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A logic of sights

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Abstract

We study labeled transition systems where at each state an agent is aware of and hence reasons about only a part of the entire system (called the 'sight'). We develop a logic for such systems: the 'logic of sights'. We explore its model theory, give an axiomatization and prove its completeness. We show that the logic is a fragment of the loosely guarded fragment of first-order logic. We show that the satisfiability problem of the logic is PSPACE-complete and the combined complexity of its model-checking problem is in PTIME. Finally we discuss its relation to other logics as well as extensions.

Keywords: Bounded rationality, modal logic, temporal logics, dynamic logics, sights, labeled transition systems.

1 Introduction

Chess, under the usual restrictions, is a finite extensive form game. Thus, at least in theory, it is possible to do a backward induction on the finite game tree so as to decide if one of the players has a winning strategy and to compute such a strategy if it exists. However, applying backward induction on the full game tree is intractable, and human and artificial players alike have to reason differently to make decisions. The extensive form game tree is simply too large for any algorithm to process. In such a situation, other types of analyses are called for. Can we look at the game from the perspective of the agents playing it? In chess, for instance, a player usually looks a few steps into the future of the current position, and then tries to play the optimal move, hoping that such a move will lead to a global optimization, and eventually to victory. In the process, there might be certain moves or paths along the game tree that the player is not aware of or does not consider possible. In other words, the player has a particular 'view' of the current game and of what can happen.

More formally, given a labeled transition system, an agent might be able to foresee only some, but not all, of the possible evolutions of the system. Things can get even more subtle if the agent has to take into account what she might be able to foresee once certain actions have been taken and the state of the system has changed. In the context of extensive games, Rêgo and Halpern [27] develop a semantic approach for such a notion of awareness (to be distinguished from an approach in which awareness depends on the syntactic resources of the language, such as, e.g. [15, 16, 22]). The main idea is that an agent is assigned a subtree out of the current subgames at every stage in the extensive games where she plays. This approach is further explored in [13].

In this article, we explore the modal logic of this concept, which [13] call 'sight'. Our semantics is as general as possible: expanding labeled transition systems with a sight function at every state consisting in an arbitrary subset of states containing the current state. Syntactically, we add an operator $[!\mathsf{N}]\varphi$, meaning that, at the current state, from the perspective of the agent, φ is true or, paraphrasing the semantics more closely, that, at the current state, in the sight of the agent at the current state, φ is true. The modal logic of these types of models is very natural; $[!\mathsf{N}]$ results in a restriction of the current model. The dynamic and restricting nature of $[!\mathsf{N}]$ makes our logic a cousin of a recent family of dynamic and/or temporal epistemic logics: Future Event Logic (FEL) [34, 35], to PAL [2, 11, 26] and to TPAL [32], PAL with protocols [7, 25]. But there are important differences, and in many respects

our system is more general. For instance, we extend temporal logics with horizons, while (T)PAL and FEL treat modally-definable restrictions or quantification over all such restrictions, respectively. We assume a given unique restriction in each state, generalizing away from any process of describing the structure of that restriction, via e.g. quantification as in FEL. Furthermore, the restrictions we are considering are in general not syntactically-definable (unlike those in (T)PAL). Hence, there is no decompositional analysis from the logic of sights back into the basic modal language. Finally, we note that while Liu et al. [21] proposes a logic of extensive games with short sights, where $v \models \langle\sigma\rangle\varphi$ holds whenever the visibly terminal state—that is terminal in the sight $s(v)$—that players would reach by playing according to the strategy profile σ is a φ-state. While the motivation behind the development of [21] and our work is similar, formally the two systems are very different. In [21], the action modalities, as well, as the sight modality receive the standard relational semantics of basic modal logic. We show below that the expressive power of such a static sight operator is incomparable with our dynamic sight operator.

We give an axiomatization for our logic, which uses induction rules corresponding to the well-known fact that universal formulas are preserved under restrictions. It follows a simple idea: every transition that can be done in the current sight can be done in general, and hence is simulated in the full model. We prove the completeness of our axiom system. We compare the expressive power of the logic with some other known logics. We also show that the satisfiability problem for the logic is PSPACE-complete. Finally, we consider some interesting extensions, and discuss the definability problem of sight-based subgame perfection.

Structure of the article: The rest of the article is organized as follows. Section 2 has notation and technical background, that might be skipped at first reading. Section 3 introduces the language of our logic of sights, $\mathsf{L}_{(!\sigma,A)}$, and its semantics. We discuss its translation into decidable fragments of FO. Section 4 discusses the expressive power of $\mathsf{L}_{(!\sigma,A)}$. In Section 5, we provide an axiomatization of $\mathsf{L}_{(!\sigma,A)}$ and prove its completeness in Section 6. Section 7 shows that the satisfiability problem for $\mathsf{L}_{(!\sigma,A)}$ is PSPACE-complete and the combined complexity of its model-checking problem is in PTIME. Section 8 discusses the relation of $\mathsf{L}_{(!\sigma,A)}$ to other logics, as well as extensions. Section 9 discusses the modal definability of sight-based subgame perfection. We conclude in Section 10.

2 Preliminaries

In this section, we give the necessary background and define the notation used in the article.

2.1 Basic notation

Let ϵ denote the empty sequence. Given a set X we let X^* be the set of finite sequences over X and let $X^+ = X^* \setminus \{\epsilon\}$. Given a relation $R \subseteq X \times X$, R^* will be the reflexive transitive closure of R. It will always be clear from the context whether we mean the reflexive transitive closure of a relation or the set of finite sequences over a set. And we define $R[x] = \{y \in X \mid (x,y) \in R\}$—for $A \subseteq X$, $R[A] = \{y \in X \mid (x,y) \in R \text{ for some } x \in A\}$. We use $R(x)$ as a notational variant of $R[x]$ and $R(A)$ as a notational variant of $R[A]$. Given a sequence (s_i), s_n is the n-th member of (s_i). Given a sequence $\vec{x}$, $|\vec{x}|$ denotes the set of elements in $\vec{x}$, $\vec{x}[k]$ its k-th-element and $\vec{x}|_k$ its initial prefix of length k and $\mathsf{len}(\vec{x})$ denotes the length of $\vec{x}$. Given a non-empty finite sequence $\vec{x}$, we let $\mathsf{last}(\vec{x}) = \vec{x}[\mathsf{len}(\vec{x})]$. We use ω for the set of natural numbers. Given a relational structure $\mathbb{M}$, we refer to its domain by $|\mathbb{M}|$, but sometimes also by $\mathbb{M}$. In Section 7, on complexity, we use $|X|$ to refer to the cardinality of X. We make sure that it is clear from the context we which one we intend. Finally, given a function f we let $\mathsf{dom}(f)$ be its domain and $\mathsf{ran}(f)$ be its range.

2.2 *LTS and the basic modal language*

DEFINITION 2.1
Given a finite non-empty set A, an A-labeled transition system (henceforth LTS(A)) is a tuple $\langle W, (\xrightarrow{a})_{a\in A}, V\rangle$ where W is a non-empty set, A is a set of labels, for each $a\in A$, $\xrightarrow{a}\subseteq(W\times W)$, and for some non-empty countable set of propositional letters PROP, $V:\text{PROP}\to\wp(W)$ is a valuation function.

When the underlying set of labels A is unimportant or is clear from the context, we simply write LTS for LTS(A). Given an LTS $\mathbb{M}=\langle W, (\xrightarrow{a})_{a\in A}, V\rangle$, the elements of W are called the states of $\mathbb{M}$. We also use the notation $|\mathbb{M}|$ to denote the set of states W of the LTS $\mathbb{M}$.

Intuitive interpretation. We would like to be as comprehensive as possible as how to interpret these models, as we believe that the concepts of sights we will discuss are interesting in many contexts or situations that can be modeled as labeled transition systems. An intuitive, yet generic, interpretation is to think of W as states of a system and A as possible events or actions that can change those states. In general, more than one action could be executable in a given state.

DEFINITION 2.2
Given an LTS $\mathbb{M}$ and a state $w\in|\mathbb{M}|$, the pair $(\mathbb{M},w)$ is called a pointed model or a pointed LTS.

A tree-like LTS (now TLTS) is such that for any $a,b\in A$ with $a\neq b$ we have $\xrightarrow{a}\cap\xrightarrow{b}=\emptyset$ and moreover $\to=\bigcup_{a\in A}\xrightarrow{a}$ is such that $\langle W,\to\rangle$ is a tree.

The basic modal language L_A over A is recursively defined as follows:

$$\varphi ::= p \mid \neg\varphi \mid \varphi\wedge\varphi \mid \langle a\rangle\varphi,$$

where p ranges over PROP and a over A. Given an LTS $\mathbb{M}$ and a state $w\in|\mathbb{M}|$, we define the satisfaction of a formula $\varphi\in\mathsf{L}_A$ at w inductively:

$$\begin{array}{lll}
\mathbb{M},w\models p & \text{iff} & w\in V(p)\\
\mathbb{M},w\models\neg\varphi & \text{iff} & \mathbb{M},w\not\models\varphi\\
\mathbb{M},w\models\varphi\wedge\psi & \text{iff} & \mathbb{M},w\models\varphi \text{ and } \mathbb{M},w\models\psi\\
\mathbb{M},w\models\langle a\rangle\varphi & \text{iff} & \text{for some } v \text{ with } w\xrightarrow{a}v \text{ we have } \mathbb{M},v\models\varphi
\end{array}$$

Given a modal language L, a formula $\varphi\in\mathsf{L}$ and a model $\mathbb{M}$, we let $||\varphi||^{\mathbb{M}}:=\{w\in|\mathbb{M}| \mid \mathbb{M},w\models\varphi\}$, be the truth set of φ in $\mathbb{M}$. We simply write $||\varphi||$ when $\mathbb{M}$ is clear from context.

2.3 *Invariance and expressivity*

Let $\mathbb{M}=\langle W, (\xrightarrow{a})_{a\in A}, V\rangle$ and $\mathbb{M}'=\langle W', (\xrightarrow{a'})_{a\in A}, V'\rangle$ be two LTS(A)s.

DEFINITION 2.3 (Simulation)
A non-empty binary relation $Z\subseteq W\times W'$ is a simulation from $\mathbb{M}$ to $\mathbb{M}'$ iff Z is such that:

1. $x'\in V'(p)$ whenever $(x,x')\in Z$ and $x\in V(p)$
2. If $x\xrightarrow{a}y$ and $(x,x')\in Z$, then there exists $y'\in W'$ such that $(y,y')\in Z$ and $x'\xrightarrow{a}{}' y'$

DEFINITION 2.4 (Bisimulation)
A non-empty binary relation $Z \subseteq W \times W'$ is a bisimulation between $\mathbb{M}$ and $\mathbb{M}'$ iff Z is such that:

1. $x \in V(p)$ iff $x' \in V'(p)$, whenever $(x, x') \in Z$
2. If $x \xrightarrow{a} y$ and $(x, x') \in Z$, then there exists $y' \in W'$ such that $(y, y') \in Z$ and $x' \xrightarrow{a}{}' y'$
3. If $x' \xrightarrow{a}{}' y'$ and $(x, x') \in Z$, then there exists $y \in W$ such that $(y, y') \in Z$ and $x \xrightarrow{a} y$

DEFINITION 2.5 (Bounded p-morphisms)
A function $f : W \to W'$ is a bounded p-morphism from $\mathbb{M}$ to $\mathbb{M}'$ iff f is such that

1. $x \in V(p)$ iff $f(x) \in V'(p)$
2. $f(x) \xrightarrow{a}{}' f(y)$ whenever $x \xrightarrow{a} y$
3. If $f(x) \xrightarrow{a}{}' y'$, then there exists y such that $x \xrightarrow{a} y$ and $f(y) = y'$

DEFINITION 2.6 (Submodel)
Let $\mathbb{M} = \langle W, (\xrightarrow{a})_{a \in A}, V \rangle$ be an LTS. Given a non-empty subset $X \subseteq W$, we let $\mathbb{M}|_X := \langle X, (\xrightarrow{a}|_X)_{a \in A}, V|_X \rangle$ where $\xrightarrow{a}|_X$ is the restriction of $\xrightarrow{a}$ to X and $V|_X$ is the restriction of V to X. If $\mathbb{M}' = \mathbb{M}|_X$ for some non-empty subset $X \subseteq W$, we say that $\mathbb{M}'$ is a submodel of $\mathbb{M}$.

DEFINITION 2.7 (Generated submodel)
If $\mathbb{M}'$ is a submodel of $\mathbb{M}$ and $y \in W'$ whenever $x \in W'$ and $x \xrightarrow{a} y$ for some $x \in W'$ and some $a \in A$, we say that $\mathbb{M}'$ is an A-generated submodel of $\mathbb{M}$. We drop A whenever it is clear from context.

DEFINITION 2.8
We say that two pointed models $\mathbb{M}, w$ and $\mathbb{M}', w'$ are L-equivalent iff for every $\varphi \in \mathsf{L}$ we have $\mathbb{M}, w \models \varphi$ iff $\mathbb{M}', w' \models \varphi$. We write $\mathbb{M}, w \leftrightsquigarrow_{\mathsf{L}} \mathbb{M}', w'$. We drop L whenever it is clear from context. We also drop $\mathbb{M}$ and $\mathbb{M}'$ whenever $\mathbb{M} = \mathbb{M}'$.

DEFINITION 2.9 (Expressivity)
We say that a language L' is at least as expressive as a language L over a class of pointed models $\mathcal{C}$ (denoted $L \leq_{\mathcal{C}} L'$) if for every pair of pointed models $(\mathbb{M}, w), (\mathbb{M}', w') \in \mathcal{C}$ if there is a formula $\varphi \in L$ with $\mathbb{M}, w \models \varphi$ but $\mathbb{M}', w' \not\models \varphi$, we also have a formula $\varphi' \in L'$ such that $\mathbb{M}, w \models \varphi'$ but $\mathbb{M}', w' \not\models \varphi'$. When the intended class of pointed models $\mathcal{C}$ is clear from context we drop the subscript.

2.4 *Guarded and bounded fragments of FO*

Given an FO-formula φ, $\mathsf{Free}(\varphi)$ is the set of free variables occurring in φ.

DEFINITION 2.10
The guarded fragment GF of FO is defined by induction as follows:

1. Every relational atomic formula belongs to GF.
2. GF is closed under propositional connectives $\neg, \wedge, \vee, \to, \leftrightarrow$.
3. If $\vec{x}, \vec{y}$ are tuples of variables, $\alpha(\vec{x}, \vec{y})$ is atomic and $\varphi(\vec{x}, \vec{y})$ is a formula in GF, such that $\mathsf{Free}(\varphi) \subseteq \mathsf{Free}(\alpha) = \{\vec{x}, \vec{y}\}$, then the formulas

$$\exists \vec{y}(\alpha(\vec{x}, \vec{y}) \wedge \varphi(\vec{x}, \vec{y})) \text{ and } \forall \vec{y}(\alpha(\vec{x}, \vec{y}) \to \varphi(\vec{x}, \vec{y}))$$

are in GF.

DEFINITION 2.11 ([29])
The loosely guarded fragment LGF of FO is defined similarly to the GF, by relaxing clause (3) as follows:

3′. If $\psi \in \mathsf{LGF}$, γ is a conjunction of relational atomic formulas, $\mathsf{Free}(\psi) \subseteq \mathsf{Free}(\gamma)$, $\vec{y}$ is a tuple of free variables of γ and moreover

- if $x \in \mathsf{Free}(\gamma)$, $y \in |\vec{y}|$ and $x \neq y$, then there is a conjunct of γ in which x, y both occur,

then $\exists \vec{y}(\gamma \wedge \psi) \in \mathsf{LGF}$.

Van Benthem [29] calls the LGF of FO, the pair-wise guarded fragment. The clause (3′) relaxes (3) in that the guard that relativizes the quantifier does not have to be an atomic formula, but can be a conjunction of atomic formulas, provided that for every quantified variable y and every free variable x from the guard there is an atom of the guard in which both x and y occur. Two examples taken from [12], will illustrate this condition.

First, an example of a formula that is loosely guarded, but not guarded, is the first-order translation of (ψ **until** φ):

$$\exists y\ (x \leq y \wedge \varphi(y) \wedge \forall z\ ((x \leq z \wedge z \leq y) \rightarrow \psi(z)))$$

By contrast the first-order translation of the transitivity axiom:

$$\forall xyz\ ((Rxy \wedge Ryz) \rightarrow Rxz)$$

is not loosely guarded.

DEFINITION 2.12
Given a relational signature, the bounded fragment BF of FO is defined by induction as follows:

1. Every relational atomic formula belongs to BF.
2. BF is closed under propositional connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$.
3. If $\vec{y}$ is a $(n-1)$-tuple of variables, t is a term with $t \notin |\vec{y}|$ and $\varphi \in \mathsf{BF}$, then for every n-ary relational symbol R in the relational signature, the formulas

$$\exists \vec{t}(Rx\vec{y} \wedge \varphi) \text{ and } \forall \vec{t}(Rx\vec{y} \rightarrow \varphi)$$

are also in BF.

THEOREM 2.13 ([8, 9])
Given a relational signature τ, a formula φ of $\mathsf{FO}(\tau)$ is equivalent to a formula in $\mathsf{BF}(\tau)$ iff φ is invariant under τ-generated submodels.

2.5 *Extensions of basic modal logic*

In this article, we refer to two well-understood extensions of basic modal logic. The first one $\mathsf{L}_{A,\cap}$ allows for operators that exploit the intersection of relations in the similarity type. The second one adds nominals, names for states, to the language, and is the basic hybrid logic H_A over A.

2.5.1 Modal logic with intersection

The modal language $\mathsf{L}_{A,\cap}$ over A is recursively defined as follows:

$$\alpha ::= a \mid \alpha \cap \alpha; \quad \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle\alpha\rangle\varphi,$$

where p ranges over PROP and a over A. R_α, the interpretation of α is defined recursively as follows:

$$\begin{aligned} R_a &= \xrightarrow{a} \\ R_{\alpha\cap\beta} &= R_\alpha \cap R_\beta \end{aligned}$$

Given an LTS $\mathbb{M}$ and a state $w \in |\mathbb{M}|$, the only new clause in definition of the satisfaction of a formula $\varphi \in \mathsf{L}_{A,\cap}$ at w is the following:

$$\mathbb{M}, w \models \langle\alpha\rangle\varphi \text{ iff } \quad \text{for some } v \text{ with } wR_\alpha v \text{ we have } \mathbb{M}, v \models \varphi$$

2.5.2 Basic hybrid logic

The basic hybrid language H_A over A, has another parameter, the set of nominals NOM, disjoint from the set of propositional letters PROP. It is interpreted over LTSs for which the valuation V has domain PROP$\uplus$NOM and such that for every $p_i \in$ NOM, $|V(p_i)| = 1$. When we need to be clear about NOM; we write H_A(NOM). The modal language H_A(NOM) is recursively defined as follows:

$$\varphi ::= p \mid p_i \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle a\rangle\varphi,$$

where p ranges over PROP, p_i over NOM and a over A. Its semantics is defined on LTSs. The satisfaction definition contains nothing new except for a clause for nominals:

$$\mathbb{M}, w \models p_i \text{ iff } V(p_i) = \{w\}$$

2.6 *Computational complexity*

We assume familiarity with basic computational complexity theory (for an introduction, see, e.g. [24]). We refer to a set of (encoding of) inputs as a language. Given a function $f : \omega \to \omega$, DSPACE(f) is the class of languages which can be recognized by a deterministic Turing machine M which uses at most $f(|x|)$ cells of the working tape for inputs x of size $|x| \geq n_0$ for some constant $n_0 \in \omega$, on all branches in the computation tree of M on x.

Alternation is a generalization of non-determinism. For a precise definition of an alternating Turing machine and corresponding notions, see [4]. Here is an informal definition:

DEFINITION 2.14 (Alternating Turing machine, see, e.g. [18])
An alternating Turing machine is a Turing machine whose states are divided into existential and universal states. The notion of acceptance is defined recursively as for non-deterministic Turing machines, except that an alternating Turing machine in a universal state accepts if there is at least one next configuration and all next configurations are accepting.

Given a function $f:\omega\to\omega$, let ATIME(f) be the class of languages which can be recognized by an alternating Turing machine M whose computation tree depth for inputs x of size $|x|\geq n_0$, for some constant $n_0\in\omega$, is bounded by $f(|x|)$. We write: $\mathsf{APTIME}=\bigcup_{k\in\omega}\mathsf{ATIME}(n^k)$ and we make use of the following result:

THEOREM 2.15 ([4])
$\mathsf{PSPACE}=\mathsf{APTIME}$

3 Dynamic logic of sights

We extend labeled transition systems with sights, which are functions associating to each state a set of states, which contains that state.

DEFINITION 3.1
A Labeled Transition System with Sights (henceforth σLTS) $\mathbb{M}$ is an LTS expanded with a function: $\mathsf{N}:W\to\wp(W)$ such that $w\in\mathsf{N}(w)$ for every $w\in W$.

Given a non-empty subset $X\subseteq W$, we write

$$\mathbb{M}|_X:=\langle X,(\xrightarrow{a}|_X)_{a\in A},V|_X\rangle$$

where $\xrightarrow{a}|_X$ is the appropriate restriction to X. Let $\mathfrak{C}_\sigma$ be the set of all labeled transition system with sights. A σTLTS is a σLTS such that its N-free reduct is a TLTS.

The language of $\mathsf{L}_{(!\sigma,A)}$ over A extends the basic modal language L_A over A with an unary operator $[!\mathsf{N}]$:

$$\varphi::=p\,|\,\neg\varphi\,|\,\varphi\wedge\varphi\,|\,\langle a\rangle\varphi\,|\,[!\mathsf{N}]\varphi,$$

where p ranges over PROP and a over A. The semantics of the Boolean and of the basic modal operators is the usual one. The semantics of $[!\mathsf{N}]\varphi$ is defined as follows:

$$\mathbb{M},w\models[!\mathsf{N}]\varphi\ \text{ iff }\ \mathbb{M}|_{\mathsf{N}(w)},w\models\varphi$$

Informally speaking, $[!\mathsf{N}]\varphi$ means that from the perspective of the agent at the current state φ holds.

As we said earlier, we would like to be as minimal as possible as how to A should be interpreted as we believe the concept of sight is relevant in different contexts that call for different interpretation of the meaning of A. If the reader would like to have some concrete intuition to rely on, she or he can think of A as possible actions or events that can change the state that the system is in. $\langle a\rangle\varphi$ would then mean that action a can be executed in the current state and in one of the possible execution of a, φ will hold afterwards. To take an example involving both the sight and the action modality, the intuitive interpretation of:

$$([!\mathsf{N}](\langle a\rangle p\wedge\neg\langle a\rangle\neg p))\wedge\langle a\rangle\neg p$$

is that 'in the sight of the agent, the action a is executable and executing a will necessary lead to the system being in a p-state, while in fact there is a possible execution of a that leads to a $\neg p$-state'. But again, we believe that the approach is more general and interesting in contexts where other interpretations of A are evoked.

3.1 Notions of complexity and modal depth for $\mathsf{L}_{(!\sigma,A)}$

Depending on the context two notions of complexity will be useful, the first one depends on the negation normal form of a formula. We write it as $\mathsf{cpx}:\mathsf{L}_{(!\sigma,A)}\to\omega$ and define it as follows:

$$\begin{array}{lcl}
\mathsf{cpx}(p) & = & \mathsf{cpx}(\neg p) \quad = \quad 1 \\
\mathsf{cpx}(\varphi\wedge\psi) & = & \max\{\mathsf{cpx}(\varphi),\mathsf{cpx}(\psi)\}+1 \\
\mathsf{cpx}(\varphi\vee\psi) & = & \max\{\mathsf{cpx}(\varphi),\mathsf{cpx}(\psi)\}+1 \\
\mathsf{cpx}(\langle a\rangle\varphi) & = & \mathsf{cpx}(\varphi)+1 \\
\mathsf{cpx}([a]\varphi) & = & \mathsf{cpx}(\varphi)+1 \\
\mathsf{cpx}([!\mathsf{N}]\varphi) & = & \mathsf{cpx}(\varphi)+1
\end{array}$$

Alternatively, we will use $\mathsf{com}:\mathsf{L}_{(!\sigma,A)}\to\omega$ with the following modified clause for negation:

$$\mathsf{com}(\neg\varphi) \quad = \quad \mathsf{com}(\varphi)+1$$

Later, we will also need a notion of A-modal depth $\mathsf{dep}_A:\mathsf{L}_{(!\sigma,A)}\to\omega$, defined as follows:

$$\begin{array}{lcl}
\mathsf{dep}_A(p) & = & 0 \\
\mathsf{dep}_A(\neg\varphi) & = & \mathsf{dep}_A(\varphi) \\
\mathsf{dep}_A(\varphi\wedge\psi) & = & \max\{\mathsf{dep}_A(\varphi),\mathsf{dep}_A(\psi)\} \\
\mathsf{dep}_A(\varphi\vee\psi) & = & \max\{\mathsf{dep}_A(\varphi),\mathsf{dep}_A(\psi)\} \\
\mathsf{dep}_A(\langle a\rangle\varphi) & = & \mathsf{dep}_A(\varphi)+1 \\
\mathsf{dep}_A([a]\varphi) & = & \mathsf{dep}_A(\varphi)+1 \\
\mathsf{dep}_A([!\mathsf{N}]\varphi) & = & \mathsf{dep}_A(\varphi)
\end{array}$$

as well as a notion of σ-modal depth $\sigma\mathsf{dep}_A:\mathsf{L}_{(!\sigma,A)}\to\omega$, which differs in only one clause from A-modal depth:

$$\sigma\mathsf{dep}_A([!\mathsf{N}]\varphi) \quad = \quad \sigma\mathsf{dep}_A(\varphi)+1$$

3.2 Standard translation of $\mathsf{L}_{(!\sigma,A)}$

We show how to translate $\mathsf{L}_{(!\sigma,A)}$ into the loosely guarded fragment of FO.

DEFINITION 3.2
Let $\mathsf{FO(N,A)}$ be the first-order language with unary relation symbols $(R_p)_{p\in\mathrm{PROP}}$ corresponding to propositional letters in $p\in\mathrm{PROP}$, binary relations $(R_a)_{a\in A}$ for each label $a\in A$ and finally a binary relation N with $\mathsf{N}xy$ to mean $y\in\mathsf{N}(x)$.

We let VAR be our set of first-order variables. An assignment in $\mathbb{M}$ is a function $g:VAR\to|\mathbb{M}|$. We let $g[x_i/w_i]$ be defined as follows:

$$g[x_i/w_i](x)=\begin{cases} g(x) & \text{if } x\neq x_i, \\ w_i & \text{if } x=x_i. \end{cases} \tag{1}$$

We also let $g[x_i/w_i,x_j/w_j]=(g[x_i/w_i])[x_j/w_j]$.

Let $x \in VAR$ be a first-order variable, $\sigma \in VAR^*$ be a finite sequence of first-order variables and consider the following translation, the standard translation of $\mathsf{L}_{(!\sigma,A)}$:

$$\mathrm{ST}_x : (\mathsf{L}_{(!\sigma,A)} \times VAR^*) \to \mathsf{FO(N,A)}$$

which is recursively defined as follows:

$$\begin{array}{lcl}
\mathrm{ST}_x(p^\epsilon) & = & R_p(x) \\
\mathrm{ST}_x(p^\sigma) & = & \mathrm{ST}_x(p) \\
\mathrm{ST}_x((\neg\varphi)^\sigma) & = & \neg\mathrm{ST}_x(\varphi^\sigma) \\
\mathrm{ST}_x((\varphi \wedge \psi)^\sigma) & = & \mathrm{ST}_x(\varphi^\sigma) \wedge \mathrm{ST}_x(\psi^\sigma) \\
\mathrm{ST}_x((\langle a\rangle\varphi)^\sigma) & = & \exists y\ (\bigwedge_{z\in|\sigma|} \mathsf{N}(z,y) \wedge R_a(x,y) \wedge \mathrm{ST}_y(\varphi^\sigma)) \\
 & & \qquad\qquad\qquad\qquad\qquad\qquad \text{[where } y \text{ is fresh]} \\
\mathrm{ST}_x(([!\mathsf{N}]\varphi)^\sigma) & = & \mathrm{ST}_x(\varphi^{\sigma x}) \\
\mathrm{ST}_x(\varphi^{\sigma xx}) & = & \mathrm{ST}_x(\varphi^{\sigma x})
\end{array}$$

For every $\mathsf{L}_{(!\sigma,A)}$ formula and every finite sequence of variables $\sigma \in VAR^*$ we have:

FACT 3.3
$|\mathrm{ST}_x(\varphi^\sigma)| < f(|\varphi^\sigma|)$ where f is a polynomial function on ω.

PROOF. By induction on the structure of φ. The atomic cases and the negation case are immediate. For $\mathrm{ST}_x((\varphi \wedge \psi)^\sigma)$, we have by induction hypothesis (IH) that $|\mathrm{ST}_x(\varphi^\sigma)|$ and $|\mathrm{ST}_x(\psi^\sigma)|$ are bounded by polynomials in $|\varphi^\sigma|$ and $|\psi^\sigma|$, respectively, and hence $|\mathrm{ST}_x((\varphi \wedge \psi)^\sigma)| \leq |\mathrm{ST}_x(\varphi^\sigma)| + |\mathrm{ST}_x(\psi^\sigma)|$ is bounded by a polynomial in $|\mathrm{ST}_x((\varphi \wedge \psi)^\sigma)|$. $|\mathrm{ST}_x(\langle a\rangle\varphi^\sigma)| \leq |\mathrm{ST}_x(\varphi^\sigma)| + c_1|\sigma| + c_2$ where c_1 and c_2 are constants and since $|\mathrm{ST}_x(\varphi^\sigma)|$ is bounded by a polynomial in $|\varphi^\sigma|$, $|\mathrm{ST}_x(\langle a\rangle\varphi^\sigma)|$ is bounded by a polynomial in $|\langle a\rangle\varphi^\sigma|$ as well. The cases for $([!\mathsf{N}]\varphi)^\sigma$ and $\varphi^{\sigma xx}$ are straightforward. ■

To prove that the above translation is correct, we need the following lemma, which intuitively shows that $ST_x(\varphi^\sigma)$ correctly captures the idea of φ being true in the restriction to $\bigwedge_{x\in\sigma} \mathsf{N}(x)$. We will use the following piece of notation. Let $Z \subseteq |\mathbb{M}|$, we write $\mathsf{N}(Z) := \bigcap_{w\in Z} \mathsf{N}(w)$. We often abuse notation and write $\vec{w}$ for $|\vec{w}|$. For example, given $\vec{w} \in |\mathbb{M}|^*$, we write $\mathsf{N}(\vec{w})$ for $\mathsf{N}(|\vec{w}|)$, and $\mathbb{M}|_{\mathsf{N}(\vec{w})}$ for $\mathbb{M}|_{\mathsf{N}(|\vec{w}|)}$. In particular, given $\vec{w} = (w_0, \ldots, w_n)$, with $w_{i+1} \in |\mathbb{M}| \cap \mathsf{N}(w_0) \cap \ldots \cap \mathsf{N}(w_i)$, for all i with $0 \leq i \leq n$, we have $\mathbb{M}|_{\mathsf{N}(\vec{w})} = (\ldots(\mathbb{M}|_{\mathsf{N}(w_0)})|\ldots)|_{\mathsf{N}(w_n)}$.

LEMMA 3.4
Let $\mathbb{M}$ be a pointed $\sigma\mathsf{LTS}$. For every $\vec{w} \in |\mathbb{M}|^*$, every $\vec{x} \in VAR^*$, every $w' \in \mathsf{N}(\vec{w})$ and every assignment $g : VAR \to |\mathbb{M}|$, we have:

$$\mathbb{M}|_{\mathsf{N}(\vec{w})}, w' \models \varphi \text{ iff } \mathbb{M}, g[x_0/w_0, \ldots, x_n/w_n, y/w'] \models ST_y((\varphi)^{\vec{x}})$$

PROOF. The proof is by induction on the complexity of φ. The base case is immediate from the definition of the standard translation. So is the induction step for negation and conjunction. Now assume that $\varphi = \langle a\rangle\psi$. Assume that $\mathbb{M}|_{\mathsf{N}(\vec{w})}, w' \models \langle a\rangle\psi$. It follows that there is a $t \in |\mathbb{M}|$ such that:

(a) for all $w_i \in \vec{w}$, $t \in \mathsf{N}(w_i)$
(b) $w' \xrightarrow{a} t$
(c) $\mathbb{M}|_{\mathsf{N}(\vec{w})}, t \models \psi$

By the IH, we have (c) iff (d) $\mathbb{M}, g[\vec{x}/\vec{w}, z/t] \models ST_z((\psi)^{\vec{x}})$. From (a), (b) and (d) it follows that:

$$\mathbb{M}, g[\vec{x}/\vec{w}, y/w', z/t] \models \bigwedge_{x_i \in \vec{x}} \mathsf{N}x_i z \wedge R_a yz \wedge ST_z((\psi)^{\vec{x}}) \tag{2}$$

By semantics of first-order logic we have thus:

$$\mathbb{M}, g[\vec{x}/\vec{w}, y/w'] \models \exists z \bigwedge_{x_i \in \vec{x}} \mathsf{N}x_i z \wedge R_a yz \wedge ST_z((\psi)^{\vec{x}}) \tag{3}$$

But this is equivalent to

$$\mathbb{M}, g[\vec{x}/\vec{w}, y/w'] \models ST_y((\langle a \rangle \psi)^{\vec{x}}) \tag{4}$$

The other direction is similar, which concludes the proof for this case.

Assume that $\mathbb{M}|_{\mathsf{N}(\vec{w})}, w' \models [!\mathsf{N}]\psi$. It follows by semantics of $[!\mathsf{N}]$ that: $\mathbb{M}|_{\mathsf{N}(\vec{w} \cup w')}, w' \models \psi$. But then by the IH it follows that $\mathbb{M}, g[\vec{x}/\vec{w}, y/w] \models ST_y((\psi)^{\vec{x} \cup y})$. Hence, $\mathbb{M}, g[\vec{x}/\vec{w}, y/w] \models ST_y(([!\mathsf{N}]\psi)^{\vec{x}})$. The argument works in both directions, concluding this case and the induction step. ■

We can now prove that the above translation is correct:

PROPOSITION 3.5
For each $\varphi \in \mathsf{L}_{(!\sigma,A)}$, each pointed $\sigma\mathsf{LTS}$ $(\mathbb{M}, s)$, and an assignment $g : VAR \to |\mathbb{M}|$ the following holds:

$$\mathbb{M}, s \models \varphi \text{ iff } \mathbb{M}, g[g(x) := s] \models \mathrm{ST}_x(\varphi)$$

PROOF. The proof is by induction on the complexity of φ.

Base case: Assume that $\varphi = p$ for some propositional letter $p \in$ PROP. We simply note that $\mathbb{M}, w \models p$ iff $w \in V(p)$ iff $\mathbb{M}, g[x/w] \models R_p(x)$ iff $\mathbb{M}, g[x/s] \models ST_x(p)$.

Induction step: Assume that the claim holds for formulas of complexity at most n. We have to prove that it holds for formulas of complexity at most $n+1$. If the main operator is a conjunction and or a negation, the claim is immediate from the definition of ST_x.

Assume that $\varphi = \langle a \rangle \psi''$ and that $\mathbb{M}, w \models \langle a \rangle \psi$. By semantics there is some state w' with

$$R_a ww' \text{ and } \mathbb{M}, w' \models \psi \tag{5}$$

Hence, by inductive hypothesis (henceforth IH), $\mathbb{M}, g[y/w'] \models ST_y(\psi)$. By definition

$$\begin{aligned} ST_x(\langle a \rangle \psi) &= ST_x((\langle a \rangle \psi)^{\epsilon}) \\ &= \exists y \, (\bigwedge_{z \in |\epsilon|} \mathsf{N}(z, y) \wedge R_a(x, y) \wedge ST_y(\psi^{\epsilon})) \\ &= \exists y \, (R_a(x, y) \wedge ST_y(\psi^{\epsilon})) \end{aligned}$$

But then we can set $g(x) := w$ and $g(y) := w'$ and have by (5) and the IH that $\mathbb{M}, g \models R_a(x, y) \wedge ST_y(\psi^{\epsilon})$. The argument is symmetric for the other direction.

Now assume that $\varphi = [!\mathsf{N}]\psi$ and assume $\mathbb{M}, w \models [!\mathsf{N}]\psi$. Note that this holds iff $\mathbb{M}|_{\mathsf{N}(w)}, w \models \psi$. But by Lemma 3.4 this holds iff $\mathbb{M}, g[x/w] \models ST_x((\varphi)^x)$. But this is equivalent to $\mathbb{M}, g[x/w] \models ST_x(([!\mathsf{N}]\varphi))$, which concludes the argument for this case and the proof. ■

Observe that the translation may not belong to the guarded fragment of FO:

FACT 3.6
$\mathrm{ST}_x([!\mathsf{N}]\langle a\rangle[!\mathsf{N}]\langle b\rangle p)$ is not guarded.

PROOF. We show that the guards in $\mathrm{ST}_x([!\mathsf{N}]\langle a\rangle[!\mathsf{N}]\langle b\rangle p)$ are not atomic.

$$\begin{aligned}
&ST_x([!\mathsf{N}]\langle a\rangle[!\mathsf{N}]\langle b\rangle p)\\
&=ST_x(\langle a\rangle[!\mathsf{N}]\langle b\rangle p^x)\\
&=\exists y(\mathsf{N}(x,y)\wedge R_a(x,y)\wedge ST_y([!\mathsf{N}]\langle b\rangle p^x))\\
&=\exists y(\mathsf{N}(x,y)\wedge R_a(x,y)\wedge ST_y(\langle b\rangle p^{xy}))\\
&=\exists y(\mathsf{N}(x,y)\wedge R_a(x,y)\wedge(\exists z(\mathsf{N}(y,z)\wedge R_b(y,z)\wedge ST_y(p^{xy}))))\\
&=\exists y(\mathsf{N}(x,y)\wedge R_a(x,y)\wedge(\exists z(\mathsf{N}(y,z)\wedge R_b(y,z)\wedge R_p(y))))
\end{aligned}$$

But the guard of the main formula $\mathsf{N}(x,y)\wedge R_a(x,y)$ is not atomic, and neither is the guard of $\exists z(\underbrace{\mathsf{N}(y,z)\wedge R_b(y,z)}_{\text{non atomic}}\wedge R_p(y))$.
Hence, $ST_x([!\mathsf{N}]\langle a\rangle[!\mathsf{N}]\langle b\rangle p)$ is not guarded. ■

However, the standard translation of $\mathsf{L}_{(!\sigma,A)}$-formula is in the loosely guarded fragment of FO. To see this, first observe that

LEMMA 3.7
For every $\sigma\in VAR^*$, $\varphi\in\mathsf{L}_{(!\sigma,A)}$ and $x\in VAR$, $\mathsf{Free}(ST_x(\varphi^\sigma))\subseteq\{x\}\cup|\sigma|$.

PROOF. The proof is by induction on the complexity of φ. **Base case:** Note that for any $\sigma\in VAR^*$ and $x\in VAR$, $\mathsf{Free}(ST_x(p^\sigma))=\mathsf{Free}(R_p(x))=\{x\}$. **Induction hypothesis;** Assume that the claim holds for formulas of complexity at most n. **Induction Step;** Now assume that φ is of complexity $n+1$. The proof is now by case. **Case** $\varphi=\neg\psi$. Simply observe that $\mathsf{Free}(ST_x(\neg\psi^\sigma))=\mathsf{Free}(ST_x(\psi^\sigma))$. The claim follows from IH. **Case** $\varphi=\psi\wedge\chi$. Observe that $\mathsf{Free}(ST_x((\psi\wedge\chi)^\sigma))=\mathsf{Free}(ST_x(\psi^\sigma)\wedge ST_x(\chi^\sigma))=\mathsf{Free}(ST_x(\psi^\sigma))\cup\mathsf{Free}(ST_x(\chi^\sigma))$. The claim follows from IH. **Case** $\varphi=\langle a\rangle\psi$. Now,

$$\begin{aligned}
&\mathsf{Free}(ST_x((\langle a\rangle\psi)^\sigma))=\\
&\mathsf{Free}(\exists y\,(\bigwedge_{z\in|\sigma|}\mathsf{N}(z,y)\wedge R_a(x,y)\wedge \mathrm{ST}_y(\psi^\sigma))),\text{ for some } y
\end{aligned}$$

That is, $\mathsf{Free}(ST_x((\langle a\rangle\psi)^\sigma))=|\sigma|\cup\{x\}\cup\mathsf{Free}(ST_y(\psi^\sigma))\setminus\{y\}$. But from IH we have $\mathsf{Free}(ST_y(\psi^\sigma))\setminus\{y\}\subseteq|\sigma|$. **Case** $\varphi=[!\mathsf{N}]\psi$. First observe that $\mathsf{Free}(ST_x([!\mathsf{N}]\psi^\sigma))=\mathsf{Free}(ST_x(\psi^{\sigma x}))$. But by IH it follows that $\mathsf{Free}(ST_x(\psi^{\sigma x}))\subseteq|\sigma x|\cup\{x\}=|\sigma|\cup\{x\}$. ■

We can now prove that:

PROPOSITION 3.8
For every formula $\varphi\in\mathsf{L}_{(!\sigma,A)}$, $x\in VAR$ and $\sigma\in VAR^*$, $\mathrm{ST}_x(\varphi^\sigma)$ is loosely guarded.

PROOF. Take an arbitrary $\sigma\in VAR^*$ and some $x\in VAR$. For the base case, simply note $ST_x(p^\sigma)$ is an atomic formula, hence loosely guarded. Negation and conjunctions are straightforward from the

closure of LGF under Booleans. Now for the $\langle a\rangle$-case. First note, that by Lemma 3.7, we have

$$\mathrm{ST}_x((\langle a\rangle\psi)^\sigma) = \underbrace{\exists y}_{\vec{y}=\{y\}} \underbrace{(\bigwedge_{z\in|\sigma|} \mathsf{N}(z,y) \wedge R_a(x,y) \wedge}_{\mathsf{Free}(\gamma)=|\sigma|\cup\{x,y\}} \underbrace{\mathrm{ST}_y(\psi^\sigma))}_{\mathsf{Free}(\psi^\sigma)\subseteq\{y\}\cup|\sigma|}$$

(where y is fresh)

And observe that all conditions of the loosely guarded quantification are respected:

- γ is a conjunction of atomic formulas
- By IH, $\psi^\sigma \in \mathsf{LGF}$
- $\mathsf{Free}(ST_y(\psi^\sigma)) \subseteq \mathsf{Free}(\gamma)$
- $\{y\} \subseteq \mathsf{Free}(\gamma)$
- for every $z \in \sigma$ and for x we have a conjunct in γ in which both x and it are occurring.

The $[!\mathsf{N}]$-case is immediate since $\mathrm{ST}_x(([!\mathsf{N}]\varphi)^\sigma) = \mathrm{ST}_x(\varphi^{\sigma x})$. ■

Since the loosely guarded fragment of FO is decidable Hodkinson [17], as an immediate corollary we have:

COROLLARY 3.9
$\mathsf{L}_{(!\sigma,A)}$ is decidable.

