HAL
open science

Uniqueness for Neumann problems for nonlinear elliptic equations

Maria Francesca F Betta, Olivier Guibé, Anna Mercaldo

To cite this version:

Maria Francesca F Betta, Olivier Guibé, Anna Mercaldo. Uniqueness for Neumann problems for nonlinear elliptic equations. Communications in Pure and Applied Analysis, 2019, 10.3934/cpaa.2019050 . hal-01659249

HAL Id: hal-01659249

https://hal.science/hal-01659249

Submitted on 8 Dec 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

UNIQUENESS FOR NEUMANN PROBLEMS FOR NONLINEAR ELLIPTIC EQUATIONS

M.F. BETTA, O. GUIBÉ, AND A. MERCALDO

Abstract. In the present paper we prove uniqueness results for solutions to a class of Neumann boundary value problems whose prototype is

$$\begin{cases} -\mathrm{div}((1+|\nabla u|^2)^{(p-2)/2}\nabla u) - \mathrm{div}(c(x)|u|^{p-2}u) = f & \text{in } \Omega, \\ \big((1+|\nabla u|^2)^{(p-2)/2}\nabla u + c(x)|u|^{p-2}u\big)\cdot \underline{n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary, $1 < p < N$, $\underline{n}$ is the outer unit normal to $\partial\Omega$, the datum f belongs to $L^{(p^*)'}(\Omega)$ or to $L^1(\Omega)$ and satisfies the compatibility condition $\int_\Omega f\,dx = 0$. Finally the coefficient $c(x)$ belongs to an appropriate Lebesgue space.

Mathematics Subject Classification:MSC 2000 : 35J25
Key words: Nonlinear elliptic equations, Neumann problems, renormalized solutions, uniqueness results

1. Introduction

In the present paper we prove uniqueness results for solutions to a class of Neumann boundary value problems whose prototype is

$$\begin{cases} -\mathrm{div}((1+|\nabla u|^2)^{(p-2)/2}\nabla u) - \mathrm{div}(c(x)|u|^{p-2}u) = f & \text{in } \Omega, \\ \big((1+|\nabla u|^2)^{(p-2)/2}\nabla u + c(x)|u|^{p-2}u\big)\cdot \underline{n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary, $1 < p < N$, $\underline{n}$ is the outer unit normal to $\partial\Omega$, the datum f belongs to $L^{(p^*)'}(\Omega)$, where $p^* = \frac{Np}{N-p}$, or to $L^1(\Omega)$ and satisfies the compatibility condition

$$\int_\Omega f\,dx = 0.$$

Finally the coefficient $c(x)$ belongs to an appropriate Lebesgue space which will be specified later.

The main difficulties in studying existence or uniqueness for this type of problems are due to the presence of a lower order term, the lower summability of the datum f and the boundary Neumann conditions.

The existence for Neumann boundary value problems with L^1-data when $c = 0$ has been treated in various contests. In [3], [14], [19], [20] and [29] the existence of a distributional solution which belongs to a suitable Sobolev space and which has null mean value is proved. Nevertheless when p is close to 1, i.e. $p \leq 2 - 1/N$, the distributional solution to problem (1.1) does not belong to a Sobolev space and in general is not a summable function; this implies that its mean value has not meaning and any existence result for distributional solution with null mean value cannot hold. This difficulty is overcome in [18] by considering solutions u which are not in $L^1(\Omega)$, but for which $\Phi(u)$ is in $L^1(\Omega)$, where $\Phi(t) = \int_0^t \frac{ds}{(1+|s|)^\alpha}$ with appropriate $\alpha > 1$.
In [1] the case where both the datum f and the domain Ω are not smooth enough is studied and the existence and continuity with respect to the data of solutions whose median is equal to zero is proved with a natural process of approximations and symmetrization techniques.
We recall that the median of u is defined by

$$\mathrm{med}(u) = \sup\{t \in \mathbb{R} : \mathrm{meas}\{u > t\} > \mathrm{meas}(\Omega)/2\}\,. \tag{1.2}$$

The existence for solutions having null median to problem (1.1) when $c \neq 0$ are proved in [8].
We explicitly remark that when the datum f has a lower summability, i.e. it is just an L^1-function, one has to give a meaning to the notion of solution; such a question has been faced already in the case where Dirichlet boundary conditions are prescribed, by introducing different notion of solutions (cf. [7], [17], [27], [28]). Such notion turn out to be equivalent, at least when the datum is an L^1-function.
In the present paper, when $f \in L^1(\Omega)$, we refer to the so-called renormalized solutions (see [16], [27], [28]) whose precise definition is recalled in Section 2.

The main novelty of this article is to prove uniqueness (up to additive constants) results for renormalized solutions to problem (1.1) having null median and whose existence has been proved in [8].

To our knowledge uniqueness results for problem (1.1) are new even in the variational case, i.e. when f belongs to $L^{(p^*)'}(\Omega)$ and the usual notion of weak solution is considered.
When $c(x) = 0$ and f is an element of the dual space of the Sobolev space $W^{1,p}(\Omega)$, the existence and uniqueness (up to additive constants) of weak solutions to problem (1.1) is consequence of the classical theory of pseudo monotone operators (cfr. [25], [26]), while existence results for weak solutions to problem (1.1) when the lower order term appears have been proved in [8].

As pointed out we will prove different results according to the summability of f, i.e. $f \in L^{(p^*)'}(\Omega)$ or $f \in L^1(\Omega)$ and to the value of p, i.e. $p \leq 2$ and $p \geq 2$. As far as p is concerned such a difference is due to the principal part of the operator, which we consider. Actually we assume that the principal part $-\mathrm{div}(a(x, Du))$ is not degenerate when $p > 2$, i.e. in the model case $-\mathrm{div}(a(x, \nabla u)) = -\mathrm{div}((1 + |\nabla u|^2)^{(p-2)/2}\nabla u)$. But such an assumption is not required when $p \leq 2$, that is for such values of p we prove uniqueness results for operators whose prototype is the so-called p-Laplace operator, $-\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u)$.

Let us explain the main ideas of our results. To this aim let us consider the simpler case of weak solutions and $p = 2$. When a Dirichlet boundary value problem is considered, following an idea of Artola [4] (see also [13, 15]), denoted by u and v two solutions, one can use the test function $T_k(u - v)$ and obtain

$$\lim_{k\to 0} \frac{1}{k^2} \int_\Omega |\nabla T_k(u - v)|^2 \, dx = 0. \tag{1.3}$$

Since u, $v \in H^1_0(\Omega)$, Poincaré inequality implies that

$$\int_\Omega |\mathrm{sign}(u - v)|^2 \, dx = \lim_{k\to 0} \frac{1}{k^2} \int_\Omega |T_k(u - v)|^2 \, dx = 0 \,, \tag{1.4}$$

from which one can deduce that $u = v$ a.e. in Ω. In contrast when we consider Neumann boundary conditions and two solutions u, $v \in H^1(\Omega)$ having null median, by using $T_k(u - v)$ we can prove equality (1.3), but Poincaré-Wirtinger inequality does not allow to get

$$\int_\Omega |\mathrm{sign}(u - v)|^2 \, dx = 0$$

and therefore that $u = v$ a.e. in Ω. However (1.3) and Poincaré-Wirtinger inequality, allow to deduce that $u = v$ a.e. in Ω either $u > v$ a.e. in Ω either $u < v$ a.e. in Ω. Then we prove that $u > v$ a.e. in Ω or $u < v$ a.e. in Ω leads to a contradiction: it is done through a new test function

$$w_{k,\delta} = \frac{T_k(u - v)}{k} \left(\frac{T_\delta(u^+)}{\delta} - \frac{T_\delta(v^-)}{\delta} \right) ,$$

where T_k denote the truncate function at height k and

$$u^+ = \max\{0 \,, u\} \,, \qquad v^- = \max\{0 \,, -v\}.$$

Neumann problems have been studied by a different point of view in [21], [22], while existence or uniqueness results for Dirichlet boundary value problems for nonlinear elliptic equations with L^1-data are treated in [11],[12] and was continued in various contributions, including [2],

[6], [7], [9], [10], [16], [5] [17], [23], [24]; mixed boundary value problems have been also studied, for example, in [6], [19].

The paper is organized as follows. In Section 2 we detail the assumptions and we give the definition of a renormalized solution to (1.1). Section 3 is devoted to prove two uniqueness results for weak solutions when the datum is in the Lebesgue space $L^{(p^*)'}(\Omega)$. In Section 4 we state our main results, Theorem 4.1, Theorem 4.2, where we prove the uniqueness of a renormalized solution to (1.1) when datum is a L^1 function.

2. Assumptions and definitions

Let us consider the following nonlinear elliptic Neumann problem

$$\begin{cases} -\mathrm{div}\,(\mathbf{a}\,(x,\nabla u)+\Phi(x,u)) = f & \text{in } \Omega, \\ (\mathbf{a}\,(x,\nabla u)+\Phi(x,u))\cdot \underline{n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$, having finite Lebesgue measure and Lipschitz boundary, $\underline{n}$ is the outer unit normal to $\partial\Omega$. We assume that p is a real number such that $1 < p < N$. The function $\mathbf{a} : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}^N$ is a Carathéodory function such that

$$\mathbf{a}(x,\xi)\cdot\xi \geq \alpha|\xi|^p, \quad \alpha > 0, \tag{2.2}$$

$$|\,\mathbf{a}(x,\xi)| \leq c[|\xi|^{p-1} + a_0(x)], \quad c > 0, \quad a_0 \in L^{p'}(\Omega), \quad a_0 \geq 0, \tag{2.3}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$. Moreover $\mathbf{a}$ is strongly monotone, that is a constant $\beta > 0$ exists such that

$$(\mathbf{a}(x,\xi) - \mathbf{a}(x,\eta))\cdot(\xi-\eta) \geq \begin{cases} \beta\dfrac{|\xi-\eta|^2}{(|\xi|+|\eta|)^{2-p}} & \text{if } 1 < p \leq 2, \\ \beta|\xi-\eta|^2(1+|\xi|+|\eta|)^{p-2} & \text{if } p \geq 2, \end{cases} \tag{2.4}$$

for almost every $x \in \Omega$ and for every $\xi, \eta \in \mathbb{R}^N$, $\xi \neq \eta$.
We assume that $\Phi : \Omega \times \mathbb{R} \mapsto \mathbb{R}^N$ is a Carathéodory function which satisfies the following "growth condition"

$$|\Phi(x,s)| \leq c(x)(1+|s|)^{p-1}, \quad c \in L^t(\Omega),\ c \geq 0, \tag{2.5}$$

with

$$t \geq \frac{N}{p-1} \tag{2.6}$$

for a.e. $x \in \Omega$ and for every $s \in \mathbb{R}$. Moreover we assume that such function is locally Lipschitz continuous with respect to the second variable,

that is

$$|\Phi(x,s)-\Phi(x,z)| \le c(x)(1+|s|+|z|)^{\tau}|s-z|, \quad \tau \ge 0, \tag{2.7}$$

for almost every $x \in \Omega$, for every s, $z \in \mathbb{R}$.

