Impact of output metric on the resolution of mode-localized MEMS resonant sensors

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***Abstract*—In this paper, we investigate how additive noise, e.g. thermomechanical noise, impacts the resolution of mode-localized resonant sensing architectures based on two weakly-coupled resonators. Existing work suggests that the resolution of these sensors can be improved by decreasing the coupling coefficient of the resonators. The present work gives an analytical proof that this result does not hold when the ratio of the motional amplitudes of the resonators is used as an output metric, and that, in this case, the sensor resolution is actually independent of the coupling strength. We then extend our proof, supported by transient simulations of a simple model, to other output metrics.**

Keywords— resonant sensors; coupled resonators; thermomechanical noise.

I. INTRODUCTION

Resonant sensing based on coupled MEMS resonators has received considerable interest in the past 5 years [1-5]. The "mode-localization" approach proposed in [1] relies on two (or more) matched resonators coupled through a "weak" restoring force (weak with respect to each resonator's own restoring force). It can be shown that, when the system is stimulated close to one of its resonance frequencies, the ratio of the motional amplitudes of the resonators is highly sensitive to any mismatch of their - uncoupled - natural frequencies (Fig. 1). Other output metrics, such as the relative shift in eigenstate, as in [1], have also been proposed [2]. The sensitivity of such measurements is inversely proportional to the coupling strength (i.e. the ratio of the coupling stiffness to each resonator's stiffness). One may use amplitude ratio measurements for high-sensitivity differential sensing, where one resonator is used as a reference, the other as a sensing cell, to detect infinitesimal changes of its stiffness or mass. In [3], it is reported that the resolution of these sensors is linearly proportional to the coupling strength and that it may consequently surpass that of "conventional" sensors based on a single resonator.

An alternative approach to differential resonant sensing consists in using mutually injection-locked oscillators (MILOs) based on matched MEMS resonators [4-5]. In this closed-loop approach, an electronic mixer is used to enforce the resonators into a synchronized oscillation state, in which the phase difference between the resonators becomes highly sensitive to any stiffness or mass mismatch and can thus provide a differential measurement of a physical quantity of interest [5]. An interesting result in [4] is that, in a MILO, decreasing the coupling strength (in that case, the ratio of the injection signal to each resonator's own feedback signal) results in an increase of the sensor sensitivity, as in mode-localized approaches, but does not improve its resolution.

In spite of many conceptual differences (reactive vs. non-reactive coupling, closed-loop vs. open-loop, etc.), the MILO-based and mode-localized approaches have several similarities. In this paper, we prove that mode-localized sensors based on the amplitude ratio output metric provide measurements whose resolution is independent on coupling strength. Thus, the gain in sensitivity made by decreasing the coupling strength is in fact compensated by an amplification of the additive noise in the system (e.g. thermomechanical), as in the MILO-based approach, and the resolution obtained with this output metric is in fact comparable to that of conventional resonant sensors. We also show that this result is valid for other output metrics such as their phase difference.

In section II, we set the framework and the notations used in our proof. In section III, we study the equations governing the steady-state of the system, and establish the sensitivity of the amplitude ratio and of the phase difference output metrics. In section IV, we study the case when the system is perturbed by additive noise sources In section IV, we show that similar results hold in the case when the phase difference of the resonators is used as an output metric. Our analytical results are validated with transient simulations of the system. In section V, we compare the results obtained in this paper to prior work, and discuss the relative merits of MILOs, mode-localized approaches, and single-oscillator approaches.