In Section 7, we show the exact complexity of the satisfiability problem for $\mathsf{L}_{(!\sigma,A)}$. Finally, note that it also follows that

COROLLARY 3.10
$\mathsf{L}_{(!\sigma,A)}$ is compact.

4 Model theory and expressive power

4.1 Logic of sights and basic modal logic

In this section, we investigate the expressive power of our modal logics. We start with a simple question: how does the basic modal language $\mathsf{L}_{(\mathsf{N},A)}$ with the relation N as a binary relation compare with $\mathsf{L}_{(!\sigma,A)}$ in terms of expressive power? The expressive powers of these two languages are incomparable, as we now show.

Formally, the language of $\mathsf{L}_{(\mathsf{N},A)}$ over A extends the basic modal language L_A over A with a unary operator $\langle\mathsf{N}\rangle$:

$$\varphi ::= p \mid \neg\varphi \mid \varphi\wedge\varphi \mid \langle a\rangle\varphi \mid \langle\mathsf{N}\rangle\varphi,$$

where p ranges over PROP and a over A. The semantics of $\langle\mathsf{N}\rangle\varphi$ is given as follows:

$$\mathbb{M}, w \models \langle\mathsf{N}\rangle\varphi \text{ iff for some } v \text{ with } v \in \mathsf{N}(w) \text{ we have } \mathbb{M}, v \models \varphi$$

We show that the expressive power of $\mathsf{L}_{(!\sigma,A)}$ and $\mathsf{L}_{(\mathsf{N},A)}$ are incomparable.

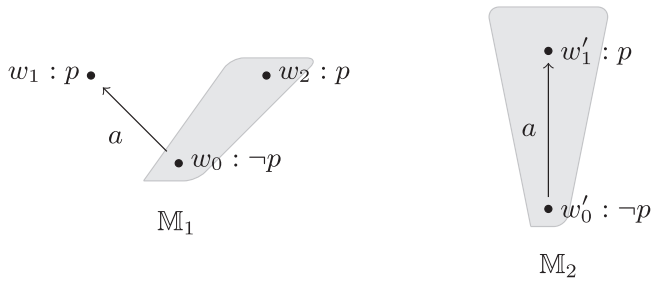

FIGURE 1. $\mathsf{L}_{(!\sigma,A)}$ is not invariant under bisimulation.

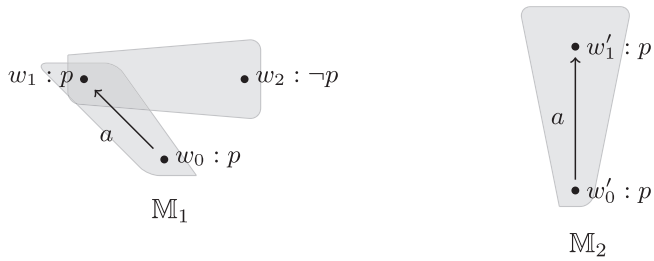

FIGURE 2. $\mathsf{L}_{(!\sigma,A)}$ is not as expressive as $\mathsf{L}_{(N,A)}$.

PROPOSITION 4.1

1. $\mathsf{L}_{(!\sigma,A)} \not\leq \mathsf{L}_{(\mathsf{N},A)}$
2. $\mathsf{L}_{(\mathsf{N},A)} \not\leq \mathsf{L}_{(!\sigma,A)}$

PROOF. (1) For any model, define $\xrightarrow{\mathsf{N}} = \{(w,w') | w' \in \mathsf{N}(w)\}$. $\mathsf{L}_{(\mathsf{N},A)}$ is clearly invariant under $(\xrightarrow{a}, \xrightarrow{\mathsf{N}})$-bisimulations. We show that $\mathsf{L}_{(!\sigma,A)}$ is not invariant under such bisimulations. Consider the two models given in Figure 1 where in $\mathbb{M}_1$, $\mathsf{N}(w_0) = \{w_0, w_2\}, \mathsf{N}(w_1) = \{w_1\}, \mathsf{N}(w_2) = \{w_2\}$ and in $\mathbb{M}_2$, $\mathsf{N}(w'_0) = \{w'_0, w'_1\}, \mathsf{N}(w'_1) = \{w'_1\}$. Note that $(\mathbb{M}_1, w_0)$ and $(\mathbb{M}_2, w'_0)$ are $(\xrightarrow{a}, \xrightarrow{\mathsf{N}})$-bisimilar. Finally, note that $\mathbb{M}_1, w_0 \not\models [!\mathsf{N}]\langle a\rangle p$ while $\mathbb{M}_2, w'_0 \models [!\mathsf{N}]\langle a\rangle p$.

(2) Consider the two models in Figure 2 where in $\mathbb{M}_1$ we have $\mathsf{N}(w_0) = \{w_0, w_1\}$, $\mathsf{N}(w_1) = \{w_1, w_2\}$, $\mathsf{N}(w_2) = \{w_2\}$ and in $\mathbb{M}_2$ we have $\mathsf{N}(w'_0) = \{w'_0, w'_1\}$, $\mathsf{N}(w'_1) = \{w'_1\}$. Clearly the formula $\langle a\rangle\langle \mathsf{N}\rangle \neg p$ is satisfied at $(\mathbb{M}_1, w_0)$, while, $\mathbb{M}_2, w'_0 \not\models \langle a\rangle\langle \mathsf{N}\rangle \neg p$.

Now we show inductively that

$$\text{for every formula } \varphi \in \mathsf{L}_{(!\sigma,A)} \text{ we have } \mathbb{M}_1, w_0 \models \varphi \text{ iff } \mathbb{M}_2, w'_0 \models \varphi \tag{6}$$

To do so, we first start by showing that for every formula $\varphi \in \mathsf{L}_{(!\sigma,A)}$,

$$\mathbb{M}_1, w_1 \models \varphi \text{ iff } \mathbb{M}_2, w'_1 \models \varphi \tag{7}$$

$$\mathbb{M}_1|_{\{w_1,w_2\}}, w_1 \models \varphi \text{ iff } \mathbb{M}_1|_{\{w_1\}}, w_1 \models \varphi \tag{8}$$

$$\mathbb{M}_1|_{\{w_1\}}, w_1 \models \varphi \text{ iff } \mathbb{M}_2|_{\{w'_1\}}, w'_1 \models \varphi \tag{9}$$

We first show (8). To see that simply note that $\mathbb{M}_1|_{\{w_1,w_2\}}, w_1$ and $\mathbb{M}_1|_{\{w_1\}}, w_1$ are A-bisimilar and that $\mathsf{N}(w_1) = \{w_1, w_2\}$. Now to see that (9), simply note that $\mathbb{M}_1|_{\{w_1\}}, w_1$ and $\mathbb{M}_2|_{\{w'_1\}}, w'_1$ are A-bisimilar,

and that $\mathsf{N}(w_1)\cap\{w_1\}=\{w_1\}$ and similarly for w_1'. To verify that (7), first note that $\mathbb{M}_1, w_1$ and $\mathbb{M}_2, w_1'$ are A-bisimilar and that $\mathsf{N}(w_1)=\{w_1, w_2\}$ and $\mathsf{N}(w_1')=\{w_1\}$. The claim follows from these observations together with (8) and (9).
We observe:

$$\mathbb{M}_2|_{\mathsf{N}(w_0')}=\mathbb{M}_2. \tag{10}$$

Since $R_a(w_1)=\emptyset$ and $w_1\in\mathsf{N}(w_0)$, we have

$$\mathbb{M}_1|_{\mathsf{N}(w_1)}, w_1\models\varphi \text{ iff } \mathbb{M}_1|_{\mathsf{N}(w_0)}, w_1\models\varphi. \tag{11}$$

And similarly

$$\mathbb{M}_2|_{\mathsf{N}(w_1')}, w_1'\models\varphi \text{ iff } \mathbb{M}_2|_{\mathsf{N}(w_0')}, w_1'\models\varphi. \tag{12}$$

Let us now show inductively that for every formula $\varphi\in\mathsf{L}_{(!\sigma,A)}$

$$\mathbb{M}_1|_{\mathsf{N}(w_0)}, w_0\models\varphi \text{ iff } \mathbb{M}_2|_{\mathsf{N}(w_0')}, w_0'\models\varphi \tag{13}$$

For the base case, $\mathbb{M}_1|_{\mathsf{N}(w_0)}, w_0\models p$ iff $\mathbb{M}_2|_{\mathsf{N}(w_0')}, w_0'\models p$, is immediate from the definition of $V(p)$. The conjunction and negation cases follow the usual argument. Now assume $\mathbb{M}_1|_{\mathsf{N}(w_0)}, w_0\models\langle a\rangle\varphi$. Since $R_a(w_0)\cap\mathsf{N}(w_0)=\{w_1\}$, it follows that $\mathbb{M}_1|_{\mathsf{N}(w_0)}, w_1\models\varphi$. From (11) it follows that $\mathbb{M}_1|_{\mathsf{N}(w_1)}, w_1\models\varphi$, and hence by (9) that $\mathbb{M}_2|_{\{w_1'\}}, w_1'\models\varphi$. By (12), we have $\mathbb{M}_2|_{\{w_0'\}}, w_1'\models\varphi$. Since $w_1'\in R_a(w_0')\cap\mathsf{N}(w_0')$ the claim follows. Finally, note that $\mathbb{M}_1|_{\mathsf{N}(w_0)}, w_0\models[!\mathsf{N}]\varphi$ iff $\mathbb{M}_1|_{\mathsf{N}(w_0)}, w_0\models\varphi$ and that $\mathbb{M}_2|_{\mathsf{N}(w_0')}, w_0'\models[!\mathsf{N}]\varphi$ iff $\mathbb{M}_2|_{\mathsf{N}(w_0')}, w_0'\models\varphi$. This concludes the induction step.
We are now ready to prove our main claim (6).

For the base case, note that $\mathbb{M}_1, w_0\models p$ iff $\mathbb{M}_2, w_0'\models p$, is immediate from the definition of $V(p)$. Conjunction and negation follows the usual argument. Now assume $\mathbb{M}_1, w_0\models\langle a\rangle\varphi$. Since $R_a(w_0)=\{w_1\}$, we have $\mathbb{M}_1, w_1\models\varphi$, and thus by (7) we have $\mathbb{M}_2, w_1'\models\varphi$. Now since $R_a(w_0')\supseteq\{w_1'\}$, we have $\mathbb{M}_2, w_1'\models\langle a\rangle\varphi$. Now assume that $\mathbb{M}_1, w_0\models[!\mathsf{N}]\varphi$, we have thus $\mathbb{M}_1|_{\mathsf{N}(w_0)}, w_0\models\varphi$. By (13), it follows that $\mathbb{M}_2|_{\mathsf{N}(w_0')}, w_0'\models\varphi$. By (10), it follows that $\mathbb{M}_2, w_0'\models\varphi$. This concludes the main induction and the proof. ■

A more direct proof of this result uses the proper notion of bisimulation for $\mathsf{L}_{(!\sigma,A)}$. To enable us to define this notion, we first prove a few facts.

4.2 Invariance non-invariance properties

From the first direction of the proof of Fact 4.1, we can also show that the dynamic logic of sights is not invariant under bounded p-morphisms.

FACT 4.2
$\mathsf{L}_{(!\sigma,A)}$ is not invariant under bounded p-morphisms.

We can also show that $\mathsf{L}_{(!\sigma,A)}$ is not invariant under $\cap$-bisimulations. Conversely, $\mathsf{L}_{(!\sigma,A)}$ is not as expressive as $\mathsf{L}_{(\mathsf{N},A),\cap}$ either. However, the logic is invariant under taking generated submodels.

FACT 4.3
$\mathsf{L}_{(!\sigma,A)}$ is not invariant under $(A,\mathsf{N},\cap)$-bisimulations.

COROLLARY 4.4
$\mathsf{L}_{(\mathsf{N},A),\cap} \not\geq \mathsf{L}_{(!\sigma,A)}$.

FACT 4.5
Let $\mathbb{M}'$ be an A-generated submodel of $\mathbb{M}$. For every φ, every $\{w_0,\ldots,w_n\} \subseteq |\mathbb{M}'|$ and for every $w \in \mathsf{N}(w_0) \cap \ldots \cap \mathsf{N}(w_n) \cap |\mathbb{M}'|$ we have $\mathbb{M}|_{\mathsf{N}(w_0)\cap\ldots\cap\mathsf{N}(w_n)}, w \models \varphi$ iff $\mathbb{M}'|_{\mathsf{N}'(w_0)\cap\ldots\cap\mathsf{N}'(w_n)}, w \models \varphi$.

PROOF. By induction on φ. The base case follows from the fact that $V'(p) = V(p) \cap |\mathbb{M}'|$. Induction steps for booleans are straightforward. We now prove the $\langle a \rangle$ and $[!\mathsf{N}]$ induction steps.

For the left to right direction of the $\langle a \rangle$-induction step, assume $\mathbb{M}|_{\mathsf{N}(w_0)\cap\ldots\cap\mathsf{N}(w_n)}, w \models \langle a \rangle \psi$. It follows that there is a state $v \in \mathsf{N}(w_0) \cap \ldots \cap \mathsf{N}(w_n)$ (0) with $wR_a v$ (1) and $\mathbb{M}|_{\mathsf{N}(w_0)\cap\ldots\cap\mathsf{N}(w_n)}, v \models \psi$ (2). From (1), the fact that $\mathbb{M}'$ is an A-generated submodel of $\mathbb{M}$, and the fact that $w \in |\mathbb{M}'|$, it follows that $v \in |\mathbb{M}'|$ (3) and $wR'_a v$ (4). From (0), the fact that $\{w_0,\ldots,w_n\} \subseteq |\mathbb{M}'|$ and construction of $\mathbb{M}'$ we have $v \in \mathsf{N}'(w_0) \cap \ldots \cap \mathsf{N}'(w_n)$ (5). From (2), (3), (5) and IH, it follows that $\mathbb{M}'|_{\mathsf{N}'(w_0)\cap\ldots\cap\mathsf{N}'(w_n)}, v \models \psi$ (6). From (4) and (6), it follows that $\mathbb{M}'|_{\mathsf{N}'(w_0)\cap\ldots\cap\mathsf{N}'(w_n)}, w \models \langle a \rangle \psi$. The other direction is trivial.

For the left ro right direction of the $[!\mathsf{N}]$-induction step, assume that $\mathbb{M}|_{\mathsf{N}(w_0)\cap\ldots\cap\mathsf{N}(w_n)}, w \models [!\mathsf{N}]\psi$. It follows that $\mathbb{M}|_{\mathsf{N}(w_0)\cap\ldots\cap\mathsf{N}(w_n)\cap\mathsf{N}(w)}, w \models \psi$ (7). By definition of $\mathbb{M}'$ it follows that $w \in \mathsf{N}'(w_0) \cap \ldots \cap \mathsf{N}'(w_n) \cap \mathsf{N}'(w)$ (8). From (7), (8) and IH it follows that $\mathbb{M}'|_{\mathsf{N}'(w_0)\cap\ldots\cap\mathsf{N}'(w_n)\cap\mathsf{N}'(w)}, w \models \psi$ (9). From (9), (8) and semantics, the claim follows. The other direction is similar. ■

The following is immediate from the previous fact.

COROLLARY 4.6
$\mathsf{L}_{(!\sigma,A)}$ is invariant under A-generated submodels.

From Feferman and Kreisel's Theorem 2.13, Corollary 4.6 and Proposition 3.8 it follows that:

COROLLARY 4.7
For every formula $\varphi \in \mathsf{L}_{(!\sigma,A)}$, $x \in VAR$ and $\sigma \in VAR^*$, $\mathrm{ST}_x(\varphi^\sigma)$ is equivalent to a formula in $\mathsf{BF(N,A)} \cap \mathsf{LGF(N,A)}$.

In words, every formula of the logic of sights is equivalent to a first-order formula that is both bounded and loosely guarded.

4.3 σ-bisimulations

As expected, the notion of bisimulation matching $\mathsf{L}_{(!\sigma,A)}$-modal invariance, is particularly natural.

DEFINITION 4.8
A $(!\mathsf{N},A)$-**bisimulation** between two $\sigma\mathsf{LTS}$ $\mathbb{S} = (\langle W, (\xrightarrow{a})_{a\in A}, \mathsf{N}, V\rangle)$ and $\mathbb{S}' = (\langle W', (\xrightarrow{a}{}')_{a\in A}, \mathsf{N}', V'\rangle)$ is a relation $Z \subseteq (\wp(W) \times W) \times (\wp(W') \times W')$ such that for some $w \in W$ and $w' \in W'$ we have $(W, w, W', w') \in Z$, and such that the following conditions hold:

1. (Atomic Harmony) if $(X, x, X', x') \in Z$ then for every $p \in \mathrm{PROP}$, we have $x \in V(p)$ iff $x' \in V'(p)$
2. (Forth) if $(X, x, X', x') \in Z$ and $xR_a y$ for some $y \in X$, then there exists some $y' \in X'$ such that $x'R_a y'$ and $(X, y)Z(X', y')$
3. (Back) if $(X, x, X', x') \in Z$ and $x'R'_a y'$ for some $y' \in X'$, then there exists some $y \in X$ such that $xR_a y$ and $(X, y)Z(X', y')$
4. (N-Forth) if $(X, x, X', x') \in Z$ and $xR_a y$ for some $y \in X \cap \mathsf{N}(x)$, then there exists some $y' \in X' \cap \mathsf{N}(x')$ such that $x'R'_a y'$ and $(X \cap \mathsf{N}(x), y)Z(X' \cap \mathsf{N}(x'), y')$

5. (N-Back) if $(X,x,X',x') \in Z$ and $x' R'_a y$ for some $y' \in X' \cap \mathsf{N}(x')$, then there exists some $y \in X \cap \mathsf{N}(x)$ such that $xR_a y$ and $(X \cap \mathsf{N}(x), y) Z (X' \cap \mathsf{N}(x'), y')$

We say that two pointed σLTS $(\mathbb{S}, w_0) = (\langle W, (\xrightarrow{a})_{a \in A}, \mathsf{N}, V \rangle, w_0)$ and $(\mathbb{S}', w'_0) = (\langle W', (\xrightarrow{a'})_{a \in A}, \mathsf{N}', V' \rangle, w'_0)$ are $(!\mathsf{N}, A)$-bisimilar (notation: $\mathbb{S}, w_0 \underline{\leftrightarrow}^{!\mathsf{N},A} \mathbb{S}', w'_0$) whenever there is bisimulation Z between $\mathbb{S}$ and $\mathbb{S}'$ such that $(W, w_0, W', w'_0) \in Z$. When A is clear from context, we write σ-bisimilar for $(!\mathsf{N}, A)$-bisimilar.

PROPOSITION 4.9 (Hennessy–Milner Theorem)
Let $\mathbb{S} = (\langle W, (\xrightarrow{a})_{a \in A}, \mathsf{N}, V \rangle)$ and $\mathbb{S}' = \langle W', (\xrightarrow{a'})_{a \in A}, \mathsf{N}', V' \rangle$ be two image-finite σLTS. For every $w \in W$ and $w' \in W'$, $w \leftrightsquigarrow_{\mathsf{L}_{(!\sigma,A)}} w'$ iff $w \underline{\leftrightarrow}^{!\mathsf{N},A} w'$

PROOF. The proof is standard. We just indicate the N-Forth direction of the left to right direction. Assume for contradiction that there are two pointed image-finite σLTS $\mathbb{S}, w_0$ and $\mathbb{S}', w'_0$, such that $\mathbb{S}, w_0 \leftrightsquigarrow \mathbb{S}', w'_0$, $w_0 R_a w_1$ and $w_1 \in \mathsf{N}[w_0]$, but there is no state $v' \in \mathbb{S}'|_{\mathsf{N}(w'_0)}$ such that $w'_0 R_a v'$ and $\mathbb{S}'|_{\mathsf{N}(w'_0)}, w_1 \leftrightsquigarrow \mathbb{S}'|_{\mathsf{N}(w'_0)}, v'$ (1). It is easy to see that $R_a[w'_0] \cap \mathsf{N}(w'_0) \neq \emptyset$. Since $\mathbb{S}'$ is image-finite, we can enumerate $R_a[w'_0] \cap \mathsf{N}(w'_0) = \{v'_1, \ldots, v'_n\}$ (for some $n \in \omega$). Now by (1), for each v'_i there is some formula φ_i such that $\mathbb{S}|_{\mathsf{N}(w_0)}, v' \models \varphi_i$ and $\mathbb{S}'|_{\mathsf{N}(w'_0)}, v'_i \not\models \varphi_i$. It follows that that $\mathbb{S}, w_0 \models [!\mathsf{N}]\langle a \rangle \bigwedge_{i \in n+1} \varphi_i$ but $\mathbb{S}', w'_0 \not\models [!\mathsf{N}]\langle a \rangle \bigwedge_{i \in n+1} \varphi_i$. Contradicting (0). ■

Now that we have the right notion of bisimulation, a shorter alternative proof for the second part of Proposition 4.1 boils down to showing that $\mathbb{M}_1, w_0 \underline{\leftrightarrow}^{!\mathsf{N},A} \mathbb{M}_2, w'_0$.

5 A complete axiomatization

In this section, we give an axiom system for the logic $\mathsf{L}_{(!\sigma,A)}$ and prove its completeness.

5.1 Existential and universal formulas

In what follows, let a universal formula be defined recursively as follows:

1. if $p \in$ PROP, then $p, \neg p$ are universal
2. if φ and ψ are universal, then $\varphi \wedge \psi$ and $\varphi \vee \psi$ are universal.
3. if φ is universal, then $[a]\varphi$ and $[!\mathsf{N}]\varphi$ are universal.

Existential formulas are defined symmetrically, using $\langle a \rangle \varphi$ instead of $[a]\varphi$. Clearly, the negation of a universal formula is an existential formula and conversely.

5.2 Preliminary observations

We start with some simple observations.

FACT 5.1
If $\models \varphi \to \psi$ then $\models \langle a \rangle \varphi \to \langle a \rangle \psi$

We recall the following standard preservation result:

THEOREM 5.2 (Łoś-Tarski)
A first-order formula is preserved under embeddings if, and only if, it is logically equivalent to an existential formula.

TABLE 1. !N axiom system

PL	$\vdash \varphi$ if φ is a substitution instance of a tautology of propositional logic
Nec	if $\vdash \varphi$, then $\vdash [a]\varphi$
K	$\vdash [a](\varphi \to \psi) \to ([a]\varphi \to [a]\psi)$
Dual	$\vdash [a]\neg\varphi \leftrightarrow \neg\langle a\rangle\varphi$
[!N]-Nec	if $\vdash \varphi$, then $\vdash [!\mathsf{N}]\varphi$
[!N]-K	$\vdash [!\mathsf{N}](\varphi \to \psi) \to ([!\mathsf{N}]\varphi \to [!\mathsf{N}]\psi)$
[!N]-p	$\vdash p \leftrightarrow [!\mathsf{N}]p$
[!N]-¬	$\vdash [!\mathsf{N}]\neg\varphi \leftrightarrow \neg[!\mathsf{N}]\varphi$
[!N]-∧	$\vdash [!\mathsf{N}](\varphi \wedge \psi) \leftrightarrow ([!\mathsf{N}]\varphi \wedge [!\mathsf{N}]\psi)$
[!N]-∨	$\vdash [!\mathsf{N}](\varphi \vee \psi) \leftrightarrow ([!\mathsf{N}]\varphi \vee [!\mathsf{N}]\psi)$
$[!\mathsf{N}]\langle a\rangle$	if φ is existential, then $\vdash [!\mathsf{N}]\varphi \to \varphi$
[!N][!N]	$\vdash [!\mathsf{N}][!\mathsf{N}]\varphi \leftrightarrow [!\mathsf{N}]\varphi$
MP	if $\vdash \varphi \to \psi$ and $\vdash \varphi$, then $\vdash \psi$

COROLLARY 5.3
If φ is existential, then $\models [!\mathsf{N}]\varphi \to \varphi$.

PROOF. Let $\varphi \in \mathsf{L}_\sigma$. By Proposition 3.5 there is an equivalent FO-formula $ST_x(\varphi)$. An easy induction shows that if φ is existential then so is $ST_x(\varphi)$. The claim follows from the right to left direction of Theorem 5.2 ■

5.3 Axiom system

Consider the axiom system !N given in Table 1. In the rest of this section, we drop reference to !N and simply use $\vdash \varphi$ to mean that φ is derivable using the axiom system !N and the term consistency to mean !N-consistency.

Interpretation of the axioms: [!N] axioms over Booleans follow from determinacy of the [!N] operator. The [!N][!N] axiom indicates that there is no non-trivial immediate nesting of [!N]. This form of introspection for sights, can be interpreted as saying that agents are aware of the continuations they are aware of. The $[!\mathsf{N}]\langle a\rangle$ axiom is saying that agents never consider as possible, evolutions of the system that are in fact impossible. Or looking, at the converse, those agents are always aware of the necessary evolutions of the system.

FACT 5.4
$\vdash [!\mathsf{N}](\varphi \to \psi) \leftrightarrow ([!\mathsf{N}]\varphi \to [!\mathsf{N}]\psi)$

FACT 5.5
If φ is universal, then $\vdash \varphi \to [!\mathsf{N}]\varphi$

5.4 Soundness and completeness

LEMMA (Soundness)

If $\vdash \varphi$ then $\models \varphi$.

PROOF. The only non-trivial case is that of axiom ($[!\mathsf{N}]\langle a\rangle$). Its soundness is immediate from Corollary 5.3. ■

In the next section, we prove the following:

LEMMA (Weak completeness)

Every finite $!\mathsf{N}$-consistent sets of $\mathsf{L}_{(!\sigma,A)}$-formulas is satisfiable.

THEOREM (Strong completeness)

Every $!\mathsf{N}$-consistent sets of $\mathsf{L}_{(!\sigma,A)}$-formulas is satisfiable.

6 Completeness

The structure of the proof is as follows. We first show how to inductively construct a finite witness model from a finite set of formulas. This model will *not* be a $\sigma\mathsf{LTS}$, but rather what we call a pseudo-$\sigma\mathsf{LTS}$ (see Section 6.1). We then show that these pseudo-$\sigma\mathsf{LTS}$s are in some sense good, allowing us to gather them back in a truth-preserving way into $\sigma\mathsf{LTS}$s. A witness pointed pseudo-$\sigma\mathsf{LTS}$ is constructed from finite maximal consistent sets ('atoms'). A truth lemma shows that membership in the set of formulas labeling a state of the witness pseudo-$\sigma\mathsf{LTS}$ is equivalent to satisfaction at that state. Putting this all together gives us weak completeness, and strong completeness follows from compactness (Corollary 3.10).

All of this is fairly abstract at the moment but we need some concepts before we can give the details in Section 6.6. To be able to guarantee that we construct good, yet finite pseudo-$\sigma\mathsf{LTS}$s, we will use a function associating a subset of our language with each word in a finite tree on A^*, the set of finite words on our set of labels. Section 6.3 is dedicated to this. The reader uninterested in the details of the construction might want to skip this section, looking only at Example 6.4.2 to see how this function is defined in a concrete case, and simply refer to it as needed.

Omitted proofs are given in Appendix B, while Appendix A contains an important numbers of intermediate facts that a play a role in some of these proofs. The proofs of these facts are also given in Appendix B.

6.1 *Pseudo-*σLTS

A pseudo-$\sigma\mathsf{LTS}$ (henceforth often $\mathsf{P}\sigma\mathsf{LTS}$) is a structure of the form:

$$\langle W, C, (R_a)_{a\in A}, \xrightarrow{!\mathsf{N}}, Val\rangle \text{ where,} \tag{14}$$

- W is a non-empty set and $C \subseteq (W \times \wp(W))$
- for each $a \in A$, $R_a \subseteq C \times C$ and $\xrightarrow{!\mathsf{N}} \subseteq C \times C$
- $Val : C \to \wp(\text{PROP})$.

The finitary witness $\mathsf{P}\sigma\mathsf{LTS}$ will be generated inductively from a consistent set of L_σ-formulas, as we will explain below.

Since $\mathsf{P}\sigma\mathsf{LTS}$s are not $\sigma\mathsf{LTS}$s, the semantics of $\mathsf{L}_{(!\sigma)}$ on $\mathsf{P}\sigma\mathsf{LTS}$s is different:

$$\begin{array}{lll}
\mathbb{M},w,X \models p & \text{iff} & p \in Val(w,X) \\
\mathbb{M},w,X \models \langle a \rangle \varphi & \text{iff} & \text{there is } s \in W \text{ with } (s,X) \in C, (w,X) R_a (s,X) \\
 & & \text{and } \mathbb{M},s,X \models \varphi \\
\mathbb{M},w,X \models [!\mathsf{N}]\varphi & \text{iff} & \text{there is } (v,Y) \in C \text{ with } (w,X) \xrightarrow{!\mathsf{N}} (v,Y) \\
 & & \text{and } \mathbb{M},v,Y \models \varphi
\end{array}$$

The semantics of Booleans is as usual. In words, this is a basic modal semantics with C being the state space.

Later in the proof, we will restrict ourselves to a class of well-behaved $\mathsf{P}\sigma\mathsf{LTS}$s that are more like $\sigma\mathsf{LTS}$s. As we will explain, our construction ensures that our witness $\mathsf{P}\sigma\mathsf{LTS}$s are in this well-behaved class. This will allow us to work with a semantics more like our original semantics, but nevertheless equivalent to the one above on the class of proper $\mathsf{P}\sigma\mathsf{LTS}$s. We return to this in Section 6.7.

6.2 Notation and terminology

Henceforth, in this section, we refer to L_σ-formulas simply as formulas, unless stated otherwise. Given a set Δ of formumas, let $\mathsf{exist}(\Delta) = \{\varphi \in \Delta | \varphi \text{ is existential.}\}$. Given a finite set of formulas $A = \{\varphi_1, \ldots, \varphi_n\}$ we let $\widehat{A} = \bigwedge A = \varphi_1 \wedge \ldots \wedge \varphi_n$. Given a formula φ, let $sub(\varphi)$ be the set of subformulas of φ. In particular $\varphi \in sub(\varphi)$. Given a set of formulas A, let $\mathsf{Sub}(A) = \bigcup\{sub(\varphi) | \varphi \in A\}$.

6.3 Linguistic closure: trees and kappas

We build up the witness model by inductively selecting appropriate atoms. We construct atoms from particular finite subsets of the language, which we describe in this section. As will become clear, we associate with each finite subset Δ of L_σ based on A, a subtree $\mathsf{Tree}(\Delta)$ of A^*. To each word $\vec{a}$ in A^* we will associate a finite set $\kappa(\vec{a}) \subseteq L_\sigma$. Atoms labeling the states in our witness models will be maximal consistent subsets of $\kappa(\vec{a})$ for some $\vec{a} \in \mathsf{Tree}(\Delta)$.

6.3.1 Properties of CN-closure

Let Δ be finite a set of formulas, we define $\mathsf{CN}(\Delta)$ as the smallest set of formulas such that:

- $\mathsf{Sub}(\Delta) \subseteq \mathsf{CN}(\Delta)$
- If $\varphi \in \mathsf{CN}(\Delta)$ and φ is neither of the form $\neg\psi$, nor of the form $[!\mathsf{N}]\psi$ then $\neg\varphi \in \mathsf{CN}(\Delta)$
- If $\varphi \in \mathsf{CN}(\Delta)$ and φ is neither of the form $[!\mathsf{N}]\psi$, nor of the form $\neg[!\mathsf{N}]\psi$ then $[!\mathsf{N}]\varphi \in \mathsf{CN}(\Delta)$

We sometimes abuse notation and write $\mathsf{CN}(\varphi)$ for $\mathsf{CN}(\{\varphi\})$. Note, that we do not allow arbitrary iteration of negation and $[!\mathsf{N}]$. This is necessary to keep the sets we are working with finite. However, we have the following:

FACT 6.1
Let Δ be a finite set. If $\varphi \in \mathsf{CN}(\Delta)$, then there is a formula $\mathsf{Norm}(\neg\varphi) \in \mathsf{CN}(\Delta)$ such that $\vdash \mathsf{Norm}(\neg\varphi) \leftrightarrow \neg\varphi$.

FACT 6.2
Let Δ be a finite set. If $\varphi \in \mathsf{CN}(\Delta)$, then there is a formula $\mathsf{Norm}([!\mathsf{N}]\varphi) \in \mathsf{CN}(\Delta)$ such that $\vdash \mathsf{Norm}([!\mathsf{N}]\varphi) \leftrightarrow [!\mathsf{N}]\varphi$.

COROLLARY 6.3
Let Δ be a finite set. If $\varphi \in \mathsf{CN}(\Delta)$ then there is a formula $\mathsf{Norm}([!\mathsf{N}]\varphi) \in \mathsf{CN}(\Delta)$ such that $\vdash \mathsf{Norm}([!\mathsf{N}]\neg\varphi) \leftrightarrow [!\mathsf{N}]\neg\varphi$.

FACT 6.4
If Δ is finite, then $\mathsf{CN}(\Delta)$ is finite as well.

6.3.2 a images, $E(\Delta, a)$ and $\Gamma(a)$

Given a finite set Δ of formulas, let $E(\Delta, a) = \{\varphi \mid \langle a \rangle \varphi \in \Delta \text{ or } \neg\langle a \rangle \varphi \in \Delta\}$.
Given a set of formulas Γ, we also write:

$$\Gamma(a) = \mathsf{CN}(E(\Gamma, a)). \tag{15}$$

That is, $\Gamma(a) = \mathsf{CN}(\{\varphi \mid \langle a \rangle \varphi \in \Gamma \text{ or } \neg\langle a \rangle \varphi \in \Gamma\})$. Hence, in particular:

COROLLARY 6.5
Let Δ be a finite set. $\mathsf{CN}(\Delta)(a) = \mathsf{CN}(\{\varphi \mid \langle a \rangle \varphi \in \mathsf{CN}(\Delta)\})$.

6.4 Initial downward cn*-labeling and the* $\mathsf{Tree}(\Delta)$ *tree*

Again, throughout this section, Δ is assumed to be finite. Definitions and results are stated under this proviso.

Now we define the following notation, for finite sequences $\vec{a} \in A^*$:

$$\mathsf{cn}(\Delta)(\vec{a}) = \begin{cases} \mathsf{CN}(\Delta) & \text{if } \vec{a} = \epsilon \\ \mathsf{cn}(\Delta)(\vec{b})(c) & \text{if } \vec{a} = \vec{b} \cdot c. \end{cases} \tag{16}$$

FACT 6.6
For every $\vec{a} \in A^*$, $\mathsf{cn}(\Delta)(\vec{a})$ is finite.

FACT 6.7
For every $\vec{a} \in A^*$, we have $\mathsf{cn}(\Delta)(\vec{a}) = \mathsf{CN}(\mathsf{cn}(\Delta)(\vec{a}))$.

We call a finite sequence $\vec{a} \in A^*$ Δ-maximal if for every $b \in A$, $\mathsf{cn}(\Delta)(\vec{a} \cdot b) = \emptyset$ but $\mathsf{cn}(\Delta)(\vec{a}) \neq \emptyset$. We also call such sequences Δ-leaves. Let $\mathsf{tree}(\Delta) = \{\vec{a} \in A^* \mid \mathsf{cn}(\Delta)(\vec{a}) \neq \emptyset\}$. Let $\sqsubseteq$ be the subsequence relation on A^*. We also write $\mathsf{SubTree}(\Delta, \vec{a}) = \{\vec{b} \in A^* \mid \vec{a} \cdot \vec{b} \in \mathsf{tree}(\Delta)\}$.

Define a function $\mathsf{rank}_\Delta : \mathsf{tree}(\Delta) \to \omega$ be defined as follows:

$$\mathsf{rank}_\Delta(\vec{a}) = \begin{cases} 0 & \text{if } \vec{a} \text{ is a } \Delta\text{-leaf} \\ n+1 & \text{if } \max(\{\mathsf{rank}(\vec{a} \cdot b) \mid \vec{a} \cdot b \in \mathsf{tree}(\Delta)\}) = n \end{cases}$$

6.4.1 The linguistic function κ

DEFINITION 6.8 (cj and da)
Let Γ be finite a set of formulas. Let $\mathsf{cj}(\Gamma)$ be defined as follows:

$$\mathsf{cj}(\Gamma) = \mathsf{CN}(\{\widehat{S} \mid S \subseteq \Gamma\}) \tag{17}$$

and let $\mathsf{da}(\Gamma, a)$ be defined as follows:

$$\mathsf{da}(\Gamma, a) = \{\langle a\rangle\varphi \mid \varphi \in \mathsf{cj}(\Gamma)\} \tag{18}$$

DEFINITION 6.9
We define a function $\kappa_\Delta : A^* \to \wp(\mathsf{L}_\sigma)$ recursively:

1. If $\mathsf{cn}(\Delta)(\vec{a}) = \emptyset$, then $\kappa_\Delta(\vec{a}) = \emptyset$.
2. If $\vec{a}$ is a Δ-leaf, then $\kappa_\Delta(\vec{a}) = \mathsf{cn}(\Delta)(\vec{a})$.
3. If $\vec{a}$ is not a Δ-leaf, then $\kappa_\Delta(\vec{a}) = \mathsf{CN}(\bigcup_{b\in A} \mathsf{da}(\kappa_\Delta(\vec{a}\cdot b), b) \cup \mathsf{cn}(\Delta)(\vec{a}))$

The maximum A-modal depth of $\kappa_\Delta(\vec{a})$ is the same as that of $\mathsf{cn}(\Delta)(\vec{a})$.

LEMMA 6.10
For every $\vec{a} \in \mathsf{tree}(\Delta)$, $\max(\mathsf{dep}(\kappa_\Delta(\vec{a}))) = \max(\mathsf{dep}(\mathsf{cn}(\Delta)(\vec{a}))) = \mathsf{rank}_\Delta(\vec{a})$.

We observe two facts: sets of the form $\kappa_\Delta(\vec{a})$ are finite and closed under CN.

FACT 6.11
For each $\vec{a} \in A^*$, $\kappa_\Delta(\vec{a})$ is finite.

FACT 6.12
For each $\vec{a} \in A^*$, $\kappa_\Delta(\vec{a}) = \mathsf{CN}(\kappa_\Delta(\vec{a}))$.

The following fact states that there is a maximal rank at which a formula φ can occur.

LEMMA 6.13 (Maximal φ-children)
If $\varphi \in \kappa_\Delta(\vec{a})$ then there is some, possibly empty, $\vec{b} \in A^*$ such that for every prefix $\vec{c}$ of $\vec{b}$ we have $\varphi \in \kappa_\Delta(\vec{a}\cdot\vec{c})$ and for every $\vec{d} \in A^*$ such that $\vec{d} \neq \epsilon$ we have $\varphi \notin \kappa_\Delta(\vec{a}\cdot\vec{b}\cdot\vec{d})$.

The following facts illustrate the exact relation between the content $\kappa_\Delta(\vec{a})$ and the κ content of $\vec{a}$'s children in the syntactic tree $\mathsf{Tree}(\Delta)$.

FACT 6.14
If $\psi \in \mathsf{CN}(\{\widehat{S} \mid S \subseteq \kappa_\Delta(\vec{a}\cdot b)\})$, then $\langle b\rangle\psi \in \kappa_\Delta(\vec{a})$.

FACT 6.15
If $\langle c\rangle\varphi \in \kappa_\Delta(\vec{a})$, then $\varphi \in \mathsf{CN}(\{\widehat{S} \mid S \subseteq \kappa_\Delta(\vec{a}\cdot c)\})$.

COROLLARY 6.16
If $\langle c\rangle\varphi \in \kappa_\Delta(\vec{a})$, then we are in one the following cases:

- $\varphi \in \kappa_\Delta(\vec{a}\cdot c)$
- $\varphi = (\psi_1 \wedge \ldots \wedge \psi_n)$ and $\psi_1, \ldots, \psi_n \in \kappa_\Delta(\vec{a}\cdot c)$, with $n \geq 2$
- $\varphi = [!\mathsf{N}](\psi_1 \wedge \ldots \wedge \psi_n)$ and $\psi_1, \ldots, \psi_n \in \kappa_\Delta(\vec{a}\cdot c)$, with $n \geq 2$
- $\varphi = \neg(\psi_1 \wedge \ldots \wedge \psi_n)$ and $\psi_1, \ldots, \psi_n \in \kappa_\Delta(\vec{a}\cdot c)$, with $n \geq 2$
- $\varphi = [!\mathsf{N}]\neg(\psi_1 \wedge \ldots \wedge \psi_n)$ and $\psi_1, \ldots, \psi_n \in \kappa_\Delta(\vec{a}\cdot c)$, with $n \geq 2$

6.4.2 An example

In Appendix C, we illustrate these syntactic concepts, by computing them for $\Delta = \{\langle a\rangle(\langle b\rangle p \wedge \langle b\rangle\neg p), [!\mathsf{N}]\langle a\rangle[!\mathsf{N}]\langle c\rangle q, [!\mathsf{N}]\langle a\rangle\neg\langle b\rangle\neg p\}$.

6.5 *Atoms and their existence*

Atoms are the building blocks of our models. They are finitary maximal consistent sets. Given a finite set Δ, κ_Δ associates to each $\vec{a} \in \mathsf{tree}(\Delta)$ a non-empty finite subset of L_σ. $(\vec{a})$-atoms are maximal consistent subsets of $\kappa_\Delta(\vec{a})$. In this section, we show that we can find enough of these building blocks and that they can be associated to build a canonical $\mathsf{pseudo}\sigma\mathsf{LTS}$. The actual construction will be discussed in Section 6.9.

6.5.1 $(\vec{a})$-Atoms and their properties

DEFINITION 6.17 ($(\Delta, \vec{a})$-Atoms)
For each $\vec{a} \in A^*$, let $At(\Delta, \vec{a})$ be the set of maximally consistent subsets of $\kappa_\Delta(\vec{a})$. We refer to elements of $At(\Delta, \vec{a})$ as $(\Delta, \vec{a})$-atoms.

First we prove a few properties about the *semantic* richness our set of atoms.

FACT 6.18
Let $\vec{a} \in A^*$ be such that $\kappa_\Delta(\vec{a}) \neq \emptyset$. For each $\varphi \in \kappa_\Delta(\vec{a})$ there is a formula $\mathsf{Norm}(\neg\varphi)$ such that $\vdash \mathsf{Norm}(\neg\varphi) \leftrightarrow \neg\varphi$ and $\mathsf{Norm}(\neg\varphi) \in \kappa_\Delta(\vec{a})$.

FACT 6.19
Let $\vec{a} \in A^*$ be such that $\kappa_\Delta(\vec{a}) \neq \emptyset$. For each $\varphi \in \kappa_\Delta(\vec{a})$ there is a formula $\mathsf{Norm}([!\mathsf{N}]\varphi)$ such that $\vdash \mathsf{Norm}([!\mathsf{N}]\varphi) \leftrightarrow \neg\varphi$ and $\mathsf{Norm}([!\mathsf{N}]\varphi) \in \kappa_\Delta(\vec{a})$.

FACT 6.20
Let $\vec{a} \in A^*$ be such that $\kappa_\Delta(\vec{a}) \neq \emptyset$. For each $\varphi \in \kappa_\Delta(\vec{a})$ there is a formula $\mathsf{Norm}([!\mathsf{N}]\neg\varphi)$ such that $\vdash \mathsf{Norm}([!\mathsf{N}]\neg\varphi) \leftrightarrow \neg\varphi$ and $\mathsf{Norm}([!\mathsf{N}]\neg\varphi) \in \kappa_\Delta(\vec{a})$.