Finally we assume that the datum f is a measurable function in a Lebesgue space $L^r(\Omega)$, $1 \le r \le +\infty$, which belongs to the dual space of the classical Sobolev space $W^{1,p}(\Omega)$ or is just an L^1- function. Moreover it satisfies the compatibility condition

$$\int_\Omega f\,dx = 0. \tag{2.8}$$

As explained in the Introduction we deal with solutions whose median is equal to zero. Let us recall that if u is a measurable function, we denote the median of u by

$$\mathrm{med}(u) = \sup\left\{t \in \mathbb{R} : \mathrm{meas}\{x \in \Omega : u(x) > t\} > \frac{\mathrm{meas}(\Omega)}{2}\right\}. \tag{2.9}$$

Let us explicitely observe that if $\mathrm{med}(u) = 0$ then

$$\mathrm{meas}\{x \in \Omega : u(x) > 0\} \le \frac{\mathrm{meas}(\Omega)}{2},$$
$$\mathrm{meas}\{x \in \Omega : u(x) < 0\} \le \frac{\mathrm{meas}(\Omega)}{2}.$$

In this case a Poincaré-Wirtinger inequality holds (see e.g. [30]):

Proposition 2.1. *If $u \in W^{1,p}(\Omega)$, then*

$$\|u - \mathrm{med}(u)\|_{L^p(\Omega)} \le C\|\nabla u\|_{(L^p(\Omega))^N} \tag{2.10}$$

where C is a constant depending on p, N, Ω.

When the datum f is not an element of the dual space of the classical Sobolev space $W^{1,p}(\Omega)$, the classical notion of weak solution does not fit. We will refer to the notion of renormalized solution to (2.1) (see [16, 28] for elliptic equations with Dirichlet boundary conditions) which we give below.

In the whole paper, T_k, $k \ge 0$, denotes the truncation at height k that is $T_k(s) = \min(k, \max(s, -k))$, $\forall s \in \mathbb{R}$.

Definition 2.2. A real function u defined in Ω is a renormalized solution to (2.1) if

$$u \text{ is measurable and finite almost everywhere in } \Omega, \tag{2.11}$$

$$T_k(u) \in W^{1,p}(\Omega), \text{ for any } k > 0, \tag{2.12}$$

$$\lim_{n\to+\infty} \frac{1}{n}\int_{\{x\in\Omega;\,|u(x)|<n\}} \mathbf{a}(x,\nabla u)\nabla u\,dx = 0 \tag{2.13}$$

and if for every function h belonging to $W^{1,\infty}(\mathbb{R})$ with compact support and for every $\varphi \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$, we have

$$\int_\Omega h(u)\,\mathbf{a}(x,\nabla u)\nabla\varphi\,dx + \int_\Omega h'(u)\,\mathbf{a}(x,\nabla u)\nabla u\varphi\,dx + \int_\Omega h(u)\Phi(x,u)\nabla\varphi\,dx + \int_\Omega h'(u)\Phi(x,u)\nabla u\varphi\,dx = \int_\Omega f\varphi h(u)\,dx. \tag{2.14}$$

Remark 2.3. A renormalized solution is not in general an $L^1_{loc}(\Omega)$-function and therefore it has not a distributional gradient. Condition (2.12) allows to define a generalized gradient of u according to Lemma 2.1 of [7], which asserts the existence of a unique measurable function v defined in Ω such that $\nabla T_k(u) = \chi_{\{|u|<k\}}v$ a.e. in Ω, $\forall k > 0$. This function v is the generalized gradient of u and it is denoted by ∇u.

Equality (2.14) is formally obtained by using in (2.1) the test function $\varphi h(u)$ and by taking into account Neumann boundary conditions. Actually in a standard way one can check that every term in (2.14) is well-defined under the structural assumptions on the elliptic operator.

Let us recall Theorem 4.1 of [8]; under assumptions (2.2)-(2.8) there exists at least one renormalized solution u having null median of problem (2.1). Moreover any renormalized solution to (2.1) verifies the following proposition

Proposition 2.4. *Under the assumptions* (2.2)-(2.8), *if u denotes any renormalized solution to problem* (2.1), *then*

$$\lim_{n\to+\infty} \frac{1}{n}\int_\Omega |\Phi(x,u)|\,|\nabla T_n(u)|dx = 0, \tag{2.15}$$

$$\int_\Omega \frac{|\nabla u|^p}{(1+|u|)^{1+m}}\,dx \le C\,, \qquad \forall m > 0, \tag{2.16}$$

where C is a positive constant depending only on m, f, Ω, α and Φ

$$|u|^{p-1} \in L^q(\Omega), \forall\, 1 < q < \frac{N}{N-p}, \tag{2.17}$$

$$|\nabla u|^{p-1} \in L^q(\Omega), \forall\, 1 < q < \frac{N}{N-1}. \tag{2.18}$$

Sketch of the proof. For the proof of (2.15) see Remark 2.4 of [8]. The estimate (2.16) is related to the Boccardo-Gallouët estimates [11], and it is obtained through a usual process. Indeed since $m > 0$,

$\displaystyle\int_0^r \frac{ds}{(1+|s|)^{1+m}} \in L^\infty(\mathbb{R}) \cap \mathcal{C}^1(\mathbb{R})$. Defining h_n by

$$h_n(s) = \begin{cases} 0 & \text{if } |s| > 2n, \\ \dfrac{2n-|s|}{n} & \text{if } n < |s| \le 2n, \\ 1 & \text{if } |s| \le n\,, \end{cases} \tag{2.19}$$

we can use the renormalized formulation (2.14) with $h = h_n$ and $\varphi = \displaystyle\int_0^{T_{2n}(u)} \frac{ds}{(1+|s|)^{1+m}}$. In view of (2.13) and (2.15), the growth condition (2.5) on Φ allows one to pass to the limit as $n \to +\infty$ and to obtain (2.16).

As far as (2.17) and (2.18) are concerned, it is sufficient to observe that (2.12), (2.16) and Poincaré-Wirtinger inequality imply (through an approximation process) that $\displaystyle\int_0^u \frac{ds}{(1+|s|)^{\frac{1+m}{p}}} \in W^{1,p}(\Omega)$. Then Sobolev embedding Theorem leads to

$$\forall m > 0,\ |u|^{\frac{p-(1+m)}{p}} \in L^{\frac{Np}{N-p}}(\Omega)$$

which is equivalent to

$$|u|^{p-1} \in L^q(\Omega),\ \forall\, 1 \le q < \frac{N}{N-p}.$$

Using again that $\displaystyle\int_0^u \frac{ds}{(1+|s|)^{\frac{1+m}{p}}} \in W^{1,p}(\Omega)$, (2.17) and Hölder inequality allow one to deduce (2.18).

3. Uniqueness results for weak solution

In this section we assume that the right-hand side f is an element of the dual space $L^{(p^*)'}(\Omega)$. In [8] an existence result for weak solution to problem (2.1) having null median has been proved. Such a weak solution u is a function such that

$$u \in W^{1,p}(\Omega),$$
$$\int_\Omega \mathbf{a}(x,\nabla u)\nabla v dx + \int_\Omega \Phi(x,u)\nabla v dx = \int_\Omega f v dx,$$

for any $v \in W^{1,p}(\Omega)$.

In this section we assume a suitable growth condition on Φ, that is a bound on τ in (2.7) is assumed and the following assumption on the datum is made

$$f \in L^{(p^*)'}(\Omega)\,. \tag{3.1}$$

Now we prove two uniqueness results depending on the values of p:

Theorem 3.1. *Let $1 < p < 2$. Assume that (2.2)–(2.7) with*

$$\tau \leq p - 1 \tag{3.2}$$

and (2.8), (3.1) hold. If u, v are two weak solutions to problem (2.1) having $\mathrm{med}(u) = \mathrm{med}(v) = 0$, then $u = v$ a.e. in Ω.

Theorem 3.2. *Let $p \geq 2$. Assume that (2.2)–(2.7) with*

$$\tau \leq \frac{Np}{N-p}\left(\frac{1}{2} - \frac{1}{t}\right), \tag{3.3}$$

$$t \geq \max\left\{2, \frac{N}{p-1}\right\} \tag{3.4}$$

and (2.8), (3.1) hold. If u, v are two weak solutions to problem (2.1) having $\mathrm{med}(u) = \mathrm{med}(v) = 0$, then $u = v$ a.e. in Ω.

Remark 3.3. We explicitely observe that if $2 \leq p \leq \frac{N+2}{2}$, we have uniqueness results under the assumption that c belongs to $L^{\frac{N}{p-1}}(\Omega)$ assumption which guarantees the existence of a solution. If $p > \frac{N+2}{2}$ the uniqueness result holds if c belongs to $L^2(\Omega)$, which means that uniqueness result holds under a stronger assumption on the summability of c.

Remark 3.4. Let us observe that the bounds on τ in the two theorems overlaps when $p = 2$.

Proof of Theorem 3.1. Since for every fixed $k > 0$, $T_k(u-v) \in W^{1,p}(\Omega)$, it can be used as test function in the equation satisfied by u and in the equation satisfied by v. Then by subtracting the two equations, we get

$$\begin{aligned} &\int_\Omega (\mathbf{a}(x, \nabla u) - \mathbf{a}(x, \nabla v)) \cdot \nabla T_k(u - v)\, dx \\ &\qquad + \int_\Omega (\Phi(x, u) - \Phi(x, v)) \cdot \nabla T_k(u - v)\, dx = 0\,. \end{aligned} \tag{3.5}$$

We proceed by dividing the proof by steps.

Step 1. We prove that

$$\lim_{k \to 0} \frac{1}{k^p} \int_\Omega |\nabla T_k(u - v)|^p\, dx = 0. \tag{3.6}$$

By the assumptions on the strong monotonicity on the operator (2.4) and the local Lipschitz condition on Φ (2.7) with τ which satisfies (3.2), we get

$$\beta \int_\Omega \frac{|\nabla T_k(u-v)|^2}{(|\nabla u|+|\nabla v|)^{2-p}}\,dx \le k\int_\Omega c(x)(1+|u|+|v|)^\tau |\nabla T_k(u-v)|\,dx\,. \tag{3.7}$$

The assumption on τ assures that the right-hand side of the previous inequality is finite. Moreover by Hölder inequality and assumption on τ, we obtain

$$\begin{aligned}\beta \int_\Omega \frac{|\nabla T_k(u-v)|^2}{(|\nabla u|+|\nabla v|)^{2-p}}\,dx &\le k\left(\int_{\{0<|u-v|<k\}} c(x)^2(1+|u|+|v|)^{2\tau}(|\nabla u|+|\nabla v|)^{2-p}\,dx\right)^{\frac12}\\ &\quad\times\left(\int_\Omega \frac{|\nabla T_k(u-v)|^2}{(|\nabla u|+|\nabla v|)^{2-p}}\,dx\right)^{\frac12}\end{aligned} \tag{3.8}$$

i.e.

$$\frac{\beta^2}{k^2}\int_\Omega \frac{|\nabla T_k(u-v)|^2}{(|\nabla u|+|\nabla v|)^{2-p}}\,dx \le \int_{\{0<|u-v|<k\}} c(x)^2(1+|u|+|v|)^{2\tau}(|\nabla u|+|\nabla v|)^{2-p}\,dx. \tag{3.9}$$

Since $\tau \le p-1 = (1-\frac{1}{p}-\frac{p-1}{N})\frac{Np}{N-p}$, Hölder inequality assures that the integral in the right-hand side is finite.
Since $\chi_{\{0<|u-v|<k\}}$ tends to 0 a.e. in Ω as k goes to 0, this implies

$$\lim_{k\to 0}\frac{1}{k^2}\int_\Omega \frac{|\nabla T_k(u-v)|^2}{(|\nabla u|+|\nabla v|)^{2-p}}\,dx = 0. \tag{3.10}$$

Moreover by Hölder inequality we get

$$\begin{aligned}\frac{1}{k^p}\int_\Omega |\nabla T_k(u-v)|^p\,dx &\le \left(\frac{1}{k^2}\int_\Omega \frac{|\nabla T_k(u-v)|^2}{(|\nabla u|+|\nabla v|)^{2-p}}\,dx\right)^{\frac p2}\left(\int_\Omega (|\nabla u|+|\nabla v|)^p\,dx\right)^{1-\frac p2}\end{aligned} \tag{3.11}$$

which implies (3.6) by (3.10).

