

Phase transition for continuum Widom-Rowlinson model with random radii

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Abstract

In this paper we study the phase transition of continuum Widom-Rowlinson measures in \mathbb{R}^d with q types of particles and random radii. Each particle x_i of type i is marked by a random radius r_i distributed by a probability measure Q_i on \mathbb{R}^+ . The particles of same type do not interact each other whereas particles x_i and x_j with different type $i \neq j$ interact via an exclusion hardcore interaction forcing $r_i + r_j$ to be smaller than $|x_i - x_j|$. In the integrable case (i.e. $\int r^d Q_i(dr) < +\infty$, $1 \leq i \leq q$), we show that the Widom-Rowlinson measures exhibit a standard phase transition providing uniqueness, when the activity is small, and co-existence of q ordered phases, when the activity is large. In the non-integrable case (i.e. $\int r^d Q_i(dr) = +\infty$, $1 \leq i \leq q$), we show another type of phase transition. We prove, when the activity is small, the existence of at least $q + 1$ extremal phases and we conjecture that, when the activity is large, only the q ordered phases subsist. We prove a weak version of this conjecture by showing that the symmetric Widom-Rowlinson measure with free boundary condition is a mixing of the q ordered phases if and only if the activity is large.

Key words: Gibbs point process, DLR equation, Boolean model, continuum percolation, random cluster model, Fortuin-Kasteleyn representation.

1 Introduction

In this paper we deal with the non-symmetric continuum Widom-Rowlinson model in \mathbb{R}^d with q types of particles and with random radii. Each type of particle $1 \leq i \leq q$ has its proper activity parameter $z_i > 0$ and a proper probability measure Q_i on \mathbb{R}^+ for the distribution of radii. Each particle x_i of type i is marked by a random radius r_i distributed by Q_i . The particles of the same type do not interact each other whereas a particle x_i and x_j with different type $i \neq j$ interact via an exclusion hardcore interaction forcing $r_i + r_j$ to be

smaller than $|x_i - x_j|$. This model can be viewed as a collection of q Boolean models, each of intensity z_i and radii distribution Q_i , $i = 1 \dots q$, conditioned to not overlap each other.

This model is a generalisation of the simple and beautiful model introduced in the late 1960' by Widom and Rowlinson [20] where $q = 2$ and the radii are deterministic. Its interest comes not only from its applicability in the description of a binary gas, but also from the fact that it was the very first continuum model for which a phase transition was rigorously proved, first by Ruelle [19] using the so-called Peierls' argument. A modern proof of this phase transition, relying on percolation arguments and a Fortuin-Kasteleyn representation, was done in [2, 10]. Involving the non-symmetric case with $q \geq 3$, phase transition results were proved in several articles such as [1, 17] using the Pirogov-Sinai theory. All these results concern the case of deterministic radii.

In the present paper, we investigate the random radii case which can be interpreted as a random media or as a size distribution of particles. We prove several phase transition results described below which are, depending on the distribution of radii, similar or different from the deterministic case.

The formal definition of Widom-Rowlinson measures is based on the standard DLR equations, which prescribe the local conditional distributions of the model, see definition 2.2. The existence and uniqueness of such solutions are not obvious. The set of solution is denoted by $WR(\mathbf{z}, \mathbf{Q})$ where $\mathbf{z} = (z_1, \dots, z_q)$ is the vector of activities and $\mathbf{Q} := (Q_1, \dots, Q_q)$ the vector of radii distributions. When the radii are uniformly bounded, a general existence result by Ruelle [18] ensures the existence of Widom-Rowlinson measures. When the radii are not bounded, a long range interaction occurs and the existence is more delicate. In a first theorem we prove the existence of Widom-Rowlinson measures without any assumption on activities or radii. The set $WR(\mathbf{z}, \mathbf{Q})$ is never empty.

We say that a phase transition occurs when the geometry of $WR(\mathbf{z}, \mathbf{Q})$ changes drastically with the choice of parameters \mathbf{z} , considering that \mathbf{Q} is fixed. In the case of deterministic radii, it is proved in papers mentioned above that $WR(\mathbf{z}, \mathbf{Q})$ is a singleton for \mathbf{z} small enough and that there exists \mathbf{z} large enough such that $WR(\mathbf{z}, \mathbf{Q})$ contains q distinct extremal ordered phases (Widom-Rowlinson measure with boundary condition full of particles with a prescribed type). Precisely in [1] it is proved that for any $(z_1, z_2, \dots, z_{q-1})$ large enough there exists z_q such that $WR(\mathbf{z}, \mathbf{Q})$ contains q distinct extremal ordered phases. This result is based on an extension of the Pirogov-Sinai theory of phase transitions in general lattice spin systems to continuum systems. In our random radii setting we do not obtain such a general result. Actually the unbounded radii seem to be a serious and difficult obstacle for using Pirogov-Sinai machinery. Nevertheless, in the integrable setting, $\int r^d Q_i(dr) < +\infty$, $1 \leq i \leq q$, we show first that $WR(\mathbf{z}, \mathbf{Q})$ is a singleton for small activities \mathbf{z} . And, in the symmetric integrable case $Q_i = Q$, $1 \leq i \leq q$, we show that for activities $\mathbf{z} = (z, z, \dots, z)$ large enough, the set $WR(\mathbf{z}, \mathbf{Q})$ contains q distinct extremal ordered phases. These results derive from a coupling result in [14] and a Fortuin-Kasteleyn representation and a percolation result developed in [15]. In conclusion, in the integrable setting, $WR(\mathbf{z}, \mathbf{Q})$ exhibits a standard phase transition similar to the deterministic radii case.

Let us now turn to the most interesting and surprising result of the present paper. In the non-integrable case (i.e. $\int r^d Q_i(dr) = +\infty$, $1 \leq i \leq q$), we show another type of phase transition on the geometry of $WR(\mathbf{z}, \mathbf{Q})$. First, it is easy to see that $WR(\mathbf{z}, \mathbf{Q})$ contains always q ordered phases corresponding each to a Poisson point process with only one type of particle whose the balls covered the full space \mathbb{R}^d . But we prove, when the activity is small enough, the existence of a $q + 1$ -th extremal phase. As far as we know, the existence of such a $q + 1$ -th extremal phase has never been observed for a continuum Widom-Rowlinson model with deterministic radii. Let us note that our result is valid in the non-symmetric setting and that the proof is not based on the Pirogov-Sinai theory. The main ingredient is a discrimination by specific entropy. We show that the Widom-Rowlinson measure with free boundary condition has a specific entropy smaller than any ordered phase.

Moreover we conjecture that in the non integrable setting, when the activity is large enough, the set $WR(\mathbf{z}, \mathbf{Q})$ is exactly the convex hull of the q ordered phases (i.e. for large activities, only the q ordered phases subsist and the disordered phase disappears). It would imply a phase transition result since the set $WR(\mathbf{z}, \mathbf{Q})$ would have exactly q extremal Gibbs measures for \mathbf{z} large and at least $q + 1$ extremal Gibbs measures for \mathbf{z} small. Our belief in this conjecture is based on a similar conjecture for the continuum random cluster presented in [7] and for which a heuristic proof is given. Moreover simulations in the sense of the conjecture have been implemented in [16].

In the present paper we prove a weak version of this conjecture by showing that the Widom-Rowlinson measure with free boundary condition is a mixing of the q ordered phases if and only if the activity is large. The proof of this last result is based on a renewal argument in a prescribed direction. It implies that the scales in thermodynamic limit are different for each direction. We believe that the result remains true with other choice of scale limits.

The paper is organized as follows. Section 2 introduces the notations, main definitions and tools. In Section 3 the results of the article are stated. The proof of Theorem 1 concerning the existence of Widom-Rowlinson measures is done in Section 4. In Section 5 we prove the existence of a $(q + 1)$ -th extremal phase in the non-integrable setting. Finally Section 6 is devoted to the proof of the weak version of the conjecture.

2 Preliminaries

2.1 Space

Let us consider the state space $S := \mathbb{R}^d \times \mathbb{R}^+$ with $d \geq 1$ being the dimension. Let Ω be the set of locally finite configurations ω on S . This means that $|\omega \cap (\Lambda \times \mathbb{R}^+)| < \infty$ for every bounded Borel set Λ of \mathbb{R}^d , with $|\omega|$ being the cardinal of the configuration ω . We write ω_Λ as a shorthand for $\omega \cap (\Lambda \times \mathbb{R}^+)$. The configuration space is embedded with the usual σ -algebra \mathcal{F} generated by the counting variables. To a configuration $\omega \in \Omega$ we associate the germ-grain

structure

$$L(\omega) := \bigcup_{X \in \omega} B(X),$$

where $B(X)$ is the closed ball associated to the marked point $X = (x, r)$, centred at x and of radius r .

Let q be an integer larger than 1 fixed through the paper, and consider the space $\Omega := \Omega^q$ of multi-index configurations $\omega := (\omega^1, \dots, \omega^q)$ embedded with the σ -algebra $\mathcal{F} := \mathcal{F}^{\otimes q}$. An element $i \in \{1, \dots, q\}$ is called a *type* or a *colour*. We write $L(\omega)$ as a shorthand for $\bigcup_{1 \leq i \leq q} L(\omega^i)$ and ω_Λ for $(\omega_\Lambda^1, \dots, \omega_\Lambda^q)$. We also write $(x, r) \in \omega$ when there exists a colour i such that $(x, r) \in \omega^i$.

