Numerical analysis with error estimates for the Korteweg-de Vries Equation

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Abstract

This article deals with the numerical analysis of the Cauchy problem for the Korteweg-de Vries equation with a finite difference scheme. We consider the Rusanov scheme for the hyperbolic flux term and a 4-points θ-scheme for the dispersive term. We prove the convergence under a hyperbolic Courant-Friedrichs-Lewy condition when $\theta \geq \frac{1}{2}$ and under an "Airy" Courant-Friedrichs-Lewy condition when $\theta < \frac{1}{2}$. More precisely, we get the first order convergence rate for strong solutions in the Sobolev space $H^s(\mathbb{R})$, $s \geq 6$ and extend this result to the non-smooth case for initial data in $H^s(\mathbb{R})$, with $s \geq \frac{3}{4}$, to the price of a loss in the convergence order. The orders of convergence seem optimal with numerical simulations in some cases, at least when $s \geq 3$.

1 Introduction

We are interested in the Korteweg-de Vries equation (called the KdV equation thereafter) which is a model for wave propagation on shallow water surfaces in a channel and was first established by D.J. Korteweg and G. de Vries in 1895 [KdV95]. We focus on the numerical analysis of the Cauchy problem

$$\begin{cases} \partial_t u(t,x) + \partial_x \left(\dfrac{u^2}{2}\right)(t,x) + \partial_x^3 u(t,x) = 0, & (t,x) \in [0,T] \times \mathbb{R}, \quad (1a) \\ u_{|t=0}(x) = u_0(x), & x \in \mathbb{R}, \quad (1b) \end{cases}$$

for which the local well-posedness in Sobolev spaces $H^s(\mathbb{R})$ is well-established. (In particular, well-posedness was proved for $s \geq 2$ in [ST76], $s > \frac{3}{2}$ in [BS75], $s > \frac{3}{4}$ in [KPV91], $s \geq 0$ in [Bou93], $s > -\frac{5}{8}$ in [KPV93], note that one of the first existence result was obtained by proving the convergence of a semi-discrete scheme [Sjö70]). Due to the conservation of the L^2 norm, this yields global well-posedness for any $s \geq 0$. Note that global well-posedness is even known below L^2 (see [CKS+03], for example). There are two antagonist effects in the KdV equation: the Burgers nonlinearity tends to create singularities (shock waves, which yield a blow up in finite time) whereas the linear term tends to smooth the solution due to dispersive effects (and creates dispersive oscillating waves of Airy type). In some sense the above global well-posedness results come from the fact that dispersive effects dominate.

Given the practical importance of the KdV equation in concrete physical situations, there exists a wide range of numerical schemes to solve it. A very classical numerical approach is *the finite difference method* which consists in approximating the exact solution u by a numerical solution $(v_j^n)_{(n,j)}$ in such a way that $v_j^n \approx u(t^n, x_j)$ in which $t^n = n\Delta t$, $x_j = j\Delta x$ with Δt and Δx respectively the time and space steps. In most cases, the convergence is ensured only if a stability condition between Δt and Δx is satisfied. Let us mention for instance the explicit leap-frog scheme designed by Zabusky and Kruskal in [ZK65] with periodic boundaries conditions, or the Lax-Friedrichs scheme studied by Vliegenthart in [Vli71]. Both are formally convergent to the second order in space under a very restrictive stability condition $\Delta t = \mathcal{O}(\Delta x^3)$. The price to pay to avoid a so restrictive stability condition $\Delta t = \mathcal{O}(\Delta x^3)$ is to design formally an implicit scheme, as in [Win80], for example,

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with a twelve-points implicit finite difference scheme with three time levels or in [TA84] with a pentagonal implicit scheme.

The analysis and the rigorous justification of the stability condition started in [Vli71], where Vliegenthart computed rigorously the amplification factor for a linearized equation. More recently, Holden, Koley and Risebro in [HKR15] prove the convergence of the Lax-Friedrichs scheme with an implicit dispersion under the stability condition $\Delta t = \mathcal{O}(\Delta x^{\frac{3}{2}})$ without obtaining convergence rates. More precisely, they obtain the strong convergence without rate of the numerical scheme towards a classical solution if $u_0 \in H^3(\mathbb{R})$ and a strong convergence towards a weak solution $L^2(0,T;L^2_{\text{loc}}(\mathbb{R}))$ if $u_0 \in L^2(\mathbb{R})$.

The aim of this paper is to prove rigorously the convergence of some finite difference schemes by analyzing the rate of convergence and in particular its dependence on the regularity of the initial datum.

Let us mention that many types of numerical methods can be used to solve the KdV equation sometimes with the aim of capturing the long time behavior or the low dispersion regime. The equation being Hamiltonian (the Hamiltonian is the energy), symplectic schemes based on compact finite differences that conserve the energy have been designed. We refer for example to [KMY12], [LV06], [AM05]. Splitting methods (the equation being split into the linear Airy part and the nonlinear Burgers part) are also widely studied. For example, a rigorous analysis of such schemes has been performed in [HKRT11], [HLR13]. One can also use spectral methods see [NS89] for example or [HS17] where a Fourier pseudo spectral method is combined with an exponential-type time-integrator. A quite widespread discretization is related to finite element type schemes, see for example [BDK83], [DK85] for Galerkin methods and the recent work [BCKX13] where a non-dissipative scheme is derived and applied to the computation of traveling waves.

In the present paper, we discretize Equation (1a) together with the initial datum (1b) in a finite difference way and our aim is to determine the convergence rate of this numerical scheme. We exhibit the error estimate on the convergence error by a method which suits both non-linear term and dispersive term of KdV.

Let us introduce some notations and present the finite difference scheme here under study.

Notations and numerical scheme We use a uniform time- and space-discretization of (1a). Let Δt be the constant time step and Δx the constant space step. We note $t^n = n\Delta t$ for all $n \in [\![0,N]\!] = \{0,1,..,N\}$ where $N = \lfloor \frac{T}{\Delta t} \rfloor$ (with $\lfloor . \rfloor$ the integer part) and $x_j = j\Delta x$ for all $j \in \mathbb{Z}$.

Numerical scheme. Let $c \in \mathbb{R}_+^*$ and $\theta \in [0,1]$. We denote by $(v_j^n)_{(n,j)\in\mathbb{N}\times\mathbb{Z}}$ the discrete unknown defined by the following scheme with parameters c and θ :

$$\frac{v_j^{n+1}-v_j^n}{\Delta t} + \frac{\left(v_{j+1}^n\right)^2 - \left(v_{j-1}^n\right)^2}{4\Delta x} + \theta \frac{v_{j+2}^{n+1} - 3v_{j+1}^{n+1} + 3v_j^{n+1} - v_{j-1}^{n+1}}{\Delta x^3}$$
$$+ (1-\theta)\frac{v_{j+2}^n - 3v_{j+1}^n + 3v_j^n - v_{j-1}^n}{\Delta x^3} = c\left(\frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{2\Delta x}\right), \quad n \in [\![0,N]\!],\ j \in \mathbb{Z} \quad (2)$$

with

$$v_j^0 = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} u_0(y)dy, \quad j \in \mathbb{Z}. \quad (3)$$

If $\theta = 0$, we recognize the explicit scheme whereas $\theta = 1$ corresponds to the implicit scheme (with respect to the dispersive term). Without the dispersive term $\theta\frac{v_{j+2}^{n+1}-3v_{j+1}^{n+1}+3v_j^{n+1}-v_{j-1}^{n+1}}{\Delta x^3}+(1-\theta)\frac{v_{j+2}^n-3v_{j+1}^n+3v_j^n-v_{j-1}^n}{\Delta x^3}$, we recognize the Rusanov scheme applied to the Burgers equation, which consists in a centered hyperbolic flux $\frac{\left(v_{j+1}^n\right)^2-\left(v_{j-1}^n\right)^2}{4\Delta x}$ and an added artificial viscosity $c\left(\frac{v_{j+1}^n-2v_j^n+v_{j-1}^n}{2\Delta x}\right)$ in order to ensure the stability of the scheme. In the following, the constant c will be called the Rusanov coefficient.

Without the non-linear term and the right hand side, we recognize the θ-right winded finite difference scheme for the Airy equation

$$\frac{v_j^{n+1}-v_j^n}{\Delta t} + \theta\frac{v_{j+2}^{n+1} - 3v_{j+1}^{n+1} + 3v_j^{n+1} - v_{j-1}^{n+1}}{\Delta x^3} + (1-\theta)\frac{v_{j+2}^n - 3v_{j+1}^n + 3v_j^n - v_{j-1}^n}{\Delta x^3} = 0, \quad n \in [\![0,N]\!],\ j \in \mathbb{Z}.$$

Remark 1. *All the results are valid with a variable time step Δt^n and a variable Rusanov coefficient c^n. For simplicity, we will keep them constant.*

Remark 2. *The choice of the right winded scheme for the dispersive part is dictated by the result in [Cou16] on numerical schemes applied to high-order dispersive equations $\partial_t u + \partial_x^{2p+1} u = 0$, with $p \in \mathbb{N}$, which brought to light that right winded schemes are stable under a Courant-Friedrichs-Lewy type condition (CFL-type condition) for p odd (including the Airy equation) and left winded schemes are stable under a CFL-type condition for p even.*

Remark 3. *This scheme (2)-(3) is a generalization of the one studied by Holden, Koley and Risebro [HKR15]. Indeed, they consider the Lax-Friedrichs scheme for the hyperbolic flux term together with the implicit scheme for the dispersive term, which consists in taking $c\Delta t = \Delta x$ and $\theta = 1$ in Scheme (2)-(3).*

Discrete operators. For notational convenience for Scheme (2)-(3), let us use the notations introduced in [HKR15] and define the following discrete operators. For any sequence $(a_j^n)_{(n,j)\in\mathbb{N}\times\mathbb{Z}}$,

$$D_-(a)_j^n = \frac{a_j^n - a_{j-1}^n}{\Delta x}, \quad D_+(a)_j^n = \frac{a_{j+1}^n - a_j^n}{\Delta x}, \quad D(a)_j^n = \frac{D_+(a)_j^n + D_-(a)_j^n}{2}. \tag{4}$$

Equation (2) rewrites

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} + D\left(\frac{v^2}{2}\right)_j^n + \theta D_+ D_+ D_- (v)_j^{n+1} + (1-\theta) D_+ D_+ D_- (v)_j^n = \frac{c\Delta x}{2} D_+ D_- (v)_j^n. \tag{5}$$

Function spaces. In the following, we denote by $H^r(\mathbb{R})$, with $r \in \mathbb{R}$, the Sobolev space whose norm is

$$||u||_{H^r(\mathbb{R})} = \left(\int_{\mathbb{R}} \left(1 + |\xi|^2\right)^r |\widehat{u}(\xi)|^2\right)^{\frac{1}{2}}, \tag{6}$$

with $\widehat{u}$ the Fourier transform of u. If there is ambiguity, an 'x' will be added in H_x^r for the Sobolev space with respect to the space variable.

We study the convergence in the discrete space $\ell^\infty([\![0,N]\!], \ell^2_\Delta(\mathbb{Z}))$ whose norm is defined by

$$||a||_{\ell^\infty([\![0,N]\!],\ell^2_\Delta(\mathbb{Z}))} = \sup_{n\in[\![0,N]\!]} ||a^n||_{\ell^2_\Delta} = \sup_{n\in[\![0,N]\!]} \left(\sum_{j\in\mathbb{Z}} \Delta x |a_j^n|^2\right)^{\frac{1}{2}}, \tag{7}$$

for all $a = (a^n)_{n\in[\![0,N]\!]} = (a_j^n)_{(n,j)\in[\![0,N]\!]\times\mathbb{Z}}$. This norm is a relevant discrete equivalent for the $L^\infty([0,T]; L^2(\mathbb{R}))$-norm.

Convergence error. Let u be the exact solution of (1a)-(1b). From u, we construct the following sequence

$$\begin{cases} [u_\Delta]_j^n = \dfrac{1}{\Delta x[\inf(t^{n+1},T) - t^n]} \displaystyle\int_{t^n}^{\inf(t^{n+1},T)} \int_{x_j}^{x_{j+1}} u(s,y)dyds, & \text{if } (n,j) \in [\![1,N]\!] \times \mathbb{Z}, \\ [u_\Delta]_j^0 = \dfrac{1}{\Delta x} \displaystyle\int_{x_j}^{x_{j+1}} u_0(y)dy, & \text{if } j \in \mathbb{Z}. \end{cases} \tag{8}$$

From the averaged exact sequence $\left([u_\Delta]_j^n\right)_{(n,j)}$ and the numerical one $\left(v_j^n\right)_{(n,j)}$, we define two piecewise constant functions u_Δ and v_Δ by, for all $n \in [\![0,N]\!]$ and $j \in \mathbb{Z}$,

$$\begin{cases} u_\Delta(t,x) = (u_\Delta)_j^n, \\ v_\Delta(t,x) = v_j^n, \end{cases} \quad \text{if } (t,x) \in [t^n, \inf\left(t^{n+1},T\right)[\times [x_j, x_{j+1}[. \tag{9}$$

We define the convergence error by the following difference

$$e_j^n = v_\Delta(t^n, x_j) - u_\Delta(t^n, x_j), \quad (n,j) \in [\![0,N]\!] \times \mathbb{Z}. \tag{10}$$

Thanks to the definition (7), the convergence error satisfies

$$||e||_{\ell^\infty([\![0,N]\!],\ell^2_\Delta(\mathbb{Z}))} = ||v_\Delta - u_\Delta||_{L^\infty([0,T],L^2(\mathbb{R}))}.$$

Consistency error. We denote by $\left(\epsilon_j^n\right)_{(n,j)\in[\![0,N]\!]\times\mathbb{Z}}$ the consistency error defined by the following relation

$$\epsilon_j^n = \frac{(u_\Delta)_j^{n+1} - (u_\Delta)_j^n}{\Delta t} + D\left(\frac{u_\Delta^2}{2}\right)_j^n + \theta D_+D_+D_-\left(u_\Delta\right)_j^{n+1} \\ + (1-\theta)D_+D_+D_-\left(u_\Delta\right)_j^n - \frac{c\Delta x}{2}D_+D_-\left(u_\Delta\right)_j^n, \quad (n,j)\in[\![0,N]\!]\times\mathbb{Z}. \quad (11)$$

Main result In our first main result we handle the case of smooth enough initial data, $u_0 \in H^s(\mathbb{R})$, $s \geq 6$.

Theorem 1 (Convergence rate in the smooth case)**.** *Let $s \geq 6$ and $u_0 \in H^s(\mathbb{R})$. Let $T > 0$ and $c > 0$ such that the unique global solution u of (1a)-(1b) verifies*

$$\sup_{t\in[0,T]} ||u(t,\cdot)||_{L^\infty(\mathbb{R})} < c. \quad (12)$$

Let $\beta_0 \in (0,1)$ and $\theta \in [0,1]$. There exists $\widehat{\omega_0} > 0$ such that, for every $\Delta x \leq \widehat{\omega}_0$ and Δt satisfying

$$\begin{cases} 4\left(1-2\theta\right)\dfrac{\Delta t}{\Delta x^3} \leq 1-\beta_0, & (13a)\\ \left[c+\dfrac{1}{2}\right]\dfrac{\Delta t}{\Delta x} \leq 1-\beta_0, & (13b)\end{cases}$$

the finite difference scheme (2)-(3) with parameters c and θ and time- and space discretization steps Δt, Δx satisfies, for any $\eta \in (0, s-\frac{3}{2}]$,

$$||e^n||_{\ell^\infty([\![0,N]\!],\ell^2_\Delta(\mathbb{Z}))} \leq \Lambda_{T,\|u_0\|_{H^{\frac{3}{4}}}}\left(1+\|u_0\|^2_{H^{\frac{1}{2}+\eta}}\right)\left(\frac{\|u_0\|_{H^6}}{c+\frac{1}{2}} + \|u_0\|_{H^4} + \|u_0\|_{H^{\frac{3}{2}+\eta}}\|u_0\|_{H^1}\right)\Delta x. \quad (14)$$

In this estimate, e^n is defined as in (10)-(8)-(9) and $\Lambda_{T,\|u_0\|_{H^{\frac{3}{4}}}}$ is defined by

$$\Lambda_{T,\|u_0\|_{H^{\frac{3}{4}}}} = \exp\left(\frac{C}{2}\left(1+c^2\right)\left(1+\frac{(1-\beta_0)^2}{(c+\frac{1}{2})^2}\right)\left(T+(T^{\frac{3}{4}}+T^{\frac{1}{2}})||u_0||_{H^{\frac{3}{4}}}e^{\kappa_{\frac{3}{4}}T}\right)\right)Ce^{\kappa T}\sqrt{T\left\{1+\frac{1-\beta_0}{c+\frac{1}{2}}\right\}}, \quad (15)$$

in which C is a constant, $\kappa_{\frac{3}{4}}$ and κ depend only on $\|u_0\|_{L^2(\mathbb{R})}$.

Remark 4. *Conditions (13a)-(13b) are Courant-Friedrichs-Lewy-type conditions (in short, CFL conditions). The hypothesis $\left[c+\frac{1}{2}\right]\frac{\Delta t}{\Delta x} \leq 1-\beta_0$ seems to be a technical hypothesis, which probably may be replaced with the classical hyperbolic CFL condition $c\Delta t \leq \Delta x$. Indeed, experimental results suit with Theorem 1 with this classical CFL condition, see Section 7.*

Remark 5. *Thereafter, η should be chosen as small as possible, then norms $||u_0||_{H^{s+\eta}(\mathbb{R})}$ should be regarded as $||u_0||_{H^{s+}(\mathbb{R})}$.*

Thus, the scheme (2)-(3) is convergent to the first order in space in the $\ell^\infty([\![0,N]\!],\ell^2_\Delta(\mathbb{Z}))$-norm.
In our second main result, we improve the previous result to handle non-smooth initial data $u_0 \in H^s(\mathbb{R})$, $s \geq 3/4$. To perform the analysis, we first have to approximate in a suitable way the initial datum. Let χ be a $\mathcal{C}^\infty$-function such that

- $0 \leq \chi \leq 1$,
- $\chi \equiv 1$ in the ball $\mathcal{B}\left(0,\frac{1}{2}\right)$ and χ is supported in the ball $\mathcal{B}\left(0,1\right)$,
- $\chi(-\xi) = \chi(\xi)$.

Let φ be such as $\widehat{\varphi}\left(\xi\right) = \chi\left(\xi\right)$, where $\widehat{\varphi}$ stands for the Fourier transform of φ, and for all $\delta > 0$, we define φ^δ such that $\widehat{\varphi^\delta}\left(\xi\right) = \chi\left(\delta\xi\right)$, which implies $\varphi^\delta = \frac{1}{\delta}\varphi\left(\frac{\cdot}{\delta}\right)$. Eventually,

- we shall still denote by u the exact solution of (1a) starting from the initial datum u_0.
- Let u^δ be the solution of (1a) with $u_0^\delta = u_0 \star \varphi^\delta$ as initial datum, where $\star$ stands for the convolution product.
- We denote then by $(v_j^n)_{(n,j)\in[\![0,N]\!]\times\mathbb{Z}}$ the numerical solution obtained by applying the numerical scheme (2) from the initial datum $(u_0^\delta)_\Delta$:

$$v_j^0 = (u_0^\delta)_\Delta = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} u_0 \star \varphi^\delta(y)dy. \tag{16}$$

Theorem 2 (Convergence rate in the non-smooth case)**.** *Let $s \geq \frac{3}{4}$ and $u_0 \in H^s(\mathbb{R})$. Let $T > 0$ and $c > 0$ such that the unique global solution u of (1a)-(1b) verifies*

$$\sup_{t\in[0,T]} ||u(t,\cdot)||_{L^\infty(\mathbb{R})} < c.$$

Let $\beta_0 \in (0,1)$ and $\theta \in [0,1]$. There exists $\widehat{\omega_0} > 0$ such that for every $\Delta x \leq \widehat{\omega_0}$ and Δt satisfying

$$\begin{cases} 4\,(1-2\theta)\,\dfrac{\Delta t}{\Delta x^3} \leq 1-\beta_0, \\ \left[c+\dfrac{1}{2}\right]\dfrac{\Delta t}{\Delta x} \leq 1-\beta_0, \end{cases} \tag{17}$$

the finite difference scheme (2)-(16) with parameters c and θ and time- and space discretization steps Δt, Δx satisfies, for any $\eta \in (0, s-\frac{1}{2}]$,

$$||e^n||_{\ell^\infty([\![0,N]\!],\ell^2_\Delta(\mathbb{Z}))} \leq \Gamma_{T,\|u_0\|_{H^{\frac{3}{4}}}}\left(1+\|u_0\|^2_{H^{\frac{1}{2}+\eta}}\right)\left(\frac{1}{c+\frac{1}{2}}+1+\|u_0\|_{H^{\min(1,s)}}\right)\|u_0\|_{H^s}\Delta x^q.$$

In this estimate, e^n is defined as in (10)-(8)-(9),

- $q = \frac{s}{12-2s}$ *if* $\frac{3}{4} \leq s \leq 3$,
- $q = \frac{\min(s,6)}{6}$ *if* $3 < s$,

and $\Gamma_{T,\|u_0\|_{H^{\frac{3}{4}}}}$ is defined by

$$\Gamma_{T,\|u_0\|_{H^{\frac{3}{4}}}} = C\left[\Lambda_{T,\|u_0\|_{H^{\frac{3}{4}}}} + \exp\left(\frac{T^{\frac{3}{4}}C_{\frac{3}{4}}e^{\kappa_{\frac{3}{4}}T}}{4}\|u_0\|_{H^{\frac{3}{4}}}\right)\right],$$

where $\Lambda_{T,\|u_0\|_{H^{\frac{3}{4}}}}$ is defined by (15), C and $C_{\frac{3}{4}}$ are two constants and $\kappa_{\frac{3}{4}}$ depends only on $\|u_0\|_{L^2(\mathbb{R})}$.

If $u_0 \in H^m(\mathbb{R})$ with $m \geq 6$, then Theorem 2 implies an order of convergence equal to 1 and we get back the result of Theorem 1. Note that the results are valid for any $T > 0$ in agreement with the fact that at this level of regularity we have global solutions keeping their regularity.

To prove Theorem 1, we prove consistency and stability of the scheme. It is in the control of the consistency error that we need the exact solution to be smooth. The most challenging part of the proof is the study of the stability of the scheme in order to take advantage of the fact that the exact solution remains smooth on the whole $[0,T]$. The main idea is to transpose at the discrete level the well known weak-strong stability property for hyperbolic conservation laws that relies on a relative entropy estimate, see [Daf10] for a detailed presentation. This method is classical for the study of hyperbolic systems, see for exemple [CMS16] for the numerical approximation of systems of conservation laws, [Tza05] for a relaxation hyperbolic system or [LV11] for the approximation of shocks and contact discontinuities. An important outcome of this approach is that in the stability estimate, the exponential amplification factor only involves the norm $\int_0^T \|\partial_x u(t,.)\|_{L^\infty}\mathrm{d}t$ of the exact solution, which is bounded thanks to the dispersive properties of the equation. This allows to get the convergence of the scheme on the full interval of time $[0,T]$ and also to handle less smooth initial data at the price of deteriorating the convergence order as stated in Theorem 2. Indeed in order to prove Theorem 2, we replace the initial data u_0 with a smoother one u_0^δ and just use the triangular inequality

$$\|v_\Delta - u_\Delta\|_{L^\infty([0,T],L^2_x)} \leq \|v_\Delta - u_\Delta^\delta\|_{L^\infty([0,T],L^2_x)} + \|u_\Delta^\delta - u_\Delta\|_{L^\infty([0,T],L^2_x)},$$

where u_Δ^δ is the discretization of the exact solution u^δ of the KdV equation with initial data u_0^δ. We then use the stability in L^2 for exact solutions of the KdV equation and the stability estimate of Theorem 1. The amplification factor $\int_0^T \|\partial_x u^\delta(t,.)\|_{L^\infty} dt$ is finite and can be bounded independently of δ as soon as the initial data is in $H^s(\mathbb{R})$, with $s \geq 3/4$ because of the Strichartz estimate that ensures that at this level of regularity, the exact solution is actually also such that $\partial_x u \in L^4([0,T], L^\infty(\mathbb{R}))$. We then end the proof by optimizing these estimates in terms of δ and Δx.

Outline of the paper In Section 2, we state precisely the results of the Cauchy theory of KdV that we shall use in this paper. Then, in Section 3, we analyze the consistency error of the scheme (postponing the more technical part to the appendix). The aim of Section 4 is to derive the crucial ℓ^2_Δ-stability inequality. We study the discrete equation verified by the convergence error and we obtain the ℓ^2_Δ estimates. Eventually, the rate of convergence is determined in Section 5.
Section 6 is devoted to the study of the convergence rate for a non smooth solution. A convolution product by mollifiers enables us to counteract the lack of regularity. It requires several general approximation estimates between initial data and regularized initial data which are gathered in Subsection 6.1. The proof of Theorem 2 is developed in Subsection 6.2. Some numerical results illustrate the theoretical rate of convergence in Section 7. Finally, in Appendix A, we prove an estimate of the consistency error in $\ell^\infty\left([\![0,N]\!], \ell^2_\Delta(\mathbb{Z})\right)$-norm, and in Appendix B, some technical lemmas of Section 4 are proved.

