State estimation of system with bounded uncertain parameters: interval multi-model approach

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Abstract

The objective of this study is the analysis of dynamic systems represented by multi-model with variable parameters. Changes in these parameters are unknown but bounded. Since it is not possible to estimate these parameters over time, the simulation of such systems requires to consider all possible values taken by these parameters. More precisely, the goal is to determine, at any moment, the smallest set containing all the possible values of the state vector simultaneously compatible with the state equations and with a priori known bounds of the uncertain parameters. This set will be characterized by two trajectories corresponding to the lower and upper limits of the state at every moment. This characterization can be realized by a direct simulation of the system, given the bounds of its parameters. It can also be implemented with a Luenberger type observer, fed with the system measurements.

Keywords: Interval Observers, Uncertain Systems, Bounded Uncertainties, Multi-Model Technique

1. Introduction

As widely known, one of the main difficulties in system simulation, control or estimation is to deal with uncertainties. These uncertainties may affect the input or output signals of the system (e.g. unknown input, disturbance, measurement noises, etc) as well as the system model itself (e.g. non modeled dynamics, unknown parameter, etc). The uncertainties may also be of different natures: total lack of information (unknown value of a parameter) or partial knowledge (upper and lower bounds, statistical properties, etc). The proposed paper deals with system with uncertain parameters.

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Even a low magnitude change in some parameters may have a significant impact on the system behavior, and namely on the system state trajectory. Uncertain parameters can be considered from two main points of view: the stochastic and the deterministic ones. In the first approach, the uncertain parameters are assumed to be the results of random process realizations. It then needs to chose the probability density functions and their parameters describing the system uncertain parameters. In the deterministic approach, no statistical models of the parameters are assumed to be available and only upper and lower bounds of the parameter values are known. This approach, also known as the interval approach, is adopted in the present paper.

The search for trajectories which are solutions of differential inclusions [32] were the starting point of a large amount of works on stability, stabilization and state estimation. In the latter topic our communication follows the pioneering work of Gouzé [21] that proposed the synthesis of interval type observer to reconstruct the system state from measurements of inputs and outputs. Since then, many results were published on designing observers adapted to nonlinear systems with uncertain but bounded parameters [11], [12], [6], [9].

The model-based state estimation techniques in the context of bounded uncertainties can be classified into two categories. The first one is based on the mechanism "prediction/correction" [24], [33]. Specifically, based on information at time k, the dynamic state equation provides the possible state set at the time $k+1$ through a one step prediction. At this $k+1$ instant, the available measures can also be used to provide a state set by inverting the output equations. The correction phase, which proceeds by searching for the intersection of the two sets obtained above, then provides the estimated state compatible with the model, its uncertainties and the measures. The second approach [7], [15], [4], [40], [36] consists in explicitly building an interval observer from the interval model of the system. Then, solving the interval state equations of the observer, upper and lower bounds of the state variable estimates are provided.

Studies on systems with bounded uncertain parameters can be found in several areas of automatic control: stability analysis [30], [18], state estimation [16] [14] [33], synthesis of iterative learning control law [1], system diagnosis [27] [22], components fault detection [25], [34] , data analysis [23], model sensitivity analysis [35], regression with abnormal values [39], classification [38]. Roughly speaking, uncertainties in the measures and in the parameters are handled in two main ways. In the first one, under robustness constraints, aims to develop analysis and synthesis techniques insensitive or partially immune to uncertainties. This leads in most cases to minimize a criterion reflecting the impact of these uncertainties on the objective to be pursued in terms of control or state estimation. In the second one, the aim is to quantify the impact of the uncertainties in the objective to pursue. The objective can be, for instance, to quantify the influence of uncertain and bounded parameters on the estimated system state, leading to analyze the uncertainty propagation throughout the control and state estimation procedures.

Up to the authors' knowledge, the use of observer adapted to real processes with bounded uncertain parameter is barely exposed in the literature, never-

theless, one may find some applications with only simulations or laboratory processes. In [26], an interval observer was synthesized to estimate the water quality in a distribution network. In [28] [31], [2] and [19] bio-processes were considered as applications for observer design, as well as in [6] with probabilistic uncertain parameters. In [8], a mechanical structure analysis with interval type parameters is proposed. In the field of transport, [20] and [40] presented the synthesis of an interval observer to estimate the vehicle position and velocity. The socio-economic field is no exception to the use of this approach, as shown for example in [39].

Our proposal also participates of this second approach, even if it differs markedly from two perspectives. First, based on bounded uncertainties, a multi-model approach provides two distinct models for the upper and lower bounds of the system state. Then, based on these two models, an interval Luenberger-like multi-observer is designed, to reconstruct the bounds of the system states from measurements of its inputs and outputs. The contributions of this paper is to develop an improved structure of an interval observer with an analytical form, to study its stability in order to compute its gains and to take into account the uncertainties affecting the system with the help of a polytopic representation.

The main required assumptions for the interval observer design to guarantee error boundedness is the exclusion of the zero value from the intervals defining the parameter uncertainties. Put in other words, it means that at least the signs of the uncertain parameters are known even if their exact values are not known. It can also be pointed out that our approach does not need to assume some structural requirements on the state matrix, such as being Metzler (off diagonal element should be positive) as often used in the literature [16], [6], [36], [40] (and references therein).

The paper is organized as follows. Section 2 is devoted to the interval formulation of a system with time varying bounded parameters. In section 3, the stability of the proposed structure is studied. This result is then exploited in section 4 to derive the observer synthesis. After some concluding remarks in section 5, an appendix details the derivation of the proposed multi-model approach.

2. Interval model of a system with uncertain bounded parameters

2.1. Structure

The discrete time representation (1) is considered, where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ the input, $y \in \mathbb{R}^p$ the measured output, A_k and B matrices of appropriate dimensions. Parametric variations Δ_k (which are not available for measurements) only affect the state matrix, taking into account changes in the B and C matrices being a simple extension.

$$\begin{cases} x_{k+1} = A_k\, x_k + B\, u_k \\ A_k = A + \Delta_k \\ y_k = C\, x_k \end{cases} \tag{1}$$

Hypothesis 1. *Parametric variations Δ_k are bounded by*

$$-\Delta \leq \Delta_k \leq \Delta \tag{2}$$

where the bound Δ is known and the inequalities in (2) are to be understood component-wise.

Hypothesis 2. *The null matrix of the same size as those of A, does not belong to the interval matrix $[A-\Delta \quad A+\Delta]$. In other words, the lower and upper bounds of each element of this interval matrix have the same sign.*

From (1), one can build the state trajectories $\underline{x}_k$ and $\overline{x}_k$ containing the set of reachable states given the parametric variations Δ_k. As it will be explained, the objective is to avoid to evaluate all the possible values of the uncertainties to determine all the possible corresponding state trajectories. What will be done is to recursively calculate only the upper and lower extremal trajectories of the state envelope and a proof will be provided to show that all the possible state trajectories are included in this envelope.

