformation [Abbildung]  $\varphi$  we entirely neglect the special character of the elements; simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting transformation  $\varphi$ , then are these elements called *natural numbers* or *ordinal numbers* or simply *numbers*, and the base-element 1 is called *the base-number of the number-series  $N$* . With reference to this freeing the elements from every other content (abstraction) we are justified in calling numbers a free creation of the human mind <sup>27</sup> (*Zahlen*, Definition 73).

One might take the term “abstraction” as being a logical procedure<sup>28</sup>, so that the structural ascent might seem to point to a logical ascent. But to which extent is it legitimate to equate “abstract” with “logical”? Is any mathematical abstract procedure *in fine* a logical move?<sup>29</sup> Anyway, one should clarify that by “logic” Dedekind refers to the structure of the operative mind – not to a structure that the mind should recognize as independent from itself. Surely, such an epistemological view does not come down to a psychological stand. The why and how of this would need to be developed in another paper.

#### 4 Logical vs mathematical structure

One cannot find the expression “the general laws of logic” either in *Stetigkeit* or in *Zahlen*. What we may find there is the expression “the laws of thought” and the terms “logic” and “logical” as qualifying the process of “building up the science

<sup>27</sup> “Wenn man bei der Betrachtung eines einfach unendlichen, durch eine Abbildung  $\varphi$  geordneten Systems  $N$  von der besonderen Beschaffenheit der Elemente gänzlich absieht, lediglich ihre Unterscheidbarkeit festhält und nur die Beziehungen auffaßt, in die sie durch die ordnende Abbildung  $\varphi$  zueinander gesetzt sind, so heißen diese Elemente *natürliche Zahlen* oder *Ordinalzahlen* oder auch schlechthin *Zahlen*, und das Grundelement 1 heißt die *Grundzahl* der *Zahlenreihe  $N$* . In Rücksicht auf diese Befreiung der Elemente von jedem anderen Inhalt (Abstraktion) kann man die Zahlen mit Recht eine freie Schöpfung des menschlichen Geistes nennen.” (Dedekind’s emphasis).

<sup>28</sup> This is W. Tait’s opinion (“Some recent essays in the history of the philosophy of mathematics: a critical review”, *Synthese* 96 [23], 293–331, and “Frege versus Cantor and Dedekind: on the concept of number”, in *Early Analytic Philosophy*, Open Court, Chicago, [24], 213–248). W. Sieg and B. Morris defend a much more qualified view. They claim that the definition 73 of *Zahlen* does not determine a structure, it determines a “structure-mère”, in the sense of Bourbaki, and therefore constitutes a “higher-level concept”. Moreover, they think that Dedekind “was deeply influenced by Lotze’s views on concept formation, in particular on abstraction”, and that he considered mathematics to be rooted in logic. But Lotze’s abstraction is based on a close connection between logic, language, and metaphysics. In this respect, it is close to Aristotle’s abstraction [*ἀφαίρεσις*]. Dedekind dropped at least two ingredients: language and metaphysics.

<sup>29</sup> My answer is No: see “Facets and Levels of Mathematical Abstraction”, *Philosophia Scientia*, 18 (1), 2014, 81–112.

of numbers” thanks to concepts more general and more primitive than the concepts usually taken as primitive in arithmetic or analysis. As for real analysis, the concept of the arithmetical/numerical real domain comes first; on it depend the notions of limit, continuity or convergence of a real function of real variables. Indeed, *Stetigkeit* shows that the Dedekindian “completeness” of the real numbers field *logically implies* its Cauchy’s completeness, once one defines a distance (a metric) on the field. Therefore the logical priority of arithmetic vis-à-vis geometry does *not* mean, as Frege holds, that geometry depends on intuition. It simply means, as Frege too maintains in his early period, that numbers and numerical operations have an *intrinsic* definition, without any recourse to geometrical notion. Indeed, Dedekind held that even geometry does not rest on intuition. He is the first mathematician who states that “continuity” (connectedness) of a space is not given to us by spatial intuition. According to him, we do not have really a visual or intuitive apprehension of the continuity of a geometric line drawn on the blackboard. We envision it as a property that “we attribute to the line” (or to space) by a convenient axiom which must be explicitly formulated as a primitive – non provable – principle, since “for a great part of the science of space the continuity of its configurations is not even a necessary condition”.<sup>30</sup> In contrast with what Frege does in his early writings, Dedekind does not put an exclusive disjunction between logic and geometry. He holds that the mathematical concept of space, taken generally, differs from that of the Euclidean space and from the physical sensible space.