II. NOTATIONS AND FRAMEWORK

Consider two nearly-identical coupled linear resonators described by the following non-dimensional model:

$$\begin{aligned} x+\frac{1}{Q}\frac{dx}{dt}+\frac{d^2x}{dt^2} &= \kappa\left(y-x\right)+f_x\left(t\right)+n_x\left(t\right) \\ \left(1+\varepsilon\right)y+\frac{1}{Q}\frac{dy}{dt}+\frac{d^2y}{dt^2} &= \kappa\left(x-y\right)+n_y\left(t\right) \end{aligned} \tag{1}$$

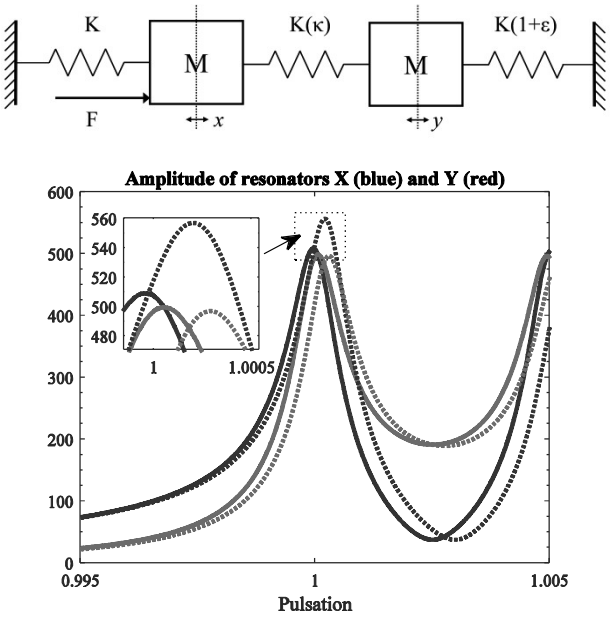


Fig. 1. Top: symbolic representation of a mode-localized sensor as a mass-spring system. Bottom: simulated amplitude responses for varying pulsation, in the case Q =1000, $\kappa = 5\times10^{-3}$, $\varepsilon = 0$ (solid line) $\varepsilon = 10^{-3}$ (dotted line).

where x and y denote the (non-dimensional) position of each resonator, Q >>1 is their quality factor, κ <<1 is the relative coupling strength (i.e. the ratio of the coupling stiffness to the nominal stiffness of the resonators), ε <<1 is the relative stiffness mismatch, $f_x = F\sin\omega t$ is an external force, and n_x and n_y are additive perturbations. These may for example represent the contributions of thermomechanical or electronic noise in the system.

When ε=0 and $\omega \approx 1$, the two resonators oscillate in phase, with the same amplitude. A small variation of ε results in a comparatively large change of the oscillation amplitudes of the resonators (as depicted in Fig. 1), and also in their phase difference. The sensitivity of these output metrics is studied in section III.

III. Sensitivity of Mode-Localized Resonant Sensors

In the absence of perturbations ($n_x = n_y = 0$), the steady-state solution of (1):

$$\begin{aligned} x(t) &= X_\varepsilon \sin(\omega t + \phi_\varepsilon) \\ y(t) &= Y_\varepsilon \sin(\omega t + \phi_\varepsilon + \psi_\varepsilon) \end{aligned} \tag{2}$$

is given by a nonlinear set of equations $\underline{\mathbf{g}}(X_\varepsilon, Y_\varepsilon, \phi_\varepsilon, \psi_\varepsilon) = \underline{\mathbf{0}}$:

$$\begin{cases} \left(1+\kappa-\omega^2\right)X_\varepsilon - \kappa Y_\varepsilon\cos\psi_\varepsilon - F\cos\phi_\varepsilon = 0 \\ \dfrac{\omega}{Q}X_\varepsilon - \kappa Y_\varepsilon\sin\psi_\varepsilon + F\sin\phi_\varepsilon = 0 \\ \left(1+\kappa+\varepsilon-\omega^2\right)Y_\varepsilon - \kappa X_\varepsilon\cos\psi_\varepsilon = 0 \\ \dfrac{\omega}{Q}Y_\varepsilon + \kappa X_\varepsilon\sin\psi_\varepsilon = 0 \end{cases} \tag{3}$$