2.2. Envelope of a state trajectory

Proposition 1 (Product of two interval variables)**.** *Let us consider the scalar $p_k = (a+\delta_k)x_k$ where $\mid \delta_k \mid \leq \delta$ and where x_k is an interval variable, namely $x_k \in [\underline{x}_k \ \overline{x}_k]$. To determine the bounds $\underline{p}_k$ and $\overline{p}_k$ of the interval variable p_k, two cases must be distinguished whether a is positive or negative:*

$$a>0 \begin{cases} \underline{p}_k &= a\underline{x}_k - \delta \mid \underline{x}_k \mid \\ &= \underline{x}_k\,(a-\delta \mathrm{sgn}(\underline{x}_k)) \\ \overline{p}_k &= a\overline{x}_k + \delta \mid \overline{x}_k \mid \\ &= \overline{x}_k\,(a+\delta \mathrm{sgn}(\overline{x}_k)) \end{cases} \tag{3}$$

$$a<0 \begin{cases} \underline{p}_k &= a\overline{x}_k - \delta \mid \overline{x}_k \mid \\ &= \overline{x}_k\,(a-\delta \mathrm{sgn}(\overline{x}_k)) \\ \overline{p}_k &= a\underline{x}_k + \delta \mid \underline{x}_k \mid \\ &= \underline{x}_k\,(a+\delta \mathrm{sgn}(\underline{x}_k)) \end{cases} \tag{4}$$

Having considered the two cases (3) and (4), p_k is expressed in respect to its two bounds as follows:

$$\begin{cases} \underline{p}_k &= \dfrac{1}{2}\left[\ (a+\mid a \mid)\underline{x}_k + (a-\mid a \mid)\overline{x}_k\right] \\ &\quad -\dfrac{1}{2}\left[\delta\,(\mid \underline{x}_k \mid + \mid \overline{x}_k \mid + \mathrm{sgn}(a)(\mid \underline{x}_k \mid - \mid \overline{x}_k \mid))\ \right] \\ \overline{p}_k &= \dfrac{1}{2}\left[\ (a-\mid a \mid)\underline{x}_k + (a+\mid a \mid)\overline{x}_k\right] \\ &\quad +\dfrac{1}{2}\left[\delta(\mid \underline{x}_k \mid + \mid \overline{x}_k \mid - \mathrm{sgn}(a)(\mid \underline{x}_k \mid - \mid \overline{x}_k \mid))\ \right] \end{cases} \tag{5}$$

These results can directly be generalized to the case of matrix products as it appears in (1), which explicitly expresses the system state in an interval form, for any time k. The coupled recurrences are obtained and allow to recursively express the two bounds of the state [10]:

$$\left\{\begin{array}{rcl} \underline{x}_{k+1} & = & \frac{1}{2}\left[(A+ \mid A \mid)\,\underline{x}_k + (A- \mid A \mid)\,\overline{x}_k\right] + Bu_k \\ & & -\frac{1}{2}\left[(\Delta + \Delta * S_A) \mid \underline{x}_k \mid - (\Delta - \Delta * S_A) \mid \overline{x}_k \mid\right] \\ \overline{x}_{k+1} & = & \frac{1}{2}\left[(A- \mid A \mid)\,\underline{x}_k + (A+ \mid A \mid)\,\overline{x}_k\right] + Bu_k \\ & & +\frac{1}{2}\left[(\Delta - \Delta * S_A) \mid \underline{x}_k \mid + (\Delta + \Delta * S_A) \mid \overline{x}_k \mid\right] \end{array}\right. \tag{6}$$

where $*$ symbolizes the Hadamard matrix product and $S_A = \mathrm{sgn}(A)$ (where the sign function is applied component-wise to the matrix A).

Proposition 2 (Interval trajectory of uncertain dynamic system)**.** *The system with bounded uncertain parameters (1) is characterized by an interval-type state which bounds are given by the following augmented system:*

$$\begin{bmatrix} \underline{x}_{k+1} \\ \overline{x}_{k+1} \end{bmatrix} = M \begin{bmatrix} \underline{x}_k \\ \overline{x}_k \end{bmatrix} + N \begin{bmatrix} \mid \underline{x}_k \mid \\ \mid \overline{x}_k \mid \end{bmatrix} + E\,B\,u_k \tag{7}$$

with

$$M = \frac{1}{2}\begin{bmatrix} A+ \mid A \mid & A- \mid A \mid \\ A- \mid A \mid & A+ \mid A \mid \end{bmatrix}, \quad E = \begin{bmatrix} I_n \\ I_n \end{bmatrix}, \quad N = \frac{1}{2}\begin{bmatrix} -\Delta - \Delta * S_A & -\Delta + \Delta * S_A \\ \Delta - \Delta * S_A & \Delta + \Delta * S_A \end{bmatrix} \tag{8}$$

Remark 1 (Computation of the i^{th} component of the interval state vector in the input free case $u_k = 0$)**.** *Let us detail the computation of this component with respect to the lower bound of the system state (the calculation is similar to the upper bound). Denoting $a_{i,j}$ and $\delta_{i,j}$ the respective elements of matrices A and Δ, from (6) one gets:*

$$\begin{array}{rcl} \underline{x}_{k+1,i} & = & \frac{1}{2}\sum_{j=1}^{n}\left((a_{i,j}+ \mid a_{i,j} \mid)\,\underline{x}_{k,j} + (a_{i,j}- \mid a_{i,j} \mid)\,\overline{x}_{k,j}\right) \\ & & -\frac{1}{2}\sum_{j=1}^{n}\left(\delta_{i,j}\left(\mid \underline{x}_{k,j} \mid + \mid \overline{x}_{k,j} \mid\right) + \mathrm{sgn}(a_{i,j})\delta_{i,j}\left(\mid \underline{x}_{k,j} \mid - \mid \overline{x}_{k,j} \mid\right)\right) \end{array} \tag{9}$$

Let us define the term $\sigma_{i,j}$ of rank j in (9) such that $\underline{x}_{k+1,i} = \sum_{j=1}^{n}\sigma_{i,j}$. One can then simplify the expression of the term $\sigma_{i,j}$ with respect to the sign of the matrix A coefficients:

- *If $a_{i,j} > 0$*

$$\begin{array}{rcl} \sigma_{i,j} & = & a_{i,j}\underline{x}_{k,j} - \frac{\delta_{i,j}}{2}\left(\left(\mid \underline{x}_{k,j} \mid + \mid \overline{x}_{k,j} \mid\right) + \left(\mid \underline{x}_{k,j} \mid - \mid \overline{x}_{k,j} \mid\right)\right) \\ & = & a_{i,j}\underline{x}_{k,j} - \delta_{i,j} \mid \underline{x}_{k,j} \mid \end{array} \tag{10}$$

- *If* $a_{i,j} < 0$

$$\begin{aligned} \sigma_{i,j} &= a_{i,j}\overline{x}_{k,j} - \frac{\delta_{i,j}}{2}\left(\left(\mid \underline{x}_{k,j} \mid + \mid \overline{x}_{k,j} \mid\right) - \left(\mid \underline{x}_{k,j} \mid - \mid \overline{x}_{k,j} \mid\right)\right) \\ &= a_{i,j}\overline{x}_{k,j} - \delta_{i,j} \mid \overline{x}_{k,j} \mid \end{aligned} \tag{11}$$

These results will be used later.