FACT 6.21
Let $\vec{a} \in A^*$ be such that $\kappa_\Delta(\vec{a}) \neq \emptyset$. Now let $D \in At(\Delta, \vec{a})$. For each $\varphi \in \kappa_\Delta(\vec{a})$ we have either $\varphi \in D$ or else there is some formula $\mathsf{Norm}(\neg\varphi)$ such that $\vdash \mathsf{Norm}(\neg\varphi) \leftrightarrow \neg\varphi$ and $\mathsf{Norm}(\neg\varphi) \in D$.

6.5.2 Existence and uniqueness properties of $(\vec{a})$-atoms

In this section, we show that we have enough atoms, or building blocks, to construct our pointed $\mathsf{P}\sigma\mathsf{LTS}$ of Δ. As usual for such finitary constructions, we will need a finite form of the Lindenbaum lemma.

FACT 6.22 (Existence of atoms)
If Δ is a finite consistent set, then there exists an atom $A \in At(\Delta, \epsilon)$, such that $\Delta \subseteq A$.

Next we show that if we an $\vec{a}$-atom containing a formula of the form $\langle b \rangle \varphi$, then we have an $\vec{a \cdot b}$-atom that can play the role of $\xrightarrow{b}$-successor.

LEMMA 6.23 (Existence Lemma)
Let Γ be a finite consistent set and $\vec{a}, (\vec{a} \cdot b) \in A^*$. Moreover, assume that $\kappa_\Gamma(\vec{a}) \neq \emptyset$ and $\kappa_\Gamma(\vec{a} \cdot b) \neq \emptyset$. Let $D \in At(\Gamma, \vec{a})$. If $\langle b \rangle \varphi \in D$, then there is an atom $E \in At(\Gamma, \vec{a} \cdot b)$ such that there are formulas $\chi_1, \ldots, \chi_n \in E$ such that $\vdash (\chi_1 \wedge \ldots \wedge \chi_n) \to \varphi$, and $\widehat{D} \wedge \langle b \rangle \widehat{E}$ is consistent.

PROOF. By Corollary 6.16 we are in one of the following cases:

1. $\varphi \in \kappa_\Delta(\vec{a} \cdot b)$

2. $\varphi = (\psi_1 \wedge \ldots \wedge \psi_n)$ and $\psi_1, \ldots, \psi_n \in \kappa_\Delta(\vec{a} \cdot b)$, with $n \geq 2$
3. $\varphi = [!\mathsf{N}](\psi_1 \wedge \ldots \wedge \psi_n)$ and $\psi_1, \ldots, \psi_n \in \kappa_\Delta(\vec{a} \cdot b)$, with $n \geq 2$
4. $\varphi = \neg(\psi_1 \wedge \ldots \wedge \psi_n)$ and $\psi_1, \ldots, \psi_n \in \kappa_\Delta(\vec{a} \cdot b)$, with $n \geq 2$
5. $\varphi = [!\mathsf{N}]\neg(\psi_1 \wedge \ldots \wedge \psi_n)$ and $\psi_1, \ldots, \psi_n \in \kappa_\Delta(\vec{a} \cdot b)$, with $n \geq 2$

We only prove Case 5 as all other cases are similar.

Case 5: By axioms it follows that for some $\psi_i \in \{\psi_1, \ldots, \psi_n\}$ we have $[!\mathsf{N}]\neg\psi_i \in D$. But since $\psi_i \in \kappa_\Delta(\vec{a} \cdot b)$ it follows by Fact 6.20 that we have a formula $\mathsf{Norm}([!\mathsf{N}]\neg\psi_i) \in \kappa_\Delta(\vec{a} \cdot b)$ such that $\vdash \mathsf{Norm}([!\mathsf{N}]\neg\psi_i) \leftrightarrow [!\mathsf{N}]\neg\psi_i$. We define $E_0 = \{\mathsf{Norm}([!\mathsf{N}]\neg\psi_i)\}$. Assume for contradiction that there is no such ψ_i such that $\widehat{D} \wedge \langle b \rangle \widehat{E_0}$ is consistent. It follows that for all $\psi_i \in \{\psi_1, \ldots, \psi_n\}$ we have $\vdash \widehat{D} \rightarrow \neg\mathsf{Norm}([!\mathsf{N}]\neg\psi_i)$, hence by axioms, $\vdash \widehat{D} \rightarrow \mathsf{Norm}([!\mathsf{N}]\psi_i)$. But then $\vdash \widehat{D} \rightarrow \neg\varphi$, contradicting the fact that D is consistent. By reduction it follows that $\widehat{D} \wedge \langle b \rangle \widehat{E_0}$ is consistent (a).

Now enumerate the formulas in $\kappa_\Delta(\vec{a} \cdot b)$ as $\alpha_1, \ldots, \alpha_m$. By Fact 6.18, we can define E_n inductively as follows:

- $E_0 = \{\mathsf{Norm}([!\mathsf{N}]\psi_i)\}$,
- $E_{n+1} = E_n \cup \{\alpha_{n+1}\}$, if $\widehat{D} \wedge \langle b \rangle (\widehat{E_n} \wedge \alpha_{n+1})$ is consistent,
- $E_{n+1} = E_n \cup \{\mathsf{Norm}(\neg\alpha_{n+1})\}$, otherwise.

Finally, define $E = E_m$. We claim that for every k with $0 \leq k \leq m$ we have that $\widehat{D} \wedge \langle b \rangle \widehat{E_k}$ is consistent. The proof is by induction on k. The base case follows from (a). For the induction step, assume that E_n has been defined and $\widehat{D} \wedge \langle b \rangle \widehat{E_n}$ is consistent. We have by basic modal logic: $\vdash \langle b \rangle \widehat{E_n} \leftrightarrow \langle b \rangle ((\widehat{E_n} \wedge \alpha_{n+1}) \vee (\widehat{E_n} \wedge \mathsf{Norm}(\neg\alpha_{n+1})))$ and thus $\vdash \langle b \rangle \widehat{E_n} \leftrightarrow (\langle b \rangle (\widehat{E_n} \wedge \alpha_{n+1}) \vee \langle b \rangle \widehat{E_n} \wedge \mathsf{Norm}(\neg\alpha_{n+1}))$. Hence either $\widehat{D} \wedge \langle b \rangle (\widehat{E_n} \wedge \alpha_{n+1})$ is consistent or $\widehat{D} \wedge \langle b \rangle (\widehat{E_n} \wedge \mathsf{Norm}(\neg\alpha_{n+1}))$ is consistent. It also follows that $\widehat{D} \wedge \langle b \rangle \widehat{E}$ is consistent and moreover by construction that E is a maximally consistent subset of $\kappa_\Delta(\vec{a} \cdot b)$. ■

Now, we show an existence lemma for $[!\mathsf{N}]$.

PROPOSITION 6.24 (Existence Lemma for $[!\mathsf{N}]$)
Let Δ be a finite set and let $\vec{a} \in A^*$ be such that $\kappa_\Delta(\vec{a}) \neq \emptyset$. Let D be a $(\Delta, \vec{a})$-atoms. There exists some $(\Delta, \vec{a})$, E, such that $\widehat{D} \wedge [!\mathsf{N}] \bigwedge \widehat{E}$ is consistent.

PROOF. By hypothesis $\kappa_\Delta(\vec{a}) \neq \emptyset$. Hence there is some formula $\varphi_0 \in D$ (0). Either $\widehat{D} \wedge [!\mathsf{N}]\varphi_0$ is consistent, and we set $E_0 = \{\varphi_0\}$, or $\vdash (\widehat{D} \wedge [!\mathsf{N}]\varphi_0) \rightarrow \bot$. But then we have by axioms $\vdash \widehat{D} \rightarrow [!\mathsf{N}]\neg\varphi_0$. But since D is an atom it follows that $\widehat{D} \wedge [!\mathsf{N}]\neg\varphi_0$ is consistent. Moreover, by Fact 6.20 there is some formula $\mathsf{Norm}([!\mathsf{N}]\neg\varphi_0) \in \kappa_\Delta(\vec{a})$ such that $\vdash \mathsf{Norm}([!\mathsf{N}]\neg\varphi) \leftrightarrow [!\mathsf{N}]\neg\varphi_0$. Moreover by (0) and Fact 6.18 we have some formula $\mathsf{Norm}(\neg\varphi_0) \in \kappa_\Delta(\vec{a})$ such that $\vdash \mathsf{Norm}(\neg\varphi) \leftrightarrow \neg\varphi_0$, and we set $E_0 := \{\mathsf{Norm}(\neg\varphi_0)\}$. Either way we can define some set E_0 such that $\widehat{D} \wedge [!\mathsf{N}]\widehat{E_0}$ is consistent (1).

Now enumerate the formulas in $\kappa_\Delta(\vec{a})$ as $\alpha_1, \ldots, \alpha_m$. By (1) and Fact 6.18, we can define E_n inductively as follows:

- E_0, defined as indicated above.
- $E_{n+1} = E_n \cup \{\alpha_{n+1}\}$, if $\widehat{D} \wedge [!\mathsf{N}](\widehat{E_n} \wedge \alpha_{n+1})$ is consistent,
- $E_{n+1} = E_n \cup \{\mathsf{Norm}(\neg\alpha_{n+1})\}$, otherwise.

Finally, define $E = E_m$. We claim that of for every k with $0 \leq k \leq m$ we have that $\widehat{D} \wedge [!\mathsf{N}]\widehat{E_k}$ is consistent. The proof is by induction on k. Base case follows from (1). For the induction step, assume that E_n has been defined and $\widehat{D} \wedge [!\mathsf{N}]\widehat{E_n}$ is consistent. Now we are in one of two cases. Either $\widehat{D} \wedge [!\mathsf{N}](\widehat{E_n} \wedge \alpha_{n+1})$ is consistent, and by construction so is $\widehat{D} \wedge [!\mathsf{N}]\widehat{E_{n+1}}$, or it is not consistent, that is $\vdash (\widehat{D} \wedge [!\mathsf{N}](\widehat{E_n} \wedge \alpha_{n+1})) \rightarrow \bot$. But in this case, we have by axioms $[!\mathsf{N}]\neg$ and $[!\mathsf{N}]\wedge$ that

$\vdash \widehat{D} \to ([!\mathsf{N}]\widehat{E_n} \to [!\mathsf{N}]\neg\alpha_{n+1})$. But then by IH, Fact 6.18 and construction we have that $\widehat{D} \wedge [!\mathsf{N}]\widehat{E_{n+1}}$ is consistent. Hence either way $\widehat{D} \wedge [!\mathsf{N}]\widehat{E_{n+1}}$ is consistent. It also follows that $\widehat{D} \wedge [!\mathsf{N}]\widehat{E}$ is consistent and moreover by construction that E is a maximally consistent subset of $\kappa_\Delta(\vec{a})$. ■

The following shows that the canonical $\xrightarrow{!\mathsf{N}}$ relation is deterministic.

PROPOSITION 6.25 (Uniqueness)
Let $\vec{a} \in A^*$ be such that $\kappa_\Delta(\vec{a}) \neq \emptyset$. Let D, E, F be $(\Delta, \vec{a})$-atoms. If $\widehat{D} \wedge [!\mathsf{N}] \bigwedge \widehat{E}$ is consistent and $\widehat{D} \wedge [!\mathsf{N}] \bigwedge \widehat{F}$ is consistent as well, then $E = F$.

PROOF. Assume for contradiction that $E \neq F$. Since E and F are $(\Delta, \vec{a})$-atoms, it follows that there is some formula $\varphi \in \kappa_\Delta(\vec{a})$ such that either $\varphi \in E$ and $\varphi \notin F$, or $\varphi \notin E$ and $\varphi \in F$. But in the first case it follows Fact 6.21 that there is some formula $\mathsf{Norm}(\neg\varphi) \in E$ such that $\vdash \mathsf{Norm}(\neg\varphi) \leftrightarrow \neg\varphi$, but then by consistency of atoms it follows that $\mathsf{Norm}(\neg\varphi) \notin B$ and clearly we have $\vdash \varphi \leftrightarrow \neg\mathsf{Norm}(\neg\varphi)$ (1). (And symmetrically in the other case). Hence we can assume that that there are two formulas $\varphi, \mathsf{Norm}(\neg\varphi) \in \kappa_\Delta(\vec{a})$ such that $\varphi \in E$ and $\mathsf{Norm}(\neg\varphi) \in F$. But since $\widehat{D} \wedge [!\mathsf{N}] \bigwedge \widehat{E}$ is consistent and $\widehat{D} \wedge [!\mathsf{N}] \bigwedge \widehat{F}$ is consistent as well, it follows that $\widehat{D} \wedge [!\mathsf{N}]\varphi$ is consistent and $\widehat{D} \wedge [!\mathsf{N}]\mathsf{Norm}(\neg\varphi)$ is consistent.

Hence by maximality of D and Corollary A.32 we have some formula $\mathsf{Norm}([!\mathsf{N}]\varphi) \in D$ with $\vdash \mathsf{Norm}([!\mathsf{N}]\varphi) \leftrightarrow [!\mathsf{N}]\varphi$ and similarly $\mathsf{Norm}([!\mathsf{N}]\mathsf{Norm}(\neg\varphi)) \in D$ with $\vdash \mathsf{Norm}([!\mathsf{N}]\mathsf{Norm}(\neg\varphi)) \leftrightarrow [!\mathsf{N}]\mathsf{Norm}(\neg\varphi)$, and thus by (1) and axioms $\vdash \mathsf{Norm}([!\mathsf{N}]\mathsf{Norm}(\neg\varphi)) \leftrightarrow [!\mathsf{N}]\neg\varphi$. But then $\vdash \widehat{D} \to ([!\mathsf{N}]\varphi \wedge [!\mathsf{N}]\neg\varphi)$, hence by $[!\mathsf{N}]\wedge$, we have $\vdash \widehat{D} \to [!\mathsf{N}](\varphi \wedge \neg\varphi)$. But then by axioms $\widehat{D}$ is inconsistent. A contradiction. ■

We show that the canonical $\xrightarrow{!\mathsf{N}}$ reaches a reflexive sink immediately.

FACT 6.26
Let $\vec{a} \in A^*$ be such that $\kappa_\Delta(\vec{a}) \neq \emptyset$. Let D, E be $(\Delta, \vec{a})$-atoms. If $\widehat{D} \wedge [!\mathsf{N}] \bigwedge \widehat{E}$ is consistent, then $\widehat{E} \wedge [!\mathsf{N}] \bigwedge \widehat{E}$ is consistent.

PROOF. Assume that $\widehat{D} \wedge [!\mathsf{N}] \bigwedge \widehat{E}$ (1) is consistent and assume for contradiction that $\widehat{E} \wedge [!\mathsf{N}] \bigwedge \widehat{E}$ is inconsistent. It follows that $\vdash \widehat{E} \to \neg[!\mathsf{N}]\widehat{E}$, by axioms it follows that $\vdash [!\mathsf{N}](\widehat{E} \to \neg\widehat{E})$. Contradicting (1). ■

6.5.3 Key lemmas

Before we can sketch the structure of the completeness proof in full detail, we need a few more lemmas that will play a key role in the proof. To do so, it will be handy to have the following piece of notation.

Assume that Γ is a finite consistent set and $\vec{a}, (\vec{a} \cdot b) \in A^*$. Moreover, assume that $\kappa_\Gamma(\vec{a}) \neq \emptyset$ and $\kappa_\Gamma(\vec{a} \cdot b) \neq \emptyset$.

- Let $D \in At(\Gamma, \vec{a})$ and let $E \in At(\Gamma, \vec{a} \cdot b)$. We write $D \xrightarrow{b} E$ iff $\widehat{D} \wedge \langle b \rangle \widehat{E}$ is consistent.
- Let $D, E \in At(\Gamma, \vec{a})$. We write $D \xrightarrow{!\mathsf{N}} E$ iff $\widehat{D} \wedge [!\mathsf{N}]\widehat{E}$ is consistent.
- Let $D, E \in At(\Gamma, \vec{a})$. We write $D \xrightarrow{\pi} E$ iff for every φ with $\varphi \in E$ such that φ is equivalent to an existential formula, we have $\varphi \in D$.

The first two items are self-explanatory. They are the equivalent of the relation between contexts corresponding to the $\langle b \rangle$ and $[!\mathsf{N}]$ operators in a $\mathsf{P}\sigma\mathsf{LTS}$. The last relation corresponds to the relation

between two contexts based on the same state but within the sight of a richer collection of states. $\xrightarrow{\pi}$ generalizes $\xrightarrow{!\mathsf{N}}$. Lemma 6.28 will make this relation precise. But we first prove the following:

LEMMA 6.27 (Existence of witnesses)
Let Γ be a finite consistent set and $\vec{a}\cdot b\in A^*$. Let $D\in At(\Gamma,\vec{a})$ and let $\langle b\rangle\varphi\in\kappa_\Gamma(\vec{a})$. We have $\langle a\rangle\varphi\in D$ iff there is some $E\in At(\Gamma,\vec{a}\cdot b)$ such that $D\xrightarrow{a}E$ and such that there are formulas $\chi_1,\ldots,\chi_n\in E$ such that $\vdash(\chi_1\wedge\ldots\wedge\chi_n)\to\varphi$.

PROOF. The left to right direction follows from Lemma 6.23. For the other direction simply observe that if $D\xrightarrow{b}E$, then $\widehat{D}\wedge\langle b\rangle\varphi$ is consistent. But since $\langle b\rangle\varphi\in\kappa_\Gamma(\vec{a})$ and D is a maximally consistent subset of $\kappa_\Gamma(\vec{a})$ we have $\langle b\rangle\varphi\in D$. ■

LEMMA 6.28
Let Γ be a finite consistent set and $\vec{a}\in A^*$. Moreover, assume that $\kappa_\Gamma(\vec{a})\neq\emptyset$. Let $D,E\in At(\Gamma,\vec{a})$ we have

$$\text{If } D\xrightarrow{!\mathsf{N}}E,\ \text{then } D\xrightarrow{\pi}E \tag{19}$$

PROOF. Assume $D\xrightarrow{!\mathsf{N}}E$, it follows that $\widehat{D}\wedge\langle a\rangle\widehat{E}$ is consistent. Now assume we have some $\varphi\in E$ such that $\vdash\varphi\leftrightarrow\psi$ (0) and ψ is existential. We have that $\vdash D\wedge[!\mathsf{N}]\varphi$ is consistent (1). By Corollary A.32 we also have some formula $\mathsf{Norm}([!\mathsf{N}])\in\kappa_\Gamma(\vec{a})$ (2) such that $\vdash\mathsf{Norm}([!\mathsf{N}]\varphi)\leftrightarrow[!\mathsf{N}]\varphi$ (3). By (1), (2) and maximality of atoms it follows that $\mathsf{Norm}([!\mathsf{N}]\varphi)\in D$ (4). But by (0) and (3) and axioms we have $\vdash\mathsf{Norm}([!\mathsf{N}]\varphi)\leftrightarrow[!\mathsf{N}]\psi$ (5). Since ψ is existential, we have by axioms, that $\vdash[!\mathsf{N}]\psi\to\psi$ (6). From (5) and (6) we have $\vdash\mathsf{Norm}([!\mathsf{N}]\varphi)\to\varphi$ (7). From (7), (4) and maximality of D it follows that $\varphi\in D$. Concluding the proof. ■

We are now ready to prove a key commutation lemma.

LEMMA 6.29 (Simulation Image)
Let Γ be a finite consistent set and $\vec{a},(\vec{a}\cdot b)\in A^*$. Moreover assume that $\kappa_\Gamma(\vec{a})\neq\emptyset$ and $\kappa_\Gamma(\vec{a}\cdot b)\neq\emptyset$. Let $D,E\in At(\Gamma,\vec{a})$ and $F\in At(\Gamma,\vec{a}\cdot b)$. If $D\xrightarrow{\pi}E$ and $E\xrightarrow{b}F$. Then there exists $G\in At(\Gamma,\vec{a}\cdot b)$ such that $G\xrightarrow{\pi}F$ and $D\xrightarrow{b}G$.

PROOF. Define $G_0=\{\varphi\in F\,|\,\varphi$ is equivalent to an existential formula.$\}$ (0). Clearly, $\widehat{G_0}$ is equivalent to an existential formula and so is $\langle a\rangle\widehat{G_0}$. Now by Fact A.33 we have $\langle b\rangle\widehat{G_0}\in\kappa_\Gamma(\vec{a})$. Hence, by maximality of E we have $\langle b\rangle\widehat{G_0}\in E$. Now since $D\xrightarrow{\pi}E$ it follows that $\langle b\rangle\widehat{G_0}\in D$. Hence $\widehat{D}\wedge\langle b\rangle\widehat{G_0}$ is consistent. (1)

$$\begin{tikzcd}
D \arrow[r, "\pi"] \arrow[d, dashed, "b"'] & E \arrow[d, "b"] \\
G \arrow[r, dashed, "\pi"'] & F
\end{tikzcd}$$

Now since F is atom, G_0 is consistent. Moreover, since $F\in At(\Gamma,\vec{a}\cdot b)$ it follows, that $G_0\subseteq\kappa_\Gamma(\vec{a}\cdot b)$ (2).

By Fact 6.11, $\kappa_\Gamma(\vec{a}\cdot b)$ is finite. Hence we can enumerate it, and let $\kappa_\Gamma(\vec{a}\cdot b)=\{\psi_1,\ldots,\psi_n\}$. By Fact 6.18 we have for each $\psi_i\in\kappa_\Gamma(\vec{a}\cdot b)$ a corresponding formula $\mathsf{Norm}(\neg\psi_i)\in\kappa_\Gamma(\vec{a}\cdot b)$ such that

$\vdash \kappa_\Gamma(\vec{a}\cdot b) \leftrightarrow \neg\psi_i$ (3). Hence, we can define a sequence $(H_i)_{0\leq i\leq n}$, as follows:

$$H_i = \begin{cases} G_0 & \text{if } i=0 \\ H_j \cup \{\psi_{n+1}\} & \text{if } i=j+1 \\ & \text{and } \widehat{D} \wedge \langle b\rangle(\widehat{H_j} \wedge \psi_{j+1}) \text{ is consistent} \\ H_j \cup \{\mathsf{Norm}(\neg\psi_{j+1})\} & \text{if } i=j+1 \\ & \text{and } \vdash (\widehat{D} \wedge \langle b\rangle(\widehat{H_j} \wedge \psi_{j+1})) \to \bot \end{cases}$$

We claim that $\widehat{D} \wedge \langle b\rangle\widehat{H_i}$ is consistent for every i with $0 \leq k \leq n$. The proof is by induction on k. The base case is immediate from (1). Now assume that $\widehat{D} \wedge \langle b\rangle\widehat{H_k}$ is consistent k (4). By propositional logic we have $\vdash \widehat{H_k} \leftrightarrow (\widehat{H_k} \wedge \psi_{k+1}) \vee (\widehat{H_k} \wedge \neg\psi_{k+1})$. By (3) we have thus $\vdash \widehat{H_k} \leftrightarrow (\widehat{H_k} \wedge \psi_{k+1}) \vee (\widehat{H_k} \wedge \mathsf{Norm}(\neg\psi_{k+1}))$. It follows that

$$\vdash (\widehat{D} \wedge \langle b\rangle\widehat{H_k}) \leftrightarrow (\widehat{D} \wedge \langle b\rangle((\widehat{H_k} \wedge \psi_{k+1}) \vee (\widehat{H_k} \wedge \mathsf{Norm}(\neg\psi_{k+1})))) \tag{20}$$

By basic modal logic we have thus

$$\vdash (\widehat{D} \wedge \langle b\rangle\widehat{H_k}) \leftrightarrow (\widehat{D} \wedge (\langle b\rangle(\widehat{H_k} \wedge \psi_{k+1}) \vee \langle b\rangle(\widehat{H_k} \wedge \mathsf{Norm}(\neg\psi_{k+1}))) \tag{21}$$

By IH $\widehat{D} \wedge \langle b\rangle\widehat{H_k}$ is consistent (5). Hence we have either that $\widehat{D} \wedge (\langle b\rangle(\widehat{H_k} \wedge \psi_{k+1})$ is consistent, but then $\widehat{D} \wedge \langle b\rangle\widehat{H_{k+1}}$ is consistent, or $\widehat{D} \wedge (\langle b\rangle(\widehat{H_k} \wedge \psi_{k+1})$ (6) is inconsistent, but in this case it follows from (5), (21) and (6) that $\widehat{D} \wedge (\langle b\rangle(\widehat{H_k} \wedge \mathsf{Norm}(\neg\psi_{k+1}))$ is consistent (7). But by (6), construction of H_{k+1} and (7) it follows that $\widehat{D} \wedge \langle b\rangle\widehat{H_{k+1}}$ is consistent. Concluding the induction step. It follows that $\widehat{D} \wedge \langle b\rangle\widehat{H_n}$ (8).

Now it is easy to see that by construction H_n is a maximally consistent subset of $\kappa_\Gamma(\vec{a}\cdot b)$, hence $H_{k+1} \in At(\Gamma, \vec{a}\cdot b)$ (9). We can set $G = H_n$. By (0) and $G_0 = H_0 \subseteq H_n = G$ we have $G \xrightarrow{\pi} F$ (10). By (8) we have $D \xrightarrow{b} G$ (11). (9), (10) and (11) is what we had to prove. ■

We have now our main ingredients and can now present the structure of the proof in details (in Section 6.6). Appendix D analyzes the structure of the set of atoms for the specific finite set of formulas considered in Appendix C.

6.6 *Structure of the proof: details*

To construct our finitary canonical $\mathsf{P}\sigma\mathsf{LTS}$ from a finite consistent set Γ we proceed as follows. (A formal version of this procedure will be given in Section 6.9.)

First we generate the κ_Γ function. Intuitively $\kappa_\Gamma(\epsilon)$ is a finite subset of L_σ that has nice closure properties. The sets of the form $\kappa_\Gamma(\vec{a})$, for $\vec{a} \in \mathsf{tree}(\Gamma)$, will be the sublanguages our witness axioms will be constructed from. Intuitively, these sublanguages allow for less and less modal depth as $\vec{a}$ becomes longer.

Next, we start generating our finitary canonical $\mathsf{P}\sigma\mathsf{LTS}$ from our set Γ. Recall that such structures have the following form

$$\langle W, C, (R_a)_{a\in A}, \xrightarrow{!\mathsf{N}}, Val\rangle \tag{22}$$

where W is a non-empty set, $C \subseteq (W \times \wp(W))$ —C can be thought of as a set of contexts—and for each $a \in A$, $R_a \subseteq C \times C$, $\xrightarrow{!\mathsf{N}} \subseteq C \times C$, $Val : C \to \wp(\text{PROP})$.

By Fact 6.22 we can construct an atom $A \in At(\Gamma, \epsilon)$, such that $\Gamma \subseteq A$. We will use a labeling function $\delta : C \to \kappa_\Gamma(\epsilon)$—we will be more formal about this later—and we set $\delta(w_0, \epsilon) = \Gamma$. The procedure will add two types of witness children: a-children and !N-children. Let $b \in A$. b-witnesses of an atom in $At(\Gamma, \vec{a})$ will be selected from $At(\Gamma, \vec{a} \cdot b)$ for some $\vec{a} \in A^*$. !N-witnesses are atoms of the same types as their !N-parent.

Selecting !N-witnesses is easy. Assume that we have a context of the form (w, S) where $w \notin S$ and $\delta(w, S) = D \in At(\Gamma, \vec{a})$. From Fact 6.25, we know that there will be a unique atom $E \in At(\Gamma, \vec{a})$ such that $\widehat{D} \wedge [!\mathsf{N}]\widehat{E}$. We take it as our witness and will define $\delta(w, S \cup \{w\}) = E$.

For a-children, we need to be more careful. First of all, we will need to make sure that we start providing a-children to the right-most context first! Intuitively, if $S \subset S'$, then (w, S') will be said to be to the right of (w', S'). The reason is that for each child a-child (w_a, S') of (w, S') we need to make sure that we have a corresponding 'related' or 'similar' child (w_a, S) of (w, S). In what sense should they be related or similar? They should be related or similar, in the sense that for each existential formula $\varphi \in \delta(w_a, S')$ we have $\varphi \in \delta(w_a, S)$. Lemma 6.27 guarantees that we can find suitable children (for our right-most contexts). Lemma 6.29 and Lemma 6.28 guarantee that we can find suitable related children for all (non right-most) parents.

This procedure will end after a finite number of steps and the resulting structure will be our finitary canonical PσLTS. From here we will proceed as follows:

1. Give a simpler semantics for $\mathsf{L}_{(!\sigma)}$ on proper PσLTS (Section 6.7).
2. Define good PσLTSs as a subclass of proper PσLTSs (Section 6.8).
3. Give a procedure to construct canonical PσLTSs, $\mathbb{M}^\Gamma$, from a finite consistent set Γ (Section 6.9).
4. Show that canonical PσLTSs are good PσLTSs (Section 6.9)
5. Prove a Truth Lemma with respect to finitary canonical PσLTSs.
6. Give a truth-preserving transformation of good PσLTSs into σLTSs

6.7 *Interpreting L_σ on* PσLTS*s*

In this section, we show that on a restricted class of PσLTSs we can use a semantics that is closer to the original semantics of [!N] on σLTSs.

We restrict ourselves to a class of well-behaved PσLTSs. Call a PσLTSs proper if for all $x, y \in W$ and $(w, X), (y, Y) \in C$ we have that:

1. $(w, \emptyset) \in C$
2. whenever $(w, X) \in C$ then $(w, X \cup \{w\}) \in C$
3. $(w, X) \xrightarrow{!\mathsf{N}} (v, Y)$ iff $w = v$ and $Y = X \cup \{w\}$.

Our construction will make sure that our canonical PσLTSs are in this class.

Recall that the syntax of the language of $\mathsf{L}_{(!\sigma)}$ over A is:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle a \rangle \varphi \mid [!\mathsf{N}]\varphi,$$

where p ranges over PROP and a over A. On a proper PσLTS we interpret $\mathsf{L}_{(\sigma,A)}$ as follows:

$$\begin{array}{lll}
\mathbb{M},w,X \models p & \text{iff} & p \in Val(w,X) \\
\mathbb{M},w,X \models \langle a \rangle \varphi & \text{iff} & \text{there is } s \in W \text{ with } (s,X) \in C, (w,X) R_a (s,X) \\
 & & \text{and } \mathbb{M},s,X \models \varphi \\
\mathbb{M},w,X \models [!\mathsf{N}]\varphi & \text{iff} & \mathbb{M},w,X \cup \{w\} \models \varphi
\end{array}$$

The semantics of Booleans is defined as usual. This simplifies the semantics for PσLTSs in Section 6.1. By restricting ourselves to proper PσLTSs the clause for $[!\mathsf{N}]\varphi$ is really nothing but that of a basic modal formula. Let us record this fact:

FACT 6.30
Let $\mathbb{M}$ be a proper PσLTSs. The following are equivalent:

1. $\mathbb{M},w,X \cup \{w\} \models \varphi$
2. there is some $(v,Y) \in C$ with $(x,X) \xrightarrow{!\mathsf{N}} (v,Y)$ and $\mathbb{M},v,Y \models \varphi$

6.8 *Good* PσLTS*s*

There are good and bad PσLTSs. A PσLTS is bad iff it is not good. Intuitively a bad PσLTS is one that does not behave like a σLTS. Given a PσLTS

$$\mathbb{M} = \langle W, C, (R_a)_{a \in A}, \xrightarrow{!\mathsf{N}}, Val \rangle$$

we say that:

1. C is reflexive if for every $w \in W$ we have $(w,\{w\}) \in C$.
2. $C \subseteq (W \times \wp(W))$ is locally closed if for every $w \in W$ and $S_1, S_2 \in \wp(W)$ we have $(w,S_1),(w,S_2) \in C$ then for every $T \subseteq S_1 \cup S_2$, $(w,T) \in C$.
3. a is vertical if for every $(w,S),(v,T) \in C$ such that $(w,S)R_a(v,T)$ we have $S = T$.
4. a is grounded if for every $w,v \in W$ whenever $(w,S)R_a(v,S)$ for some $S \in \wp(W)$ then for all $S \in \wp(W)$ such that $(w,T),(v,T) \in C$ we have $(w,T)R_a(v,T)$.
5. Val is W-based if for every $p \in$ PROP whenever $p \in Val(w,X)$ for some $w \in W$ and $(w,X) \in C$, then for all $(w,Y) \in C$, $p \in Val(w,Y)$.

DEFINITION 6.31 (Good PσLTSs)
A PσLTS

$$\mathbb{M} = \langle W, C, (R_a)_{a \in A}, \xrightarrow{!\mathsf{N}}, Val \rangle$$

is good iff it satisfies the following properties:

- $\mathbb{M}$ is proper;
- C is reflexive and locally closed;
- for all $a \in A$, a is vertical and grounded; and
- Val is W-based.

Since a good PσLTS is proper, we can take it to be a structure of the form $\langle W, C, (R_a)_{a \in A}, Val \rangle$, taking $\xrightarrow{!\mathsf{N}}$ as a defined notion. Namely, $\xrightarrow{!\mathsf{N}} := \{(w,X,v,Y) \in C \times C \mid w = v \text{ and } Y = X \cup \{w\}\}$.

6.9 *Canonical* PσLTS*s over* Γ *are* good

We now give the formal definition of the procedure generating our finitary canonical PσLTSs from finite consistent sets. We then show that the resulting PσLTSs are good in the sense just defined in the previous section.

Let Γ be a finite consistent set of $\mathsf{L}_{(\sigma,A)}$ formulas. During the procedure we will use a number of functions.

- Linguistic function. $\eta : W \to A^*$. It associates to each state a sequence from the Tree.
- Labeling function. $\delta : C \to \wp(\mathsf{L}_{(\sigma,A)})$ and $\delta(w,X) \in At(\Gamma, \eta(w))$. It associates to each context in C an atom.
- Marking functions.
  - $m_0 : C \to \{0,1\}$. If $m_0(C) = 1$, we say that C is m_0-marked. Otherwise we say that C is m_0-unmarked.
  - $m_1 : C \to \{0,1\}$. If $m_1(C) = 1$, we say that C is m_1-marked. Otherwise we say that C is m_1-unmarked.

Structures and functions will be indexed with natural numbers. The index will be incremented after each run of the procedure until a fixed point is reached. Since Γ is a finite consistent set of $\mathsf{L}_{(\sigma,A)}$ formulas, it follows by Fact 6.22 that there is an atom $A_0 \in At(\Gamma, \epsilon)$ such that $\Gamma \subseteq A_0$. We initiate our procedure by setting $\mathbb{M}_0 := \langle W_0, C_0 \rangle$ and η, δ, m_0 and m_1 as follows:

- $W_0 = \{w_0\}$; $C_0 = \{(w_0, \emptyset)\}$
- for each $a \in A$, set $R_a = \emptyset$
- $\eta(w_0) = \epsilon$
- $\delta(w_0, \emptyset) = A_0$
- m_0 and m_1 are the constant functions with value 0.

In the procedures below, unless stated otherwise, structures and functions remain stable from k to $k+1$. Sets and relations are expanded unless stated otherwise. We often simply say that we 'add' an element, to indicate that we define the corresponding set indexed by its immediate successor as the union of that set and of the singleton containing this element. When it is clear we sometimes also drop the subscript k.

The sequence of procedures:

Execute the following procedure until $\mathbb{M}^{k+1} = \mathbb{M}^k$.

Procedure 1(!N saturation): check if it is necessary to add a !N-witness to any context, that is check if there is some w such that $(w,S) \in C_k$, $w \notin S$ and $(w, S \cup \{w\})$.

- If there is such a (w,S), then define $C_{k+1} := C_k \cup \{(w, S \cup \{w\})\}$. By Proposition 6.25, there is a unique $(\Gamma, \eta_k(w))$-atom $B_{k+1} \in$ such that $\delta_k(w,S) \xrightarrow{!\mathsf{N}} B_{k+1}$. Set $\delta_{k+1}(w,S) := B_{k+1}$. Increment index of everything else. Repeat the sequence of procedures with procedure 1.
- If there is no such context, move to procedure 2.

Procedure 2: find all the right-most m_0-unmarked contexts.

- If there is no such context, move to procedure 3.
- If there is such a context, pick one. Say (w,X). Enumerate A as $a_1, \ldots, a_n$. For each $a_i \in A$, check if there is a formula of the form $\langle a_i \rangle \varphi \in \delta(w,X)$.

- if there is no such formula, do nothing.
- if there is such a formula, enumerate $\{B \in At(\Gamma, (\eta_k(w) \cdot a_i)) | \delta(w,X) \xrightarrow{a_i} B\}$ as $B_1, \ldots, B_m$. For each B_j add a state $w \cdot (a_i, j)$ to W_{k+1} and add $(w \cdot (a_i, j), X)$ to C. Add $((w \cdot (a_i, j), X), (w \cdot (a_i, j), X))$ to R_a. Define $\delta((w \cdot (a_i, j), X)) = B_j$

m_0-Mark (w, X). Restart the sequence with procedure 1.

Procedure 3: find all the right-most m_1-unmarked contexts.

- If there is no such context, move to procedure 4.
- If there is such a context, pick one. Say (w, X). Enumerate $\{S \in \wp(W) \setminus \{X\} |$ there is some $v \in W$ such that $S = X \cup \{v\}\}$ as $S_1, \ldots, S_m$ and enumerate A as $a_1, \ldots, a_n$. For each each S_i and each a_j, check if $R_{a_j}[w, S_i] = \emptyset$.
  - If $R_{a_j}[w, S_i] = \emptyset$, do nothing.
  - Else enumerate $R_{a_j}[w, S_i]$ as $(w \cdot (a_j, 1), S_i), \ldots, (w \cdot (a_j, q), S_i)$. By Lemma 6.28, Lemma 6.29 and an inductive argument, for each $(w \cdot (a_j, h), S_i)$ there is an atom $G \in At(\Gamma, \eta(w \cdot (a_j, h)))$ such that $G \xrightarrow{\pi} \delta(w \cdot (a_j, h), S_i)$ and $\delta(w, X) \xrightarrow{a_j} G$. We add $(w \cdot (a_j, h), X)$ to C, $((w, X), (w \cdot (a_j, h), X))$ to R_{a_j} and define $\delta(w \cdot (a_j, h), X) = G$.

m_1-Mark (w, X). Restart the sequence with procedure 1.

Procedure 4: For each $(w, X) \in C$, define $Val(w, X) = \{ p \in \text{PROP} \mid p \in \delta(w, X)\}$.

FACT 6.32
The procedure is well defined.

FACT 6.33
The procedure terminates.

We call the structure output by the procedure $\mathbb{M}^\Gamma$ with

$$\mathbb{M}^\Gamma = W^\Gamma, C^\Gamma, (R_a^\Gamma)_{a \in A}, \xrightarrow{!\mathsf{N}^\Gamma}, Val^\Gamma \rangle$$

We also let η^Γ and δ^Γ be our output labeling functions.

FACT 6.34
Let Γ be a consistent set. $\mathbb{M}^\Gamma$ is proper.

PROOF. (1) Similar argument as in the proof of Fact 6.35.

(2) Assume that $(w, X) \in C$, either $w \in S$ and thus $(w, X \cup \{w\}) \in C$, or else, Procedure 1 has introduced a context $(w, X \cup \{w\})$ in C. Either way we have $(w, X \cup \{w\}) \in C$.

(3) This fact is hard-wired in the fact that we are not using a !N-relation and will be using instead an alternative semantics. But it is easy to see that Procedure 1 can be adapted by letting it add elements to a relation $\xrightarrow{!\mathsf{N}}$ under the same conditions, hence if by construction $((w, X), (v, Y))$ is added only whenever $x = v$ and $Y = X \cup \{w\}$. ■

FACT 6.35
Let Γ be a consistent set. C^Γ is reflexive.

PROOF. Let $w \in W$. w is either our starting state, but then $\{(w_0, \emptyset)\} \in C$ and by Procedure 1 we have $\{(w_0, \{w_0\})\} \in C$, or w has been introduced by Procedure 2 or Procedure 3. In this case, w has been

introduced with a context $(w,S)\in C$ as a R_a-successor to some state v with $(v,S)\in C$ for some $a\in A$, with w actually with a name of the form $(v\cdot(a,\mathsf{index}(w))$.

Either $w\in S$ or else Procedure 1 has introduced a context $(w,S\cup\{w\})$ in C. Hence we have a context of the form $(w,\{w_1\ldots,w_n\}\cup\{w\})\in C$. By recursive application of Procedure 3, Lemma 6.28 and Lemma 6.29 we see that Procedure 3 will add a chain of contexts

$$((w,\{w_1\ldots,w_n\}\cup\{w\}),(w,\{w_1\ldots,w_{n-1}\}\cup\{w\}),\ldots \\ \ldots,(w,\{w_1\}\cup\{w\}),(w,\{w\}))$$

such that

$$\delta((w,\{w_1\ldots,w_n\}\cup\{w\}))\xrightarrow{\pi}\delta((w,\{w_1\ldots,w_{n-1}\}\cup\{w\}))\xrightarrow{\pi}\ldots \\ \xrightarrow{\pi}\delta((w,\{w_1\}\cup\{w\}))\xrightarrow{\pi}\delta(w,\{w\}))$$

It follows in particular that $(w,\{w\})\in C^\Gamma$. ■

FACT 6.36
Let Γ be a consistent set. C^Γ is locally closed.

PROOF. Either w is the starting state, but then by construction if $(w,S),(w,T)\in C^\Gamma$ then $S,T\in\{\emptyset,\{w\}\}$. Local closure at w follows.

Assume instead that w is not the starting state; w was introduced with a context $(w,S)\in C$ as a R_a-successor to some state v with $(v,S)\in C$ for some $a\in A$. We have in particular a chain of contexts

$$((w,\{w_1\ldots,w_n\}\cup\{w\}),(w,\{w_1\ldots,w_{n-1}\}\cup\{w\}),\ldots \\ \ldots,(w,\{w_1\}\cup\{w\}),(w,\{w\}))$$

such that

$$\delta((w,\{w_1\ldots,w_n\}\cup\{w\}))\xrightarrow{\pi}\delta((w,\{w_1\ldots,w_{n-1}\}\cup\{w\}))\xrightarrow{\pi}\ldots \\ \xrightarrow{\pi}\delta((w,\{w_1\}\cup\{w\}))\xrightarrow{\pi}\delta(w,\{w\}))$$

where $\{w_1\ldots,w_n\}$ is the context of introduction of w. Now we have $(w,X)\in C$ iff $X\subseteq\{w_1\ldots,w_n,w\}$. Local closure follows. ■

FACT 6.37
Let Γ be a consistent set. R_a^Γ is vertical.

PROOF. Simply observe that the last clauses of Procedure 2 and 3 only add to R_a pairs of contexts such that their second components are equal. ■

FACT 6.38
Let Γ be a consistent set. R_a^Γ is grounded.