Step 2. We prove that either

$$\begin{cases} u = v & \text{a.e. in } \Omega, \\ u < v & \text{a.e. in } \Omega, \\ u > v & \text{a.e. in } \Omega. \end{cases}$$

Since $\frac{T_k(u-v)}{k}$ belongs to $W^{1,p}(\Omega)$ Poincaré-Wirtinger inequality yields

$$\int_\Omega \left| \frac{T_k(u-v)}{k} - \text{med}\left(\frac{T_k(u-v)}{k}\right)\right|^p dx \le C \int_\Omega \left| \frac{\nabla T_k(u-v)|}{k}\right|^p dx\,. \tag{3.12}$$

Therefore, by Step 1, we deduce that

$$\lim_{k\to 0} \int_\Omega \left| \frac{T_k(u-v)}{k} - \text{med}\left(\frac{T_k(u-v)}{k}\right)\right|^p dx = 0. \tag{3.13}$$

Since $\left|\frac{T_k(u-v)}{k}\right| \le 1$, we obtain

$$\left|\text{med}\left(\frac{T_k(u-v)}{k}\right)\right| \le 1\,,\ k > 0$$

and, up to a subsequence, by (3.13)

$$\lim_{k\to 0} \text{med}\left(\frac{T_k(u-v)}{k}\right) = \gamma$$

for a suitable constant $\gamma \in \mathbb{R}$, $|\gamma| \le 1$. On the other hand, we have

$$\lim_{k\to 0} \frac{T_k(u-v)}{k} = \text{sign}\ (u-v)\,.$$

Therefore, up to subsequence, by (3.13) we get

$$\int_\Omega |\text{sign}\ (u-v) - \gamma|^p\, dx = 0$$

which implies

$$\gamma = 0 \qquad \text{or} \qquad \gamma = -1 \qquad \text{or} \qquad \gamma = 1\,.$$

This means that either

$$\begin{cases} u = v & \text{a.e. in } \Omega, \\ u < v & \text{a.e. in } \Omega, \\ u > v & \text{a.e. in } \Omega. \end{cases}$$

Step 3. We prove that $u < v$, a.e. in Ω or $u > v$, a.e. in Ω can not occur.

We assume that

$$u > v\,, \quad \text{a.e. in } \Omega \tag{3.14}$$

and we prove that this yields a contradiction. The same arguments prove that $u < v$ a.e. in Ω can not be verified.

Since $\mathrm{med}(v) = 0$, $\mathrm{meas}\{x \in \Omega \,:\, v(x) < 0\} \le \dfrac{\mathrm{meas}(\Omega)}{2}$, then

$$\mathrm{meas}\{x \in \Omega \,:\, v(x) \ge 0\} \ge \frac{\mathrm{meas}(\Omega)}{2}\,. \tag{3.15}$$

On the other hand, we have

$$\begin{aligned} &\{x \in \Omega \,:\, u(x) > 0\} \\ &= \{x \in \Omega \,:\, u(x) > 0\,, v(x) \ge 0\} \cup \{x \in \Omega \,:\, u(x) > 0\,, v(x) < 0\}. \end{aligned}$$

Since we assume (3.14), then we deduce

$$\{x \in \Omega \,:\, u(x) > 0\,, v(x) \ge 0\} = \{x \in \Omega \,:\, v(x) \ge 0\}$$

Therefore we get

$$\begin{aligned} &\{x \in \Omega \,:\, u(x) > 0\} \\ &= \{x \in \Omega \,:\, v(x) \ge 0\} \cup \{x \in \Omega \,:\, u(x) > 0,\, v(x) < 0\}. \end{aligned}$$

Moreover, since (3.15) holds true and since

$$\mathrm{meas}\{x \in \Omega \,:\, u(x) > 0\} \le \frac{\mathrm{meas}(\Omega)}{2}\,,$$

we conclude that

$$\mathrm{meas}\{x \in \Omega \,:\, u(x) > 0\,, v(x) < 0\} = 0\,.$$

This means that "u and v have the same sign".

Now let us consider the test function

$$w_{k,\delta} = \frac{T_k(u-v)}{k}\left(\frac{T_\delta(u^+)}{\delta} - \frac{T_\delta(v^-)}{\delta}\right)\,, \tag{3.16}$$

for fixed $k > 0$, $\delta > 0$, where

$$u^+ = \max\{0\,, u\}\,, \qquad v^- = \max\{0\,, -v\}.$$

Since $T_k(u-v) > 0$ a.e. in Ω, one can verify that

$$\{x \in \Omega \,:\, w_{k,\delta}(x) > 0\} = \{x \in \Omega \,:\, u(x) > 0\}$$

and

$$\{x \in \Omega \,:\, w_{k,\delta}(x) < 0\} = \{x \in \Omega \,:\, v(x) < 0\}.$$

Moreover $T_\delta(u^+)\,, T_\delta(v^-) \in W^{1,p}(\Omega)$ and hence, since $\mathrm{med}\,(u)\,, \mathrm{med}\,(v) = 0$, we conclude that

$$\mathrm{meas}\{x \in \Omega \,:\, w_{k,\delta}(x) > 0\} \le \frac{\mathrm{meas}(\Omega)}{2}\,,$$

$$\text{meas}\{x \in \Omega \,:\, w_{k,\delta}(x) < 0\} \le \frac{\text{meas}(\Omega)}{2}\,,$$

this means

$$\text{med}\,(w_{k,\delta}) = 0\,.$$

Therefore by Poincaré-Wirtinger inequality we deduce

$$\int_\Omega |w_{k,\delta}|^p\,dx \le C\int_\Omega |\nabla w_{k,\delta}|^p\,dx\,. \tag{3.17}$$

We now evaluate the gradient of $w_{k,\delta}$,

$$\begin{aligned}\nabla w_{k,\delta} &= \frac{\nabla T_k(u-v)}{k}\left(\frac{T_\delta(u^+)}{\delta} - \frac{T_\delta(v^-)}{\delta}\right)\\ &+\frac{T_k(u-v)}{k}\left(\frac{\nabla u}{\delta}\chi_{\{\,0<u<\delta\}} + \frac{\nabla v}{\delta}\chi_{\{\,-\delta<v<0\}}\right) \quad \text{a.e. in } \Omega.\end{aligned} \tag{3.18}$$

Since u and v "have the same sign", then, for every fixed $k > 0$, it results

$$0 < \frac{T_k(u-v)}{k\delta}\chi_{\{\,0<u<\delta\}} \le \frac{1}{k}\chi_{\{\,0<u<\delta\}}\,,$$

$$0 < \frac{T_k(u-v)}{k\delta}\chi_{\{\,-\delta<v<0\}} \le \frac{1}{k}\chi_{\{\,-\delta<v<0\}}\,,$$

then for fixed $k > 0$, we have

$$\lim_{\delta\to 0}\frac{T_k(u-v)}{k\delta}\chi_{\{\,0<u<\delta\}} = 0 \qquad \text{a.e. in } \Omega\,.$$

$$\lim_{\delta\to 0}\frac{T_k(u-v)}{k\delta}\chi_{\{\,-\delta<v<0\}} = 0 \qquad \text{a.e. in } \Omega\,.$$

Moreover we have also

$$\left|\frac{T_k(u-v)}{k\delta}\nabla u\chi_{\{\,0<u<\delta\}}\right| \le \frac{1}{k}|\nabla u|\chi_{\{\,0<u<\delta\}}\,,$$

$$\left|\frac{T_k(u-v)}{k\delta}\nabla v\chi_{\{\,-\delta<v<0\}}\right| \le \frac{1}{k}|\nabla v|\chi_{\{\,-\delta<v<0\}}\,,$$

and since $|\nabla u|, |\nabla v| \in L^p(\Omega)$, we can apply Lebesgue dominated convergence Theorem, i.e.

$$\lim_{\delta\to 0}\int_\Omega |\nabla w_{k,\delta}|^p\,dx = \int_\Omega \left|\frac{\nabla T_k(u-v)}{k}\left(\chi_{\{\,u>0\}} - \chi_{\{\,v<0\}}\right)\right|^p\,dx\,.$$

Since

$$\int_\Omega \left|\frac{\nabla T_k(u-v)}{k}\left(\chi_{\{\,u>0\}} - \chi_{\{\,v<0\}}\right)\right|^p\,dx \le \int_\Omega \left|\frac{\nabla T_k(u-v)}{k}\right|^p\,dx,$$

by Step 1, we conclude that

$$\lim_{k\to 0}\lim_{\delta\to 0}\int_\Omega |\nabla w_{k,\delta}|^p\,dx = 0\,.$$

Now we can pass to the limit in (3.17) as $\delta\to 0$ first and then as $k\to 0$ and we get

$$\int_\Omega \left|\text{sign }(u-v)\left(\chi_{\{\,u>0\}}-\chi_{\{\,v<0\}}\right)\right|^p\,dx = \int_\Omega |\text{sign}(u)|^p dx = 0\,.$$

We deduce that $\chi_{\{\,u>0\}} = \chi_{\{\,v<0\}}$ a.e. in Ω; this yields a contradiction since we have proved that u and v have the same sign.

The same arguments yield that we can not have $u<v$ a.e. in Ω. The conclusion follows. □

Proof of Theorem 3.2. As in the previous proof we arrive to equality (3.5) and we divide the proof by 3 steps.
Step 1. We prove that

$$\lim_{k\to 0}\frac{1}{k^2}\int_\Omega |\nabla T_k(u-v)|^2\,dx = 0. \tag{3.19}$$

By the assumptions on the strong monotonicity on the operator (2.4) and the local Lipschitz condition on Φ (2.7) with τ which satisfies (3.3), we get

$$\begin{aligned}\beta\int_\Omega (1+|\nabla u|+|\nabla v|)^{p-2}|\nabla T_k(u-v)|^2\,dx \\ \le k\int_\Omega c(x)(1+|u|+|v|)^\tau|\nabla T_k(u-v)|\,dx\,.\end{aligned} \tag{3.20}$$

Since $p\ge 2$, $T_k(u-v)$ belongs to $W^{1,2}(\Omega)$. Then by Hölder inequality and assumption on τ, we obtain

$$\begin{aligned}\beta\int_\Omega |\nabla T_k(u-v)|^2\,dx \\ \le k\|c\|_{L^t(\{0<|u-v|<k\})}\|1+|u|+|v|\|^\tau_{L^{p^*}}\|\nabla T_k(u-v)\|_{L^2}\,dx\,,\end{aligned} \tag{3.21}$$

i.e.

$$\frac{\beta^2}{k^2}\int_\Omega |\nabla T_k(u-v)|^2\,dx \le \|c\|^2_{L^t(\{0<|u-v|<k\})}\|1+|u|+|v|\|^{2\tau}_{L^{p^*}}\,. \tag{3.22}$$

Since $\chi_{\{x\,:\,0<|u-v|<k\}}\to 0$ a.e. in Ω, Lebesgue dominated convergence theorem implies that (3.19) holds.