Dedekind deals with logical dependence or independence, not with logical laws ruling *the dependence relation*. Whereas Dedekind is attentive to defining everything which can be defined and to proving any proposition which is provable<sup>31</sup>, in order to obtain the most simple concepts and the very primitive propositions, and hence to make clear the logical connections between mathematical propositions, he does not tackle directly the very questions, which form the object of Frege’s investigations:

- what constitutes a logical inference in and for itself?
- what is or how must be an adequate definition?
- what is a correct proof ?
- what is a true proposition?<sup>32</sup>

<sup>30</sup> *Stetigkeit*, § 3: “Die Annahme dieser Eigenschaft [Stetigkeit] der Linie ist nichts als ein Axiom, durch welches wir erst die Stetigkeit in die Linie hineindenken. Hat überhaupt der Raum eine reale Existenz, so braucht er doch nicht notwendig stetig zu sein; unzählige seiner Eigenschaften würden dieselben bleiben, wenn er auch unstetig wäre.”.

<sup>31</sup> “In science nothing capable of proof ought to be accepted without proof” (“Was beweisbar ist, soll in der Wissenschaft nicht ohne Beweis geglaubt werden”), *Zahlen*, first sentence.

<sup>32</sup> “The word “true” indicates the aim of logic ... All sciences have truth as their goal; but logic is also concerned with it in a quite different way from this. ... To discover truths is the task of all sciences; it falls to logic to discern the laws of truth ... Rules for asserting, thinking, judging, inferring, follow from the laws of truth.” English translation of “Der Gedanke. Eine logische Untersuchung”, *Mind*, 65, July [14], 289–311 (“Wie das Wort “schön” der Ästhetik und “gut” der Ethik, so weist “wahr” der Logik die Richtung. Zwar haben alle Wissenschaften Wahrheit als Ziel; aber die Logik beschäftigt sich noch in ganz anderer Weise mit ihr. Sie verhält sich zur Wahrheit etwa so wie die Physik zur Schwere oder zur Wärme. Wahrheiten zu entdecken, ist Aufgabe aller Wissenschaften: der Logik kommt es zu, die Gesetze des Wahrseins zu erkennen. [...] Aus den Gesetzen des Wahrseins ergeben sich nun Vorschriften für das Fürwahrhalten, das Denken, Urteilen, Schließen. Und so spricht man wohl auch von Denkgesetzen.”).

Moreover, in Dedekind's view, the most fundamental law of thought is "the *ability* of the mind to relate things to things, to let a thing correspond to a thing, or to represent a thing by a thing, an ability without which no thinking is possible"<sup>33</sup>. The most fundamental law of thought is also the *Abbilden*-ability. The *Abbilden*-ability is a dynamic rational process resulting into mathematical innovations and progress, because it permits *taking one thing for another thing playing the same role*. What matters is not about identity of individual things but about *analogy, similarity* in Dedekind's wording<sup>34</sup>, between domains equipped with operations on elements which may be numbers or things which are not numbers.

## 5 Conclusions

1. Dedekind's work contains *neither* a specifically logical development *nor* a body of articulated philosophical views. Nevertheless his scattered remarks about the nature of number or the essence [Wesen] of continuity form a reasonably coherent picture, which pertains to knowledge and epistemology rather than to being and ontology.<sup>35</sup> Dedekind endorses the Kantian split between epistemology and ontology, whereas Frege renews the ancient ontological questions about the being and the truth.
2. "Dedekind's logic" has nothing to do with an analysis of language, nor with a theory of inference, nor with a theory of mathematical truth, nor with a theory of definition. Now, if one persists in taking *à la lettre* the assertion that arithmetic is a part of logic, one should make clear that the building tools, viz. *System* and *Abbildung*, are not logical in Frege's sense of the word. Frege himself highlighted that view.<sup>36</sup> Indeed, Dedekind's *Abbildung* conflicts frontally with Frege's concern for strict identity. As Dedekind's work shows everywhere, identity is much less fruitful than *analogy* (similar representation in Dedekind's wording), which provides not a *substantial* but a *functional* identity. Dealing with unspecified elements

<sup>33</sup> *Zahlen*, Preface to the first edition (emphasis added): "Verfolgt man genau, was wir bei dem Zählen der Menge oder Anzahl von Dingen tun, so wird man auf die Betrachtung der Fähigkeit des Geistes geführt, Dinge auf Dinge zu beziehen, einem Dinge ein Ding entsprechen zu lassen, oder ein Ding durch ein Ding abzubilden, ohne welche Fähigkeit überhaupt kein Denken möglich ist." Let me stress again that "numbers" are not restricted to the natural numbers, they mean any kind of numbers.