2.2 Poisson point processes

For $z > 0$ and Q a probability measure on \mathbb{R}^+ , let $\pi^{z, Q}$ be the distribution on Ω of a Poisson point process with intensity measure $z\mathcal{L}^d \otimes Q$, where \mathcal{L}^d stands for the Lebesgue measure on \mathbb{R}^d . Recall that it means

- for every bounded Borel set Λ , the distribution of the number of points in $\Lambda \times \mathbb{R}^+$ under $\pi^{z, Q}$ is a Poisson distribution with parameter $z\mathcal{L}^d(\Lambda)$;
- given the number of points in every bounded Λ , the points are independent and uniformly distributed in Λ . Each point is marked by a mark distributed by Q and all the marks are independent.

We refer to [4] for details on Poisson point process.

For multi-index $\mathbf{z} = (z_1, \dots, z_q)$ and $\mathbf{Q} := (Q_1, \dots, Q_q)$, $\pi^{\mathbf{z}, \mathbf{Q}} := \pi^{z_1, Q_1} \otimes \dots \otimes \pi^{z_q, Q_q}$ is the distribution of a multi-index Poisson point process. For $\Lambda \subseteq \mathbb{R}^d$ bounded, we denote by $\pi_\Lambda^{\mathbf{z}, \mathbf{Q}}$ (respectively $\pi_\Lambda^{z, Q}$) the restriction of $\pi^{\mathbf{z}, \mathbf{Q}}$ (respectively $\pi^{z, Q}$) on $\Lambda \times \mathbb{R}^+$ (respectively $(\Lambda \times \mathbb{R}^+)^q$). Note that measures $\pi^{z, Q}$ and $\pi^{\mathbf{z}, \mathbf{Q}}$ are *stationary*, which means they are invariant under all translations of vector $x \in \mathbb{R}^d$.

The connectivity properties of the Poisson point process play a crucial role in this study. It changes drastically depending on an integrability condition, formalised in the following definition.

Definition 2.1. *A family \mathbf{Q} is said to satisfy the integrability assumption if for every colour i ,*

$$\int_{\mathbb{R}^+} r^d Q_i(dr) < \infty. \quad (2.1)$$

If not satisfied we say that \mathbf{Q} is not integrable (or in the extreme case). We say that \mathbf{Q} is completely non-integrable if for every $1 \leq i \leq q$, $\int_{\mathbb{R}^+} r^d Q_i(dr) = \infty$.

2.3 Widom-Rowlinson measures

The Widom-Rowlinson measures are defined with standard DLR equations requiring the probability measures to have prescribed conditional probabilities.

But first let us define the event \mathcal{A} of *authorized* (or *allowed*) configurations

$$\mathcal{A} = \{\omega \in \Omega, \forall 1 \leq i < j \leq q, L(\omega^i) \cap L(\omega^j) = \emptyset\}.$$

The Widom-Rowlinson specification on a bounded $\Lambda \subseteq \mathbb{R}^d$ with boundary condition ω_{Λ^c} is

$$\mathcal{P}_{\Lambda, \omega_{\Lambda^c}}^{wr}(d\omega'_\Lambda) := \frac{\mathbb{1}_{\mathcal{A}}(\omega'_\Lambda \cup \omega_{\Lambda^c})}{Z(\Lambda, \omega_{\Lambda^c})} \pi_{\Lambda}^{z, \mathbf{Q}}(d\omega'_\Lambda),$$

with

$$Z(\Lambda, \omega_{\Lambda^c}) := \int_{\Omega} \mathbb{1}_{\mathcal{A}}(\omega'_\Lambda \cup \omega_{\Lambda^c}) \pi_{\Lambda}^{z, \mathbf{Q}}(d\omega'_\Lambda).$$

Definition 2.2. *A probability measure \mathbf{P} on Ω is a Widom-Rowlinson measure of parameters \mathbf{z} and \mathbf{Q} , written $\mathbf{P} \in WR(\mathbf{z}, \mathbf{Q})$, if \mathbf{P} is stationary and if for every bounded Borel set $\Lambda \subseteq \mathbb{R}^d$ and every bounded measurable function f ,*

$$Z(\Lambda, \omega_{\Lambda^c}) > 0 \quad \mathbf{P}(d\omega) - a.s.; \quad (2.2a)$$

$$\int_{\Omega} f d\mathbf{P} = \int_{\Omega} \int_{\Omega} f(\omega'_\Lambda \cup \omega_{\Lambda^c}) \mathcal{P}_{\Lambda, \omega_{\Lambda^c}}^{wr}(d\omega'_\Lambda) \mathbf{P}(d\omega). \quad (2.2b)$$

For every Λ the equations (2.2b) are called DLR equations, named after Dobrushin, Lanford and Ruelle. Thanks to [9], a Widom-Rowlinson measure is a mixture of ergodic Widom-Rowlinson measures.

2.4 Stochastic domination

Let us discuss stochastic domination, which is going to be a key element of several proofs of the paper. Recall that an event $E \in \mathcal{F}$ is said *increasing* if for $\omega' \in E$ and $\omega \supseteq \omega'$, we have $\omega \in E$. Finally if P and P' are two probability measures on Ω , the measure P is said to *stochastically dominate* the measure P' , written $P' \preceq P$, if $P'(E) \leq P(E)$ for every increasing event $E \in \mathcal{F}$. Those definitions naturally extend to the case of the multi-index configuration space Ω .

The following proposition is a direct application of [11, Theorem 1.1] and gives a comparison between a Widom-Rowlinson measure and a Poisson point process.

Proposition 2.1 (Stochastic domination). *For every bounded $\Lambda \subseteq \mathbb{R}^d$ and every boundary condition ω_{Λ^c} , we have*

$$\mathcal{P}_{\Lambda, \omega_{\Lambda^c}}^{wr}(d\omega'_\Lambda) \preceq \pi_{\Lambda}^{z, \mathbf{Q}}.$$

Furthermore for every $\mathbf{P} \in WR(\mathbf{z}, \mathbf{Q})$

$$\mathbf{P} \preceq \pi^{z, \mathbf{Q}}.$$

3 Results

This section states the main results of the present article.

3.1 Existence

The first question of interest in statistical physics where the objects are defined through prescribed conditional equations, namely the DLR equations (2.2b), is the question of the existence of a probability measure solving those equations. The following theorem gives a positive answer to this question.

Theorem 1. *For all parameters z and Q , the set $WR(z, Q)$ is not empty.*

The proof of this theorem follows a standard scheme used for several models [5, 6, 7]. First, using the specific entropy, we build an accumulation point of a sequence of finite volume Gibbs measures. The difficulty is then to prove that the accumulation point satisfies the DLR equations (2.2b). This is done using the stochastic domination result of Proposition 2.1. A detailed proof is given in Section 4.

Remark 3.1. *The existence of a Widom-Rowlinson measure was already known in several cases. First in the cases of bounded radii, the Widom-Rowlinson interaction is finite range and therefore the existence is a consequence of a general result of Ruelle [18]. Second in the symmetric case where $z_1 = \dots = z_q$ and $Q_1 = \dots = Q_q$, a Widom-Rowlinson measure can be build from a Continuum Random Cluster Model where each connected component is assigned a colour uniformly over the q choices. This relation is known as the Fortuin-Kasteleyn representation. The existence of the Continuum Random Cluster Model with random radii has been recently proved in [7], yielding the existence of a Widom-Rowlinson measure.*

3.2 Phase transition in the integrable case

Now that Theorem 1 proves the existence of at least one Widom-Rowlinson measure, the second question concerns the uniqueness, non-uniqueness and consequently the phase transition between both regimes. It is usual, for Gibbs point processes with different type of particles, to show the uniqueness for small activities and non-uniqueness for large activity. In the integrable case (see definition 2.1), we recover these both regimes.

First, the following proposition proves the uniqueness for small activities.

Proposition 3.1. *Write $z_i = z\alpha_i$ where $\alpha := (\alpha_1, \dots, \alpha_q)$ is a discrete probability measure. If Q satisfies the integrability assumption (definition 2.1), then there exists a unique Widom-Rowlinson measure $WR(z\alpha, Q)$ as soon as $z \leq z_c(d, \sum_i \alpha_i Q_i)$, where $z_c(d, Q) > 0$ is the percolation threshold of the Poisson Boolean model in \mathbb{R}^d of radius measure Q .*

Proof. From Proposition 2.1 we have the stochastic domination $\mathcal{P}_{\Lambda, \omega_{\Lambda^c}}^{wr}(d\omega'_{\Lambda}) \preceq \pi_{\Lambda}^{z, Q}$. Therefore as a direct consequence of Theorem 3.2 in [14] we have uniqueness of the Widom-Rowlinson measure as soon as the "single-type" Poisson

Boolean model $\pi^{z, \sum_i \alpha_i Q_i}$ does not percolate. Thus the result. Let us note that $z_c(d, Q)$ is positive for every Q satisfying the integrability assumption $\int_{\mathbb{R}^+} r^d Q(dr) < \infty$ [13]. \square

In the symmetric case a non-uniqueness result was proved, initially in [2, 10] in the case of deterministic radii. A non-trivial generalisation to the case of unbounded radii was proved in [15] and is stated in the following proposition.