From (3), it is trivial to establish that:

$$\begin{cases} \tan\psi_\varepsilon = -\dfrac{\omega}{Q\left(1+\kappa+\varepsilon-\omega^2\right)} \\ \dfrac{X_\varepsilon}{Y_\varepsilon} = \dfrac{1}{\kappa}\left(\left(1+\kappa+\varepsilon-\omega^2\right)^2 + \dfrac{\omega^2}{Q^2}\right)^{1/2} \end{cases}, \tag{4}$$

For a given value of the pulsation ω, the sensitivity to mismatch of the phase difference or of the amplitude ratio can be derived from (4) by differentiation with respect to ε. In particular, when ε=0 and ω=1, we find:

$$\begin{cases} S_{\psi_0} = \dfrac{Q}{1+Q^2\kappa^2} \approx \dfrac{1}{Q\kappa^2} \\ S_{X_0/Y_0} = \dfrac{Q}{\left(1+Q^2\kappa^2\right)^{1/2}} \approx \dfrac{1}{\kappa} \end{cases}, \tag{5}$$

where the approximations hold provided κQ 1>>, i.e. the two peaks in the frequency response of either of the resonators are well-resolved.

IV. Resolution of Mode-Localized Resonant Sensors

When the system is perturbed by additive noise sources n_x and n_y, one may look for a solution of (1) of the form:

$$\begin{aligned} x(t) &= \left(X_0 + \delta X(t)\right)\sin\left(\omega t + \phi_0 + \delta\phi(t)\right) \\ y(t) &= \left(Y_0 + \delta Y(t)\right)\sin\left(\omega t + \phi_0 + \psi_0 + \delta\psi(t)\right) \end{aligned}, \tag{6}$$

where the amplitude and phase fluctuations are assumed to be small and slowly-varying. Differential equations governing the slow dynamics (hence the spectra) of these fluctuations may be derived using a number of perturbation methods, as in [4]. However, in the present paper, our interest lies in the near-DC terms of these fluctuations (corresponding to close to the carrier fluctuations of x and y), from which the "ultimate" resolution of the different output metrics may be derived. These near-DC terms are governed by a linear set of equations:

$$\underline{\underline{\mathbf{J}}} \times \left[\delta X \quad \delta Y \quad \delta\phi \quad \delta\psi\right]^T = \left[n_x^s \quad n_x^c \quad n_y^s \quad n_y^c\right]^T, \tag{7}$$

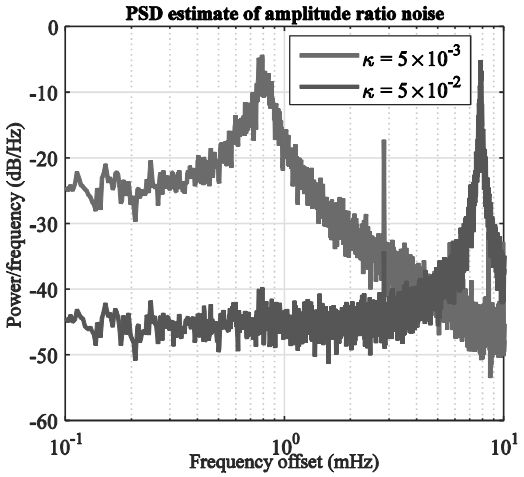


Fig. 2. Spectrum of amplitude ratio fluctuations. Simulation parameters are ω=1, Q=1000, $\varepsilon = 0$. The peak in the response corresponds to the resonance of the out-of-phase mode at frequency offset $\Delta f = \kappa / \pi$.

where $\underline{\underline{\mathbf{J}}}$ is the Jacobian of $\underline{\mathbf{g}}$ at steady-state $\left(X_\varepsilon, Y_\varepsilon, \phi_\varepsilon, \psi_\varepsilon\right)$, and the right-hand term corresponds to the projections of the additive noise on sin(ωt) and cos(ωt). This linear system can be solved analytically. The resolution of each output metric can then be obtained as the ratio of the magnitude of its fluctuations on a (near-DC) frequency band to its sensitivity.