2.3. Validation of the envelope of the state trajectory

In (6) the two bounds $\underline{x}_k$ and $\overline{x}_k$ of the state were recursively defined and they were claimed to define an envelope of the original system with interval parameters (1). It still remains to prove that the envelope does contain the actual state of the system (1). For this, the following property is proven by induction on k:

$$\mathcal{H}_k : \underline{x}_k \leq x_k \leq \overline{x}_k \tag{12}$$

By definition of $\underline{x}_0$ and $\overline{x}_0$ in (6), it holds that $x_0 \in [\underline{x}_0, \quad \overline{x}_0]$, so the property is satisfied for $k = 0$. Assuming that for a given positive k the property $\mathcal{H}_k$ is true, then let us show that $\underline{x}_{k+1} \leq x_{k+1}$ is true (the other inequality in (12) is proved similarly) implying that $\mathcal{H}_{k+1}$ is also true. Given (6) and (1), the distance between the actual state and the lower bound is deduced:

$$\begin{aligned} \underline{\tilde{x}}_{k+1} &= x_{k+1} - \underline{x}_{k+1} \\ &= (A + \Delta_k)\, x_k - \tfrac{1}{2}\,(A+ \mid A \mid)\, \underline{x}_k - \tfrac{1}{2}\,(A- \mid A \mid)\, \overline{x}_k \\ &\quad + \tfrac{1}{2}\,(\Delta + \Delta * S_A) \mid \underline{x}_k \mid - \tfrac{1}{2}\,(\Delta - \Delta * S_A) \mid \overline{x}_k \mid \end{aligned} \tag{13}$$

As reported in remark 1, the analysis of this difference can be made for the i^{th} component of $\tilde{x}_{k+1}$. To do this, substitute (9) and (1) in (13) to obtain:

$$\begin{aligned} \underline{\tilde{x}}_{k+1,i} &= x_{k+1,i} - \underline{x}_{k+1,i} \\ &= \textstyle\sum_{j=1}^{n} \left((a_{i,j} + \delta_{k,i,j})x_{k,j} - \sigma_{i,j}\right) \end{aligned} \tag{14}$$

where $\delta_{k,i,j}$ denotes the (i,j) entry of the matrix Δ_k. In the sum appearing in (14), only the term $\tilde{\sigma}_{i,j}$ of rank j is explained in view of the expression (10) and (10) of $\sigma_{i,j}$:

- If $a_{i,j} > 0$

$$\begin{aligned} \tilde{\sigma}_{i,j} &= (a_{i,j} + \delta_{k,i,j})x_{k,j} - a_{i,j}\underline{x}_{k,j} + \delta_{i,j} \mid \underline{x}_{k,j} \mid \\ &\geq (a_{i,j} + \delta_{k,i,j})\underline{x}_{k,j} - a_{i,j}\underline{x}_{k,j} + \delta_{i,j} \mid \underline{x}_{k,j} \mid \\ &\geq \delta_{k,i,j}\, \underline{x}_{k,j} + \delta_{i,j} \mid \underline{x}_{k,j} \mid \\ &\geq 0 \end{aligned} \tag{15}$$

- If $a_{i,j} < 0$

$$\begin{aligned} \tilde{\sigma}_{i,j} &= (a_{i,j} + \delta_{k,i,j})x_{k,j} - a_{i,j}\overline{x}_{k,j} + \delta_{i,j} \mid \overline{x}_{k,j} \mid \\ &\geq (a_{i,j} + \delta_{k,i,j})\overline{x}_{k,j} - a_{i,j}\overline{x}_{k,j} + \delta_{i,j} \mid \overline{x}_{k,j} \mid \\ &\geq \delta_{k,i,j}\, \overline{x}_{k,j} + \delta_{i,j} \mid \overline{x}_{k,j} \mid \\ &\geq 0 \end{aligned} \tag{16}$$

Finally, according to (14), (15) and (16), $\underline{\tilde{x}}_{k+1,i}$ is a sum of non negative terms and thus $x_{k+1,i} \geq \underline{x}_{k+1,i}$, i.e. $x_{k+1} \geq \underline{x}_{k+1}$. Thus, the property $\mathcal{H}_{k+1}$ has been demonstrated and the induction principle ensures that for positive k, $x_k \geq \underline{x}_k$. In the same way, by analyzing the quantity $\tilde{\overline{x}}_k = \overline{x}_k - x_k$, it can be established that $x_k \leq \overline{x}_k$. It can thus be concluded that (6) is an envelope of the system state trajectory.

2.4. Example

The considered example is a system (1) with $m = 2$ inputs and $n = 2$ state variables, defined by the matrices A and B given in (17). The parametric variations Δ_k affecting the state matrix take values between -0.01 and 0.01 and accordingly its bound (2) is defined by the matrix Δ in (17). Figure 1 displays the two inputs (top of the figure) and the two states (bottom of the figure). For each state variable, the red line is a possible state trajectory and the two others are the upper and lower bounds of the state trajectories according to the considered parameter uncertainties.

$$A = \begin{bmatrix} 0.925 & -0.05 \\ -0.25 & 0.25 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad \Delta = \begin{bmatrix} 0.01 & 0.01 \\ 0.01 & 0.01 \end{bmatrix} \tag{17}$$

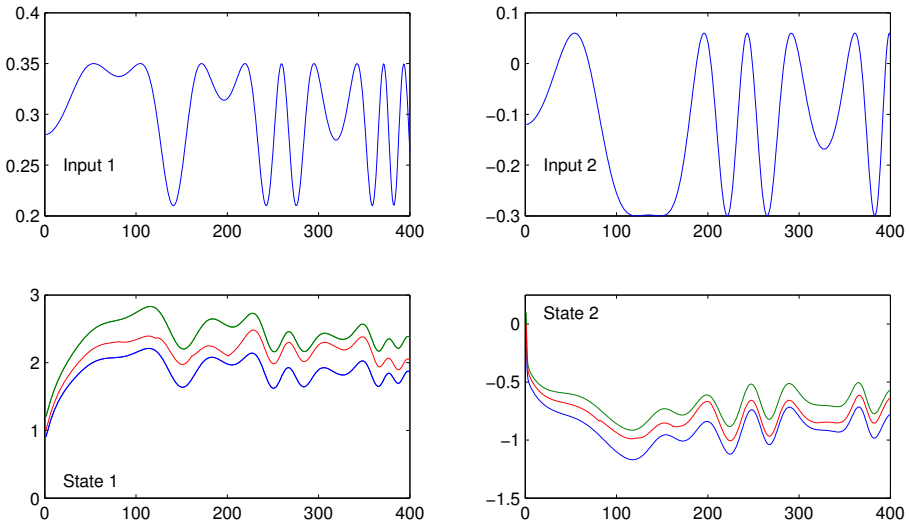


Figure 1: Simulation of a system with interval parameters

3. Stability analysis

In this section, the stability of interval systems with bounded uncertain parameters is studied with the second Lyapunov method, which is known to be devoted to the analysis of internal stability of dynamic systems. The principle of this method is to study the convergence of the system state x to the origin.