Proposition 3.2. *We are in the symmetric case $z := z_1 = \dots = z_q$ and $Q := Q_1 = \dots = Q_q$. If Q satisfies the integrability assumption and $Q(\{0\}) = 0$, then for activities z large enough, there exist q distinct ergodic Widom-Rowlinson measures.*

This result is a consequence of the Fortuin-Kasteleyn representation and the percolation of the Continuum Random Cluster Model for large activities z . As usual, the q distinct ergodic Widom-Rowlinson measures corresponds to the distinct Gibbs measures with boundary condition $1 \leq i \leq q$.

Remark 3.2. *The assumption $Q(\{0\}) = 0$ is an artefact of the proof of the percolation of the Continuum Random Cluster Model developed in [15]. In this paper the author emphasizes that the proof would carry out the same for radius measures Q having small atoms in 0, and he conjectures that the results would be true with the maximal assumption $Q(\{0\}) < 1$. We do not investigate this generalization here.*

In the case of deterministic radii r_1, r_2, \dots, r_q , it is proved in [1] that for large activity z_1, z_2, \dots, z_{q-1} there exists $z_q > 0$ such that the set of Widom-Rowlinson measures $WR((z_1, z_2, \dots, z_q), (\delta_{r_1}, \delta_{r_2}, \dots, \delta_{r_q}))$ exhibits at least q extremal phases. This result is based on an extension of the Pirogov-Sinai theory of phase transitions in general lattice spin systems to continuum systems. In the case of non-symmetric random radii we do not know if such results holds. However it is reasonable to believe that in the case of bounded random radii, the Pirogov-Sinai machinery is feasible and similar results could be obtained.

3.3 Existence of $q + 1$ extremal phases in the non integrable setting

The main results of the paper are presented in this section where we investigate the phase diagram in the non-integrable setting. A central notion here is the monochromaticity or polychromaticity of Widom-Rowlinson measures. This is defined as follows.

Definition 3.1. *Let $Mono$ be the event of monochromatic configurations $\omega \in \Omega$ such that $\omega_i = \emptyset$ for all $1 \leq i \leq q$ excepted one index. Let $Poly$ be the set of polychromatic configurations, meaning that $Poly = Mono^c$. A probability measure \mathbf{P} of Ω is said monochromatic (respectively polychromatic) if $\mathbf{P}(Mono) = 1$ (respectively $\mathbf{P}(Poly) = 1$).*

Let us note that, in the case of monochromatic \mathbf{P} , the index i such that $\omega_i \neq \emptyset$ can be random. In the case of radii \mathbf{Q} satisfying the integrability assumption (2.1), it is clear that every Widom-Rowlinson measure $\mathbf{P} \in WR(z, \mathbf{Q})$

is polychromatic. Therefore the question of monochromaticity is relevant only in this non-integrable setting. Moreover, we know that such monochromatic Widom-Rowlinson measures exist in the non-integrable setting as mentioned in the next proposition. The proof is obvious and is not detailed here.

Proposition 3.3. *For every $1 \leq i \leq q$, such that $\int_{\mathbb{R}^+} r^d Q_i(dr) = +\infty$, the Poisson point process $\pi^{\bar{z}^i, \mathbf{Q}}$ with $\bar{z}^i = (0 \dots, 0, z_i, 0, \dots, 0)$, is an extreme phase of $WR(z, \mathbf{Q})$.*

In particular, if \mathbf{Q} is completely non-integrable (i.e. $\int_{\mathbb{R}^+} r^d Q_i(dr) = +\infty$ for every $1 \leq i \leq q$) then $WR(z, \mathbf{Q})$ has q extremal monochromatic Widom-Rowlinson measures which correspond to the usual q ordered phases. In the next theorem we prove that, if the activity is small enough, there always exists a polychromatic Widom-Rowlinson measure. Therefore, the existence of a $q+1$ -th extremal phase follows in the completely non-integrable setting.

Theorem 2. *Write $z_i = z\alpha_i$ with $\alpha := (\alpha_i)_i$ being a discrete probability distribution. Then,*

1. *for every α such that $\max_i \alpha_i < 1$, there exists z_c^α such that for all $z < z_c^\alpha$, the set $WR(z\alpha, \mathbf{Q})$ contains at least one polychromatic Widom-Rowlinson measure;*
2. *the constant z_c^α can be chosen uniform in α satisfying for some $0 < \alpha_{\max} < 1$,*

$$\forall i \in \{1, \dots, q\}, \quad \alpha_i \leq \alpha_{\max}. \quad (3.1)$$

The sketch of the proof is as follows. We build first an accumulation point of the sequence of finite volume Widom-Rowlinson measures with free boundary condition. Then we show, for small activity, that the specific entropy of this measure is smaller than the specific entropy of every monochromatic stationary probability measure. Therefore this measure is not monochromatic and the theorem follows. The details of the proof are given in Section 5. The assumption $\max_i \alpha_i < 1$ (respectively $\alpha_{\max} < 1$) ensures that at least two of the α_i are positive. This is a natural assumption in order to have polychromaticity.

Corollary 3.1. *We assume that $\alpha := (\alpha_i)_i$ is a discrete probability with non-null coordinate and that \mathbf{Q} is completely non integrable. Then $WR(z\alpha, \mathbf{Q})$ has at least $q+1$ extreme phases for z small enough.*

The assumption "z small" appears in the proof of Theorem 2 as an artificial assumption needed to ensure that the accumulation point is not monochromatic. However it is our belief that this assumption is crucial and that for large activities all Widom-Rowlinson measures would be monochromatic. This is formalized in the following conjecture.

Conjecture. *In the non-integrable case, for activities z large enough, every Widom-Rowlinson measures is monochromatic.*

Note that if the conjecture is true, it would imply a phase transition result since the set $WR(\mathbf{z}, \mathbf{Q})$ would have exactly q extremal Gibbs measures for \mathbf{z} large and at least $q + 1$ extremal Gibbs measures for \mathbf{z} small. Our belief in this conjecture is based on a similar conjecture for the continuum random cluster presented in [7] and for which a heuristic proof is given. Moreover simulations in the sense of the conjecture have been implemented in [16, Section III.2].

We have not succeed to prove the conjecture but our last result is a first step towards it, by proving a weaker version of the conjecture.

Indeed we show that the symmetric Widom-Rowlinson measure on \mathbb{R}^d with free boundary condition and non-integrable radii is monochromatic if and only the activity z is large enough. Note that the scale we use in the thermodynamic limit is not symmetric since one direction is favoured. Unfortunately we are not able to extend the result for every thermodynamic limit, in particular, when the volume Λ_n is simply an hypercube.

For every $k > 0$, let $\Lambda_\infty^{(k)} := \mathbb{R} \times [0, k]^{d-1}$, $\Lambda_\infty^{(k+)} := \mathbb{R}^+ \times [0, k]^{d-1}$ and $\Lambda_n^{(k)} :=] - n, n[\times [0, k]^{d-1}$. Let us fix a sequence $(k_n)_{n \geq 1}$ of positive integers such that $k_n \rightarrow +\infty$ and (k_n) is negligible with respect to $(\log(n))_{n \geq 1}$. We are considering, in the same fashion as in the proof of Theorem 1, $\mathbf{P}_n^{\text{free}}(d\omega) := \frac{\mathbb{1}_A(\omega)}{Z_n} \pi_{\Lambda_n^{(k_n)}}^{\mathbf{z}, \mathbf{Q}}(d\omega)$, $\hat{\mathbf{P}}_n^{\text{free}} := \otimes_{x \in I_n} \mathbf{P}_n^{\text{free}} \circ \tau_x^{-1}$ and $\bar{\mathbf{P}}_n^{\text{free}} := \frac{1}{\mathcal{L}^d(\Lambda_n^{(k_n)})} \int_{\Lambda_n^{(k_n)}} \hat{\mathbf{P}}_n^{\text{free}} \circ \tau_x^{-1} dx$, where $I_n := 2n\mathbb{Z} \times (k_n\mathbb{Z})^{d-1}$ and where τ_x is the translation operator of vector $x \in \mathbb{R}^d$. Using the specific entropy (defined in Section 4), it is easy to prove that the sequence $(\bar{\mathbf{P}}_n^{\text{free}})$ admits at least one accumulation point \mathbf{P}^{free} , with respect to the local convergence topology.

The following theorem is our phase transition result.

Theorem 3. *We assume that we are in the symmetric case $z := z_1 = \dots = z_q$, $Q := Q_1 = \dots = Q_q$, with Q satisfying the two following conditions*

$$\int_1^\infty \exp\left(-\int_1^u Q(]r, \infty[) dr\right) du < \infty; \quad (3.2a)$$

$$Q(\{0\}) < \frac{1}{q}. \quad (3.2b)$$

Then, for z large enough, $(\bar{\mathbf{P}}_n^{\text{free}})$ converges (without passing by a subsequence) to \mathbf{P}^{free} which is the mixture $\sum_{i=1}^q \frac{1}{q} \pi^{\bar{\mathbf{z}}^i, \mathbf{Q}}$, with $\bar{\mathbf{z}}^i = (0, \dots, 0, z_i, 0, \dots, 0)$ (i.e. \mathbf{P}^{free} is monochromatic with equal probability of having any color). In opposite, when z is small enough, every accumulation point of $(\bar{\mathbf{P}}_n^{\text{free}})$ is not monochromatic and therefore it is not a mixture of the monochromatic phase $\pi^{\bar{\mathbf{z}}^i, \mathbf{Q}}$, $1 \leq i \leq q$.