Notation Thereafter, the letter C represents a positive number that may differ from line to line and that can be chosen independently of Δt, Δx, u, u_0, T and δ. We denote by κ all numbers depending only on $\|u_0\|_{L^2(\mathbb{R})}$.

2 Known results on the Cauchy problem for the KdV equation

Let us recall the definition of Bourgain spaces. A tempered distribution $u(t,x)$ on $\mathbb{R}^2$ belongs to $X^{s,b}$ if its following norm is finite

$$||u||_{X^{s,b}} = \left(\int_{\mathbb{R}}\int_{\mathbb{R}} (1+|\xi|)^{2s} \left(1+|\tau-\xi^3|\right)^{2b} |\tilde{u}(\tau,\xi)|^2 d\xi d\tau\right)^{\frac{1}{2}},$$

where $\tilde{u}$ is the space and time Fourier transform of u. We shall also use a localized version of this space, $u \in X^{s,b}(I)$ where $I \subset \mathbb{R}$ is an interval if $\|u\|_{X^{s,b}(I)} < +\infty$ where

$$\|u\|_{X^{s,b}(I)} = \inf\{\|\overline{u}\|_{X^{s,b}},\ \overline{u}_{/I} = u\}.$$

By using results from [KPV91], [Bou93], [KPV93], see for example the book [LP15], we get the following theorem.

Theorem 3. *Consider $s \geq 0$, $1 > b > 1/2$. There exists a unique global solution u of (1a)-(1b), with $u_0 \in H^s(\mathbb{R})$, such that for every $T \geq 0$, $u \in \mathcal{C}([0,T], H^s(\mathbb{R})) \cap X^{s,b}([0,T])$. Moreover, there exists $\kappa_s > 0$, depending only on s and on the norm $\|u_0\|_{L^2}$, and $C_s > 0$, depending only on s, such that, for any $T \geq 0$,*

- $\sup_{t\in[0,T]} \|u(t)\|_{H^s(\mathbb{R})} \leq C_s \|u_0\|_{H^s(\mathbb{R})} e^{\kappa_s T}$,
- *if* $s \geq \dfrac{3}{4}$, $\quad \|\partial_x u\|_{L^i([0,T],L^\infty(\mathbb{R}))} \leq T^{\frac{4-i}{4i}} \|u_0\|_{H^{\frac{3}{4}}(\mathbb{R})} C_{\frac{3}{4}} e^{\kappa_{\frac{3}{4}} T}$, *for* $i \in \{1,2\}$.

The growth rate in the above estimates is not optimal.

Note that a local well-posedness result for $s > 3/4$ follows directly from [KPV91]. In the present paper, we will be only interested in $s \geq 3/4$, nevertheless, to get the global well-posedness for $s \in [3/4, 1)$, we need to go through the L^2 local well-posedness result.

Proof. Let us just briefly explain how we can organize now classical arguments to get the result. We refer for example to [KPV93], [LP15] for the details. The existence is proven by a fixed point argument on the following truncated problem:

$$v \mapsto F(v)$$

such that

$$F(v)(t) = \chi(t)e^{-t\partial_x^3}u_0 - \chi(t)\int_0^t e^{-(t-\tau)\partial_x^3}\partial_x\left(\chi\left(\frac{\tau}{\delta}\right)\frac{v^2}{2}(\tau)\right)d\tau,$$

where $\chi \in [0,1]$ is a smooth compactly supported function which is equal to 1 on $[-1,1]$ and supported in $[-2,2]$. For $|t| \le \delta \le 1/2$, a fixed point of the above equation gives a solution of the original Cauchy problem, denoted by u.

Fix $C > 0$, that does not depend on u, such that

$$\|\chi(t)e^{-t\partial_x^3}u_0\|_{X^{0,b}} \le C\|u_0\|_{L^2}.$$

We can first prove that F is a contraction on a suitable ball of $X^{0,b}$, provided $8C^2\|u_0\|_{L^2}\delta^\beta \le 1$ for some $\beta > 0$ (that is related to $1 > b > 1/2$) that does not depend on δ nor u. In particular, for the fixed point, we can ensure that

$$\|v\|_{X^{0,b}} \le 2C\|u_0\|_{L^2}.$$

Next, by using again the Duhamel formula, we can obtain, for $s \ge 0$,

$$\|v\|_{X^{s,b}} \le c_s\|u_0\|_{H^s} + c_s\delta^\beta\|v\|_{X^{0,b}}\|v\|_{X^{s,b}} \le c_s\|u_0\|_{H^s} + 2c_sC\|u_0\|_{L^2}\delta^\beta\|v\|_{X^{s,b}},$$

where c_s depends only on s. In particular, by choosing δ, possibly smaller than previously, such that $2c_sC\|u_0\|_{L^2}\delta^\beta \le 1/2$, we thus obtain that

$$\|v\|_{X^{s,b}} \le 2c_s\|u_0\|_{H^s}.$$

Next, by using that the $X^{s,b}$ norm for $b > 1/2$ controls the $\mathcal{C}(\mathbb{R}, H^s)$ norm (see for example [Tao06] lemma 2.9 page 100), we obtain that

$$\|u\|_{\mathcal{C}([0,\delta],H^s(\mathbb{R}))} \le \|v\|_{\mathcal{C}(\mathbb{R},H^s(\mathbb{R}))} \le B_s\|u_0\|_{H^s(\mathbb{R})},$$

where B_s depends only on s. Since the existence time δ depends only on the L^2-norm of the initial datum and that the L^2-norm is conserved for the KdV equation, we can iterate the above argument to get a global solution. Moreover, in a quantitative way, by choosing $n = \lfloor T/\delta \rfloor + 1$ and iterating n times, we obtain that

$$\|u\|_{\mathcal{C}([0,T],H^s)} + \|u\|_{X^{s,b}[0,T]} \le B_s^n\|u_0\|_{H^s} \le C_s\|u_0\|_{H^s}e^{\kappa_s T},$$

where κ_s depends only on s and $\|u_0\|_{L^2}$ while C_s depends only on s.

Finally, since the Strichartz estimate in the KdV context (see [KPV91]) reads

$$\||\partial_x|^{\frac{1}{4}}e^{-t\partial_x^3}u_0\|_{L_t^4(\mathbb{R},L_x^\infty)} \le C\|u_0\|_{L^2},$$

by using the embedding properties of the Bourgain spaces (see again [Tao06] lemma 2.9 page 100), we obtain that

$$\|\partial_x u\|_{L_t^4([0,\delta],L_x^\infty)} \le \|\partial_x v\|_{L_t^4(\mathbb{R},L_x^\infty)} \le \|v\|_{X^{\frac{3}{4},b}} \le C\|u_0\|_{H^{\frac{3}{4}}}.$$

Again by iterating this estimate, we finally obtain that

$$\|\partial_x u\|_{L_t^4([0,T],L_x^\infty)} \le C_{\frac{3}{4}}\|u_0\|_{H^{\frac{3}{4}}}e^{\kappa_{\frac{3}{4}}T}$$

and the desired estimate follows from the Hölder inequality.

□

3 Consistency error estimate

This section is devoted to the computation of the consistency error defined by Equation (11). As a starting point, by using Theorem 3, we obtain the following estimates on the averaged solution u_Δ.

Lemma 1. *Let u be the exact solution of (1a)-(1b) from $u_0 \in H^s(\mathbb{R})$, $s > \frac{1}{2}$ and u_Δ be defined by (9). Then there exists $C > 0$, depending only on s, and $\kappa_s > 0$, depending only on s and $\|u_0\|_{L^2}$, such that, for any $T \geq 0$ and any $n \in [\![0, N]\!]$ with $N = \lfloor \frac{T}{\Delta t} \rfloor$,*

- $|| (u_\Delta)^n ||_{\ell^\infty} \leq C e^{\kappa_s T} \|u_0\|_{H^s}$,
- *if* $s \geq \dfrac{3}{4}$, $\Delta t || D_+ (u_\Delta)^n ||^i_{\ell^\infty} \leq \displaystyle\int_{t^n}^{t^{n+1}} ||\partial_x u(s,.)||^i_{L^\infty_x} ds \leq T^{\frac{4-i}{4i}} C e^{\kappa_{\frac{3}{4}} T} \|u_0\|_{H^{\frac{3}{4}}(\mathbb{R})}$, *for $i \in \{1, 2\}$.* (18)

Proof. The Sobolev embedding $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, for $s > \frac{1}{2}$ yields the inequality

$$|| (u_\Delta)^n ||_{\ell^\infty} \leq \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} ||u(t,.)||_{L^\infty(\mathbb{R})} dt \leq C \sup_{t \in [0,T]} ||u(t,.)||_{H^s(\mathbb{R})}.$$

Theorem 3 implies

$$|| (u_\Delta)^n ||_{\ell^\infty} \leq C C_s \|u_0\|_{H^s(\mathbb{R})} e^{\kappa_s T},$$

which proves the first estimate of Lemma 1.
To prove Equation (18) for $i = 1$, we use a Taylor expansion:

$$\begin{aligned}
\Delta t \, ||D_+ (u_\Delta)^n||_{\ell^\infty} &= \Delta t \left|\left| \frac{1}{\Delta t \Delta x^2} \int_{t^n}^{t^{n+1}} \int_{x_j}^{x_{j+1}} u(s, y + \Delta x) - u(s,y) dy ds \right|\right|_{\ell^\infty} \\
&= \Delta t \left|\left| \frac{1}{\Delta x^2 \Delta t} \int_{t^n}^{t^{n+1}} \int_{x_j}^{x_{j+1}} \int_y^{y+\Delta x} \partial_x u(s,z) dz dy ds \right|\right|_{\ell^\infty} \\
&\leq \int_{t^n}^{t^{n+1}} ||\partial_x u(s,.)||_{L^\infty_x} ds.
\end{aligned}$$

For $i = 2$, the same Taylor expansion gives, thanks to the Cauchy-Schwarz inequality,

$$\begin{aligned}
\Delta t \, ||D_+ (u_\Delta)^n||^2_{\ell^\infty} &= \Delta t \left|\left| \frac{1}{\Delta x^2 \Delta t} \int_{t^n}^{t^{n+1}} \int_{x_j}^{x_{j+1}} \int_y^{y+\Delta x} \partial_x u(s,z) dz dy ds \right|\right|^2_{\ell^\infty} \\
&\leq \frac{\Delta t}{\Delta x^4 \Delta t^2} \Delta t \Delta x^2 \int_{t^n}^{t^{n+1}} \int_{x_j}^{x_{j+1}} \int_y^{y+\Delta x} ||\partial_x u(s,.)||^2_{L^\infty_x} dz dy ds \\
&\leq \int_{t^n}^{t^{n+1}} ||\partial_x u(s,.)||^2_{L^\infty_x} ds.
\end{aligned}$$

Theorem 3 concludes the proof. $\square$

Remark 6. *The Sobolev regularity of the initial datum is at least $H^{\frac{3}{4}}(\mathbb{R})$ in Theorem 2 because we need to control $\int_0^T ||\partial_x u(t,.)||^i_{L^\infty(\mathbb{R})} dt$, for $i \in \{1, 2\}$ in some of the proofs. This is explicitly needed in Lemma 1, Theorem 3 and in the definition of $\Lambda_{T, \|u_0\|_{\frac{3}{4}}}$ in (15).*

As a consequence, we control the ℓ^2_Δ-norm of the consistency error ϵ^n defined in (11) in terms of the initial datum thanks to the following proposition.

Proposition 1. *Consider $u_0 \in H^s(\mathbb{R})$, with $s \geq 6$. There exist $C > 0$ and $\kappa > 0$, depending only on $\|u_0\|_{L^2}$, such that for any $\eta \in (0, s - \frac{3}{2}]$ and for any $T > 0$,*

$$||\epsilon^n||_{\ell^\infty([\![0,N]\!];\ell^2_\Delta(\mathbb{Z}))} \leq Ce^{\kappa T}\left(1 + ||u_0||^2_{H^{\frac{1}{2}+\eta}}\right)\left\{\Delta t\ ||u_0||_{H^6} + \Delta x\ \left[||u_0||_{H^4} + ||u_0||_{H^{\frac{3}{2}+\eta}}||u_0||_{H^1}\right]\right\}, \tag{19}$$

Proof. The consistency error satisfies the following inequality (*cf.* Appendix A for the proof):

$$||\epsilon^n||_{\ell^\infty([\![0,N]\!],\ell^2_\Delta)} \leq B_1\left\{\Delta t \sup_{t\in[0,T]}\left[\left(1 + ||u||^2_{L^\infty_x}\right)||u||_{H^6_x}\right] + \Delta x \sup_{t\in[0,T]}\left[\left(1 + ||u||_{L^\infty_x}\right)||u||_{H^4_x} + ||\partial_x u||_{L^\infty_x}||u||_{H^1_x}\right]\right\},$$

with B_1 a constant that does not depend on u, u_0, T, Δt and Δx. By using the Sobolev embedding $H^{\frac{1}{2}+\eta}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, with $\eta > 0$, we obtain

$$||\epsilon^n||_{\ell^\infty([\![0,N]\!],\ell^2_\Delta)} \leq B_1\left\{\Delta t \sup_{t\in[0,T]}\left[\left(1 + ||u||^2_{H_x^{\frac{1}{2}+\eta}}\right)||u||_{H^6_x}\right] + \Delta x \sup_{t\in[0,T]}\left[\left(1 + ||u||_{H_x^{\frac{1}{2}+\eta}}\right)||u||_{H^4_x} + ||u||_{H_x^{\frac{3}{2}+\eta}}||u||_{H^1_x}\right]\right\}.$$

Theorem 3 enables to rewrite

$$\begin{aligned}||\epsilon^n||_{\ell^\infty([\![0,N]\!],\ell^2_\Delta)} \leq \Delta t\ B_1 C_6 C^2_{\frac{1}{2}+\eta} e^{(2\kappa_{\frac{1}{2}+\eta}+\kappa_6)T}&\left[\left(1 + ||u_0||^2_{H^{\frac{1}{2}+\eta}}\right)||u_0||_{H^6}\right]\\ &+ \Delta x\ \overline{C}e^{\overline{\kappa}T}\left[\left(1 + ||u_0||_{H^{\frac{1}{2}+\eta}}\right)||u_0||_{H^4} + ||u_0||_{H^{\frac{3}{2}+\eta}}||u_0||_{H^1}\right],\end{aligned}$$

with $\overline{C} = \max\left(B_1 C_{\frac{1}{2}+\eta}C_4, B_1 C_{\frac{3}{2}+\eta}C_1, B_1 C_4\right)$ and $\overline{\kappa} = \max\left(\kappa_{\frac{1}{2}+\eta} + \kappa_4, \kappa_{\frac{3}{2}+\eta} + \kappa_1, \kappa_4\right)$.
Inequality (19) follows from the fact that there exists a constant $B_2 \geq 2$ such that

$$\left(1 + ||u_0||_{H^{\frac{1}{2}+\eta}}\right) \leq B_2\left(1 + ||u_0||^2_{H^{\frac{1}{2}+\eta}}\right).$$

We fix $C = \max\left(B_1 C_6 C^2_{\frac{1}{2}+\eta}, B_2\overline{C}\right)$ and $\kappa = \max\left(2\kappa_{\frac{1}{2}+\eta} + \kappa_6, \overline{\kappa}\right)$. $\square$

4 Stability estimate

The stability property will be proved in stating a discrete weak-strong stability type inequality : Equation (39). This inequality gives an upper bound of the convergence error at time $n+1$ with convergence error at time n. However, some terms B_i of (39) are in factor of derivatives on the convergence error $(e^n_j)_{(n,j)}$, which are not controlled at this first step: we will cancel those terms in Section 5.

4.1 Preliminary results

We here collect some discrete "Leibniz's rules" (Lemma 2), ℓ^2-norm identities (Lemma 3) and discrete integrations by parts formulas (Lemma 4) which will be used in Subsection 4.2. Their proofs lie in Appendix B.

Lemma 2. *Let $(a_j)_{j\in\mathbb{Z}}$ and $(b_j)_{j\in\mathbb{Z}}$ be two sequences and let D, D_+, D_- be the discrete operators defined in (4). One has, for any $j \in \mathbb{Z}$:*

- $$D_+D_-(a)_j = D_-D_+(a)_j, \tag{20}$$

- $$\begin{cases} D_+(ab)_j = a_{j+1}D_+(b)_j + b_jD_+(a)_j, & (21a)\\ D_-(ab)_j = a_{j-1}D_-(b)_j + b_jD_-(a)_j. & (21b)\end{cases}$$

- $$D(ab)_j = D(a)_jb_{j+1} + a_{j-1}D(b)_j, \tag{22}$$

- $$D(ab)_j = b_j D(a)_j + \frac{a_{j+1}}{2} D_+(b)_j + \frac{a_{j-1}}{2} D_-(b)_j, \tag{23}$$

- $$\begin{cases} a_j D_+(a)_j = \frac{1}{2} D_+\left(a^2\right)_j - \frac{\Delta x}{2}\left(D_+(a)_j\right)^2, & (24a)\\ a_j D_-(a)_j = \frac{1}{2} D_-\left(a^2\right)_j + \frac{\Delta x}{2}\left(D_-(a)_j\right)^2. & (24b)\end{cases}$$

Lemma 3. *For $(a_j)_{j\in\mathbb{Z}}$ a sequence in $\ell^2_\Delta(\mathbb{Z})$, one has*

- $$\sum_{j\in\mathbb{Z}} \left[D_+(a)_j\right]^2 = \sum_{j\in\mathbb{Z}} \left[D_-(a)_j\right]^2, \tag{25}$$

- $$\sum_{j\in\mathbb{Z}} \left[D\left(\frac{a^2}{2}\right)_j\right]^2 = \sum_{j\in\mathbb{Z}} \left[D(a)_j\right]^2 \left(\frac{a_{j+1}+a_{j-1}}{2}\right)^2, \tag{26}$$

- $$\sum_{j\in\mathbb{Z}} \left[D_+D_-(a)_j\right]^2 = \frac{4}{\Delta x^2}\sum_{j\in\mathbb{Z}} \left[D_+(a)_j\right]^2 - \frac{4}{\Delta x^2}\sum_{j\in\mathbb{Z}} \left[D(a)_j\right]^2. \tag{27}$$

Applying (27) to $D_+(a)_j$ rather than a_j enables to state

Corollary 1. *Let $(a_j)_{j\in\mathbb{Z}}$ be a sequence in $\ell^2_\Delta(\mathbb{Z})$. One has*

$$\sum_{j\in\mathbb{Z}} \left[D_+D_+D_-(a)_j\right]^2 = \frac{4}{\Delta x^2}\sum_{j\in\mathbb{Z}} \left[D_+D_-(a)_j\right]^2 - \frac{4}{\Delta x^2}\sum_{j\in\mathbb{Z}} \left[D_+D(a)_j\right]^2. \tag{28}$$

Lemma 4. *Let $(a_j)_{j\in\mathbb{Z}}$ and $(b_j)_{j\in\mathbb{Z}}$ be two sequences in $\ell^2_\Delta(\mathbb{Z})$. One has*

- $$\sum_{j\in\mathbb{Z}} D_+(a)_j\, b_j = -\sum_{j\in\mathbb{Z}} a_j D_-(b)_j\,, \tag{29}$$

- $$\sum_{j\in\mathbb{Z}} D(a)_j\, b_j = -\sum_{j\in\mathbb{Z}} a_j D(b)_j\,, \tag{30}$$

- $$\sum_{j\in\mathbb{Z}} a_j D_+(a)_j = -\sum_{j\in\mathbb{Z}} \frac{\Delta x}{2}\left[D_+(a)_j\right]^2, \tag{31}$$

- $$\sum_{j\in\mathbb{Z}} D_+(a)_j\, a_j a_{j+1} = -\sum_{j\in\mathbb{Z}} \frac{\Delta x^2}{3}\left[D_+(a)_j\right]^3, \tag{32}$$

- $$\sum_{j\in\mathbb{Z}} D(a)_j\, a_{j-1} a_{j+1} = -\sum_{j\in\mathbb{Z}} \frac{4\Delta x^2}{3}\left[D(a)_j\right]^3, \tag{33}$$

- $$\sum_{j\in\mathbb{Z}} a_j D(ab)_j = \sum_{j\in\mathbb{Z}} D_+(b)_j \frac{a_j a_{j+1}}{2}, \tag{34}$$

- $$\sum_{j\in\mathbb{Z}} D_+D_-(a)_j\, D(ab)_j = -\sum_{j\in\mathbb{Z}} \frac{1}{\Delta x^2} D_+(b)_j\, a_j a_{j+1} + \sum_{j\in\mathbb{Z}} \frac{1}{\Delta x^2} D(b)_j\, a_{j-1} a_{j+1}. \tag{35}$$

With (32) and (33), taking $(b)_{j\in\mathbb{Z}} = (\frac{a_j}{2})_{j\in\mathbb{Z}}$ in (34) and (35) gives the following corollary.