This is achieved through a scalar positive definite function of the state, denoted V, and called the Lyapunov candidate function.

Since no specific Lyapunov functions have already been proposed for interval models (6) defined by a generalized state encompassing the upper and lower bounds of the state as well as its absolute value, it is here proposed to transform the absolute value function using a nonlinear sector bounded approach. Section 3.1 is dedicated to this approach, which then allows the stability study in section 3.2.

3.1. New structure of the system model

Defining the generalized state $z_k = \begin{bmatrix} \underline{x}_k^T & \overline{x}_k^T \end{bmatrix}^T$ the system (7) becomes:

$$z_{k+1} = M\, z_k + N \mid z_k \mid + E\, B\, u_k \tag{18}$$

with M and N defined in (8).

As mentioned in the preamble, the stability analysis is eased by writing the nonlinearity under a polytopic form depending on $\mid z_k \mid$.

For any scalar α, the following decomposition holds:

$$\begin{cases} \mid \alpha \mid &= \mu_1(\alpha)\,.\,(\alpha) + \mu_2(\alpha)\,.\,(-\alpha) \\ \mu_1(\alpha) &= \dfrac{1 + \operatorname{sgn}(\alpha)}{2} \\ \mu_2(\alpha) &= 1 - \mu_1(\alpha) \end{cases} \tag{19}$$

which can also be written as:

$$\begin{cases} \mid \alpha \mid &= \sum_{i=1}^{2} \mu_i(\alpha)\, E_i\, \alpha \\ E_1 &= 1 \\ E_2 &= -1 \end{cases} \tag{20}$$

As detailed in the appendix Appendix A, this writing directly extends to a vector $z \in \mathbb{R}^{2n}$ (Appendix A):

$$\mid z_k \mid = \sum_{j=1}^{r} \mu_j(z_k)\, E_j\, z_k, \quad r = 2^{2n} \tag{21}$$

Finally, with (21), the interval model (18) is expressed under the polytopic form:

$$z_{k+1} = \sum_{j=1}^{r} \mu_j(z_k)(M + N\, E_j) z_k + E\, B\, u_k \tag{22}$$

where the activating functions μ_j satisfy the so-called convex sum property:

$$\sum_{j=1}^{r} \mu_j(z_k) = 1, \quad 0 \leq \mu_j(z_k) \leq 1, \forall k \in \mathbb{N}, \forall j = 1, \ldots, r \tag{23}$$

3.2. Stability criteria

Theorem 1. *The asymptotical stability of the input free system* (22) *(i.e. with* $u_k = 0$*) is guaranteed if there exists matrices* $P = P^T > 0$ *and* G *such that the LMI* (24) *holds.*

$$\begin{bmatrix} -P & (M + N\,E_j)^T\,G \\ G^T(M + N\,E_j) & P - G - G^T \end{bmatrix} < 0, \quad j = 1, \ldots, r \tag{24}$$

Proof. Defining a quadratic Lyapunov function $V(z_k) = z_k^T P z_k$ with $P > 0$, its variation along the state trajectory of (22), defined by $\Delta V_k = V(z_{k+1}) - V(z_k)$, is given by $\Delta V_k = z_k^T \mathcal{M} z_k$, with:

$$\mathcal{M} = \left(M + N \sum_{j=1}^{r} \mu_j(z_k) E_j \right)^T P \left(M + N \sum_{j=1}^{r} \mu_j(z_k) E_j \right) - P \tag{25}$$

Obviously, the negativity of the variation ΔV_k is implied if there exists $P = P^T > 0$ satisfying $\mathcal{M} < 0$. From lemma 3 of [13], it is known that finding $P = P^T > 0$ satisfying $\mathcal{M} < 0$ is equivalent to find $P = P^T > 0$ and G satisfying

$$\begin{bmatrix} -P & (M + N \sum_{j=1}^{r} \mu_j(z_k) E_j)^T G \\ G^T(M + N \sum_{j=1}^{r} \mu_j(z_k) E_j) & P - G - G^T \end{bmatrix} < 0 \tag{26}$$

The positivity of the functions μ_j ensures that (24) are sufficient conditions to (26) and thus to $\Delta V_k < 0$ and to the input free asymptotical stability of (22). □

Corollary 1. *The system (22) is asymptotically stable if there exists matrices* $P_i = P_i^T > 0$ *and* G *such that:*

$$\mathcal{M}_{i,j} + \mathcal{M}_{j,i} < 0, \;\; i, j = 1, \ldots, r \tag{27}$$

where $\mathcal{M}_{i,j}$ *is defined by:*

$$\mathcal{M}_{i,j} = \begin{bmatrix} -P_i & (M + N\,E_i)^T\,G \\ G^T(M + N\,E_i) & P_j - G - G^T \end{bmatrix} \tag{28}$$

Proof. The proof is similar to the one of Theorem 1, with the Lyapunov candidate function defined by $V(z_k) = z_k^T \left(\sum_{i=1}^{r} \mu_i(z_k) P_i \right) z_k$ and the factorization of [13]. □

4. Observer design for a system with bounded uncertain parameters

4.1. Observer structure

Before detailing the proposed observer, let us recall that the state equation of the system (1) may also be described by its lower and upper bounds (6).

Given the structure of (6), the proposed observer (29) is designed to provide an interval state estimate $[\underline{\hat{x}}_k \;\; \hat{\overline{x}}_k]$ which gives an envelope of the actual system state, based only on the input and output measurements of the system (1). Namely, it reduces to find the observer gains L_1 and L_2 such that $x_k \in [\underline{\hat{x}}_k \; \hat{\overline{x}}_k]$, for every instant k and to impose some dynamics to the state reconstruction error.

$$\left\{\begin{array}{rcl} \underline{\hat{x}}_{k+1} & = & \frac{1}{2}\left[(A+ \mid A \mid)\,\underline{\hat{x}}_k + (A- \mid A \mid)\,\hat{\overline{x}}_k\right] - (\Delta + \Delta * S_A) \mid \underline{\hat{x}}_k \mid \\ & & -\,(\Delta - \Delta * S_A) \mid \hat{\overline{x}}_k \mid + L_1(y_k - C\underline{\hat{x}}_k) + B\,u_k \\ \hat{\overline{x}}_{k+1} & = & \frac{1}{2}\left[(A- \mid A \mid)\,\underline{\hat{x}}_k + (A+ \mid A \mid)\,\overline{x}_k\right] + (\Delta - \Delta * S_A) \mid \underline{\hat{x}}_k \mid \\ & & +\,(\Delta + \Delta * S_A) \mid \hat{\overline{x}}_k \mid + L_2(y_k - C\hat{\overline{x}}_k) + B\,u_k \end{array}\right. \tag{29}$$

4.2. Rationale for observer structure

The rationale for this structure is done in two steps. First, it must be shown that $\underline{\hat{x}}_k \leq x_k \leq \hat{\overline{x}}_k$. Secondly, the observer gains L_1 and L_2 must be adjusted to specify the temporal characteristics of state reconstruction error. Each of these steps is addressed in Theorem 2 and 3 respectively.