The proof is based on the Fortuin-Kasteleyn representation in order to transfer the problem from the Widom-Rowlinson setting to the Continuum Random Cluster setting. We will prove that the sequence of finite volume Continuum Random Cluster measures converges towards the Poisson Boolean model which

covers the all space \mathbb{R}^d , proving that \mathbf{P}^{free} is monochromatic with equal probability for each color by symmetry of the model. This is done by bounding the mean number of connected component of a Continuum Random Cluster measure, using a fine renewal argument. The detailed proof is given in Section 6.

Remark 3.3. *In the case of dimension $d = 1$, then the sequence $(k_n)_n$ plays no role and the probability measures $\mathbf{P}_n^{\text{free}}$, $\hat{\mathbf{P}}_n^{\text{free}}$ and $\bar{\mathbf{P}}_n^{\text{free}}$ are just the measures \mathbf{P}_n , $\hat{\mathbf{P}}_n$ and $\bar{\mathbf{P}}_n$ which will be introduced in the proof of Theorem 1 and Theorem 2, see Section 4 and Section 5.*

4 Proof of Theorem 1

First let us consider the extreme case where at least one radius measure, let say Q_i , satisfy $\int_{\mathbb{R}_+} r^d Q_i(dr) = \infty$. It is known, see [3] for instance, that the Poisson Boolean model covers almost surely the all space \mathbb{R}^d . In particular for every bounded $\Lambda \subseteq \mathbb{R}^d$ and π^{z_i, Q_i} -almost every configuration ω , we have $L(\omega_{\Lambda^c}) = \mathbb{R}^d$. Let us consider the probability measure on Ω with one marginal being π^{z_i, Q_i} and the others producing almost surely empty configurations. This is a Poisson Point process $\pi^{\bar{z}^i, \mathbf{Q}}$ with only one non-zero intensity z_i , i.e. $\bar{z}^i := (0, \dots, 0, z_i, 0, \dots, 0)$. This probability measure trivially satisfies conditions (2.2) of Definition 2.2 and is therefore a Widom-Rowlinson measure of parameters z and \mathbf{Q} .

So from now on we consider the case of \mathbf{Q} satisfying the integrability assumption (2.1).

4.1 Construction of a good cluster point

To build a Widom-Rowlinson measure, consider a sequence of Widom-Rowlinson measures on the bounded boxes Λ_n with free boundary condition defined as

$$\mathbf{P}_n(d\omega) = \mathcal{P}_{\Lambda_n, \emptyset}^{wr}(d\omega) = \frac{\mathbb{1}_{\mathcal{A}}(\omega)}{\mathbf{Z}_n} \pi_{\Lambda_n}^{z, \mathbf{Q}}(d\omega_{\Lambda_n}),$$

with $\Lambda_n :=]-n, n]^d$ and $\mathbf{Z}_n = \int_{\Omega} \mathbb{1}_{\mathcal{A}}(\omega) \pi_{\Lambda_n}^{z, \mathbf{Q}}(d\omega_{\Lambda_n})$. Then consider $\hat{\mathbf{P}}_n = \otimes_{i \in 2n\mathbb{Z}^d} \mathbf{P}_n \circ \tau_i^{-1}$ and $\bar{\mathbf{P}}_n = \frac{1}{\mathcal{L}^d(\Lambda_n)} \int_{\Lambda_n} (\hat{\mathbf{P}}_n \circ \tau_x^{-1}) dx$, where τ_x is the translation operator of vector $x \in \mathbb{R}^d$. The measures $\bar{\mathbf{P}}_n$ are by construction stationary. The aim is to find an accumulation point for the sequence (\mathbf{P}_n) with respect to the local convergence topology defined in the next definition.

Definition 4.1. *A measurable function $f : \Omega \rightarrow \mathbb{R}$ is said local if there exists a bounded $\Lambda \subseteq \mathbb{R}^d$ such that $f(\omega) = f(\omega_{\Lambda})$ for all $\omega \in \Omega$.*

A sequence $(\tilde{\mathbf{P}}_n)$ converges, with respect to the local convergence topology, towards $\tilde{\mathbf{P}}$, if for every bounded local function f we have

$$\int_{\Omega} f(\omega) \tilde{\mathbf{P}}_n(d\omega) \xrightarrow{n \rightarrow \infty} \int_{\Omega} f(\omega) \tilde{\mathbf{P}}(d\omega).$$

A very convenient tool for proving the existence of an accumulation point is the specific entropy. It was introduced in [9] and is defined in the following definition.

Definition 4.2. For a stationary probability measure \mathbf{P} we define the specific entropy of \mathbf{P} , written $\mathcal{I}(\mathbf{P})$, as the following limit:

$$\mathcal{I}(\mathbf{P}) = \lim_{n \rightarrow \infty} \frac{1}{\mathcal{L}^d(\Lambda_n)} \mathcal{I}_{\Lambda_n}(\mathbf{P}|\pi^{z,\mathbf{Q}}), \quad (4.1)$$

with $\mathcal{I}_{\Lambda_n}(\mathbf{P}|\pi^{z,\mathbf{Q}})$ being the relative entropy of \mathbf{P} , with respect to $\pi^{z,\mathbf{Q}}$, defined as

$$\mathcal{I}_{\Lambda_n}(\mathbf{P}|\pi^{z,\mathbf{Q}}) = \begin{cases} \int_{\Omega} g \log(g) d\pi_{\Lambda_n}^{z,\mathbf{Q}} & \text{if } \mathbf{P}_{\Lambda_n} \ll \pi_{\Lambda_n}^{z,\mathbf{Q}}, g = \frac{d\mathbf{P}_{\Lambda_n}}{d\pi_{\Lambda_n}^{z,\mathbf{Q}}} \\ +\infty & \text{else} \end{cases}.$$

The stationarity ensures the convergence of the limit in (4.1). This is the reason why we introduce a stationary version $\bar{\mathbf{P}}_n$ of the finite volume Widom-Rowlinson measure \mathbf{P}_n . The following proposition ensures the compactness of the level sets of the specific entropy.

Proposition 4.1 (Proposition 2.6 [12]). *With respect to the local convergence topology induced on the stationary probability measures on Ω , we have*

1. \mathcal{I} is affine;
2. \mathcal{I} is lower semi-continuous;
3. the set $\{\mathbf{P} \text{ stationary, } \mathcal{I}(\mathbf{P}) \leq C\}$ is compact for every positive real number C .

Proposition 4.2. *For all n we have*

$$\mathcal{I}(\bar{\mathbf{P}}_n) \leq z_1 + \cdots + z_q.$$

Proof. First using the fact that the specific entropy is affine, see Proposition 4.1, we have

$$\mathcal{I}(\bar{\mathbf{P}}_n) = \frac{1}{\mathcal{L}^d(\Lambda_n)} \mathcal{I}_{\Lambda_n}(\mathbf{P}_n|\pi^{z,\mathbf{Q}}).$$

Now using the definition of the specific entropy and standard bounds on the partition function, we get

$$\mathcal{I}_{\Lambda_n}(\mathbf{P}_n|\pi^{z,\mathbf{Q}}) = -\log(\mathbf{Z}_n) \leq (z_1 + \cdots + z_q)\mathcal{L}^d(\Lambda_n),$$

which leads to the expected result. \square

Using Proposition 4.1 and Proposition 4.2, we obtain the existence of an accumulation point \mathbf{P} of the sequence $(\bar{\mathbf{P}}_n)$. For the rest of the proof we are, for convenience of notation, omitting to take a subsequence when making n

go to infinity. We now have a good candidate for being a Widom-Rowlinson measure.

Then we have to prove equations (2.2). This is done for the symmetric case in the PhD manuscript [16] and we are here adapting the proof to the non-symmetric case.

In the next proposition we prove that the measure \mathbf{P} produces almost surely authorized configuration. This trivially implies that condition (2.2a) is fulfilled.