Corollary 2. *Let $(a_j)_{j\in\mathbb{Z}}$ be a sequence in $\ell^2_\Delta(\mathbb{Z})$. One has*

- $$\sum_{j\in\mathbb{Z}} a_j D\left(\frac{a^2}{2}\right)_j = -\sum_{j\in\mathbb{Z}} \frac{\Delta x^2}{12}\left[D_+(a)_j\right]^3, \tag{36}$$

- $$\sum_{j\in\mathbb{Z}} D\left(\frac{a^2}{2}\right)_j D_+D_-(a)_j = \sum_{j\in\mathbb{Z}} \frac{1}{6}\left[D_+(a)_j\right]^3 - \frac{2}{3}\left[D(a)_j\right]^3. \tag{37}$$

4.2 The ℓ^2_Δ-stability inequality

We focus on the derivation of the ℓ^2_Δ-stability inequality (39), which corresponds to a discrete equivalent of a weak-strong estimate.
Combining (5), (10) and (11), we obtain

$$\begin{aligned} & e_j^{n+1} + \theta\Delta t D_+D_+D_-\left(e\right)_j^{n+1} \\ & = e_j^n - (1-\theta)\Delta t D_+D_+D_-\left(e\right)_j^n - \Delta t D\left(\frac{e^2}{2}\right)_j^n - \Delta t D\left(u_\Delta e\right)_j^n + \frac{c\Delta x\Delta t}{2}D_+D_-\left(e\right)_j^n - \Delta t\epsilon_j^n,\ (n,j)\in[\![0,N]\!]\times\mathbb{Z}. \end{aligned} \tag{38}$$

Proposition 2 (ℓ^2_Δ-stability inequality)**.** *Let $(e_j^n)_{(j,n)}$ be the convergence error defined by (10) with respect to Scheme (2)-(3). For every $\theta\in[0,1], \Delta t>0$ and $\Delta x>0$, for every $(n,j)\in[\![0,N]\!]\times\mathbb{Z}$ and $\gamma\in[0,\frac{1}{2})$ and $\sigma\in\{0,1\}$, one has*

$$\begin{aligned} & \left|\left|e^{n+1} + \theta\Delta t D_+D_+D_-\left(e\right)^{n+1}\right|\right|^2_{\ell^2_\Delta} \le ||e^n + \theta\Delta t D_+D_+D_-\left(e\right)^n||^2_{\ell^2_\Delta} + \Delta t B_a||e^n||^2_{\ell^2_\Delta} + \Delta t\,||e^n - (1-\theta)\Delta t D_+D_+D_-\left(e\right)^n||^2_{\ell^2_\Delta} \\ & + \Delta t B_b\,||D_+\left(e\right)^n||^2_{\ell^2_\Delta} + \Delta t^2 B_c\,||D\left(e\right)^n||^2_{\ell^2_\Delta} + \Delta t B_d\,||D_+D_-\left(e\right)^n||^2_{\ell^2_\Delta} + \Delta t B_e\,||D_+D\left(e\right)^n||^2_{\ell^2_\Delta} + \Delta t B_f\,||D_+D_+D_-\left(e\right)^n||^2_{\ell^2_\Delta} \\ & + \Delta t||\epsilon^n||^2_{\ell^2_\Delta}\left\{1 + 4\frac{\Delta t}{\Delta x} + \Delta t\right\}, \end{aligned} \tag{39}$$

where the coefficients B_i, for $i\in\ \{a,b,c,d,e,f\}$, are defined in Equations (40a)-(40f).

$$\begin{aligned} B_a = ||u^n_\Delta||^2_{\ell^\infty} + ||D_+\left(u_\Delta\right)^n||_{\ell^\infty}\left(2-\theta+\frac{\Delta t}{\Delta x}\left[2c+\frac{2}{3}||e^n||_{\ell^\infty}+\frac{3}{2}||(u_\Delta)^n||_{\ell^\infty}\right]\right) & \\ + \frac{\Delta t^2}{\Delta x^2}||D_+(u_\Delta)^n||^2_{\ell^\infty} + \frac{\Delta t}{\Delta x}(||u^n_\Delta||^2_{\ell^\infty}+2c^2), & \quad (40a) \end{aligned}$$

$$B_b = \left(\frac{\Delta x}{6}D_+\left(e\right)_j^n - c\right)(\Delta x - c\Delta t) + (1-\theta)\Delta t||D_+\left(u_\Delta\right)^n||^{2-\sigma}_{\ell^\infty}, \tag{40b}$$

$$B_c = ||e^n||^2_{\ell^\infty}\left[1+\Delta x\right] + ||(u_\Delta)^n||^2_{\ell^\infty} - c^2 + 2||e^n||_{\ell^\infty}||(u_\Delta)^n||_{\ell^\infty} + \frac{2c}{3}||e^n||_{\ell^\infty}, \tag{40c}$$

$$B_d = (1-\theta)\Delta t\left[||D_+(u_\Delta)^n||^\sigma_{\ell^\infty} + \frac{\Delta x}{2}||D_-\left(u_\Delta\right)^n||_{\ell^\infty}\right], \tag{40d}$$

$$B_e = 2(1-\theta)\Delta t\left\{||\left(u_\Delta\right)^n||_{\ell^\infty} + ||e^n||_{\ell^\infty} + \left[\frac{\Delta x^{\frac{1}{2}-\gamma} + ||e^n||_{\ell^\infty} + 9||e^n||^2_{\ell^\infty}\Delta x^{\gamma-\frac{1}{2}}}{2}\right]\right\} - \Delta x, \tag{40e}$$

$$\begin{aligned} B_f = \Delta t\left\{(1-2\theta) + \frac{(1-\theta)\Delta x^2}{2}\left[c + \frac{\Delta x^{\frac{1}{2}-\gamma} + ||e^n||_{\ell^\infty} + 9||e^n||^2_{\ell^\infty}\Delta x^{\gamma-\frac{1}{2}}}{2}\right]\right. & \\ \left. + \Delta t(1-\theta)||D_+\left(u_\Delta\right)^n||_{\ell^\infty}\right\} - \frac{\Delta x^3}{4}. & \quad (40f) \end{aligned}$$

Remark 7. *One of our purposes, here below, will be to control the right hand side terms B_i with $i\in\{b,c,d,e,f\}$ only in terms of u_Δ and not v. This is why, this inequality can be viewed as a weak-strong inequality.*

Proof. Thanks to (38), one has

$$\left(e_j^{n+1} + \theta\Delta t D_+D_+D_-(e)_j^{n+1}\right)^2 = \left(\mathrm{RHS}_j^n\right)_a^2 + \left(\mathrm{RHS}_j^n\right)_b^2 + \left(\mathrm{RHS}_j^n\right)_c^2$$

with

$$\left(\mathrm{RHS}_j^n\right)_a^2 = \left(e_j^n\right)^2 + (1-\theta)^2\Delta t^2\left[D_+D_+D_-\left(e\right)_j^n\right]^2 + \Delta t^2\left[D\left(\frac{e^2}{2}\right)_j^n\right]^2 + \Delta t^2\left[D\left(u_\Delta e\right)_j^n\right]^2 + \frac{c^2\Delta t^2\Delta x^2}{4}\left[D_+D_-\left(e\right)_j^n\right]^2,$$

$$
\begin{aligned}
\left(\mathrm{RHS}_j^n\right)_b^2 =& -2(1-\theta)\Delta t e_j^n D_+D_+D_-\left(e\right)_j^n - 2\Delta t e_j^n D\left(\frac{e^2}{2}\right)_j^n - 2\Delta t e_j^n D\left(u_\Delta e\right)_j^n + c\Delta x\Delta t e_j^n D_+D_-\left(e\right)_j^n \\
&+ 2(1-\theta)\Delta t^2 D_+D_+D_-\left(e\right)_j^n D\left(u_\Delta e\right)_j^n + 2(1-\theta)\Delta t^2 D_+D_+D_-\left(e\right)_j^n D\left(\frac{e^2}{2}\right)_j^n \\
&- c\Delta x\Delta t^2(1-\theta)D_+D_+D_-\left(e\right)_j^n D_+D_-\left(e\right)_j^n + 2\Delta t^2 D\left(\frac{e^2}{2}\right)_j^n D\left(u_\Delta e\right)_j^n - c\Delta x\Delta t^2 D\left(\frac{e^2}{2}\right)_j^n D_+D_-\left(e\right)_j^n \\
&- c\Delta x\Delta t^2 D\left(u_\Delta e\right)_j^n D_+D_-\left(e\right)_j^n,
\end{aligned}
\tag{41}
$$

and

$$
\begin{aligned}
\left(\mathrm{RHS}_j^n\right)_c^2 = -2\Delta t\epsilon_j^n\left(e_j^n - (1-\theta)\Delta t D_+D_+D_-\left(e\right)_j^n\right) + 2\Delta t^2 D\left(\frac{e^2}{2}\right)_j^n\epsilon_j^n + 2\Delta t^2 D\left(u_\Delta e\right)_j^n\epsilon_j^n - c\Delta x\Delta t^2 D_+D_-\left(e\right)_j^n\epsilon_j^n \\
+ \Delta t^2\left(\epsilon_n^j\right)^2.
\end{aligned}
$$

Right Hand Side $\left(\mathrm{RHS}_j^n\right)_a^2$ We here will bound $\sum_{j\in\mathbb{Z}}\Delta x\left(\mathrm{RHS}_j^n\right)_a^2$.

- To this aim, we use the discrete integrations by parts formulas of Subsection 4.1, to see that, thanks to Identity (26),

$$
\sum_{j\in\mathbb{Z}}\Delta x\Delta t^2\left[D\left(\frac{e^2}{2}\right)_j^n\right]^2 = \sum_{j\in\mathbb{Z}}\Delta x\Delta t^2\left[D\left(e\right)_j^n\right]^2\left(\frac{e_{j+1}^n + e_{j-1}^n}{2}\right)^2.
$$

- To bound $\sum_{j\in\mathbb{Z}}\Delta x\Delta t^2\left[D\left(u_\Delta e\right)_j^n\right]^2$, we use the following result, proved in Appendix B.

Lemma 5. *Let $(a_j)_{j\in\mathbb{Z}}$ and $(b_j)_{j\in\mathbb{Z}}$ be two sequences in $\ell^2_\Delta(\mathbb{Z})$, then they satisfy the following inequality*

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}}\left[D\left(ab\right)_j\right]^2 \leq& \sum_{j\in\mathbb{Z}}\left\{(b_j)^2 + \frac{\Delta t}{2}\left[(D_+b_j)^2 + (D_-b_j)^2\right]\right\}(Da_j)^2 \\
&+ \sum_{j\in\mathbb{Z}}\frac{1}{2}\left\{\frac{(b_{j-1})^2 + (b_{j+1})^2}{\Delta t} + \frac{3}{4}(D_+b_j)^2 + \frac{3}{4}(D_-b_j)^2\right\}(a_j)^2.
\end{aligned}
\tag{42}
$$

Relation (42) gives

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}}\Delta x\Delta t^2\left[D\left(u_\Delta e\right)_j^n\right]^2 \leq& \sum_{j\in\mathbb{Z}}\Delta x\Delta t^2\left\{\left([u_\Delta]_j^n\right)^2 + \frac{\Delta t}{2}\left(D_+\left(u_\Delta\right)_j^n\right)^2 + \frac{\Delta t}{2}\left(D_-\left(u_\Delta\right)_j^n\right)^2\right\}\left(De_j^n\right)^2 \\
&+ \sum_{j\in\mathbb{Z}}\frac{\Delta t\Delta x}{2}\left\{\left([u_\Delta]_{j-1}^n\right)^2 + \left([u_\Delta]_{j+1}^n\right)^2 + \frac{3\Delta t}{4}\left(D_+\left(u_\Delta\right)_j^n\right)^2 + \frac{3\Delta t}{4}\left(D_-\left(u_\Delta\right)_j^n\right)^2\right\}\left(e_j^n\right)^2.
\end{aligned}
$$

We turn our attention to the term $\sum_{j\in\mathbb{Z}}\frac{\Delta x\Delta t^3}{2}\left\{\left(D_+(u_\Delta)_j^n\right)^2 + \left(D_-(u_\Delta)_j^n\right)^2\right\}\left(De_j^n\right)^2$. Definition of De_j^n gives

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}}\frac{\Delta x\Delta t^3}{2}\left\{\left(D_+(u_\Delta)_j^n\right)^2 + \left(D_-(u_\Delta)_j^n\right)^2\right\}\left(De_j^n\right)^2 &\leq \sum_{j\in\mathbb{Z}}\Delta x\Delta t^3||D_+(u_\Delta)^n||_{\ell^\infty}^2\left(\frac{|e_j^n|}{\Delta x}\right)^2 \\
&\leq \sum_{j\in\mathbb{Z}}\Delta x\frac{\Delta t^2}{\Delta x^2}\Delta t||D_+\left(u_\Delta\right)^n||_{\ell^\infty}^2|e_j^n|^2.
\end{aligned}
$$

- Thanks to Relation (27), one has

$$\sum_{j\in\mathbb{Z}} \frac{c^2\Delta t^2\Delta x^3}{4}\left[D_+D_-(e)_j^n\right]^2 = \sum_{j\in\mathbb{Z}} c^2\Delta t^2\Delta x\left[D_+\left(e\right)_j^n\right]^2 - \sum_{j\in\mathbb{Z}} c^2\Delta t^2\Delta x\left[D\left(e\right)_j^n\right]^2.$$

All this yields

$$\sum_{j\in\mathbb{Z}}\Delta x\left(\mathrm{RHS}_j^n\right)_a^2 \le \sum_{j\in\mathbb{Z}}\Delta x\Delta t^2\left[D_+D_+D_-\left(e\right)_j^n\right]^2\left(\theta^2+(1-2\theta)\right) + \sum_{j\in\mathbb{Z}} c^2\Delta t^2\Delta x\left[D_+\left(e\right)_j^n\right]^2$$
$$+\sum_{j\in\mathbb{Z}}\Delta x\Delta t^2\left[D\left(e\right)_j^n\right]^2\left\{\left(\frac{e_{j+1}^n+e_{j-1}^n}{2}\right)^2+\left[\left(u_\Delta\right)_j^n\right]^2-c^2\right\}$$
$$+\sum_{j\in\mathbb{Z}}\Delta x\left(e_j^n\right)^2\left\{1+\frac{\Delta t}{2}\left[\left([u_\Delta]_{j-1}^n\right)^2+\left([u_\Delta]_{j+1}^n\right)^2+\frac{3\Delta t}{4}\left(D_+\left(u_\Delta\right)_j^n\right)^2+\frac{3\Delta t}{4}\left(D_-\left(u_\Delta\right)_j^n\right)^2+2\frac{\Delta t^2}{\Delta x^2}||D_+(u_\Delta)^n||_{\ell^\infty}^2\right]\right\}.$$

Right Hand Side $\left(\mathrm{RHS}_j^n\right)_b^2$ We next focus on $\sum_{j\in\mathbb{Z}}\Delta x\left(\mathrm{RHS}_j^n\right)_b^2$ and on its different ten terms.

- By Relations (29) and (31), one sees that

$$\sum_{j\in\mathbb{Z}} -2(1-\theta)\Delta t\Delta x e_j^n D_+D_+D_-(e)_j^n = \sum_{j\in\mathbb{Z}} 2\theta\Delta t\Delta x e_j^n D_+D_+D_-\left(e\right)_j^n + \sum_{j\in\mathbb{Z}} 2\Delta t\Delta x D_-\left(e\right)_j^n D_+D_-\left(e\right)_j^n,$$
$$= \sum_{j\in\mathbb{Z}} 2\theta\Delta t\Delta x e_j^n D_+D_+D_-\left(e\right)_j^n - \sum_{j\in\mathbb{Z}}\Delta t\Delta x^2\left[D_+D_-\left(e\right)_j^n\right]^2.$$

  Equality (28) enables to write

$$\sum_{j\in\mathbb{Z}} -2(1-\theta)\Delta t\Delta x e_j^n D_+D_+D_-(e)_j^n = \sum_{j\in\mathbb{Z}} 2\theta\Delta t\Delta x e_j^n D_+D_+D_-\left(e\right)_j^n - \sum_{j\in\mathbb{Z}}\frac{\Delta t\Delta x^4}{4}\left[D_+D_+D_-\left(e\right)_j^n\right]^2$$
$$-\sum_{j\in\mathbb{Z}}\Delta t\Delta x^2\left(D_+D\left(e\right)_j^n\right)^2.$$

- Thanks to Identity (36), one has

$$\sum_{j\in\mathbb{Z}} -2\Delta x\Delta t e_j^n D\left(\frac{e^2}{2}\right)_j^n = \sum_{j\in\mathbb{Z}}\frac{\Delta x^3\Delta t}{6}\left[D_+\left(e\right)_j^n\right]^3.$$

- Identity (34) gives

$$\sum_{j\in\mathbb{Z}} -2\Delta x\Delta t e_j^n D\left(u_\Delta e\right)_j^n = \sum_{j\in\mathbb{Z}} -\Delta x\Delta t D_+\left(u_\Delta\right)_j^n e_j^n e_{j+1}^n \le \sum_{j\in\mathbb{Z}}\Delta x\Delta t\left(e_j^n\right)^2||D_+\left(u_\Delta\right)^n||_{\ell^\infty}.$$

- Moreover, Relations (20) and (29) imply

$$\sum_{j\in\mathbb{Z}} c\Delta x^2\Delta t e_j^n\left[D_+D_-(e)_j^n\right] = \sum_{j\in\mathbb{Z}} -c\Delta x^2\Delta t\left[D_+\left(e\right)_j^n\right]^2.$$

- To bound $\sum_{j\in\mathbb{Z}} 2(1-\theta)\Delta t^2\Delta x D_+D_+D_-(e)_j^n D\left(u_\Delta e\right)_j^n$, we use Lemma 6, whose proof lies in Appendix B.

Lemma 6. *Let $(a_j)_{j\in\mathbb{Z}}$, $(b_j)_{j\in\mathbb{Z}}$ be two sequences in $\ell^2_\Delta(\mathbb{Z})$ and $\sigma \in \{0,1\}$. One has*

$$\sum_{j\in\mathbb{Z}} D_+D_+D_-(a)_j\, D(ab)_j \le \sum_{j\in\mathbb{Z}} \frac{\Delta t}{4}\left\{|D_+(b)_j| + |D_-(b)_j|\right\}\left(D_+D_+D_-(a)_j\right)^2$$
$$+\sum_{j\in\mathbb{Z}}\frac{1}{2}\left\{||D_+(b)||^\sigma_{\ell^\infty} - \frac{\Delta x}{2}D_-(b)_j\right\}\left(D_+D_-(a)_j\right)^2 + \sum_{j\in\mathbb{Z}}\frac{1}{2}||D_+(b)||^{2-\sigma}_{\ell^\infty}\left(D_+(a)_j\right)^2 - \sum_{j\in\mathbb{Z}} b_j\left(D_+D(a)_j\right)^2$$
$$+\sum_{j\in\mathbb{Z}}\frac{1}{4\Delta t}\left\{|D_-(b)_j| + |D_+(b)_j|\right\}(a_j)^2. \quad (43)$$

Thanks to this lemma applied with $a_j = e^n_j$ and $b_j = (u_\Delta)^n_j$, one has

$$\sum_{j\in\mathbb{Z}} 2(1-\theta)\Delta t^2\Delta x D_+D_+D_-(e)^n_j D(u_\Delta e)^n_j \le \sum_{j\in\mathbb{Z}}\frac{\Delta t^3\Delta x}{2}(1-\theta)\left\{|D_+(u_\Delta)^n_j| + |D_-(u_\Delta)^n_j|\right\}\left(D_+D_+D_-(e)^n_j\right)^2$$
$$+\sum_{j\in\mathbb{Z}}\frac{\Delta t\Delta x}{2}(1-\theta)\left\{|D_-(u_\Delta)^n_j| + |D_+(u_\Delta)^n_j|\right\}(e^n_j)^2$$
$$+\sum_{j\in\mathbb{Z}}(1-\theta)\Delta t^2\Delta x\left\{||D_+(u_\Delta)^n||^\sigma_{\ell^\infty} - \frac{\Delta x}{2}D_-(u_\Delta)^n_j\right\}\left(D_+D_-(e)^n_j\right)^2$$
$$+\sum_{j\in\mathbb{Z}}(1-\theta)\Delta x\Delta t^2||D_+(u_\Delta)^n||^{2-\sigma}_{\ell^\infty}\left(D_+(e)^n_j\right)^2 - \sum_{j\in\mathbb{Z}}2(1-\theta)\Delta x\Delta t^2(u_\Delta)^n_j\left(D_+D(e)^n_j\right)^2,$$

for $\sigma \in \{0,1\}$.

- To bound $\sum_{j\in\mathbb{Z}} 2(1-\theta)\Delta t^2\Delta x D_+D_+D_-(e)^n_j D\left(\frac{e^2}{2}\right)^n_j$, we use Lemma 7 (proved in Appendix B).

Lemma 7. *Let $(a_j)_{j\in\mathbb{Z}}$ be a sequence in $\ell^2_\Delta(\mathbb{Z})$ and $\gamma \in [0, \frac{1}{2})$, one has*

$$\sum_{j\in\mathbb{Z}} D_+D_+D_-(a)_j D\left(\frac{a^2}{2}\right)_j \le \sum_{j\in\mathbb{Z}}\frac{\Delta x^{\frac{1}{2}-\gamma} + ||a||_{\ell^\infty} + 9||a||^2_{\ell^\infty}\Delta x^{\gamma-\frac{1}{2}}}{2}\left(D_+D_-(a)_j\right)^2$$
$$+\sum_{j\in\mathbb{Z}}||a||_{\ell^\infty}\left(D_+D(a)_j\right)^2.$$

Applying this inequality to $a_j = e^n_j$, one gets

$$\sum_{j\in\mathbb{Z}} 2(1-\theta)\Delta t^2\Delta x D_+D_+D_-(e)^n_j D\left(\frac{e^2}{2}\right)^n_j$$
$$\le \sum_{j\in\mathbb{Z}}\Delta t^2\Delta x(1-\theta)\left(\Delta x^{\frac{1}{2}-\gamma} + ||e^n||_{\ell^\infty} + 9||e^n||^2_{\ell^\infty}\Delta x^{\gamma-\frac{1}{2}}\right)\left(D_+D_-(e)^n_j\right)^2$$
$$+\sum_{j\in\mathbb{Z}}2(1-\theta)\Delta t^2\Delta x||e^n||_{\ell^\infty}\left(D_+D(e)^n_j\right)^2.$$

Once again, Relation (28) transforms $\sum_{j\in\mathbb{Z}}\Delta t^2\Delta x(1-\theta)\left(\Delta x^{\frac{1}{2}-\gamma} + ||e^n||_{\ell^\infty} + 9||e^n||^2_{\ell^\infty}\Delta x^{\gamma-\frac{1}{2}}\right)\left(D_+D_-(e)^n_j\right)^2$ to obtain

$$\sum_{j\in\mathbb{Z}} 2(1-\theta)\Delta t^2\Delta x D_+D_+D_-(e)^n_j D\left(\frac{e^2}{2}\right)^n_j$$

$$\leq \sum_{j\in\mathbb{Z}} \Delta t^2 \Delta x(1-\theta)\left[\Delta x^{\frac{1}{2}-\gamma} + ||e^n||_{\ell^\infty} + 9||e^n||^2_{\ell^\infty}\Delta x^{\gamma-\frac{1}{2}}\right]\left(D_+ D\,(e)^n_j\right)^2$$
$$+\sum_{j\in\mathbb{Z}}(1-\theta)\frac{\Delta t^2\Delta x^3}{4}\left[\Delta x^{\frac{1}{2}-\gamma} + ||e^n||_{\ell^\infty} + 9||e^n||^2_{\ell^\infty}\Delta x^{\gamma-\frac{1}{2}}\right]\left(D_+D_+D_-(e)^n_j\right)^2$$
$$+\sum_{j\in\mathbb{Z}} 2(1-\theta)\Delta t^2\Delta x||e^n||_{\ell^\infty}\left(D_+D(e)^n_j\right)^2.$$

Remark 8. *Thereafter, a_j will be replaced by the unknown e^n_j whereas b_j will be replaced by the exact solution $[u_\Delta]^n_j$. We could not use Lemma 6 with $b_j = \frac{a_j}{2}$ instead of Lemma 7 because $D_+(b)_j$ in Lemma 6 will be replaced by $D_+(\frac{a}{2})_j = D_+(\frac{e}{2})^n_j$ which is always unknown.*

- Relation (31) gives

$$\sum_{j\in\mathbb{Z}} -c\Delta x^2\Delta t^2(1-\theta)D_+D_+D_-(e)^n_j D_+D_-(e)^n_j = \sum_{j\in\mathbb{Z}}(1-\theta)c\frac{\Delta x^3\Delta t^2}{2}\left[D_+D_+D_-\,(e)^n_j\right]^2.$$

- To deal with $\sum_{j\in\mathbb{Z}} 2\Delta x\Delta t^2 D\left(\frac{e^2}{2}\right)^n_j D\,(u_\Delta e)^n_j$, we use Lemma 8 (see Appendix B for its proof)

Lemma 8. *Let $(a_j)_{j\in\mathbb{Z}}$ and $(b_j)_{j\in\mathbb{Z}}$ be two sequences in $\ell^2_\Delta(\mathbb{Z})$, then one has*

$$\sum_{j\in\mathbb{Z}} D\,(ab)_j\, D\left(\frac{a^2}{2}\right)_j = \sum_{j\in\mathbb{Z}}\left[D\,(a)_j\right]^2\frac{a_{j+1}b_{j+1}+a_{j-1}b_{j-1}}{2}$$
$$-\sum_{j\in\mathbb{Z}} D\,(b)_j\left[D\,(a)_j\right]^3\frac{4\Delta x^2}{3} - \sum_{j\in\mathbb{Z}}\frac{1}{3}DD\,(b)_j\, a_j^3. \quad (44)$$

Identity (44) with $a_j = e^n_j$ and $b_j = (u_\Delta)^n_j$ gives

$$\sum_{j\in\mathbb{Z}} 2\Delta x\Delta t^2 D\left(\frac{e^2}{2}\right)^n_j D\,(u_\Delta e)^n_j = \sum_{j\in\mathbb{Z}}\Delta x\Delta t^2\left[D\,(e)^n_j\right]^2\left[(u_\Delta)^n_{j+1}e^n_{j+1} + (u_\Delta)^n_{j-1}e^n_{j-1}\right]$$
$$-\sum_{j\in\mathbb{Z}}\frac{8\Delta x^3\Delta t^2}{3}D\,(u_\Delta)^n_j\left[D\,(e)^n_j\right]^3 - \sum_{j\in\mathbb{Z}}\frac{2\Delta x\Delta t^2}{3}DD\,(u_\Delta)^n_j\left(e^n_j\right)^3.$$

- Relation (37) yields

$$\sum_{j\in\mathbb{Z}} -c\Delta x^2\Delta t^2 D\left(\frac{e^2}{2}\right)^n_j D_+D_-(e)^n_j = \sum_{j\in\mathbb{Z}} -\frac{c\Delta x^2\Delta t^2}{6}\left[D_+\,(e)^n_j\right]^3 + \sum_{j\in\mathbb{Z}}\frac{2c\Delta x^2\Delta t^2}{3}\left[D\,(e)^n_j\right]^3.$$

- Relation (35) implies

$$\sum_{j\in\mathbb{Z}} -c\Delta x^2\Delta t^2 D\,(u_\Delta e)^n_j\, D_+D_-(e)^n_j = \sum_{j\in\mathbb{Z}} c\Delta t^2 D_+\,(u_\Delta)^n_j\, e^n_j e^n_{j+1} - \sum_{j\in\mathbb{Z}} c\Delta t^2 D\,(u_\Delta)^n_j\, e^n_{j-1}e^n_{j+1}.$$

Thus, thanks to the Cauchy-Schwarz inequality,

$$\sum_{j\in\mathbb{Z}} -c\Delta x^2\Delta t^2 D\,(u_\Delta e)^n_j\, D_+D_-(e)^n_j \leq \sum_{j\in\mathbb{Z}} c\Delta t^2||D_+\,(u_\Delta)^n||_{\ell^\infty}\left(e^n_j\right)^2 + \sum_{j\in\mathbb{Z}} c\Delta t^2||D\,(u_\Delta)^n||_{\ell^\infty}\left(e^n_j\right)^2.$$

Gathering all these relations yields the following inequality, for $\sigma \in \{0,1\}$.

$$\sum_{j\in\mathbb{Z}} \Delta x \left(\mathrm{RHS}_j^n\right)_b^2 \leq \sum_{j\in\mathbb{Z}} 2\theta\Delta t\Delta x e_j^n D_+D_+D_-\left(e\right)_j^n + \sum_{j\in\mathbb{Z}}(1-\theta)\Delta t^2\Delta x \left[||D_+(u_\Delta)^n||_{\ell^\infty}^\sigma + \frac{\Delta x}{2}||D_-\left(u_\Delta\right)^n||_{\ell^\infty}\right]\left(D_+D_-e_j^n\right)^2$$
$$+\sum_{j\in\mathbb{Z}} \Delta x\Delta t\left\{||D_+u_\Delta^n||_{\ell^\infty} - \frac{2\Delta t}{3}DD\left(u_\Delta\right)_j^n e_j^n + \frac{c\Delta t}{\Delta x}||D_+u_\Delta^n||_{\ell^\infty} + \frac{c\Delta t}{\Delta x}||Du_\Delta^n||_{\ell^\infty} + \frac{(1-\theta)}{2}\left[|D_+u_\Delta^n| + |D_-u_\Delta^n|\right]\right\}\left(e_j^n\right)^2$$
$$+\sum_{j\in\mathbb{Z}} \Delta x\Delta t\left\{\frac{\Delta x^2}{6}D_+\left(e\right)_j^n - c\Delta x - \frac{c\Delta t\Delta x}{6}D_+\left(e\right)_j^n + (1-\theta)\Delta t||D_+\left(u_\Delta\right)^n||_{\ell^\infty}^{2-\sigma}\right\}\left[D_+\left(e\right)_j^n\right]^2$$
$$+\sum_{j\in\mathbb{Z}} \Delta x\Delta t^2\left[D\left(e\right)_j^n\right]^2\left\{\left(u_\Delta\right)_{j+1}^n e_{j+1}^n + \left(u_\Delta\right)_{j-1}^n e_{j-1}^n - \frac{8\Delta x^2}{3}D\left(u_\Delta\right)_j^n D\left(e\right)_j^n + \frac{2c\Delta x}{3}D\left(e\right)_j^n\right\}$$
$$+\sum_{j\in\mathbb{Z}} \Delta t\Delta x\left\{-\Delta x - 2(1-\theta)\Delta t\left(u_\Delta\right)_j^n + 2(1-\theta)\Delta t||e^n||_{\ell^\infty} + \Delta t(1-\theta)\left[\Delta x^{\frac{1}{2}-\gamma} + ||e^n||_{\ell^\infty} + 9||e^n||_{\ell^\infty}^2\Delta x^{\gamma-\frac{1}{2}}\right]\right\}\left(D_+De_j^n\right)^2$$
$$+\sum_{j\in\mathbb{Z}} \Delta t\Delta x\left\{-\frac{\Delta x^3}{4} + c\frac{(1-\theta)\Delta x^2\Delta t}{2} + \frac{\Delta t^2(1-\theta)}{2}\left[|D_+\left(u_\Delta\right)_j^n| + |D_-\left(u_\Delta\right)_j^n|\right]\right.$$
$$\left.+(1-\theta)\frac{\Delta t\Delta x^2}{4}\left(\Delta x^{\frac{1}{2}-\gamma} + ||e^n||_{\ell^\infty} + 9||e^n||_{\ell^\infty}^2\Delta x^{\gamma-\frac{1}{2}}\right)\right\}\left[D_+D_+D_-\left(e\right)_j^n\right]^2.$$

Right Hand Side $\left(\mathrm{RHS}_j^n\right)_c$ Let us now focus on $\sum_{j\in\mathbb{Z}}\Delta x\left(\mathrm{RHS}_j^n\right)_c^2$ and its four different terms.