PROOF. A state v different from the initial state, is such that v is introduced by procedure 2 in context (v,X) such that there is no $Y\in\wp(w)$ with $X\cup\{v\}\subset Y$ and $(v,Y)\in C^\Gamma$ with $(w,X)R_a(v,X)$. Now for each context (v,T) introduced by procedure 3 we have $T\subset X$ and $\delta(w,T)\xrightarrow{\pi}(w,X)$ and procedure 3 will $(w,T)R_a(v,T)$, hence R_a is grounded for these contexts. Further contexts introduced will be of the form $T\cup\{v\}$ but then $(w,T\cup\{v\})\notin C^\Gamma$, hence R_a is trivially grounded for these contexts. ■

FACT 6.39
Let Γ be a consistent set. Val^{Γ} is W-based.

PROOF. By selection of atoms in procedures 3 and 1 it is easy to see that if $p \in \delta(w,X)$ then for every Y such that $(w,Y) \in C^{\Gamma}$ then $p \in \delta(w,Y)$. ■

LEMMA 6.40
Let Γ be finite consistent set. $\mathbb{M}^{\Gamma}$ is good.

PROOF. Follows from the previous sequence of facts. ■

6.9.1 An example

Appendix E shows the procedure at work with a concrete example. For, $\Delta = \{\neg\langle a\rangle\neg p, \langle a\rangle q, [!\mathsf{N}]\langle a\rangle\neg q\}$, the procedure outputs the $\mathsf{P}\sigma\mathsf{LTS}$ $\mathbb{M}^{\Delta}$ in Figure 3.

6.10 Truth Lemma for the canonical $\mathsf{P}\sigma\mathsf{LTS}$*s over* Γ

Let $\mathbb{M}^{\Gamma} = W^{\Gamma}, C^{\Gamma}, (R_a^{\Gamma})_{a\in A}, \xrightarrow{!\mathsf{N}^{\Gamma}}, Val^{\Gamma}\rangle$. Let η^{Γ} and δ^{Γ} be our output labeling functions.

For each $(w,X) \in C^{\Gamma}$, $\lambda^{\Gamma}(w,X) = \{\varphi \in \mathsf{cj}(\kappa_{\Gamma}(\eta(w))) | \vdash \widehat{\delta^{\Gamma}(w,X)} \to \varphi\}$.

LEMMA 6.41 (Truth Lemma)
For every $(w,X) \in C^{\Gamma}$ and $\varphi \in \mathsf{cj}(\kappa_{\Gamma}(\eta(w)))$, we have

$$\mathbb{M}^{\Gamma}, w, X \models \varphi \text{ iff } \varphi \in \lambda^{\Gamma}(w,X)$$

PROOF. The proof is by induction of the complexity of φ.

Base case: From left to right. Assume that $\mathbb{M}^{\Gamma}, w, X \models p$. By semantics, it follows that $p \in Val^{\Gamma}(w,X)$. Hence by construction $p \in \delta^{\Gamma}(w,X)$ (1). By construction we have $\delta(w,X) \in At(\Gamma, \eta^{\Gamma}(w))$ and thus $p \in \mathsf{cj}(\kappa_{\Gamma}(\eta(w))$ (2). The claim follows from (1) and (2).

From right to left. Assume that $p \in \lambda^{\Gamma}(w,X)$. It follows by definition of λ, that $p \in \kappa_{\Gamma}(\eta(w))$ (1) and $\vdash \widehat{\delta^{\Gamma}(w,X)} \to p$ (2). From (1), (2) and maximality of atoms we have $p \in \delta^{\Gamma}(w,X)$. By construction it follows that $p \in Val^{\Gamma}(w,X)$. The claim follows from semantics.

Induction step:
Case $\neg$: From left fo right. Assume $\mathbb{M}^{\Gamma}, w, X \models \neg\psi$. It follows from semantics that $\mathbb{M}^{\Gamma}, w, X \not\models \psi$. By IH we have $\psi \notin \lambda^{\Gamma}(w,X)$ (0). But by hypothesis $\neg\psi \in \mathsf{cj}(\kappa_{\Gamma}(\eta(w)))$ (1), hence $\psi \in \mathsf{cj}(\kappa_{\Gamma}(\eta(w)))$ (2). By definition of cj and CN there is a finite subset $\{\chi_1, \ldots, \chi_n\} \subseteq \kappa_{\Gamma}(\eta(w))$ such that $\psi = \chi_1 \wedge \ldots \wedge \chi_n$ (3). By (0) and (3) it follows that there is some $\chi_i \in \{\chi_1, \ldots, \chi_n\}$ such that $\chi_i \notin \delta(w,X)$. But then $\vdash \widehat{\delta(w,X)} \to \neg\psi$ (4). From (1) and (4) we have thus by definition of λ, $\neg\psi \in \lambda^{\Gamma}(w,X)$.

From right to left. Assume that $\neg\psi \in \lambda^{\Gamma}(w,X)$. From definition of λ, it follows that $\neg\psi \in \mathsf{cj}(\kappa_{\Gamma}(\eta(w)))$ and $\vdash \widehat{\delta^{\Gamma}(w,X)} \to \neg\psi$. By consistency of atoms it follows that $\not\vdash \widehat{\delta^{\Gamma}(w,X)} \to \psi$, hence $\psi \notin \lambda^{\Gamma}(w,X)$. But then by IH, we have $\mathbb{M}^{\Gamma}, w, X \not\models \psi$. Hence by semantics $\mathbb{M}^{\Gamma}, w, X \models \neg\psi$.

Case $\wedge$: From left fo right. Assume $\mathbb{M}^{\Gamma}, w, X \models \psi_1 \wedge \psi_2$. It follows from semantics that $\mathbb{M}^{\Gamma}, w, X \models \psi_1$ and $\mathbb{M}^{\Gamma}, w, X \models \psi_2$. By IH we have $\psi_1, \psi_2 \in \lambda^{\Gamma}(w,X)$ (1). But by hypothesis $\psi_1 \wedge \psi_2 \in \mathsf{cj}(\kappa_{\Gamma}(\eta(w)))$ (2). The claim follows from (1) and (2).

From left fo right. $\varphi_1 \wedge \varphi_2 \in \lambda^{\Gamma}(w,X)$. It follows that $\varphi_1, \varphi_2 \in \delta(w,X)$, hence by definition of λ and IH $\mathbb{M}^{\Gamma}, w, X \models \psi_1$ and $\mathbb{M}^{\Gamma}, w, X \models \psi_2$. The claim follows from semantics.

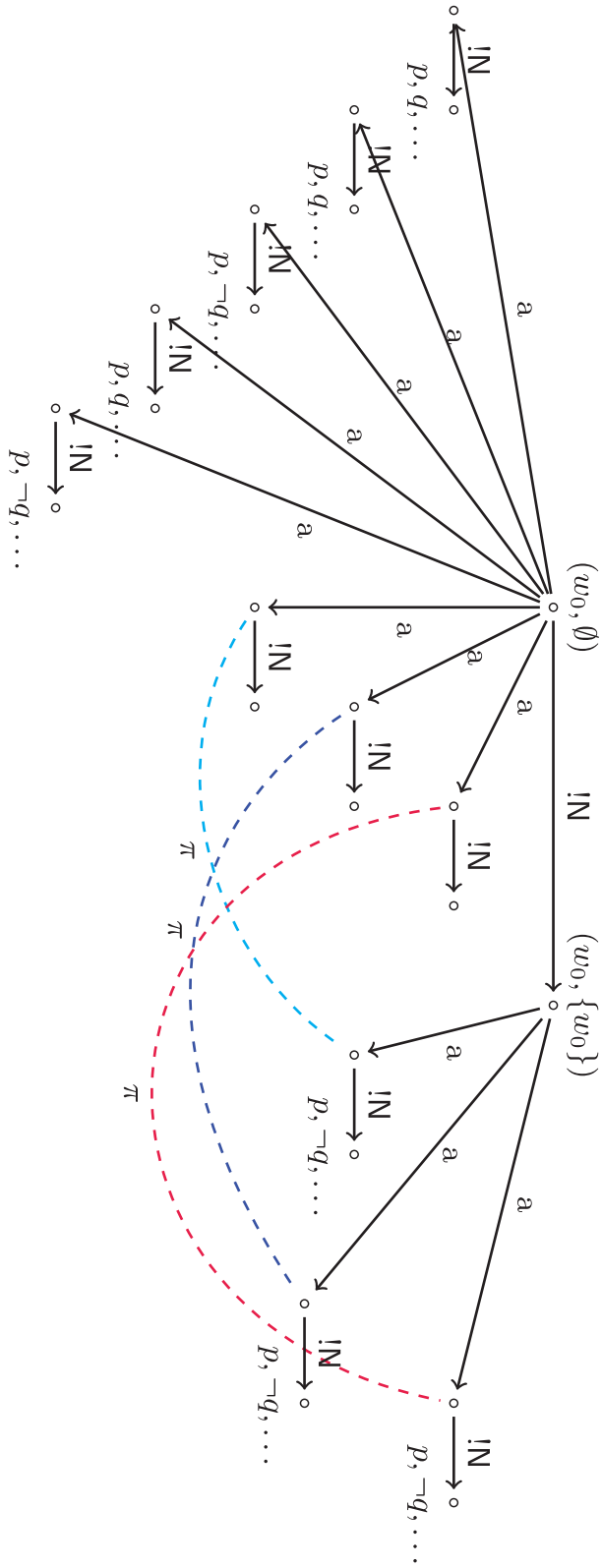

FIGURE 3. The PσLTS $\mathbb{M}^{\Delta}$ for $\Delta = \{\neg\langle a\rangle\neg p, \langle a\rangle q, [!\mathsf{N}]\langle a\rangle\neg q\}$.

Case $\langle a\rangle$**:** From left to right. Assume $\mathbb{M}^\Gamma, w, X \models \langle a\rangle\psi$. It follows from semantics that there is some $s \in W^\Gamma$, $(s,X) \in C^\Gamma$ such that $(w,X)R_a(s,X)$ (1) and $\mathbb{M}^\Gamma, s, X \models \psi$ (2). By (1) it follows from construction that $\widehat{\delta(w,X)} \wedge \langle a\rangle\widehat{\delta(s,X)}$ is consistent (3). From (2) by IH we have $\psi \in \lambda^\Gamma(s,X)$, it follows that $\vdash \widehat{\delta^\Gamma(s,X)} \to \psi$ (4). But by hypothesis $\langle a\rangle\psi \in \mathsf{cj}(\kappa_\Gamma(\eta(w)))$, hence, $\langle a\rangle\psi \in \kappa_\Gamma(\eta(w))$ (5). From (5), (4), (3) and maximality of atoms it follows that $\langle a\rangle\psi \in \delta(w,X)$ and hence $\langle a\rangle\psi \in \lambda^\Gamma(w,X)$.

From right to left. Assume that $\langle a\rangle\psi \in \lambda^\Gamma(w,X)$. By hypothesis we have $\langle a\rangle\psi \in \mathsf{cj}(\kappa_\Gamma(\eta(w)))$, hence by definition of cj we have $\langle a\rangle\psi \in \kappa_\Gamma(\eta(w))$ (a). By definition of λ, construction and maximality of atoms that $\langle a\rangle\psi \in \delta^\Gamma(w,X)$ (b).

It follows by Lemma 6.27 that there is an atom $E \in At(\Gamma, \eta^\Gamma(w)\cdot a)$ (6) such that there are $\chi_1, \ldots, \chi_n \in E$ (7) such that $\widehat{\delta^\Gamma(w,X)} \wedge \langle a\rangle\widehat{E}$ is consistent (8) and $\vdash (\chi_1, \ldots, \chi_n) \to \psi$ (9). But then by construction there is a state $w\cdot(a, \mathsf{index}(E)) \in W^\Gamma$ and $(w\cdot(a, \mathsf{index}(E)), X) \in C^\Gamma$ with $(w,X)R_a(w\cdot(a, \mathsf{index}(E)), X)$ (10) and $\delta(w\cdot(a, \mathsf{index}(E)), X) = E$ (11).

From (a) and Corollary 6.16 we are in one of the following cases:

1. $\psi \in \kappa_\Delta(\eta(w)\cdot a)$
2. $\psi = (\alpha_1 \wedge \ldots \wedge \alpha_n)$ and $\alpha_1, \ldots, \alpha_n \in \kappa_\Delta(\eta(w)\cdot a)$, with $n \geq 2$
3. $\psi = [!\mathsf{N}](\alpha_1 \wedge \ldots \wedge \alpha_n)$ and $\alpha_1, \ldots, \alpha_n \in \kappa_\Delta(\eta(w)\cdot a)$, with $n \geq 2$
4. $\psi = \neg(\alpha_1 \wedge \ldots \wedge \alpha_n)$ and $\alpha_1, \ldots, \alpha_n \in \kappa_\Delta(\eta(w)\cdot a)$, with $n \geq 2$
5. $\psi = [!\mathsf{N}]\neg(\alpha_1 \wedge \ldots \wedge \alpha_n)$ and $\alpha_1, \ldots, \alpha_n \in \kappa_\Delta(\eta(w)\cdot a)$, with $n \geq 2$

In either case, $\psi \in \mathsf{cj}(\kappa_\Gamma(\eta(w)\cdot a))$ (12). From (7), (9), (12) and (11) it follows that $\psi \in \lambda^\Gamma(w\cdot(a, \mathsf{index}(E)), X)$. But then by IH $\mathbb{M}^\Gamma, w\cdot(a, \mathsf{index}(E)), X \models \psi$ (13). From (13), (10) and semantics we have thus $\mathbb{M}^\Gamma, w, X \models \langle a\rangle\psi$.

Case $[!\mathsf{N}]$**:** From left fo right. Assume $\mathbb{M}^\Gamma, w, X \models [!\mathsf{N}]\psi$. It follows from semantics that $\mathbb{M}^\Gamma, w, X \cup \{w\} \models \psi$. But then by IH, we have $\psi \in \lambda^\Gamma(w, X \cup \{w\})$ (1). It follows by definition of λ that $\vdash \bigwedge \delta^\Gamma(w, X \cup \{w\}) \to \psi$. By axioms we have $\vdash [!\mathsf{N}]\bigwedge \delta^\Gamma(w, X \cup \{w\}) \to [!\mathsf{N}]\psi$ and $\vdash \bigwedge\{[!\mathsf{N}]\chi \mid \chi \in \delta^\Gamma(w, X \cup \{w\})\} \to [!\mathsf{N}]\psi$ (2).

By construction $\delta^\Gamma(w, X \cup \{w\})$ is an atom in $At(\Gamma, \eta^\Gamma(w))$ (3). From construction it follows that $\widehat{\delta^\Gamma(w,X)} \wedge [!\mathsf{N}]\bigwedge \delta^\Gamma(w, X \cup \{w\})$ is consistent (a). But then by Fact 6.26 we have that $\bigwedge \delta^\Gamma(w, X \cup \{w\}) \wedge [!\mathsf{N}]\bigwedge \delta^\Gamma(w, X \cup \{w\})$ is consistent (4). Moreover by (1), (3) and Fact 6.19, it follows that for every $\chi \in \delta^\Gamma(w, X \cup \{w\})$, there is a formula $Norm([!\mathsf{N}]\chi) \in \kappa_\Gamma(\eta^\Gamma(w))$ such that $\vdash Norm([!\mathsf{N}]\chi) \leftrightarrow [!\mathsf{N}]\chi$ (5). From (a), (5) and maximality of atoms we have that $\{Norm([!\mathsf{N}]\chi) \mid \chi \in \delta^\Gamma(w, X \cup \{w\})\} \subseteq \delta^\Gamma(w,X)$ (6). By hypothesis we have $[!\mathsf{N}]\psi \in \mathsf{cj}(\kappa_\Gamma(\eta(w)))$ (7). But from (7), (2), (5) and (6) it follows that $[!\mathsf{N}]\psi \in \lambda^\Gamma(w,X)$.

From right to left. Assume that $[!\mathsf{N}]\psi \in \lambda^\Gamma(w,X)$. By definition of λ we have $\widehat{\delta^\Gamma(w,X)} \to [!\mathsf{N}]\psi$ (7) and $[!\mathsf{N}]\psi \in \mathsf{cj}(\kappa_\Gamma(\eta(w)))$ (8). By construction we have that $\widehat{\delta^\Gamma(w,X)} \wedge [!\mathsf{N}]\widehat{\delta^\Gamma(w, X \cup \{w\})}$ is consistent (9).

By definition of cj and Fact 6.11 it follows that there is a finite subset $\{\chi_1, \ldots, \chi_n\} \subseteq \kappa_\Gamma(\eta(w))$ such that $\psi = \chi_1 \wedge \ldots \wedge \chi_n$ (10). From (7), (10) and axioms we have for each $\chi_i \in \{\chi_1, \ldots, \chi_n\}$, $\widehat{\delta^\Gamma(w,X)} \to [!\mathsf{N}]\chi_i$ (11), and from (10), Fact 6.12 and Fact 6.19 we have a formula $Norm([!\mathsf{N}]\chi_i) \in \kappa_\Gamma(\eta(w))$ such that $\vdash Norm([!\mathsf{N}]\chi_i) \leftrightarrow [!\mathsf{N}]\chi_i$ (12). But from (11), (12), construction and maximality of atoms we have $Norm([!\mathsf{N}]\chi_i) \in \delta^\Gamma(w,X)$ (13). By (10), (8) and Fact 6.12 we have $\chi_i \in \kappa_\Gamma(\eta(w))$ (14). From (9), (13), (12), construction and maximality of atoms it follows that $\chi_i \in \delta^\Gamma(w, X \cup \{w\})$. But by IH it follows that $\mathbb{M}^\Gamma, w, X \cup \{w\} \models \chi_i$. Since this holds for every $\chi_i \in \{\chi_1, \ldots, \chi_n\}$ it follows from semantics

that $\mathbb{M}^{\Gamma}, w, X \cup \{w\} \models \chi_1 \wedge \ldots \wedge \chi_n$, that is by (10), $\mathbb{M}^{\Gamma}, w, X \cup \{w\} \models \psi$. But from semantics we have then $\mathbb{M}^{\Gamma}, w, X \models [!\mathsf{N}]\psi$. ■

6.11 *Truth-preserving* gathering *of good* PσLTS*s into* σLTS

Given a good PσLTSs $\mathbb{M} = \langle W, C, (R_a)_{a \in A}, \xrightarrow{!\mathsf{N}}, Val \rangle$ we construct a σLTS

$$\mathsf{gather}(\mathbb{M}) = \langle S, (\xrightarrow{a})_{a \in A}, \mathsf{N}, V \rangle, \text{ as follows:}$$

- $S = W$
- for each $a \in A$ and $w, v \in W$, $(w, v) \in \xrightarrow{a}$ iff $(w, \emptyset) R_a (v, \emptyset)$
- for each $w \in S$, $\mathsf{N}(w) = \{v \in S \mid (v, \{w\}) \in C\}$
- for each $p \in \text{PROP}$, $V(p) = \{w \in S \mid p \in Val(w, \emptyset)\}$

FACT 6.42
If $\mathbb{M}$ is a good PσLTS, then gather($\mathbb{M}$) is a well-defined σLTS.

PROOF. $S = W$ is non-empty. Assume that $w, v \in W$, since C is reflexive it follows that $(w, \{w\}) \in C$ and $(v, \{v\}) \in C$. Since C is locally closed it follows that we have $(w, \emptyset) \in C$ and $(v, \emptyset) \in C$ (1). It follows by definition of $\xrightarrow{a}$, that it is well defined. By construction $\mathsf{N}(w)$ is well defined for each $w \in S$. Finally by (1) it follows that $V(p)$ is well defined for every $p \in \text{PROP}$. ■

FACT 6.43
Let $\mathbb{M} = \langle W, C, (R_a)_{a \in A}, \xrightarrow{!\mathsf{N}}, Val \rangle$ be a good PσLTS. And let $\mathsf{gather}(\mathbb{M}) = \langle S, (\xrightarrow{a})_{a \in A}, \mathsf{N}, V \rangle$. For every $w_1, \ldots, w_n, v \in S$ we have $v \in \mathsf{N}(w_1) \cap \ldots \cap \mathsf{N}(w_n)$ iff $(v, \{w_1, \ldots, w_n\}) \in C$

PROOF. For the left to right direction. Assume that $v \in \mathsf{N}(w_1) \cap \ldots \cap \mathsf{N}(w_n)$. It follows by construction that for every $w_i \in \{w_1, \ldots, w_n\}$ we have $(v, \{w_i\}) \in C$. Now since C is locally closed it follows that $(v, \{w_1, \ldots, w_n\}) \in C$.

For the right to left direction. Assume that $(v, \{w_1, \ldots, w_n\}) \in C$. Since C is locally closed it follows that for every $w_i \in \{w_1, \ldots, w_n\}$ we have $(v, \{w_i\}) \in C$. But then by construction we have $v \in \mathsf{N}(w_1), \ldots, v \in \mathsf{N}(w_n)$. Hence $v \in \mathsf{N}(w_1) \cap \ldots \cap \mathsf{N}(w_n)$. ■

FACT 6.44
Let $\mathbb{M} = \langle W, C, (R_a)_{a \in A}, \xrightarrow{!\mathsf{N}}, Val \rangle$ be a good PσLTS. And let $\mathsf{gather}(\mathbb{M}) = \langle S, (\xrightarrow{a})_{a \in A}, \mathsf{N}, V \rangle$. For every $w_1, \ldots, w_n, v, t \in S$ and $a \in A$ we have $v, t \in \mathsf{N}(w_1) \cap \ldots \cap \mathsf{N}(w_n)$ and $(v, t) \in \xrightarrow{a}$, iff $(v, \{w_1, \ldots, w_n\}) R_a (t, \{w_1, \ldots, w_n\})$.

PROOF. For the left to right direction. Assume that $v, t \in \mathsf{N}(w_1) \cap \ldots \cap \mathsf{N}(w_n)$ (1) and $(v, t) \in \xrightarrow{a}$ (2). From (1) and Fact 6.43 it follows that $(v, \{w_1, \ldots, w_n\}) \in C$ and $(t, \{w_1, \ldots, w_n\}) \in C$ (3). From (2) and construction, it follows that $(w, \emptyset) R_a (v, \emptyset)$ (4). But since a is grounded, it follows from (3) and (4) that $(v, \{w_1, \ldots, w_n\}) R_a (t, \{w_1, \ldots, w_n\})$ (5).

For the right to left direction. Assume that $(v, \{w_1, \ldots, w_n\}) R_a (t, \{w_1, \ldots, w_n\})$ (1). It follows that $(v, \{w_1, \ldots, w_n\}), (t, \{w_1, \ldots, w_n\}) \in C$ (2). Hence by Fact 6.43 $v, t \in \mathsf{N}(w_1) \cap \ldots \cap \mathsf{N}(w_n)$ (a). Since C is locally closed, it follows from (2), that $(v, \emptyset), (t, \emptyset) \in C$ (3). From (1), (3) and the fact that a is grounded, it follows that $(v, \emptyset) R_a (t, \emptyset)$. But then by construction $(v, t) \in \xrightarrow{a}$ (4). The claim follows from (4) and (a). ■

LEMMA 6.45 (Truth Preservation Lemma)
Let $\mathbb{M}$ be a good $\mathsf{P}\sigma\mathsf{LTS}$. For every $\varphi \in L_\sigma$ and $v, w_1, \ldots w_n \in W$ we have:

$$\mathsf{gather}(\mathbb{M})|_{\mathsf{N}(w_1)\cap\ldots\cap\mathsf{N}(w_n)}, v \models \varphi \text{ iff } \mathbb{M}^\Gamma, v, \{w_1, \ldots, w_n\} \models \varphi$$

PROOF.
Base case: From left to right. Assume that $\mathsf{gather}(\mathbb{M})|_{\mathsf{N}(w_1)\cap\ldots\cap\mathsf{N}(w_n)}, v \models p$ (1) and $v \in \mathsf{N}(w_1)\cap\ldots\cap\mathsf{N}(w_n)$ (2). By semantics it follows from (1) that $v \in V(p)$. Hence by construction $p \in Val(v, \emptyset)$ (3). From (2) and Fact 6.43 it follows that $(v, \{w_1, \ldots, w_n\}) \in C$ (4). Since Val is W-based, it follows from (3) and (4), that $p \in Val(v, \{w_1, \ldots, w_n\})$. Hence by semantics $\mathbb{M}^\Gamma, v, \{w_1, \ldots, w_n\} \models p$.

From right to left. Assume that $\mathbb{M}^\Gamma, v, \{w_1, \ldots, w_n\} \models p$. It follows by the semantics that $p \in Val(v, \{w_1, \ldots, w_n\})$ (1) and $(v, \{w_1, \ldots, w_n\}) \in C$ (2). Since C is locally closed it follows from (2) that $(v, \emptyset) \in C$, hence since Val is W-based we have from (1) $p \in Val(v, \emptyset)$ (3). But from (2) we have by Fact 6.43 that $v \in \mathsf{N}(w_1)\cap\ldots\cap\mathsf{N}(w_n)$ (4). From (3) and construction we have $v \in V(p)$, hence by (4) and semantics we have $\mathsf{gather}(\mathbb{M})|_{\mathsf{N}(w_1)\cap\ldots\cap\mathsf{N}(w_n)}, v \models p$.

Induction Step:
Case $\neg$ and case $\wedge$: Immediate from IH and semantics.

Case $\langle a \rangle$: From left to right. Assume that $\mathsf{gather}(\mathbb{M})|_{\mathsf{N}(w_1)\cap\ldots\cap\mathsf{N}(w_n)}, v \models \langle a \rangle \varphi$ (0). It follows from semantics that there is $v, t \in \mathsf{N}(w_1)\cap\ldots\cap\mathsf{N}(w_n)$ (2) with $v \xrightarrow{a} t$ (3) and $\mathsf{gather}(\mathbb{M})|_{\mathsf{N}(w_1)\cap\ldots\cap\mathsf{N}(w_n)}, t \models \varphi$ (4). From (2) and Fact 6.43 it follows that $(v, \{w_1, \ldots, w_n\}) \in C$ (5) and $(t, \{w_1, \ldots, w_n\}) \in C$ (6). From (3) and construction it follows that $(v, \emptyset) R_a (t, \emptyset)$ (7). Since it follows from (5), (6), (7) and the fact that a is grounded we have $(v, \{w_1, \ldots, w_n\}) R_a (t, \{w_1, \ldots, w_n\})$ (8). From (4) and IH it follows that $\mathbb{M}^\Gamma, t, \{w_1, \ldots, w_n\} \models \varphi$ (9). From (8), (9) and semantics it follows that $\mathbb{M}^\Gamma, v, \{w_1, \ldots, w_n\} \models \langle a \rangle \varphi$.

From right to left. Assume that $\mathbb{M}^\Gamma, v, \{w_1, \ldots, w_n\} \models \langle a \rangle \varphi$, since a is vertical, it follows that $v, \{w_1, \ldots, w_n\} \in C$ and by semantics it follows that there is some t with $t, \{w_1, \ldots, w_n\} \in C$ (0), $(v, \{w_1, \ldots, w_n\}) R_a (t, \{w_1, \ldots, w_n\})$ (1) and $\mathbb{M}^\Gamma, t, \{w_1, \ldots, w_n\} \models \varphi$ (2). Since C is locally closed we have $(v, \emptyset) \in C$ and $(t, \emptyset) \in C$, hence since a is grounded it follows from (1) that $(v, \emptyset) R_a (t, \emptyset)$. Hence by construction we have $v \xrightarrow{a} t$ (3). From (0) and Fact 6.43 it follows that $t \in \mathsf{N}(w_1)\cap\ldots\cap\mathsf{N}(w_n)$ (4). From (2) and IH it follows that $\mathsf{gather}(\mathbb{M})|_{\mathsf{N}(w_1)\cap\ldots\cap\mathsf{N}(w_n)}, t \models \varphi$. Hence by (4), (3) and semantics we have $\mathsf{gather}(\mathbb{M})|_{\mathsf{N}(w_1)\cap\ldots\cap\mathsf{N}(w_n)}, v \models \langle a \rangle \varphi$.

Case $[!\mathsf{N}]$: From left to right. Assume that $\mathsf{gather}(\mathbb{M})|_{\mathsf{N}(w_1)\cap\ldots\cap\mathsf{N}(w_n)}, v \models [!\mathsf{N}]\varphi$ (0). It follows that $v \in \mathsf{N}(w_1)\cap\ldots\cap\mathsf{N}(w_n)$ (1) and from semantics that $\mathsf{gather}(\mathbb{M})|_{\mathsf{N}(w_1)\cap\ldots\cap\mathsf{N}(w_n)\cap\mathsf{N}(v)}, v \models \varphi$ (2). Since $v \in \mathsf{N}(v)$ it follows from construction that $(v, \{v\}) \in C$ (3). By (1) and Fact 6.43 it follows that $(v, \{w_1, \ldots, w_n\}) \in C$. Hence since C is locally closed we have from (3) that $(v, \{w_1, \ldots, w_n, v\}) \in C$ (4). From (4), (2) and IH it follows that $\mathbb{M}^\Gamma, v, \{w_1, \ldots, w_n, v\} \models \varphi$. Hence from semantics we have $\mathbb{M}^\Gamma, v, \{w_1, \ldots, w_n\} \models [!\mathsf{N}]\varphi$.

From right to left. Assume that $\mathbb{M}^\Gamma, v, \{w_1, \ldots, w_n\} \models [!\mathsf{N}]\varphi$ it follows from semantics that $(v, \{w_1, \ldots, w_n, v\}) \in C$ (0)—hence since C is locally closed that $(v, \{v\}) \in C$ (1)—and that $\mathbb{M}^\Gamma, v, \{w_1, \ldots, w_n, v\} \models \varphi$ (2). From (1) and construction it follows that $v \in N(v)$ (3) and from (0), by Fact 6.43, that $v \in \mathsf{N}(w_1)\cap\ldots\cap\mathsf{N}(w_n)$ (4). From (3), (4), (2) and IH it follows that $\mathsf{gather}(\mathbb{M})|_{\mathsf{N}(w_1)\cap\ldots\cap\mathsf{N}(w_n)\cap\mathsf{N}(v)}, v \models \varphi$. Hence, by semantics, $\mathsf{gather}(\mathbb{M})|_{\mathsf{N}(w_1)\cap\ldots\cap\mathsf{N}(w_n)}, v \models [!\mathsf{N}]\varphi$. ■

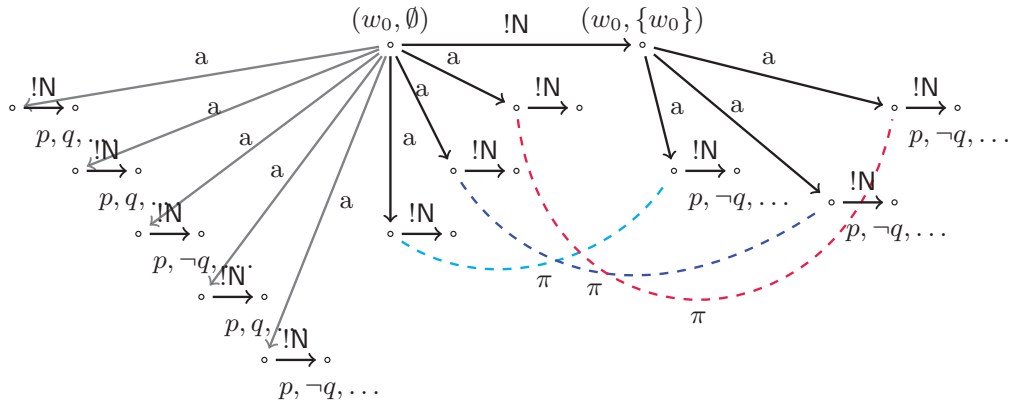

FIGURE 4. PσLTS from the example in Section 6.9.1.

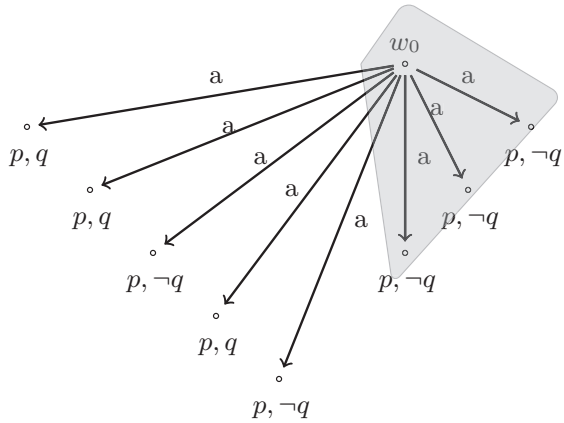

FIGURE 5. σLTS obtained by gathering the PσLTS from Figure 4.

6.11.1 Example, continued

Gathering our model from the example in Section 6.9.1 (see Figure 4), we get the σLTS displayed in Figure 5.

6.12 Completeness theorem

LEMMA 6.46 (Weak completeness)
Every finite !N-consistent sets of $\mathsf{L}_{(!\sigma,A)}$-formulas is satisfiable.

PROOF. Assume that Γ is a finite !N-consistent sets of $\mathsf{L}_{(!\sigma,A)}$-formulas. By Lemma 6.40 and Truth Lemma (Lemma6.41) it follows that there is some good pointed pseudoσLTS, $\mathbb{M}^\Gamma, w_0, \emptyset$ such that for every $\varphi \in \Gamma$ we have $\mathbb{M}^\Gamma, w_0, \emptyset \models \varphi$. But then by Fact 6.42 and Lemma 6.45 it follows that we have a pointed σLTS gather$(\mathbb{M}^\Gamma), w_0$ such that for every $\varphi \in \Gamma$, we have gather$(\mathbb{M}^\Gamma), w_0 \models \varphi$. ■

Our main result follows:

THEOREM 6.47 (Strong Completeness)
Every !N-consistent sets of $\mathsf{L}_{(!\sigma,A)}$-formulas is satisfiable.

This follows immediately from Lemmas 3.10 and 6.46.

7 Decidability and complexity

We have already indicated that there exists a polynomial translation from $\mathsf{L}_{(!\sigma,A)}$ into the loosely guarded fragment of first-order logic (Fact 3.3, Proposition 3.8 and Proposition 3.5). It follows that the satisfiability problem of $\mathsf{L}_{(!\sigma,A)}$ is decidable, and in DEXPTIME. But this is certainly not a tight upper bound. In this section, we show that the satisfiability problem of $\mathsf{L}_{(!\sigma,A)}$ is PSPACE-complete. We give an alternating algorithm for checking satisfiability of a formula φ in $\mathsf{L}_{(!\sigma,A)}$ and prove that our algorithm runs in time polynomial in the size of the input formula φ. This results in a proof of the fact that the decidability of the logic $\mathsf{L}_{(!\sigma,A)}$ is PSPACE-complete.

7.1 Algorithm for SAT

We give an *alternating* algorithm for deciding the satisfiability of a formula φ in the logic $\mathsf{L}_{(!\sigma,A)}$. Recall that Alternating Turing machines (Definition 2.14, see, e.g. [18]) generalize non-deterministic Turing machines. An ATM is a Turing machine with both universal states and existential states. An alternating Turing machine in a universal state accepts if there is at least one next configuration *and all next configurations are accepting*, while an ATM in an existential state accepts if there is *at least one next configuration that is accepting*. Algorithm 1 is alternating: it has both existential states (step 16) and universal states (step 27). At step 27, 'Universally choose $\langle a\rangle\varphi' \in \Delta$' means that the algorithm should accept the input if it accepts no matter which formula of the form $\langle a\rangle\varphi'$ (for some $a \in A$ and φ') is chosen from Δ'. At step 16, 'Existentially guess $\Delta' \in \{\Delta_1, \Delta_2\}$' means that the algorithm should accept the input if it accepts for at least one of Δ_1 or Δ_2.

Throughout this section we assume that formulas are in negation normal form. It is easy to see that we can always put a formula φ in negation normal form in time polynomial in $|\varphi|$. Moreover, recall from Section 3.2, that given $Z \subseteq |\mathbb{M}|$, we write $\mathsf{N}(Z) := \bigcap_{w\in Z} \mathsf{N}(Z)$. The main procedure is called SAT and it takes three arguments as input:

1. A finite multiset of formulas Δ in $\mathsf{L}_{(!\sigma,A)}$.
2. A state w of the potential model $\mathbb{M}$ of Δ.
3. A function ν which associates to every formula $\varphi \in \Delta$, a set of state $X \subseteq |\mathbb{M}|$ such that φ has to be satisfied at $\mathbb{M}|_{\mathsf{N}(X)}, w$.

Initially the procedure is called with $\{\varphi\}, w_0$ and ν_0 where $\nu_0(\varphi) = \epsilon$. $\mathsf{SAT}(\{\varphi\}, w_0, \nu_0)$ returns 1 if and only if φ is satisfiable.

The subprocedures do as their names say. More specifically, `is_not_normalized`(Δ) returns **1** if there is a formula in Δ with the main operator being either $\wedge, \vee$ or $[!\mathsf{N}]$ and returns **0** otherwise. `contains_only_literals`(Δ) checks if Δ consists of only atomic literals. `is_atomically_inconsistent`(Δ, PROP) returns **1** if there exist both p and $\neg p$ in Δ for some propositional letter $p \in$ PROP and returns **0** otherwise. `contains_no_diamond`(Δ) returns **1** if there is no formula of the form $\langle a\rangle\varphi'$ in Δ and **0** otherwise. Note that `is_not_normalized`(Δ), `contains_only_literals`(Δ) and `contains_no_diamond`(Δ) runs in time linear in the size of the set Δ while `is_atomically_inconsistent`(Δ, PROP) takes time at most quadratic in the size of Δ.

The procedure States(ν) takes as argument the current function ν and returns its range which is a set of states of the model constructed so far. Note that it runs in time linear in the size of ν.

LEMMA 7.1
For any finite set $\Delta \subseteq \mathsf{L}_{(!\sigma,A)}$, w and ν, we have $\mathsf{SAT}(\Delta, w, \nu)$=1 iff there is a pointed σLTS, $\mathbb{M}, w$ with $\mathsf{ran}(\nu) \cup \{w\} \subseteq |\mathbb{M}|$ such that for each $\varphi \in \Delta$, $\mathbb{M}|_{\mathsf{N}(\nu(\varphi))}, w \models \varphi$.

Algorithm 1 $\mathsf{SAT}(\Delta, w, \nu)$

1: **if** `is_not_normalized`(Δ) **then**
2: **for** $\varphi_1 \wedge \varphi_2 \in \Delta$ **do**
3: Let $\Delta \leftarrow \Delta \setminus \{\varphi_1 \wedge \varphi_2\} \cup \{\varphi_1, \varphi_2\}$
4: $\nu(\varphi_1) \leftarrow \nu(\varphi_1 \wedge \varphi_2)$
5: $\nu(\varphi_2) \leftarrow \nu(\varphi_1 \wedge \varphi_2)$
6: **return** $\mathsf{SAT}(\Delta, w, \nu)$
7: **end for**
8: **for** $[!\mathsf{N}]\varphi' \in \Delta$ **do**
9: Let $\Delta = \Delta \setminus \{[!\mathsf{N}]\varphi'\} \cup \{\varphi'\}$
10: Let $\nu(\varphi') = \nu([!\mathsf{N}]\varphi') \cup \{w\}$
11: **return** $\mathsf{SAT}(\Delta, w, \nu)$
12: **end for**
13: **for** $\varphi_1 \vee \varphi_2 \in \Delta$ **do**
14: Let $\Delta_1 = \Delta \setminus \{\varphi_1 \vee \varphi_2\} \cup \{\varphi_1\}$
15: Let $\Delta_2 = \Delta \setminus \{\varphi_1 \vee \varphi_2\} \cup \{\varphi_2\}$
16: Let $\nu(\varphi_1) = \nu(\varphi_2) = \nu(\varphi_1 \vee \varphi_2)$
17: `Existentially guess` $\Delta' \in \{\Delta_1, \Delta_2\}$
18: **return** $\mathsf{SAT}(\Delta', w, \nu)$
19: **end for**
20: **else if** `is_atomically_inconsistent`(Δ, PROP) **then**
21: **return 0**
22: **else if** `contains_only_literals`(Δ) **then**
23: **return 1**
24: **else if** `contains_no_diamond`(Δ) **then**
25: **return 1**
26: **else**
27: Universally choose $\langle a \rangle \varphi' \in \Delta$
28: Let $\Delta' = \{\varphi'\} \cup \{\varphi'' \mid [a]\varphi'' \in \Delta \ \& \ \nu([a]\varphi'') \subseteq \nu(\langle a \rangle \varphi')\}$
29: Let $\nu'(\varphi') = \nu(\langle a \rangle \varphi')$
30: Let $\nu'(\varphi'') = \nu([a]\varphi'')$
31: Let $w' \notin$ `states`(ν)
32: **return** $\mathsf{SAT}(\Delta', w', \nu')$
33: **end if**

PROOF. The proof is by induction on the maximum complexity of formulas in Δ.

For the base case, assume that Δ is a set of literals. It is easy to see that for any w and ν, $\mathsf{SAT}(\Delta, w, \nu)$ returns 1 iff it is a consistent set of literals, hence a satisfiable one. The claim follows from the fact that we can simply put $w \in \mathsf{N}(w_i)$ for the relevant w_i's.

We prove the induction step sequentially.

$\wedge$-step. Let $\Delta = \Gamma \cup (\varphi_1 \wedge \varphi_2)$ with $\mathsf{cpx}(\Gamma) \leq n$ and $\mathsf{cpx}(\varphi_1 \wedge \varphi_2) = n+1$. Now assume that $\mathsf{SAT}(\Gamma \cup \{\varphi_1 \wedge \varphi_2\}, w, \nu) = 1$. By steps (2–6) of the procedure, it follows that $\mathsf{SAT}(\Gamma, \varphi_1, \varphi_2), w, \nu) = 1$ but by IH and the semantics of $\wedge$, the claim follows. The claim holds for any number of conjunctions of complexity at most $n+1$.

$\vee$-step. Let $\Delta = \Gamma_1 \cup \Gamma_2 \cup (\varphi_1 \vee \varphi_2)$ with $\mathsf{cpx}(\Gamma_1) \leq n$ and $\Gamma_2 = \{\psi_1 \wedge \chi_1, \ldots, \psi_n \wedge \chi_n\}$, with $\max \mathsf{cpx}(\{\psi_1, \chi_1, \ldots, \psi_n, \chi_n\}) = n$. Now assume that $\mathsf{cpx}(\varphi_1 \vee \varphi_2) = n+1$ and assume that

$\mathsf{SAT}(\Gamma_1 \cup \Gamma_2 \cup \{\varphi_1 \vee \varphi_2\}, w, \nu) = 1$. By steps (12–18) of the procedure, we have either $\mathsf{SAT}(\Gamma_1 \cup \Gamma_2 \cup (\varphi_1), w, \nu) = 1$ or $\mathsf{SAT}(\Gamma_1 \cup \Gamma_2 \cup (\varphi_1 \vee \varphi_2), w, \nu) = 1$. The claim follows front the previous step and semantics of $\vee$. The claim holds for any number of disjunctions of complexity at most $n+1$.