Step 2. We prove that either

$$\begin{cases} u = v & \text{a.e. in } \Omega, \\ u < v & \text{a.e. in } \Omega, \\ u > v & \text{a.e. in } \Omega. \end{cases}$$

By Poincaré-Wirtinger inequality, we get

$$\int_\Omega \left| \frac{T_k(u-v)}{k} - \text{med}\left(\frac{T_k(u-v)}{k}\right) \right|^2 dx \leq C \int_\Omega \left| \frac{\nabla T_k(u-v)|}{k} \right|^2 dx\,. \tag{3.23}$$

Therefore, by Step 1. we deduce that

$$\lim_{k\to 0} \int_\Omega \left| \frac{T_k(u-v)}{k} - \text{med}\left(\frac{T_k(u-v)}{k}\right) \right|^2 dx = 0 \tag{3.24}$$

Since $\left|\frac{T_k(u-v)}{k}\right| \leq 1$, we obtain

$$\left| \text{med}\left(\frac{T_k(u-v)}{k}\right) \right| \leq 1\,,$$

and, up to a subsequence,

$$\lim_{k\to 0} \text{med}\left(\frac{T_k(u-v)}{k}\right) = \gamma$$

for a suitable constant $\gamma \in \mathbb{R}$, $|\gamma| \leq 1$. On the other hand, we have

$$\lim_{k\to 0} \frac{T_k(u-v)}{k} = \text{sign}\ (u-v)\,,$$

Therefore, up to subsequence, we get

$$\int_\Omega |\text{sign}\ (u-v) - \gamma|^2\, dx = 0$$

which implies

$$\gamma = 0 \qquad \text{or} \qquad \gamma = -1 \qquad \text{or} \qquad \gamma = 1\,.$$

This means that either

$$u = v\,, \quad \text{a.e. in } \Omega \quad \text{or} \quad u < v\,, \quad \text{a.e. in } \Omega \quad \text{or} \quad u > v\,, \quad \text{a.e. in } \Omega\,.$$

Step 3. Arguing as in Step 3 of the previous theorem, we prove that the last two possibilities can not occur. Then conclusion follows. □

Remark 3.5. In [8] we estabilished the existence of a weak solution when $\mathbf{a}(x,\xi)$ is replaced by a Leray-Lions operator $\mathbf{a}(x,r,\xi)$ which depends on x, r and ξ and verifies the standard conditions (see [25]). In the Dirichlet case and $1 < p \leq 2$ it is well known (see [13] [15]) that under suitable assumptions on $\mathbf{a}(x,r,\xi)$ the weak solution is unique. In view of the proofs of Theorem 3.1 and Theorem 3.2 it is possible to obtain the uniqueness of the weak solution having null median of the problem

$$\begin{cases} -\mathrm{div}\,(\mathbf{a}\,(x,u,\nabla u)+\Phi(x,u)) = f & \text{in } \Omega, \\ (\mathbf{a}\,(x,u,\nabla u)+\Phi(x,u))\cdot \underline{n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.25}$$

If we assume that $\mathbf{a}(x,r,\xi)$ is a Carathéodory function which verifies

$$\mathbf{a}(x,s,\xi)\cdot\xi \geq \alpha|\xi|^p, \quad \alpha > 0, \tag{3.26}$$

$$|\,\mathbf{a}(x,s,\xi)| \leq c_1[|\xi|^{p-1}+|s|^{p-1}+a_0(x)], \tag{3.27}$$

$$c_1 > 0,\ a_0 \in L^{p'}(\Omega),\ a_0 \geq 0,$$

(3.28)

$$(\mathbf{a}(x,s,\xi)-\mathbf{a}(x,s,\eta))\cdot(\xi-\eta) \geq \begin{cases} \beta\dfrac{|\xi-\eta|^2}{(|\xi|+|\eta|)^{2-p}} & \text{if } 1 \leq p \leq 2, \\ \beta|\xi-\eta|^2(1+|\xi|+|\eta|)^{p-2} & \text{if } p \geq 2, \end{cases}$$

and moreover $\mathbf{a}(x,r,\xi)$ satisfies a Lipschitz condition with respect to r

(3.29)

$$|\,\mathbf{a}(x,s,\xi)-\mathbf{a}(x,r,\xi)| \leq c_2|s-r|(|\xi|^{p-1}+|s|^{p-1}+|r|^{p-1}+h(x)),$$

$$c_2 > 0,\ h \in L^{p'}(\Omega),\ h \geq 0,$$

for almost every $x \in \Omega$, $s \in \mathbb{R}$ and for every $\xi \in \mathbb{R}^N$, then Theorem 3.1 and Theorem 3.2 hold true. Indeed the methods developped in [13] allow one to prove *Step 1* in Theorem 3.1 namely

$$\lim_{k\to 0}\frac{1}{k^p}\int_\Omega |\nabla T_k(u-v)|^p\,dx = 0.$$

and *Step 1* in Theorem 3.2 namely

$$\lim_{k\to 0}\frac{1}{k^2}\int_\Omega |\nabla T_k(u-v)|^2\,dx = 0.$$

In both cases the *Step 2* and *Step 3* remain unchanged.

Remark 3.6. In [8] and in the present paper we have chosen to deal with solutions to (2.1) with null median value instead of null mean value. As explained in Introduction this choice allows one to consider solution to (2.1) for $f \in L^1(\Omega)$ even if the solution u does not belong

to $L^1(\Omega)$. When $f \in L^{(p^*)'}(\Omega)$ a simply examination of the proof of [8] leads to the existence of solutions to (2.1) such that $\displaystyle\int_\Omega u\,dx = 0$. Assuming that (2.2)–(2.7) are in force similar arguments to the one developped in the proof of Theorem 3.1 and Theorem 3.2 yield the uniqueness of solution to (2.1) having a null mean value. Let us explain briefly the case $p = 2$. *Step 1* remains unchanged so that if u and v are two solutions of (2.1) then we have

$$\lim_{k\to 0} \frac{1}{k^2} \int_\Omega |\nabla T_k(u-v)|^2\,dx = 0.$$

Poincaré-Wirtinger inequality leads to

$$\lim_{k\to 0} \int_\Omega \left| \frac{T_k(u-v)}{k} - \frac{1}{|\Omega|} \int_\Omega \frac{T_k(u-v)}{k} dy \right|^2 dx = 0.$$

so that, up to subsequence, there exists $\gamma \in [-1, 1]$ such that

$$\int_\Omega |\text{sign }(u-v) - \gamma|^2\,dx = 0.$$

As in *Step 2*

$$\gamma = 0 \qquad \text{or} \qquad \gamma = -1 \qquad \text{or} \qquad \gamma = 1$$

and

$$\begin{cases} u = v & \text{a.e. in } \Omega, \\ u < v & \text{a.e. in } \Omega, \\ u > v & \text{a.e. in } \Omega. \end{cases}$$

We now show that $u < v$ a.e. in Ω or $u > v$ a.e. in Ω can not occur. The method is similar to *Step 3* of the proof of Theorem 3.1:

$$w_{k,\delta} = \frac{T_k(u-v)}{k} \left(\frac{T_\delta(u^+)}{\delta} - \frac{T_\delta(v^-)}{\delta} \right),$$

belongs to $H^1(\Omega)$ while

$$\lim_{k\to 0} \lim_{\delta\to 0} \|\nabla w_{k,\delta}\|_{L^2(\Omega)} = 0.$$

Poincaré-Wirtinger inequality yields

$$\lim_{k\to 0} \lim_{\delta\to 0} \int_\Omega \left| w_{k,\delta} - \frac{1}{|\Omega|} \int_\Omega w_{k,\delta} dy \right|^2 dx = 0.$$

In the case $u > v$ a.e. in Ω, the Lebesque dominated Theorem allows one to conclude that

$$\int_\Omega \left| \text{sign}(u) - \frac{1}{|\Omega|} \int_\Omega \text{sign}(u) dy \right|^2 dx = 0\,,$$

and then u has a constant sign. Recalling that $\int_\Omega u\,dx = \int_\Omega v\,dx = 0$ gives a contradiction. Therefore $u = v$ a.e. in Ω.

4. Uniqueness result for renormalized solution

In this section we prove the uniqueness of the renormalized solution to problem (2.1), when the following assumption on datum is made

$$f \in L^1(\Omega)\,. \tag{4.1}$$

As in Section 3 we state two uniqueness theorems depending on the values of p:

Theorem 4.1. *Let $1 < p < 2$. Assume that (2.2)–(2.7) with*

$$\tau \le p - \frac{3}{2} + \left[\frac{p-1}{N} - \frac{1}{t}\right]\frac{N(p-1)}{N-p} \tag{4.2}$$

and (2.8), (4.1) hold. If u, v are two renormalized solutions to problem (2.1) having $\mathrm{med}(u) = \mathrm{med}(v) = 0$, then $u = v$ a.e. in Ω.

Theorem 4.2. *Let $p \ge 2$. Assume that (2.2)–(2.7) with*

$$\tau \le \frac{N(p-1)}{N-p}\left(\frac{1}{2} - \frac{1}{t}\right) \tag{4.3}$$

$$t \ge \max\left\{2, \frac{N}{p-1}\right\} \tag{4.4}$$

and (2.8), (4.1) hold. If u, v are two renormalized solutions to problem (2.1) having $\mathrm{med}(u) = \mathrm{med}(v) = 0$, then $u = v$ a.e. in Ω.

Proof of Theorem 4.1. Let u and v be two renormalized solutions to (2.1). Let h_n defined by (2.19). Since for any $k > 0$, $h_n(u)T_k(u-v) = h_n(v)T_k(T_{2n}(u) - T_{2n+k}(u)) \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$, we can use $h = h_n(u)$ and $\varphi = h_n(v)T_k(u-v)$ in (2.14) written in u, and we can use $h = h_n(v)$ and $\varphi = h_n(u)T_k(u-v)$ in (2.14) written in v. By substracting the two equations, we get

$$\begin{aligned}
&\int_\Omega h_n(u)h_n(v)(\mathbf{a}(x,\nabla u) - \mathbf{a}(x,\nabla v)) \cdot \nabla T_k(u-v)\,dx \\
&\quad + \int_\Omega h_n(u)h_n(v)(\Phi(x,u) - \Phi(x,v)) \cdot \nabla T_k(u-v)\,dx \\
&+ \int_\Omega h_n'(u)h_n(v)T_k(u-v)(a(x,\nabla u) + \Phi(x,u) - a(x,\nabla v) - \Phi(x,v)) \cdot \nabla u\,dx \\
&+ \int_\Omega h_n(u)h_n'(v)T_k(u-v)(a(x,\nabla u) + \Phi(x,u) - a(x,\nabla v) - \Phi(x,v)) \cdot \nabla v\,dx = 0\,.
\end{aligned} \tag{4.5}$$

We proceed by dividing the proof into 3 steps.