Theorem 2. *The system (29) is an observer for (1), i.e. the state of (1) belongs to the envelope defined by (29).*

Proof. The proof proceeds by induction. Let $\mathcal{H}_k$ be the following property:

$$\mathcal{H}_k : \underline{\hat{x}}_k \leq x_k \leq \hat{\overline{x}}_k \tag{30}$$

The gaps between the state x_k and the lower and upper bounds of its estimate are defined by:

$$\underline{e}_k = x_k - \underline{\hat{x}}_k \qquad \text{and} \qquad \overline{e}_k = \hat{\overline{x}}_k - x_k \tag{31}$$

The first one is expressed at time $k+1$, from (1) and (29), by:

$$\begin{array}{rcl} \underline{e}_{k+1} & = & (A + \Delta_k)\,x_k - \frac{1}{2}\left[(A+ \mid A \mid)\,\underline{\hat{x}}_k + (A- \mid A \mid)\,\hat{\overline{x}}_k\right] \\ & & -L_1(y_k - C\,\underline{\hat{x}}_k) - \frac{1}{2}\left[(\Delta + \Delta * S_A) \mid \underline{\hat{x}}_k \mid + (\Delta - \Delta * S_A) \mid \hat{\overline{x}}_k \mid\right] \end{array} \tag{32}$$

The i^{th} component of this difference is given by:

$$\begin{array}{rcl} \underline{e}_{k+1,i} & = & \displaystyle\sum_{j=1}^{n}(a_{i,j} + \delta_{k,i,j})\,x_{k,j} - \ell_{1,i,j}(x_{k,j} - \underline{\hat{x}}_{k,j}) \\ & & \displaystyle -\frac{1}{2}\sum_{j=1}^{n}\left((a_{i,j}+ \mid a_{i,j} \mid)\,\underline{\hat{x}}_{k,j} + (a_{i,j}- \mid a_{i,j} \mid)\,\hat{\overline{x}}_{k,j}\right) \\ & & \displaystyle -\frac{1}{2}\sum_{j=1}^{n}\left(\delta_{i,j}\left(\mid \underline{x}_{k,j} \mid + \mid \overline{x}_{k,j} \mid\right) + \mathrm{sgn}(a_{i,j})\delta_{i,j}\left(\mid \underline{\hat{x}}_{k,j} \mid - \mid \overline{x}_{k,j} \mid\right)\right) \end{array} \tag{33}$$

where $\ell_{1,i,j}$ are the coefficients of the L_1 matrix. In the previous sum, let us analyse the j^{th} term whether $a_{i,j,}$ is positive or negative:

- If $a_{i,j} > 0$

$$\begin{array}{rcl} \underline{e}_{k+1,i,j} & = & (a_{i,j} + \delta_{k,i,j})\, x_{k,j} - a_{i,j}\underline{\hat{x}}_{k,j} + \delta_{i,j} \mid \underline{\hat{x}}_{k,j} \mid -\ell_{1,i,j}(x_{k,j} - \underline{\hat{x}}_{k,j}) \\ & = & (a_{i,j} + \delta_{k,i,j} - \ell_{1,i,j})\, x_{k,j} - a_{i,j}\underline{\hat{x}}_{k,j} + \delta_{i,j} \mid \underline{\hat{x}}_{k,j} \mid +\ell_{1,i,j}\underline{\hat{x}}_{k,j} \\ & \geq & (a_{i,j} + \delta_{k,i,j} - \ell_{1,i,j})\, \underline{\hat{x}}_{k,j} - a_{i,j}\underline{\hat{x}}_{k,j} + \delta_{i,j} \mid \underline{\hat{x}}_{k,j} \mid +\ell_{1,i,j}\underline{\hat{x}}_{k,j} \\ & \geq & \delta_{k,i,j}\, \underline{\hat{x}}_{k,j} + \delta_{i,j} \mid \underline{\hat{x}}_{k,j} \mid \\ & \geq & 0 \end{array} \tag{34}$$

- If $a_{i,j} < 0$

$$\begin{array}{rcl} \underline{e}_{k+1,i,j} & = & (a_{i,j} + \delta_{k,i,j})\, x_{k,j} - a_{i,j}\hat{\overline{x}}_{k,j}\delta_{i,j} \mid \hat{\overline{x}}_{k,j} \mid -\ell_{1,i,j}(x_{k,j} - \underline{\hat{x}}_{k,j}) \\ & = & (a_{i,j} + \delta_{k,i,j} - \ell_{1,i,j})\, x_{k,j} - a_{i,j}\hat{\overline{x}}_{k,j} + \delta_{i,j} \mid \hat{\overline{x}}_{k,j} \mid +\ell_{1,i,j}\underline{\hat{x}}_{k,j} \\ & \geq & (a_{i,j} + \delta_{k,i,j} - \ell_{1,i,j})\, \hat{\overline{x}}_{k,j} - a_{i,j}\hat{\overline{x}}_{k,j} + \delta_{i,j} \mid \hat{\overline{x}}_{k,j} \mid +\ell_{1,i,j}\underline{\hat{x}}_{k,j} \\ & \geq & \delta_{k,i,j}\hat{\overline{x}}_{k,j} + \delta_{i,j} \mid \hat{\overline{x}}_{k,j} \mid \\ & \geq & 0 \end{array} \tag{35}$$

This shows that $\underline{e}_{k+1,i}$ is a sum of non negative terms and then $x_{k+1,i} \geq \underline{\hat{x}}_{k+1,i}$, i.e. $x_{k+1} \geq \underline{\hat{x}}_{k+1}$. A similar calculus leads to the result $x_k \leq \hat{\overline{x}}_k$. Therefore, the property $\mathcal{H}_{k+1}$ is demonstrated and the induction principle ensures that for all $k > 0$, we have $\underline{\hat{x}}_k \leq x_k \leq \hat{\overline{x}}_k$ which achieves the proof. □