- From Young's inequality,

$$-\sum_{j\in\mathbb{Z}} 2\Delta x\Delta t\left(e_j^n - (1-\theta)\Delta t D_+D_+D_-\left(e\right)_j^n\right)\epsilon_j^n \leq \sum_{j\in\mathbb{Z}}\Delta x\Delta t\left(e_j^n - (1-\theta)\Delta t D_+D_+D_-\left(e\right)_j^n\right)^2$$
$$+\sum_{j\in\mathbb{Z}}\Delta x\Delta t\left(\epsilon_j^n\right)^2.$$

- Once again, we apply Young's inequality to obtain

$$\sum_{j\in\mathbb{Z}} 2\Delta x\Delta t^2 D\left(\frac{e^2}{2}\right)_j^n\epsilon_j^n \leq \sum_{j\in\mathbb{Z}}\Delta t^2\left[\epsilon_j^n\right]^2 + \sum_{j\in\mathbb{Z}}\Delta t^2\Delta x^2\left[D\left(\frac{e^2}{2}\right)_j^n\right]^2.$$

  Then, Identity (26) gives

$$\sum_{j\in\mathbb{Z}} 2\Delta x\Delta t^2 D\left(\frac{e^2}{2}\right)_j^n\epsilon_j^n \leq \sum_{j\in\mathbb{Z}}\Delta t^2\left[\epsilon_j^n\right]^2 + \sum_{j\in\mathbb{Z}}\Delta t^2\Delta x^2\left[D\left(e\right)_j^n\right]^2\left(\frac{e_{j+1}^n + e_{j-1}^n}{2}\right)^2,$$

- One also has

$$\sum_{j\in\mathbb{Z}} 2\Delta x\Delta t^2 D\left(u_\Delta e\right)_j^n\epsilon_j^n \leq \sum_{j\in\mathbb{Z}}\Delta t^2\left(e_j^n\right)^2||(u_\Delta)^n||_{\ell^\infty}^2 + \sum_{j\in\mathbb{Z}}\Delta t^2\left(\epsilon_j^n\right)^2.$$

- Finally, we see that, thanks to Young's inequality,

$$\sum_{j\in\mathbb{Z}} -c\Delta x^2\Delta t^2 D_+D_-\left(e\right)_j^n\epsilon_j^n \leq \sum_{j\in\mathbb{Z}} 2c^2\Delta t^2\left(e_j^n\right)^2 + \sum_{j\in\mathbb{Z}} 2\Delta t^2\left(\epsilon_j^n\right)^2.$$

Thus, we have

$$\sum_{j\in\mathbb{Z}}\Delta x\left(\mathrm{RHS}_j^n\right)_c^2 \leq \Delta t||e^n||_{\ell_\Delta^2}^2\left\{\frac{\Delta t}{\Delta x}\left[||(u_\Delta)^n||_{\ell^\infty}^2 + 2c^2\right]\right\} + \Delta t||\epsilon^n||_{\ell_\Delta^2}^2\left\{1 + 4\frac{\Delta t}{\Delta x} + \Delta t\right\}$$
$$+\sum_{j\in\mathbb{Z}}\Delta x\Delta t\left(e_j^n - (1-\theta)\Delta t D_+D_+D_-\left(e\right)_j^n\right)^2 + \sum_{j\in\mathbb{Z}}\Delta t^2\Delta x^2\left[D\left(e\right)_j^n\right]^2\left(\frac{e_{j+1}^n + e_{j-1}^n}{2}\right)^2.$$

Final inequality To conclude, the convergence error satisfies the following inequality

$$\sum_{j\in\mathbb{Z}} \Delta x \left(e_j^{n+1} + \theta\Delta t D_+D_+D_- (e)_j^{n+1}\right)^2 \le \sum_{j\in\mathbb{Z}} \Delta x \left(e_j^n + \theta\Delta t D_+D_+D_- (e)_j^n\right)^2 + \Delta t ||e^n||_{\ell^2_\Delta}^2 A_a$$
$$+ \sum_{j\in\mathbb{Z}} \Delta t\Delta x \left(e_j^n - (1-\theta)\Delta t D_+D_+D_- (e)_j^n\right)^2 + \sum_{j\in\mathbb{Z}} \Delta x\Delta t A_b \left[D_+ (e)_j^n\right]^2 + \sum_{j\in\mathbb{Z}} \Delta x\Delta t^2 A_c \left[D (e)_j^n\right]^2$$
$$+ \sum_{j\in\mathbb{Z}} \Delta x\Delta t A_d \left(D_+D_- (e)_j^n\right)^2 + \sum_{j\in\mathbb{Z}} \Delta x\Delta t A_e \left(D_+D (e)_j^n\right)^2 + \sum_{j\in\mathbb{Z}} \Delta t\Delta x A_f \left[D_+D_+D_- (e)_j^n\right]^2$$
$$+ \Delta t||\epsilon^n||_{\ell^2_\Delta}^2 \left\{1 + 4\frac{\Delta t}{\Delta x} + \Delta t\right\},$$

with

$$A_a = \frac{\left([u_\Delta]_{j-1}^n\right)^2}{2} + \frac{\left([u_\Delta]_{j+1}^n\right)^2}{2} + \frac{\Delta t}{2}\left[\frac{3}{4}\left(D_- (u_\Delta)_j^n\right)^2 + \frac{3}{4}\left(D_+ (u_\Delta)_j^n\right)^2\right] + \frac{\Delta t^2}{\Delta x^2}||D_+(u_\Delta)^n||_{\ell^\infty}^2$$
$$+ \frac{(1-\theta)}{2}\left[|D_- (u_\Delta)_j^n| + |D_+ (u_\Delta)_j^n|\right] + ||D_+ (u_\Delta)^n||_{\ell^\infty}\left(1 + \frac{c\Delta t}{\Delta x}\right)$$
$$+ \frac{c\Delta t}{\Delta x}||D (u_\Delta)^n||_{\ell^\infty} - \frac{2\Delta t}{3} DD (u_\Delta)_j^n e_j^n + \frac{\Delta t}{\Delta x}\left(||(u_\Delta)^n||_{\ell^\infty}^2 + 2c^2\right),$$

$$A_b = c^2\Delta t + \frac{\Delta x^2}{6} D_+ (e)_j^n - c\Delta x - \frac{c\Delta x\Delta t}{6} D_+ (e)_j^n + (1-\theta)\Delta t||D_+ (u_\Delta)^n||_{\ell^\infty}^{2-\sigma},$$

$$A_c = \left(\frac{e_{j+1}^n + e_{j-1}^n}{2}\right)^2 [1+\Delta x] + \left([u_\Delta]_j^n\right)^2 - c^2 + (u_\Delta)_{j+1}^n e_{j+1}^n + (u_\Delta)_{j-1}^n e_{j-1}^n - \frac{8\Delta x^2}{3} D (u_\Delta)_j^n D (e)_j^n + \frac{2c\Delta x}{3} D (e)_j^n,$$

$$A_d = (1-\theta)\Delta t\left[||D_+(u_\Delta)^n||_{\ell^\infty}^\sigma + \frac{\Delta x}{2}||D_- (u_\Delta)^n||_{\ell^\infty}\right],$$

$$A_e = -2(1-\theta)\Delta t (u_\Delta)_j^n + 2(1-\theta)\Delta t||e^n||_{\ell^\infty} - \Delta x + \Delta t(1-\theta)\left[\Delta x^{\frac{1}{2}-\gamma} + ||e^n||_{\ell^\infty} + 9||e^n||_{\ell^\infty}^2 \Delta x^{\gamma-\frac{1}{2}}\right],$$

and

$$A_f = \Delta t\left[(1-2\theta) + \frac{c(1-\theta)\Delta x^2}{2} + \frac{\Delta t(1-\theta)}{2}\left[|D_+ (u_\Delta)_j^n| + |D_- (u_\Delta)_j^n|\right]\right.$$
$$\left. + (1-\theta)\frac{\Delta x^2}{4}\left(\Delta x^{\frac{1}{2}-\gamma} + ||e^n||_{\ell^\infty} + 9||e^n||_{\ell^\infty}^2 \Delta x^{\gamma-\frac{1}{2}}\right)\right] - \frac{\Delta x^3}{4}.$$

• Since $||DD (u_\Delta)^n||_{\ell^\infty} \le \frac{1}{\Delta x}||D (u_\Delta)^n||_{\ell^\infty}$, $||D (u_\Delta)^n||_{\ell^\infty} \le ||D_+ (u_\Delta)^n||_{\ell^\infty}$ and $\Delta t||D_+ (u_\Delta)^n||_{\ell^\infty} \le \frac{2\Delta t}{\Delta x}||u_\Delta^n||_{\ell^\infty}$, then

$$A_a \le B_a,$$

where B_a is defined by (40a).
• For A_b, we recognize the definition (40b) of B_b.
• For the term A_c, we have

$$A_c \le ||e^n||_{\ell^\infty}^2 [1+\Delta x] + ||(u_\Delta)^n||_{\ell^\infty}^2 - c^2 + \frac{1}{3} e_{j+1}^n (u_\Delta)_{j+1}^n + \frac{1}{3} e_{j-1}^n (u_\Delta)_{j-1}^n + \frac{2}{3}(u_\Delta)_{j+1}^n e_{j-1}^n + \frac{2}{3}(u_\Delta)_{j-1}^n e_{j+1}^n$$
$$+ \frac{2c}{3}||e^n||_{\ell^\infty}.$$

Thus, one has $A_c \le B_c$ (40c).
• Furthermore, from (40d) and (40e)

$$A_d = B_d$$

and

$$A_e = B_e.$$

- At last, we see that $A_f \leq B_f$ defined by (40f).

$\square$

5 Rate of convergence

In the left hand side of the ℓ^2_Δ-stability inequality (39), e_j^{n+1} appears in the operator $I + \Delta t\theta D_+D_+D_-$. The study of this dispersive operator is the aim of Subsection 5.1.

In the right hand side of (39), $D_+(e)_j^n$, $D_+D_-(e)_j^n$ appear in factor of some terms B_i. Since we have no control on these derivatives of the convergence error, we reorganize terms B_i in Subsection 5.2 to obtain non-positive terms : the C_i and E_i terms of Corollaries 3 and 4. We notice that the reorganization is not exactly the same for $\theta \geq \frac{1}{2}$ and $\theta < \frac{1}{2}$, therefore, we state two corollaries.

In Subsection 5.3, the correct CFL hypothesis enables to cancel extra terms C_i and E_i and an induction method concludes the convergence proof.

5.1 Properties of the operator $I + \theta\Delta t D_+D_+D_-$

For more simplicity, we denote by $\mathcal{A}$ the operator

$$\mathcal{A} = I + \theta\Delta t D_+D_+D_-, \tag{45}$$

where I is the identity operator in $\ell^2_\Delta(\mathbb{Z})$.

Proposition 3. *For every $\Delta t > 0$ and $\Delta x > 0$, $\mathcal{A}$ is*

- *continuous (with a norm depending on $\frac{\Delta t}{\Delta x^3}$) from ℓ^2_Δ to ℓ^2_Δ,*
- *invertible.*

Moreover, one has the following inequalities, for any sequence $(a_j)_{j\in\mathbb{Z}} \in \ell^2_\Delta(\mathbb{Z})$

$$||a||^2_{\ell^2_\Delta} \leq ||\mathcal{A}a||^2_{\ell^2_\Delta} \leq \left\{1 + \frac{16\theta\Delta t}{\Delta x^3}\left[1 + \frac{4\theta\Delta t}{\Delta x^3}\right]\right\}||a||^2_{\ell^2_\Delta}. \tag{46}$$

Remark 9. *Inequality (46) implies that the inverse of $\mathcal{A}$ is continuous from ℓ^2_Δ to ℓ^2_Δ with a norm independent of $\frac{\Delta t}{\Delta x^3}$.*

Proof. Given $a \in \ell^2_\Delta(\mathbb{Z})$, we may define the function $\widehat{a} \in L^2(0,1)$ by

$$\widehat{a}(\xi) = \sum_{k\in\mathbb{Z}} a_k e^{2i\pi k\xi}, \quad \xi \in (0,1),$$

(the sequence a is seen as the Fourier-series of the function $\widehat{a}$). Parseval identity yields

$$\sum_{j\in\mathbb{Z}} \Delta x|a_j|^2 = \Delta x\int_0^1 |\widehat{a}(\xi)|^2 d\xi. \tag{47}$$

We define furthermore the shift operator $\mathcal{S}^\ell$ with $\ell \in \mathbb{Z}$ by

$$\mathcal{S}^\ell a = (a_{j+\ell})_{j\in\mathbb{Z}},$$

the associated function verifies

$$\widehat{\mathcal{S}^\ell a}(\xi) = e^{-2i\pi\ell\xi}\widehat{a}(\xi), \quad \xi \in (0,1).$$

The function associated to $\mathcal{A}a$ is

$$\begin{aligned}\widehat{\mathcal{A}a}\,(\xi) &= \widehat{a} + \theta\frac{\Delta t}{\Delta x^3}\widehat{a}\left(e^{-4i\pi\xi} - 3e^{-2i\pi\xi} + 3 - e^{2i\pi\xi}\right), \quad \xi \in (0,1),\\ &= \widehat{a}\left\{1 + \theta\frac{\Delta t}{\Delta x^3}\left[-2ie^{-i\pi\xi}\sin(3\pi\xi) + 6ie^{-i\pi\xi}\sin(\pi\xi)\right]\right\}, \quad \xi \in (0,1).\end{aligned}$$

As $\sin(3\pi\xi) = 3\sin(\pi\xi) - 4\sin^3(\pi\xi)$, we obtain

$$\widehat{\mathcal{A}a}\,(\xi) = \widehat{a}\left\{1 + 8i\theta\frac{\Delta t}{\Delta x^3}e^{-i\pi\xi}\sin^3(\pi\xi)\right\}.$$

The operator $\mathcal{A}$ is thus inversible and its inverse is defined by $\widehat{\mathcal{A}^{-1}a}(\xi) = \frac{1}{1+8i\theta\frac{\Delta t}{\Delta x^3}e^{-i\pi\xi}\sin^3(\pi\xi)}\widehat{a}(\xi)$. Moreover, this operator and its inverse are continuous since

$$||\mathcal{A}a||^2_{\ell^2_\Delta} = \Delta x\int_0^1\left|1 + 8i\theta\frac{\Delta t}{\Delta x^3}e^{-i\pi\xi}\sin^3(\pi\xi)\right|^2|\widehat{a}(\xi)|^2 d\xi,$$

and the module $\left|1 + 8i\theta\frac{\Delta t}{\Delta x^3}e^{-i\pi\xi}\sin^3(\pi\xi)\right|^2$ satisfies

$$\begin{aligned}\left|1 + 8i\theta\frac{\Delta t}{\Delta x^3}e^{-i\pi\xi}\sin^3(\pi\xi)\right|^2 &= \left(1 + 8\theta\frac{\Delta t}{\Delta x^3}\sin^4(\pi\xi)\right)^2 + \left(8\theta\frac{\Delta t}{\Delta x^3}\cos(\pi\xi)\sin^3(\pi\xi)\right)^2\\ &= 1 + 16\theta\frac{\Delta t}{\Delta x^3}\sin^4(\pi\xi)\left(1 + 4\theta\frac{\Delta t}{\Delta x^3}\sin^2(\pi\xi)\right)\\ &\in [1, 1 + 16\theta\frac{\Delta t}{\Delta x^3}\left(1 + 4\theta\frac{\Delta t}{\Delta x^3}\right)].\end{aligned}$$

Thus, the operator $\mathcal{A}$ verifies

$$\Delta x\int_0^1|\widehat{a}(\xi)|^2 d\xi \le ||\mathcal{A}a||^2_{\ell^2_\Delta} \le \left\{1 + 16\theta\frac{\Delta t}{\Delta x^3}\left(1 + 4\theta\frac{\Delta t}{\Delta x^3}\right)\right\}\Delta x\int_0^1|\widehat{a}(\xi)|^2 d\xi.$$

We conclude by using Identity (47). $\square$

Remark 10. *The norm of the inverse operator $\mathcal{A}^{-1}$ is upper bounded by 1 (independent of $\frac{\Delta t}{\Delta x^3}$). This independence is crucial to be able to impose a hyperbolic Courant-Friedrichs-Lewy condition ($[c + \frac{1}{2}]\frac{\Delta t}{\Delta x} < 1$) for $\theta \ge \frac{1}{2}$, to establish Equation (66) for example.*

The operator $\mathcal{A}$ enables us to control not only the ℓ^2_Δ-norm (as proved in Proposition 3) but also an h^2_Δ-discrete norm and h^3_Δ-discrete norm as in the following proposition.

Proposition 4. *Let $\mathcal{A}$ be the operator defined by (45), then for any sequence $(a_j)_{j\in\mathbb{Z}}$, one has*

$$||\mathcal{A}a||^2_{\ell^2_\Delta} = ||a||^2_{\ell^2_\Delta} + \theta\Delta t\Delta x||D_+D_-(a)||^2_{\ell^2_\Delta} + \theta^2\Delta t^2||D_+D_+D_-(a)||^2_{\ell^2_\Delta}.$$

Proof. We develop the square of the ℓ^2_Δ-norm of $(\mathcal{A}a_j)_{j\in\mathbb{Z}}$:

$$\sum_{j\in\mathbb{Z}}\Delta x\left(a_j + \theta\Delta t D_+D_+D_-(a)_j\right)^2 = \sum_{j\in\mathbb{Z}}\Delta x\,(a_j)^2 + \sum_{j\in\mathbb{Z}}2\theta\Delta x\Delta t a_j D_+D_+D_-(a)_j + \sum_{j\in\mathbb{Z}}\theta^2\Delta x\Delta t^2\left(D_+D_+D_-(a)_j\right)^2.$$

Let us focus on the cross term. Discrete integration by parts (29) together with (31) (with $D_-(a)_j$ instead of a_j) give

$$\sum_{j\in\mathbb{Z}}2\theta\Delta t\Delta x a_j D_+D_+D_-(a)_j = -\sum_{j\in\mathbb{Z}}2\theta\Delta t\Delta x D_-(a)_j D_+D_-(a)_j = \sum_{j\in\mathbb{Z}}\theta\Delta t\Delta x^2\left(D_+D_-(a)_j\right)^2,$$

which concludes the proof. $\square$

The following proposition enables to deal with the term $e_j^n - (1-\theta)\Delta t D_+D_+D_-\left(e\right)_j^n$ in Equation (39).

Proposition 5. *For $\theta \in [0,1]$, assume the CFL condition $\Delta t(1-2\theta) \leq \frac{\Delta x^3}{4}$ is satisfied. Then, for any sequence $(a_j)_{j\in\mathbb{Z}}$, it holds*

$$\sum_{j\in\mathbb{Z}} \Delta x \left(a_j - (1-\theta)\Delta t D_+D_+D_-\left(a\right)_j\right)^2 \leq ||\mathcal{A}a||^2_{\ell^2_\Delta}. \tag{48}$$

Proof. We develop the expression:

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \Delta x \left(a_j - (1-\theta)\Delta t D_+D_+D_-\left(a\right)_j\right)^2 = &\sum_{j\in\mathbb{Z}} \Delta x \left(a_j + \theta\Delta t D_+D_+D_-\left(a\right)_j\right)^2 - \sum_{j\in\mathbb{Z}} 2\Delta x\Delta t a_j D_+D_+D_-\left(a\right)_j \\ &+ \sum_{j\in\mathbb{Z}} \Delta x\Delta t^2(1-2\theta)\left(D_+D_+D_-\left(a\right)_j\right)^2.\end{aligned}$$

By applying Relations (29) and (31) (with $D_-\left(a\right)_j$ instead of a_j), the previous equation becomes

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \Delta x \left(a_j - (1-\theta)\Delta t D_+D_+D_-\left(a\right)_j\right)^2 = &\sum_{j\in\mathbb{Z}} \Delta x \left(a_j + \theta\Delta t D_+D_+D_-\left(a\right)_j\right)^2 - \sum_{j\in\mathbb{Z}} \Delta x^2\Delta t \left(D_+D_-\left(a\right)_j\right)^2 \\ &+ \sum_{j\in\mathbb{Z}} \Delta x\Delta t^2(1-2\theta)\left(D_+D_+D_-\left(a\right)_j\right)^2.\end{aligned}$$

If $\theta \geq \frac{1}{2}$, Proposition 5 is proved.
If $\theta < \frac{1}{2}$, thanks to Identity (28), we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \Delta x \left(a_j - (1-\theta)\Delta t D_+D_+D_-\left(a\right)_j\right)^2 = &\sum_{j\in\mathbb{Z}} \Delta x \left(a_j + \theta\Delta t D_+D_+D_-\left(a\right)_j\right)^2 - \sum_{j\in\mathbb{Z}} \Delta x^2\Delta t \left(D_+D_-\left(a\right)_j\right)^2 \\ &+ \sum_{j\in\mathbb{Z}} \frac{4\Delta t^2(1-2\theta)}{\Delta x}\left(D_+D_-\left(a\right)_j\right)^2 - \sum_{j\in\mathbb{Z}} \frac{4\Delta t^2(1-2\theta)}{\Delta x}\left(D_+D\left(a\right)_j\right)^2.\end{aligned}$$

Since $\Delta t(1-2\theta) \leq \frac{\Delta x^3}{4}$, the term $\frac{4\Delta t^2(1-2\theta)}{\Delta x}$ is upper bounded by $\Delta t\Delta x^2$, which transforms the previous equation into

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \Delta x \left(a_j - (1-\theta)\Delta t D_+D_+D_-\left(a\right)_j\right)^2 \leq &\sum_{j\in\mathbb{Z}} \Delta x \left(a_j + \theta\Delta t D_+D_+D_-\left(a\right)_j\right)^2 - \sum_{j\in\mathbb{Z}} \Delta x^2\Delta t \left(D_+D_-\left(a\right)_j\right)^2 \\ &+ \sum_{j\in\mathbb{Z}} \Delta t\Delta x^2\left(D_+D_-\left(a\right)_j\right)^2 - \sum_{j\in\mathbb{Z}} \frac{4\Delta t^2(1-2\theta)}{\Delta x}\left(D_+D\left(a\right)_j\right)^2.\end{aligned}$$

The conclusion of the proposition is a straightforward consequence, since $1-2\theta > 0$.