[!N]-step. Let $\Delta = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \{[!\mathsf{N}]\varphi\}$ with $\mathsf{cpx}(\Gamma_1) \leq n$, $\Gamma_2 = \{\psi_1 \wedge \chi_1, \ldots, \psi_n \wedge \chi_n\}$, $\Gamma_3 = \{\alpha_1 \vee \beta_1, \ldots, \alpha_n \vee \beta_n\}$ with $\max \mathsf{cpx}(\{\psi_1, \chi_1, \alpha_1, \beta_1 \ldots, \psi_n, \chi_n, \alpha_n, \beta_n\}) = n$. Now assume that $\mathsf{SAT}(\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \{[!\mathsf{N}]\varphi\}, w, \nu) = 1$. Let $\nu' = \nu[\varphi / \nu([!\mathsf{N}]\varphi) \cup \{w\}]$ be the expansion of ν, with domain $\mathsf{dom}(\nu) \cup \{\varphi\}$ and $\nu'(\varphi) := \nu([!\mathsf{N}]\varphi) \cup \{w\}$. By steps (7–11) we have $\mathsf{SAT}(\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \{\varphi\}, w, \nu') = 1$. But then by previous step we have a model $\mathbb{M}$ such that for every $\psi \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \{\varphi\}$, we have $\mathbb{M}|_{\mathsf{N}(\nu'(\psi))}, w_0 \models \psi$. In particular, we have $\mathbb{M}|_{\mathsf{N}(\nu'(\varphi))}, w_0 \models \varphi$. Hence by semantics of [!N] and definition of ν' we have $\mathbb{M}|_{\mathsf{N}(\nu(\varphi))}, w_0 \models [!\mathsf{N}]\varphi$. The claim holds for any number of [!N]-formulas of complexity at most $n+1$.

$\langle a \rangle, [a]$-step. When the $\langle a \rangle, [a]$-step is reached we have $\Delta = \Gamma_1 \cup \Gamma_2$ with $\Gamma_1 = \{\langle a \rangle \varphi_1, \ldots, \langle a \rangle \varphi_n, [a]\psi_1, \ldots, [a]\psi_m\}$ and $\Gamma_2 = \{p_1, \ldots, p_k, \neg q_1, \ldots, \neg q_r\}$, with $p_1, \ldots, p_k, \neg q_1, \ldots, \neg q_r$ being literals. Now assume that $\mathsf{SAT}(\Delta, w, \nu) = 1$. By steps (27–33), we have for every $i = 1, \ldots, n$, $\mathsf{SAT}(\Delta_i, w'_i, \nu') = 1$ where

$$\Delta_i = \{\varphi_i\} \cup \{\psi_j | [a]\psi_j \in \Delta \text{ and } \nu([a]\psi_j) \subseteq \nu(\langle a \rangle \varphi_i)\}$$

and $\nu'(\varphi_i) = \nu(\varphi_i)$ (0) and for every j with $\nu([a]\psi_j) \subseteq \nu(\langle a \rangle \varphi_i)$ we have $\nu'(\psi_j) = \nu([a]\psi_j)$ (1).

By IH, (0) and (1) there is a pointed model $\mathbb{M}_i, w'_i$ with $\mathsf{ran}(\nu) \cup \{w'_i\} \subseteq |\mathbb{M}_i|$ and $\mathbb{M}_i|_{\mathsf{N}(\nu(\varphi_i))}, w'_i \models \varphi_i$ and for every j with $\nu([a]\psi_j) \subseteq \nu(\langle a \rangle \varphi_i)$ we have $\mathbb{M}_i|_{\mathsf{N}(\nu(\psi_j))}, w' \models \psi_j$ (2).

Now let $Rename_i(\mathbb{M}_i)$ be identical to $\mathbb{M}_i$ except that we rename all states occurring in $\mathbb{M}_i$ with names not occurring in $\mathbb{M}$. Let $\mathsf{newname}_i$ be the natural isomorphism from $\mathbb{M}_i$ into $Rename(\mathbb{M}_i)$. From steps (31) of the procedure, $\mathsf{newname}_i(w'_i) = w'_i$. Now take the $\{w'_i\}$, A-generated submodel of $Rename_i(\mathbb{M}_i)$. Call it $\mathbb{N}_i$. By Corollary 4.6 and (2) we have $\mathbb{N}_i|_{\mathsf{N}(\mathsf{newname}_i(\nu(\varphi_i)))}, w'_i \models \varphi_i$ (3) and for every j with $\nu([a]\psi_j) \subseteq \nu(\langle a \rangle \varphi_i)$ we have $\mathbb{N}_i|_{\mathsf{N}(\mathsf{newname}_i(\nu(\psi_j)))}, w'_i \models \psi_j$ (4). And let $P(\biguplus_i \mathbb{N}_i)$ be the smallest elementary extension of $\biguplus_i \mathbb{N}_i$ such that:

1. $|P(\mathbb{N}_i)| = |\biguplus_i \mathbb{N}_i| \cup \{States(\nu)\}$
2. $R_a^{P(\mathbb{N}_i)} = R_a^{|\biguplus_i \mathbb{N}_i} \cup \biguplus_i \{(w, w'_i)\}$
3. For every $v \in States(\nu)$, $\mathsf{N}^{P(\mathbb{N}_i)}(v) = \{\mathsf{N}(\mathsf{newname}_i(v)) \cup \{w'_i\} | v \in \nu(\langle a \rangle \varphi_i)\} \cup \{w\}$
4. $w \in V^{P(\mathbb{N}_i)}(p)$ iff $p \in \Delta$

From (3) and construction we have $P(\mathbb{N})_i|_{\mathsf{N}^{P(\mathbb{N}_i)}(\nu(\varphi_i))}, w'_i \models \varphi_i$. From construction and semantics of $\langle a \rangle$ we have $P(\mathbb{N})_i|_{\mathsf{N}^{P(\mathbb{N}_i)}(\nu(\varphi_i))}, w \models \langle a \rangle \varphi_i$.

Moreover from (4) and construction we have for every j with $\nu([a]\psi_j) \subseteq \nu(\langle a \rangle \varphi_i)$ that $P(\mathbb{N})_i|_{\mathsf{N}^{P(\mathbb{N}_i)}(\nu(\psi_j))}, w'_i \models \psi_j$. From construction and semantics of $[a]$ we have $P(\mathbb{N})_i|_{\mathsf{N}^{P(\mathbb{N}_i)}(\nu(\psi_j))}, w \mathbb{M}[a]\psi_j$. Truth of literals is guaranteed by the last part of the preceding construction. The claim follows. ■

COROLLARY 7.2
For any finite set $\Delta \subseteq \mathsf{L}_{(!\sigma,A)}$, Δ is satisfiable in a $\sigma\mathsf{LTS}$ iff $\mathsf{SAT}(\Delta, w_0, \mathsf{N}_0) = 1$, for $\mathsf{N}_0(\Delta) = \epsilon$.

PROOF. Immediate from the previous lemma. ■

The observation that given an initial formula $\varphi \in \mathsf{L}_{(!\sigma,A)}$, an initial state w_0 of the pointed model to be constructed and an initial context fucntion N_0 where $\mathsf{N}_0(\varphi) = \epsilon$, $\mathsf{SAT}(\{\varphi\}, w_0, \mathsf{N}_0)$ runs in time polynomial in $|\varphi|$ is also quite straightforward.

LEMMA 7.3
$\mathsf{SAT}(\{\varphi\}, \mathsf{w_0}, \mathsf{N_0})$ is in $\mathsf{APTIME}(|\varphi|)$.

PROOF. The If-condition at the beginning checks if the set of formulas Δ is in the required 'normal' form. That is, if the main operator of every formulas is none among $\wedge, \vee$ or [!N]. If not, then it manipulates Δ to convert it to the required form. Note that given an initial formula φ, $|\varphi|$ recursive calls of the SAT() procedure is sufficient to achieve this normal form.

In each such recursive call, one of the following happens.

- Either the set Δ is not in the normal form and the procedure SAT is called again. This is done after updating the set Δ and/or after updating the context function N. The function N may grow in the process. However, note that for every formula φ' in the current target set Δ, at most one new state is added to $\mathsf{N}(\varphi')$. Now since there can be at most $|\varphi|$ recursive calls to SAT, the size of the function N does not exceed $|\varphi|^2$.
- Or Δ is in the required normal form. In that case the call bypasses the initial If-condition. The second, third and the fourth If-conditions, respectively, check whether Δ is consistent, has only literals or has a formula of the form $\langle a \rangle \varphi'$. As we already observed, each of these procedures is polynomial in the size of $|\varphi|$. The final Else-condition updates Δ to a new set Δ'. Note that as the size of Δ was polynomial in $|\varphi|$, and the constructed Δ' is at most as large as Δ, Δ' is also polynomial in $|\varphi|$. The function N changes but does not grow in size. Finally to initialize the fresh state w' the procedure States is called which as we observed is linear in the size of N.

Thus, in each recursive call of SAT, the algorithm does only polynomially many operations in the size of the initial formula φ. Hence, the overall algorithm runs in time polynomial in $|\varphi|$ which means it is in $\mathsf{APTIME}(|\varphi|)$. ■

THEOREM 7.4
The satisfiability problem for the logic $\mathsf{L}_{(!\sigma,A)}$ is PSPACE-complete.

PROOF. Corollary 7.2 and Lemma 7.3 shows that the satisfiability problem is in APTIME. Since by [4] we know that APTIME = PSPACE we have that the problem is in PSPACE. PSPACE-hardness follows from the fact that $\mathsf{L}_{(!\sigma,A)}$ properly embeds the modal system K which is known to be PSPACE-complete [20]. ■

7.2 Model-checking

In this section, we give an algorithm for model-checking a formula $\varphi \in \mathsf{L}_{(!\sigma,A)}$ and show that the complexity of the model-checking problem is in time polynomial in the size of the input (the size of the σLTS and the length of φ). Formally, the model-checking problem is the following: given a σLTS $\mathbb{M}$, a state $w \in \mathbb{M}$ and a formula $\varphi \in \mathsf{L}_{(!\sigma,A)}$ decide if $\mathbb{M}, w \models \varphi$.

7.2.1 Why the bottom-up algorithm will not do

Given a pointed LTS $\mathbb{M}, w_0$, with $\mathbb{M} = \langle W, (\xrightarrow{a})_{a \in A}, V \rangle$, and $w_0 \in W$ and a basic modal formula φ, a standard bottom-up algorithm (see, for instance, [5] or [3]) can decide whether $\mathbb{M}, w \models \varphi$ recursively as follows. Enumerate the set of subformulas of φ, and for each $\psi \in \mathsf{sub}(\varphi)$, define $||\psi||^{\mathbb{M}}$ recursively as follows:

1. $||\top||^{\mathbb{M}} = W$

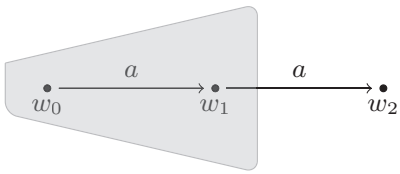

FIGURE 6. $\mathbb{M}_4$.

2. $||p||^{\mathbb{M}} = V(p)$ for $p \in \text{PROP}$
3. $||\neg\psi||^{\mathbb{M}} = W \setminus ||\psi||^{\mathbb{M}}$
4. $||\psi \wedge \chi||^{\mathbb{M}} = ||\psi||^{\mathbb{M}} \cap ||\chi||^{\mathbb{M}}$
5. $||\langle a\rangle\psi||^{\mathbb{M}} = \{w \in W \mid (\xrightarrow{a}[w] \cap |||\psi||^{\mathbb{M}}) \neq \emptyset\}$

If $w_0 \in \varphi$ then $\mathbb{M}, w \models \varphi$, otherwise $\mathbb{M}, w \not\models \varphi$.

This bottom-up algorithm for model-checking a formula φ in the basic modal language in an LTS (see, for instance Boolos et al. [3]) does not work for the logic $\mathsf{L}_{(!\sigma,A)}$. To see this consider the $\sigma\mathsf{LTS}$ $\mathbb{M}_4$ shown in Figure 6 and the $\mathsf{L}_{(!\sigma,A)}$ formula $\varphi := [!\mathsf{N}]\langle a\rangle\langle a\rangle\top$. We have $\mathbb{M}_4, w_0 \not\models \varphi$. However, the bottom-up algorithm would work as follows. Enumerate, the subformula closure of φ, as follows $\{\top, \langle a\rangle\top, \langle a\rangle\langle a\rangle\top, [!\mathsf{N}]\langle a\rangle\langle a\rangle\top\}$. The algorithm computes as follows:

1. $||\top||^{\mathbb{M}} = \{w_0, w_1, w_2\}$
2. $||\langle a\rangle\top||^{\mathbb{M}} = \{w \in W \mid \xrightarrow{a}[w] \neq \emptyset\} = \{w_0, w_1\}$
3. $||\langle a\rangle\langle a\rangle\top|| = \{w \in W \mid (\xrightarrow{a}[w] \cap \{w_0, w_1\}) \neq \emptyset\} = \{w_0\}$

However, in the next round the algorithm has to decide whether $w_0 \in ||[!\mathsf{N}]\langle a\rangle\langle a\rangle\top||^{\mathbb{M}}$ but it cannot decide correctly simply by looking at whether $w \in ||\psi||^{\mathbb{M}}$ for each $w \in \mathsf{N}(w_0)$ and each strict subformula ψ of $[!\mathsf{N}]\langle a\rangle\langle a\rangle\top$, that is for each $\psi \in (\mathsf{sub}([!\mathsf{N}]\langle a\rangle\langle a\rangle\top) \setminus \{[!\mathsf{N}]\langle a\rangle\langle a\rangle\top\})$.

7.2.2 Model-checking in the logic of sights

We see that a model-checking procedure for the logic $\mathsf{L}_{(!\sigma,A)}$ should, at every iteration, also keep track of the restricted model as dictated by the function N. Algorithm 2 is a recursive algorithm for this purpose. Note that in our recursive algorithm we are making use of the Boolean functions $\neg$, $\wedge$ and $\vee$, defined as usual. We also make use of the **switch** statement, standard in most programming languages, allowing to declare *case by case*, depending on the main operator of the current formula, what the algorithm has to do.

To prove that the above algorithm is correct, we show the following invariant:

PROPOSITION 7.5
Algorithm 2, in every recursive step $\mathsf{ModelCheck}(\mathbb{M}_{\text{cur}}, w_{\text{cur}}, \varphi_{\text{cur}})$ returns 1 iff $\mathbb{M}_{\text{cur}}, w_{\text{cur}} \models \varphi_{\text{cur}}$.

PROOF. The proof is by induction on the structure of φ_{cur}. The base case follows from the definition and the Boolean cases are straightforward. Now suppose $\varphi_{\text{cur}} = [!\mathsf{N}]\varphi$. By semantics,

$$\mathbb{M}_{\text{cur}}, w_{\text{cur}} \models [!\mathsf{N}]\varphi \text{ iff } \mathbb{M}_{\text{cur}}|_{\mathsf{N}(w_{\text{cur}})}, w_{\text{cur}} \models \varphi$$

Algorithm 2 ModelCheck($\mathbb{M}_{\text{cur}}, w_{\text{cur}}, \varphi_{\text{cur}}$)

```
1:  if φ_cur = [!N]φ then
2:      M ← M|_{N(w_cur)}
3:      ModelCheck(M, w_cur, φ)
4:  else
5:      M ← M_cur, w ← w_cur, φ ← φ_cur
6:      switch φ:
7:          case φ = p
8:              if p ∈ V(w) then
9:                  return 1
10:             else
11:                 return 0
12:             end if
13:         case φ = ¬p
14:             if p ∉ V(w) then
15:                 return 1
16:             else
17:                 return 0
18:             end if
19:         case φ = ¬φ'
20:             return ¬ModelCheck(M, w, φ')
21:         case φ = φ_1 ∧ φ_2
22:             return ModelCheck(M, w, φ_1) ∧ ModelCheck(M, w, φ_2)
23:         case φ = φ_1 ∨ φ_2
24:             return ModelCheck(M, w, φ_1) ∨ ModelCheck(M, w, φ_2)
25:         case φ = ⟨a⟩φ'
26:             if ∃w' ∈ M, w -a-> w' then
27:                 return ⋁_{w':w -a-> w'} ModelCheck(M, w', φ')
28:             else
29:                 return 0
30:             end if
31:         case φ = [a]φ'
32:             if ∄w' ∈ M, w -a-> w' then
33:                 return 1
34:             else
35:                 return ⋀_{w':w -a-> w'} ModelCheck(M, w', φ')
36:             end if
37:     end switch
38: end if
```

Lines 1–2 of the algorithm performs the model restriction $\mathbb{M}_{\text{cur}}|_{\mathsf{N}(w_{\text{cur}})}$. By IH ModelCheck($\mathbb{M}_{\text{cur}}|_{\mathsf{N}(w_{\text{cur}})}, w_{\text{cur}}, \varphi$) returns 1 iff $\mathbb{M}_{\text{cur}}|_{\mathsf{N}(w_{\text{cur}})}, w_{\text{cur}} \models \varphi$ and hence ModelCheck($\mathbb{M}_{\text{cur}}, w_{\text{cur}}, \varphi_{\text{cur}}$) returns 1 iff $\mathbb{M}_{\text{cur}}, w_{\text{cur}} \models [!\mathsf{N}]\varphi$.

Now, let $\varphi_{\text{cur}} = \langle a \rangle \varphi$. By semantics

$$\mathbb{M}_{\text{cur}}, w_{\text{cur}} \models \langle a \rangle \varphi \text{ iff } \exists w' : w_{\text{cur}} \xrightarrow{a} w', \mathbb{M}_{\text{cur}}, w' \models \varphi$$

By IH, $\mathsf{ModelCheck}(\mathbb{M}_{\text{cur}}, w', \varphi)$ returns 1 iff $\mathbb{M}_{\text{cur}}, w' \models \varphi$. Hence in line 27,

$$\bigvee_{w' : w \xrightarrow{a} w'} \mathsf{ModelCheck}(\mathbb{M}, w', \varphi') \text{ is } 1 \text{ iff } \exists w' : w_{\text{cur}} \xrightarrow{a} w', \mathbb{M}_{\text{cur}}, w' \models \varphi.$$

The argument for $\varphi_{\text{cur}} = [a]\varphi$ is similar. ■

We thus have the correctness of Algorithm 2

COROLLARY 7.6
Given a pointed $\sigma\mathsf{LTS}$ $(\mathbb{M}, w)$ and a formula $\varphi \in \mathsf{L}_{(!\sigma,A)}$, $\mathsf{ModelCheck}(\mathbb{M}, w, \varphi)$ returns 1 if and only if $\mathbb{M}, w \models \varphi$.

From the above algorithm, we also have that the model-checking problem for $\mathsf{L}_{(!\sigma,A)}$ is polynomial-time solvable in the size of the input model and formula. More precisely, let $|\mathbb{M}|$ denote the size of the σLTS $\mathbb{M}$ and $|\varphi|$ denote the length of a formula $\varphi \in \mathsf{L}_{(!\sigma,A)}$. We have

PROPOSITION 7.7
$\mathsf{ModelCheck}(\mathbb{M}, w, \varphi)$ runs in time $\mathcal{O}(|\mathbb{M}| \cdot |\varphi|)$.

PROOF. The proof is rather straightforward. First note that φ can have at most $|\varphi|$ subformulas. Every recursive step of $\mathsf{ModelCheck}(\mathbb{M}, w, \varphi)$ is on a subformula of φ which is strictly smaller than the current subformula. Hence there are at most $|\varphi|$ recursive steps. In a recursive step, the current model can be of size at most as large as the original model, that is $|\mathbb{M}|$. The steps other than the recursive calls run in time linear in the size of the current model. ■

8 Extensions and relations to other logics

We now turn to some extensions and comparisons to other logics. We discuss multi-agent generalizations of the logic of sights, as well as temporal and epistemic extensions. We discuss how the logic of sights relates to the logic of relativization, and to what extent it can be generalized as a multi-dimensional modal logic.

8.1 Minimal multi-agent versions

The simplest extension is a *multi-agent* logic of sights, which consists in having sights defined for all agents in some finite set $\mathcal{N}$ at every state of an LTS. To axiomatize this extension, the only important axiom is the commutativity axiom $[!1][!2]\varphi \leftrightarrow [!2][!1]\varphi$, for every $1, 2 \in \mathcal{N}$. Checking the soundness of this axiom is straightforward from the semantics and the fact that restriction is associative. We believe that the main change in the completeness proof, will be in the statement of Proposition 6.25, that should be generalized as follows:

CONJECTURE 8.1 (Commutativity)
Let Δ be a finite set and let $\vec{a} \in A^*$ be such that $\kappa_\Delta(\vec{a}) \neq \emptyset$. Let D, E, F be $(\Delta, \vec{a})$-atoms. If $\widehat{D} \wedge [!\mathsf{N}_1] \bigwedge \widehat{E}$ is consistent and $\widehat{E} \wedge [!\mathsf{N}_2] \bigwedge \widehat{F}$ is consistent as well, then there is some $(\Delta, \vec{a})$-atom, G, such that $\widehat{D} \wedge [!\mathsf{N}_2] \bigwedge \widehat{G}$ is consistent and $\widehat{G} \wedge [!\mathsf{N}_1] \bigwedge \widehat{F}$ is consistent.

We will check the details in future work.

8.2 *Temporal extensions*

So far we have considered the logic of sights as an extension of the basic modal language. This is a natural starting point. However extending temporal logics such as Linear-Time Temporal Logic (LTL), Computation Tree Logic (CTL), dynamic logics such as Propositional Dynamic Logic (PDL, [36]), or multi-agent temporal logics such as Alternating-Time Temporal Logic (ATL) or STIT, is a next step with natural applications.

We first discuss the question of the extension of PDL with a sight operator and then discuss the question of the extension of ATL with a sight operator.

8.2.1 PDL with sights

If we extend our language with formulas such as $\langle a^* \rangle \varphi$ with the usual semantics, we believe that the traditional axiomatization of PDL using a least fixed-point axiom and an induction axiom on top of the axiomatization given in Section 5.3 will be complete for PDL with sights on the class σLTS.

Proving this would require an adaption of our proof. We believe, this can be done as follows. Instead of our CN-closure and of the usual Fisher-Ladner closure used in completeness proofs of PDL, we would need to use a notion of closure that satisfies the richness properties of both, while remaining finite. Next, we need to construct a PσLTS as we have done here. The idea is then to define the interpretation of complex programs semantically from the atomic programs. To recover the truth lemma for the $\langle \alpha^* \rangle \varphi$ case, we need to extend our proof. Precisely, we need to show if there are atoms D, E such that, then $D \xrightarrow{\alpha^*} E$ then there is a sequence of atoms $A_0 \xrightarrow{\alpha} A_1 \xrightarrow{\alpha} \ldots \xrightarrow{\alpha} A_n \xrightarrow{\alpha} A_{n+1}$ with $A_0 = D$ and $A_{n+1} = E$. This should follow from the richness of the set of atoms and axioms. But, we need to check the details in a separate paper.

8.2.2 ATL with sights

Our initial discussion of sights was motivated by examples of strategic interaction. Section 9 says a bit more about the relation between the logic of sights and games. But let us look briefly at a possible extension of ATL [1] with sights. The following definitions are variations on that of [1].

DEFINITION 8.2 (Alternating Transition Systems (ATS), [1])
An ATS is a tuple $\mathbb{S} = \langle W, Z, A, Ag, \rho, \alpha, \delta, V \rangle$ where W is a non-empty set, $Z \subseteq W$, $A = \{a_1, \ldots, a_m\}$ is a finite set of actions, $Ag = \{1, \ldots, n\}$ is a finite set of agents, $\rho : W \to \wp(Ag)$, with $\rho(Z) = \{\emptyset\}$, $\alpha : (Ag \times W) \to \wp(A) \setminus \{\emptyset\}$, $V : \text{PROP} \to \wp(W)$ and $\delta : (W \times Ag \times A) \to \wp(W)$.

Let a limitation on α be a function $\alpha' : (Ag \times W) \to \wp(A) \setminus \{\emptyset\}$ such that for every $(i, w) \in \mathsf{dom}(\alpha')$, $\alpha'(i, w) \subseteq \alpha(i, w)$. An $\mathbb{S}$-choice function c is a function $c : (W \times Ag) \to A$ such that for every $(w, i) \in \mathsf{dom}(c)$ we have $c(w, i) \in \alpha(i, w)$.

DEFINITION 8.3 (Safety)
We say that an ATS $\mathbb{S} = \langle W, Z, A, Ag, \rho, \alpha, \delta, V \rangle$ is safe whenever

- for every $w \in W \setminus Z$, $i \notin \rho(w)$ and $a \in \alpha(i, w)$ we have $\delta(w, i, a) = \wp(W)$ (only agents in $\rho(w)$ have non-trivial power at w)
- for every $w \in Z$, i and $a \in \alpha(i, w)$ we have $\delta(w, i, a) = \{w\}$ (terminal nodes are safe sink states)
- For every $w \in W$ and every $\mathbb{S}$-choice function c we have: $\bigcap_{i \in Ag} \delta(w, i, c(i)) \neq \emptyset$ (safety, the transition function is non-blocking)

DEFINITION 8.4 (Coalitions)
Given an ATS, a coalition is a non-empty subset of Ag.

DEFINITION 8.5 (Strategies)
Given a coalition $C \subseteq Ag$, a strategy for C is a function $S_C : (W^+ \times C) \to A$ such that for every $(\vec{w}, i) \in \mathsf{dom}(S_C)$ we have $S_C(\vec{w}, i) \in \alpha(i, \mathsf{last}(\vec{w}))$.

DEFINITION 8.6 (Positional Strategies)
Given a coalition $C \subseteq Ag$, a positional strategy for C is a function $S_C : (W \times C) \to A$ such that for every $(w, i) \in \mathsf{dom}(S_C)$ we have $S_C(w, i) \in \alpha(i, w)$.

FACT 8.7
Every positional strategy for C induces a strategy for C.

Given a limitation α' on α, we say that S_C is an α'-strategy iff for every $i \in C$ and $\vec{w} \in \mathsf{dom}(S_C)$ we have $S_C(\vec{w}, i) \in \alpha'(i, \mathsf{last}(\vec{w}))$.

DEFINITION 8.8 (Compatible runs)
Let a run be a sequence $\vec{w} \in (W^* \cup W^\omega)$. We say that a run $\vec{w}$ is compatible with a profile of strategies S_{Ag} at a state w_0—for short compatible with (w_0, S_{Ag})—iff $\vec{w}[1] = w_0$ and for every $k, k+1 \leq \mathsf{len}(\vec{w})$ we have $\vec{w}[k+1] \in \bigcap_{i \in Ag} \delta(\vec{w}[k], i, S_{Ag}(\vec{w}|_k))$. We say that $v \in W$ is an eventuality at w_0 iff there is a run $\vec{w}$ that is compatible with (w_0, S_{Ag}) and some $k \leq \mathsf{len}(\vec{w})$ such that $\vec{w}[k] = v$.

We say that a run $\vec{w}$ is compatible with the limitation α' at w_0 iff there is some α'-strategy S_{Ag} for Ag such that $\vec{w}$ is compatible with (w_0, S_{Ag}). We say that $v \in W$ is an α'-eventuality at w_0 iff there is a run $\vec{w}$ that is compatible with the limitation α' at w_0 and some $k \leq \mathsf{len}(\vec{w})$ such that $\vec{w}[k] = v$.

DEFINITION 8.9 (Immediately C-achievable sets)
Let $\mathbb{S} = \langle W, Z, A, Ag, \rho, \alpha, \delta, V \rangle$ be an ATS, let C be a non-empty subset of Ag and let $w_0 \in W$. We say that $X \subseteq W$ is an immediately C-achievable next set at w_0 whenever there is a positional strategy for C, S_C such that for every positional strategy S_{-C} for $Ag \setminus C$ we have $\delta(w_0, S_C \cup S_{-C}) \subseteq X$.

DEFINITION 8.10 (Locking)
Let $\mathbb{S} = \langle W, Z, A, Ag, \rho, \alpha, \delta, V \rangle$ be an ATS, let C be a non-empty subset of Ag and let $w_0 \in W$. We say that a strategy S_C for C locks $\mathbb{S}$ in $X \subseteq W$ at w_0 iff for every strategy S_{-C} for $Ag \setminus C$, and for every run $\vec{w} \in W^* \cup W^\omega$ which is compatible with $(w_0, S_C \cup S_{-C})$ and every $n \leq \mathsf{len}(\vec{w})$ we have $\vec{w}[n] \in X$.

We say that C can lock $\mathbb{S}$ in $X \subseteq W$ at w_0 whenever there is a strategy S_C for C that locks $\mathbb{S}$ in $X \subseteq W$ at w_0.

DEFINITION 8.11 (C-visitable sets)
Let $\mathbb{S} = \langle W, Z, A, Ag, \rho, \alpha, \delta, V \rangle$ be an ATS, let C be a non-empty subset of Ag and let $w_0 \in W$. We say that $X \subseteq W$ is a C-visitable set at w_0 whenever there is a strategy S_C for C such that for every strategy S_{-C} for $Ag \setminus C$, and for every run $\vec{w} \in W^* \cup W^\omega$ which is compatible with $(w_0, S_C \cup S_{-C})$, we have some $n \leq \mathsf{len}(\vec{w})$ such that $\vec{w}[n] \in X$.

In such a system, we have operators of the form $\langle\langle C \rangle\rangle \bigcirc \varphi$ and $\langle\langle C \rangle\rangle \mathsf{G}\varphi$ and $\langle\langle C \rangle\rangle \mathsf{F}\varphi$, with the intuitive meaning C can enforce that the system moves into a φ-state, C can make sure that φ holds permanently from now on and C can make sure that φ will eventually hold, respectively. These operators have the following semantics:

$$
\begin{array}{lll}
\mathbb{S}, w \models \langle\langle C \rangle\rangle \bigcirc \varphi & \text{iff} & ||\varphi|| \text{ is an immediately } C\text{-achievable next set at } w \\
\mathbb{S}, w \models \langle\langle C \rangle\rangle \mathsf{G}\varphi & \text{iff} & C \text{ can lock } \mathbb{S} \text{ in } ||\varphi|| \text{ at } w \\
\mathbb{S}, w \models \langle\langle C \rangle\rangle \mathsf{F}\varphi & \text{iff} & ||\varphi|| \text{ is a } C\text{-visitable set at } w
\end{array}
$$

How should safe ATS be expanded with sights? We could simply add sight functions on top of ATS. But in general, the restriction of a safe ATS will not be a safe ATS, since the transition function might fail to be non-blocking (an existential property). Whether or not this form of safety should be preserved under taking sights. We could of course simply add either way.

Consider a safe pointed ATS $(\mathbb{S}, w_0)$ with $\mathbb{S} = \langle W, Z, A, Ag, \rho, \alpha, \delta, V \rangle$ and let α' be a limitation on α. We let $\mathsf{N}_{\alpha'}(w_0)$ be the smallest subset of W containing all α'-eventualities and w_0. This subset need not be a proper subset of W. We refer to $\mathsf{N}_{\alpha'}(w_0)$ as the (w_0, α')-induced sight. We show the following:

PROPOSITION 8.12
Given a safe pointed ATS $(\mathbb{S}, w_0)$ with $\mathbb{S} = \langle W, Z, A, Ag, \rho, \alpha, \delta, V \rangle$ and a limitation α' on α $(\mathbb{S}, w_0)[\alpha/\alpha']|_{\mathsf{N}_{\alpha'}(w_0)}$ is a safe pointed ATS.

PROOF. We only give the idea of the proof. Safety requires three properties to be satisfied: that agents not in $\rho(w)$ have trivial power at w, that terminal nodes are safe sink states, and that the transition function is non-blocking. Satisfaction of the first condition follows from the fact that ρ and δ are defined as restrictions. For the second condition, observe that a limitation α' of α can never map a pair (i, w) in the original model to the empty set and that it has map (i, w) to a subset of $\alpha(i, w)$. Hence, terminal safe sink states that survive the restriction will still be safe sinks. Moreover, by the same conditions and the fact that $\mathsf{N}_{\alpha'}(w_0)$ contains all α' eventualities (from w_0), the resulting set Z' (after restriction) will be a non-empty subset of Z. Finally, the third condition follows from the definition for $\mathsf{N}_{\alpha'}(w_0)$ just mentioned, and the fact that the new transition function δ is defined as a restriction and that the original δ was non-blocking. ■

We can now define a safe σATS as follows:

DEFINITION 8.13 (Safe σATS)
A safe σ ATS is a structure $\mathbb{S} = \langle W, Z, A, Ag, \rho, \alpha, \delta, V, (\sigma_i)_{i \in Ag} \rangle$ such that $\mathbb{S} \setminus (\sigma_i)_{i \in Ag}$ is a safe ATS and for each $w \in W$ and $i \in Ag$, $\sigma_i(w)$ is a restriction on α. Given a state w_0 and an agent i, we let $\sigma(w_0, i, \mathbb{S}) = (\mathbb{S}, w_0)[\alpha/\alpha']|_{\mathsf{N}_{\sigma_i(w_0)}(w_0)}$.

We can now expand our language with operators of the form $[!\mathsf{N}_i]\varphi$ with semantics:

$$\mathbb{S}, w \models [!\mathsf{N}_i]\varphi \text{ iff } \sigma(w, i, \mathbb{S}), w \models \varphi$$

We will study σATS and its logic in another paper. We simply observe that due to the fact that the semantics of the operator $\langle\langle C \rangle\rangle \bigcirc \varphi$ contains a non-decomposable quantifier alternation $\exists\forall$, the axiomatization of $[!\mathsf{N}] + \langle\langle C \rangle\rangle \bigcirc$ will not follow trivially from the axiomatization of L_σ over σLTS.

8.3 Epistemic extensions

We think that sight operators have interesting interactions with epistemic operators. But, before we discuss any extensions of our logic with epistemic operators, we first discuss whether we can give a natural epistemic or doxastic interpretation to $[!\mathsf{N}]$ itself.

First note that $\models [!\mathsf{N}]p \leftrightarrow p$. As an epistemic/doxastic operator, it would mean that the agent is never mistaken about basic facts. Moreover the agent is never mistaken about existential modal facts (e.g., $\models [!\mathsf{N}]\langle a \rangle p \leftrightarrow \langle a \rangle p$, but might be wrong about non-existential formulas. Conversely, the agent is fully knowledgeable of universal facts (e.g. $\models [a]p \leftrightarrow [!\mathsf{N}][a]p$).

We believe these properties are not intuitively plausible as properties of a reasonable epistemic or doxastic operator. They are, however, relatively natural as describing the awareness an agent has of the possible evolution of a system. If I am aware that a could happen, then a could happen. If

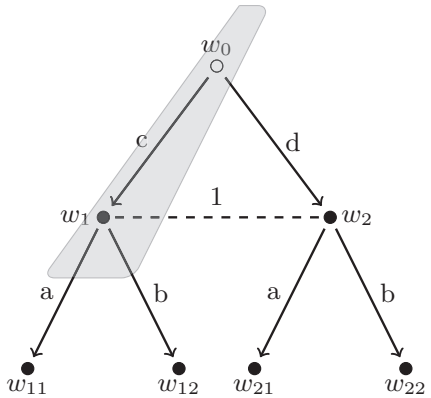

FIGURE 7. Interaction between epistemic uncertainty and sights.

executing a can only lead to p-states, then I cannot be aware of an a-execution that would lead to non-p state. Of course, agents can be wrong, but this is why it is interesting to expand the logic of sights with epistemic operators, or, as most will think of it, expand an epistemic or doxastic signature with sight operators.

Many interesting questions arise when concerned with the interaction of information uncertainty and sights. While we reserve a systematic analysis of these interactions to a follow-up paper, consider, nevertheless, the following example. Is a model, where w_1 is in the sight of w_0, while w_2 is not, but $w_1 \sim w_2$ an acceptable model? (Figure 7 is partial representation of such a situation.)

Note that this is different from a situation in which perfect recall—see [38] for a formal discussion of alternative mathematical definitions of the concept and their relation—is violated. The question is essentially whether past sights (or past awareness, or past expectations) should be preserved after actions are executed. Whether we consider the above model as acceptable or not depends on whether an agent can be aware of certain transitions in the system, but yet, when continuations she was considering possible do happen, is not able to distinguish whether one of the unexpected actions has taken place.

We discussed the possibility of treating $[!\mathsf{N}]$ as an epistemic operator. But we could also treat our underlying transition systems as an epistemic model. Doing so, however, requires a slightly different approach to the semantics of sights. Consider the following example:

EXAMPLE 8.14
Naive car buyer Georg_1 takes a shiny, clean car (p) to be well kept (q) by its owner and a dusty, dirty ($\neg p$) car to be not so well kept ($\neg q$); skeptical buyer David_2 does not make such an assumption, but is not a car expert and cannot decide whether the car has been well kept or not.

Intuitively, we expect that in the state of the world where the car is not so well kept but shiny and clean, it is common knowledge that the car is well kept in the sight of Georg (i). But this is incompatible with both the assumption that for every w we have $w \in \mathsf{N}(w)$ and the standard assumption that the epistemic relation is an equivalence relation.

Another intuitive requirement is an uniformity one: whenever $w \sim_1 v$ then $\mathsf{N}_1(w) = \mathsf{N}_2(w)$ (ii). If we relax the assumption that for every w we have $w \in \mathsf{N}(w)$ to the fact that for every w we have $\mathsf{N}(w) \neq \emptyset$, we could hope to accommodate (i) and (ii) with the model in Figure 8.

But if we leave the semantics of $[!\mathsf{N}]$ unchanged, it will not be well defined any longer. For example, whether $[!\mathsf{N}_1]\varphi$ is satisfied in the model of Figure 8 at w_2 would be an ill-defined question. At best

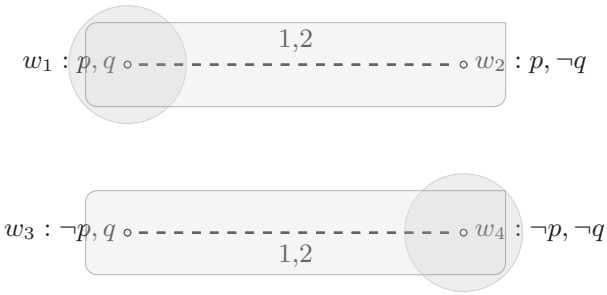

FIGURE 8. Circles represent N_1 Rectangles represent N_2.

we could try to patch the semantics to have:

$$w_2 \models [!\mathsf{N}_1]\bot$$

But, clearly, this is not something we would like to have either. Let $K_i[w] = \{v | w \sim_i v\}$. We also write $\langle i \rangle \varphi := \neg K_i \neg \varphi$. In addition to our previous assumptions, we require that for every w we have $(\mathsf{N}_i(w) \cap K_i[w]) \neq \emptyset$. Together with uniform sights and the assumption that $\sim_i$ is an equivalence relation, we can use the following semantics:

$$\mathbb{M}, w \models [!\mathsf{N}_i]\varphi \text{ iff for every } v \in K_i[w] \cap \mathsf{N}_i(w) \text{ we have } \mathbb{M}|_{\mathsf{N}_i(v)}, v \models \varphi$$

Now the meaning of $[!\mathsf{N}_i]\varphi$ is that from agent i's perspective, i knows that φ. This is different from a belief operator. For example, we would not have $\mathbb{M}, w_2 \models B_i C_{1,2}\varphi$ with a reasonable belief operator. Moreover, with uniform sights and epistemic equivalence relations we have, with the above semantics, not only $\models [!\mathsf{N}_i]\varphi \to K_i[!\mathsf{N}_i]\varphi$, but also $\models [!\mathsf{N}_i]\varphi \to [!\mathsf{N}_i]K_i\varphi$. While the first assumption is generally unproblematic for most standard semantics of belief, the second one—usually referred to as positive certainty—is easily shown to be incompatible with having a non-trivial **KD45**-belief operator, an **S5**-knowledge operator and the assumption that knowledge implies belief. The reader can consult e.g. Halpern [14] for details.

Our remarks indicate that the details of the relation between knowledge, beliefs, sights and their dynamics will be interesting but complex. We plan to discuss these in a separate paper.

8.4 Changes in sights

Sights need not be stable over time. Some events might make agents consider options they did not consider before (sight extension) or it might make them stop considering what they considered before (sight restriction). In either case, we might have:

- either have a semantically driven change and ask what happens if this or that subset of the model is added to the sight of the agent, or,
- a syntactically driven change and ask what happens if someone makes the agent consider the possibility that φ and if someone makes that agent only focused on possibilities that are compatible with φ.

We only look at the second case. Assume that we are in the left-hand side model in Figure 9 at state w_0. Now the agent is being convinced to consider the possibility that $\langle a \rangle \langle b \rangle \top$. For example, a chess

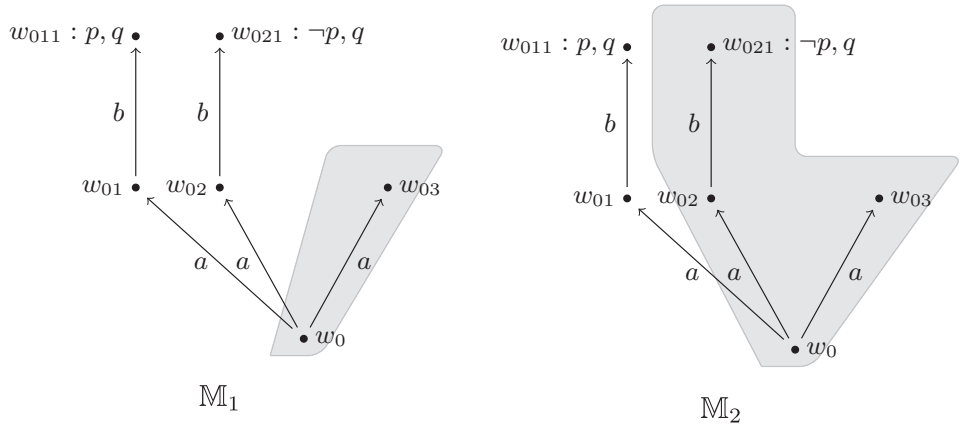

FIGURE 9. The agent starts to consider that $\langle a\rangle\langle b\rangle\top$.

player is being suggested to look at a certain sequence of moves she or he was not paying attention to. Let $[+\langle a\rangle\langle b\rangle\top]$ be the syntactic counterpart to this yet-to-be-defined operation. What should we expect about the resulting model? We should probably expect that $[+\langle a\rangle\langle b\rangle\top][!\mathsf{N}]\langle a\rangle\langle b\rangle\top$. Of course, if only the sight of our agent is changing, we either need a precondition to the execution of $+\langle a\rangle\langle b\rangle\top$ or we need to restrict its success to cases where it is possible to succeed. So what we really would like is something like

$$\langle a\rangle\langle b\rangle\top \to [+\langle a\rangle\langle b\rangle\top][!\mathsf{N}]\langle a\rangle\langle b\rangle\top.$$

The right-hand side model in Figure 9 represents one of the satisfying extension of $\mathsf{N}(w_0)$. The two other being $\mathsf{N}(w_0)=\{w_0, w_{03}, w_{01}, w_{01}\}$ and $\mathsf{N}(w_0)=|\mathbb{M}_2|$. The first two are in some sense minimal, but it is hard to argue for either one, without having some form underlying notion of plausibility (or preferences) on either states or subsets of the model. Instead, we could simply require truth in all models $\mathbb{M}'$ which are isomorphic to $\mathbb{M}$ except for $\mathsf{N}^{\mathbb{M}'}(w_0)$ and in which $\mathsf{N}^{\mathbb{M}'}(w_0)$ is a minimal extension of $\mathsf{N}^{\mathbb{M}_1}(w_0)$ such that $\mathbb{M}'|_{\mathsf{N}(w_0)}, w_0 \models \langle a\rangle\langle b\rangle\top$. We would then have

$$\mathbb{M}, w_0 \models [+\langle a\rangle\langle b\rangle\top][!\mathsf{N}]\langle a\rangle\langle b\rangle q$$

but the same would not be true for $\langle a\rangle\langle b\rangle p$ or $\langle a\rangle\langle b\rangle\neg p$

$$\mathbb{M}, w_0 \not\models [+\langle a\rangle\langle b\rangle\top][!\mathsf{N}]\langle a\rangle\langle b\rangle p$$

Finally in the preceding model for an unsatisfiable condition we would have

$$\mathbb{M}, w_0 \not\models [+\langle c\rangle\top]\bot$$

A similar semantics can be given for the other operation, using minimal restrictions instead of minimal extensions.