In the following, the state observer equations are denoted:

$$\begin{array}{l} \underline{\hat{x}}_{k+1} = H_{11}\, \underline{\hat{x}}_k + H_{12}\, \hat{\overline{x}}_k + H_{13} \mid \underline{\hat{x}}_k \mid + H_{14} \mid \hat{\overline{x}}_k \mid + L_1(y_k - C\underline{\hat{x}}_k) + B\, u_k \\ \hat{\overline{x}}_{k+1} = H_{21}\, \underline{\hat{x}}_k + H_{22}\, \hat{\overline{x}}_k + H_{23} \mid \underline{\hat{x}}_k \mid + H_{24} \mid \hat{\overline{x}}_k \mid + L_2(y_k - C\hat{\overline{x}}_k) + B\, u_k \end{array} \tag{36}$$

with:

$$\begin{array}{lll} H_{11} = \dfrac{1}{2}(A+ \mid A \mid), & H_{12} = \dfrac{1}{2}(A- \mid A \mid), & H_{13} = \dfrac{-1}{2}(\Delta + \Delta * S_A), \\ H_{14} = \dfrac{1}{2}(-\Delta + \Delta * S_A) & H_{21} = \dfrac{1}{2}(A- \mid A \mid), & H_{22} = \dfrac{1}{2}(A+ \mid A \mid), \\ H_{23} = \dfrac{1}{2}(\Delta - \Delta * S_A), & H_{24} = \dfrac{1}{2}(\Delta + \Delta * S_A) & \end{array}$$

To characterize the observer's performance, let rewrite the deviations (31) as:

$$\begin{array}{rcl} \underline{e}_{k+1} & = & (A + \Delta_k)\, x_k - H_{11}\, \underline{\hat{x}}_k - H_{12}\, \hat{\overline{x}}_k - H_{13} \mid \underline{\hat{x}}_k \mid - H_{14} \mid \hat{\overline{x}}_k \mid - L_1(y_k - C\underline{\hat{x}}_k) \\ & = & (H_{11} - L_1 C)\underline{e}_k - H_{12}\overline{e}_k - \Delta_k\, x_k - H_{13} \mid \underline{\hat{x}}_k \mid - H_{14} \mid \hat{\overline{x}}_k \mid \\ \overline{e}_{k+1} & = & H_{21}\, \underline{\hat{x}}_k + H_{22}\, \hat{\overline{x}}_k - (A + \Delta_k)\, x_k + H_{23} \mid \underline{\hat{x}}_k \mid + H_{23} \mid \hat{\overline{x}}_k \mid + L_2(y_k - C\hat{\overline{x}}_k) \\ & = & -H_{21}\underline{e}_k + (H_{22} - L_2 C)\, \overline{e}_k - \Delta_k\, x_k - H_{23} \mid \underline{\hat{x}}_k \mid - H_{24} \mid \hat{\overline{x}}_k \mid \end{array} \tag{37}$$

The two deviations defined in (37) are gathered in the following augmented state estimation error: $e_k^T = [\underline{e}_k^T \;\; \overline{e}_k^T]$ and ruled by the following discrete time system:

$$e_{k+1} = (A_0 - L\, C_0)\, e_k + B_0\, \omega_k \tag{38}$$

where the matrices A_0, L, C_0 and B_0 and the input signal ω_k are defined by

$$A_0 = \begin{bmatrix} H_{11} & -H_{12} \\ -H_{21} & H_{22} \end{bmatrix}, \quad B_0 = \begin{bmatrix} I_n & -H_{13} & -H_{14} \\ -I_n & -H_{23} & -H_{24} \end{bmatrix}, \quad L = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix},$$
$$C_0 = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \quad \omega_k = \begin{bmatrix} (\Delta_k\, x_k)^T & |\, \underline{\hat{x}}_k\, |^T & |\, \overline{\hat{x}}_k\, |^T \end{bmatrix}^T \tag{39}$$

It is then necessary to analyze the dynamics of the deviations in (38) to adjust the gains L_1 and L_2 in L to ensure the boundedness and the positivity of $\underline{e}_k$ and $\overline{e}_k$. This can be done as the result of an LMI optimisation procedure, as stated in the following theorem.

Theorem 3. *Given a scalar $\gamma > 0$, the system (29) is an observer of system (1), if there exists a matrix $P = P^T > 0 \in I\!R^{2n \times 2n}$, matrices K_1 and $K_2 \in I\!R^{n \times p}$ and invertible matrices G_1 and $G_2 \in I\!R^{n \times n}$, such that the following LMI holds*

$$\begin{bmatrix} I_{2n} - P & 0 & A_0^T G - C_0^T K^T \\ 0 & -\gamma^2 I_m & B_0^T G \\ G^T A_0 - K C_0 & G^T B_0 & P - G^T - G \end{bmatrix} < 0 \tag{40}$$

where G_0 and K_0 are the block diagonal matrices defined by:

$$G = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \quad K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \tag{41}$$

After solving the LMI (40), the observer gain is defined by:

$$L = (G^T)^{-1} K \tag{42}$$

Proof. The evolution of the deviations is governed by (38), depending on the gains L_1 and L_2. The objective of the synthesis of the parameters L_1 and L_2 is twofold: ensure the stability of the matrix $A_0 - L\,C_0$ and reduce the influence of the disturbance-like term ω on the error e. This is translated as follows:

$$\begin{array}{l} \lim_{k \to \infty} e_k = 0 \quad \text{if} \quad \omega_k = 0 \\ \dfrac{\| e_k \|_2}{\| \omega_k \|_2} < \gamma \quad \text{if} \quad \omega_k \neq 0 \end{array} \tag{43}$$

Let $V(e_k) = e_k^T P e_k$ be a Lyapunov candidate function. The state estimation error convergence is guaranteed and the gain $\mathcal{L}_2$ of the transfer from ω_k to e_k is bounded by γ if:

$$V(e_{k+1}) - V(e_k) + e_k^T e_k - \gamma^2 \omega_k^T \omega_k < 0 \tag{44}$$

Defining $\Psi = A_0 - L\,C_0$ and using (38), one obtains:

$$V(e_{k+1}) - V(e_k) = e_k^T \Psi^T P \Psi e_k + 2 e_k^T \Psi^T\, P\, B_0\, \omega_k + \omega_k^T\, B_0^T\, P\, B_0\, \omega_k - e_k^T P e_x \tag{45}$$

and, with (45), (44) becomes:

$$e_k^T (\Psi^T P \Psi - P + I) e_k + 2 e_k^T \Psi^T\, P\, B_0\, \omega_k + \omega_k^T B_0^T P B_0 \omega_k - \gamma^2 \omega_k^T \omega_k < 0 \tag{46}$$

The constraint (46) is ensured if there exists matrices P and L satisfying:

$$\begin{bmatrix} I-P & 0 \\ 0 & -\gamma^2 I_m \end{bmatrix} + \begin{bmatrix} \Psi^T \\ B_0^T \end{bmatrix} P \begin{bmatrix} \Psi & B_0 \end{bmatrix} < 0 \tag{47}$$