$\square$

5.2 Simplification of Inequality (39)

The previous study of the dispersive operator $\mathcal{A}$ enables us to reorganize terms in ℓ^2_Δ-stability inequality (39) in a way simpler to study : signs of new terms are easier to identify. The reorganization is not exactly the same for $\theta \geq \frac{1}{2}$ and $\theta < \frac{1}{2}$, as seen in the two following corollaries of Proposition 2.

Corollary 3 (Corollary of Proposition 2)**.** *Consider Scheme (2)-(3). Let $(e_j^n)_{(j,n)}$ be the convergence error defined by (10). Then, for every $n \in [\![0,N]\!]$, $\gamma \in [0,\frac{1}{2})$ and $\theta \geq \frac{1}{2}$, one has*

$$\begin{aligned}||\mathcal{A}e^{n+1}||^2_{\ell^2_\Delta} \leq &||\mathcal{A}e^n||^2_{\ell^2_\Delta}\left[1+\Delta t F_a\right] + \Delta t C_b\left|\left|D_+\left(e\right)^n\right|\right|^2_{\ell^2_\Delta} + \Delta t^2 C_c\left|\left|D\left(e\right)^n\right|\right|^2_{\ell^2_\Delta} + \Delta t C_e\left|\left|D_+D\left(e\right)^n\right|\right|^2_{\ell^2_\Delta} \\ &+ \Delta t C_f\left|\left|D_+D_+D_-\left(e\right)^n\right|\right|^2_{\ell^2_\Delta} + \Delta t||\epsilon^n||^2_{\ell^2_\Delta}\left\{1+4\frac{\Delta t}{\Delta x}+\Delta t\right\}.\end{aligned} \tag{49}$$

with

$$F_a = ||u_\Delta^n||_{\ell^\infty}^2 \left(1+\frac{\Delta t}{\Delta x}\right) + ||D_+\left(u_\Delta\right)^n||_{\ell^\infty}\left(7+\frac{\Delta t}{\Delta x}\left[2c+\frac{2}{3}||e^n||_{\ell^\infty}+\frac{3}{2}||(u_\Delta)^n||_{\ell^\infty}\right]\right)$$
$$+ ||D_+(u_\Delta)^n||_{\ell^\infty}^2\left[\sqrt{2}\frac{\sqrt{\Delta t}}{\sqrt{\Delta x}}+\frac{\Delta t^2}{\Delta x^2}\right] + 1 + 2c^2\frac{\Delta t}{\Delta x}, \tag{50a}$$

$$C_b = \left(\frac{\Delta x}{6}D_+\left(e\right)_j^n - c\right)\left(\Delta x - c\Delta t\right), \tag{50b}$$

$$C_c = ||(u_\Delta)^n||_{\ell^\infty}^2 + \left\{||e^n||_{\ell^\infty}^2\left[1+\Delta x\right] + 2||e^n||_{\ell^\infty}||(u_\Delta)^n||_{\ell^\infty} + \frac{2c}{3}||e^n||_{\ell^\infty}\right\} - c^2, \tag{50c}$$

$$C_e = 2(1-\theta)\Delta t\left\{||\left(u_\Delta\right)^n||_{\ell^\infty} + ||e^n||_{\ell^\infty} + \frac{1}{2} + \left[\frac{\Delta x^{\frac{1}{2}-\gamma} + ||e^n||_{\ell^\infty} + 9||e^n||_{\ell^\infty}^2\Delta x^{\gamma-\frac{1}{2}}}{2}\right]\right\} - \Delta x, \tag{50d}$$

$$C_f = \Delta t\left\{(1-2\theta) + \frac{(1-\theta)\Delta x^2}{2}\left[c+\frac{1}{2}+\frac{\Delta x^{\frac{1}{2}-\gamma} + ||e^n||_{\ell^\infty} + 9||e^n||_{\ell^\infty}^2\Delta x^{\gamma-\frac{1}{2}}}{2}\right]\right\} - \frac{\Delta x^3}{4}. \tag{50e}$$

Remark 11. *Corollary 3 is, in fact, true for all $\theta \neq 0$ (if $\theta < \frac{1}{2}$ we have to add the dispersive CFL condition hypothesis $\Delta t(1-2\theta) \leq \frac{\Delta x^3}{4}$), but we essentially use it for $\theta \geq \frac{1}{2}$.*

Proof. We choose $\sigma = 0$ in Inequality (39).
• First, we upper bound $||e^n - (1-\theta)\Delta t D_+D_+D_-(e)^n||_{\ell^2_\Delta}^2$ in (39) by $||e^n + \theta\Delta t D_+D_+D_-(e)^n||_{\ell^2_\Delta}^2$ thanks to Proposition 5.
• We tranform B_b in (40b) into

$$B_b = C_b + (1-\theta)\Delta t||D_+(u_\Delta)^n||_{\ell^\infty}^2,$$

with

$$C_b = \left(\frac{\Delta x}{6}D_+\left(e\right)_j^n - c\right)\left(\Delta x - c\Delta t\right). \tag{51}$$

The B_b-term in (39) thus is

$$\sum_{j\in\mathbb{Z}}\Delta x\Delta t B_b\left(D_+e_j^n\right)^2 = \sum_{j\in\mathbb{Z}}\Delta x\Delta t C_b\left(D_+e_j^n\right)^2 + (1-\theta)\Delta t^2||D_+u_\Delta^n||_{\ell^\infty}^2||D_+e^n||_{\ell^2_\Delta}^2. \tag{52}$$

For any sequence $(a_j)_{j\in\mathbb{Z}}$, the following Gagliardo-Nirenberg inequality

$$||D_+(a)||_{\ell^2_\Delta}^2 \leq ||a||_{\ell^2_\Delta}||D_+D_-(a)||_{\ell^2_\Delta}$$

is valid even with the ℓ^2_Δ-norm. We will use it on $||D_+(e)^n||_{\ell^2_\Delta}^2$ in (52), to obtain

$$(1-\theta)\Delta t^2||D_+(u_\Delta)^n||_{\ell^\infty}^2||D_+e^n||_{\ell^2_\Delta}^2 \leq (1-\theta)\Delta t^2||D_+(u_\Delta)^n||_{\ell^\infty}^2\frac{||e^n||_{\ell^2_\Delta}\sqrt{\theta\Delta t\Delta x}||D_+D_-(e)^n||_{\ell^2_\Delta}}{\sqrt{\theta\Delta t\Delta x}}.$$

Proposition 4 enables to make $||\mathcal{A}e^n||_{\ell^2_\Delta}^2$ appear and

$$(1-\theta)\Delta t^2||D_+(u_\Delta)^n||_{\ell^\infty}^2||D_+e^n||_{\ell^2_\Delta}^2 \leq (1-\theta)\Delta t^2||D_+(u_\Delta)^n||_{\ell^\infty}^2\frac{||e^n+\theta\Delta t D_+D_+D_-(e)^n||_{\ell^2_\Delta}^2}{\sqrt{\theta\Delta t\Delta x}}$$
$$\leq \frac{(1-\theta)}{\sqrt{\theta}}\frac{\sqrt{\Delta t}}{\sqrt{\Delta x}}\Delta t||D_+(u_\Delta)^n||_{\ell^\infty}^2||e^n+\theta\Delta t D_+D_+D_-(e)^n||_{\ell^2_\Delta}^2.$$

• As a third step, we transform the B_d-term of (39) (recall that $\sigma = 0$):

$$\sum_{j\in\mathbb{Z}}\Delta x\Delta t B_d\left(D_+D_-(e)_j^n\right)^2 = \sum_{j\in\mathbb{Z}}(1-\theta)\Delta t^2\Delta x\left(D_+D_-(e)_j^n\right)^2 + \frac{(1-\theta)}{2\theta}\Delta t||D_+(u_\Delta)^n||_{\ell^\infty}\theta\Delta t\Delta x||D_+D_-(e)^n||_{\ell^2_\Delta}^2.$$

Relation (28) allows to rewrite the term $\sum_{j\in\mathbb{Z}}(1-\theta)\Delta t^2\Delta x\left(D_+D_-(e)_j^n\right)^2$:

$$\sum_{j\in\mathbb{Z}}(1-\theta)\Delta t^2\Delta x\left(D_+D_-(e)_j^n\right)^2=\sum_{j\in\mathbb{Z}}(1-\theta)\Delta t^2\Delta x\left(D_+D(e)_j^n\right)^2+\sum_{j\in\mathbb{Z}}(1-\theta)\frac{\Delta t^2\Delta x^3}{4}\left(D_+D_+D_-(e)_j^n\right)^2.$$

Proposition 4 gives

$$\frac{(1-\theta)}{2\theta}\Delta t||D_+(u_\Delta)^n||_{\ell^\infty}\theta\Delta t\Delta x||D_+D_-(e)^n||^2_{\ell^2_\Delta}\le\frac{(1-\theta)}{2\theta}\Delta t||D_+(u_\Delta)^n||_{\ell^\infty}||e^n+\theta\Delta tD_+D_+D_-(e)^n||^2_{\ell^2_\Delta}.$$

• Eventually, we focus on the B_f-term in (39). We decompose B_f into

$$B_f=B_g+\Delta t^2(1-\theta)||D_+(u_\Delta)^n||_{\ell^\infty}$$

with

$$B_g=\Delta t\left\{(1-2\theta)+\frac{(1-\theta)\Delta x^2}{2}\left[c+\frac{\Delta x^{\frac12-\gamma}+||e^n||_{\ell^\infty}+9||e^n||^2_{\ell^\infty}\Delta x^{\gamma-\frac12}}{2}\right]\right\}-\frac{\Delta x^3}{4}\tag{53}$$

which leads to the following inequality (thanks to Proposition 4):

$$\begin{aligned}\Delta tB_f||D_+D_+D_-(e)^n||^2_{\ell^2_\Delta}&=\Delta tB_g||D_+D_+D_-(e)^n||^2_{\ell^2_\Delta}+\frac{(1-\theta)}{\theta^2}\Delta t||D_+(u_\Delta)^n||_{\ell^\infty}||\theta\Delta tD_+D_+D_-(e)^n||^2_{\ell^2_\Delta}\\&\le\Delta tB_g||D_+D_+D_-(e)^n||^2_{\ell^2_\Delta}+\frac{(1-\theta)}{\theta^2}\Delta t||D_+(u_\Delta)^n||_{\ell^\infty}||e^n+\theta\Delta tD_+D_+D_-(e)^n||^2_{\ell^2_\Delta}.\end{aligned}$$

Thanks to all the previous relations, we rewrite Inequality (39) as

$$\begin{aligned}||\mathcal{A}e^{n+1}||^2_{\ell^2_\Delta}\le&||\mathcal{A}e^n||^2_{\ell^2_\Delta}\left[1+\Delta tC_a\right]+\Delta tC_b\,||D_+\,(e)^n||^2_{\ell^2_\Delta}+\Delta t^2B_c\,||D\,(e)^n||^2_{\ell^2_\Delta}+\Delta t\left[B_e+(1-\theta)\Delta t\right]||D_+D\,(e)^n||^2_{\ell^2_\Delta}\\&+\Delta t\left[B_g+(1-\theta)\frac{\Delta t\Delta x^2}{4}\right]||D_+D_+D_-\,(e)^n||^2_{\ell^2_\Delta}+\Delta t||\epsilon^n||^2_{\ell^2_\Delta}\left\{1+4\frac{\Delta t}{\Delta x}+\Delta t\right\},\end{aligned}$$

with

$$\begin{aligned}C_a=&||u_\Delta^n||^2_{\ell^\infty}\left(1+\frac{\Delta t}{\Delta x}\right)+||D_+\,(u_\Delta)^n\,||_{\ell^\infty}\left(2-\theta+\frac{1-\theta}{2\theta}+\frac{1-\theta}{\theta^2}+\frac{\Delta t}{\Delta x}\left[2c+\frac23||e^n||_{\ell^\infty}+\frac32||(u_\Delta)^n||_{\ell^\infty}\right]\right)\\&+||D_+(u_\Delta)^n||^2_{\ell^\infty}\left[\frac{(1-\theta)}{\sqrt{\theta}}\frac{\sqrt{\Delta t}}{\sqrt{\Delta x}}+\frac{\Delta t^2}{\Delta x^2}\right]+1+2c^2\frac{\Delta t}{\Delta x}.\end{aligned}$$

For $\theta\in[\frac12,1]$, one has $C_a\le F_a$ with F_a defined in (50a). Finally, we define $C_c:=B_c$ and $C_e:=B_e+(1-\theta)\Delta t$ and $C_f:=B_g+(1-\theta)\frac{\Delta t\Delta x^2}{4}$.

□

Corollary 4 (Corollary of Proposition 2)**.** *Consider Scheme* (2)-(3)*. Let* $(e_j^n)_{(j,n)}$ *be the convergence error defined by* (10)*. Then, for every* $n\in[\![0,N]\!]$*,* $\gamma\in[0,\frac12)$ *and* $\theta<\frac12$*, one has, if* $\Delta t(1-2\theta)\le\frac{\Delta x^3}{4}$

$$\begin{aligned}||\mathcal{A}e^{n+1}||^2_{\ell^2_\Delta}\le||\mathcal{A}e^n||^2_{\ell^2_\Delta}\left[1+F_a\Delta t\right]+\Delta tE_b||D_+(e)^n||^2_{\ell^2_\Delta}+\Delta t^2E_c||D(e)^n||^2_{\ell^2_\Delta}+\Delta tE_d||D_+D_-(e)^n||^2_{\ell^2_\Delta}\\+\Delta tE_e||D_+D(e)^n||^2_{\ell^2_\Delta}+\Delta t||\epsilon^n||^2_{\ell^2_\Delta}\left\{1+4\frac{\Delta t}{\Delta x}+\Delta t\right\},\end{aligned}$$

with

$$\begin{aligned}F_a=||u_\Delta^n||^2_{\ell^\infty}\left(1+\frac{\Delta t}{\Delta x}\right)+||D_+\,(u_\Delta)^n\,||_{\ell^\infty}\left(7+\frac{\Delta t}{\Delta x}\left[2c+\frac23||e^n||_{\ell^\infty}+\frac32||(u_\Delta)^n||_{\ell^\infty}\right]\right)\\+||D_+(u_\Delta)^n||^2_{\ell^\infty}\left[\sqrt{2}\frac{\sqrt{\Delta t}}{\sqrt{\Delta x}}+\frac{\Delta t^2}{\Delta x^2}\right]+1+2c^2\frac{\Delta t}{\Delta x},\end{aligned}\tag{54a}$$

$$E_b = \left(\frac{\Delta x}{6} D_+ (e)_j^n - c\right)(\Delta x - c\Delta t) + (1-\theta)\Delta t ||D_+ (u_\Delta)^n||_{\ell^\infty}, \tag{54b}$$

$$E_c = ||(u_\Delta)^n||_{\ell^\infty}^2 + \left\{ ||e^n||_{\ell^\infty}^2 [1+\Delta x] + 2||e^n||_{\ell^\infty} ||(u_\Delta)^n||_{\ell^\infty} + \frac{2c}{3}||e^n||_{\ell^\infty} \right\} - c^2, \tag{54c}$$

$$E_d = \frac{4}{\Delta x^2}\left\{ \Delta t \left[(1-2\theta) + \frac{(1-\theta)\Delta x^2}{2}\left[c + \frac{\Delta x^{\frac12 - \gamma} + ||e^n||_{\ell^\infty} + 9||e^n||_{\ell^\infty}^2 \Delta x^{\gamma - \frac12}}{2}\right] \right.\right.$$
$$\left.\left. + \Delta t(1-\theta)||D_+ (u_\Delta)^n||_{\ell^\infty} + \frac{(1-\theta)\Delta x^2}{4}\left\{ ||D_+ (u_\Delta)^n||_{\ell^\infty} + \frac{\Delta x}{2}||D_- (u_\Delta)^n||_{\ell^\infty} \right\}\right] - \frac{\Delta x^3}{4} \right\}, \tag{54d}$$

$$E_e = 2(1-\theta)\Delta t\left\{ ||(u_\Delta)^n||_{\ell^\infty} + ||e^n||_{\ell^\infty} + \left[\frac{\Delta x^{\frac12-\gamma} + ||e^n||_{\ell^\infty} + 9||e^n||_{\ell^\infty}^2 \Delta x^{\gamma-\frac12}}{2}\right]\right\} - \frac{4\Delta t}{\Delta x^2}\left\{ (1-2\theta) \right.$$
$$\left. + \frac{(1-\theta)\Delta x^2}{2}\left[c + \frac{\Delta x^{\frac12-\gamma} + ||e^n||_{\ell^\infty} + 9||e^n||_{\ell^\infty}^2 \Delta x^{\gamma-\frac12}}{2}\right] + \Delta t (1-\theta)||D_+ (u_\Delta)^n||_{\ell^\infty} \right\}. \tag{54e}$$

Remark 12. *The variables F_a are identical in both previous corollaries. It is noticed that Corollary 4 is valid for all θ but thereafter, it will be mainly used for $\theta < \frac{1}{2}$.*

Proof. We choose $\sigma = 1$ in Inequality (39).
• From Relation (28), we transform the B_f-term in Inequality (39) into

$$\sum_{j\in\mathbb{Z}} \Delta t \Delta x B_f \left(D_+ D_+ D_- e_j^n\right)^2 = \sum_{j\in\mathbb{Z}} \Delta t \Delta x B_f \left[\frac{4}{\Delta x^2}\left(D_+ D_- e_j^n\right)^2 - \frac{4}{\Delta x^2}\left(D_+ D e_j^n\right)^2\right].$$

• We upper bound $||e^n - (1-\theta)\Delta t D_+ D_+ D_- (e)^n||_{\ell^2_\Delta}^2$ by $||e^n + \theta \Delta t D_+ D_+ D_- (e)^n||_{\ell^2_\Delta}^2$ thanks to Proposition 5, to obtain, instead of Inequality (39),

$$\begin{aligned}||e^{n+1} + \theta\Delta t D_+ D_+ D_- (e)^n||_{\ell^2_\Delta}^2 \le{}& ||e^n + \Delta t \theta D_+ D_+ D_- (e)^n||_{\ell^2_\Delta}^2 [1 + B_a \Delta t + \Delta t] + \Delta t B_b ||D_+ (e)^n||_{\ell^2_\Delta}^2 \\ &+ \Delta t^2 B_c ||D(e)^n||_{\ell^2_\Delta}^2 + \Delta t\left\{B_d + \frac{4B_f}{\Delta x^2}\right\}||D_+ D_- (e)^n||_{\ell^2_\Delta}^2 \\ &+ \Delta t \left\{B_e - \frac{4B_f}{\Delta x^2}\right\}||D_+ D(e)^n||_{\ell^2_\Delta}^2 + \Delta t ||\epsilon^n||_{\ell^2_\Delta}^2\left\{1 + 4\frac{\Delta t}{\Delta x} + \Delta t\right\}.\end{aligned}$$

We note $E_a := B_a + 1$ and verify $E_a \le F_a$. Finally, we fix $E_b := B_b$ with $\sigma = 1$, $E_c := B_c$, $E_d := B_d + \frac{4B_f}{\Delta x^2}$ with $\sigma = 1$ and $E_e := B_e - \frac{4B_f}{\Delta x^2}$.

□

In the following, we will have to show that C_i and E_i are non-positive to loop the estimates.

5.3 Induction method

We are now able to prove, by induction, the main result for a smooth initial datum: Theorem 1.

Proof of Theorem 1. Let $T > 0$ and $s \ge 6$ with $u_0 \in H^s(\mathbb{R})$. Let the Rusanov coefficient c be such that (12) is true. This choice is possible because of Theorem 3 which proves that the exact solution belongs to L^∞_x for $t \in [0, T]$.

Remark 13. *Thanks to Hypothesis (12) : $\sup_{t\in[0,T]} ||u(t,\cdot)||_{L^\infty(\mathbb{R})} < c$, there exists a constant $\alpha_0 > 0$ such that, for all $\Delta t > 0$, $\Delta x > 0$ and for all $n \in [\![0, N]\!]$,*

$$||(u_\Delta)^n||_{\ell^\infty(\mathbb{Z})} + \alpha_0 \le ||u_\Delta||_{\ell^\infty([\![0,N]\!];\ell^\infty(\mathbb{Z}))} + \alpha_0 \le \sup_{t\in[0,T]} ||u(t,\cdot)||_{L^\infty(\mathbb{R})} + \alpha_0 \le c.$$

Let $\beta_0 \in (0,1)$, $\theta \in [0,1]$ and $\gamma \in (0,\frac{1}{2})$. We fix $\tilde{\omega}_0 > 0$ such that

$$\tilde{\omega}_0 = \left[\Lambda_{T,\|u_0\|_{H^{\frac{3}{4}}}} \left(1+\|u_0\|^2_{H^{\frac{1}{2}+\eta}}\right)\left(\frac{\|u_0\|_{H^6}}{c+\frac{1}{2}} + \|u_0\|_{H^4} + \|u_0\|_{H^{\frac{3}{2}+\eta}}\|u_0\|_{H^1}\right)\right]^{-\frac{1}{\gamma}}, \tag{55}$$

with $\Lambda_{T,\|u_0\|_{H^{\frac{3}{4}}}}$ defined in (15).
We also fix $\omega_0 > 0$ such that following Inequalities (56) and (57a)-(57d) if $\theta \geq \frac{1}{2}$ and following Inequalities (56) and (58a)-(58d) if $\theta < \frac{1}{2}$ are verified

$$\omega_0^{\frac{1}{2}-\gamma} \leq 3c, \tag{56}$$

- for $\theta \geq \frac{1}{2}$,

$$\begin{cases} \omega_0^{\frac{1}{4}-\frac{\gamma}{2}}\sqrt{\left[\omega_0^{\frac{1}{2}-\gamma}+\omega_0^{\frac{3}{2}-\gamma}\right]+2\sup\limits_{t\in[0,T]}||u(t,\cdot)||_{L^\infty(\mathbb{R})}+\dfrac{2c}{3}} \leq \alpha_0, & (57a)\\ \dfrac{13(1-\beta_0)}{2c+1}\omega_0^{\frac{1}{2}-\gamma} \leq \beta_0, & (57b)\\ (1-2\theta)+\dfrac{(1-\theta)\omega_0^2}{2}\left[c+\dfrac{1}{2}+\dfrac{11}{2}\omega_0^{\frac{1}{2}-\gamma}\right] \leq 0, \quad \text{if } \theta > \dfrac{1}{2}, & (57c)\\ \dfrac{11(1-\beta_0)}{2c+1}\omega_0^{\frac{1}{2}-\gamma} \leq \beta_0, \quad \text{if } \theta = \dfrac{1}{2}, & (57d) \end{cases}$$

- for $\theta < \frac{1}{2}$,

$$\begin{cases} \omega_0^{\frac{1}{4}-\frac{\gamma}{2}}\sqrt{\left[\omega_0^{\frac{1}{2}-\gamma}+\omega_0^{\frac{3}{2}-\gamma}\right]+2\sup\limits_{t\in[0,T]}||u(t,\cdot)||_{L^\infty(\mathbb{R})}+\dfrac{2c}{3}} \leq \alpha_0, & (58a)\\ 12\omega_0^{\frac{1}{2}-\gamma} \leq \alpha_0, & (58b)\\ \dfrac{(1-\theta)(1-\beta_0)}{2(1-2\theta)c}||(u_\Delta)^n||_{\ell^\infty}\omega_0+\dfrac{(1-\beta_0)}{3c+\frac{3}{2}}\omega_0^{\frac{1}{2}-\gamma}+\dfrac{\omega_0^{\frac{1}{2}-\gamma}}{3c} \leq \beta_0, & (58c)\\ \dfrac{(1-\theta)(1-\beta_0)}{2(1-2\theta)}\omega_0^2\left[c+\dfrac{11}{2}\omega_0^{\frac{1}{2}-\gamma}\right]+(1-\theta)||(u_\Delta)^n||_{\ell^\infty}\dfrac{(1-\beta_0)}{(1-2\theta)}\left[\dfrac{(1-\beta_0)}{2(1-2\theta)}\omega_0^2+\dfrac{\omega_0(2+\omega_0)}{4}\right] \leq \beta_0. & (58d) \end{cases}$$

Remark 14. *These conditions on ω_0 are very likely not optimal.*

Let us prove by induction on $n \in [\![0,N]\!]$ that

$$\text{if } \Delta x \leq \min(\tilde{\omega}_0,\omega_0) \text{ and if CFL conditions (13a)} - \text{(13b) hold, one has } ||e^n||_{\ell^\infty} \leq \Delta x^{\frac{1}{2}-\gamma}, \text{ for all } n \in [\![0,N]\!]$$

Initialization : For $n = 0$, the inequality $||e^0||_{\ell^\infty} \leq \Delta x^{\frac{1}{2}-\gamma}$ is true because Expressions (3) and (8) imply

$$e_j^0 = 0, \quad j \in \mathbb{Z}.$$

Heredity : Let us assume that

$$\text{if } \Delta x \leq \min(\tilde{\omega}_0,\omega_0) \text{ and if CFL conditions (13a)} - \text{(13b) hold, one has } ||e^k||_{\ell^\infty} \leq \Delta x^{\frac{1}{2}-\gamma}, \text{ for all } k \leq n. \tag{59}$$

Then our goal is to prove that

$$\text{if } \Delta x \leq \min(\tilde{\omega}_0,\omega_0) \text{ and if CFL conditions (13a)} - \text{(13b) hold, one has } ||e^{n+1}||_{\ell^\infty} \leq \Delta x^{\frac{1}{2}-\gamma}.$$

Step 1 : simplification of Corollaries 3 and 4. Let us prove in this first step that $\Delta x \leq \min(\tilde{\omega}_0, \omega_0)$ and CFL conditions (13a)-(13b) imply the non-positivity of extra terms C_i and E_i in Corollaries 3 and 4. We dissociate two cases according to the value of θ.