Step 1. By passing to the limit in (4.5) first as $n \to +\infty$, then as $k \to 0$ this step is to devoted to prove that

$$\lim_{k\to 0} \frac{1}{k^2} \int_\Omega \frac{|\nabla T_k(u-v)|^2}{(|\nabla u| + |\nabla v|)^{2-p}}\, dx = 0. \tag{4.6}$$

We first study the behaviour of the last two integrals in (4.5) as n goes to $+\infty$ by showing

$$\lim_{n\to+\infty} \int_\Omega h_n'(u) h_n(v) T_k(u-v)(\mathbf{a}(x,\nabla u)+\Phi(x,u)-\mathbf{a}(x,\nabla v)-\Phi(x,v))\cdot\nabla u\, dx = 0 \tag{4.7}$$

and by symmetry with respect to u and v

$$\lim_{n\to+\infty} \int_\Omega h_n'(v) h_n(u) T_k(u-v)(\mathbf{a}(x,\nabla u)+\Phi(x,u)-\mathbf{a}(x,\nabla v)-\Phi(x,v))\cdot\nabla v\, dx = 0. \tag{4.8}$$

By (2.13) of Definition 2.2 and (2.15) of Proposition 2.4, we get

$$\lim_{n\to+\infty} \int_\Omega h_n'(u) h_n(v) T_k(u-v)\, \mathbf{a}(x,\nabla u)\cdot\nabla u\, dx = 0, \tag{4.9}$$

$$\lim_{n\to+\infty} \int_\Omega h_n'(u) h_n(v) T_k(u-v) \Phi(x,u)\cdot\nabla u\, dx = 0. \tag{4.10}$$

By assumption (2.3) and Hölder inequality we have

$$\left| \int_\Omega h_n'(u) h_n(v) T_k(u-v)\, \mathbf{a}(x,\nabla v)\cdot\nabla u\, dx \right|$$
$$\le \frac{ck}{n} \left(\int_{\{|v|\le 2n\}} (|a_0(x)| + |\nabla v|^{p-1})^{\frac{p}{p-1}}\, dx \right)^{\frac{p-1}{p}} \left(\int_{\{|u|\le 2n\}} |\nabla u|^p\, dx \right)^{\frac{1}{p}}.$$

Using (2.2) and (2.13) we deduce that

$$\lim_{n\to+\infty} \frac{1}{n} \int_{\{|u|\le 2n\}} |\nabla u|^p\, dx = 0. \tag{4.11}$$

Therefore recalling that $a_0 \in L^{p'}(\Omega)$ we conclude that

$$\lim_{n\to+\infty} \left| \int_\Omega h_n'(u) h_n(v) T_k(u-v)\, \mathbf{a}(x,\nabla v)\cdot\nabla u\, dx \right| = 0. \tag{4.12}$$

To prove that (4.7) holds it remains to control $\int_\Omega h_n'(u)h_n(v)T_k(u-v)\Phi(x,v)\cdot\nabla u\,dx$. By assumption (2.5) and Hölder inequality we have

$$\left|\int_\Omega h_n'(u)h_n(v)T_k(u-v)\Phi(x,v)\cdot\nabla u\,dx\right|$$
$$\leq k\left(\frac{1}{n}\int_{\{|v|\leq 2n\}}|c(x)|^{\frac{p}{p-1}}(1+|v|)^p\,dx\right)^{\frac{p-1}{p}}\left(\frac{1}{n}\int_{\{|u|\leq 2n\}}|\nabla u|^p\,dx\right)^{\frac{1}{p}}.$$

Since $c\in L^t(\Omega)$ with $t\geq N/(p-1)$ (see Assumption (2.6)) we get

$$\int_{\{|v|\leq 2n\}}|c(x)|^{\frac{p}{p-1}}(1+|v|)^p\leq\Big(\int_\Omega|c|^{N/(p-1)}\,dx\Big)^{p/N}\|1+T_{2n}(v)\|^p_{L^{p^*}(\Omega)}$$

and Poincaré-Wirtinger inequality leads to

$$\int_{\{|v|\leq 2n\}}|c(x)|^{\frac{p}{p-1}}(1+|v|)^p\leq C\Big(\int_\Omega|c|^{N/(p-1)}\,dx\Big)^{p/N}\Big(1+\int_\Omega|\nabla T_{2n}(v)|^p\,dx\Big),$$

where $C>0$ is independent of n and k. It follows that

$$\left|\int_\Omega h_n'(u)h_n(v)T_k(u-v)\Phi(x,v)\cdot\nabla u\,dx\right|\leq Ck\|c\|_{L^{N/(p-1)}(\Omega)}$$
$$\times\Big(\frac{1}{n}+\frac{1}{n}\int_{\{|v|\leq 2n\}}|\nabla v|^p\,dx\Big)^{\frac{p-1}{p}}\Big(\frac{1}{n}\int_{\{|u|\leq 2n\}}|\nabla u|^p\,dx\Big)^{\frac{1}{p}}.$$

Therefore (4.11) leads to

$$\lim_{n\to+\infty}\left|\int_\Omega h_n'(u)h_n(v)T_k(u-v)\Phi(x,v)\cdot\nabla u\,dx\right|=0, \tag{4.13}$$

and then (4.7) holds. We observe that (4.8) is obtained by analogous argument.

Then by (4.5), (4.7), (4.8), using the assumptions on the strong monotonicity on the operator (2.4), the local Lipschitz condition on Φ (2.7) with τ which satisfies (4.2) and Young inequality we get

$$\beta\int_\Omega h_n(u)h_n(v)\frac{|\nabla T_k(u-v)|^2}{(|\nabla u|+|\nabla v|)^{2-p}}\,dx\leq 2\omega_k(n) \tag{4.14}$$
$$+\frac{2k^2}{\beta}\int_{\{0<|u-v|<k\}}h_n(u)h_n(v)|c(x)|^2(1+|u|+|v|)^{2\tau}(|\nabla u|+|\nabla v|)^{2-p}\,dx,$$

where $\lim_n\omega_k(n)=0$.

We now prove that

$$|c(x)|^2(1+|u|+|v|)^{2\tau}(|\nabla u|+|\nabla v|)^{2-p}\in L^1(\Omega), \tag{4.15}$$

so that we can pass to the limit in (4.14) as $n \to +\infty$. By Hölder inequality we get

$$\begin{aligned}
(4.16)\qquad & \int_\Omega h_n(u)h_n(v)|c(x)|^2(1+|u|+|v|)^{2\tau}(|\nabla u|+|\nabla v|)^{2-p}\,dx \\
&\le \left(\int_{\{|u|<2n,\,|v|<2n\}} h_n(u)h_n(v)|c(x)|^t\,dx\right)^{\frac{2}{t}} \\
&\times\left(\int_{\{|u|<2n,\,|v|<2n\}} h_n(u)h_n(v)(1+|u|+|v|)^{\nu}\,dx\right)^{\frac{2\tau}{\nu}} \\
&\times\left(\int_{\{|u|<2n,\,|v|<2n\}} h_n(u)h_n(v)(|\nabla u|+|\nabla v|)^{\mu}\,dx\right)^{\frac{2-p}{\mu}}
\end{aligned}$$

with

$$(4.17)\qquad \frac{1}{t}+\frac{2\tau}{\nu}+\frac{2-p}{\mu}\le 1,\ \nu<\frac{N(p-1)}{N-p},\ \mu<\frac{N(p-1)}{N-1},$$

This choice is possible since (4.2) holds and in view of (2.17) and (2.18) of Proposition 2.4 we have

$$(1+|u|+|v|)^{\nu}\in L^1(\Omega),\quad (|\nabla u|+|\nabla v|)^{\mu}\in L^1(\Omega).$$

Passing to the limit as n goes to $+\infty$, assumption (2.6) on c and Fatou Lemma yield that

$$|c(x)|^2(1+|u|+|v|)^{2\tau}(|\nabla u|+|\nabla v|)^{2-p}\in L^1(\Omega).$$

Then we can pass to the limit as $n \to +\infty$ in (4.14), and dividing (4.14) by k^2 and using Fatou Lemma we get

$$\begin{aligned}
(4.18)\qquad & \frac{1}{k^2}\int_\Omega \frac{|\nabla T_k(u-v)|^2}{(|\nabla u|+|\nabla v|)^{2-p}}\,dx \\
&\le \frac{1}{\beta^2}\int_{\{0<|u-v|<k\}} |c(x)|^2(1+|u|+|v|)^{2\tau}(|\nabla u|+|\nabla v|)^{2-p}\,dx\,.
\end{aligned}$$

Recalling that $\chi_{\{0<|u-v|<k\}}$ converges to 0 a.e. as k goes to zero, Lebesgue dominated Theorem and (4.15) allow one to conclude that (4.6) holds.

Step 2. We prove that either

$$\begin{cases} u=v & \text{a.e. in } \Omega,\\ u<v & \text{a.e. in } \Omega,\\ u>v & \text{a.e. in } \Omega.\end{cases}$$

Observe that for $k < n$

$$h_n(u)\frac{T_k(u-v)}{k} = h_n(u)\frac{T_k(T_{3n}(u)-T_{3n}(v))}{k} \in L^\infty(\Omega)\cap W^{1,p}(\Omega).$$

Then by Poincaré-Wirtinger inequality, we get

$$\int_\Omega \left| h_n(u)\frac{T_k(u-v)}{k} - \operatorname{med}\left(h_n(u)\frac{T_k(u-v)}{k}\right)\right|^p dx \tag{4.19}$$

$$\leq C\int_\Omega \left|\nabla\left(h_n(u)\frac{T_k(u-v)}{k}\right)\right|^p dx.$$

Let us evaluate the integral at the right-hand side. We show that it goes to zero first as $k \to 0$ and then as $n \to +\infty$.

Since

$$\nabla\left(h_n(u)\frac{T_k(u-v)}{k}\right) \tag{4.20}$$

$$= h_n'(u)\nabla u\frac{T_k(u-v)}{k} + h_n(u)\frac{\nabla T_k(u-v)}{k} \quad \text{a.e. in } \Omega,$$

$\left|\frac{T_k(u-v)}{k}\right| \leq 1$ and $h_n'(u) \leq \frac{1}{n}$, we get

$$\int_\Omega \left|\nabla\left(h_n(u)\frac{T_k(u-v)}{k}\right)\right|^p dx \tag{4.21}$$

$$\leq \frac{1}{n^p}\int_\Omega |\nabla T_{2n}(u)|^p\,dx + \frac{1}{k^p}\int_\Omega h_n(u)^p\,|\nabla T_k(u-v)|^p\,dx\,.$$

Let us evaluate the second integral in the right hand side of (4.21). By Hölder inequality we obtain

$$\frac{1}{k^p}\int_\Omega h_n(u)^p\,|\nabla T_k(u-v)|^p\,dx$$

$$\leq \left(\frac{1}{k^2}\int_\Omega \frac{|\nabla T_k(u-v)|^2}{(|\nabla u|+|\nabla v|)^{2-p}}\,dx\right)^{\frac{p}{2}}\left(\int_\Omega h_n(u)^{\frac{2p}{2-p}}\chi_{\{|u-v|<k\}}(|\nabla u|+|\nabla v|)^p\,dx\right)^{\frac{2-p}{2}}$$

$$\leq C\left(\frac{1}{k^2}\int_\Omega \frac{|\nabla T_k(u-v)|^2}{(|\nabla u|+|\nabla v|)^{2-p}}\,dx\right)^{\frac{p}{2}}\left(\int_\Omega (|\nabla T_{2n}(u)|^p + |\nabla T_{2n+k}(v)|^p)\,dx\right)^{\frac{2-p}{2}},$$

that is, if n is fixed,

$$\frac{1}{k^p}\int_\Omega h_n(u)^p\,|\nabla T_k(u-v)|^p\;dx \le C_n\left(\frac{1}{k^2}\int_\Omega \frac{|\nabla T_k(u-v)|^2}{(|\nabla u|+|\nabla v|)^{2-p}}\,dx\right)^{\frac{p}{2}},$$

where $C_n>0$ is a constant depending on n (and independent of k). Therefore, by Step 1, we deduce that

$$\lim_{k\to 0}\frac{1}{k^p}\int_\Omega h_n(u)^p\,|\nabla T_k(u-v)|^p\;dx = 0\,. \tag{4.22}$$

Since $\left|\dfrac{T_k(u-v)}{k}\right|$ converges to $\operatorname{sign}(u-v)$ in $L^\infty(\Omega)$ weak-$*$, we deduce from (4.20) and (4.22) that for fixed n, as $k\to 0$

$$\nabla\left(h_n(u)\frac{T_k(u-v)}{k}\right)\longrightarrow h_n'(u)\nabla u\operatorname{sign}(u-v)\,,\ \text{in } (L^p(\Omega))^N.$$

We now pass to the limit as $n\to+\infty$. By the definition of h_n we have

$$\int_\Omega |h_n'(u)\nabla u|^p\,dx \le \frac{1}{n^p}\int_\Omega |\nabla T_{2n}(u)|^p\,dx$$

so that (2.13) and (4.22) lead to

$$\lim_{n\to+\infty}\lim_{k\to 0}\int_\Omega\left|\nabla\left(h_n(u)\frac{T_k(u-v)}{k}\right)\right|^p\;dx = 0.$$

Therefore using (4.19), we deduce that

$$\lim_{n\to+\infty}\lim_{k\to 0}\int_\Omega\left|h_n(u)\frac{T_k(u-v)}{k}-\operatorname{med}\left(h_n(u)\frac{T_k(u-v)}{k}\right)\right|^p\;dx = 0. \tag{4.23}$$

Since $\left|h_n(u)\frac{T_k(u-v)}{k}\right|\le 1$, we obtain

$$\left|\operatorname{med}\left(h_n(u)\frac{T_k(u-v)}{k}\right)\right|\le 1\,,\ k>0.$$

It follows that, up to a subsequence

$$\lim_{n\to+\infty}\lim_{k\to 0}\operatorname{med}\left(h_n(u)\frac{T_k(u-v)}{k}\right)=\gamma\,.$$

for a suitable constant $\gamma\in\mathbb{R}$, $|\gamma|\le 1$.