In an epistemic context with sights, we need to restrict the acceptable extensions of $\mathsf{N}(w_0)$ to be in some sense compatible with $K[w_0]$: if $K[w_0]$ encodes the agent's information and a reasonable agent will only expand her sight to 'accommodate suggestions' that are compatible with her information. By contrast, this would not need to be the case with respect to belief sets, but we will not get into that in this article. In epistemic temporal models, we have to be careful about the exact meaning of the

previous restrictions. One possible formalization could be that the acceptable extensions of $\mathsf{N}(w_0)$ should be subsets of $\xrightarrow{a}{}^{*}(K[w_0])$.

8.5 *Logic of sights and relativization*

There is a natural relation between the logic of sights and relativization. Relativization is the syntactic counterpart to restricting models to the truth set of a formula. Let φ, ψ be formulas of the basic modal language. The relativization of φ to ψ, written $(\varphi)^{\psi}$, is a modal formula defined recursively as follows:

$$\begin{array}{lcl} p^{\psi} & = & p \wedge \psi \\ (\neg\varphi)^{\psi} & = & \neg\varphi^{\psi} \\ (\varphi_1 \wedge \varphi_2)^{\psi} & = & \varphi_1^{\psi} \wedge \varphi_2^{\psi} \\ (\langle a\rangle\varphi)^{\psi} & = & \langle a\rangle(\psi \wedge \varphi) \end{array}$$

Consider a finite $\sigma\mathsf{LTS}$ $\mathbb{M}$ with domain $\{w_0, \ldots, w_{n-1}\}$. For each, w_i, expand the basic modal language with a surjective set of fresh nominals $\textsc{nom}^{\mathbb{M}}$, that is with a set of fresh propositional letters such that for every $p_i \in \textsc{nom}^{\mathbb{M}}$, $|V(p_i)| = 1$ and $V(\textsc{nom}^{\mathbb{M}}) = \{w_0, \ldots, w_{n-1}\}$, call the resulting model $\mathbb{M}^{+\textsc{nom}}$. For every w_i, define a formula $\pi_i := \bigvee_{w_j \in N(w_i)} p_j$. Note that for basic modal formulas φ, we have:

$$\mathbb{M}, w_i \models [!\mathsf{N}]\varphi \text{ iff } \mathbb{M}^{+\textsc{nom}}, w_i \models \varphi^{\pi_i}$$

Thus, given a finite $\sigma\mathsf{LTS}$ $\mathbb{M}$, we can recursively define a function $f_{\mathbb{M}} : \mathsf{L}_{(!\sigma,A)}(\textsc{prop}) \to \mathrm{H}_{(A)}(\textsc{prop} \cup \textsc{nom}^{\mathbb{M}})$ giving us for every state $w \in \mathbb{M}$ and every formula $\varphi \in \mathsf{L}_{(!\sigma,A)}(\textsc{prop})$, a formula $f_{\mathbb{M},w_i}(\varphi)$ as follows:

$$\begin{array}{lcl} f_{\mathbb{M},w_i}(\varphi) & = & f_{\mathbb{M},w_i}(\varphi^{\top}) \\ f_{\mathbb{M},w_i}(p^{\top}) & = & p \text{ for both propostional letters and nominals.} \\ f_{\mathbb{M},w_i}(p^{\psi}) & = & (p \wedge f_{\mathbb{M},w_i}(\psi)) \\ f_{\mathbb{M},w_i}((\varphi \wedge \chi)^{\psi}) & = & f_{\mathbb{M},w_i}(\varphi^{\psi}) \wedge f_{\mathbb{M},w_i}(\chi^{\psi}) \\ f_{\mathbb{M},w_i}((\varphi \vee \chi)^{\psi}) & = & f_{\mathbb{M},w_i}(\varphi^{\psi}) \vee f_{\mathbb{M},w_i}(\chi^{\psi}) \\ f_{\mathbb{M},w_i}((\neg\varphi)^{\psi}) & = & \neg f_{\mathbb{M},w_i}(\varphi^{\psi}) \\ f_{\mathbb{M},w_i}((\langle a\rangle\varphi)^{\psi}) & = & \langle a\rangle \bigvee\limits_{w_j \in R_a[w_i]} f_{\mathbb{M},w_j}(\psi \wedge \varphi^{\psi}) \\ f_{\mathbb{M},w_i}(([!\mathsf{N}]\varphi)^{\psi}) & = & f_{\mathbb{M},w_i}(\varphi^{\pi_i \wedge \psi}) \end{array}$$

We now have to show that this translation can simulate the sight operator over finite models. First let us make a few observations:

FACT 8.15
For any finite pointed LTS $\mathbb{M}, w$ and any formula $\varphi \in \mathsf{L}_{(!\sigma,A)}$, $f_{\mathbb{M},w_i}(\varphi) \in \mathrm{H}_{(A)}(\textsc{prop} \cup \textsc{nom}^{\mathbb{M}})$.

PROOF. Note that ultimately only propositional letters will be exponentiated at which stage it is transformed in a simple conjunction. Moreover, at every stage the exponent φ is a conjunction of

clauses of nominals. Finiteness of the translation of $\langle a\rangle\varphi$ formulas follows from finiteness of the model. ■

FACT 8.16
For every formula $\varphi \in \mathsf{L}_{(!\sigma,A)}(\text{PROP})$ and every finite pointed model $\mathbb{M}, w_i$, we have

$$\mathbb{M}^{+\text{NOM}}, w_i \models \varphi \text{ iff } \mathbb{M}, w_i \models \varphi$$

PROOF. Trivial. ■

FACT 8.17
For every $\varphi \in \mathsf{L}_{(!\sigma,A)}(\text{PROP}\cup\text{NOM})$, every finite pointed model $\mathbb{M}, w_i$ and finite subset $W_0 \subseteq |\mathbb{M}|$, such that $w_i \in \bigcap_{w\in W_0} N(w)$ we have:

$$\mathbb{M}^{+\text{NOM}}_{\bigcap_{w_k\in W_0} N(w_k)}, w_i \models \varphi \text{ iff } \mathbb{M}^{+\text{NOM}}|_{||\bigwedge_{w_k\in W_0}\pi_k||}, w_i \models \varphi$$

PROOF. Immediate from the definition of π_k. ■

LEMMA 8.18
For every formula $\varphi \in \mathsf{L}_{(!\sigma,A)}(\text{PROP}\cup\text{NOM})$, and every finite pointed model $\mathbb{M}, w_i$ and finite subset $\Psi \subseteq \text{NOM}\cup\text{PROP}$ with $w_i \in ||\bigwedge\Psi||$, we have:

$$\mathbb{M}^{+\text{NOM}}|_{||\bigwedge\Psi||}, w_i \models \varphi \text{ iff } \mathbb{M}^{+\text{NOM}}, w_i \models f_{\mathbb{M},w_i}(\varphi^{\bigwedge\Psi})$$

PROOF. Proof is by induction on the complexity of φ. If φ is a propositional letter p, since $\Psi \subseteq \text{NOM}\cup\text{PROP}$ and Ψ is finite, we have $f_{\mathbb{M},w_i}(p^{\bigwedge\Psi}) = p \wedge \bigwedge\Psi$. The argument is the same for nominals. Now for the induction step. Booleans are straightforward from definition.

Now assume that $\mathbb{M}^{+\text{NOM}}|_{||\bigwedge\Psi||}, w_i \models [!\mathsf{N}]\varphi$ (0). First note that we have $||\pi_i \wedge \bigwedge\Psi|| = ||\pi_i|| \cap ||\bigwedge\Psi||$ (1). Note that, by definition of π_i, $||\pi_i|| = N(w_i)$ (2). Hence we have $\mathbb{M}^{+\text{NOM}}|_{||\bigwedge\Psi||}|_{N(w_i)} = \mathbb{M}^{+\text{NOM}}|_{||\pi_i\wedge\bigwedge\Psi||}$. Also note that $\pi_i \wedge \bigwedge\Psi$ is equivalent to some formula of the form $\bigwedge\Psi'$, where $\Psi' \subseteq \text{NOM}\cup\text{PROP}$ and Ψ' is finite. Hence $\mathbb{M}^{+\text{NOM}}|_{||\pi_i\wedge\bigwedge\Psi||}, w_i \models \varphi$, and thus by IH we have $\mathbb{M}^{+\text{NOM}}, w_i \models f_{\mathbb{M},w_i}(\varphi^{\pi_i\wedge\bigwedge\Psi})$ (3). But by definition of $f_{\mathbb{M},w_i}$, we have $f_{\mathbb{M},w_i}(([!\mathsf{N}]\varphi)^{\bigwedge\Psi}) = f_{\mathbb{M},w_i}(\varphi^{\pi_i\wedge\bigwedge\Psi})$ (4). The claim follows.

Now assume that $\mathbb{M}^{+\text{NOM}}|_{||\bigwedge\Psi||}, w_i \models \langle a\rangle\varphi$ (5). It follows that there is some state $w_j \in R_a(w_i)$ (6) such that $w_j \in ||\bigwedge\Psi||$ (7) and $\mathbb{M}^{+\text{NOM}}|_{||\bigwedge\Psi||}, w_i \models \varphi$ (8). By IH we have thus $\mathbb{M}^{+\text{NOM}}, w_j \models f_{\mathbb{M},w_j}(\varphi^{\bigwedge\Psi})$. By (6) and (7) we have thus $\mathbb{M}^{+\text{NOM}}, w_i \models \langle a\rangle f_{\mathbb{M},w_j}(\varphi^{\bigwedge\Psi})$, hence $\mathbb{M}^{+\text{NOM}}, w_i \models \langle a\rangle \bigvee_{w_j\in R_a[w_i]} f_{\mathbb{M},w_j}(\varphi^{\bigwedge\Psi})$, that is, $\mathbb{M}^{+\text{NOM}}, w_i \models f_{\mathbb{M},w_i}((\langle a\rangle\varphi)^{\bigwedge\Psi}$. The converse direction is similar. ■

LEMMA 8.19
For every $\varphi \in \mathsf{L}_{(!\sigma,A)}(\text{PROP})$, and every finite pointed model $\mathbb{M}, w_i$ and finite subset $W_0 \subseteq |\mathbb{M}|$, such that $w_i \in \bigcap_{w\in W_0} N(w)$ we have:

$$\mathbb{M}_{\bigcap_{w_j\in W_0} N(w_j)}, w_i \models \varphi \text{ iff } \mathbb{M}^{+\text{NOM}}, w_i \models f_{\mathbb{M},w_i}(\varphi^{\bigwedge_{j\in\{k|w_k\in W_0\}}\pi_j})$$

PROOF. Immediate from Lemma 8.18 and the two previous facts. ■

Our main claim follows directly:

PROPOSITION 8.20
For every formula $\varphi \in \mathsf{L}_{(!\sigma,A)}(\text{PROP})$, and every finite pointed model $\mathbb{M}, w_i$ we have

$$\mathbb{M}, w_i \models \varphi \text{ iff } \mathbb{M}^{+\text{NOM}}, w_i \models f_{\mathbb{M},w_i}(\varphi)$$

PROOF. The proof is by induction on the complexity of φ. The base case is immediate. Booleans are immediate from IH. Now assume that, $\mathbb{M}, w_i \models [!\mathsf{N}]\varphi$, that is, $\mathbb{M}|_{N(w_i)}, w_i \models \varphi$. By Lemma 8.19, we have $\mathbb{M}^{+\text{NOM}}, w_i \models f_{\mathbb{M},w_i}(\varphi^{\pi_i})$. But $f_{\mathbb{M},w_i}([!\mathsf{N}]\varphi) = f_{\mathbb{M},w_i}(([!\mathsf{N}]\varphi)^{\top}) = f_{\mathbb{M},w_i}(\varphi^{\pi_i})$. Finally assume that $\mathbb{M}, w_i \models \langle a\rangle\varphi$, it follows that there is some $w_j \in R_a[w_i]$ (0) and $\mathbb{M}, w_j \models \varphi$. Hence by IH, $\mathbb{M}^{+\text{NOM}}, w_j \models f_{\mathbb{M},w_j}(\varphi)$ (1). But from (0) and (1) we have: $\mathbb{M}^{+\text{NOM}}, w_j \models \bigvee_{w_j \in R_a[w_i]} f_{\mathbb{M},w_j}(\varphi)$ and thus $\mathbb{M}^{+\text{NOM}}, w_i \models \langle a\rangle \bigvee_{w_j \in R_a[w_i]} f_{\mathbb{M},w_j}(\varphi)$. But $f_{\mathbb{M},w_i}(\langle a\rangle\varphi) = f_{\mathbb{M},w_i}((\langle a\rangle\varphi)^{\top}) = \langle a\rangle \bigvee_{w_j \in R_a[w_i]} f_{\mathbb{M},w_j}(\top \wedge \varphi^{\top}) = \langle a\rangle \bigvee_{w_j \in R_a[w_i]} f_{\mathbb{M},w_j}(\varphi^{\top}) = \langle a\rangle \bigvee_{w_j \in R_a[w_i]} f_{\mathbb{M},w_j}(\varphi)$. ■

The notion of bisimilarity corresponding to the basic hybrid language with nominals only is the following:

DEFINITION 8.21 (H_A-Bisimulation, see, e.g. [28], ch. 4)
An H_A-bisimulation between $\mathbb{M} = \langle W, (\xrightarrow{a})_{a\in A}, V\rangle$ and $\mathbb{M}' = \langle W', (\xrightarrow{a}{}')_{a\in A}, V'\rangle$ is a bisimulation satisfying:

1. $x \in V(p_i)$ iff $x' \in V'(p_i)$, whenever $(x, x') \in Z$, for every $p_i \in \text{NOM}$

H_A-bisimilarity does not imply σ-bisimilarity as such. However, consider the basic hybrid language enriched with the $\langle N\rangle$ operator discussed at the beginning of Section 4.1. We refer to it as $\mathsf{H}_{A,N}(\text{NOM})$. Consider two finite pointed LTSs $\mathbb{M}, w_i$ and $\mathbb{M}', w_i'$. We first claim the following:

CLAIM 8.22
Whenever $\mathbb{M}, w_i$ and $\mathbb{M}', w_i'$ are $\mathsf{H}_{A,N}$-bisimilar, then for any $\varphi \in \mathsf{L}_{(!\sigma,A)}$, $f_{\mathbb{M},w_i}(\varphi)$ is logically equivalent to $f_{\mathbb{M}',w_i'}(\varphi)$.

PROOF. First, observe that at every stage of the translation we have an exponentiated formula of the form φ^{ψ}, where ψ is a conjunction of the form $\pi_{i_1} \wedge \ldots \wedge \pi_{i_n}$, where each $\pi_{i_k} := \bigvee_{w_j \in N(w_i)} p_j$. Moreover, it is easy to see that such conjunctions of clauses of nominals are invariant under $\mathsf{H}_{A,N}$-bisimulations.

The proof is by induction on the complexity of φ in φ^{ψ}. Base case is for formulas of the form p^{φ} where φ is a Boolean formula over NOM and PROP, where $p \in \text{PROP}$. Hence by definition of the translation, its translation in the same in both models.

Now for the induction step. First note, that the translation is only model-dependent for $[!\mathsf{N}]$ and $\langle a\rangle$ clauses. Hence induction steps for Booleans are trivial.

Now consider an exponentiated formula of the form $(\langle a\rangle\varphi)^{\psi}$. By definition of the translation $f_{\mathbb{M},w_i}((\langle a\rangle\varphi)^{\psi}) = \langle a\rangle \bigvee_{w_j \in R_a[w_i]} f_{\mathbb{M},w_j}(\psi \wedge \varphi^{\psi})$. Now by definition of a bisimulation, for each $w_j \in R_a[w_i]$ there is a state $w_j' \in R_a[w_i']$ such that w_j and w_j' are themselves $\mathsf{H}_{A,N}$-bisimilar. But then by IH $f_{\mathbb{M},w_j}(\psi \wedge \varphi^{\psi})$ is logically equivalent to $f_{\mathbb{M},w_j'}(\psi \wedge \varphi^{\psi})$. And conversely using the other direction in the definition of a bisimulation. By finiteness, it follows that $f_{\mathbb{M},w_i}((\langle a\rangle\varphi)^{\psi})$ and $f_{\mathbb{M},w_i'}((\langle a\rangle\varphi)^{\psi})$ are of the form $\langle a\rangle \bigvee_{\chi\in\psi} \Psi$ and $\langle a\rangle \bigvee_{\chi'\in\Psi'} \Psi'$, such that Ψ and Ψ' are finite collections of formulas, such that for each formula in one collection, there is a logically equivalent one in the other. Hence the two formulas are logically equivalent.

The [!N]-case is immediate from the definition of the translation $f_{\mathbb{M},w_i}(([!\mathsf{N}]\varphi)^{\psi}) = f_{\mathbb{M},w_i}(\varphi^{\pi_i \wedge \psi})$. It is sufficient to note that if ψ was a Boolean combination of nominals, then so is $\pi_i \wedge \psi$, and that if π_i and π_i' must be logically equivalent, for otherwise there would be a state in either $N(w_i)$ or $N(w_i')$ satisfying a nominal, that is not satisfied by any state in the $N(w_i')$ or, respectively, in $N(w_i)$, contradicting the hypothesis that w_i and w_i' are $\mathsf{H}_{A,N}$-bisimilar. Concluding the proof for this case and the proof of the claim. ■

Now, let C be a class of pointed finite LTSs. We say that NOM is surjective with respect to C, provided that for every $\mathbb{M} \in \mathsf{C}$, $V_{\mathbb{M}}(\text{NOM}) = |\mathbb{M}|$. As a Corollary to Proposition 8.20 we now can observe that:

COROLLARY 8.23
Let C be a class of finite pointed LTSs and let NOM be a set of nominals, surjective with respect to C. We have $\mathsf{L}_{(!\sigma,A)} \leq_{\mathsf{C}} \mathsf{H}_{A,N}(\text{NOM})$.

PROOF. Assume for contradiction that we have two finite pointed LTSs, $\mathbb{M}, w_i$ and $\mathbb{M}', w_i'$ that are $\mathsf{H}_{A,N}$-bisimilar and such that we have a formula $\varphi \in \mathsf{L}_{(!\sigma,A)}$ with $\mathbb{M}, w_i \models \varphi$ and $\mathbb{M}', w_i \not\models \varphi$. By Claim 8.22 it follows that $f_{\mathbb{M},w_i}(\varphi)$ is logically equivalent to $f_{\mathbb{M}',w_i'}(\varphi)$. Since NOM is surjective with respect to C, we have by Fact 8.15, that the translation is into $\mathsf{H}_A(\text{NOM})$, hence certainly in $\mathsf{H}_{A,N}(\text{NOM})$. Take $F(\varphi) := f_{\mathbb{M},w_i}(\varphi) \wedge f_{\mathbb{M}',w_i'}(\varphi)$, which is still in $\mathsf{H}_{A,N}(\text{NOM})$. By Proposition 8.20 it follows, that $\mathbb{M}, w_i \models F(\varphi)$ and $\mathbb{M}', w_i \not\models F(\varphi)$, contradicting the assumption that $\mathbb{M}, w_i$ and $\mathbb{M}', w_i'$ are $\mathsf{H}_{A,N}$-bisimilar. By reduction, it follows that $\mathbb{M}, w_i$ and $\mathbb{M}', w_i'$ are not $\mathsf{H}_{A,N}$-bisimilar, but since both are finite, they are also $\mathsf{H}_{A,N}$-distinguishable. Concluding our proof. ■

8.6 *Logic of sights as a multi-dimensional modal logic*

If we look back at our SAT-algorithm, we can see that we have implicitly treated our logic as a *multi-dimensional* modal logic. But the logic itself is really unidimensional; when we move to a restriction the semantics does not require that we keep track of the original model to interpret the rest of the formula. If we would like, however, to be able to switch back and forth between different perspectives, we would require a truly multi-dimensional modal logic. Consider e.g. the addition of an 'abstract away' unary operator that we call $[(!\mathsf{N})^{-1}]\varphi$. To accommodate it, we change our original semantics. Our models are still $\sigma\mathsf{LTS}$ of the form:

$$\mathbb{M} := \langle W, (\xrightarrow{a})_{a \in A}, N, V \rangle$$

where $N : W \to \wp(W)$ where $w \in N(w)$, for every $w \in W$.

The syntax of the language of $\mathsf{L}_{(!\sigma,(!\sigma)^{-1},A)}$ over A is:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle a \rangle \varphi \mid [!\mathsf{N}]\varphi \mid [(!\mathsf{N})^{-1}]\varphi,$$

where p ranges over PROP and a over A. Given a model

$$\mathbb{M} := \langle W, (\xrightarrow{a})_{a \in A}, N, V \rangle,$$

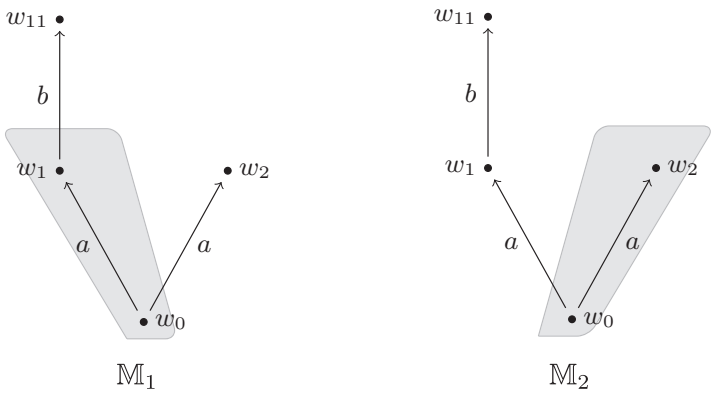

FIGURE 10. $\mathsf{L}_{(!\sigma,A)}$ is not as expressive as $\mathsf{L}_{(!\sigma,(!\sigma)^{-1},A)}$.

a state $w \in W$ and a finite, possibly empty, set $X \subseteq W$, our semantics is given as follows:

$$\begin{array}{lll} \mathbb{M}, w, X \models \langle a \rangle \varphi & \text{iff} & \text{there is } s \in (\bigcap_{t \in X} N(t)) \cap R_a[w] \\ & & \text{with } \mathbb{M}, s, X \models \varphi \\ \mathbb{M}, w, X \models [!\mathsf{N}]\varphi & \text{iff} & \mathbb{M}, w, X \cup \{w\} \models \varphi \\ \mathbb{M}, w, X \models [(!\mathsf{N})^{-1}]\varphi & \text{iff} & \mathbb{M}, w, X \setminus \{w\} \models \varphi \end{array}$$

Note that our two operations are still very restrictive: they only allow us to add or remove the current state from the last part of the context. However, a logic with these operations is strictly more expressive than our logic. First, consider the two models given in Figure 10. Note that $\mathbb{M}_1, w_0, \emptyset \models [!\mathsf{N}]\langle a \rangle [(!\mathsf{N})^{-1}]\langle b \rangle \top$ but $\mathbb{M}_2, w_0, \emptyset \not\models [!\mathsf{N}]\langle a \rangle [(!\mathsf{N})^{-1}]\langle b \rangle \top$. But the two are σ-bisimilar. (Showing that $\mathsf{L}_{(!\sigma,(!\sigma)^{-1},A)}$ is at least as expressive as $\mathsf{L}_{(!\sigma,A)}$ is straightforward.)

There are of course other multi-dimensional modal logics along those lines. $\mathsf{L}_{(!\sigma,A)}$, the logic of sight we have discussed in this paper, can be said to be internal: $[!\mathsf{N}]\varphi$ means that φ is true in the restriction to $N(w)$, where further $[!\mathsf{N}]$ operators are recursively interpreted in the relevant submodels. If instead we would like an operator, call it $[\downarrow \mathsf{N}]$, setting the perspective to be that of the agent at the current state—hence a logic of sight from an external perspective— we can draw on multi-dimensional semantics as explained below.

Our models are still $\sigma\mathsf{LTS}$ of the form:

$$\mathbb{M} := \langle W, (\xrightarrow{a})_{a \in A}, N, V \rangle$$

where $N : W \to \wp(W)$ where $w \in N(w)$, for every $w \in W$.

The syntax of the language of $\mathsf{L}_{(\downarrow\sigma,A)}$ over A is:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle a \rangle \varphi \mid [\downarrow \mathsf{N}]\varphi,$$

where p ranges over PROP and a over A. Given a model

$$\mathbb{M} := \langle W, (\xrightarrow{a})_{a \in A}, N, V \rangle,$$

a pair in $(w, v) \in W \times W \cup \{\epsilon\}$, and defining $N(\epsilon) := W$, our semantics is given as follows:

$$\begin{array}{lll} \mathbb{M}, w, v \models \langle a \rangle \varphi & \text{iff} & \text{there is } s \in N(v) \cap R_a[w] \\ & & \text{with } \mathbb{M}, s, v \models \varphi \\ \mathbb{M}, w, v \models [\downarrow \mathsf{N}]\varphi & \text{iff} & \mathbb{M}, w, w \models \varphi \end{array}$$

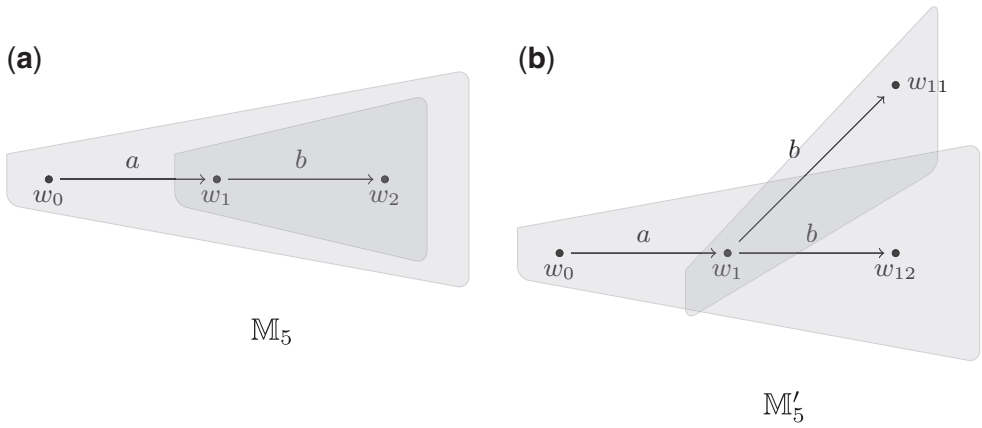

FIGURE 11. $\mathsf{L}_{(\downarrow\sigma,A)}$ is not as expressive as $\mathsf{L}_{(!\sigma,A)}$.

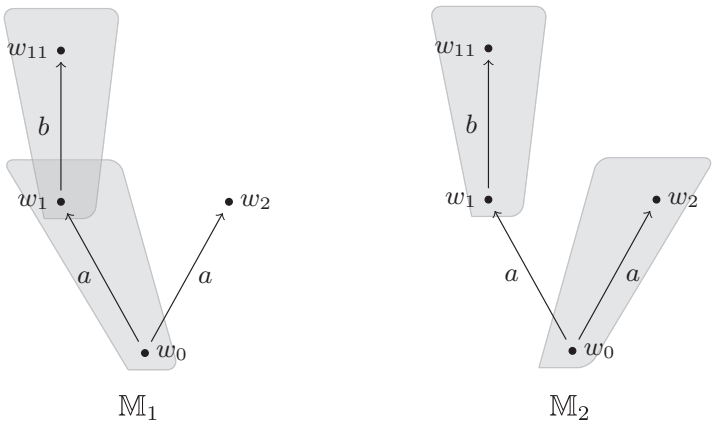

FIGURE 12. $\mathsf{L}_{(!\sigma,A)}$ is not as expressive as $\mathsf{L}_{(\downarrow\sigma,A)}$.

$\mathbb{M}_5$, the left-hand side model in Figure 11 illustrates the difference between the $[!\mathsf{N}]$ and the $[\downarrow\mathsf{N}]$ operator. Indeed, we have $\mathbb{M}_5, w_0 \models [\downarrow\mathsf{N}]\langle a\rangle[\downarrow\mathsf{N}]\langle a\rangle\top$ but $\mathbb{M}_5, w_0 \models [!\mathsf{N}]\langle a\rangle[!\mathsf{N}]\neg\langle a\rangle\top$. Moreover, we can see from Figure 12 that $\mathsf{L}_{(!\sigma,A)}$ is not as expressive as $\mathsf{L}_{(\downarrow\sigma,A)}$. Indeed the two w_0-pointed models are σ-bisimilar, but $\mathbb{M}_1, w_0 \models [\downarrow\mathsf{N}]\langle a\rangle[\downarrow\mathsf{N}]\langle b\rangle\top$ while $\mathbb{M}_2, w_0 \not\models [\downarrow\mathsf{N}]\langle a\rangle[\downarrow\mathsf{N}]\langle b\rangle\top$.

The model-theoretic notion of invariance corresponding to modal invariance for $\mathsf{L}_{(\downarrow\sigma,A)}$ is based on the following notion of bisimulation. (The proof of this correspondence is the usual one.)

DEFINITION 8.24
A $(\downarrow\mathsf{N},A)$-bisimulation between two σLTS $\mathbb{S}=(\langle W,(\xrightarrow{a})_{a\in A},\mathsf{N},V\rangle)$ and $\mathbb{S}'=(\langle W',(\xrightarrow{a}{}')_{a\in A},\mathsf{N}',V'\rangle)$ is a relation $Z\subseteq(W\times(W\cup\{\epsilon\}))\times(W'\times(W'\cup\{\epsilon\}))$ such that for some $w\in W$ and $w'\in W'$ we have $(w,\epsilon,w',\epsilon)\in Z$, and such that the following conditions hold:

1. (Atomic Harmony) if $(x,y,x',y')\in Z$ then for every $p\in$ PROP, we have $x\in V(p)$ iff $x'\in V'(p)$
2. (Forth) if $(x,y,x',y')\in Z$ and xR_az for some $z\in\mathsf{N}(y)$, then there exists some $z'\in\mathsf{N}'(y')$ such that $x'R'_az'$ and $(z,y)Z(z',y')$
3. (Back) if $(x,y,x',y')\in Z$ and $x'R'_az'$ for some $z'\in\mathsf{N}'(y')$, then there exists some $z\in\mathsf{N}(y)$ such that xR_az and $(z,y)Z(z',y')$

4. ($\downarrow$N-Forth) if $(x,y,x',y') \in Z$ and $xR_a z$ for some $z \in \mathsf{N}(y)$, then there exists some $z' \in \mathsf{N}'(y')$ such that $x'R'_a z'$ and $(z,z)Z(z',z')$
5. ($\downarrow$N-Back) if $(x,y,x',y') \in Z$ and $x'R'_a z'$ for some $z' \in \mathsf{N}'(y')$, then there exists some $z \in \mathsf{N}(y)$ such that $xR_a z$ and $(z,z)Z(z',z')$

We say that two pointed σLTS $(\mathbb{S},w_0)=(\langle W,(\overset{a}{\to})_{a\in A},\mathsf{N},V\rangle,w_0)$ and $(\mathbb{S}',w'_0)=(\langle W',(\overset{a}{\to}{}')_{a\in A},\mathsf{N}',V'\rangle,w'_0)$ are ($\downarrow$N,A)-bisimilar (notation $\mathbb{S},w_0) \underline{\leftrightarrow}^{\downarrow\mathsf{N},A}(\mathbb{S}',w'_0)$ whenever there is bisimulation Z between $\mathbb{S}$ and $\mathbb{S}'$ such that $(w_0,\epsilon,w'_0,\epsilon) \in Z$. When A is clear from context, we write ($\downarrow$N)-bisimilar for ($\downarrow$N,A)-bisimilar.

Note the difference with σ-bisimulations. While specific back and forth conditions for σ-bisimulations are cumulative, the ones for ($\downarrow$N,A)-bisimulations are reseting the sight to that of the successor. Figure 11 gives an example of two models that are ($\downarrow$N)-bisimilar, but not σ-bisimilar. Hence, $\mathsf{L}_{(\downarrow\sigma,A)}$ is not as expressive as $\mathsf{L}_{(!\sigma,A)}$. The expressive power of the two languages are thus incomparable.

9 Definability of sight-based subgame perfection

We can finally come back to what was one of our original motivations: game-theoretical analysis of strategic situations involving limited sights. We started the article with the example of a chess game, claiming that both human and artificial players will only take into account a limited part of the (sub)game tree into account before deciding which move to take.

Magnus Carlsen, 2013 world No. 1 chess player, claims to read up to 15–20 moves ahead (*Spiegel*, 19 March 2010). This limit in the depth of his sight, has to be paired not only with a limit in the number of moves considered at each stage, but with the selection of only a small subset of relevant sequences, even assuming an ability to go through positions at blazing speed. In general, both players will have some form of limited sight. In particular a player, say black, might expect another (white) to make decisions within a shorter sight than she (black) is, and try to use that to her advantage. The converse is of course not very helpful, even if white knows that black is able to read deeper, she still has to rely on her own reading ability, that is her own depth of reading.

9.1 Sequential game models

We follow the logical analysis of sequential games of perfect information developed in van Benthem and Gheerbrant [31] and take our models to be finite σ-treeLTSs extended with a turn function ρ and a preference orderings for each player. Concretely we consider structures of the following form:

DEFINITION 9.1 (model of a sequential game with sights (σSG))
$\mathbb{M}=\langle W,Z,Ag,(\overset{a}{\to})_{a\in A},(\mathsf{N}_i)_{i\in Ag},\rho,(\leq_i)_{i\in Ag},V\rangle$ where $\langle W,(\overset{a}{\to})_{a\in A},(\mathsf{N}_i)_{i\in Ag},V\rangle$ is a σTLTS, Z is the set of leafs, $\rho:(W\setminus Z)\to Ag$ and for each $i\in Zg$, $\leq_i$ is a total pre-order on W. We refer to a structure of the previous type as a model of a sequential game with sights, σSG for short.

Unlike models of sequential games without sights, it is necessary to require agents' preferences to be defined on the whole tree and not only on leaves. To go back to the chess analogy, Carlsen states that he believes the important part (when reading ahead in a chess game) 'is to correctly assess the position at the end of the calculation'. In other words, being able to see ahead is pointless for the purpose of decision-making if you are not able to assess intermediate positions. In general, for arbitrary games, the task is more delicate than for chess, because a player must be able to assess

such intermediate stages of the games for other players as well. We let for each $i \in Ag$, turn_i be a propositional letter with semantics:

$$\mathbb{M}, w \models \mathsf{turn}_i \text{ iff } w \notin Z \text{ and } \rho(w) = i$$

We also write end for $[\rightarrow]\bot$, that is $w \in ||\mathsf{end}||$ iff $w \in Z$.

A strategy for i, is a function $f_i : \rho^{-1}(i) \rightarrow W$ with $f_i(w) \in \rightarrow [w]$. A sight-compatible strategy for i is a strategy f_i such that $f_i(w) \in \mathsf{N}_i(w)$.

9.2 Backward induction with sights

A subgame perfect equilibrium (see e.g. [23]) is a profile of strategies such that the restriction of the profile to any subgame is still a Nash-equilibrium for that subgame. This is the usual solution concept for extensive games of perfect information. We refer to it as an SPE. On generic games, that is games such that there is no indifference for any player between end-nodes, there exists a unique SPE. On the class of generic games, backward induction (henceforth BI) can be seen as a deterministic procedure selecting a unique action at each non-terminal node (or history) in the game. When we allow for indifference, there exists generally more than one SPE and the set of pure SPE is generally not closed under union. In other words, subgame perfection or equivalently backward induction *cannot* be defined inductively *as a relation* on arbitrary games. However, this can be done for generic games and this can also be done for some generalizations of backward induction, allowing for FO with fixed points definability, as well as modal definability. The reader is referred to [31] for details on such definability results. We say that a σSG is generic iff the preference ordering of each agent is a strict total order. For the sake of simplicity, we will henceforth speak of the BI relation, but the reader is now aware that this expression should be taken with a grain of salt.

In the context of sequential games with sights, [13] proposes a concept of solution they refer to as 'sight-compatible' SPE.

Definition 9.2 (non-recursive sight-compatible SPE, [13])
A profile of strategy is a non-recursive sight-compatible SPE if at every non-terminal stage w the action it prescribes at h is compatible with an SPE of the restriction of the game to $\mathsf{N}_{\{\rho(w)\}}(w)$.

We refer to it as non-recursive sight-compatible SPE, because in this solution concept, as stressed by Grossi and Turrini [13], players are not taking into account the sights restrictions under which other players will be making decisions. Given the reader's favorite notion of a BI relation, we define its corresponding non-recursive sight-compatible relational counterpart as follows:

Definition 9.3 (non-recursive sight-compatible BI relation)
Given a (generic) σSG, $\mathbb{M}$, the non-recursive sight-compatible BI relation, NRσBI is the subset of $(W \setminus Z) \times W$ such that $(x, y) \in \mathsf{NR}\sigma\mathsf{BI}$ iff $(x, y) \in \mathsf{BI}(\mathbb{M}|_{\sigma(\rho(w))})$.

An important difference with the classical setting, is that we are really dealing with a family of relations $\{\mathsf{BI}(\mathbb{M}|_{\sigma(\rho(w))}) | w \in |\mathbb{M}| \setminus Z\}$. Concretely, for non-terminal x we have a backward induction relation BI_x defined with respect to the restriction of the game model to $\sigma(\rho(x))$.

9.3 Modal definability

We will be working with a relational generalization of BI defined in [31], but a similar recipe can be applied to the reader's favorite modal characterization of the (reader's favorite notion of a) BI relation

(see e.g. [31, 33, 37]) in some modal language L_{BI}. Van Benthem and Gheerbrant [31] characterize the BI relation as follows:

$$\mathsf{turn}_i \wedge \langle \mathsf{BI} \rangle [\mathsf{BI}^*](\mathsf{end} \rightarrow p) \rightarrow [\rightarrow]\langle \mathsf{BI}^* \rangle (\mathsf{end} \wedge \langle \leq_i \rangle p) \tag{23}$$

The crucial ingredients are the use of a preference $\langle \leq_i \rangle$ modality for each agent and an iteration program construct $*$. Now the locality of the backward induction relation BI_x makes it necessary for modal characterization purposes that the syntax is able to refer to the particular node in the game tree at which this sight-dependent BI relation is being defined. A way to do this is to use a sight modality $[!\mathsf{N}_i]$ for each agent, defined as previously in the article, in an hybrid setting with binder. Putting it all together, our language will then be a multi-agent version of L_σ expanded with preference modalities $\langle \leq_i \rangle$ for each agent, an iteration program construct $*$, state variables $\textsc{svar} := x, y, z, \ldots$ and a binder $\downarrow x., \downarrow y., \ldots$ which binds state variables to the current state (see [28], ch. 9, for a complete introduction).

Concretely, we have the programs:

$$\alpha ::= a \mid \rightarrow \mid \alpha^* \quad \beta ::= \alpha \mid \leq_i$$

and the following formulas:

$$\varphi ::= p \mid \mathsf{turn}_i \mid x \mid @_x\varphi \mid \downarrow x.\varphi \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle \alpha \rangle \varphi \mid [!\mathsf{N}_i],$$

where i ranges over Ag, p over $\textsc{prop}$ and x over $\textsc{svar}$. We write $[!\mathsf{N}_\rho]\varphi$ for $\bigvee_{i \in Ag}(\mathsf{turn}_i \wedge [!\mathsf{N}_i]\varphi)$.

Programs and formulas will be interpreted on pointed $\sigma\mathsf{SG}$ (cf. Definition 9.1) together with assignment functions $g : \textsc{svar} \rightarrow W$. Programs are interpreted in the obvious way:

$$\begin{aligned} ||a|| &= \xrightarrow{a} \\ ||\rightarrow|| &= \bigcup_{a \in A} \xrightarrow{a} \\ ||\alpha^*|| &= ||\alpha||^* \\ ||\leq_i|| &= \leq_i \end{aligned}$$

Now, we give the interesting cases for formulas:

$$\begin{array}{lll} \mathbb{M}, w, g \models x & \text{iff} & g(x) = w \\ \mathbb{M}, w, g \models \mathsf{turn}_i & \text{iff} & \rho(x) = i \\ \mathbb{M}, w, g \models @_x\varphi & \text{iff} & g(x) \in |\mathbb{M}| \text{ and } \mathbb{M}, g(x), g \models \varphi \\ \mathbb{M}, w, g \models \downarrow x.\varphi & \text{iff} & \mathbb{M}, w, g[g(x) := w] \models \varphi \\ \mathbb{M}, w, g \models [!\mathsf{N}_i]\varphi & \text{iff} & \mathbb{M}|_{\mathsf{N}_i(w)}, w, g \models \varphi \\ \mathbb{M}, w, g \models \langle \alpha \rangle \varphi & \text{iff} & \text{there exists some } v \text{ with } (w, v) \in ||\alpha||^{\mathbb{M}} \\ & & \text{and } \mathbb{M}, v, g \models \varphi \end{array}$$

There are a few subtleties in the above semantics, which have to do with the interaction of assignment functions and model restriction operations. The first question is whether, when we interpret a formula of the form $[!\mathsf{N}_i]\varphi$, we should restrict the assignment function as well. We will not restrict the assignment function but instead modify the semantics of formulas of the form $@_x\varphi$ at $\mathbb{M}|_{\mathsf{N}_i(w)}, v, g$ in the case that $g(x) \notin \mathsf{N}_i(w)$. There are two ways to strengthen the semantics of $@_x\varphi$,

making it either an 'existential' modality or a 'universal' modality. In the first case, we would have $\mathbb{M}|_{\mathsf{N}_i(w)}, v, g \not\models @_x\varphi$ whenever $g(x) \notin \mathsf{N}_i(w)$. This interpretation has our preference. $@_x\varphi$ will then intuitively mean that there is some state currently named x in the domain such that φ holds there. The universal interpretation would make such formulas true. It is essentially a matter of preference which of these strengthenings one chooses. Another way of dealing with this issue who be to keep track of the full domain as e.g. a disjoint part of the model, whenever interpreting an $[!\mathsf{N}_i]\varphi$ modality. But it does not seem natural to us to modify the semantics $[!\mathsf{N}_i]$ to resolve this issue. Finally, having a subspeficied semantics is not an option we see as desirable either. In general, a systematic analysis of logics allowing for restrictions, binders and hybrid @ operators is interesting, but we leave the complete analysis to another paper.