Proposition 4.3. *We have $\mathbf{P}(\mathcal{A}) = 1$, and therefore*

$$\mathbf{Z}(\Lambda, \omega_{\Lambda^c}) \geq \exp(-z\mathcal{L}^d(\Lambda)) \mathbf{P}(d\omega) - \text{almost surely.}$$

Proof. The event \mathcal{A} is not local and we cannot use directly the local convergence. But this event could be called "almost local" since for every configuration ω ,

$$\mathbb{1}_{\mathcal{A}}(\omega_{\Lambda_k}) \xrightarrow[k \rightarrow \infty]{} \mathbb{1}_{\mathcal{A}}(\omega).$$

Therefore we have

$$\begin{aligned} \mathbf{P}(\mathcal{A}) &= \int_{\Omega} \mathbb{1}_{\mathcal{A}}(\omega) \mathbf{P}(d\omega) = \lim_{k \rightarrow \infty} \int_{\Omega} \mathbb{1}_{\mathcal{A}}(\omega_{\Lambda_k}) \mathbf{P}(d\omega) \\ &= \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\Omega} \mathbb{1}_{\mathcal{A}}(\omega_{\Lambda_k}) \bar{\mathbf{P}}_n(d\omega) \end{aligned}$$

with

$$\begin{aligned} \int_{\Omega} \mathbb{1}_{\mathcal{A}}(\omega_{\Lambda_k}) \bar{\mathbf{P}}_n(d\omega) &= \frac{1}{\mathcal{L}^d(\Lambda_n)} \int_{\Lambda_n} \int_{\Omega} \mathbb{1}_{\mathcal{A}}(\omega_{\Lambda_k}) \hat{\mathbf{P}}_n \circ \tau_x^{-1}(d\omega) dx \\ &= \frac{1}{\mathcal{L}^d(\Lambda_n)} \int_{\Lambda_n} \int_{\Omega} \mathbb{1}_{\mathcal{A}}(\omega_{\tau_x(\Lambda_k)}) \hat{\mathbf{P}}_n(d\omega) dx. \end{aligned}$$

For $n > k$, we have $\tau_x(\Lambda_k) \subseteq \Lambda_n$ as soon as $x \in [k-n, n-k]^d$ and so

$$\frac{1}{\mathcal{L}^d(\Lambda_n)} \int_{\Lambda_n} \int_{\Omega} \mathbb{1}_{\mathcal{A}}(\omega_{\tau_x(\Lambda_k)}) \hat{\mathbf{P}}_n(d\omega) \geq \frac{\mathcal{L}^d([k-n, n-k]^d)}{\mathcal{L}^d(\Lambda_n)},$$

which tends to 1 as n goes to infinity. The result is proved. \square

4.2 The cluster point satisfies (2.2b)

Starting now we fix $\Lambda \subseteq \mathbb{R}^d$. We are going to prove that \mathbf{P} satisfies the DLR(Λ) equation. To do so we need to modify the sequence $(\bar{\mathbf{P}}_n)$, defining a new sequence $(\tilde{\mathbf{P}}_n^\Lambda)$. This new sequence will be asymptotically equivalent to the former one and each $\tilde{\mathbf{P}}_n^\Lambda$ will satisfy the DLR(Λ) equation (2.2b). Finally by considering good "localizing" events, we will be able to pass the DLR property through the limit.

Consider

$$\tilde{\mathbf{P}}_n^\Lambda = \frac{1}{\mathcal{L}^d(\Lambda_n)} \int_{\Lambda_n} \mathbb{1}_{\Lambda \subseteq \tau_x(\Lambda_n)} \times (\mathbf{P}_n \circ \tau_x^{-1}) dx.$$

The measures $\tilde{\mathbf{P}}_n^\Lambda$ are not probability measures but satisfies good property, see the following proposition.

Proposition 4.4. For each bounded local function $f : \Omega \rightarrow \mathbb{R}$ we have

$$\left| \int_{\Omega} f d\tilde{P}_n^{\Lambda} - \int_{\Omega} f dP_n \right| \rightarrow 0,$$

which implies that \mathbf{P} is an accumulation point of the sequence (\tilde{P}_n^{Λ}) .

Furthermore the measure \tilde{P}_n^{Λ} satisfies the DLR(Λ) equation (2.2b).

We are omitting the proof of this standard result. The first point is done in [5] for the *Quermass-interaction model*, and the second point is a consequence of the compatibility of the Gibbs specification $\mathcal{P}_{\Lambda, \omega_{\Lambda^c}}^{wr}$. And both points are done in the PhD manuscript [16].

Now let fix a measurable function f bounded by 1. By the structure of \mathcal{F} we can consider f to be local. We are going to prove that for each $\epsilon > 0$, the quantity

$$\delta = \left| \int_{\Omega} f dP - \int_{\Omega} \int_{\Omega} f(\omega'_{\Lambda} + \omega_{\Lambda^c}) \frac{\mathbb{1}_{\mathcal{A}}(\omega'_{\Lambda} + \omega_{\Lambda^c})}{Z(\Lambda, \omega_{\Lambda^c})} \pi_{\Lambda}^{z, Q}(d\omega'_{\Lambda}) P(d\omega) \right|$$

is bounded by 7ϵ .

The function $f_{\Lambda}(\omega) := \int_{\Omega} f(\omega'_{\Lambda} + \omega_{\Lambda^c}) \frac{\mathbb{1}_{\mathcal{A}}(\omega'_{\Lambda} + \omega_{\Lambda^c})}{Z(\Lambda, \omega_{\Lambda^c})} \pi_{\Lambda}^{z, Q}(d\omega'_{\Lambda})$ is in general not local, which is the main obstacle in proving the result. We cannot directly use the local convergence, and the piecewise convergence used in the proof of Proposition 4.3 is not good enough as we need a more uniform convergence.

Let us consider the event

$$U_{r_1} = \{\omega, \forall (x, r) \in \omega_{\Lambda}, r \leq r_1\}.$$

Lemma 4.1. For r_1 large enough and for all $\omega \in \mathcal{A}$ we have

$$\left| f_{\Lambda}(\omega) - \int_{\Omega} f(\cdot + \omega_{\Lambda^c}) \mathbb{1}_{U_{r_1}}(\cdot) \frac{\mathbb{1}_{\mathcal{A}}(\cdot + \omega_{\Lambda^c})}{Z_{r_1}(\Lambda, \omega_{\Lambda^c})} d\pi_{\Lambda}^{z, Q} \right| \leq \epsilon,$$

where $Z_{r_1}(\Lambda, \omega_{\Lambda^c}) := \int_{\Omega} \mathbb{1}_{U_{r_1}}(\omega'_{\Lambda}) \mathbb{1}_{\mathcal{A}}(\omega'_{\Lambda} + \omega_{\Lambda^c}) \pi_{\Lambda}^{z, Q}(d\omega'_{\Lambda})$ is the modified partition function.

The constant r_1 can be chosen to also have $\pi_{\Lambda}^{z, Q}(U_{r_1}^c) \leq \epsilon$.

The proof of Lemma 4.1 is done in Section 4.3. Using Lemma 4.1 we get

$$\delta \leq \epsilon + \left| \int_{\Omega} f dP - \int_{\Omega} \int_{\Omega} f(\omega'_{\Lambda} + \omega_{\Lambda^c}) \frac{\mathbb{1}_{U_{r_1}}(\omega'_{\Lambda}) \mathbb{1}_{\mathcal{A}}(\omega'_{\Lambda} + \omega_{\Lambda^c})}{Z_{r_1}(\Lambda, \omega_{\Lambda^c})} d\pi_{\Lambda}^{z, Q} dP \right|.$$

Now in order to "localize" (with respect to ω) the functions $\mathbb{1}_{\mathcal{A}}$ and $Z_{r_1}(\Lambda, \omega_{\Lambda^c})$ we introduce the events

$$\Upsilon_k = \{\omega \in \Omega, \forall (x, r) \in \omega, r \leq \frac{|x|}{2} + k\}.$$

Lemma 4.2. For k large enough we have:

1. $P(\Upsilon_k^c) \leq \epsilon$;

2. $\tilde{\mathbf{P}}_n^\Lambda(\Upsilon_k^c) \leq \epsilon$ for each n ;

3. $\pi^{z, \mathbf{Q}}(\Upsilon_k^c) \leq \epsilon$.

The proof of Lemma 4.2 is done in Section 4.3. Using this lemma we have

$$\delta \leq 2\epsilon + \left| \int_{\Omega} f d\mathbf{P} - \int_{\Omega} \int_{\Omega} f(\omega'_\Lambda + \omega_{\Lambda^c}) \mathbb{1}_{U_{R_1}}(\omega'_\Lambda) \mathbb{1}_{\Upsilon_k}(\omega) \frac{\mathbb{1}_{\mathcal{A}}(\omega'_\Lambda + \omega_{\Lambda^c})}{\mathbf{Z}_{r_1}(\Lambda, \omega_{\Lambda^c})} d\pi_\Lambda^{z, \mathbf{Q}} d\mathbf{P} \right|.$$

Now with the introduction of the events U_{r_1} and Υ_k , the next lemma enables us to "localize" the integrated functions.

Lemma 4.3. *There exists Δ bounded, depending on r_1 and k , such that for every $\omega'_\Lambda \in U_{R_1}$ and $\omega \in \Upsilon_k \cap \mathcal{A}$,*

$$\mathbb{1}_{\mathcal{A}}(\omega'_\Lambda + \omega_{\Lambda^c}) = \mathbb{1}_{\mathcal{A}}(\omega'_\Lambda + \omega_{\Delta \setminus \Lambda}) \text{ and } \mathbf{Z}_{r_1}(\Lambda, \omega_{\Lambda^c}) = \mathbf{Z}_{R_1}(\Lambda, \omega_{\Delta \setminus \Lambda}).$$