For example, close to in-phase resonance and supposing $\kappa Q >> 1$, we find, when ε=0:

$$\begin{cases} \delta\psi \approx \dfrac{1}{\kappa Q}\dfrac{2n_y^c}{F} \\ \dfrac{\delta X - \delta Y}{X_0} \approx -\dfrac{1}{\kappa Q}\dfrac{2n_y^s}{F} \end{cases}, \tag{8}$$

where the first line corresponds to the phase difference fluctuations, and the second line to the amplitude ratio fluctuations. Both quantities are inversely proportional to κ, as confirmed by transient simulations of (1): for example, Fig. 2 shows the spectrum of the amplitude ratio fluctuations obtained by simulation for two values of κ. The power of the near-DC fluctuations is multiplied by 100 when κ is divided by 10, as predicted by (8). The simulations also show that the spectrum of the phase difference fluctuations is superposed to that of the amplitude ratio fluctuations.

From (5) and (8), the resolution of the amplitude ratio output metric in a frequency band Δf may be expressed as:

$$R_{X_0/Y_0} = \frac{2}{Q}\frac{N}{F}\Delta f^{1/2}, \tag{9}$$

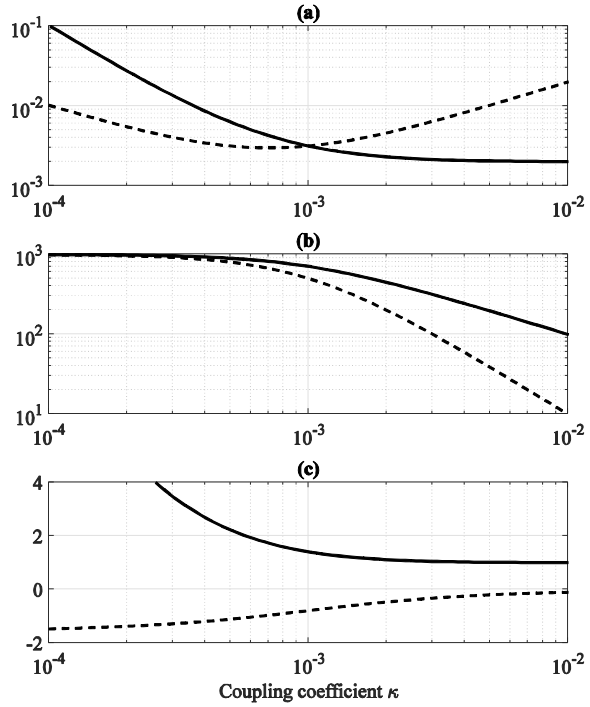


Fig. 3. Resolution (a), sensitivity (b) and value (c) of the amplitude ratio (full lines) and phase difference (dashed lines) output metrics vs. coupling coefficient κ, resulting from the analytical solution of (7). Calculation parameters are ω=1, Q=1000, $\varepsilon = 0$, F=1. The resolution is calculated for a unit noise magnitude and frequency band (N=1, Δf=1).

where N corresponds to the magnitude of the additive perturbations. Hence, the resolution of this output metric is independent of κ. Likewise, the resolution of phase difference measurements can be expressed as:

$$R_{\psi_0} = 2\kappa\frac{N}{F}\Delta f^{1/2}. \tag{10}$$

Thus, the resolution of this output metric is proportional to κ. However, note that (8-10) are established under the assumption that $\kappa Q \gg$. Consequently, as far as metrological performance is considered and provided the assumption is verified, the amplitude ratio is a better output metric than the phase difference.

We show in Fig. 3 how both output metrics behave in terms of resolution and sensitivity for different values of κ. These results are obtained by solving (3) and (7) without any simplifying assumption. It is remarkable that, when $\kappa Q \le 1$, the phase difference becomes a better output metric than the amplitude ratio. One should also stress that the resolution in Fig. 3-a is plotted for a unit frequency band, regardless of the value of κ, even though our simulations (Fig. 2) show that the bandwidth of amplitude ratio or phase difference fluctuations is clearly dependent of κ.