$\boxed{\text{CASE } \theta \geq \frac{1}{2} :}$

We show the non-positivity of coefficients C_i in Corollary 3, for $i \in \{b, c, e, f\}$.

- **Sign of C_b**: We get by developing $D_+(e)_j^n$

$$\frac{\Delta x}{6} D_+ (e)_j^n \leq \frac{||e^n||_{\ell^\infty}}{3}.$$

However, by induction hypothesis, one has $\Delta x \leq \omega_0$ (with ω_0 verifying, among others, Inequality (56)) and $||e^n||_{\ell^\infty} \leq \Delta x^{\frac{1}{2}-\gamma}$. It gives

$$\frac{||e^n||_{\ell^\infty}}{3} \leq \frac{\Delta x^{\frac{1}{2}-\gamma}}{3} \leq \frac{\omega_0^{\frac{1}{2}-\gamma}}{3} \leq c.$$

Due to the CFL condition (13b), one has

$$\Delta x - c\Delta t \geq 0.$$

Thus, $C_b \leq 0$.

- **Sign of C_c**: For the term C_c, thanks to the hypothesis $||e^n||_{\ell^\infty} \leq \Delta x^{\frac{1}{2}-\gamma}$, we obtain

$$C_c \leq ||(u_\Delta)^n||^2_{\ell^\infty} + \left\{ \left[\Delta x^{1-2\gamma} + \Delta x^{2-2\gamma}\right] + 2\Delta x^{\frac{1}{2}-\gamma}||(u_\Delta)^n||_{\ell^\infty} + \frac{2c\Delta x^{\frac{1}{2}-\gamma}}{3} \right\} - c^2.$$

As $c \geq \alpha_0 + || (u_\Delta)^n ||_{\ell^\infty}$ (see Remark 13) and $\Delta x \leq \omega_0$ (with ω_0 satisfying Inequality (57a)) by induction hypothesis, one has

$$C_c \leq ||(u_\Delta)^n||^2_{\ell^\infty} + \left\{ \left[\omega_0^{1-2\gamma} + \omega_0^{2-2\gamma}\right] + 2\omega_0^{\frac{1}{2}-\gamma}||(u_\Delta)^n||_{\ell^\infty} + \frac{2c\omega_0^{\frac{1}{2}-\gamma}}{3} \right\} - c^2 \leq 0.$$

- **Sign of C_e**: since we suppose $||e^n||_{\ell^\infty} \leq \Delta x^{\frac{1}{2}-\gamma}$, the term C_e satisfies

$$C_e \leq 2(1-\theta)\Delta t \left\{ ||(u_\Delta)^n||_{\ell^\infty} + \frac{1}{2} + \frac{13}{2}\Delta x^{\frac{1}{2}-\gamma} \right\} - \Delta x.$$

As $\theta \geq \frac{1}{2}$, then $2(1-\theta) \leq 1$, and, thanks to the choice of c (12), one has

$$C_e \leq \Delta t \left\{ c + \frac{1}{2} + \frac{13}{2}\Delta x^{\frac{1}{2}-\gamma} \right\} - \Delta x = \Delta x \left\{ \frac{\Delta t}{\Delta x}\left[c + \frac{1}{2}\right] - 1 + \frac{13}{2}\frac{\Delta t}{\Delta x}\Delta x^{\frac{1}{2}-\gamma} \right\}.$$

Using $\Delta x \leq \omega_0$ and using hyperbolic CFL (13b), one has

$$\frac{13}{2}\frac{\Delta t}{\Delta x}\Delta x^{\frac{1}{2}-\gamma} \leq \frac{13}{2}\frac{(1-\beta_0)}{c+\frac{1}{2}}\Delta x^{\frac{1}{2}-\gamma} \leq \frac{13(1-\beta_0)}{2c+1}\omega_0^{\frac{1}{2}-\gamma}$$

which is less than β_0 thanks to Inequality (57b). Thus one has

$$C_e \leq 0.$$

- **Sign of C_f**: the dispersive CFL-type condition (13a) together with hypothesis $||e^n||_{\ell^\infty} \leq \Delta x^{\frac{1}{2}-\gamma}$ give

$$C_f \leq \Delta t \left\{ (1-2\theta) + \frac{(1-\theta)\Delta x^2}{2}\left[c + \frac{1}{2} + \frac{11}{2}\Delta x^{\frac{1}{2}-\gamma}\right] \right\} - \frac{\Delta x^3}{4},$$

which is non-positive if $\Delta x \leq \omega_0$. Indeed,

– if $\theta > \frac{1}{2}$, one has chosen ω_0 such that

$$(1-2\theta) + \frac{(1-\theta)}{2}\Delta x^2 \left[c + \frac{1}{2} + \frac{11}{2}\Delta x^{\frac{1}{2}-\gamma}\right] \leq (1-2\theta) + \frac{(1-\theta)}{2}\omega_0^2 \left[c + \frac{1}{2} + \frac{11}{2}\omega_0^{\frac{1}{2}-\gamma}\right] \leq 0,$$

thanks to Inequality (57c),

– if $\theta = \frac{1}{2}$,

$$C_f \leq \frac{\Delta t \Delta x^2}{4}\left[c + \frac{1}{2} + \frac{11}{2}\Delta x^{\frac{1}{2}-\gamma}\right] - \frac{\Delta x^3}{4} = \frac{\Delta x^3}{4}\left\{\frac{\Delta t}{\Delta x}\left[c + \frac{1}{2}\right] - 1 + \frac{11\Delta t}{2\Delta x}\Delta x^{\frac{1}{2}-\gamma}\right\},$$

and Condition (13b) together with $\Delta x \leq \omega_0$ for ω_0 verifying Inequality (57d) enable us to conclude about the non-positivity of C_f.

$\boxed{\text{CASE } \theta < \frac{1}{2}\ :}$

In the same way, from Corollary 4, we show the non-positivity of E_i, for $i \in \{b, c, d, e\}$.

- **Sign of** E_b: one has, by definition of E_b and by hypothesis $||e^n||_{\ell^\infty} \leq \Delta x^{\frac{1}{2}-\gamma}$

$$\begin{aligned}
E_b &\leq \left(\frac{\Delta x}{6}D_+\left(e\right)_j^n - c\right)(\Delta x - c\Delta t) + 2(1-\theta)\frac{\Delta t}{\Delta x}||(u_\Delta)^n||_{\ell^\infty} \\
&\leq \frac{\Delta x ||e^n||_{\ell^\infty}}{3} + \frac{c\Delta t ||e^n||_{\ell^\infty}}{3} - c\Delta x + c^2\Delta t + 2(1-\theta)\frac{\Delta t}{\Delta x}||(u_\Delta)^n||_{\ell^\infty} \\
&\leq c\left[c\Delta t\left(1 + \frac{\Delta x^{\frac{1}{2}-\gamma}}{3c}\right) - \Delta x\left(1 - \frac{\Delta x^{\frac{1}{2}-\gamma}}{3c} - 2(1-\theta)\frac{\Delta t}{\Delta x^2 c}||(u_\Delta)^n||_{\ell^\infty}\right)\right] \\
&\leq c\Delta x\left[c\frac{\Delta t}{\Delta x} + \frac{\Delta t}{\Delta x}\frac{\Delta x^{\frac{1}{2}-\gamma}}{3} - 1 + \frac{\Delta x^{\frac{1}{2}-\gamma}}{3c} + 2(1-\theta)\frac{\Delta t}{\Delta x^2 c}||(u_\Delta)^n||_{\ell^\infty}\right].
\end{aligned}$$

The hyperbolic CFL condition (13b) and the dispersive one (13a) (we recall that $1-2\theta > 0$ in that case) imply

$$E_b \leq c\Delta x\left[1 - \beta_0 + \frac{(1-\beta_0)\Delta x^{\frac{1}{2}-\gamma}}{3c + \frac{3}{2}} - 1 + \frac{\Delta x^{\frac{1}{2}-\gamma}}{3c} + (1-\theta)\frac{\Delta x(1-\beta_0)}{2c(1-2\theta)}||(u_\Delta)^n||_{\ell^\infty}\right].$$

The choice of ω_0 small enough to satisfy Inequalities (58c) implies $E_b \leq 0$.

- **Sign of** E_c: since $E_c = C_c$, we follow exactly the same proof as for $\theta \geq \frac{1}{2}$ to show $E_c \leq 0$.
- **Sign of** E_d: thanks to Definition (54d), one has

$$\begin{aligned}
E_d = \frac{4}{\Delta x^2}&\left\{\Delta t\left[(1-2\theta) + \frac{(1-\theta)\Delta x^2}{2}\left[c + \frac{\Delta x^{\frac{1}{2}-\gamma} + ||e^n||_{\ell^\infty} + 9||e^n||_{\ell^\infty}^2 \Delta x^{\gamma-\frac{1}{2}}}{2}\right]\right.\right. \\
&\left.\left. + \Delta t(1-\theta)||D_+\left(u_\Delta\right)^n||_{\ell^\infty} + \frac{(1-\theta)\Delta x^2}{4}\left\{||D_+(u_\Delta)^n||_{\ell^\infty} + \frac{\Delta x}{2}||D_-\left(u_\Delta\right)^n||_{\ell^\infty}\right\}\right] - \frac{\Delta x^3}{4}\right\}
\end{aligned}$$

Since $||e^n||_{\ell^\infty} \leq \Delta x^{\frac{1}{2}-\gamma}$, it becomes, thanks to dispersive CFL (13a),

$$
\begin{aligned}
E_d &= \Delta x \left\{ \frac{4\Delta t}{\Delta x^3}(1-2\theta) + \frac{2\Delta t}{\Delta x}(1-\theta)\left[c + \frac{11\Delta x^{\frac{1}{2}-\gamma}}{2}\right] \right.\\
&\qquad \left. +8\frac{\Delta t^2}{\Delta x^4}(1-\theta)||u_\Delta^n||_{\ell^\infty} + 2(1-\theta)\frac{\Delta t}{\Delta x^2}||u_\Delta^n||_{\ell^\infty} + (1-\theta)\frac{\Delta t}{\Delta x}||u_\Delta^n||_{\ell^\infty} - 1\right\}\\
&\le \Delta x \left\{ \frac{4\Delta t}{\Delta x^3}(1-2\theta) + \frac{\Delta x^2(1-\beta_0)}{2(1-2\theta)}(1-\theta)\left[c + \frac{11\Delta x^{\frac{1}{2}-\gamma}}{2}\right] + \frac{(1-\beta_0)^2\Delta x^2}{2(1-2\theta)^2}(1-\theta)||u_\Delta^n||_{\ell^\infty} \right.\\
&\qquad \left. +(1-\theta)\frac{(1-\beta_0)\Delta x}{2(1-2\theta)}||u_\Delta^n||_{\ell^\infty} + (1-\theta)\frac{\Delta x^2(1-\beta_0)}{4(1-2\theta)}||u_\Delta^n||_{\ell^\infty} - 1\right\}\\
&= \Delta x \left\{ \frac{4\Delta t}{\Delta x^3}(1-2\theta) + \frac{\Delta x^2(1-\beta_0)}{2(1-2\theta)}(1-\theta)\left[c + \frac{11\Delta x^{\frac{1}{2}-\gamma}}{2}\right] \right.\\
&\qquad \left. +(1-\theta)||u_\Delta^n||_{\ell^\infty}\frac{(1-\beta_0)}{(1-2\theta)}\left[\frac{(1-\beta_0)}{2(1-2\theta)}\Delta x^2 + \frac{\Delta x(2+\Delta x)}{4}\right] - 1\right\}
\end{aligned}
$$

Thanks to $\Delta x \le \omega_0$, with ω_0 verifying (58d) and thanks to the CFL condition (13a), one has

$$E_d \le 0.$$

- **Sign of E_e:** we develop E_e to obtain

$$
\begin{aligned}
E_e &\le 2(1-\theta)\Delta t\left\{||(u_\Delta)^n||_{\ell^\infty} + \frac{13}{2}\Delta x^{\frac{1}{2}-\gamma}\right\} - \frac{4\Delta t}{\Delta x^2}(1-2\theta) - 2(1-\theta)\Delta t\left[c - \frac{11\Delta x^{\frac{1}{2}-\gamma}}{2}\right]\\
&\quad - \frac{8\Delta t^2}{\Delta x^3}(1-\theta)||\left(u_\Delta\right)^n||_{\ell^\infty}\\
&\le 2(1-\theta)\Delta t\left\{||(u_\Delta)^n||_{\ell^\infty} + 12\Delta x^{\frac{1}{2}-\gamma} - c\right\} - \frac{4\Delta t}{\Delta x^2}\left[(1-2\theta) + \frac{2\Delta t}{\Delta x}(1-\theta)||(u_\Delta)^n||_{\ell^\infty}\right].
\end{aligned}
$$

Since $\theta < \frac{1}{2}$, one has $1-2\theta > 0$ then $-\frac{4\Delta t}{\Delta x^2}\left[(1-2\theta) + \frac{2\Delta t}{\Delta x}(1-\theta)||(u_\Delta)^n||_{\ell^\infty}\right] \le 0$. The hypothesis $\Delta x \le \omega_0$, with ω_0 satisfying (58b) and the choice of c (12) give $E_e \le 0$.

ALL IN ALL :

We have proved that, under the induction hypothesis, the following equality holds, for all $\theta \in [0,1]$

$$||\mathcal{A}e^{n+1}||^2_{\ell^2_\Delta} \le ||\mathcal{A}e^n||^2_{\ell^2_\Delta}\left\{1 + \Delta t F_a\right\} + \Delta t||\epsilon^n||^2_{\ell^2_\Delta}\left\{1 + 4\frac{\Delta t}{\Delta x} + \Delta t\right\}, \tag{60}$$

with F_a defined by (50a).

Step 2 : From e^n to e^{n+1} thanks to discret Grönwall lemma. By splitting F_a and using the first inequality of (18) to upper bound $\Delta t||D_+\left(u_\Delta\right)^n||_{\ell^\infty}$ and $\Delta t||D_+\left(u_\Delta\right)^n||^2_{\ell^\infty}$, Inequality (60) becomes

$$||\mathcal{A}e^{n+1}||^2_{\ell^2_\Delta} \le ||\mathcal{A}e^n||^2_{\ell^2_\Delta}\left\{1 + \Delta t F_b^n + \sum_{i=1}^{2}\left(\int_{t^n}^{t^{n+1}}||\partial_x u(s,.)||^i_{L^\infty_x}ds\right)F^n_{c,i}\right\} + \Delta t||\epsilon^n||^2_{\ell^2_\Delta}\left\{1 + 4\frac{\Delta t}{\Delta x} + \Delta t\right\},$$

with

$$F_b^n = \left[||u_\Delta^n||^2_{\ell^\infty}\left(1 + \frac{\Delta t}{\Delta x}\right) + 1 + 2c^2\frac{\Delta t}{\Delta x}\right] \le \left[1 + ||u_\Delta||^2_{\ell^\infty_n\ell^\infty}(1 + \frac{\Delta t}{\Delta x}) + 2\frac{\Delta t}{\Delta x}c^2\right]$$

and

$$F^n_{c,1} = \left[7 + \frac{\Delta t}{\Delta x}\left(2c + \frac{2}{3}\Delta x^{\frac{1}{2}-\gamma} + \frac{3}{2}||(u_\Delta)^n||_{\ell^\infty}\right)\right] \le \left[7 + \frac{\Delta t}{\Delta x}\left(2c + \frac{2}{3}\Delta x^{\frac{1}{2}-\gamma} + \frac{3}{2}||u_\Delta||_{\ell^\infty\ell^\infty_n}\right)\right]$$

and

$$F^n_{c,2} = \left[\sqrt{2}\frac{\sqrt{\Delta t}}{\sqrt{\Delta x}} + \frac{\Delta t^2}{\Delta x^2}\right].$$

Due to the CFL condition, we have, denoting by C a number independent of c, u^n_Δ, Δt and Δx

$$F^n_b \leq C\left(1 + c^2\left(1 + \frac{\Delta t}{\Delta x}\right)\right) =: F_b, \tag{61}$$

$$F^n_{c,1} \leq C\left(1 + \frac{\Delta t}{\Delta x}[1+c]\right) =: F_{c,1} \tag{62}$$

and

$$F^n_{c,2} = \left[\sqrt{2}\frac{\sqrt{\Delta t}}{\sqrt{\Delta x}} + \frac{\Delta t^2}{\Delta x^2}\right] =: F_{c,2}. \tag{63}$$

We can now apply a discrete Grönwall Lemma (noticing that $e^0_j = 0, \quad j \in \mathbb{Z}$). It provides, for every $n \in [\![0, N-1]\!]$,

$$||\mathcal{A}e^{n+1}||^2_{\ell^2_\Delta} \leq \exp\left(t^{n+1}F_b + \sum_{i=1}^{2}\int_0^{t^{n+1}} ||\partial_x u(s,.)||^i_{L^\infty_x(\mathbb{R})}F_{c,i}\right) \sup_{n\in[\![0,N]\!]} ||\epsilon^n||^2_{\ell^2_\Delta} T\left\{1 + 4\frac{\Delta t}{\Delta x} + \Delta t\right\}. \tag{64}$$

Finally, Theorem 3 and Proposition 1 give, for $0 < \eta \leq 6 - \frac{3}{2}$,

$$||\mathcal{A}e^{n+1}||^2_{\ell^2_\Delta} \leq M^2\left(1 + \|u_0\|^2_{H^{\frac{1}{2}+\eta}}\right)^2 \left\{\Delta t^2\|u_0\|^2_{H^6} + \Delta x^2\left[\|u_0\|^2_{H^4} + \|u_0\|^2_{H^{\frac{3}{2}+\eta}}\|u_0\|^2_{H^1}\right] \right\}, \tag{65}$$

with

$$\begin{aligned} M^2 &= \exp\left(TF_b + \|u_0\|_{H^{\frac{3}{4}}} C_{\frac{3}{4}} e^{\kappa_{\frac{3}{4}}T}\left[F_{c,1}T^{\frac{3}{4}} + F_{c,2}T^{\frac{1}{2}}\right]\right) C^2 e^{2\kappa T} T\left\{1 + 4\frac{\Delta t}{\Delta x} + \Delta t\right\} \\ &\leq \exp\left(C\left(1+c^2\right)\left(1 + \frac{\Delta t^2}{\Delta x^2}\right)\left(T + (T^{\frac{3}{4}} + T^{\frac{1}{2}})||u_0||_{H^{\frac{3}{4}}} e^{\kappa_{\frac{3}{4}}T}\right)\right) C^2 e^{2\kappa T} T\left\{1 + \frac{\Delta t}{\Delta x}\right\}, \end{aligned}$$

with C independent of u_0 and κ, $\kappa_{\frac{3}{4}}$ dependent only on $||u_0||_{L^2}$. Thanks to the CFL condition (13b), an upper bound for M is

$$M^2 \leq \Lambda^2_{T,||u_0||_{H^{\frac{3}{4}}}}$$

with

$$\Lambda^2_{T,||u_0||_{H^{\frac{3}{4}}}} = \exp\left(C\left(1+c^2\right)\left(1 + \frac{(1-\beta_0)^2}{(c+\frac{1}{2})^2}\right)\left(T + (T^{\frac{3}{4}} + T^{\frac{1}{2}})||u_0||_{H^{\frac{3}{4}}} e^{\kappa_{\frac{3}{4}}T}\right)\right) C^2 e^{2\kappa T} T\left\{1 + \frac{1-\beta_0}{c+\frac{1}{2}}\right\}.$$

Since $||e^{n+1}||^2_{\ell^2_\Delta} \leq ||\mathcal{A}e^{n+1}||^2_{\ell^2_\Delta}$ (Proposition 3), Inequality (65) gives

$$\begin{aligned} ||e^{n+1}||^2_{\ell^2_\Delta} &\leq \Lambda^2_{T,\|u_0\|_{H^{\frac{3}{4}}}}\left(1 + \|u_0\|^2_{H^{\frac{1}{2}+\eta}}\right)^2 \left\{\Delta t^2\|u_0\|^2_{H^6} + \Delta x^2\left[\|u_0\|^2_{H^4} + \|u_0\|^2_{H^{\frac{3}{2}+\eta}}\|u_0\|^2_{H^1}\right]\right\} \\ &\leq \Lambda^2_{T,\|u_0\|_{H^{\frac{3}{4}}}}\left(1 + \|u_0\|^2_{H^{\frac{1}{2}+\eta}}\right)^2 \left(\frac{\|u_0\|^2_{H^6}}{(c+\frac{1}{2})^2} + \|u_0\|^2_{H^4} + \|u_0\|^2_{H^{\frac{3}{2}+\eta}}\|u_0\|^2_{H^1}\right)\Delta x^2, \end{aligned} \tag{66}$$

where the last inequality is obtained thanks to the CFL condition (13b).

Conclusion : It remains to verify the induction hypothesis (59) at step $n+1$. The definition of the ℓ^2_Δ-norm, Identity (7), together with the inclusion $\ell^2 \subset \ell^\infty$, holds

$$||e^n||_{\ell^\infty} \leq \frac{||e^n||_{\ell^2_\Delta}}{\sqrt{\Delta x}}.$$

According to the upper bound (66), the ℓ^∞-norm is bounded as follow

$$||e^{n+1}||_{\ell^\infty} \leq \Lambda_{T,\|u_0\|_{H^{\frac{3}{4}}}} \left(1+\|u_0\|^2_{H^{\frac{1}{2}+\eta}}\right)\left(\frac{\|u_0\|_{H^6}}{c+\frac{1}{2}} + \|u_0\|_{H^4} + \|u_0\|_{H^{\frac{3}{2}+\eta}}\|u_0\|_{H^1}\right)\sqrt{\Delta x}.$$

The choice of a small Δx satisfying $\Delta x \leq \min(\tilde{\omega}_0, \omega_0)$ with $\tilde{\omega}_0$ defined in (55) implies thus $||e^{n+1}||_{\ell^\infty} \leq \Delta x^{\frac{1}{2}-\gamma}$. The induction hypothesis is then true for $n+1$. $\square$

Thus, we have proved Equation (14) with $\Lambda_{T,||u_0||_{H^{\frac{3}{4}}}}$ defined by (15) and $\widehat{\omega_0} = \min(\omega_0, \tilde{\omega}_0)$.

Remark 15. *The choice of a time average in the definition of u_Δ, Equation (9), is dictated by the discrete Grönwall Lemma on (64). Indeed, applying discrete Grönwall Lemma introduces the following term $\sum_{n=0}^N \Delta t||D_+\left(u_\Delta\right)^n||^i_{\ell^\infty}$ which is controlled thanks to the estimate (18), where the time integral plays a crucial role.*
Regarding the space average in the definition of u_Δ, its necessity comes from controlling the sum on $j \in \mathbb{Z}$ in the consistency estimates (75).