On the other hand since u is finite a.e. we have

$$\lim_{k\to 0} h_n(u)\frac{T_k(u-v)}{k} = h_n(u)\operatorname{sign}(u-v)\,,\quad \text{a.e. and } L^\infty(\Omega)\text{ weak-}*,$$

$$\lim_{n\to+\infty} h_n(u)\operatorname{sign}(u-v) = \operatorname{sign}(u-v)\,,\quad \text{a.e. and } L^\infty(\Omega)\text{ weak-}*.$$

Then, up to subsequence, by (4.23) we get

$$\int_\Omega |\text{sign }(u-v)-\gamma|^p\,dx = 0.$$

This implies

$$\gamma = 0 \qquad \text{or} \qquad \gamma = -1 \qquad \text{or} \qquad \gamma = 1,$$

and means that either

$$u = v\,,\ \text{a.e. in } \Omega \text{ or } \ u < v\,,\ \text{a.e. in } \Omega \text{ or } \ u > v\,,\ \text{a.e. in } \Omega\,.$$

Step 3. We prove that $u < v$, a.e. in Ω or $u > v$, a.e. in Ω can not occur.
We assume that

$$u > v\,, \quad \text{a.e. in } \Omega \tag{4.24}$$

and we prove that this yields a contradiction.

The arguments used in Step 3 of Theorem 3.1 allow us to prove that

"u and v have the same sign".

Let us consider the test function

$$w_{n,k,\delta} = h_n(u)\frac{T_k(u-v)}{k}\left(\frac{T_\delta(u^+)}{\delta} - \frac{T_\delta(v^-)}{\delta}\right), \tag{4.25}$$

for fixed $n > 0$, $k > 0$, $\delta > 0$, where

$$u^+ = \max\{0\,,u\}\,, \qquad v^- = \max\{0\,,-v\}.$$

Observe that, since for $k < n$ $h_n(u)\frac{T_k(u-v)}{k} \in L^\infty(\Omega)\cap W^{1,p}(\Omega)$ we have

$$w_{n,k,\delta} \in L^\infty(\Omega)\cap W^{1,p}(\Omega).$$

We now evaluate the gradient of $w_{n,k,\delta}$:

$$\begin{aligned}\nabla w_{n,k,\delta} &= \nabla\left(h_n(u)\frac{T_k(u-v)}{k}\right)\left(\frac{T_\delta(u^+)}{\delta} - \frac{T_\delta(v^-)}{\delta}\right)\\ &+h_n(u)\frac{T_k(u-v)}{k}\left(\frac{\nabla u}{\delta}\chi_{\{\,0<u<\delta\}} - \frac{\nabla v}{\delta}\chi_{\{\,-\delta<v<0\}}\right) \ \text{a.e. in } \Omega.\end{aligned} \tag{4.26}$$

and we study the limit as $\delta \to 0$, $k \to 0$ and then $n \to +\infty$. We firstly show that $\text{med}(w_{n,k,\delta}) = 0$. Let η such that $0 < \eta < \frac{1}{2}$.

$$\begin{aligned}\{x\in\Omega\,:\,w_{n,k,\delta}(x) > \eta\} &= \{x\in\Omega\,:\,w_{n,k,\delta}(x) > \eta,\ 0 < u < 2n\}\\ &\subset \left\{x\in\Omega\,:\,\frac{T_\delta(u^+)}{\delta} > \eta\right\} = \{x\in\Omega\,:\,u^+ > \eta\delta\}.\end{aligned}$$

Since $\mathrm{med}(u) = 0$, we have

$$\mathrm{meas}\{x \in \Omega \,:\, u(x) > \eta\delta\} < \frac{\mathrm{meas}(\Omega)}{2}\,.$$

It follows that $\forall \eta < \frac{1}{2}$,

$$\mathrm{meas}\{x \in \Omega \,:\, w_{n,k,\delta}(x) > \eta\} < \frac{\mathrm{meas}(\Omega)}{2}\,,$$

which means $\mathrm{med}(w_{n,k,\delta}) \leq 0$.

On the other hand since

$$\forall \eta > 0\,,\ \{x \in \Omega \,:\, w_{n,k,\delta}(x) > -\eta\} \supset \{x \in \Omega \,:\, u \geq 0\}$$

and

$$\mathrm{meas}\{x \in \Omega \,:\, u \geq 0\} \geq \frac{\mathrm{meas}(\Omega)}{2}\,,$$

we deduce that

$$\mathrm{meas}\{x \in \Omega \,:\, w_{n,k,\delta}(x) > -\eta\} \geq \frac{\mathrm{meas}(\Omega)}{2}\,,\ \forall \eta > 0,$$

which means $\mathrm{med}(w_{n,k,\delta}) \geq 0$. We can conclude that

$$\mathrm{med}(w_{n,k,\delta}) = 0.$$

Then from Poincaré-Wirtinger inequality, by using (4.21) and (4.26), we obtain

$$\begin{aligned}\int_\Omega |w_{n,k,\delta}|^p\,dx &\leq C\int_\Omega |\nabla w_{n,k,\delta}|^p\,dx \\ &\leq C\Big\{\frac{1}{n^p}\int_\Omega |\nabla T_{2n}(u)|^p\,dx + \frac{1}{k^p}\int_\Omega h_n^p(u)\,|\nabla T_k(u-v)|^p\,dx \\ &+\frac{1}{\delta^p k^p}\int_\Omega h_n^p(u)\,|T_k(u-v)|^p\,\big(\big|\nabla T_\delta(u^+)\big| + \big|\nabla T_\delta(v^-)\big|\big)^p\,dx\Big\}\,.\end{aligned} \tag{4.27}$$

We now prove that

$$\lim_{n\to+\infty}\frac{1}{n^p}\int_\Omega |\nabla T_{2n}(u)|^p\,dx = 0, \tag{4.28}$$

$$\lim_{k\to 0}\frac{1}{k^p}\int_\Omega h_n^p(u)\,|\nabla T_k(u-v)|^p\,dx = 0, \tag{4.29}$$

$$\lim_{\delta\to 0}\frac{1}{\delta^p k^p}\int_\Omega h_n^p(u)\,|T_k(u-v)|^p\,\big|\nabla T_\delta(u^+)\big|^p\,dx = 0, \tag{4.30}$$

$$\lim_{\delta\to 0}\frac{1}{\delta^p k^p}\int_\Omega h_n^p(u)\,|T_k(u-v)|^p\,\big|\nabla T_\delta(v^-)\big|^p\,dx = 0. \tag{4.31}$$

Clearly (4.28) is a consequence of (2.13) in Definition 2.2. As far as (4.29) is concerned, by Hölder inequality we have

$$\frac{1}{k^p}\int_\Omega h_n^p(u)\,|\nabla T_k(u-v)|^p\,dx \le \left(\frac{1}{k^2}\int_{\{0<|u-v|<k\}}\frac{|\nabla u-\nabla v|^2}{(|\nabla u|+|\nabla v|)^{2-p}}dx\right)^{\frac{p}{2}}$$
$$\times\left(\int_{\{0<|u-v|<k\}} h_n^{\frac{2p}{2-p}}(u)(|\nabla u|+|\nabla v|)^p dx\right)^{\frac{1}{2}}$$

and in view of the definition of h_n, if n is fixed, for any $k<1$ we have

$$\begin{aligned}\int_{\{0<|u-v|<k\}} & h_n^{\frac{2p}{2-p}}(u)(|\nabla u|+|\nabla v|)^p dx\\ &\le \int_\Omega (|\nabla T_{2n}(u)|+|\nabla T_{2n+1}(v)|)^p dx.\\ &\le C_n,\end{aligned}$$

where $C_n>0$ is a constant depending on n (and independent of k). From (4.6) it follows that for any fixed $n>0$ (4.29) holds.

We now turn to (4.30) and (4.31). Observe that

$$\frac{1}{k^p\delta^p}|T_k(u-v)|^p\left|\nabla T_\delta(u^+)\right|^p = \frac{1}{k^p\delta^p}\,|T_k(u-v)|^p\,|\nabla u|^p\,\chi_{\{\,0<u<\delta\}}$$

a.e. in Ω. Since $u>v$ a.e. in Ω and meas$\{x\in\Omega\,:\,u>0,\,v<0\}=0$, we get

$$|T_k(u-v)|^p\,\chi_{\{0<u<\delta\}}\le\delta^p,$$

and then

$$\frac{1}{k^p\delta^p}|T_k(u-v)|^p\left|\nabla T_\delta(u^+)\right|^p\le\frac{1}{k^p}\,|\nabla u|^p\,\chi_{\{0<u<\delta\}}.$$

The Lebesgue dominated Theorem gives for fixed $k>0$,

$$\frac{1}{k}|\nabla u|\chi_{\{0<u<\delta\}}\to 0\qquad\text{strongly in } L^p(\Omega),\ \text{as } \delta\to 0.$$

We deduce (4.30). In analogous way we get (4.31).

By collecting (4.30), (4.31), (4.29), (4.28) and (4.27) we can conclude that

$$\lim_{n\to+\infty}\lim_{k\to 0}\lim_{\delta\to 0}\int_\Omega |w_{n,k,\delta}|^p\,dx=0\,,$$

which gives, via Lebesgue dominated Theorem,

$$\left|\text{sign }(u-v)\left(\chi_{\{u>0\}}-\chi_{\{v<0\}}\right)\right|=0\,.$$

This implies that $\chi_{\{u>0\}}=\chi_{\{v<0\}}$ a.e. in Ω; this yields a contradiction since we have proved that u and v have the same sign.

The same arguments yield that we can not have $u<v$ a.e. in Ω. The conclusion follows.

□

Proof of Theorem 4.2. Arguing as in the previous theorem we obtain (4.5) and we proceed by dividing the proof by steps. The main difference with respect to the proof of Theorem 4.1 is that for $p > 2$ we have to control quadratic terms in $u - v$ (see (4.32)) while u and v are solutions to a p-growth problem.