The proof of Lemma 4.3 is done in Section 4.3. With this Lemma we have

$$\begin{aligned} \delta &\leq 2\epsilon + \left| \int_{\Omega} f d\mathbf{P} - \int_{\Omega} \int_{\Omega} f(\omega'_\Lambda + \omega_{\Lambda^c}) \mathbb{1}_{U_{r_1}}(\omega'_\Lambda) \mathbb{1}_{\Upsilon_k}(\omega) \frac{\mathbb{1}_{\mathcal{A}}(\omega'_\Lambda + \omega_{\Delta \setminus \Lambda})}{\mathbf{Z}_{r_1}(\Lambda, \omega_{\Delta \setminus \Lambda})} d\pi_\Lambda^{z, \mathbf{Q}} d\mathbf{P} \right| \\ &\leq 4\epsilon + \left| \int_{\Omega} f d\tilde{\mathbf{P}}_n^\Lambda - \int_{\Omega} \int_{\Omega} f(\omega'_\Lambda + \omega_{\Lambda^c}) \mathbb{1}_{U_{r_1}}(\omega'_\Lambda) \frac{\mathbb{1}_{\mathcal{A}}(\omega'_\Lambda + \omega_{\Delta \setminus \Lambda})}{\mathbf{Z}_{r_1}(\omega_{\Delta \setminus \Lambda})} d\pi_\Lambda^{z, \mathbf{Q}} d\tilde{\mathbf{P}}_n^\Lambda \right|, \end{aligned}$$

where the last inequality comes from Lemma 4.2 and the local convergence of Proposition 4.4, for n large enough fixed from now on. Now using again Lemma 4.1 and Lemma 4.2, we obtain

$$\delta \leq 7\epsilon + \underbrace{\left| \int_{\Omega} f d\tilde{\mathbf{P}}_n^\Lambda - \int_{\Omega} \int_{\Omega} f(\omega'_\Lambda + \omega_{\Lambda^c}) \frac{\mathbb{1}_{\mathcal{A}}(\omega'_\Lambda + \omega_{\Lambda^c})}{\mathbf{Z}(\Lambda, \omega_{\Lambda^c})} \pi_\Lambda^{z, \mathbf{Q}}(d\omega'_\Lambda) \tilde{\mathbf{P}}_n^\Lambda(d\omega) \right|}_{=0 \text{ thanks to Proposition 4.4}}.$$

4.3 Proof of the lemmas

4.3.1 Proof of Lemma 4.1

First from standard computation we have

$$\pi^{z, \mathbf{Q}}(U_{r_1}^c) = 1 - \exp\left(-\mathcal{L}^d(\Lambda) \sum_i z_i Q_i([r_1, \infty[)\right) \quad (4.2)$$

which indeed can be as small as needed when r_1 is large enough.

$$\begin{aligned}
& \left| f_\Lambda(\omega) - \int_{\Omega} f(\cdot + \omega_{\Lambda^c}) \mathbb{1}_{U_{r_1}}(\cdot) \frac{\mathbb{1}_{\mathcal{A}}(\cdot + \omega_{\Lambda^c})}{Z_{r_1}(\Lambda, \omega_{\Lambda^c})} d\pi_{\Lambda}^{z, \mathbf{Q}} \right| \\
& \leq \mathcal{P}_{\Lambda, \omega_{\Lambda^c}}^{wr}(U_{r_1}^c) + \frac{Z(\Lambda, \omega_{\Lambda^c}) - Z_{r_1}(\Lambda, \omega_{\Lambda^c})}{Z(\Lambda, \omega_{\Lambda^c})} \\
& \leq \pi^{z, \mathbf{Q}}(U_{r_1}^c) + \exp(\mathcal{L}^d(\Lambda) \sum_i z_i) \pi^{z, \mathbf{Q}}(U_{r_1}^c), \quad (4.3)
\end{aligned}$$

where the last inequality comes from Proposition 2.1 and Proposition 4.3. So by choosing r_1 large enough both quantities (4.2) and (4.3) are smaller than ϵ .

4.3.2 Proof of Lemma 4.2

The events Υ_k^c are increasing. Therefore from Proposition 2.1 we have $\mathbf{P}(\Upsilon_k^c) \leq \pi^{z, \mathbf{Q}}(\Upsilon_k^c)$. Furthermore we have

$$\tilde{\mathbf{P}}_n^\Lambda(\Upsilon_k^c) \leq \mathbf{P}_n(\Upsilon_k^c) \leq \pi_{\Lambda}^{z, \mathbf{Q}}(\Upsilon_k^c) \leq \pi^{z, \mathbf{Q}}(\Upsilon_k^c),$$

where the third inequality is a consequence of Proposition 2.1 applied to $\mathbf{P}_n = \mathcal{P}_{\Lambda_n, \emptyset}^{wr}$. So points 1 and 2 from the lemma are a direct consequence of point 3.

The point 3 is proved in [16, Lemme I.3.22] or can be adapted from the proof of Lemma 3.5 in [5].

4.3.3 Proof of Lemma 4.3

Since $\omega \in \mathcal{A}$, it is enough to check that balls $(x, r) \in \omega$ centred far enough can not overlap $L(\omega'_\Lambda)$. But since $\omega'_\Lambda \in U_{R_1}$, we have $L(\omega'_\Lambda) \subseteq \Lambda \oplus B(0, r_1)$. Finally $\omega \in \Upsilon_k$ and therefore we have for a set Δ large enough (which can be made explicit) that for $(x, r) \in \omega_{\Delta^c}$, $B(x, r) \cap \Lambda \oplus B(0, R_1) = \emptyset$. This concludes the proof of the Lemma.

5 Proof of Theorem 2

The first assertion is a consequence of the second which we prove now. Consider the sequence (\mathbf{P}_n) of finite-volume Widom-Rowlinson measures with free boundary condition, and $(\bar{\mathbf{P}}_n)$ the stationary modification, defined in the proof of Theorem 1, see Section 4. As in the proof of Theorem 1, the sequence $(\bar{\mathbf{P}}_n)$ admits an accumulation point denoted by \mathbf{P} , which satisfies $\mathbf{P}(\mathcal{A}) = 1$. But the proof of the DLR equations done for the proof of Theorem 1 in Section 4 is not valid in this case since Lemma 4.2 is false in the extreme case. In fact to prove the DLR equations we will use the polychromaticity which we show now using the specific entropy.

5.1 Lower bound of the specific entropy of monochromatic measures

Consider a monochromatic stationary probability measure \mathbf{P}^{mono} . Without loss of generality consider \mathbf{P}^{mono} to be ergodic. This implies in particular that the

colour of \mathbf{P}^{mono} is deterministic, and let us call it i . Let us compute the specific entropy of \mathbf{P}^{mono} . Let $n \geq 0$. If $\mathbf{P}_{\Lambda_n}^{mono}$ is not absolutely continuous with respect to $\pi_{\Lambda_n}^{z, \mathbf{Q}}$, then $\mathcal{I}_{\Lambda_n}(\mathbf{P}^{mono} | \pi^{z, \mathbf{Q}}) = \infty$. Otherwise $\mathbf{P}_{\Lambda_n}^{mono} \ll \pi_{\Lambda_n}^{z, \mathbf{Q}}$ but, since \mathbf{P}^{mono} is monochromatic of colour i , then we also have $\mathbf{P}_{\Lambda_n}^{mono} \ll \pi_{\Lambda_n}^{\bar{z}^i, \mathbf{Q}}$ where $\bar{z}^i := (0, \dots, 0, z_i, 0, \dots, 0)$ is the vector where the only non-zero coordinate being z_i at the i -th position. Therefore

$$\begin{aligned} \mathcal{I}_{\Lambda_n}(\mathbf{P}^{mono} | \pi^{z, \mathbf{Q}}) &= \mathcal{I}_{\Lambda_n}(\mathbf{P}^{mono} | \pi^{\bar{z}^i, \mathbf{Q}}) + \mathcal{I}_{\Lambda_n}(\pi^{\bar{z}^i, \mathbf{Q}} | \pi^{z, \mathbf{Q}}) \\ &\geq \mathcal{I}_{\Lambda_n}(\pi^{\bar{z}^i, \mathbf{Q}} | \pi^{z, \mathbf{Q}}), \end{aligned} \quad (5.1)$$

where the inequality (5.1) comes from the positivity of the relative entropy. But a direct computation leads to

$$\frac{d\pi_{\Lambda_n}^{\bar{z}^i, \mathbf{Q}}}{d\pi_{\Lambda_n}^{z, \mathbf{Q}}}(\omega) = \prod_{j \neq i} \exp(z_j \mathcal{L}^d(\Lambda_n)) \mathbb{1}_{\omega^j = \emptyset}$$

which implies

$$\mathcal{I}_{\Lambda_n}(\pi^{\bar{z}^i, \mathbf{Q}} | \pi^{z, \mathbf{Q}}) = \mathcal{L}^d(\Lambda_n) \sum_{j \neq i} z_j \geq \mathcal{L}^d(\Lambda_n) \min_i \left(\sum_{j \neq i} z_j \right).$$

The last inequality together with (5.1) implies

$$\mathcal{I}(\mathbf{P}^{mono}) \geq \min_i \left(\sum_{j \neq i} z_j \right) = z \left(1 - \max_i \alpha_i \right). \quad (5.2)$$

Remark 5.1. Inequality (5.2) cannot be improve since inequality (5.1) becomes an equality in the case where $\mathbf{P}^{mono} = \pi^{\bar{z}^i, \mathbf{Q}}$.

5.2 Upper bound of the specific entropy of \mathbf{P}

Let us now look into the specific entropy of \mathbf{P} . If we prove that $\mathcal{I}(\mathbf{P}) < z(1 - \max_i \alpha_i)$, then, by the lower bound (5.2), \mathbf{P} is not monochromatic. The bound from Proposition 4.2 is not good enough for the purpose of the current proof and we now improve it. Divide the cube Λ_n into copies of the smaller cube Λ_m with $m < n$. We denote by k_n the number of such copies of Λ_m in Λ_n and we fix $\epsilon < 1 - \alpha_{\max} < 1$.