Remark 16. *This method is a process to find the CFL condition which suits also for the Airy equation*

$$\partial_t u(t,x) + \partial_x^3 u(t,x) = 0, \quad (t,x) \in [0,T]\times\mathbb{R},$$

with the finite difference scheme

$$\frac{v_j^{n+1}-v_j^n}{\Delta t} + \theta\frac{v_{j+2}^{n+1}-3v_{j+1}^{n+1}+3v_j^{n+1}-v_{j-1}^{n+1}}{\Delta x^3} + (1-\theta)\frac{v_{j+2}^{n}-3v_{j+1}^{n}+3v_j^{n}-v_{j-1}^{n}}{\Delta x^3} = 0. \tag{67}$$

The analogue of Equation (39) is here

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\Delta x\left(e_j^{n+1}+\theta\Delta t D_+D_+D_-\left(e\right)_j^{n+1}\right)^2 &\leq \{1+\Delta t\}\sum_{j\in\mathbb{Z}}\left(e_j^n+\theta\Delta t D_+D_+D_-\left(e\right)_j^n\right)^2\\ &+\sum_{j\in\mathbb{Z}}\Delta x\Delta t\underbrace{\{1+\Delta t\}\left\{(1-2\theta)\Delta t-\frac{\Delta x^3}{4}\right\}}_{C_f^{\text{Airy}}}\left[D_+D_+D_-\left(e\right)_j^n\right]^2+\sum_{j\in\mathbb{Z}}\Delta x\Delta t\{1+\Delta t\}\left(\epsilon_j^n\right)^2.\end{aligned}$$

Imposing $C_f^{\text{Airy}} \leq 0$ (which corresponds to Step 1 in the previous proof of Theorem 1) leads to

$$\Delta t(1-2\theta) \leq \frac{\Delta x^3}{4}.$$

This so-called Courant-Friedrichs-Lewy condition, in the case $\theta = 0$, is exactly the one which is obtained in [Men83] with a computation of the zeros of the amplification factor in [Men83] and the one obtained by the Fourier method. Indeed, the amplification factor obtained by Fourier analysis on Airy equation is

$$\frac{1-8\frac{(1-\theta)\Delta t}{\Delta x^3}\sin^4(\pi\xi)-8i\frac{(1-\theta)\Delta t}{\Delta x^3}\sin^3(\pi\xi)\cos(\pi\xi)}{1+8\frac{\theta\Delta t}{\Delta x^3}\sin^4(\pi\xi)+8i\frac{\theta\Delta t}{\Delta x^3}\sin^3(\pi\xi)\cos(\pi\xi)}, \quad \xi\in(0,1).$$

Requiring that its modulus is less than 1 yields

$$\Delta t\sin^2(\pi\xi)(1-2\theta) \leq \frac{\Delta x^3}{4}, \text{ for all } \xi\in(0,1).$$

Remark 17. *For a Rusanov finite difference scheme applied to the non-linear term of the KdV equation: the Burgers equation*

$$\partial_t u(t,x) + \partial_x \left(\frac{u^2}{2}\right)(t,x) = 0, \quad (t,x) \in [0,T] \times \mathbb{R},$$

which corresponds to the discrete equation

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} + \frac{\left(v_{j+1}^n\right)^2 - \left(v_{j-1}^n\right)^2}{4\Delta x} = c\left(\frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{2\Delta x}\right), \quad (n,j) \in [\![0,N]\!] \times \mathbb{Z}, \tag{68}$$

the analogue of Equation (39) *would be*

$$||e^{n+1}||_{\ell^2_\Delta}^2 \leq ||e^n||_{\ell^2_\Delta}^2 \left\{1 + \Delta t F_a^{\text{Burgers}}\right\} + \sum_{j\in\mathbb{Z}} \Delta x \Delta t C_b^{\text{Burgers}} \left[D_+ (e)_j^n\right]^2 + \sum_{j\in\mathbb{Z}} \Delta x \Delta t^2 C_c^{\text{Burgers}} \left[D\,(e)_j^n\right]^2 + \sum_{j\in\mathbb{Z}} \Delta x \Delta t \left(\epsilon_j^n\right)^2 \left\{4\frac{\Delta t}{\Delta x} + \Delta t\right\},$$

with

$$F_a^{\text{Burgers}} = ||u_\Delta^n||_{\ell^\infty}^2 + ||D_+ (u_\Delta)^n||_{\ell^\infty} \left(1 + \frac{\Delta t}{\Delta x}\left[2c + \frac{2}{3}||e^n||_{\ell^\infty} + \frac{3}{2}||u_\Delta^n||_{\ell^\infty}\right]\right) + \frac{\Delta t^2}{\Delta x^2}||D_+ (u_\Delta)^n||_{\ell^\infty}^2 + \frac{\Delta t}{\Delta x}\left(||(u_\Delta)^n||_{\ell^\infty}^2 + 2c^2\right),$$

$$C_b^{\text{Burgers}} = \left(\frac{\Delta x}{6} D_+ (e)_j^n - c\right)(\Delta x - c\Delta t),$$

and

$$C_c^{\text{Burgers}} = ||e^n||_{\ell^\infty}^2 [1 + \Delta x] + ||u_\Delta^n||_{\ell^\infty}^2 - c^2 + 2||e^n||_{\ell^\infty}||u_\Delta^n||_{\ell^\infty} + \frac{2c}{3}||e^n||_{\ell^\infty}.$$

Therefore, for $u_0 \in H^{\frac{3}{2}}(\mathbb{R})$ and for Δx small enough, the well-known CFL condition is verified

$$c\Delta t \leq \Delta x,$$

(thanks to the condition $C_b^{\text{Burgers}} \leq 0$) and the well-known condition for the Rusanov coefficient is verified

$$||u_\Delta^n||_{\ell^\infty} < c,$$

(thanks to the condition $C_c^{\text{Burgers}} \leq 0$).

Remark 18. *For Burgers equation, we know a natural bound for the convergence error: thanks to the maximum principle one has $||e^n||_{\ell^\infty} \leq 2||u_0||_{L^\infty}$.*

6 Convergence for less smooth initial data

In this section, we relax the hypothesis $u_0 \in H^6(\mathbb{R})$ and adapt the previous proof for any solution in $H^{\frac{3}{4}}(\mathbb{R})$ to obtain Theorem 2. When u_0 is not smooth enough to verify $u_0 \in H^6(\mathbb{R})$, we regularize it thanks to mollifiers $\left(\varphi^\delta\right)_{\delta>0}$, as explained in Introduction. Recall that we denote the mollifiers by $(\varphi^\delta)_{\delta>0}$, whose construction is based on χ a $\mathcal{C}^\infty$-function such that $\chi \equiv 1$ on $[-\frac{1}{2}, \frac{1}{2}]$, χ is supported in $[-1,1]$ and $\chi(\xi) = \chi(-\xi)$. We denote the exact solution from u_0 by u, the exact solution from $u_0 \star \varphi^\delta$ by u^δ and the numerical solution from (16) by $(v_j^n)_{(n,j)\in[\![0,N]\!]\times\mathbb{Z}}$.

6.1 Approximation results

We need to quantify the dependence of the Sobolev norms of the solution u^δ on δ. That result is gathered in Proposition 6 whose proof needs the following lemma.

Lemma 9. *Assume $(m,s)\in\mathbb{R}^2$ with $m\geq s\geq 0$. There exists a constant $C>0$ such that, if $u_0\in H^s(\mathbb{R})$ and $\delta>0$ and u_0^δ is such as $u_0^\delta=u_0\star\varphi^\delta$, then*

$$||u_0^\delta||_{H^m(\mathbb{R})}\leq\frac{C}{\delta^{m-s}}||u_0||_{H^s(\mathbb{R})}. \tag{69}$$

Proof. According to (6), the $H^m(\mathbb{R})$-norm of u_0^δ verifies

$$\begin{aligned}||u_0\star\varphi^\delta||^2_{H^m(\mathbb{R})}&=\int_{\mathbb{R}}\left(1+|\xi|^2\right)^m|\chi\left(\delta\xi\right)|^2|\widehat{u_0}\left(\xi\right)|^2d\xi\\&\leq\int_{\mathbb{R}}\left(1+|\xi|^2\right)^s|\widehat{u_0}|^2\left(1+|\xi|^2\right)^{m-s}|\chi\left(\delta\xi\right)|^2d\xi.\end{aligned}$$

By hypothesis on χ and its support, one has $|\chi\left(\delta\xi\right)|\leq 1$ and there exists a constant $C>0$ such that $\left(1+|\xi|^2\right)^{m-s}|\chi(\delta\xi)|^2\leq\frac{C}{\delta^{2(m-s)}}$, which concludes the proof. □

We are now able to estimate the Sobolev norms of u^δ.

Proposition 6. *Assume $m\geq s\geq 0$ and $u_0\in H^s(\mathbb{R})$ then,*

$$\sup_{t\in[0,T]}||u^\delta(t,.)||_{H^m(\mathbb{R})}\leq Ce^{\kappa_m T}\frac{||u_0||_{H^s(\mathbb{R})}}{\delta^{m-s}},$$

where C is a number which depends on m and κ_m depends on $\|u_0\|_{L^2}$ and m. Both are independent of δ.

Proof. We combine Theorem 3 and Lemma 9. □

We need then to know the rate of convergence of u_0^δ toward u_0 with respect to δ (as δ tends to 0), which is summarized as follows.

Lemma 10. *Assume $u_0\in H^s(\mathbb{R})$ with $0\leq\ell\leq s$, then, there exists a number C independent of δ such that*

$$||u_0-u_0^\delta||_{H^\ell(\mathbb{R})}\leq C\delta^{s-\ell}||u_0||_{H^s(\mathbb{R})}.$$

Proof. By definition of the $H^\ell(\mathbb{R})$-norm, we have, for $s\geq\ell$:

$$\begin{aligned}||u_0-u_0^\delta||^2_{H^\ell(\mathbb{R})}&=\int_{\mathbb{R}}(1+|\xi|^2)^\ell|\widehat{u_0}(\xi)|^2\left(1-\chi(\delta\xi)\right)^2d\xi\\&=\delta^{2(s-\ell)}\int_{\mathbb{R}}(1+|\xi|^2)^\ell|\widehat{u_0}(\xi)|^2\left(\frac{1-\chi(\delta\xi)}{(\delta\xi)^{s-\ell}}\right)^2\xi^{2(s-\ell)}d\xi.\end{aligned}$$

Hypothesis on χ implies that $\sup_{z\in\mathbb{R}}\left|\frac{1-\chi(z)}{z^{s-\ell}}\right|\leq C_2$ for a certain constant C_2. Hence, by using the inequality $(1+|\xi|^2)^\ell|\xi|^{2(s-\ell)}\leq C(1+|\xi|^2)^s$, with C a constant,

$$\begin{aligned}||u_0-u_0^\delta||^2_{H^\ell(\mathbb{R})}&\leq\delta^{2(s-\ell)}CC_2^2\int_{\mathbb{R}}\left(1+|\xi|^2\right)^s|\widehat{u_0}(\xi)|^2d\xi\\&\leq CC_2^2\delta^{2(s-\ell)}||u_0||^2_{H^s(\mathbb{R})}.\end{aligned}$$

□

6.2 Proof of Theorem 2

Let $s \geq \frac{3}{4}$. Assume $u_0 \in H^s(\mathbb{R})$, $T > 0$ and c such that (12) is true, which implies the existence of α_0 as in Remark 13. We construct $u_0^\delta = u_0 \star \varphi^\delta$ as previously.
Let $\beta_0 \in (0,1)$, $\theta \in [0,1]$ and $(v_j^n)_{(n,j)\in[\![0,N]\!]\times\mathbb{Z}}$ the unknown of the numerical scheme (2)-(16). Thanks to Theorem 1, there exists $\widehat{\omega_0} > 0$ such that for every $\Delta x \leq \widehat{\omega_0}$ and Δt satisfying CFL conditions (13a)-(13b), one has

$$||v^n - (u_\Delta^\delta)^n||_{\ell^2_\Delta} \leq \Lambda_{T,\|u_0^\delta\|_{H^{\frac{3}{4}}}} \left(1 + \|u_0^\delta\|^2_{H^{\frac{1}{2}+\eta}}\right)\left(\frac{\|u_0^\delta\|_{H^6}}{c+\frac{1}{2}} + \|u_0^\delta\|_{H^4} + \|u_0^\delta\|_{H^{\frac{3}{2}+\eta}}\|u_0^\delta\|_{H^1}\right)\Delta x,$$

with $\Lambda_{T,\|u_0^\delta\|_{H^{\frac{3}{4}}}}$ defined by (15).

Remark 19. *Thanks to Theorem 1, $\widehat{\omega_0}$ is such that $\widehat{\omega_0} \leq \tilde{\omega}_0$ with*

$$\tilde{\omega}_0 = \left[\Lambda_{T,\|u_0^\delta\|_{H^{\frac{3}{4}}}} \left(1 + \|u_0^\delta\|^2_{H^{\frac{1}{2}+\eta}}\right)\left(\frac{\|u_0^\delta\|_{H^6}}{c+\frac{1}{2}} + \|u_0^\delta\|_{H^4} + \|u_0^\delta\|_{H^{\frac{3}{2}+\eta}}\|u_0^\delta\|_{H^1}\right)\right]^{-\frac{1}{\gamma}}, \tag{70}$$

where $\Lambda_{T,\|u_0^\delta\|_{H^{\frac{3}{4}}}}$ is defined in (15).

By using a triangle inequality between the analytical solution starting from u_0 and the one starting from u_0^δ, the global error is upper bounded by

$$||e^n||_{\ell^2_\Delta} = ||v^n - (u_\Delta)^n||_{\ell^2_\Delta} \leq \sqrt{[B_1]^n} + \sqrt{[B_2]^n},$$

with

$$[B_1]^n = \sum_{j\in\mathbb{Z}} \Delta x \left(\frac{1}{\Delta x[\inf(t^{n+1},T) - t^n]}\int_{t^n}^{\inf(t^{n+1},T)}\int_{x_j}^{x_{j+1}} u(s,x) - u^\delta(s,x)dxds\right)^2 = \sum_{j\in\mathbb{Z}} \Delta x \left((u_\Delta)_j^n - [u_\Delta^\delta]_j^n\right)^2,$$

with the notation (9), and

$$[B_2]^n = \sum_{j\in\mathbb{Z}} \Delta x \left(\frac{1}{\Delta x[\inf(t^{n+1},T) - t^n]}\int_{t^n}^{\inf(t^{n+1},T)}\int_{x_j}^{x_{j+1}} u^\delta(s,x)dxds - v_j^n\right)^2 = \sum_{j\in\mathbb{Z}} \Delta x \left([u_\Delta^\delta]_j^n - v_j^n\right)^2.$$

Let us first focus on term $[B_1]^n$. The Cauchy-Schwarz inequality implies $[B_1]^n \leq \sup_{t\in[0,T]} ||u(t,.) - u^\delta(t,.)||^2_{L^2(\mathbb{R})}$, which imposes to study the difference between u and u^δ.
Since u and u^δ are two solutions of the initial equation (1a), one has

$$\partial_t\left(u - u^\delta\right) + \partial_x^3\left(u - u^\delta\right) + u\partial_x\left(u - u^\delta\right) + \left(u - u^\delta\right)\partial_x u^\delta = 0.$$

Multiplying by $\left(u - u^\delta\right)$, integrating the equation and changing u^δ in $u - \left(u - u^\delta\right)$ in the latest term yield

$$\frac{d}{dt}\int_{\mathbb{R}} \frac{\left(u(t,x) - u^\delta(t,x)\right)^2}{2}dx - \int_{\mathbb{R}} \partial_x u(t,x)\frac{\left(u(t,x) - u^\delta(t,x)\right)^2}{2}dx$$
$$+ \int_{\mathbb{R}} \left(u(t,x) - u^\delta(t,x)\right)^2 \partial_x\left[u(t,x) - \left(u(t,x) - u^\delta(t,x)\right)\right]dx = 0,$$

thus

$$\frac{d}{dt}\frac{||u(t,.) - u^\delta(t,.)||^2_{L^2(\mathbb{R})}}{2} \leq \frac{||\partial_x u(t,.)||_{L^\infty(\mathbb{R})}}{2}||u(t,.) - u^\delta(t,.)||^2_{L^2(\mathbb{R})}.$$

The previous inequality looks like the 'weak-strong uniqueness' of DiPerna [DiP79] or Dafermos [Daf79, Daf10]. The $L^2(\mathbb{R})$-norm of the difference $u - u^\delta$ is then upper bounded by

$$||u(t,.) - u^\delta(t,.)||^2_{L^2(\mathbb{R})} \leq \exp\left(\int_0^t \frac{||\partial_x u(s,.)||_{L^\infty(\mathbb{R})}}{2}ds\right)||u_0 - u_0^\delta||^2_{L^2(\mathbb{R})}$$
$$\leq \exp\left(\frac{T^{\frac{3}{4}}C_{\frac{3}{4}}e^{\kappa_{\frac{3}{4}}T}}{2}\|u_0\|_{H^{\frac{3}{4}}}\right)||u_0 - u_0^\delta||^2_{L^2(\mathbb{R})},$$

where $\kappa_{\frac{3}{4}}$ and $C_{\frac{3}{4}}$ are defined in Theorem 3. Then

$$[B_1]^n \leq \sup_{t\in[0,T]} ||u(t,.) - u^\delta(t,.)||^2_{L^2(\mathbb{R})} \leq \exp\left(\frac{T^{\frac{3}{4}} C_{\frac{3}{4}} e^{\kappa_{\frac{3}{4}} T}}{2} ||u_0||_{H^{\frac{3}{4}}}\right) ||u_0 - u_0^\delta||^2_{L^2(\mathbb{R})}.$$

Lemma 10 implies

$$[B_1]^n \leq C^2 \delta^{2s} ||u_0||^2_{H^s(\mathbb{R})} \exp\left(\frac{T^{\frac{3}{4}} C_{\frac{3}{4}} e^{\kappa_{\frac{3}{4}} T}}{2} ||u_0||_{H^{\frac{3}{4}}}\right). \tag{71}$$

In the other hand, the term $[B_2]^n$ corresponds to the estimate (66) derived in Subsection 5.3 with a smooth initial datum. It remains us to quantify the dependency of it upper bound with respect to δ. Thanks to Theorem 1, one has

$$\sqrt{[B_2]^n} \leq \Lambda_{T,\|u_0^\delta\|_{H^{\frac{3}{4}}}} \left(1 + \|u_0^\delta\|^2_{H^{\frac{1}{2}+\eta}}\right) \left(\frac{\|u_0^\delta\|_{H^6}}{c + \frac{1}{2}} + \|u_0^\delta\|_{H^4} + \|u_0^\delta\|_{H^{\frac{3}{2}+\eta}} \|u_0^\delta\|_{H^1}\right) \Delta x,$$

with $\Lambda_{T,\|u_0^\delta\|_{H^{\frac{3}{4}}}}$ defined by (15). As u_0 belongs to $H^s(\mathbb{R})$ with $s \geq \frac{3}{4}$, then $||u_0^\delta||_{H^{\frac{3}{4}}} = ||u_0||_{H^{\frac{3}{4}}}$ and $||u_0^\delta||_{H^{\frac{1}{2}+\eta}} = ||u_0||_{H^{\frac{1}{2}+\eta}}$.

Lemma 11. *For every $s \geq \frac{3}{4}$, there exists C, depending only on s and on $\|u_0\|_{L^2}$, such that, if $u_0 \in H^s(\mathbb{R})$,*

$$\frac{\|u_0^\delta\|_{H^6}}{c + \frac{1}{2}} + \|u_0^\delta\|_{H^4} + \|u_0^\delta\|_{H^{\frac{3}{2}+\eta}} \|u_0^\delta\|_{H^1} \leq \frac{\|u_0\|_{H^s}}{\delta^{6-s}} C \left(\frac{1}{c + \frac{1}{2}} + 1 + \|u_0\|_{H^{\min(1,s)}}\right).$$

Proof. We apply Lemma 9 with $s = 6, 4, \frac{3}{2} + \eta, 1$ and the biggest power of δ is $\frac{1}{\delta^{6-s}}$. $\square$

Thus, an upper bound for $[B_2]^n$ is

$$\sqrt{[B_2]^n} \leq \Lambda_{T,\|u_0\|_{H^{\frac{3}{4}}}} \left(1 + \|u_0\|^2_{H^{\frac{1}{2}+\eta}}\right) \left(\frac{1}{c + \frac{1}{2}} + 1 + \|u_0\|_{H^{\min(1,s)}}\right) C \frac{\|u_0\|_{H^s}}{\delta^{6-s}} \Delta x.$$

However, for Theorem 1 to be applied, we need to choose a small Δx such that $\Delta x \leq \widehat{\omega_0} \leq \tilde{\omega}_0$ where $\tilde{\omega}_0$ is defined by (70). With the above lemma, this condition rewrites

$$\Delta x \leq \left[\Lambda_{T,\|u_0\|_{H^{\frac{3}{4}}}} \left(1 + \|u_0\|^2_{H^{\frac{1}{2}+\eta}}\right) \left(\frac{1}{c + \frac{1}{2}} + 1 + \|u_0\|_{H^{\min(1,s)}}\right) C \frac{\|u_0\|_{H^s}}{\delta^{6-s}}\right]^{-\frac{1}{\gamma}} = \left(\frac{\tilde{C}}{\delta^{6-s}}\right)^{-\frac{1}{\gamma}}. \tag{72}$$

Hence, if $\Delta x \leq \left(\frac{\tilde{C}}{\delta^{6-s}}\right)^{-\frac{1}{\gamma}}$, with $\tilde{C} > 0$ a constant depending on T, c, β_0, $||u_0||_{H^s}$ and $||u_0||_{H^{\frac{3}{4}}}$ and if CFL conditions (13a)-(13b) are verified, the convergence error $(e_j^n)_{(n,j)}$ is upper bounded by

$$\begin{aligned} &||e^n||_{\ell^2_\Delta} \\ &\leq C\left[\Lambda_{T,\|u_0\|_{H^{\frac{3}{4}}}} \left(1 + \|u_0\|^2_{H^{\frac{1}{2}+\eta}}\right) \left(\frac{1}{c + \frac{1}{2}} + 1 + \|u_0\|_{H^{\min(1,s)}}\right) + \exp\left(\frac{T^{\frac{3}{4}} C_{\frac{3}{4}} e^{\kappa_{\frac{3}{4}} T}}{4} ||u_0||_{H^{\frac{3}{4}}}\right)\right] \|u_0\|_{H^s} \left[\frac{\Delta x}{\delta^{6-s}} + \delta^s\right], \end{aligned} \tag{73}$$

for $n \in [\![0, N]\!]$.

The final key point is to find the optimal δ, in other words, the parameter δ which makes both terms δ^s, coming from $\sqrt{[B_1]^n}$ and $\frac{\Delta x}{\delta^{6-s}}$ coming from $\sqrt{[B_2]^n}$ in (73) equal while respecting the constraint (72). Defining $\delta = \Delta x^a$ summarizes the problem in the following system

$$\begin{cases} \text{Find } a \text{ such that } : \Delta x^{as} = \dfrac{\Delta x}{\Delta x^{a(6-s)}}, \\ \text{under the constraint } : \dfrac{1}{\Delta x^{a(6-s)}} < \dfrac{1}{\Delta x^\gamma}. \end{cases}$$

Three cases have to be considered:

- if $\frac{3}{4} \leq s \leq 6-6\gamma$, the constraint is binding and we have to choose a which transforms the constraint inequality in an equality : $a = \frac{\gamma}{6-s}$. In that case, the rate of convergence is given by the smallest term between Δx^{as} and $\frac{\Delta x}{\Delta x^{a(6-s)}}$ *i.e.* $\Delta x^{\frac{\gamma s}{6-s}}$.
- If $6-6\gamma \leq s \leq 6$, $a = \frac{1}{6}$ enables both terms Δx^{as} and $\frac{\Delta x}{\Delta x^{a(6-s)}}$ to be equal without violating the constraint. This choice of a gives a rate of convergence of $\Delta x^{\frac{s}{6}}$.
- If $s \geq 6$, the result of the Theorem 1 applies.

Since γ is in $(0, \frac{1}{2})$ (cf. Lemma 7 and induction hypothesis (59)), we take the optimal γ : $\gamma = \frac{1}{2} - \eta$ with η small and $\eta > 0$. The conclusion of the theorem is straightforward consequence.

Remark 20. *The choice of δ is independent of the regularity s of the initial datum, if $3 \leq s \leq 6$.*

7 Numerical results

In this section, the previous results are illustrated numerically by some examples and the numerical convergence rates are computed for the KdV equation.

7.1 Convergence rates

Through the rest of the paper, the computations are performed with an implicit scheme $\theta = 1$ in order to avoid the dispersive CFL condition. Our purpose is to gauge the relevance of our theoretical results on the rate of convergence with respect to Δx. To this end, the time step is chosen according to the hyperbolic CFL condition. More precisely, c is numerically chosen such that $c^n = \max\limits_{k\in[\![0,n]\!], j\in[\![1,J]\!]} |v_j^k|$ and $\Delta t^n = \frac{\Delta x}{c^n}$. This choice seems surprising related to the CFL of Theorems 1 and 2 but, as explain in Remark 4, the condition $[c + \frac{1}{2}]\Delta t < \Delta x$ seems technical and may be replaced by the classical one $c\Delta t \leq \Delta x$. Eventually, we fix the final time $T = 0.1$.