From lemma 3 of [13], searching P and L satisfying (47) is known to be equivalent to searching P, G and L satisfying

$$\begin{bmatrix} I-P & 0 & \Psi^T G \\ 0 & -\gamma^2 I_m & B_0^T G \\ G^T\Psi & G^T B_0 & P-G-G^T \end{bmatrix} < 0 \tag{48}$$

which is the LMI (40), up to the variable change $G^T L = K$. □

The Theorem 3 provides an observer design fulfilling the constraints (43). However, this approach suffers from a certain conservatism mainly due to the structure of the equations (36) expressing the evolution of reconstruction errors. Indeed, in (36), and thus in (38), the dependence between the variables x_k and $|\, x_k \,|$ is not taken into account. Accordingly to (21), the magnitude of the lower and upper bounds of the state estimate can be expressed as a multi-model based on

$$\begin{array}{rcl} |\, \underline{\hat{x}}_k \,| & = & \sum_{i=1}^{r} \mu_i^1(\underline{\hat{x}}_k)\, E_i\, \underline{\hat{x}}_k \\ |\, \overline{\hat{x}}_k \,| & = & \sum_{j=1}^{r} \mu_j^2(\overline{\hat{x}}_k)\, E_j\, \overline{\hat{x}}_k \end{array} \tag{49}$$

where the activating functions μ_i^1 and μ_j^2 satisfy the convex sum property (23) and $r = 2^n$. Using this multi-model expression of the magnitude of the lower and upper bounds of the state estimation error (37), these equations become

$$\begin{array}{rcl} \underline{e}_{k+1} = & (H_{11} - L_1 C)\underline{e}_k - H_{12}\overline{e}_k - \Delta\, x_k - H_{13}\displaystyle\sum_{i=1}^{r}\mu_i^1(\underline{\hat{x}})\, E_i\, \underline{\hat{x}} - H_{14}\sum_{j=1}^{r}\mu_j^2(\overline{\hat{x}})\, E_j\, \overline{\hat{x}} \\ \overline{e}_{k+1} = & -H_{21}\underline{e}_k + (H_{22} - L_2 C)\,\overline{e}_k - \Delta_k\, x_k - H_{23}\displaystyle\sum_{i=1}^{r}\mu_i^1(\underline{\hat{x}})\, E_i\, \underline{\hat{x}} - H_{24}\sum_{j=1}^{r}\mu_j^2(\overline{\hat{x}})\, E_i\, \overline{\hat{x}} \end{array} \tag{50}$$

Since the functions μ_i^1 and μ_j^2 satisfy (23), these two equations can be written as

$$\begin{array}{rcl} \underline{e}_{k+1} & = & \displaystyle\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i^1(\underline{\hat{x}})\mu_j^2(\overline{\hat{x}})\,((H_{11} - L_1 C + H_{13}E_i)\underline{e}_k \\ & & -(H_{12} + H_{14}E_j)\overline{e}_k - (\Delta + H_{13}E_i + H_{14}E_j)x_k) \\ \overline{e}_{k+1} & = & \displaystyle\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i^1(\underline{\hat{x}})\mu_j^2(\overline{\hat{x}})\,(-(H_{21} - H_{23}E_i)\underline{e}_k \\ & & +(H_{22} - L_2 C - H_{24}E_j)\overline{e}_k - (\Delta + H_{23}E_i + H_{24}E_j)x_k) \end{array} \tag{51}$$

With (51), the augmented system (38) becomes

$$e_{k+1} = \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i^1(\underline{\hat{x}})\mu_j^2(\overline{\hat{x}})\,((A_{i,j} - L\,C_0)\,e_k + B_{i,j}\,x_k) \tag{52}$$

where L and C_0 are already defined in (39) and A_{ij} and B_{ij} are given by

$$\begin{array}{rcl} A_{i,j} & = & \begin{bmatrix} H_{11}+H_{13}E_i & -(H_{12}+H_{14}E_j) \\ -(H_{21}-H_{23}E_i) & H_{22}-H_{24}E_j \end{bmatrix} \\ B_{i,j} & = & -\begin{bmatrix} \Delta+H_{13}E_i+H_{14}E_j \\ \Delta+H_{23}E_i+H_{24}E_j \end{bmatrix} \end{array} \tag{53}$$

This leads to the following theorem.

Theorem 4. *Given a scalar $\gamma > 0$, the system (29) is an observer of system (1), if there exists a matrix $P = P^T > 0 \in \mathbb{R}^{2n\times 2n}$, matrices K_1 and $K_2 \in \mathbb{R}^{n\times p}$ and invertible matrices G_1 and $G_2 \in \mathbb{R}^{n\times n}$, such that*

$$\mathcal{M}_{ij} + \mathcal{M}_{ji} < 0,\ i,j = 1,\ldots,r \tag{54}$$

where $\mathcal{M}_{ij}$ is defined by

$$\mathcal{M}_{ij} = \begin{bmatrix} I_{2n}-P & 0 & A_{ij}^T G - C_0^T K^T \\ 0 & -\gamma^2 I_n & B_{ij}^T G \\ G^T A_{ij} - K C_0 & G^T B_{ij} & P - G^T - G \end{bmatrix}$$

and where G_0 and K_0 are the block diagonal matrices defined in (41)*. After solving the LMI (54), the observer gain is defined by* (42)*.*

Proof. The proof follows the same lines of the one of Theorem 3: firstly the quadratic Lyapunov function $V(e_k) = e_k^T P e_k$ is defined and then its decreasing along the trajectory of (52) is studied, taking benefits of the positiveness of the activating functions satisfying (23). □

Remark 2. *In Theorems 3 and 4, the attenuation level γ can be minimized, by considering the classical variable change $\bar{\gamma} = \gamma^2$ and by minimizing $\bar{\gamma}$ under the LMI constraints of Theorems 3 or 4.*

Remark 3. *The corollary 1 can be used to introduce some additional degrees of freedom in the LMI optimization and thus reduce the conservatism implied by a simple quadratic Lyapunov function. Nevertheless a trade-off should thus be made between relaxing the conservatism and limiting the computational burden of seeking numerous LMI variables.*

4.3. Example

The example 2.4 is continued with the same system matrices (17) and an output defined by $C = [1\ 0]$. Figure 2 depicts the evolution of the two inputs and the two states. As in example 2.4, the red plot is a possible state trajectory given the parameter uncertainty and the green and blue ones are the upper and lower bounds of each state trajectory. Solving the LMI conditions (54), the

following solutions are obtained

$$P = \begin{pmatrix} 3.011 & 0.328 & 1.724 & 0.636 \\ 0.328 & 12.266 & 0.636 & 0.727 \\ 1.724 & 0.636 & 3.011 & 0.328 \\ 0.636 & 0.727 & 0.328 & 12.266 \end{pmatrix},$$

$$G_1 = \begin{pmatrix} 2.766 & .424 \\ 0.424 & 18.359 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 2.766 & 0.424 \\ 0.424 & 18.359 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 2.522 & -3.168 \end{pmatrix}^T, \quad K_2 = \begin{pmatrix} 2.522 & -3.168 \end{pmatrix}^T$$

resulting in the observer gains (42)

$$L_1 = \begin{bmatrix} 0.942 & -0.194 \end{bmatrix}^T, \quad L_2 = \begin{bmatrix} 0.942 & -0.194 \end{bmatrix}^T$$

With these gains, the observer can be simulated with (29) and one can see, on the figure 3, that the actual system state (red plot) appears to be in the envelope provided by the interval observer (green and red lines). It can also be pointed out that the envelope provided by the interval observer gives a thinner estimate of the actual state of the system. The comparison of figures 2 and 3 clearly shows the interest of the interval observer: firstly it reconstructs the interval system state from the measured inputs and outputs, secondly the obtained radius of the envelope is smaller than the one provided by the direct simulation of the state equation interval.