We are now ready, to come back to the issue of modal definability of sight-dependent concepts of subgame perfection. The following characterize the local character of the BI_x relation.

$$\langle \mathsf{BI}_x\rangle p \leftrightarrow \downarrow y.@_x[!\mathsf{N}_\rho]\langle \rightarrow^*\rangle(y \wedge \langle \mathsf{BI}_x\rangle p \wedge \langle \rightarrow\rangle p) \tag{loc}$$

Now the counterpart of van Benthem and Gheerbrant's (23), for sight-dependent subgame perfection, is the following:

$$@_x[!\mathsf{N}_\rho][\rightarrow^*]((\mathsf{turn}_i \wedge \langle \mathsf{BI}_x\rangle[\mathsf{BI}_x^*](\mathsf{end} \rightarrow q)) \rightarrow [\rightarrow]\langle \mathsf{BI}_x^*\rangle(\mathsf{end} \wedge \langle \leq_i\rangle q)) \tag{opt}$$

Finally, let us introduce the following axiom corresponding to the global solution concept:

$$\langle R\rangle r \leftrightarrow \downarrow x.\langle \mathsf{BI}_x\rangle r \tag{glo}$$

We can now give a modal characterization of the non-recursive sight-compatible BI relation.

PROPOSITION 9.4
Let $\mathbb{M}$ be a finite $\sigma\mathsf{SG}$, let $\mathcal{F} = \mathbb{M} \setminus \langle V\rangle$ be the frame of $\mathbb{M}$ and let R be a program. The following are equivalent:

1. R is the non-recursive sight-compatible BI relation on $\mathcal{F}$, that is $||R||^{\mathcal{F}} = \mathsf{NR}\sigma\mathsf{BI}$.
2. $\mathcal{F} \models$ (loc) $\wedge$ (opt) $\wedge$ (glo).

We are not including the details of this correspondence. We will provide a systematic exploration of solution concepts for games with sights and their definability in an another paper. A particularly exciting next step is to explore the definability of a fully recursive concept of sight-dependent subgame perfection. By fully recursive, we mean that agents are considering the fact that other agents will make decision within their own sight and will themselves anticipate other agent to make decision within their own sight. The corresponding concept can be certainly be defined by simultaneous inflationary inductions. Their definability in modal languages such as modal μ-calculus [19] or modal iteration calculus [6] is something we still need to explore. The results in [10, 30] could shed some light on this question.

10 Conclusion

We began with the problem of representing how a limited agent can reason about games she cannot solve analytically using methods such as backward induction. Following [13] and [27] we considered agents who reason about only a limited part of the game, the part that is in her 'sight'. We have

developed this concept in a more general and abstract formal setting, using LTSs with sight functions. We introduced a simple modal language augmenting the classical modal system K with an operator that allows us to consider truth in certain submodels given by the sights.

Our language $\mathsf{L}_{(!\sigma,A)}$ has a translation into the loosely guarded fragment of first-order logic, from which decidability of its satisfiability problem follows. We have gone further, however, and showed that its satisfiability problem is PSPACE-complete and that the combined complexity of its model-checking problem is in PTIME. In terms of expressive power, our language is incomparable with a modal language using a standard unary modal operator to scan sights. We have given an axiomatization for our logic of sights and proved its completeness. In the last sections, we have offered some potential avenues to explore in future research and have indicated relations with other logical formalisms, and with game theory.

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Appendix

A Preliminary facts towards completeness

In this section we state an important number of intermediate facts that we are using in the completeness proof. The proofs of these facts can be found in Appendix B. All these facts are stated under the assumption that the sets Δ, Δ_1 and Δ_2 are finite.

A.1 Facts about CN

FACT A.1
If $\varphi \in (\mathsf{CN}(\Delta) \setminus Sub(\Delta))$, then there is a formula $\psi \in Sub(\Delta)$ such that one the following holds

1. $\varphi = \neg\psi$ and ψ is neither of the form $\neg\chi$, nor of the form $[!\mathsf{N}]\chi$
2. $\varphi = [!\mathsf{N}]\psi$ and ψ is neither of the form $[!\mathsf{N}]\chi$, nor of the form $\neg[!\mathsf{N}]\chi$
3. $\varphi = [!\mathsf{N}]\neg\psi$ and ψ is neither of the form $\neg\chi$, nor of the form $[!\mathsf{N}]\chi$

FACT A.2
$Sub(\mathsf{CN}(\Delta)) = \mathsf{CN}(\Delta)$.

FACT A.3
$\mathsf{CN}(\Delta_1 \cup \Delta_2) = \mathsf{CN}(\Delta_1) \cup \mathsf{CN}(\Delta_2)$

FACT A.4
$\mathsf{CN}(\mathsf{CN}(\Delta)) = \mathsf{CN}(\Delta)$.

FACT A.5
If $\Delta_1 \subseteq \Delta_2$, then $\mathsf{CN}(\Delta_1) = \mathsf{CN}(\Delta_2)$.

FACT A.6
If $\varphi_1 \in \mathsf{CN}(\varphi_2)$ and $\varphi_2 \in \mathsf{CN}(\varphi_3)$, then $\varphi_1 \in \mathsf{CN}(\varphi_3)$

FACT A.7
For every n with $1 < n \in \omega$ we have $\mathsf{CN}(\varphi_1 \wedge \ldots \wedge \varphi_n) = \{\varphi_1 \wedge \varphi_n, \neg(\varphi_1 \wedge \varphi_n), [!\mathsf{N}](\varphi_1 \wedge \varphi_n), [!\mathsf{N}]\neg(\varphi_1 \wedge \varphi_n)\} \cup \mathsf{CN}(\{\varphi_1, \ldots, \varphi_n\})$.

FACT A.8
$\max(\{\mathsf{dep}_A(\varphi) | \varphi \in \mathsf{CN}(\Delta)\}) = \max(\{\mathsf{dep}_A(\varphi) | \varphi \in \Delta\})$

A.2 Facts about $\mathsf{CN}(\Delta)(a)$

FACT A.9
$\mathsf{CN}(\Delta)(a) = \mathsf{CN}(E(\mathsf{CN}(\Delta), a))$.

FACT A.10
$E(\mathsf{CN}(\Delta), a) = \{\varphi \mid \langle a \rangle \varphi \in \mathsf{CN}(\Delta)\}$.

FACT A.11
Assume that $\mathsf{CN}(\Delta)(a) \neq \emptyset$. $\max(\mathsf{dep}(\mathsf{CN}(\Delta)(a))) < \max(\mathsf{dep}(\mathsf{CN}(\Delta)))$.

A.3 Facts about $\mathsf{cn}(\Delta)(\vec{a})$

FACT A.12
For every $\vec{a}, \vec{a \cdot b} \in A^*$, we have $\mathsf{cn}(\Delta)(\vec{a} \cdot b) = \mathsf{CN}(\{\varphi \mid \langle b \rangle \varphi \in \mathsf{cn}(\Delta)(\vec{a})\})$.

FACT A.13
If $\varphi \in \mathsf{cn}(\vec{a} \cdot b)$, then there is some formula χ such that $\langle b \rangle \chi \in \mathsf{cn}(\vec{a})$ with $\varphi \in \mathsf{CN}(\chi)$.

FACT A.14
If $\varphi \in \mathsf{cn}(\vec{a} \cdot \vec{b} \cdot c)$, then there is a formula χ such that $\langle c \rangle \chi \in \mathsf{cn}(\vec{a})$ and $\varphi \in \mathsf{CN}(\chi)$.

COROLLARY A.15
If $\varphi \in \mathsf{cn}(\vec{a} \cdot \vec{b} \cdot c)$, then $\varphi \in \mathsf{cn}(\vec{a} \cdot c)$.

FACT A.16
If $\mathsf{cn}(\Delta)(\vec{b} \cdot c) \neq \emptyset$, then $\max(\mathsf{dep}(\mathsf{cn}(\Delta)(\vec{b} \cdot c))) < \max(\mathsf{dep}(\mathsf{cn}(\Delta)(\vec{b})))$

A.4 Facts about $\mathsf{tree}(\Delta)$

COROLLARY A.17 (Bounded Depth)
The depth of $\mathsf{tree}(\Delta)$ is bounded by $\max \mathsf{dep}(\Delta)$.

FACT A.18
If Δ is finite, then $\mathsf{tree}(\Delta)$ is finitely branching.

A.5 Facts about cj and da

FACT A.19
If $\Delta_1 \subseteq \Delta_2$, then $\mathsf{cj}(\Delta_1) \subseteq \mathsf{cj}(\Delta_2)$

FACT A.20
If $\Delta_1 \subseteq \Delta_2$, then for every $a \in A$, $\mathsf{da}(\Delta_1, a) \subseteq \mathsf{da}(\Delta_2, a)$

A.6 Facts about $\kappa_\Delta(\vec{a})$

FACT A.21
If $\vec{a}$ is a Δ-leaf, then $\max(\mathsf{dep}(\kappa_\Delta(\vec{a}))) = 0$.

FACT A.22
$\mathsf{da}(\kappa_\Delta(\vec{a}\cdot b), b) = \{\langle b\rangle\varphi \mid \varphi \in \mathsf{CN}(\{\widehat{S} \mid S \subseteq \kappa_\Delta(\vec{a}\cdot b)\})\}$

FACT A.23
If $\vec{a}$ is a Δ-leaf, then $\kappa_\Delta(\vec{a})$ is finite.

FACT A.24
If $\mathsf{tree}(\Delta)$ is finitely branching then for every $\vec{a} \in \mathsf{tree}(\Delta)$, if for all $b \in A$, $\kappa_\Delta(\vec{a}\cdot b)$ is finite, then $\kappa_\Delta(\vec{a})$ is finite.

A.7 Properties of κ on $\mathsf{Tree}(\Delta)$

FACT A.25 (Downward cn-monotonicity)
Let $\vec{a}, \vec{b}, \vec{a}\cdot\vec{c}, \vec{b}\cdot\vec{c} \in \mathsf{Tree}(\Delta)$. If $\mathsf{cn}(\Delta)(\vec{a}) \subseteq \mathsf{cn}(\Delta)(\vec{b})$ then $\mathsf{cn}(\Delta)(\vec{a}\cdot\vec{c}) \subseteq \mathsf{cn}(\Delta)(\vec{b}\cdot\vec{c})$.

FACT A.26 (Tree Monotonicity)
Let $\vec{a}, \vec{b} \in \mathsf{Tree}(\Delta)$. If $\mathsf{cn}(\Delta)(\vec{a}) \subseteq \mathsf{cn}(\Delta)(\vec{b})$ then $\mathsf{SubTree}(\Delta, \vec{a}) \subseteq \mathsf{SubTree}(\Delta, \vec{b})$.

FACT A.27 (Immediate Upward κ-Monotonicity)
Let $\vec{a}, \vec{b} \in \mathsf{Tree}(\Delta)$. Assume that for every c such that $\vec{a}\cdot c \in \mathsf{Tree}(\Delta)$ we have $\vec{b}\cdot c \in \mathsf{Tree}(\Delta)$. If for every $c \in A$ such that $\vec{a}\cdot c \in \mathsf{Tree}(\Delta)$ we have $\kappa_\Delta(\vec{a}\cdot c) \subseteq \kappa_\Delta(\vec{b}\cdot c)$ and moreover $\mathsf{cn}(\vec{a}) \subseteq \mathsf{cn}(\vec{b})$, then we have $\kappa_\Delta(\vec{a}) \subseteq \kappa_\Delta(\vec{b})$.

FACT A.28 (cn to κ Monotonicity)
Let $\vec{a}, \vec{b} \in \mathsf{Tree}(\Delta)$. If $\mathsf{cn}(\Delta)(\vec{a}) \subseteq \mathsf{cn}(\Delta)(\vec{b})$ and for every $c \in A$, $\kappa_\Delta(\vec{a}) \subseteq \kappa_\Delta(\vec{b})$.

COROLLARY A.29 (Upward Monotonicity)
Let Δ be a finite set. Let $\vec{a}, \vec{a}\cdot b \in \mathsf{Tree}(\Delta)$. We have $\kappa_\Delta(\vec{a}\cdot b) \subseteq \kappa_\Delta(\vec{a})$.

COROLLARY A.30 (Interior Monotonicity)
Let Δ be a finite set. Let $\vec{a}\cdot\vec{b}\cdot c, \vec{a}\cdot c \in \mathsf{Tree}(\Delta)$. We have $\kappa_\Delta(\vec{a}\cdot\vec{b}\cdot c) \subseteq \kappa_\Delta(\vec{a}\cdot c)$.

A.8 Important properties of κ

FACT A.31
If $\vec{a}$ is not a Δ-leaf and $\varphi \in \kappa_\Delta(\vec{a})$, then we are in one of two cases

a. there is some $b \in A$ and $\varphi \in \mathsf{CN}(\mathsf{da}(\kappa_\Delta(\vec{a}\cdot b), b))$
b. $\varphi \in \mathsf{CN}(\mathsf{cn}(\Delta)(\vec{a}))$

COROLLARY A.32
Let Δ be a finite set. For each $\vec{a} \in A^*$, if $\varphi \in \kappa_\Delta(\vec{a})$, then there is some formula $\mathsf{Norm}([!\mathsf{N}]\varphi) \in \kappa_\Delta(\vec{a})$ such that $\vdash \mathsf{Norm}([!\mathsf{N}]\varphi) \leftrightarrow [!\mathsf{N}]\varphi$.

FACT A.33
If $\varphi_1, \ldots, \varphi_n \in \kappa_\Delta(\vec{a}\cdot b)$, then $\langle b\rangle(\varphi_1 \wedge \ldots \wedge \varphi_n) \in \kappa_\Delta(\vec{a})$.

COROLLARY A.34
Let Δ be a finite set. If $\varphi \in \kappa_\Delta(\vec{a}\cdot\vec{b})$ then $\varphi \in \kappa_\Delta(\vec{a})$

COROLLARY A.35
Let Δ be a finite set. If $\varphi \in \mathsf{CN}(\{\widehat{S} \mid S \subseteq \kappa_\Delta(\vec{a}\cdot\vec{b}\cdot c)\})$ then $\langle c\rangle\varphi \in \kappa_\Delta(\vec{a})$.

FACT A.36
If $\langle c\rangle\varphi \in \kappa_\Delta(\vec{a})$ then we have either $\varphi \in \mathsf{CN}(\{\widehat{S} | S \subseteq \kappa_\Delta(\vec{a}\cdot c)\})$ or $\langle c\rangle\varphi \in \kappa_\Delta(\vec{a}\cdot b)$ for some $b \neq c$.

B Proof of preliminary facts to the completeness proof

In this section, we list the proofs that were omitted in either the main part of the article or Appendix A. The proofs are listed in order of occurrence in the papers, hence proofs of facts stated in Appendix A will be given after the proofs of facts stated in the main part of the article.

PROOF OF FACT 6.1. Assume that φ is neither of the form $\neg\psi$, nor of the form $[!\mathsf{N}]\psi$, then by definition of $\mathsf{CN}(\Delta)$ we have $\neg\varphi \in \mathsf{CN}(\Delta)$. Now assume instead that $\varphi = \neg\psi$. ψ is a subformula of φ, hence we have by construction $\varphi \in \mathsf{CN}(\Delta)$. Finally assume instead that $\varphi = [!\mathsf{N}]\psi$. We are in one of two cases.

Case 1. ψ is neither of the form $\neg\chi$, nor of the form $[!\mathsf{N}]\chi$ (1). By construction $\chi \in \mathsf{CN}(\Delta)$ and hence by (1) we have $\neg\chi \in \mathsf{CN}(\Delta)$ as well.

Case 2. ψ is of the form $(\neg^*[!\mathsf{N}]^*)^*\chi$ for some χ, where χ is neither of the form $\neg\chi_2$, nor of the form $[!\mathsf{N}]\chi_2$ (2). Note that by axioms $[!\mathsf{N}][!\mathsf{N}]$ and $[!\mathsf{N}]\neg$, ψ is either equivalent to $[!\mathsf{N}]\chi$ or to $[!\mathsf{N}]\neg\chi$ (3). Moreover, we have $\chi \in \mathsf{CN}(\Delta)$ (4). Hence by (2) we have $\neg\chi \in \mathsf{CN}(\Delta)$ (5). From (3), (4) and (5) the conclusion follows.

■

PROOF OF FACT 6.2. Assume that $\varphi \in \mathsf{CN}(\Delta)$ and φ is neither of the form $[!\mathsf{N}]\psi$, nor of the form $\neg[!\mathsf{N}]\psi$ then $[!\mathsf{N}]\varphi \in \mathsf{CN}(\Delta)$. Assume instead that φ is of the form $[!\mathsf{N}]\psi$ and simply observe that $\vdash [!\mathsf{N}][!\mathsf{N}]\psi \leftrightarrow [!\mathsf{N}]\psi$. Finally assume that φ is of the form $\neg[!\mathsf{N}]\psi$. Again it is easy to see that by axioms $[!\mathsf{N}][!\mathsf{N}]$ and $[!\mathsf{N}]\neg$ we have $\vdash [!\mathsf{N}]\neg[!\mathsf{N}]\psi \leftrightarrow \neg[!\mathsf{N}]\psi$. ■

PROOF OF COROLLARY 6.3. Immediate from the previous two facts. ■

PROOF OF FACT 6.4. First observe that the subformula closure of a finite set, is a finite set. Now enumerate the formulas in $Sub(\Delta) = \{\psi_1, \ldots, \psi_n\}$. Next observe that to satisfy the two last closure conditions we might have to add at most three new formulas for each formula ψ_i in $Sub(\Delta)$, namely $[!\mathsf{N}]\psi_i$, $\neg\psi_i$ and $[!\mathsf{N}]\neg\psi_i$, but it is easy to see that the resulting set is closed under subformulas, and it is also easy to see that the two last closure conditions are satisfied. All conditions are then satisfied by the finite set we have just constructed. But CN is the smallest such set. ■

PROOF OF FACT 6.6. Straightforward. ■

PROOF OF FACT 6.7. If $\vec{a} = \epsilon$, then by definition of cn we have $\mathsf{cn}(\Delta)(\vec{a}) = \mathsf{CN}(\Delta)$. Hence by Fact A.4 $\mathsf{cn}(\Delta)(\vec{a}) = \mathsf{CN}(\mathsf{cn}(\Delta)(\vec{a}))$. Instead if $\vec{a} = (\vec{b}\cdot c)$. We have

$$\mathsf{cn}(\Delta)(\vec{b}\cdot c) = (\mathsf{cn}(\Delta)(\vec{b}))(c) \tag{B.1}$$

$$= \mathsf{CN}(E(\mathsf{cn}(\Delta)(\vec{b}), c)) \tag{B.2}$$

But by Fact 6.6 $\mathsf{cn}(\Delta)(\vec{b})$ is finite, hence so is $E(\mathsf{cn}(\Delta)(\vec{b}), c)$. But then Fact A.4 we have $\mathsf{cn}(\Delta)(\vec{b}\cdot c) = \mathsf{CN}(\mathsf{cn}(\Delta)(\vec{b}\cdot c))$. ■

PROOF OF LEMMA 6.10. The proof is by induction on the rank of $\vec{a}$. Assume that $\mathsf{rank}_\Delta(\vec{a}) = 0$. It follows that $\vec{a}$ is Δ-leaf, hence by definition of κ_Δ we have $\kappa_\Delta(\vec{a}) = \mathsf{cn}(\Delta)(\vec{a})$. Assume for contradiction that there is a formula $\varphi \in \mathsf{cn}(\Delta)(\vec{a})$ such that $\mathsf{dep}(\varphi) > 0$. It follows by Fact 6.7 that

there is a formula of the form $\langle b\rangle\psi \in \mathsf{cn}(\Delta)(\vec{a})$ for some $b \in A$ and ψ. Hence by Fact A.12 it follows that $\mathsf{cn}(\Delta)(\vec{a}\cdot b) \neq \emptyset$, contradicting the fact that $\vec{a}$ is Δ-leaf. Hence $\max(\mathsf{dep}(\mathsf{cn}(\Delta)(\vec{a}))) = 0$.

Induction step. Assume that $\mathsf{rank}_\Delta(\vec{a}) = n+1$ (1).

Claim 1: $\max(\mathsf{dep}(\kappa_\Delta(\vec{a}))) \leq n+1$

It follows from (1) that for every $b \in A$, $\mathsf{rank}_\Delta(\vec{a}\cdot b) \leq n$. Hence by IH for every $b \in A$, we have $\max(\mathsf{dep}(\kappa_\Delta(\vec{a}\cdot b))) = \max(\mathsf{dep}(\mathsf{cn}(\Delta)(\vec{a}\cdot b))) \leq n$ (2).

Assume for contradiction that there is a formula $\varphi \in \mathsf{cn}(\Delta)(\vec{a})$ such that $\mathsf{dep}(\varphi) > n+1$. It follows by Fact 6.7 that there is a formula of the form $\langle b\rangle\psi \in \mathsf{cn}(\Delta)(\vec{a})$ for some $b \in A$ and some ψ with $\mathsf{dep}(\varphi) > n$. But by Fact A.12, $\varphi \in \mathsf{cn}(\Delta)(\vec{a}\cdot b)$, contradicting (2). Hence $\max(\mathsf{dep}(\mathsf{cn}(\Delta)(\vec{a}))) \leq n+1$ (a).

Now assume, for contradiction that there is a formula $\varphi \in \kappa_\Delta(\vec{a})$ (3) such that $\mathsf{dep}(\varphi) > n+1$ (4). By definition of κ_Δ it follows that $\varphi \in \mathsf{CN}(\bigcup_{b\in A} \mathsf{da}(\kappa_\Delta(\vec{a}\cdot b), b) \cup \mathsf{cn}(\Delta)(\vec{a}))$ (5). From (3), (4), (5) and Fact A.3 it follows that there is a formula ψ and some $c \in A$ such that $\mathsf{dep}(\psi) > n$ (6) and such that one the following holds:

i. either $\langle c\rangle\psi \in \mathsf{CN}(\bigcup_{b\in A} \mathsf{da}(\kappa_\Delta(\vec{a}\cdot b), b))$
ii. $\langle c\rangle\psi \in \mathsf{CN}(\mathsf{cn}(\Delta)(\vec{a}))$

Note (6) and (ii) contradicts (a). Hence we have $\langle c\rangle\psi \in \mathsf{CN}(\bigcup_{b\in A} \mathsf{da}(\kappa_\Delta(\vec{a}\cdot b), b))$. By Fact A.3, it follows that for some $b \in A$ we have $\langle c\rangle\psi \in \mathsf{CN}(\mathsf{da}(\kappa_\Delta(\vec{a}\cdot b), b))$. By Fact A.22, It follows that that one the following holds:

iii. $b = c$ and $\psi \in \mathsf{CN}(\{\widehat{S} | \kappa_\Delta(\vec{a}\cdot c))$
iv. $b \neq c$ and $\langle c\rangle\psi \in \mathsf{CN}(\{\widehat{S} | \kappa_\Delta(\vec{a}\cdot b))$

If we are in case (iii), then by Fact A.3 and Fact A.7 it follows that $\psi \in \mathsf{CN}(\kappa_\Delta(\vec{a}\cdot c))$. Hence by (6) and Fact A.8, it follows that $\max(\mathsf{dep}(\kappa_\Delta(\vec{a}\cdot c))) > n$, contradicting (2). Hence by reduction $\max(\mathsf{dep}(\kappa_\Delta(\vec{a}))) \leq n+1$. If instead we are in case (iv), then by Fact A.7 it follows that $\langle c\rangle\psi \in \mathsf{CN}(\kappa_\Delta(\vec{a}\cdot b))$. Hence by (6) and Fact A.8, it follows that $\max(\mathsf{dep}(\kappa_\Delta(\vec{a}\cdot b))) > n+1 > n$, contradicting (2). Hence by reduction $\max(\mathsf{dep}(\kappa_\Delta(\vec{a}))) \leq n+1$. Hence in either case we have $\max(\mathsf{dep}(\kappa_\Delta(\vec{a}))) \leq n+1$.

Claim 2: $\max(\mathsf{dep}(\kappa_\Delta(\vec{a}))) > n$

It follows from (1) that there is some $b \in A$ with $\mathsf{rank}_\Delta(\vec{a}\cdot b) = n$. Hence by IH $\max(\mathsf{dep}(\kappa_\Delta(\vec{a}\cdot b))) = \max(\mathsf{dep}(\mathsf{cn}(\Delta)(\vec{a}\cdot b))) = n$. But by Fact A.16, it follows that $\max(\mathsf{dep}(\mathsf{cn}(\Delta)(\vec{a}))) > n$. But by definition of κ_Δ it follows that $\max(\mathsf{dep}(\kappa_\Delta(\vec{a}))) > n$. ■

Proof of Fact 6.11. Since Δ is finite, it follows by Corollary A.17, the depth $\mathsf{tree}(\Delta)$ is bounded by $\max(\mathsf{dep}(\Delta))$ and that by Fact A.4 $\mathsf{tree}(\Delta)$ is finitely branching. Now by induction on the $\mathsf{rank}_\Delta(\vec{a})$, we can show that for each $\vec{a} \in \mathsf{tree}(\Delta)$, $\kappa_\Delta(\vec{a})$ is finite. (If instead $\vec{a} \notin \mathsf{tree}(\Delta)$, the claim follows trivially, by emptiness of $\kappa_\Delta(\vec{a})$.) The base induction follows immediately from Fact A.23. For the induction step, we simply use Fact A.4 and Fact A.24. ■

Proof of Fact 6.12. Follows from Fact 6.11, definition of κ_Δ and Fact A.4. ■

Proof of Lemma 6.13. Assume that $\varphi \in \kappa_\Delta(\vec{a})$ and $\mathsf{dep}(\varphi) = n$. By Fact 6.10 we have $\mathsf{rank}_\Delta(\vec{a}) \geq n$. If $\mathsf{rank}_\Delta(\vec{a}) = n$, then for every $b \in A$, $\varphi \notin \kappa_\Delta(\vec{a}\cdot b)$ and the claim follows trivially. Assume instead that $\mathsf{rank}_\Delta(\vec{a}) > n$. Consider a branch $\vec{a}\cdot\vec{b}$ in $\mathsf{tree}(\Delta)$ with $\mathsf{rank}_\Delta(\vec{a}\cdot b) < n$ By Fact 6.10, $\varphi \notin \kappa_\Delta(\vec{a}\cdot\vec{b})$. By Corollary A.29 it follows that for any descendant $\vec{a}\cdot\vec{b}\cdot\vec{c}$ of $\vec{a}\cdot\vec{b}$ with $\vec{c} \neq \emptyset$ we have $\varphi \notin \kappa_\Delta(\vec{a}\cdot\vec{b}\cdot\vec{c})$. Now enumerate $\vec{b}$ as $(b_1, \ldots, b_n)$. We have $\varphi \notin \kappa_\Delta(\vec{a}\cdot b_1 \cdot \ldots \cdot b_n)$. By Corollary A.29 it follows that

there is some $\vec{d} \sqsubseteq (b_1 \cdot \ldots \cdot b_n)$ such that $\varphi \in \kappa_\Delta(\vec{a} \cdot \vec{d})$, for every $\vec{e} \sqsubseteq \vec{d}$, we have $\varphi \in \kappa_\Delta(\vec{a} \cdot \vec{e})$ and for every $\vec{f}$ such that $\vec{d} \sqsubset \vec{f}$ we have $\varphi \notin \kappa_\Delta(\vec{a} \cdot \vec{f})$. ■

PROOF OF FACT 6.14. Since $\psi \in \mathsf{CN}(\{\widehat{S} | S \subseteq \kappa_\Delta(\vec{a} \cdot b)\})$, it follows by definition of $\kappa_\Delta(\vec{a} \cdot b)$ that $\mathsf{cn}(\Delta)(\vec{a} \cdot b) \neq \emptyset$ and so $\vec{a}$ is not a Δ-leaf. (0) Now since $\psi \in \mathsf{CN}(\{\widehat{S} | S \subseteq \kappa_\Delta(\vec{a} \cdot b)\}) = \mathsf{cj}(\kappa_\Delta(\vec{a} \cdot b))$, we have $\langle b \rangle \psi \in \mathsf{da}(\kappa_\Delta(\vec{a} \cdot b), b)$. Hence $\langle b \rangle \psi \in \bigcup_{b \in A} \mathsf{da}(\kappa_\Delta(\vec{a} \cdot b), b)$. (1). From (0) and (1) we have $\langle b \rangle \psi \in \kappa_\Delta(\vec{a})$. ■

PROOF OF FACT 6.15. Assume that $\langle c \rangle \varphi \in \kappa_\Delta(\vec{a})$. By Fact A.36 we have either $\varphi \in \mathsf{CN}(\{\widehat{S} | S \subseteq \kappa_\Delta(\vec{a} \cdot c)\})$ or $\langle c \rangle \varphi \in \kappa_\Delta(\vec{a} \cdot b_1)$ for some $b_1 \neq c$. If we are in the first case, we are done, so assume instead we are in the second case. By Fact 6.13 there is some $\vec{b} = (b_1, \ldots, b_n)$ such that $\langle c \rangle \varphi \in \kappa_\Delta(\vec{a} \cdot \vec{b})$ (1) and for every $\vec{d} \neq \epsilon$ we have $\langle c \rangle \varphi \notin \kappa_\Delta(\vec{a} \cdot \vec{b} \cdot \vec{d})$ (2). From (1) and Fact A.36, it follows that we have either $\varphi \in \mathsf{CN}(\{\widehat{S} | S \subseteq \kappa_\Delta(\vec{a} \cdot \vec{b} \cdot c)\})$ (a) or $\langle c \rangle \varphi \in \kappa_\Delta(\vec{a} \cdot \vec{b} \cdot d_1)$ for some $d_1 \neq c$ (b). But (b) contradicts (2), hence we are in case (a). But by Fact A.30 and Fact A.19, the claim follows. ■

PROOF OF COROLLARY 6.16. Straightforward from Fact 6.15 and Fact A.1. ■

PROOF OF FACT 6.18. Immediate from Fact 6.12 and Fact 6.1. ■

PROOF OF FACT 6.19. Immediate from Fact 6.12 and Fact 6.2. ■

PROOF OF FACT 6.20. Immediate from Fact 6.12 and Fact 6.3. ■

PROOF OF FACT 6.21. Immediate from Fact 6.18 and maximality of atoms. ■

PROOF OF FACT 6.22. Folklore. ■

PROOF OF FACT A.1. Straightfoward. ■

PROOF OF FACT A.2. The right to left inclusion is from the definition of *Sub*. Now assume $\varphi \in Sub(\mathsf{CN}(\Delta))$ it follows that there is a formula $\psi \in \mathsf{CN}(\Delta)$ such that $\varphi \in sub(\psi)$. Now either $\psi \in Sub(\Delta)$, but then it is easy to see that $\varphi \in Sub(\Delta) \subseteq \mathsf{CN}(\Delta)$. Or else $\psi \in (\mathsf{CN}(\Delta) \setminus Sub(\Delta))$. But then by Fact A.1, there is a formula $\chi \in Sub(\Delta)$ such that one the following holds

a. $\psi = \neg\chi$ and χ is neither of the form $\neg\alpha$, nor of the form $[!\mathsf{N}]\alpha$
b. $\psi = [!\mathsf{N}]\chi$
c. $\psi = [!\mathsf{N}]\neg\chi$ and χ is neither of the form $\neg\alpha$, nor of the form $[!\mathsf{N}]\alpha$

It follows that one of the following holds

1. $\varphi = \chi$
2. $\varphi = \neg\chi$ and χ is neither of the form $\neg\alpha$, nor of the form $[!\mathsf{N}]\alpha$
3. $\varphi \in sub(\chi)$

In case 1, we simply have to observe that $\varphi = \chi \in Sub(\Delta) \subseteq \mathsf{CN}(\Delta)$. In case 2, we have that by definition of *CN* that $\neg\chi \in \mathsf{CN}(\Delta)$, since χ is neither of the form $\neg\alpha$, nor of the form $[!\mathsf{N}]\alpha$, hence $\varphi = \neg\chi \in \mathsf{CN}(\Delta)$. In case 3, we have $\varphi \in sub(\chi) \subseteq Sub(\Delta) \subseteq \mathsf{CN}(\Delta)$. ■

PROOF OF FACT A.3. Straightforward. ■

PROOF OF FACT A.4. The right to left inclusion follows from the fact that $\mathsf{CN}(\Delta) \subseteq Sub(\mathsf{CN}(\Delta))$ but by definition of *CN* we have $Sub(\mathsf{CN}(\Delta)) \subseteq \mathsf{CN}(\mathsf{CN}(\Delta))$. For left to right inclusion, assume that $\varphi \in \mathsf{CN}(\mathsf{CN}(\Delta))$, then either $\varphi \in Sub(\mathsf{CN}(\Delta))$, but then by Fact A.2, or else, by Fact A.2 there is some

$\psi \in Sub(\mathsf{CN}(\Delta))$ (1) such that one the following holds:

a. $\varphi = \neg\psi$ and ψ is neither of the form $\neg\chi$, nor of the form $[!\mathsf{N}]\chi$
b. $\varphi = [!\mathsf{N}]\psi$ and ψ is neither of the form $[!\mathsf{N}]\chi$, nor of the form $\neg[!\mathsf{N}]\chi$
c. $\varphi = [!\mathsf{N}]\neg\psi$ and ψ is neither of the form $\neg\chi$, nor of the form $[!\mathsf{N}]\chi$

But by (1) and Fact A.2 we have $\psi \in \mathsf{CN}(\Delta)$. But then in cases (a)–(c) we have by definition of CN, $\varphi \in \mathsf{CN}(\Delta)$. ■

Proof of Fact A.5. Assume that $\varphi \in \mathsf{CN}(\Delta_1)$ and $\Delta_1 \subseteq \Delta_2$ (1). From (1) we have $Sub(\Delta_1) \subseteq Sub(\Delta_2)$ (2). Then there is a formula $\psi \in Sub(\Delta_1)$ (3) such that either $\varphi = \psi$, but then by (2) and (3) we have $\varphi = \psi \in Sub(\Delta_2)$, or else one the following holds:

a. $\varphi = \neg\psi$ and ψ is neither of the form $\neg\chi$, nor of the form $[!\mathsf{N}]\chi$
b. $\varphi = [!\mathsf{N}]\psi$ and ψ is neither of the form $[!\mathsf{N}]\chi$, nor of the form $\neg[!\mathsf{N}]\chi$
c. $\varphi = [!\mathsf{N}]\neg\psi$ and ψ is neither of the form $\neg\chi$, nor of the form $[!\mathsf{N}]\chi$

but then, by definition of CN, we have in each case $\varphi \in \mathsf{CN}(\Delta_2)$. ■

Proof of Fact A.6. Assume that $\varphi_1 \in \mathsf{CN}(\varphi_2)$. It follows that there is some $\psi_2 \in sub(\varphi_2)$ such that one of the following holds:

a. $\varphi_1 = \psi_2$
b. $\varphi_1 = \neg\psi_2$ and ψ_2 is neither of the form $\neg\chi$, nor of the form $[!\mathsf{N}]\chi$
c. $\varphi_1 = [!\mathsf{N}]\psi_2$ and ψ_2 is neither of the form $[!\mathsf{N}]\chi$, nor of the form $\neg[!\mathsf{N}]\chi$
d. $\varphi_1 = [!\mathsf{N}]\neg\psi_2$ and ψ_2 is neither of the form $\neg\chi$, nor of the form $[!\mathsf{N}]\chi$

But since $\varphi_2 \in \mathsf{CN}(\varphi_3)$ and $\psi_2 \in sub(\varphi_2)$, we have $\psi_2 \in \mathsf{CN}(\varphi_3)$. Hence in each of the previous cases $\varphi_1 \in \mathsf{CN}(\varphi_3)$. ■

Proof of Fact A.7. Trivial. ■

Proof of Fact A.8. The $\geq$ is trivial, since $\Delta \subseteq \mathsf{CN}(\Delta)$. For the $\leq$ direction observe that $\mathsf{dep}_A([!\mathsf{N}]\varphi) = \mathsf{dep}_A([!\mathsf{N}]\neg\varphi) = \mathsf{dep}_A(\neg\varphi) = \mathsf{dep}_A(\varphi)$ (1). Next observe that for any φ, ψ if $\varphi \in sub(\psi)$ then $\mathsf{dep}_A(\varphi) \leq \mathsf{dep}_A(\psi)$ (2). Now assume that $\varphi \in \mathsf{CN}(\Delta)$, then there is some $\chi \in \Delta$ (3) and $\psi \in sub(\chi)$ such that φ is of the form $[!\mathsf{N}]^*\neg^*\psi$. Note that by (1), $\mathsf{dep}_A(\varphi) = \mathsf{dep}_A(\psi)$ (3). But by (4) we have $\mathsf{dep}_A(\chi) \geq \mathsf{dep}_A(\psi)$ (5). By (3), (4), (5) and the fact ψ was arbitrary, the claim follows. ■

Proof of Fact A.9. Immediate by the application of the definition of $\Gamma(a)$ to the case where $\Gamma = \mathsf{CN}(\Delta)$. ■

Proof of Fact A.10. The right to left inclusion is immediate from the definition of E. The left to right inclusion follows from Fact A.2. ■

Proof of Fact A.11. By Corollary 6.5 we have $\mathsf{CN}(\Delta)(a) = \mathsf{CN}(\{\varphi \mid \langle a \rangle \varphi \in \mathsf{CN}(\Delta)\})$. Hence, by Fact A.8:

$$\max(\mathsf{dep}(\mathsf{CN}(\Delta)(a))) = \max(\mathsf{dep}(\{\varphi \mid \langle a \rangle \varphi \in \mathsf{CN}(\Delta)\})) \tag{B.3}$$

Moreoever, we have

$$\max(\mathsf{dep}(\{\langle a\rangle\varphi \mid \langle a\rangle\varphi \in \mathsf{CN}(\Delta)\})) \leq \max(\mathsf{dep}(\mathsf{CN}(\Delta))) \tag{B.4}$$

And by Fact A.8

$$\max(\mathsf{dep}(\mathsf{CN}(\Delta))) = \max(\mathsf{dep}(\Delta)) \tag{B.5}$$

It follows that

$$\max(\mathsf{dep}(\{\langle a\rangle\varphi \mid \langle a\rangle\varphi \in \mathsf{CN}(\Delta)\})) \leq \max(\mathsf{dep}(\Delta)) \tag{B.6}$$

Therefore, it is enough to show that

$$\max(\mathsf{dep}(\{\varphi \mid \langle a\rangle\varphi \in \mathsf{CN}(\Delta)\})) < \max(\mathsf{dep}(\{\langle a\rangle\varphi \mid \langle a\rangle\varphi \in \mathsf{CN}(\Delta)\})) \tag{B.7}$$

But this is immediate from Fact 6.4 and the definition of dep. ■

PROOF OF FACT A.12.

$$\begin{aligned}\mathsf{cn}(\Delta)(\vec{a}\cdot b) &= (\mathsf{cn}(\Delta)(\vec{a}))(b)\\ &= \mathsf{CN}(E(\mathsf{cn}(\Delta)(\vec{a}), b))\end{aligned}$$

But then, by Fact 6.7 we have

$$\mathsf{cn}(\Delta)(\vec{a}\cdot b) = \mathsf{CN}(E(\mathsf{CN}(\mathsf{cn}(\Delta)(\vec{a}), b)))$$

Hence, by Fact A.10 we have

$$\mathsf{cn}(\Delta)(\vec{a}\cdot b) = \mathsf{CN}(\{\varphi \mid \langle b\rangle\varphi \in \mathsf{CN}(\mathsf{cn}(\Delta)(\vec{a})\}))$$

By Fact 6.7 we have thus

$$\mathsf{cn}(\Delta)(\vec{a}\cdot b) = \mathsf{CN}(\{\varphi \mid \langle b\rangle\varphi \in \mathsf{cn}(\Delta)(\vec{a})\})$$

■

PROOF OF FACT A.13. Assume that $\varphi \in \mathsf{cn}(\vec{a}\cdot b)$. From Fact A.12 it follows that $\varphi \in \mathsf{CN}(\{\varphi \mid \langle b\rangle\varphi \in \mathsf{cn}(\Delta)(\vec{a})\})$. Hence by Fact 6.6 and Fact A.3 there is some formula χ such that $\langle b\rangle\chi \in \mathsf{cn}(\Delta)(\vec{a})$ and $\varphi \in \mathsf{CN}(\chi)$. ■

PROOF OF FACT A.14. If $\vec{b} = \emptyset$ the claim holds trivially. So assume instead that $\vec{b} = (b_1 \cdot \ldots \cdot b_n)$.