Step 1. By passing to the limit in (4.5) first as $n \to +\infty$, then as $k \to 0$ this step is to devoted to prove that

$$\lim_{k\to 0} \frac{1}{k^2} \int_\Omega |\nabla T_k(u-v)|^2 \, dx = 0. \tag{4.32}$$

We pass to the limit in (4.5) first as $n \to +\infty$, then as $k \to 0$. Arguing as in Step 1 of the previous theorem we get that

$$\begin{aligned}
&\lim_{n\to+\infty} \int_\Omega h_n'(u) h_n(v) T_k(u-v) \\
&\qquad \times (\mathbf{a}(x,\nabla u) + \Phi(x,u) - \mathbf{a}(x,\nabla v) - \Phi(x,v)) \cdot \nabla u \, dx = 0 \\
&\lim_{n\to+\infty} \int_\Omega h_n(u) h_n'(v) T_k(u-v) \\
&\qquad \times (\mathbf{a}(x,\nabla u) + \Phi(x,u) - \mathbf{a}(x,\nabla v) - \Phi(x,v)) \cdot \nabla v \, dx = 0\,.
\end{aligned}$$

Then, using the assumptions on the strong monotonicity on the operator (2.4), the local Lipschitz condition on Φ (2.7) with τ which satisfies (3.3) and Young inequality we get

$$\begin{aligned}
&\frac{\beta}{2} \int_\Omega h_n(u) h_n(v) (1 + |\nabla u| + |\nabla v|)^{p-2} |\nabla T_k(u-v)|^2 \, dx \\
&\le \omega_k(n) + \frac{k^2}{2\beta} \int_{\{0<|u-v|<k\}} h_n(u) h_n(v) |c(x)|^2 (1 + |u| + |v|)^{2\tau} \, dx,
\end{aligned}$$

where $\lim_n \omega_k(n) = 0$. We then obtain

$$\begin{aligned}
\frac{\beta}{2} \int_\Omega h_n(u) h_n(v) |\nabla T_k(u-v)|^2 \, dx &\le \omega_k(n) \\
&+ \frac{k^2}{2\beta} \int_{\{0<|u-v|<k\}} h_n(u) h_n(v) |c(x)|^2 (1 + |u| + |v|)^{2\tau} \, dx.
\end{aligned} \tag{4.33}$$

By Hölder inequality and assumptions on the data we get

$$
\begin{aligned}
(4.34)\qquad & \int_\Omega h_n(u)h_n(v)|c(x)|^2(1+|u|+|v|)^{2\tau}\,dx \\
&\le \left(\int_{\{x\in\Omega:|u|<2n,\,|v|<2n\}} h_n(u)h_n(v)|c(x)|^t\,dx\right)^{\frac{2}{t}} \\
&\quad\times\left(\int_{\{x\in\Omega:|u|<2n,\,|v|<2n\}} h_n(u)h_n(v)(1+|u|+|v|)^{\nu}\,dx\right)^{\frac{t-2}{2t}}
\end{aligned}
$$

where $\nu=\frac{2t\tau}{t-2}$. According to the assumption on τ we have

$$
\frac{2t\tau}{t-2}<\frac{N(p-1)}{N-p}
$$

which implies $(1+|u|+|v|)^{\nu}\in L^1(\Omega)$. Making use of Fatou Lemma and (4.34) we obtain

$$
|c(x)|^2(1+|u|+|v|)^{2\tau}\in L^1(\Omega). \tag{4.35}
$$

We can pass to the limit as $n\to+\infty$ in (4.33), then using Fatou Lemma we get

$$
\begin{aligned}
(4.36)\qquad & \frac{1}{k^2}\int_\Omega |\nabla T_k(u-v)|^2\,dx \\
&\qquad\le \frac{1}{\beta^2}\int_{\{0<|u-v|<k\}} |c(x)|^2(1+|u|+|v|)^{2\tau}\,dx\,.
\end{aligned}
$$

Recalling that $\chi_{\{0<|u-v|<k\}}$ converges to 0 a.e. as k goes to zero Lebesgue dominated Theorem and (4.35) allow one to conclude that (4.32) holds.

Step 2. We prove that either

$$
\begin{cases} u=v & \text{a.e. in } \Omega,\\ u<v & \text{a.e. in } \Omega,\\ u>v & \text{a.e. in } \Omega.\end{cases}
$$

Let us consider the function $h_n(u)\frac{T_k(u-v)}{k}$ and observe that for $k<n$

$$
h_n(u)\frac{T_k(u-v)}{k}=h_n(u)\frac{T_k(T_{3n}(u)-T_{3n}(v))}{k}\in L^\infty(\Omega)\cap W^{1,p}(\Omega).
$$

Since $p \geq 2$ the function $h_n(u)\frac{T_k(u-v)}{k}$ belongs to $H^1(\Omega)$ by Poincaré-Wirtinger inequality we get

$$\int_\Omega \left| h_n(u)\frac{T_k(u-v)}{k} - \operatorname{med}\big(h_n(u)\frac{T_k(u-v)}{k}\big)\right|^2 dx \leq C\int_\Omega \left|\nabla\left(h_n(u)\frac{T_k(u-v)}{k}\right)\right|^2 dx. \tag{4.37}$$

Let us evaluate the integral at the right-hand side. We show that it goes to zero first as $k \to 0$ then as $n \to +\infty$.

Since

$$\nabla\left(h_n(u)\frac{T_k(u-v)}{k}\right) = h_n'(u)\nabla u\frac{T_k(u-v)}{k} + h_n(u)\frac{\nabla T_k(u-v)}{k} \quad \text{a.e. in } \Omega$$

and $\left|\frac{T_k(u-v)}{k}\right| \leq 1$ we get

$$\int_\Omega \left|\nabla\left(h_n(u)\frac{T_k(u-v)}{k}\right)\right|^2 dx \tag{4.38}$$

$$\leq \int_\Omega |h_n'(u)\nabla u|^2\, dx + \frac{1}{k^2}\int_\Omega h_n(u)^2\, |\nabla T_k(u-v)|^2\, dx\,.$$

It is easy to verify that for fixed n, as $k \to 0$

$$\nabla\left(h_n(u)\frac{T_k(u-v)}{k}\right) \longrightarrow h_n'(u)\nabla u\,\operatorname{sign}(u-v)\,, \text{ in } (L^2(\Omega))^N$$

Moreover by the definition of h_n

$$\int_\Omega |h_n'(u)\nabla u|^2\, dx \leq \frac{1}{n^2}\int_\Omega |\nabla T_{2n}(u)|^2\, dx$$

so that (2.13), (4.32) and (4.38) lead to

$$\lim_{n\to+\infty}\lim_{k\to 0}\int_\Omega \left|\nabla\left(h_n(u)\frac{T_k(u-v)}{k}\right)\right|^2 dx = 0\,.$$

Then, using (4.37), we deduce

$$\lim_{n\to+\infty}\lim_{k\to 0}\int_\Omega \left| h_n(u)\frac{T_k(u-v)}{k} - \operatorname{med}\left(h_n(u)\frac{T_k(u-v)}{k}\right)\right|^2 dx = 0. \tag{4.39}$$

Since $\left|h_n(u)\frac{T_k(u-v)}{k}\right| \le 1$, we obtain

$$\left|\text{med}\left(h_n(u)\frac{T_k(u-v)}{k}\right)\right| \le 1\,,\ \ k>0.$$

It follows that, up to a subsequence, by (4.39)

$$\lim_{n\to+\infty}\lim_{k\to 0}\text{med}\left(h_n(u)\frac{T_k(u-v)}{k}\right) = \gamma\,.$$

for a suitable constant $\gamma \in \mathbb{R}$, $|\gamma| \le 1$.

On the other hand since u is finite a.e.

$$\lim_{k\to 0} h_n(u)\frac{T_k(u-v)}{k} = h_n(u)\,\text{sign}(u-v)\,, \quad \text{a.e. and } L^\infty(\Omega) \text{ weak-}*,$$

$$\lim_{n\to+\infty} h_n(u)\,\text{sign}(u-v) = \text{sign}(u-v)\,, \quad \text{a.e. and } L^\infty(\Omega) \text{ weak-}*$$

Then, up to subsequence, by (4.39) we get

$$\int_\Omega |\text{sign }(u-v)-\gamma|^2\,dx = 0$$

This implies

$$\gamma = 0 \qquad \text{or} \qquad \gamma = -1 \qquad \text{or} \qquad \gamma = 1\,.$$

This means that either

$$u = v\,,\ \text{a.e. in } \Omega \text{ or } \ u < v\,,\ \text{a.e. in } \Omega \text{ or } \ u > v\,,\ \text{a.e. in } \Omega\,.$$

Arguing as in Step 3 of the previous theorem, we can prove that the last two possibilities can not occur. Then conclusion follows.

□

Remark 4.3. As in the case of weak solutions, the existence of renormalized solutions hold for a class of more general problems (3.25) where f belongs to $L^1(\Omega)$, Φ verifies growth conditions and $\mathbf{a}(x,r,\xi)$ is a Leray-Lions operator which depends on x, s and ξ (see [8]).
Due to the lack of regularity of u in the L^1 case by using the techniques developped in the present paper it seems not possible to obtain uniqueness result when $\mathbf{a}$ verifies (3.26)-(3.29). Let us explain the main obstacle in the case $p=2$ and what kind of stronger assumptions on $\mathbf{a}$ insures the uniqueness of the renormalized solution. In view of the proof of Theorem 4.1 and Theorem 4.2 the only new difficulty when $\mathbf{a}$ depends on x, r, ξ is to prove *Step 1* which is when $p=2$

$$\lim_{k\to 0}\frac{1}{k^2}\int_\Omega |\nabla T_k(u-v)|^2\,dx = 0. \tag{4.40}$$

In *Step 2* and *Step 3* the structure of the operator does not play any role. Equation (4.5) in which we pass to the limit first as $n \to +\infty$ and then as $k \to 0$ to derive (4.40) becomes

$$
\begin{aligned}
(4.41)\qquad & \int_\Omega h_n(u)h_n(v)(\mathbf{a}(x,u,\nabla u)-\mathbf{a}(x,v,\nabla v))\cdot\nabla T_k(u-v)\,dx \\
& + \int_\Omega h_n(u)h_n(v)(\Phi(x,u)-\Phi(x,v))\cdot\nabla T_k(u-v)\,dx \\
& + \int_\Omega h_n'(u)h_n(v)T_k(u-v)(\mathbf{a}(x,u,\nabla u) \\
& \qquad\qquad + \Phi(x,u)-\mathbf{a}(x,v,\nabla v)-\Phi(x,v))\cdot\nabla u\,dx \\
& + \int_\Omega h_n(u)h_n'(v)T_k(u-v)(\mathbf{a}(x,u,\nabla u) \\
& \qquad\qquad + \Phi(x,u)-\mathbf{a}(x,v,\nabla v)-\Phi(x,v))\cdot\nabla v\,dx = 0\,.
\end{aligned}
$$

Since the operator is pseudo-monotone the main obstacle is the control of the first term of (4.41).

$$
\begin{aligned}
& \int_\Omega h_n(u)h_n(v)(\mathbf{a}(x,u,\nabla u)-\mathbf{a}(x,v,\nabla v))\cdot\nabla T_k(u-v)\,dx \\
= & \int_\Omega h_n(u)h_n(v)(\mathbf{a}(x,u,\nabla u)-\mathbf{a}(x,u,\nabla v))\cdot\nabla T_k(u-v)\,dx \\
+ & \int_\Omega h_n(u)h_n(v)(\mathbf{a}(x,u,\nabla v)-\mathbf{a}(x,v,\nabla v))\cdot\nabla T_k(u-v)\,dx \\
& \geq \beta\int_\Omega h_n(u)h_n(v)\,|\nabla T_k(u-v)|^2\,dx \\
+ & \int_\Omega h_n(u)h_n(v)(\mathbf{a}(x,u,\nabla v)-\mathbf{a}(x,v,\nabla v))\cdot\nabla T_k(u-v)\,dx \\
& \geq \frac{\beta}{2}\int_\Omega h_n(u)h_n(v)\,|\nabla T_k(u-v)|^2\,dx \\
- & \int_{\{|u-v|<k\}} h_n(u)h_n(v)\,|\mathbf{a}(x,u,\nabla v)-\mathbf{a}(x,v,\nabla v))|^2\,dx.
\end{aligned}
$$