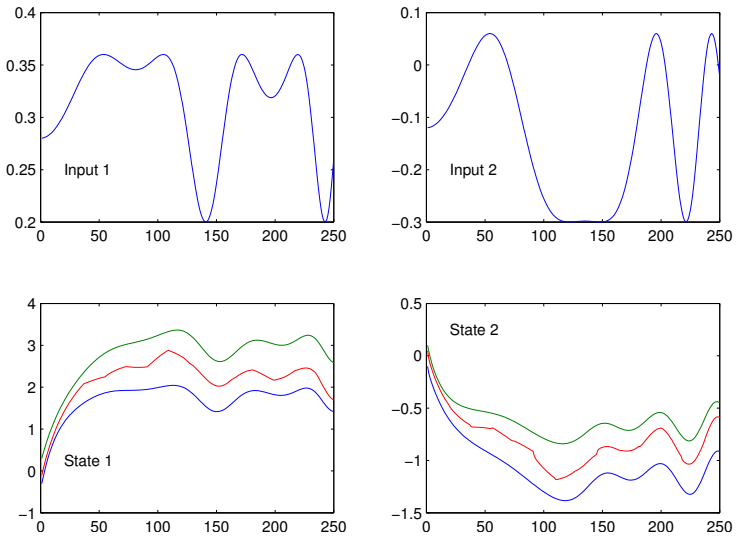


Figure 2: Simulation of a system with interval parameters

Remark 4. *Following [5], additional degrees of freedom can be introduced in the observer design by tuning an extra parameter, denoted F, in the following*

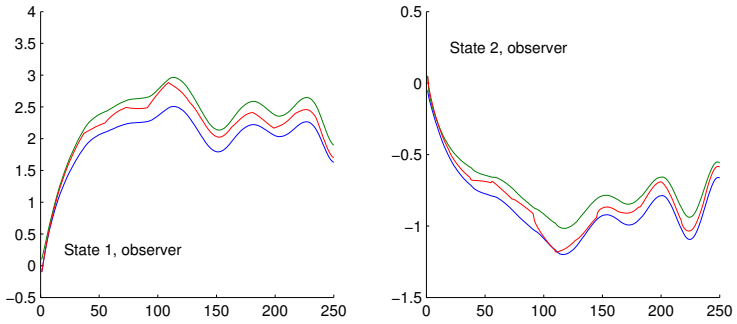


Figure 3: State reconstruction of a system with interval parameters

observer definition:

$$\begin{cases} \underline{\hat{x}}_{k+1} &= \frac{1}{2}\left[(A+ \mid A \mid)\,\underline{\hat{x}}_k + (A- \mid A \mid)\,\overline{\hat{x}}_k - (\Delta + \Delta * S_A) \mid \underline{x}_k \mid \right. \\ & \quad \left. - (\Delta - \Delta * S_A) \mid \overline{\hat{x}}_k \mid\right] + L_1(y_k - C\underline{\hat{x}}_k) - F(\overline{\hat{x}}_k - \underline{\hat{x}}_k) + B\,u_k \\ \overline{\hat{x}}_{k+1} &= \frac{1}{2}\left[(A- \mid A \mid)\,\underline{\hat{x}}_k + (A+ \mid A \mid)\,\overline{x}_k + (\Delta - \Delta * S_A) \mid \underline{\hat{x}}_k \mid \right. \\ & \quad \left. + (\Delta + \Delta * S_A) \mid \overline{\hat{x}}_k \mid\right] + L_2(y_k - C\overline{\hat{x}}_k) + F(\overline{\hat{x}}_k - \underline{\hat{x}}_k) + B\,u_k \end{cases} \tag{55}$$

5. Conclusion

The proposed approach provides a systematic state estimation for a system with uncertain but bounded parameters. It significantly differs from earlier work in showing that the system model can be used to simultaneously model the lower and upper bounds of the system state trajectory. Both models are structured in a polytopic form and allow to synthesize an interval observer. This approach can easily be extended to other situations. On the one hand, the inclusion of uncertainty affecting all system state matrices appears as a direct extension. On the other hand, one can consider changing the observer interval to encompass the case of interval measures of the inputs and outputs.

Appendix A. Some details on the multi-model form of the absolute value function

The simulation of the interval model (18) does not pose any particular difficulty, except the fact that the equation number is twice the one of the initial system. However, the presence of the absolute value operator can make tedious the stability analysis and the observer design. For this reason, it is proposed to reformulate the interval model by replacing the absolute value operator by a polytopic model that is more easily handled.

To begin with, let us consider a two component vector $z = [z_1 \quad z_2]^T$. Its component wise absolute value is then defined by $|z| = [|z_1| \quad |z_2|]^T$ and it can be expressed by:

$$\begin{cases} \mid z_i \mid = \mu_{i1}(z_i) \,.\, (z_i) + \mu_{i2}(z_i) \,.\, (-z_i) \\ \mu_{i1}(z_i) = \dfrac{1 + \operatorname{sgn}(z_i)}{2} \\ \mu_{i2}(z_i) = 1 - \mu_{i1}(z_i) \end{cases} , \; i = 1, 2 \tag{A.1}$$

Each scalar $\mid z_i \mid$ is expressed in terms of both its own weight functions μ_{i1} and μ_{i2}. In order to express all of these scalars with the same weight function set, they can be equivalently written as:

$$\mid z_i \mid = (\mu_{i1} + \mu_{i2}) \left(\mu_{i1} \,.\, (z_i) + \mu_{i2} \,.\, (-z_i)\right), \; i = 1, 2 \tag{A.2}$$

Defining $\mu_1 = \mu_{11}\mu_{21}$, $\mu_2 = \mu_{11}\mu_{22}$, $\mu_3 = \mu_{12}\mu_{21}$, $\mu_4 = \mu_{12}\mu_{22}$ and

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} E_3 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad E_4 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{A.3}$$

the absolute values $\mid z_1 \mid$ and $\mid z_2 \mid$ are expressed under a polytopic form which depends on the same weight function set, as follows :

$$\mid z \mid \;\; = \;\; \sum_{i=1}^{4} \mu_i \, E_i z \tag{A.4}$$

Based on the previous developments in the two dimensional case, the polytopic writing can easily be generalized to any vector $z \in \mathbb{R}^r$, as described in [37] or [29]. The common weighting functions $\mu_i(z)$ are defined by products of the weighting functions depending on each vector component (namely $\mu_{i1}(z_i)$ and $\mu_{i2}(z_i)$) and the matrices E_i are diagonal matrices which entries are equal to the extremal values of $\mu_{i1}(z_i)$ and $\mu_{i2}(z_i)$, i.e. 1 or -1.

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