One example of authorized configuration is when each copies of Λ_m contains only one type of particles and when the ball does not overlap the outside of the cube. This leads to the following inequality.

$$\pi_{\Lambda_n}^{z\alpha, \mathbf{Q}}(\mathcal{A}) \geq \exp\left(-z\mathcal{L}^d(\Lambda_n)\right) \left(1 + \sum_i \sum_{l \geq 1} \frac{z^l \alpha_i^l \mathcal{L}^d(\Lambda_m)^l \phi_{m,i}^l}{l!}\right)^{k_n}, \quad (5.3)$$

with $\phi_{m,i} := \frac{1}{\mathcal{L}^d(\Lambda_m)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^+} \mathbf{1}_{B(x,r) \subseteq \Lambda_m} Q_i(dr) dx$. Fix $\gamma < 1 - \epsilon - \alpha_{\max}$. By choosing m large enough, we have $\phi_{m,i} \geq 1 - \gamma$ for all i . Furthermore we have

$$\begin{aligned} \mathcal{I}(\bar{\mathbf{P}}_n) &= -\frac{\log(\pi_{\Lambda_n}^{z\alpha, \mathcal{Q}}(\mathcal{A}))}{\mathcal{L}^d(\Lambda_n)} \\ &\leq z - \frac{k_n}{\mathcal{L}^d(\Lambda_n)} \log \left(1 - q + \sum_i \exp(z\alpha_i \mathcal{L}^d(\Lambda_m) \phi_{m,i}) \right). \end{aligned} \quad (5.4)$$

Fix $\beta < 1$ satisfying $\epsilon + \alpha_{\max} \leq \beta(1 - \gamma)$. Using the fact that $\mathcal{L}^d(\Lambda_n)/k_n \xrightarrow[n \rightarrow \infty]{} \mathcal{L}^d(\Lambda_m)$, we have for n large enough that

$$\mathcal{I}(\bar{\mathbf{P}}_n) \leq z - \frac{\beta}{\mathcal{L}^d(\Lambda_m)} \log \left(1 - q + \sum_i \exp(z\alpha_i \mathcal{L}^d(\Lambda_m) \phi_{m,i}) \right). \quad (5.5)$$

The bound from (5.5) is not depending on n . It remains to prove that the following function of z is negative close to the origin, uniformly in α satisfying (3.1),

$$\Psi_\alpha(z) := z \max_i \alpha_i - \frac{\beta}{\mathcal{L}^d(\Lambda_m)} \log \left(1 - q + \sum_i \exp(z\alpha_i \mathcal{L}^d(\Lambda_m) \phi_{m,i}) \right).$$

It satisfies $\Psi_\alpha(0) = 0$ and

$$\begin{aligned} \Psi'_\alpha(z) &= \max_i \alpha_i - \beta \frac{\sum_i \alpha_i \phi_{m,i} \exp(z\alpha_i \mathcal{L}^d(\Lambda_m) \phi_{m,i})}{1 - q + \sum_i \exp(z\alpha_i \mathcal{L}^d(\Lambda_m) \phi_{m,i})} \\ &\leq \alpha_{\max} - \frac{\beta(1 - \gamma)}{1 - q + \sum_i \exp(z\alpha_{\max} \mathcal{L}^d(\Lambda_m) \phi_{m,i})}, \end{aligned}$$

where the last bound does not depend on α . Therefore by the choice of the parameters we have $\Psi'_\alpha(0) \leq -\epsilon$ and thus for z small enough, uniform in α satisfying (3.1), we have $\Psi'_\alpha(z) \leq -\frac{\epsilon}{2}$. Therefore using the lower semi-continuity of the specific entropy, we get that $\mathcal{I}(\mathbf{P}) < z(1 - \max_i \alpha_i)$, which implies together with (5.2) that $\mathbf{P}(\text{Poly}) > 0$.

Now by conditioning on the event *Poly*, one can prove that the conditioned probability measure satisfies the DLR equations (2.2b). This was done in details in [7] for the symmetric case.

6 Proof of Theorem 3

6.1 Fortuin-Kasteleyn representation

We consider the sequence $(\bar{\mathbf{P}}_n^{\text{free}})$ defined in Section 3 which admits, thanks to Proposition 4.1, at least one cluster point denoted \mathbf{P}^{free} . The second part of the theorem, which claims that, for small activity, \mathbf{P}^{free} is polychromatic, is similar to the proof of Theorem 2. We do not give the details here. So it remains to prove the first part of the theorem on the monochromaticity.

In a same fashion as Section 4, one can easily prove that $\mathbf{P}^{\text{free}}(\mathcal{A}) = 1$. So one way of proving that \mathbf{P}^{free} is monochromatic is to prove that \mathbf{P}^{free} covers the all space \mathbb{R}^d with one giant connected component.

The first step of the proof is to transfer the problem from multi-type Widom-Rowlinson measure in Ω to "single-type" Continuum Random Cluster measure in Ω .

To a measure \mathbf{P} on Ω we associate its *color-blind* measure, denoted by P , which is a probability measure on Ω defined by

$$\forall E \in \mathcal{F}, P(E) = \mathbf{P}(\{\omega \mid \cup_i \omega^i \in E\}).$$

The specific entropy and the local convergence topology can be defined the same as for Ω . In particular \mathbf{P}^{free} is still an accumulation point of the sequence $(\bar{P}_n^{\text{free}})$.

Proposition 6.1 (Fortuin-Kasteleyn representation). *The measure P_n^{free} is a Continuum Random Cluster measure on the bounded box $\Lambda_n^{(k_n)}$ with free boundary condition and with parameters q, z and Q . This means*

$$P_n^{\text{free}}(d\omega) = \frac{q^{N_{cc}(\omega)}}{Z_n^{(k_n)}} \pi_{\Lambda_n^{(k_n)}}^{z, Q}(d\omega),$$

where $N_{cc}(\omega)$ denotes the number of connected components of the structure $L(\omega)$, and where $Z_n^{(k_n)}$ is the normalizing constant.

This proposition is a standard result, known as the *Fortuin-Kasteleyn representation* or *grey representation*, proved and used in a lot of articles such as [2, 10, 7, 15].

The next proposition is the key element of the proof of Theorem 3.

Proposition 6.2. *In the assumptions of Theorem 3, there exist $\tilde{z} < \infty$ such that for all $z > \tilde{z}$ and for all n*

$$\int_{\Omega} N_{cc}(\omega) P_n^{\text{free}}(d\omega) \leq D \times C^{k_n},$$

where $C, D > 1$ are finite constants not depending on n .

Remark 6.1. *In the case $d = 1$ one can prove in the same fashion the bound*

$$\int_{\Omega} N_{cc}(\omega) P_n^{\text{free}}(d\omega) \leq C$$

for large enough z and for a constant $C > 1$ independent of n .

Before proving this proposition, let us show how it leads to the expected result. By the computation done for the proof of Theorem 1, see Section 4, we

have

$$\begin{aligned}
\mathcal{I}(\bar{P}_n^{\text{free}}) &= \frac{1}{\mathcal{L}^d(\Lambda_n^{(k_n)})} \mathcal{I}\left(P_n^{\text{free}} | \pi_{\Lambda_n^{(k_n)}}^{z, Q}\right) \\
&= \frac{1}{\mathcal{L}^d(\Lambda_n^{(k_n)})} \left(-\log(Z_n^{(k_n)}) + \log(q) \int_{\Omega} N_{cc} dP_n^{\text{free}} \right) \\
&\leq \frac{D \times C^{k_n} \log(q)}{2nk_n^{d-1}} \\
&\leq D \log(q) C^{k_n - \log(n)/\log(C)}.
\end{aligned}$$

Since (k_n) is negligible with respect to $\log(n)$, this upper-bound tends to zero when n goes to infinity. So by the lower semi-continuity of the specific entropy, we have $\mathcal{I}(P^{\text{free}}) = 0$ which implies that $P^{\text{free}} = \pi^{z, Q}$. Noting that the condition (3.2a) implies in particular that $\int_{\mathbb{R}^+} r^d Q(dr) = \infty$, so the Poisson Boolean model covers almost surely the whole space \mathbb{R}^d with one connected component. Therefore, since $\mathbf{P}^{\text{free}}(\mathcal{A}) = 1$, the measure \mathbf{P}^{free} has no choice but to be monochromatic. By symmetry of the model, each color appears with probability $1/q$.

6.2 Proof of Proposition 6.2

First let $k > 0$ which is taken for simplicity as the inverse of a positive integer. A condition on k will appear later. Furthermore let us consider $\bar{q} > q$ such that $Q(\{0\}) < 1/\bar{q}$.

Since $Z_n^{(k_n)} \geq 1$, we have for a constant $D > 1$ large enough,

$$\begin{aligned}
\int_{\Omega} N_{cc} dP_n^{\text{free}} &\leq \int_{\Omega} N_{cc} \bar{q}^{N_{cc}} d\pi_{\Lambda_n^{(k_n)}}^{z, Q} \\
&\leq D \int_{\Omega} \bar{q}^{N_{cc}} d\pi_{\Lambda_n^{(k_n)}}^{z, Q} \\
&\leq D \left(\int_{\Omega} \bar{q}^{N_{cc}} d\pi_{\Lambda_n^{(k)}}^{z, Q} \right)^{k_n/k}, \tag{6.1}
\end{aligned}$$

where the last inequality is a consequence of the stationarity of the Poisson Boolean model and the subadditivity of the number of connected components, i.e. $N_{cc}(\omega_{\Delta_1 \cap \Delta_2}) \leq N_{cc}(\omega_{\Delta_1}) + N_{cc}(\omega_{\Delta_2})$ for disjoint Δ_1 and Δ_2 .