But then by Fact A.13 there is a set of formulas $\{\chi_c, \chi_n, \ldots, \chi_1\}$ such that:

$$\begin{gathered}\varphi \in \mathsf{cn}(\Delta)(\vec{a}\cdot\vec{b}\cdot c)\\ \langle c\rangle\chi_c \in \mathsf{cn}(\Delta)(\vec{a}\cdot b_1\cdot\ldots\cdot b_n) \text{ and } \varphi \in \mathsf{CN}(\chi_c)\\ \langle b_n\rangle\chi_n \in \mathsf{cn}(\Delta)(\vec{a}\cdot b_1\cdot\ldots\cdot b_{n-1}) \text{ and } \langle c\rangle\chi_c \in \mathsf{CN}(\chi_n)\\ \langle b_{n-1}\rangle\chi_{n-1} \in \mathsf{cn}(\Delta)(\vec{a}\cdot b_1\cdot\ldots\cdot b_{n-2}) \text{ and } \langle b_n\rangle\chi_n \in \mathsf{CN}(\chi_{n-1})\\ \vdots\\ \langle b_2\rangle\chi_2 \in \mathsf{cn}(\Delta)(\vec{a}\cdot b_1) \text{ and } \langle b_3\rangle\chi_3 \in \mathsf{CN}(\chi_2)\\ \langle b_1\rangle\chi_1 \in \mathsf{cn}(\Delta)(\vec{a}) \text{ and } \langle b_2\rangle\chi_2 \in \mathsf{CN}(\chi_1)\end{gathered}$$

First observe that by Fact A.6 we have $\langle c\rangle\chi_c \in \mathsf{CN}(\chi_1)$ (1). But by Fact 6.7 we have $\chi_1 \in \mathsf{cn}(\vec{a})$ (2). From (1), (2) and Fact 6.7 we have thus $\langle c\rangle\chi_c \in \mathsf{cn}(\vec{a})$ and $\varphi \in \mathsf{CN}(\chi_c)$ for some χ_c. ■

PROOF OF COROLLARY A.15. Assume that $\varphi \in \mathsf{cn}(\vec{a}\cdot\vec{b}\cdot c)$. By Fact A.14 we have some χ such that $\langle c\rangle\chi \in \mathsf{cn}(\vec{a})$ and $\varphi \in \mathsf{CN}(\chi)$ (1) for some χ. But then by Fact A.12 we have $\chi \in \mathsf{cn}(\vec{a}\cdot c)$. Hence by (1) and Fact 6.7, $\varphi \in \mathsf{cn}(\vec{a}\cdot c)$. ■

PROOF OF FACT A.16. Follows from definition of cn and Fact A.11. ■

PROOF OF COROLLARY A.17. Follows from Fact A.16. ■

PROOF OF FACT A.18. Since Δ is finite, there is only a finite subset $A_0 \subseteq A$ occurring in Δ. Let $A_0 = \{a_1, \ldots, a_n\}$. The claim follows from the definition of $\mathsf{tree}(\Delta)$. ■

PROOF OF FACT A.19. Assume $\Delta_1 \subseteq \Delta_2$. It follows that $\wp(\Delta_1) \subseteq \wp(\Delta_2)$. It follows that $\{\widehat{S} | S \subseteq \Delta_1\} \subseteq \{\widehat{S} | S \subseteq \Delta_2\}$. The claim follows from Fact A.5. ■

PROOF OF FACT A.20. Immediate from Fact A.19 and the definition of $\mathsf{da}(\Delta_1, a)$ and $\mathsf{da}(\Delta_2, a)$. ■

PROOF OF FACT A.21. If $\vec{a}$ is a Δ-leaf then $\kappa_\Delta(\vec{a}) = \mathsf{cn}(\Delta)(\vec{a})$ (1). Now assume for contradiction that $\max(\mathsf{dep}(\mathsf{cn}(\Delta)(\vec{a}))) > 0$. It follows that there is a formula of the form $\langle b\rangle\varphi \in \mathsf{cn}(\Delta)(\vec{a})$. But then by Fact A.12, $\mathsf{cn}(\Delta)(\vec{a}\cdot b) \neq \emptyset$, contradicting the assumption that $\vec{a}$ is a Δ-leaf. Hence $\max(\mathsf{dep}(\mathsf{cn}(\Delta)(\vec{a}))) = 0$. Hence by (1), $\max(\mathsf{dep}(\kappa_\Delta(\vec{a}))) = 0$. ■

PROOF OF FACT A.22. Immediate from definitions. ■

PROOF OF FACT A.23. Immediate from the definition of κ_Δ at Δ-leaves and Fact 6.6. ■

PROOF OF FACT A.24. Assume that $\mathsf{tree}(\Delta)$ is finitely branching. Let $\vec{a} \in \mathsf{tree}(\Delta)$ with $\vec{a}$ not a Δ-leaf. It follows that there is a finite set $A_0 \subseteq A$ such that $(\vec{a}\cdot b) \in \mathsf{tree}(\Delta)$ iff $b \in A_0$ (1). Now assume that for each $b \in A_0$, $\kappa_\Delta(\vec{a}\cdot b)$ is finite. It follows that $\{\widehat{S} | S \subseteq \kappa_\Delta(\vec{a}\cdot b)\}$ is finite as well. Hence by Fact 6.4 $\mathsf{CN}(\{\widehat{S} | S \subseteq \kappa_\Delta(\vec{a}\cdot b)\})$ is finite. It follows by Fact A.22, that $\mathsf{da}(\kappa_\Delta(\vec{a}\cdot b), b)$ is finite (2). By the definition of κ_Δ at non-leaves and by (1), (2), Fact 6.6 and Fact A.22 it follows that $\kappa_\Delta(\vec{a})$ is finite. ■

PROOF OF FACT A.25. From Fact A.12 and Fact 6.7. ■

PROOF OF FACT A.26. From Fact A.12 and Fact 6.7. ■

PROOF OF FACT A.27. Straightforward from the definition $\kappa_\Delta(\vec{a})$, Fact A.20 and $\kappa_\Delta(\vec{b})$. ■

PROOF OF FACT A.28. Let $L(\vec{a})$ be the maximal words in $\mathsf{SubTree}(\Delta, \vec{a})$. By Fact A.26 we have $L(\vec{a}) \subseteq \mathsf{SubTree}(\Delta, \vec{b})$. By construction we have for every $\vec{c} \in L(\vec{a})$, Fact A.25 and the definition of κ_Δ on leaves, that $\kappa_\Delta(\vec{a}\cdot\vec{c}) \subseteq \kappa_\Delta(\vec{b}\cdot\vec{c})$. The claim follows by an easy inductive argument using Fact A.25 and Fact A.27. ■

PROOF OF COROLLARY A.29. From Fact A.28 and Fact A.13. ■

PROOF OF COROLLARY A.30. From Fact A.28 and Corollary A.15. ■

PROOF OF FACT A.31. Since $\vec{a}$ is not a Δ-leaf, we have $\kappa_\Delta(\vec{a}) = \mathsf{CN}(\bigcup_{b\in A} \mathsf{da}(\kappa_\Delta(\vec{a}\cdot b), b) \cup \mathsf{cn}(\Delta)(\vec{a}))$. But then by Fact A.3 we have $\mathsf{CN}(\bigcup_{b\in A} \mathsf{da}(\kappa_\Delta(\vec{a}\cdot b), b) \cup \mathsf{cn}(\Delta)(\vec{a})) = \mathsf{CN}(\bigcup_{b\in A} \mathsf{da}(\kappa_\Delta(\vec{a}\cdot b), b)) \cup$

$\mathsf{CN}(\mathsf{cn}(\Delta)(\vec{a}))$. Hence we are in one of two cases:

a. there is some $b \in A$ and $\varphi \in \mathsf{CN}(\mathsf{da}(\kappa_\Delta(\vec{a} \cdot b), b))$
b. $\varphi \in \mathsf{CN}(\mathsf{cn}(\Delta)(\vec{a}))$ ■

PROOF OF COROLLARY A.32. Immediate from Fact 6.12, Fact 6.11 and Fact 6.2. ■

PROOF OF FACT A.33. Assume $\varphi_1, \ldots, \varphi_n \in \kappa_\Delta(\vec{a} \cdot b)$. It follows that $(\varphi_1 \wedge \ldots \wedge \varphi_n) \in \{\widehat{S} | S \subseteq \kappa_\Delta(\vec{a} \cdot b)\}$. Hence $(\varphi_1 \wedge \ldots \wedge \varphi_n) \in \mathsf{CN}(\{\widehat{S} | S \subseteq \kappa_\Delta(\vec{a} \cdot b)\})$. But then by Fact 6.14 we have $\langle b \rangle (\varphi_1 \wedge \ldots \wedge \varphi_n) \in \kappa_\Delta(\vec{a})$. ■

PROOF OF COROLLARY A.34. Immediate from Fact A.33 and 6.12 and a very simple inductive argument. ■

PROOF OF COROLLARY A.35. Immediate from Fact 6.14, Fact A.34 and 6.12 and a very simple inductive argument. ■

PROOF OF FACT A.36. Assume that $\langle c \rangle \varphi \in \kappa_\Delta(\vec{a})$. By Fact A.21 it follows, that $\vec{a}$ is not a Δ-leaf. Moreover by Fact A.31 it follows that we are in one of two cases:

1. there is some $b \in A$ and $\langle c \rangle \varphi \in \mathsf{CN}(\mathsf{da}(\kappa_\Delta(\vec{a} \cdot b), b))$
2. $\langle c \rangle \varphi \in \mathsf{CN}(\mathsf{cn}(\Delta)(\vec{a}))$

From (2), it follows by Fact 6.7 and Fact A.12 that $\varphi \in \mathsf{cn}(\Delta)(\vec{a} \cdot c)$. Hence by definition of κ_Δ we have $\varphi \in \kappa_\Delta(\vec{a} \cdot c)$. But by definition of CN we have $\varphi \in \mathsf{CN}(\{\varphi\}) \subseteq \mathsf{CN}(\{\widehat{S} | S \subseteq \kappa_\Delta(\vec{a} \cdot c)\})$.

Assume instead that we are in case (1). There is some $b \in A$ and $\langle c \rangle \varphi \in \mathsf{CN}(\mathsf{da}(\kappa_\Delta(\vec{a} \cdot b), b))$

By definition of da we have $\mathsf{da}(\kappa_\Delta(\vec{a} \cdot b), b) = \{\langle b \rangle \psi | \psi \in \mathsf{CN}(\{\widehat{S} | S \subseteq \kappa_\Delta(\vec{a} \cdot b)\})\}$. Hence $\langle c \rangle \varphi \in \mathsf{CN}(\{\langle b \rangle \psi | \psi \in \mathsf{CN}(\{\widehat{S} | S \subseteq \kappa_\Delta(\vec{a} \cdot b)\})\})$. There are two subcases: either $b = c$ and $\varphi \in \mathsf{CN}(\{\widehat{S} | S \subseteq \kappa_\Delta(\vec{a} \cdot c)\})$, or $b \neq c$ and then there is some $\psi \in \mathsf{CN}(\{\widehat{S} | S \subseteq \kappa_\Delta(\vec{a} \cdot b)\})$ such that $\langle c \rangle \varphi \in \mathsf{CN}(\psi)$, that is $\langle c \rangle \varphi \in \mathsf{CN}(\{\widehat{S} | S \subseteq \kappa_\Delta(\vec{a} \cdot b)\})$.

But then there is some $S \subseteq \kappa_\Delta(\vec{a} \cdot b)$ with $S = \{\chi_1, \ldots, \chi_n\}$ such that for some $\chi_i \in \{\chi_1, \ldots, \chi_n\}$, $\langle c \rangle \varphi \in \mathsf{CN}(\chi_i)$. But then $\langle c \rangle \varphi \in \kappa_\Delta(\vec{a} \cdot b)$. ■

C Computing $\mathsf{CN}(\Delta)$, cn and κ_Δ

In this section, we illustrate the important syntactic definitions used in the completeness proof, such as $\mathsf{CN}(\Delta)$, the cn function and the κ_Δ function. To illustrate their computation concretely let us consider the set of formulas $\Delta = \{\langle a \rangle (\langle b \rangle p \wedge \langle b \rangle \neg p), [!\mathsf{N}] \langle a \rangle [!\mathsf{N}] \langle c \rangle q, [!\mathsf{N}] \langle a \rangle \neg \langle b \rangle \neg p\}$.

We start by computing $\mathsf{Sub}(\Delta)$ and $\mathsf{CN}(\Delta)$.

$$\begin{array}{llll}
\mathsf{Sub}(\Delta) = & \{ \quad \langle a \rangle (\langle b \rangle p \wedge \langle b \rangle \neg p), & [!\mathsf{N}] \langle a \rangle [!\mathsf{N}] \langle c \rangle q, & [!\mathsf{N}] \langle a \rangle \neg \langle b \rangle \neg p, \\
 & \langle b \rangle p \wedge \langle b \rangle \neg p, & \langle a \rangle [!\mathsf{N}] \langle c \rangle q, & \langle a \rangle \neg \langle b \rangle \neg p, \\
 & \langle b \rangle p, \quad \langle b \rangle \neg p, & [!\mathsf{N}] \langle c \rangle q, & \neg \langle b \rangle \neg p, \\
 & p, \quad \neg p, & \langle c \rangle q, q \} &
\end{array}$$

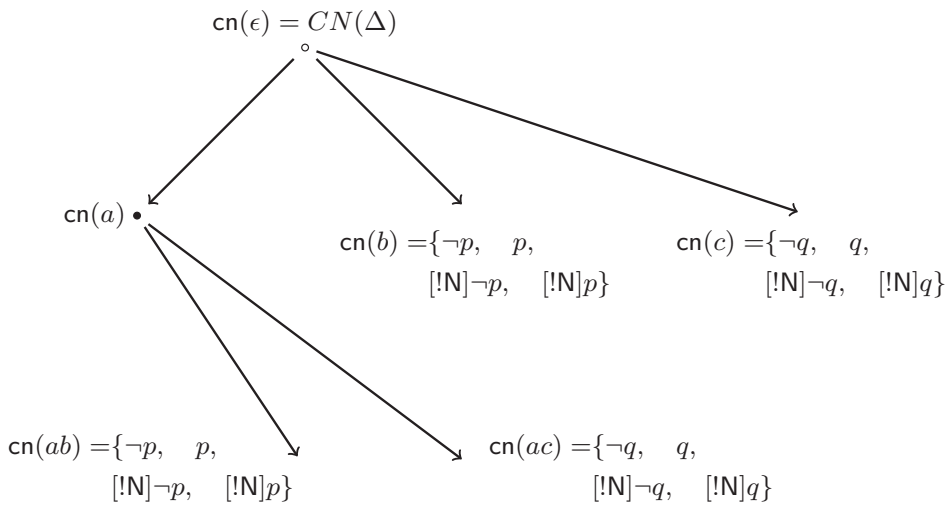

FIGURE C1. Inductive definition of the cn function.

$$
\begin{aligned}
\mathsf{CN}(\Delta) = \{ \quad & \langle a\rangle(\langle b\rangle p \wedge \langle b\rangle\neg p), && \neg\langle a\rangle(\langle b\rangle p \wedge \langle b\rangle\neg p),\\
& [!\mathsf{N}]\langle a\rangle(\langle b\rangle p \wedge \langle b\rangle\neg p), && [!\mathsf{N}]\neg\langle a\rangle(\langle b\rangle p \wedge \langle b\rangle\neg p),\\
& \langle b\rangle p \wedge \langle b\rangle\neg p, && \neg(\langle b\rangle p \wedge \langle b\rangle\neg p),\\
& [!\mathsf{N}](\langle b\rangle p \wedge \langle b\rangle\neg p), && [!\mathsf{N}]\neg(\langle b\rangle p \wedge \langle b\rangle\neg p),\\
& [!\mathsf{N}]\langle a\rangle[!\mathsf{N}]\langle c\rangle q && \langle a\rangle[!\mathsf{N}]\langle c\rangle q,\\
& [!\mathsf{N}]\neg\langle a\rangle[!\mathsf{N}]\langle c\rangle q && \neg\langle a\rangle[!\mathsf{N}]\langle c\rangle q,\\
& [!\mathsf{N}]\langle a\rangle\neg\langle b\rangle\neg p, && [!\mathsf{N}]\neg\langle a\rangle\neg\langle b\rangle\neg p,\\
& \langle a\rangle\neg\langle b\rangle\neg p, && \neg\langle a\rangle\neg\langle b\rangle\neg p,\\
& \langle b\rangle p, \neg\langle b\rangle p, [!\mathsf{N}]\langle b\rangle p, && [!\mathsf{N}]\neg\langle b\rangle p, \langle b\rangle\neg p,\\
& \neg\langle b\rangle\neg p, [!\mathsf{N}]\neg\langle b\rangle\neg p, && p, \neg p, [!\mathsf{N}]p, [!\mathsf{N}]\neg p,\\
& [!\mathsf{N}]\langle c\rangle q, [!\mathsf{N}]\neg\langle c\rangle q, \langle c\rangle q, \neg\langle c\rangle q, && [!\mathsf{N}]q, [!\mathsf{N}]\neg q, q, \neg q \quad \}
\end{aligned}
$$

We can now define cn, inductively, from the root, forming the labeled tree in Figure C1.

$$
\begin{aligned}
\mathsf{cn}(a) = \{ & \langle b\rangle p \wedge \langle b\rangle\neg p, \ \neg(\langle b\rangle p \wedge \langle b\rangle\neg p), [!\mathsf{N}](\langle b\rangle p \wedge \langle b\rangle\neg p),\\
& [!\mathsf{N}]\neg(\langle b\rangle p \wedge \langle b\rangle\neg p), [!\mathsf{N}]\neg\langle b\rangle\neg p, \ \neg\langle b\rangle\neg p, \ \langle b\rangle\neg p, [!\mathsf{N}]\neg\langle b\rangle p,\\
& [!\mathsf{N}]\langle b\rangle p, \ \neg\langle b\rangle p, \ \langle b\rangle p,\\
& [!\mathsf{N}]\langle c\rangle q, \neg[!\mathsf{N}]\langle c\rangle q, \langle c\rangle q, [!\mathsf{N}]\neg\langle c\rangle q, \neg\langle c\rangle q,\\
& \neg p, p, [!\mathsf{N}]\neg p, [!\mathsf{N}]p, \neg q, q, [!\mathsf{N}]\neg q, [!\mathsf{N}]q\}
\end{aligned}
$$

It should be clear that this tree is a syntactic tree: a useful means to describe the syntactic content that we will use to construct atoms (and hence of the canonical model). It is *not* a semantic structure.

Now that we have defined our cn function, we can construct the function κ_Δ, inductively, from the leaves, back to the root (see Figure C2). We do not give the full syntactic extension of $\kappa_\Delta(a)$, which also contains, for example, $[!\mathsf{N}]\langle c\rangle([!\mathsf{N}]\neg q \wedge q \wedge [!\mathsf{N}]q$ or $\langle c\rangle(q \wedge [!\mathsf{N}]q)$, but rather an equivalent

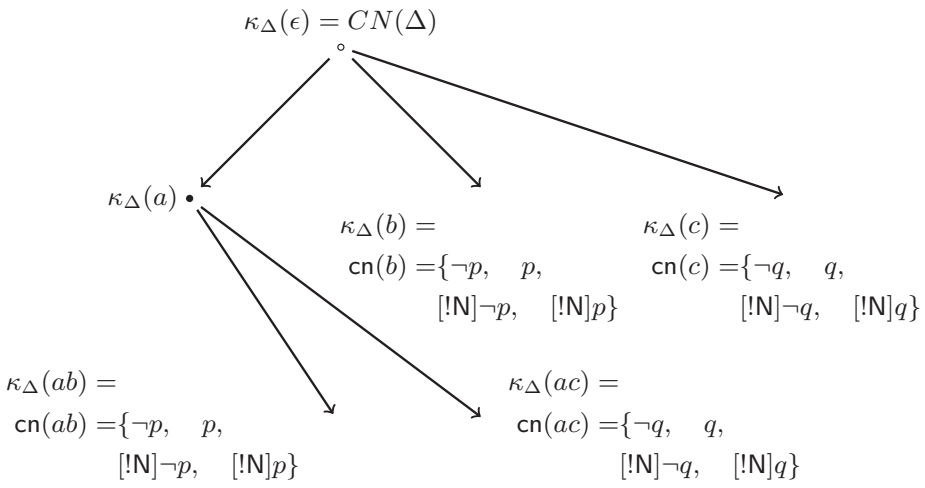

FIGURE C2. Example of a construction tree of κ_Δ.

slightly more compact representation:

$$
\begin{aligned}
\kappa_\Delta(a) \equiv\ & \mathsf{cn}(a)\ \cup \\
& \{[!\mathsf{N}]\langle c\rangle[!\mathsf{N}]\neg q, \neg[!\mathsf{N}]\langle c\rangle[!\mathsf{N}]\neg q, \langle c\rangle[!\mathsf{N}]\neg q, [!\mathsf{N}]\neg\langle c\rangle[!\mathsf{N}]\neg q, \neg\langle c\rangle[!\mathsf{N}]\neg q, \\
& \{[!\mathsf{N}]\langle c\rangle[!\mathsf{N}]q, \neg[!\mathsf{N}]\langle c\rangle[!\mathsf{N}]q, \langle c\rangle[!\mathsf{N}]q, [!\mathsf{N}]\neg\langle c\rangle[!\mathsf{N}]q, \neg\langle c\rangle[!\mathsf{N}]q, \\
& \{[!\mathsf{N}]\langle c\rangle\neg q, \neg[!\mathsf{N}]\langle c\rangle\neg q, \langle c\rangle\neg q, [!\mathsf{N}]\neg\langle c\rangle\neg q, \neg\langle c\rangle\neg q, \\
& \{[!\mathsf{N}]\langle c\rangle\bot, \neg[!\mathsf{N}]\langle c\rangle\bot, \langle c\rangle\bot, [!\mathsf{N}]\neg\langle c\rangle\bot, \neg\langle c\rangle\bot, \\
& \{[!\mathsf{N}]\langle b\rangle[!\mathsf{N}]\neg p, \neg[!\mathsf{N}]\langle b\rangle[!\mathsf{N}]\neg p, \langle b\rangle[!\mathsf{N}]\neg p, [!\mathsf{N}]\neg\langle b\rangle[!\mathsf{N}]\neg p, \neg\langle b\rangle[!\mathsf{N}]\neg p, \\
& \{[!\mathsf{N}]\langle b\rangle[!\mathsf{N}]p, \neg[!\mathsf{N}]\langle b\rangle[!\mathsf{N}]p, \langle b\rangle[!\mathsf{N}]p, [!\mathsf{N}]\neg\langle b\rangle[!\mathsf{N}]p, \neg\langle b\rangle[!\mathsf{N}]p, \\
& \{[!\mathsf{N}]\langle b\rangle\neg p, \neg[!\mathsf{N}]\langle b\rangle\neg p, \langle b\rangle\neg p, [!\mathsf{N}]\neg\langle b\rangle\neg p, \neg\langle b\rangle\neg p, \\
& \{[!\mathsf{N}]\langle b\rangle\bot, \neg[!\mathsf{N}]\langle b\rangle\bot, \langle b\rangle\bot, [!\mathsf{N}]\neg\langle b\rangle\bot, \neg\langle b\rangle\bot\}
\end{aligned}
$$

As for $\kappa_\Delta(\epsilon)$ simply note that it contains in particular every formula of the form $\langle a\rangle(\varphi_1 \wedge \ldots \wedge \varphi_n)$, where $\varphi_1, \ldots, \varphi_n \in \kappa_\Delta(a)$. For example $\langle a\rangle(\langle b\rangle\neg p \wedge \langle c\rangle[!\mathsf{N}]q) \in \kappa_\Delta(\epsilon)$. Note also that $\kappa_\Delta(\epsilon)$ is CN-closed.

D Looking at the structure of a concrete set of atoms

In this section, we return to the example from Appendix C, where

$$\Delta = \{\langle a\rangle(\langle b\rangle p \wedge \langle b\rangle\neg p), [!\mathsf{N}]\langle a\rangle[!\mathsf{N}]\langle c\rangle q, [!\mathsf{N}]\langle a\rangle\neg\langle b\rangle\neg p\}$$

and look at the candidate atoms that could play the role of building blocks for a pointed $\mathsf{P}\sigma\mathsf{LTS}$ of Δ.

Atoms for the Δ-leaves are fairly obvious. $At(\Delta, c) = At(\Delta, ac) = \{\{q, [!\mathsf{N}]q\}, \{\neg q, [!\mathsf{N}]\neg q\}\}$ and $At(\Delta, b) = At(\Delta, ab) = \{\{p, [!\mathsf{N}]p\}, \{\neg p, [!\mathsf{N}]\neg p\}\}$.

Let us now look at $At(\Delta, a)$. To do so we start by cleaning up $\kappa_\Delta(a)$ from all inconsistent formulas. First there are three 'parameters' for the candidate atoms, a $\langle b\rangle$, a $\langle c\rangle$ parameter and a propositional parameter. Consider the $\langle b\rangle$ parameters. Intuitively, each of them corresponds to one of the maximal

consistent subsets of $\{\langle b\rangle p, \neg\langle b\rangle p, \langle b\rangle\neg p, \neg\langle b\rangle\neg p, [!\mathsf{N}]\langle b\rangle p, [!\mathsf{N}]\neg\langle b\rangle p, [!\mathsf{N}]\langle b\rangle\neg p, [!\mathsf{N}]\neg\langle b\rangle\neg p\}$. By axioms, those are the following:

	$\langle b\rangle p, \langle b\rangle\neg p$	$\langle b\rangle p, \neg\langle b\rangle\neg p$	$\neg\langle b\rangle p, \langle b\rangle\neg p$	$\neg\langle b\rangle p, \neg\langle b\rangle\neg p$
$[!\mathsf{N}]\langle b\rangle p, [!\mathsf{N}]\langle b\rangle\neg p$		$\bot$	$\bot$	$\bot$
$[!\mathsf{N}]\langle b\rangle p, [!\mathsf{N}]\neg\langle b\rangle\neg p$			$\bot$	$\bot$
$[!\mathsf{N}]\neg\langle b\rangle p, [!\mathsf{N}]\langle b\rangle\neg p$		$\bot$		$\bot$
$[!\mathsf{N}]\neg\langle b\rangle p, [!\mathsf{N}]\neg\langle b\rangle\neg p$				

Hence we have nine possible values for the $\langle b\rangle$ parameter, and similarly for $\langle c\rangle$. Finally, we have four possible values for the propositional parameter. These are $4\times(9^2)=324$ atoms in $At(\Delta,a)$. Before we describe $At(\Delta,\epsilon)$, let us explain the structure behind the preceding value.

Consider a finite tree on A^*. Assume that $\vec{a}\in A^*$ is not a leaf. Assume in particular that for each $b\in A_0\subseteq A$ with $A_0\neq\emptyset$, there are n_b possible values for $s(\vec{a}\cdot b)$. Now let $s(\vec{a},b)$ be the selection of a subset of all possible values for $s(\vec{a}\cdot b)$ and of a subset of that subset. There are then $\sum_{k=0}^{n_b}\binom{n_b}{k}2^k$ possible values for $s(\vec{a}\cdot b)$.

Now observe, that the b-parameter of an atom in $At(\Delta,\vec{a})$ boils down to deciding which of the atoms in $At(\Delta,\vec{a}\cdot b)$ would make a consistent successor. Hence,

$$|At(\Delta,\vec{a})|=2^{|\mathrm{PROP}(\kappa_\Delta(\vec{a}))|}\cdot|\prod_{b\in A}\sum_{k=0}^{|At(\Delta,\vec{a}\cdot b)|}\binom{|At(\Delta,\vec{a}\cdot b)|}{k}2^k \tag{B.8}$$

In particular, we have in our example,

$$\begin{aligned}|At(\Delta,a)|&=2^{|\mathrm{PROP}(\kappa_\Delta(a))|}\cdot|\prod_{d\in A}\sum_{k=0}^{|At(\Delta,\vec{a}\cdot d)|}\binom{|At(\Delta,\vec{a}\cdot d)|}{k}2^k\\&=2^2\cdot\left(\sum_{k=0}^{2}\binom{2}{k}2^k\right)^2\\&=2^2\cdot\left(1\cdot 2^0+2\cdot 2^1+1\cdot 2^2\right)^2\\&=4\cdot(1+4+4)^2\\&=324\end{aligned}$$

and similarly:

$$\begin{aligned}|At(\Delta,\epsilon)|&=2^{|\mathrm{PROP}(\kappa_\Delta(\epsilon))|}\cdot|\prod_{d\in A}\sum_{k=0}^{|At(\Delta,\epsilon\cdot d)|}\binom{|At(\Delta,\epsilon\cdot d)|}{k}2^k\\&=2^2\cdot\left(\sum_{k=0}^{324}\binom{324}{k}2^k\right)\cdot\left(\sum_{k=0}^{2}\binom{2}{k}2^k\right)\cdot\left(\sum_{k=0}^{2}\binom{2}{k}2^k\right)\end{aligned}$$

E Illustrating the procedure used in the completeness proof

In this section, we illustrate the construction used in the proof of the completeness of the logic $\mathsf{L}_{(!\sigma,A)}$ by an example worked out through the series of figures E1 to E9.

Consider the following set $\Delta = \{\neg\langle a\rangle\neg p, \langle a\rangle q, [!\mathsf{N}]\langle a\rangle\neg q\}$.
We have $\kappa_\Delta(a) = \{p, \neg p, q, \neg q, [!\mathsf{N}]p, [!\mathsf{N}]\neg p, [!\mathsf{N}]q, [!\mathsf{N}]\neg q\}$ and we have

$$
\begin{aligned}
\kappa_\Delta(\epsilon) \equiv \quad & \kappa_\Delta(a) \\
& \cup\{\langle a\rangle p, \langle a\rangle q, \langle a\rangle\neg p, \langle a\rangle\neg q\} \\
& \cup\{\langle a\rangle(p\wedge q), \langle a\rangle(p\wedge\neg q), \langle a\rangle(\neg p\wedge q), \langle a\rangle(\neg p\wedge\neg q)\} \\
& \cup\{\neg\langle a\rangle p, \neg\langle a\rangle q, \neg\langle a\rangle\neg p, \neg\langle a\rangle\neg q\} \\
& \cup\{[!\mathsf{N}]\langle a\rangle p, [!\mathsf{N}]\langle a\rangle q, [!\mathsf{N}]\langle a\rangle\neg p, [!\mathsf{N}]\langle a\rangle\neg q\} \\
& \cup\{[!\mathsf{N}]\neg\langle a\rangle p, [!\mathsf{N}]\neg\langle a\rangle q, [!\mathsf{N}]\neg\langle a\rangle\neg p, [!\mathsf{N}]\neg\langle a\rangle\neg q\}
\end{aligned}
$$

We select as our first atom:

$$
\begin{aligned}
A_0 \equiv \quad & \{p, q\} \\
& \cup\{\langle a\rangle p, \langle a\rangle q, \neg\langle a\rangle\neg p, \langle a\rangle\neg q\} \\
& \cup\{\langle a\rangle(p\wedge q), \langle a\rangle(p\wedge\neg q), \neg\langle a\rangle(\neg p\wedge q), \neg\langle a\rangle(\neg p\wedge\neg q)\} \\
& \cup\{[!\mathsf{N}]\langle a\rangle p, [!\mathsf{N}]\neg\langle a\rangle q, [!\mathsf{N}]\neg\langle a\rangle\neg p, [!\mathsf{N}]\langle a\rangle\neg q\} \\
& \cup\{[!\mathsf{N}]\neg\langle a\rangle(p\wedge q), [!\mathsf{N}]\langle a\rangle(p\wedge\neg q), \\
& [!\mathsf{N}]\neg\langle a\rangle(\neg p\wedge q), [!\mathsf{N}]\neg\langle a\rangle(\neg p\wedge\neg q)\}
\end{aligned}
$$

$$
\begin{aligned}
& (w_0, \emptyset) \\
& \circ \\
\eta(w_0) = \epsilon, \delta(w_0, \emptyset) &= A_0 \\
m_0(w_0, \emptyset) &= 0 \\
m_1(w_0, \emptyset) &= 0
\end{aligned}
$$

FIGURE E1. Starting the $\mathsf{P}\sigma\mathsf{LTS}$ at its root.

$$
\begin{array}{ccc}
(w_0, \emptyset) & \xrightarrow{\quad !\mathsf{N} \quad} & (w_0, \{w_0\}) \\
\circ & & \circ \\
\eta(w_0) = \epsilon, \delta(w_0, \emptyset) = A_0 & & \eta(w_0) = \epsilon, \delta(w_0, \emptyset) = B_0 \\
m_0(w_0, \emptyset) = 0 & & m_0(w_0, \emptyset) = 0 \\
m_1(w_0, \emptyset) = 0 & & m_1(w_0, \emptyset) = 0 \\
 & & \langle a\rangle p, \langle a\rangle\neg q, \langle a\rangle(p \wedge \neg q)
\end{array}
$$

FIGURE E2. Adding an !N-successor to $(w_0, \emptyset)$.

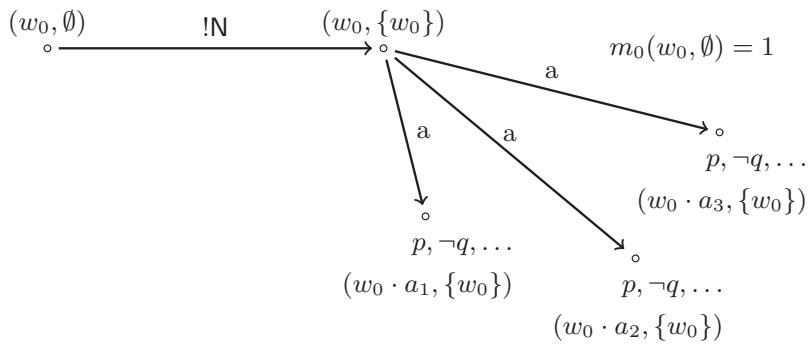

FIGURE E3. Adding $\xrightarrow{a}$-successors to $(w_0, \{w_0\})$.

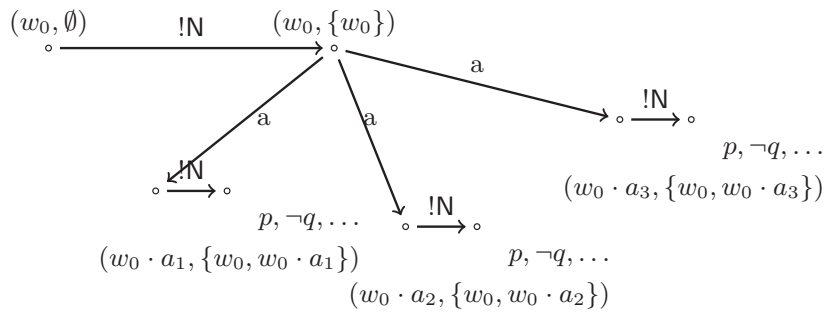

FIGURE E4. Adding !N-successors.

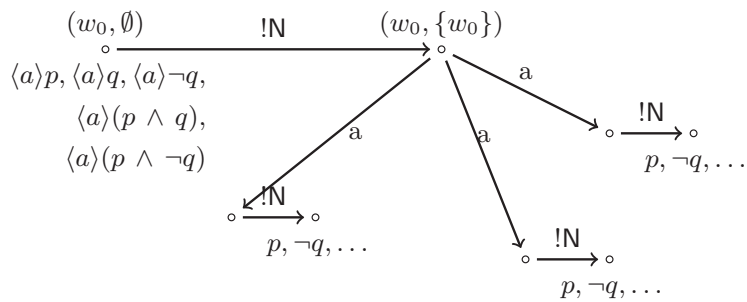

FIGURE E5. $(w_0, \emptyset)$ is now the rightmost m_0-unmarked context.

The unique atom $B_0 \in At(\Delta, \epsilon)$ such that $\widehat{A_0} \wedge [!\mathsf{N}]\widehat{B_0}$ is

$$
\begin{aligned}
B_0 \equiv \quad & \{p, q\} \\
& \cup \{\langle a \rangle p, \neg \langle a \rangle q, \neg \langle a \rangle \neg p, \langle a \rangle \neg q\} \\
& \cup \{\neg \langle a \rangle (p \wedge q), \langle a \rangle (p \wedge \neg q), \neg \langle a \rangle (\neg p \wedge q), \neg \langle a \rangle (\neg p \wedge \neg q)\} \\
& \cup \{[!\mathsf{N}] \langle a \rangle p, [!\mathsf{N}] \neg \langle a \rangle q, [!\mathsf{N}] \neg \langle a \rangle \neg p, [!\mathsf{N}] \langle a \rangle \neg q\} \\
& \cup \{[!\mathsf{N}] \neg \langle a \rangle (p \wedge q), [!\mathsf{N}] \langle a \rangle (p \wedge \neg q), \\
& [!\mathsf{N}] \neg \langle a \rangle (\neg p \wedge q), [!\mathsf{N}] \neg \langle a \rangle (\neg p \wedge \neg q)\}
\end{aligned}
$$

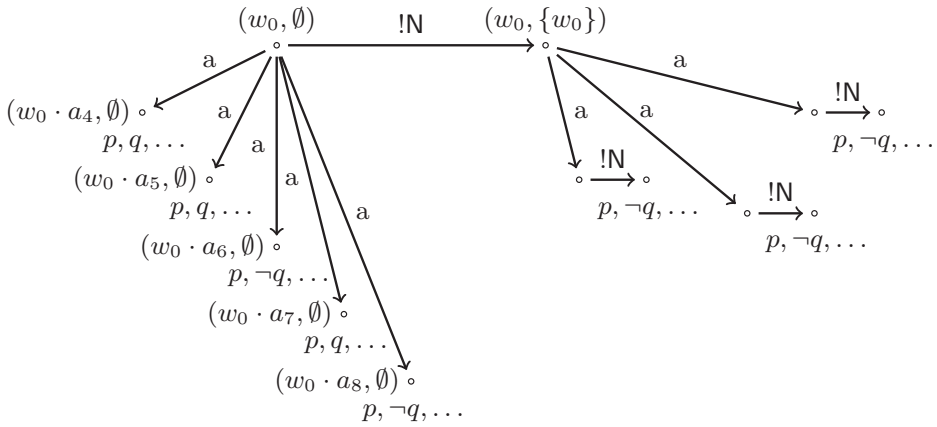

FIGURE E6. Adding $\xrightarrow{a}$-successors to $(w_0, \emptyset)$.

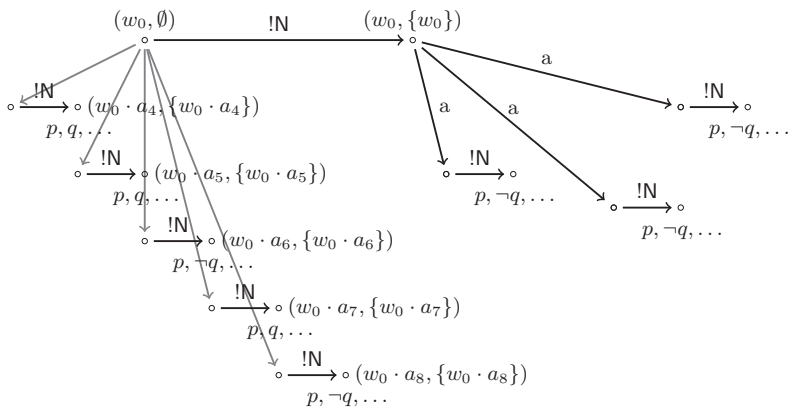

FIGURE E7. Adding !N-successors.

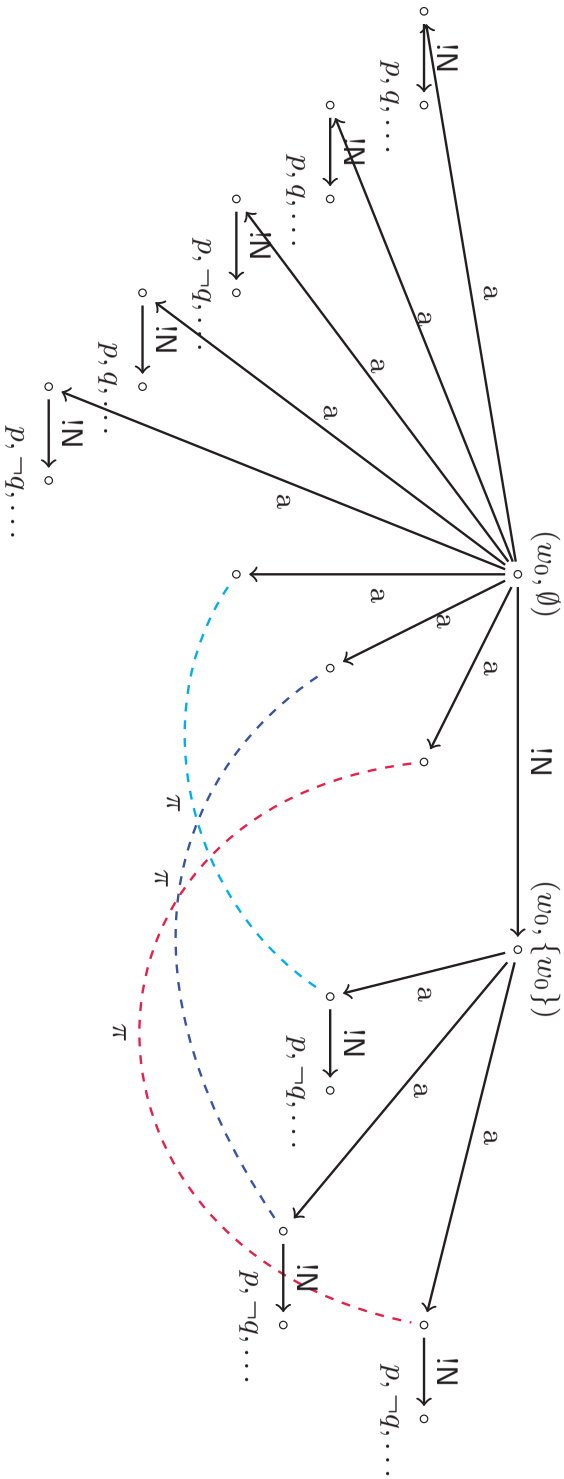

FIGURE E8. Adding π^{-1}-ancestors as a-successors of $(w_0, \emptyset)$.

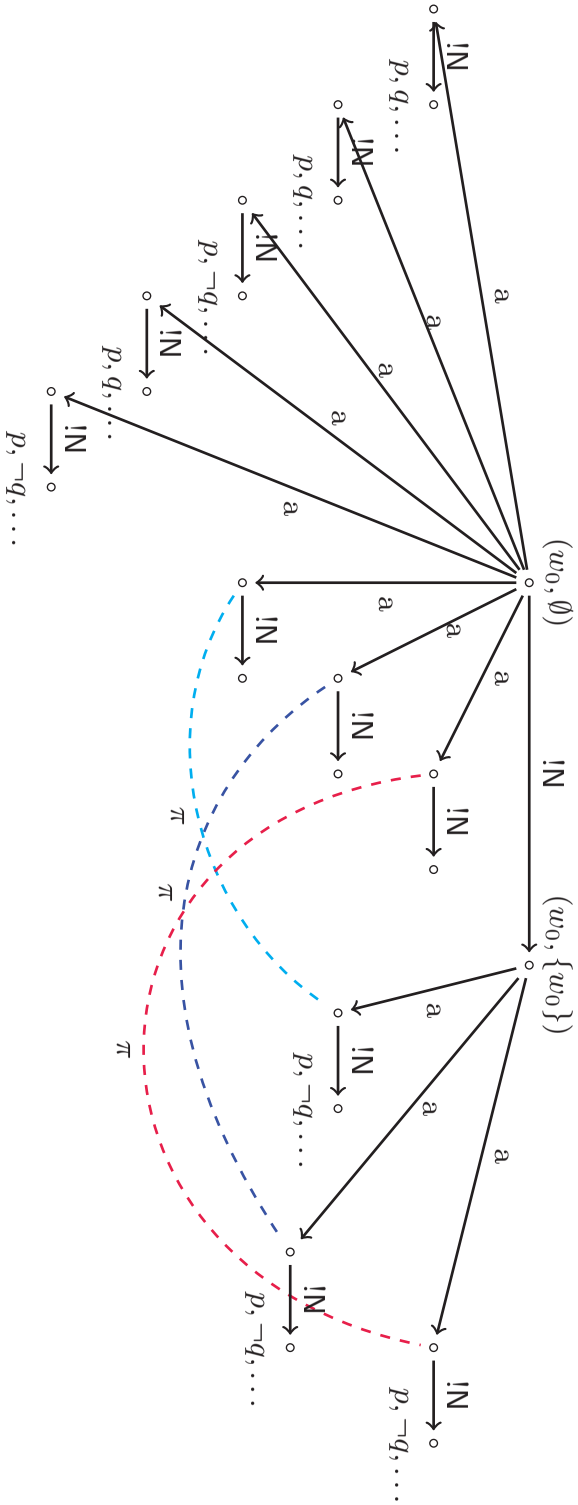

FIGURE E9. The PσLTS $\mathbb{M}^{\Delta}$ for $\Delta = \{\neg\langle a\rangle\neg p, \langle a\rangle q, [!\mathsf{N}]\langle a\rangle\neg q\}$.

The PσLTS $\mathbb{M}^{\Delta}$ is lastly gathered (Section 6.11) into a σLTS (Figure E10).

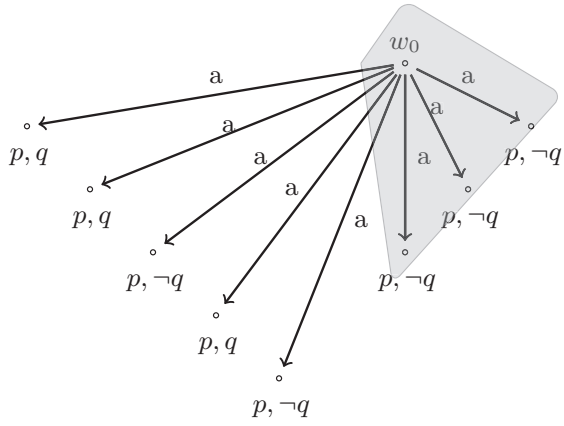

FIGURE E10. Gathering the PσLTS $\mathbb{M}^{\Delta}$ into a σLTS.