Passing first as $n \to +\infty$ and then as $k \to 0$ requires to have $\chi_{\{0<|u-v|<k\}}\,|\mathbf{a}(x,u,\nabla v)-\mathbf{a}(x,v,\nabla v))|^2 \in L^1(\Omega)$. If $\mathbf{a}$ verifies

$$
|\,\mathbf{a}(x,s,\xi)-\mathbf{a}(x,r,\xi)| \leq |s-r||\xi|,
$$

then

$$
\chi_{\{0<|u-v|<k\}}\,|\mathbf{a}(x,u,\nabla v)-\mathbf{a}(x,v,\nabla v))|^2 \leq k^2|\nabla v|^2
$$

and we cannot expect to have $|\nabla v|^2 \in L^1(\Omega)$ for L^1 data. However by assuming a stronger control of the Lipschitz coefficient of $\mathbf{a}(x,\,r,\,\xi)$

with respect to r, namely

$$|\,\mathbf{a}(x,s,\xi)-\mathbf{a}(x,r,\xi)| \le \frac{|s-r|}{(1+|s|+|r|)^{\lambda}}|\xi|,$$

with $\lambda > \frac{1}{2}$, we have

$$\chi_{\{0<|u-v|<k\}}\,|\mathbf{a}(x,u,\nabla v)-\mathbf{a}(x,v,\nabla v))|^2 \le k^2\frac{|\nabla v|^2}{(1+|v|)^{2\lambda}}$$

and since $2\lambda > 1$, estimate (2.16) implies that

$$\chi_{\{0<|u-v|<k\}}\,|\mathbf{a}(x,u,\nabla u)-\mathbf{a}(x,v,\nabla v))|^2 \in L^1(\Omega).$$

It follows that

$$\chi_{\{0<|u-v|<k\}}\frac{1}{k^2}\,|a(x,u,\nabla v)-a(x,v,\nabla v))|^2 \to 0,\ \text{in } L^1(\Omega).$$

Since the other terms in (4.41) can be controlled by similar methods to the one used in Theorem 4.2 we are able to conclude that (4.40) holds and then that $u = v$ a.e. in Ω.

We now give the complete version of Theorem 4.1 and 4.2 for problem (3.25). As in the weak case we assume that $\mathbf{a}(x,r,\xi)$ is a Carathéodory function which verifies (3.26), (3.27), (3.28), $f \in L^1(\Omega)$ and Φ verifies (2.5) and (2.7).

When $1 < p < 2$, if $\tau \le p - \frac{3}{2}$ and if $\mathbf{a}(x,r,\xi)$ satisfies

(4.42)

$$|\,\mathbf{a}(x,s,\xi)-\mathbf{a}(x,r,\xi)| \le C\frac{|s-r|}{(1+|s|+|r|)^{\lambda}}\left(|\xi|^{p-1}+|s|^{p-1}+|r|^{p-1}+h(x)\right),$$

with $\lambda > \frac{1}{2}$, $h \ge 0$ and $h \in L^{p'}(\Omega)$, then the renormalized solution u with null median of (3.25) is unique.

When $p \ge 2$ if

$$\tau \le \frac{N(p-1)}{N-p}\left(\frac{1}{2}-\frac{1}{t}\right)$$

and if $\mathbf{a}(x,r,\xi)$ satisfies

$$\begin{aligned}(4.43)\qquad &|\,\mathbf{a}(x,s,\xi)-\mathbf{a}(x,r,\xi)|\\ &\le C\frac{|s-r|}{(1+|s|+|r|)^{\lambda}}\left(|\xi|^{p-1}+|s|^{\frac{p(N-1)}{2(N-p)}}+|r|^{\frac{p(N-1)}{2(N-p)}}+h(x)\right),\end{aligned}$$

with $\lambda > \frac{1}{2}$, $h \ge 0$ and $h \in L^2(\Omega)$, then the renormalized solution u with null median of (3.25) is unique. It is worth noting that (4.42) and (4.43) are similar except in the power of $|s|$ and $|r|$ and the regularity of h. The main reason is that for $p \ge 2$ we use quadratic method for a p-growth equation.

Acknowledgement

Research partially supported by Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM), "Programma triennale della Ricerca dellUniversità degli Studi di Napoli "Parthenope" - Sostegno alla ricerca individuale 2015-2017". This work was done during the visits made by the third author to Laboratoire de Mathématiques "Raphaël Salem" de l'Université de Rouen and by the second author to Dipartimento di Matematica e Applicazioni "R. Caccioppoli" of University of Naples Federico II. Hospitality and support of all these institutions are gratefully acknowledged.

References

[1] A. Alvino, A. Cianchi, V. G. Maz'ya, and A. Mercaldo. Well-posed elliptic Neumann problems involving irregular data and domains. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 27(4):1017–1054, 2010.

[2] A. Alvino and A. Mercaldo. Nonlinear elliptic problems with L^1 data: an approach via symmetrization methods. *Mediterr. J. Math.*, 5(2):173–185, 2008.

[3] F. Andreu, J. M. Mazón, S. Segura de León, and J. Toledo. Quasi-linear elliptic and parabolic equations in L^1 with nonlinear boundary conditions. *Adv. Math. Sci. Appl.*, 7(1):183–213, 1997.

[4] M. Artola. Sur une classe de problèmes paraboliques quasi-linéaires. *Boll. Un. Mat. Ital. B (6)*, 5(1):51–70, 1986.

[5] G. Barles, G. Diaz, and J. I. Diaz. Uniqueness and continuum of foliated solutions for a quasilinear elliptic equation with a non-Lipschitz nonlinearity. *Comm. Partial Differential Equations*, 17(5-6):1037–1050, 1992.

[6] M. Ben Cheikh Ali and O. Guibé. Nonlinear and non-coercive elliptic problems with integrable data. *Adv. Math. Sci. Appl.*, 16(1):275–297, 2006.

[7] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and J. L. Vázquez. An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 22(2):241–273, 1995.

[8] M. F. Betta, O. Guibé, and A. Mercaldo. Neumann problems for nonlinear elliptic equations with L^1 data. *J. Differential Equations*, 259(3):898–924, 2015.

[9] M. F. Betta, A. Mercaldo, F. Murat, and M. M. Porzio. Existence and uniqueness results for nonlinear elliptic problems with a lower order term and measure datum. *C. R. Math. Acad. Sci. Paris*, 334(9):757–762, 2002.

[10] M. F. Betta, A. Mercaldo, F. Murat, and M. M. Porzio. Existence of renormalized solutions to nonlinear elliptic equations with a lower-order term and right-hand side a measure. *J. Math. Pures Appl. (9)*, 82(1):90–124, 2003. Corrected reprint of J. Math. Pures Appl. (9) **8**1 (2002), no. 6, 533–566 [MR1912411 (2003e:35075)].

[11] L. Boccardo and T. Gallouët. Nonlinear elliptic and parabolic equations involving measure data. *J. Funct. Anal.*, 87(1):149–169, 1989.

[12] L. Boccardo and T. Gallouët. Nonlinear elliptic equations with right-hand side measures. *Comm. Partial Differential Equations*, 17(3-4):641–655, 1992.

[13] L. Boccardo, T. Gallouët, and F. Murat. Unicité de la solution de certaines équations elliptiques non linéaires. *C. R. Acad. Sci. Paris Sér. I Math.*, 315(11):1159–1164, 1992.

[14] J. Chabrowski. On the Neumann problem with L^1 data. *Colloq. Math.*, 107(2):301–316, 2007.

[15] M. Chipot and G. Michaille. Uniqueness results and monotonicity properties for strongly nonlinear elliptic variational inequalities. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 16(1):137–166, 1989.

[16] G. Dal Maso, F. Murat, L. Orsina, and A. Prignet. Renormalized solutions of elliptic equations with general measure data. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 28(4):741–808, 1999.

[17] A. Dall'Aglio. Approximated solutions of equations with L^1 data. Application to the H-convergence of quasi-linear parabolic equations. *Ann. Mat. Pura Appl. (4)*, 170:207–240, 1996.

[18] A. Decarreau, J. Liang, and J.-M. Rakotoson. Trace imbeddings for T-sets and application to Neumann-Dirichlet problems with measures included in the boundary data. *Ann. Fac. Sci. Toulouse Math. (6)*, 5(3):443–470, 1996.

[19] J. Droniou. Solving convection-diffusion equations with mixed, Neumann and Fourier boundary conditions and measures as data, by a duality method. *Adv. Differential Equations*, 5(10-12):1341–1396, 2000.

[20] J. Droniou and J.-L. Vázquez. Noncoercive convection-diffusion elliptic problems with Neumann boundary conditions. *Calc. Var. Partial Differential Equations*, 34(4):413–434, 2009.

[21] V. Ferone and A. Mercaldo. A second order derivation formula for functions defined by integrals. *C. R. Acad. Sci. Paris Sér. I Math.*, 326(5):549–554, 1998.

[22] V. Ferone and A. Mercaldo. Neumann problems and Steiner symmetrization. *Comm. Partial Differential Equations*, 30(10-12):1537–1553, 2005.

[23] O. Guibé and A. Mercaldo. Existence and stability results for renormalized solutions to noncoercive nonlinear elliptic equations with measure data. *Potential Anal.*, 25(3):223–258, 2006.

[24] O. Guibé and A. Mercaldo. Existence of renormalized solutions to nonlinear elliptic equations with two lower order terms and measure data. *Trans. Amer. Math. Soc.*, 360(2):643–669 (electronic), 2008.

[25] J. Leray and J.-L. Lions. Quelques résulatats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder. *Bull. Soc. Math. France*, 93:97–107, 1965.

[26] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod, 1969.

[27] P.L. Lions and F. Murat. Sur les solutions renormalisées d'équations elliptiques non linéaires. In *manuscript*.

[28] F. Murat. Equations elliptiques non linéaires avec second membre L^1 ou mesure. In *Compte Rendus du 26ème Congrès d'Analyse Numérique*, les Karellis, 1994.

[29] A. Prignet. Conditions aux limites non homogènes pour des problèmes elliptiques avec second membre mesure. *Ann. Fac. Sci. Toulouse Math. (6)*, 6(2):297–318, 1997.

[30] W. P. Ziemer. *Weakly differentiable functions*, volume 120 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1989. Sobolev spaces and functions of bounded variation.

Maria Francesca Betta
Dipartimento di Ingegneria,
Università degli Studi di Napoli Parthenope,
Centro Direzionale, Isola C4 80143 Napoli, Italy
E-mail address: francesca.betta@uniparthenope.it

Olivier Guibé
Laboratoire de Mathématiques Raphaël Salem,
UMR 6085 CNRS-Université de Rouen
Avenue de l'Université, BP.12
76801 Saint-Étienne-du-Rouvray, France
E-mail address: olivier.guibe@univ-rouen.fr

Anna Mercaldo
Dipartimento di Matematica e Applicazioni "R. Caccioppoli",
Università degli Studi di Napoli Federico II,
Complesso Monte S. Angelo, Via Cintia, 80126 Napoli, Italy
E-mail address: mercaldo@unina.it