<sup>34</sup> Let me recall that Dedekind explained to Keferstein that the definition of an infinite system  $S$  is not based on the identity (equality) [Identität (Gleichheit)] but on the similarity [Ähnlichkeit] between  $S$  and one of its proper parts  $A$  (German text in *Revue d'histoire des sciences*, 27/3, 1974, p. 261). Dedekind's concept of "ähnliche Abbildung" (bijection) suffices to characterize the infinity of a system  $S$ , without appeal to the proper concept of number.

<sup>35</sup> Stein [22, p. 247] rightly points out that Dedekind's work is "quite free of the preoccupations with "ontology" that so dominated Frege, and had so fascinated later philosophers" ("Logos, logic, and logistiké: Some philosophical remarks on nineteenth century transformation of mathematics", in W. Aspray and P. Kitcher eds, University of Minnesota Press, [22]).

<sup>36</sup> *Gg I*, Vorwort: "Herr Dedekind ist der Meinung, dass die Lehre von den Zahlen ein Theil der Logik sei; aber seine Schrift trägt kaum dazu bei, diese Meinung zu erhärten, weil die von ihm angewendeten Ausdrücke "System", "ein Ding gehört zu einem Dinge" in der Logik nicht üblich sind und nicht auf anerkannt Logisches zurückgeführt werden."

is, from a mathematical point of view, not a flaw *vis-à-vis* the questions of definition and truth, but the very condition for structures that may apply to domains of different elements to be brought to light. In Dedekind's mind arithmetic constitutes *by itself* a very fundamental framework. Therefore there is no need to reduce arithmetic to logic.

3. Arithmetic is fundamental not only because numbers are applied everywhere, but because we can, following arithmetical laws, calculate with things which are not numbers. In Dedekind's view arithmetic is a formal structure of our experience of the world.<sup>37</sup> Numbers serve to organize our perception of external things. They play the role given by Kant to space and time. General arithmetic laws replace the transcendental Aesthetic.<sup>38</sup> The "logic of the mind" is arithmetic taken generally. According to me, the famous phrase "arithmetic is a part of logic" means that arithmetic itself provides a purely rational (logical) *norm* of thinking.
4. Dedekind's construction of the natural numbers shows, in fact, that Dedekind assimilates logic to set theory – rather than the reverse –, a merger that Frege refused for good reasons<sup>39</sup>. However, the demarcation between logic and set theory stressed by Frege is still in debate among philosophers of mathematics and logic. For instance, in a semantic approach, which is based on set theory, different invariance criteria across structures are proposed for capturing logical notions. In a syntactic proof-theoretic approach, logicity is defined in terms of some set of basic inference patterns. And in face of the Curry-Howard structural correspondence between proofs and programs, logic and arithmetic appear to be two faces of the same process.

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<sup>37</sup> Kitcher, "Frege, Dedekind, and the philosophy of mathematics", in Haaparanta L. and Hintikka J. (eds.), *Frege synthesized*, D. Reidel, Dordrecht [16], pp. 299–343].

<sup>38</sup> *Zahlen*, Vorwort zur ersten Auflage : "Die Zahlen sind freie Schöpfungen des menschlichen Geistes, sie dienen als ein Mittel, um die Verschiedenheit – the same word as used by Kant in *Critique of pure reason* B320 – der Dinge leichter und schärfer aufzufassen."

<sup>39</sup> This expression is borrowed from Charles Parsons, "Frege's theory of number", in Black M. (ed.), ed., *Philosophy in America*, Allen & Unwin, London, [17], §§VI, VII. Reprinted with Postscript in *Mathematics in Philosophy: Selected Essays*. Ithaca and London: Cornell University Press, [17]. Paperback edition 2005.

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