V. DISCUSSION

It may seem surprising that the above results, in particular equation (9), are in contradiction with those in [3], which predict that the resolution of the amplitude ratio output metric is proportional to κ (is, in fact, on the order of κ times the resolution we predict). However, upon close reading, there seems to be a mistake in the derivation of equation (21) in [3]: the authors are clearly interested in determining thermomechanically-induced "close to the carrier" fluctuations of the modal coordinates of the weakly-coupled system. Yet, they integrate the corresponding spectral densities in an angular frequency band $\Delta\omega$ close to ω=0, instead of ω=1 (so that near-DC modal coordinate fluctuations are estimated, rather than close-to-the-carrier ones or near-DC amplitude fluctuations). Consequently, the noise in the system is underestimated by a factor Q. This error, combined with other approximations, leads to an erroneous prediction, in [3], of the resolution of mode-localized sensors.

Within the limits of the framework of the present paper, it appears that mode-localized sensors do not provide any particular advantage in terms of resolution compared to classical resonant sensors based on the measurement of the oscillation frequency of a single resonator [6], [7]. As a basis for comparison, consider an oscillator governed by:

$$(1+\varepsilon)x + \frac{1}{Q}\frac{dx}{dt} + \frac{d^2x}{dt^2} = f_x(t) + n_x(t), \tag{11}$$

where f_x is a feedback-generated (harmonic) force in quadrature with x. With our notations, the sensitivity to ε and the resolution of the angular frequency of such a system are:

$$S_{\omega_0} = \frac{1}{2}, \tag{12}$$

$$\delta\omega = \frac{1}{Q}\frac{n_s^x}{F} \Rightarrow R_{\omega_0} = \frac{1}{Q}\frac{N}{F}\Delta f^{1/2}. \tag{13}$$

Likewise, consider a MILO-based sensor relying on active "strong" coupling of two resonators through two actuation forces generated with a nonlinear mixer [4] (Fig. 4). With the digital mixing scheme studied in [4-5], the sensitivity to ε and the resolution of the phase difference are:

$$S_{\psi_0} = Q, \tag{14}$$

$$\delta\psi = \frac{1}{Q}\frac{n_s^y - n_s^x}{F} \Rightarrow R_{\psi_0} = \frac{\sqrt{2}}{Q}\frac{N}{F}\Delta f^{1/2}. \tag{15}$$

Thus, all these architectures have output metrics with comparable resolutions (9) (13) (15), even though their sensitivities (5) (12) (14) differ vastly. In fact, one may design "weakly-coupled" MILO architectures with sensitivities much larger than (14): however, as mentioned in the introduction, this does not improve the resolution of the system, and raises several realization issues (implementation of accurate analog gains, for example) and metrological issues as well (reduction of the locking range of the MILO, and increase of the sensor response time) [4].

In fact, it should be noted that the increase in sensitivity of (mode-localized or MILO-based) coupled sensors usually comes at the cost of a decrease of the range of the measurement, compared to the single resonator case. However, this drawback may be compensated for with proper feedback control techniques (e.g. adjusting the stiffness of resonator x by changing its bias voltage to keep track of the variations of ε), but entails added complexity to the system.

It is our opinion that, more than their large sensitivity, the main interest of sensors based on (actively or passively) coupled resonators is that they can provide differential measurements (i.e. that are insensitive to drift at first order) of the physical quantity of interest. On the other hand, the design of differential architectures based on two nominally-identical, *uncoupled* oscillator loops is quite challenging. In fact, the closer the oscillators are to each other (in order to better eliminate drift), the more likely it becomes that unwanted, parasitic couplings (electrical, mechanical, etc.) affect the normal behavior of the system (through modulation, frequency pulling and locking, etc.). As we have shown in this paper, MILOs and mode-localized approaches are two solutions to this issue with comparable metrological performance.

To be complete, our analysis of mode-localized sensor output metrics should be expanded with a study of their dynamic characteristics. Other application- and context-dependent issues (ease of implementation, compatibility of these approaches with VLSI) should also be considered.

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