Remark 6.2. *In the case $d = 1$ there is no need to introduce k_n and k and we directly have*

$$\int_{\Omega} N_{cc} dP_n^{\text{free}} \leq D \int_{\Omega} \bar{q}^{N_{cc}} d\pi_{\Lambda_n}^{z, Q},$$

where $\Lambda_n =]-n, n[$.

Therefore, it is enough to show that for a fixed k small enough the sequence

$$\int_{\Omega} \bar{q}^{N_{cc}} d\pi_{\Lambda_n^{(k)}}^{z, Q}$$

is uniformly bounded in $n \geq 1$.

We are going to introduce the notion of *connected component to the right*. Thanks to a renewal phenomena, the number of connected components to the right will have a geometric law. Then we will prove, using the geometric law, that this number of connected components to the right admits exponential moments for z large enough.

More precisely we have

$$\int_{\Omega} \bar{q}^{N_{cc}} d\pi_{\Lambda_n^{(k)}}^{z,Q} \leq \left(\int_{\Omega} \bar{q}^{N_{cc}} d\pi_{\Lambda_n^{(k+)}}^{z,Q} \right)^2 \quad (6.2)$$

where this inequality comes from the stationarity and the independence property of $\pi^{z,Q}$.

Now the problem is that when n grows, a ball can appear and cover all existing connected components. So seeing n as the time, the present is depending on the yet unknown future. To overcome this issue let us do the following. To a radius r we associate the following transformed radius

$$\tilde{r} = \sqrt{r^2 - (d-1)k^2} \mathbb{1}_{r^2 \geq (d-1)k^2}.$$

This way if (x, r) is a marked point, the set

$$T(x, \tilde{r}) = [x_1, x_1 + \tilde{r}] \times [0, k]^{d-1}$$

does not cover anything left of x and is entirely covered by the ball $B(x, r)$.

Remark 6.3. *In dimension $d = 1$ we simply consider $\tilde{r} = r$.*

Now we are defining the *number of connected component to the right*, denoted by $N_{cc}^r(\omega)$, as the number of connected components of $\bigcup_{(x,r) \in \omega} T(x, \tilde{r})$. We have $N_{cc} \leq N_{cc}^r$ and N_{cc}^r is increasing with respect to n , which implies

$$\int_{\Omega} \bar{q}^{N_{cc}} d\pi_{\Lambda_n^{(k+)}}^{z,Q} \leq \int_{\Omega} \bar{q}^{N_{cc}^r} d\pi_{\Lambda_n^{(k+)}}^{z,Q} \leq \int_{\Omega} \bar{q}^{N_{cc}^r} d\pi_{\Lambda_{\infty}^{(k+)}}^{z,Q} \quad (6.3)$$

where the right-hand side in (6.3) does not depend on n . Therefore in order to prove Proposition 6.2 we have to prove that the right-hand side of (6.3) is finite.

The first thing to do is to prove that the quantity $N_{cc}^r(\omega)$ is $\pi_{\Lambda_{\infty}^{(k+)}}^{z,Q}$ almost surely finite. Then we will be able to show that this quantity admits exponential moments for z large enough. This comes out to study the number of connected components of a segment model where the left extremities of the segments are the points of a Poisson point process (on the real line) of intensity zk^d , and where the lengths of the segments are independent random variables distributed accordingly to the measure \tilde{Q} satisfying

$$\tilde{Q}([0, r]) = Q([0, \sqrt{r^2 + (d-1)k^2}]).$$

Condition (3.2a) transfers trivially to \tilde{Q} . By taking k small enough, we also have $\tilde{Q}(\{0\}) < 1/\bar{q}$.

The next lemma, proved in [8, Corollary 4], ensures that $N_{cc}^r(\omega)$ is $\pi_{\Lambda_{\infty}^{(k+)}}^{z,Q}$ almost surely finite.

Lemma 6.1. *For every $z \geq k^{-d}$, the quantity $N_{cc}^r(\omega)$ is $\pi_{\Lambda_\infty^{(k+)}}^{z,Q}$ almost surely finite.*

This lemma, working thanks to the condition (3.2a), ensures in particular that up to a random point \mathcal{Y} , the "cylinder" $[\mathcal{Y}, \infty] \times [0, k]^{d-1}$ is covered by $\bigcup_{(x,r) \in \omega_{\Lambda_\infty^{(k+)}}} T(x, \tilde{r})$. So when we go through $\Lambda_\infty^{(k+)}$ from the origin, each connected component encountered has a positive probability $p(z)$ of being the infinite one, independently of all finite connected components already encountered. Therefore the random variable $N_{cc}^r(\omega)$ is geometric of mean $p(z)^{-1}$ satisfying

$$p(z)^{-1} = \int_{\Omega} N_{cc}^r(\omega) \pi_{\Lambda_\infty^{(k+)}}^{z,Q}(d\omega)$$

and

$$p(z) = \pi_{\Lambda_\infty^{(k+)}}^{z,Q}(N_{cc}^r = 1).$$

So an explicit computation of the right-hand side of (6.3) leads to

$$\int_{\Omega} \bar{q}^{N_{cc}^r} d\pi_{\Lambda_\infty^{(k+)}}^{z,Q} = \sum_{\alpha \geq 0} \bar{q}^\alpha p(z) (1 - p(z))^{\alpha-1} \quad (6.4)$$

which is finite as soon as $p(z) > 1 - 1/\bar{q}$.

In the following lemma we prove that for large activities z , $p(z)$ is as close to 1 as we need.

Lemma 6.2. *For z large enough*

$$p(z) > 1 - 1/\bar{q}.$$

With this lemma we have for z large enough that

$$C := \left(\int_{\Omega} \bar{q}^{N_{cc}^r} d\pi_{\Lambda_\infty^{(k+)}}^{z,Q} \right)^{2/k} < \infty$$

which combined with (6.1), (6.2), (6.3) and (6.4) concludes the proof of Proposition 6.2.

The last thing needed is to prove Lemma 6.2.

6.3 Proof of Lemma 6.2

We know that $N_{cc}^r(\omega)$ is a geometric random variable. We have

$$\begin{aligned} p(z) &= \pi_{\Lambda_\infty^{(k+)}}^{z,Q}(N_{cc}^r = 1) \\ &\geq \pi_{\Lambda_\infty^{(k+)}}^{z,Q}\left(N_{cc}^r(\omega) = 1, [x + y, \infty[\text{ is right-covered in } \omega_{]x, \infty[\times]0, k]^{d-1}}\right), \end{aligned}$$

where x and y are positive real numbers and we say that $]x, \infty[$ is right-covered in ω if

$$]x, \infty[\times]0, k]^{d-1} \subseteq \bigcup_{(a,r) \in \omega} T(a, \tilde{r}).$$

Therefore we have

$$\begin{aligned}
p(z) &\geq (1 - e^{-zx}) \tilde{Q}([x + y, \infty[) \\
&\quad \pi^{z,Q}([x + y, \infty[\text{ is right-covered in } \omega_{]x, \infty[\times]0, k]^{d-1}) \\
&= (1 - e^{-zx}) \tilde{Q}([x + y, \infty[) \\
&\quad \pi^{z,Q}([y, \infty[\text{ is right-covered in } \omega_{]0, \infty[\times]0, k]^{d-1}),
\end{aligned}$$

where the last equality comes from the stationarity of $\pi^{z,Q}$.

Lemma 6.3. *For $y > 0$ fixed, the probability*

$$\pi^{z,Q}([y, \infty[\text{ is right-covered in } \omega_{]0, \infty[\times]0, k]^{d-1})$$

converges to 1 when z grows to infinity.

Proof of Lemma 6.3. Consider the realisation of a Poisson point process on the half line of intensity z and with segment length of law \tilde{Q} . Thanks to Lemma 6.1, we know that this segment model percolates for $z \geq k^{-d}$. Therefore it exists a infinite connected component starting from a random point \mathcal{Y} . If y is inside this infinite connected component then there is nothing to do. Otherwise by increasing z we are adding Poisson point and \mathcal{Y} is translating to the left, converging almost surely to 0 when z grows to infinity, and overlapping y after a finite random time. From this almost surely convergence we deduce the convergence of the probability of Lemma 6.3. \square

Let us now conclude the proof of Lemma 6.2.

For x, y small enough, thanks to the construction of \tilde{Q} satisfying $\tilde{Q}(\{0\}) < 1/\bar{q}$, we have $\tilde{Q}([x + y, \infty[) > 1 - 1/\bar{q}$. Now that x and y are fixed, thanks to Lemma 6.3 we have for z large enough that the same is true for

$$(1 - e^{-zx}) \tilde{Q}([x + y, \infty[) \pi^{z,Q}([y, \infty[\text{ is right-covered in } \omega_{]0, \infty[\times]0, k]^{d-1}).$$

Thus for z large enough $p(z) > 1 - 1/\bar{q}$ and Lemma 6.2 is proved.

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