We can not simulate numerical solutions on $\mathbb{Z}$ as done in the theoretical results. We have to take into account numerical boundaries: we use periodic initial data. We fix the space domain to $[0, L]$ with $L = 50$ (except for cnoidal wave where $L = 1$) and fix $J = \lfloor \frac{L}{\Delta x} \rfloor$.

Remark 21. *Notice that the theoretical results do not apply rigorously since the solutions do not belong to $H^s(\mathbb{R})$ because of their periodicity.*

The convergence rate r_J is computed with

$$r_J = \frac{\log(E_J) - \log(E_{2J})}{\log(2)}.$$

When the exact solution is known (e.g. for the cnoidal-wave solution), the variable E_J corresponds to the following difference

$$E_J = \sup_{n\in[\![0,N]\!]} \left|\left| \left(e_j^n\right)_{j\in\ [\![0,J]\!]} \right|\right|_{\ell^2_\Delta} = \sup_{n\in[\![0,N]\!]} \left|\left| \left(v_j^n\right)_{j\in[\![0,J]\!]} - \left([u_\Delta]_j^n\right)_{j\in\ [\![0,J]\!]} \right|\right|_{\ell^2_\Delta},$$

with $(v_j^n)_{j\in[\![0,J]\!]}$ the numerical solution computed with J cells in space and $\left([u_\Delta]_j^n\right)_{j\in\ [\![0,J]\!]}$ the J-piecewise constant function from the analytical solution. However, for test cases with an unknown exact solution, the convergence error is computed from two numerical solutions with a different mesh and E_J is replaced by the following $\tilde{E}_J$

$$\tilde{E}_J = \sup_{n\in[\![0,N]\!]} \left|\left| \left(v_j^n\right)_{j\in[\![0,J]\!]} - \left(\tilde{v}_j^n\right)_{j\in[\![0,J]\!]} \right|\right|.$$

Previously, $\tilde{v}_j^n := w_{2j}^n$ for all $j \in [\![0, J]\!]$, where $\left(w_j^n\right)_{j\in[\![0,2J]\!]}$ is a numerical solution computed with a refined grid with $2J$ space meshes. In that case, $\left(\tilde{v}_j^n\right)_{j\in[\![0,J]\!]}$, computed from the refined numerical solution $\left(w_j^n\right)_{j\in[\![0,2J]\!]}$, plays the role of the exact one $\left([u_\Delta]_j^n\right)_{j\in[\![0,J]\!]}$.

Remark 22. *We have considered exactly $2J$ space meshes for the refined numerical solution. Moreover, in all cases, 0 is a grid node, therefore, the error $\tilde{E}_J$ is nothing but the difference between every two consecutive values for $\left(w_j^n\right)_{j\in[\![0,2J]\!]}$ and every value for $\left(v_j^n\right)_{j\in[\![0,J]\!]}$.*

7.2 Smooth initial data

To assess the optimality of Theorem 1, the corresponding test cases are carried out with two smooth periodic initial data, either the sinusoidal initial datum

$$u_0(x) = \cos\left(\frac{2\pi}{L}x\right),$$

or the so-called cnoidal-wave initial datum. This cnoidal-wave solution represents a periodic solitary wave solution of the Korteweg-de Vries equation whose analytical expression is known as follow:

$$u(t,x) = \frac{1}{\mu^{\frac{1}{5}}} a\,\mathrm{cn}^2\left(4K(m)\left(\mu^{\frac{2}{5}}\left(x-\frac{L}{2}\right) - v\mu^{\frac{1}{5}}t\right)\right),$$

where $\mu = \frac{1}{24^2}$ and $\mathrm{cn}(z) = \mathrm{cn}(z:m)$ is the Jacobi elliptic function with modulus $m \in (0,1)$ (we choose $m = 0.9$) and the parameters have the values $a = 192m\mu K(m)^2$ and $v = 64\mu(2m-1)K(m)^2$. $K(m)$ is the complete elliptic integral of the first kind (cf [BCKX13]).
Both results are gathered in Figure 1 for sinusoidal solution and Figure 2 for cnoidal-wave solution. We display the values of r with respect to J in the left table and post the corresponding graph in logarithmic scale on the right. The first order is confirmed for both initial data whether in tables or in graphs.

Sinusoidal solution $u_0(x)=\cos(\frac{2\pi}{L}x)$			
J	Δx	error in $\ell^\infty(0,T,\ell^2_\Delta(\mathbb{Z}))$ computed with E_J	numerical order
1600	$3,1250.10^{-2}$	$6,2062.10^{-5}$	
3200	$1,5625.10^{-2}$	$3,1033.10^{-5}$	0.9999
6400	$7,8125.10^{-3}$	$1,5517.10^{-5}$	0.9999
12800	$3,9063.10^{-3}$	$8,0795.10^{-6}$	0.9415
25600	$1,9531.10^{-3}$	$4,1435.10^{-6}$	0.9634
51200	$9,7656.10^{-4}$	$1,9974.10^{-6}$	1.0527

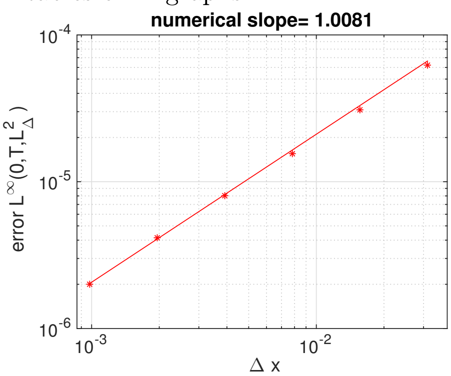

Figure 1: Experimental rate of convergence for sinusoidal solution

Cnoidal-wave			
J	Δx	error in $\ell^\infty(0,T,\ell^2_\Delta(\mathbb{Z}))$ computed with E_J	numerical order
1600	$6.2500.10^{-4}$	$8.9875.10^{-4}$	
3200	$3.1250.10^{-4}$	$4.5253.10^{-4}$	0.9899
6400	$1.5625.10^{-4}$	$2.2636.10^{-4}$	0.9994
12800	$7.8125.10^{-5}$	$1.1292.10^{-4}$	1.0034
25600	$3.9062.10^{-5}$	$5.7102.10^{-5}$	0.9837

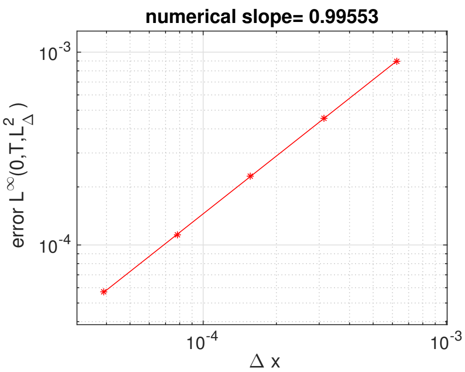

Figure 2: Experimental rate of convergence for cnoidal-wave solution

7.3 For a regularity between $H^3(\mathbb{R})$ and $H^{10}(\mathbb{R})$

To illustrate numerically Theorem 2, we initialize the scheme with a less regular initial datum. We test two kinds of periodic initial data in $H^s(\mathbb{R})$, with $s \geq 3$. The first one is computed from some integrations of the indicator function and leads to initial data in $H^s(\mathbb{R})$ with s a half-integer. The second one is computed from the periodized square root function and leads to initial data in $H^s(\mathbb{R})$ with s an integer.

Tests achieved from the indicator function. Since the indicator function $\mathbb{1}_{[0,\frac{L}{2}]}$ belongs to $H^s([0,L])$ for all $s < \frac{1}{2}$, an idea to construct a periodic function in $H^{s+\ell}([0,L])$, with $s < \frac{1}{2}$ and $\ell \in \mathbb{N}^*$ is to integrate ℓ times the periodic indicator function. For instance, after a first integration, the initial datum

$$u_0(x) = x\mathbb{1}_{[0,\frac{L}{2}]} + (L-x)\mathbb{1}_{[\frac{L}{2},L]}$$

is periodic and "almost" in $H^{\frac{3}{2}}([0,L])$. By reiterating the process of periodization and integration, we obtain initial data in $H^s([0,L])$, with $s = \frac{7}{2}^-, \frac{9}{2}^-, \frac{11}{2}^-$... The numerical results are summarized in the following table.

Sobolev regularity	$s = \frac{7}{2}^-$	$s = \frac{9}{2}^-$	$s = \frac{11}{2}^-$
Experimental convergence rate	0.66016	0.78307	0.97340
Theoretical convergence rate	0.58333	0.75000	0.91667

We detail some tests below : $s = \frac{7}{2}^-$ in Figure 3, and $s = \frac{9}{2}^-$ in Figure 4. For each figure, the graph on the right illustrates the error in $\ell^\infty(0,T,\ell^2_\Delta(\mathbb{Z}))$ with respect to the mesh size Δx in logarithmic scale. Those errors and those mesh sizes are detailed in the table on the left. On the graph, we have displayed the points cloud but also the least squares regression and the numerical slope of that regression line is considered to be the experimental convergence rate. For more simplicity, we denote $H^{s-}(\mathbb{R})$ the space $H^{s-\ell}(\mathbb{R})$ with $\ell > 0$ as small as possible.

$u_0 \in H^{\frac{7}{2}-}([0,L])$			
J	Δx	error in $\ell^\infty(0,T,\ell^2_\Delta(\mathbb{Z}))$ computed with $\tilde{E}_J$	numerical order
12800	$3.9063.10^{-3}$	$8.8562.10^{-6}$	
25600	$1.9531.10^{-3}$	$5.5291.10^{-6}$	0.6796
51200	$9.7656.10^{-4}$	$3.4903.10^{-6}$	0.6637
102400	$4.8828.10^{-4}$	$2.2460.10^{-6}$	0.6360

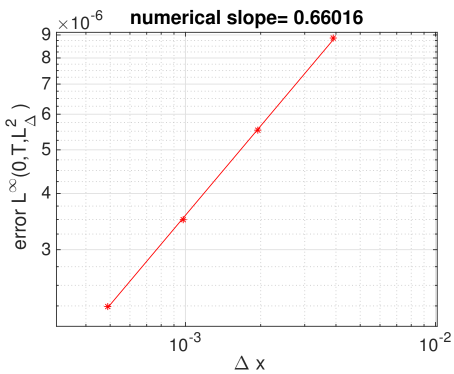

Figure 3: Experimental rate of convergence for $u_0 \in H^{\frac{7}{2}-}([0,L])$

$u_0 \in H^{\frac{9}{2}-}([0,L])$			
J	Δx	error in $\ell^\infty(0,T,\ell^2_\Delta(\mathbb{Z}))$ computed with $\tilde{E}_J$	numerical order
1600	$3.1250.10^{-2}$	$6.5105.10^{-3}$	
3200	$1.5625.10^{-2}$	$3.9541.10^{-3}$	0.71941
6400	$7.8125.10^{-3}$	$2.2620.10^{-3}$	0.80574
12800	$3.9063.10^{-3}$	$1.3091.10^{-3}$	0.78909
25600	$1.9531.10^{-3}$	$7.4923.10^{-4}$	0.80504
51200	$9.7656.10^{-4}$	$4.4105.10^{-4}$	0.764470

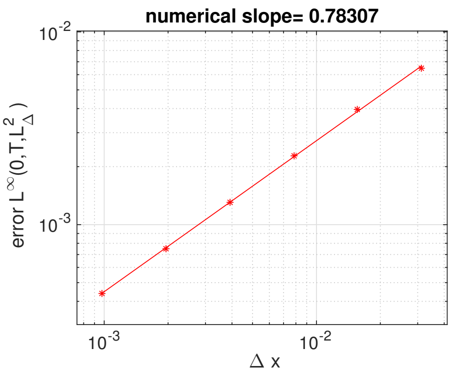

Figure 4: Experimental rate of convergence for $u_0 \in H^{\frac{9}{2}-}([0,L])$

Remark 23. *The relative error between the experimental rate and the theoretical one is sometimes significant, for example, this relative error is more than 12% in the case $s = \frac{7}{2}-$. However, the theoretical rate is an asymptotic result for Δx and Δt small enough. As seen in the table on the left of Figure 3, the numerical order appears to be decreasing when Δx decreases, which agrees closely with the theoretical results.*

Tests achieved from the square root function. Since the square root function is in $H^{1-}([0, L])$ we construct a $H^{s-}([0, L])$ function by integrating the square root function $s - 1$ times. However, we need, in addition, a periodic initial datum, this is why we add the beginning of a Taylor expansion for the function and its derivatives up to $(s - 1)$-th to be continuous and periodic. More precisely, we search the coefficients b_i, $i \in [\![1, s]\!]$ such that the function

$$x^{s-1+\frac{1}{2}} - b_1 x - \frac{b_2}{2}x^2 - \frac{b_3}{3!}x^3 ... - \frac{b_s}{s!}x^s$$

and all its derivatives up to $(s - 1)$-th be equal for $x = 0$ and for $x = L$. To find those coefficients, we just have to solve the matrix system

$$\begin{pmatrix} L & \frac{L^2}{2} & \frac{L^3}{3!} & \frac{L^4}{4!} & \dots & \frac{L^s}{s!} \\ 0 & L & \frac{L^2}{2} & \frac{L^3}{3!} & \dots & \frac{L^{s-1}}{(s-1)!} \\ 0 & 0 & L & \frac{L^2}{2} & \dots & \frac{L^{s-2}}{(s-2)!} \\ & & & \ddots & & \\ 0 & 0 & 0 & 0 & \dots & L \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_s \end{pmatrix} = \begin{pmatrix} L^{s-\frac{1}{2}} \\ (s-\frac{1}{2})L^{s-\frac{3}{2}} \\ (s-\frac{1}{2})(s-\frac{3}{2})L^{s-\frac{5}{2}} \\ \vdots \\ \vdots \\ (s-\frac{1}{2})(s-\frac{3}{2})(s-\frac{5}{2})\dots\frac{3}{2}L^{1/2} \end{pmatrix}.$$

The following tests correspond to $s \in [\![4, 10]\!]$.

The comparison between the experimental rates of convergence and the theoretical ones are very closed and gathered in the following table.

Sobolev regularity	$s = 4$	$s = 5$	$s = 6$	$s = 7$	$s = 8$	$s = 9$	$s = 10$
Experimental convergence rate	0.67225	0.86032	0.98708	0.99485	1.0060	1.0148	1.0062
Theoretical convergence rate	0.66667	0.83333	1.0000	1.0000	1.0000	1.0000	1.0000

Some results are developed below ($u_0 \in H^{4-}(\mathbb{R})$ in Figure 5 and $u_0 \in H^{5-}(\mathbb{R})$ in Figure 6).

$u_0 \in H^{4-}([0,L])$			
J	Δx	error in $\ell^\infty(0,T,\ell^2_\Delta(\mathbb{Z}))$ computed with $\tilde{E}_J$	numerical order
1600	$3.1250.10^{-2}$	$4.6454.10^{-3}$	
3200	$1.5625.10^{-2}$	$2.8109.10^{-3}$	0.72476
6400	$7.8125.10^{-3}$	$1.7147.10^{-3}$	0.71307
12800	$3.9063.10^{-3}$	$1.0892.10^{-3}$	0.65474
25600	$1.9531.10^{-3}$	$6.8793.10^{-4}$	0.66290
51200	$9.7656.10^{-4}$	$4.3185.10^{-4}$	0.67172

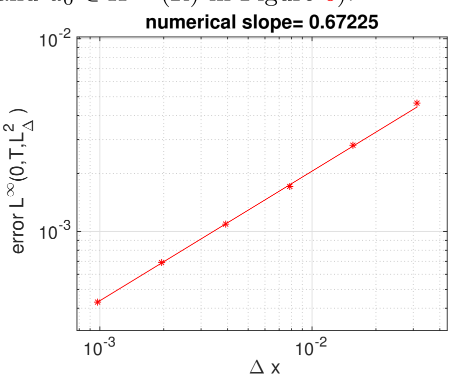

Figure 5: Experimental rate of convergence for $u_0 \in H^{4-}([0, L])$

$u_0 \in H^{5-}([0,L])$			
J	Δx	error in $\ell^\infty(0,T,\ell^2_\Delta(\mathbb{Z}))$ computed with $\tilde{E}_J$	numerical order
1600	$3.1250.10^{-2}$	$1.1066.10^{-2}$	
3200	$1.5625.10^{-2}$	$6.0878.10^{-3}$	0.86208
6400	$7.8125.10^{-3}$	$3.3067.10^{-3}$	0.88054
12800	$3.9063.10^{-3}$	$1.8609.10^{-3}$	0.82941
25600	$1.9531.10^{-3}$	$1.0240.10^{-3}$	0.86182
51200	$9.7656.10^{-4}$	$5.5475.10^{-4}$	0.88424

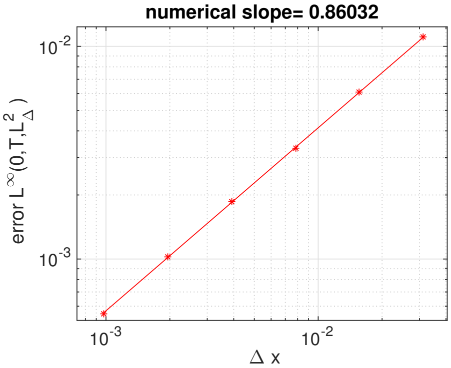

Figure 6: Experimental rate of convergence for $u_0 \in H^{5-}([0,L])$

7.4 For a regularity between $L^2(\mathbb{R})$ and $H^3(\mathbb{R})$

Theoretically, the necessity to bound $\int_0^T ||\partial_x u(s,.)||^i_{L^\infty(\mathbb{R})} ds$ in (64) forces to choose $s \geq \frac{3}{4}$. In addition, the necessity to bound $||e^n||_{\ell^\infty}$ in F_a in (50a) in order to apply the Grönwall lemma leads to choose Δx such as Equation (55) be true which forces to have the constraint $\frac{1}{\delta^{6-s}} < \frac{1}{\Delta x^\gamma}$ in (72). However, those restrictions may be only technical and the rate of convergence seems to be $\Delta x^{\frac{s}{6}}$ for all $s \in [0,3)$. This is based on the following numerical results.

Tests achieved from the indicator function. The integration from the indicator function enables us to obtain examples of functions in $H^s(\mathbb{R})$ with $s = \frac{1}{2}^-, \frac{3}{2}^-, \frac{5}{2}^-$, whose results are listed below.

Sobolev regularity	$s = \frac{1}{2}^-$	$s = \frac{3}{2}^-$	$s = \frac{5}{2}^-$
Experimental convergence rate	0.08795	0.25500	0.42595
"Conjectured" convergence rate	0.08333	0.25000	0.41667

More precisions are given in Figures 7 (for $u_0 \in H^{\frac{1}{2}-}(\mathbb{R})$), 8 (for $u_0 \in H^{\frac{3}{2}-}(\mathbb{R})$) and 9 (for $u_0 \in H^{\frac{5}{2}-}(\mathbb{R})$).

$u_0 \in H^{\frac{1}{2}-}([0,L])$			
J	Δx	error in $\ell^\infty(0,T,\ell^2_\Delta(\mathbb{Z}))$ computed with $\tilde{E}_J$	numerical order
3200	$1.5625.10^{-2}$	$1.0567.10^{-2}$	
6400	$7.8125.10^{-3}$	$9.8843.10^{-3}$	0.0964
12800	$3.9063.10^{-3}$	$9.2992.10^{-3}$	0.0880
25600	$1.9531.10^{-3}$	$8.7490.10^{-3}$	0.0879
51200	$9.7656.10^{-4}$	$8.2289.10^{-3}$	0.0885
102400	$4.8828.10^{-4}$	$7.7468.10^{-3}$	0.0871

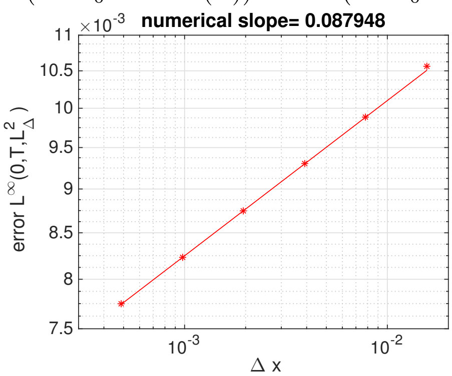

Figure 7: Experimental rate of convergence for $u_0 \in H^{\frac{1}{2}-}([0,L])$

$u_0 \in H^{\frac{3}{2}-}([0,L])$			
J	Δx	error in $\ell^\infty(0,T,\ell^2_\Delta(\mathbb{Z}))$ computed with $\tilde{E}_J$	numerical order
3200	$1.5625.10^{-2}$	$2.6584.10^{-4}$	
6400	$7.8125.10^{-3}$	$2.2108.10^{-4}$	0.2660
12800	$3.9063.10^{-3}$	$1.8624.10^{-4}$	0.2474
25600	$1.9531.10^{-3}$	$1.5632.10^{-4}$	0.2526
51200	$9.7656.10^{-4}$	$1.3074.10^{-4}$	0.2578
102400	$4.8828.10^{-4}$	$1.0949.10^{-4}$	0.2560

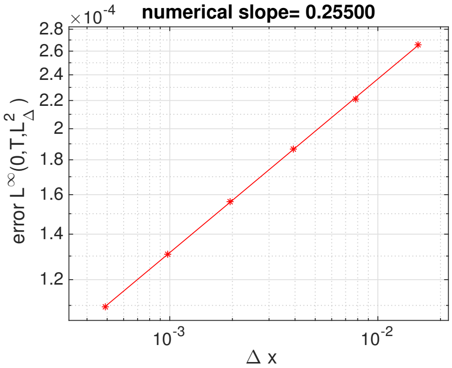

Figure 8: Experimental rate of convergence for $u_0 \in H^{\frac{3}{2}-}([0,L])$

$u_0 \in H^{\frac{5}{2}-}([0,L])$			
J	Δx	error in $\ell^\infty(0,T,\ell^2_\Delta(\mathbb{Z}))$ computed with $\tilde{E}_J$	numerical order
3200	$1.5625.10^{-2}$	$8.4546.10^{-5}$	
6400	$7.8125.10^{-3}$	$6.1968.10^{-5}$	0.4482
12800	$3.9063.10^{-3}$	$4.6531.10^{-5}$	0.4133
25600	$1.9531.10^{-3}$	$3.4745.10^{-5}$	0.4214
51200	$9.7656.10^{-4}$	$2.5795.10^{-5}$	0.4297
102400	$4.8828.10^{-4}$	$1.9199.10^{-5}$	0.4260

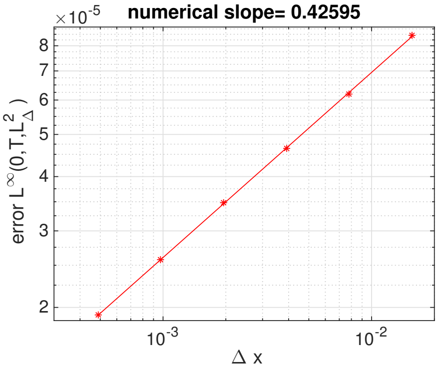

Figure 9: Experimental rate of convergence for $u_0 \in H^{\frac{5}{2}-}([0,L])$

Tests achieved from the square root function. As before, the second test consists in periodising the square root function to create an initial datum in $H^{s-}(\mathbb{R})$ with $s = 1, 2, 3$.

Sobolev regularity	$s = 1$	$s = 2$	$s = 3$
Experimental convergence rate	0.16984	0.33806	0.50173
"Conjectured" convergence rate	0.16667	0.33333	0.50000

Some precisions for those tests are given in Figures 10, 11 and 12.

$u_0 \in H^{1-}([0,L])$			
J	Δx	error in $\ell^\infty(0,T,\ell^2_\Delta(\mathbb{Z}))$ computed with $\tilde{E}_J$	numerical order
1600	$3.1250.10^{-2}$	$2.6762.10^{-2}$	
3200	$1.5625.10^{-2}$	$2.3501.10^{-2}$	0.18748
6400	$7.8125.10^{-3}$	$2.0793.10^{-2}$	0.17660
12800	$3.9063.10^{-3}$	$1.8595.10^{-2}$	0.16119
25600	$1.9531.10^{-3}$	$1.6602.10^{-2}$	0.16360
51200	$9.7656.10^{-4}$	$1.4787.10^{-2}$	0.16701

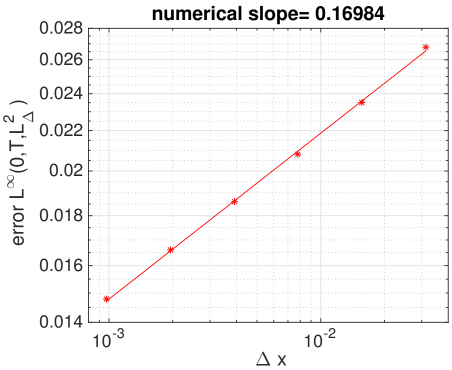

Figure 10: Experimental rate of convergence for $u_0 \in H^{1-}([0,L])$

$u_0 \in H^{2-}([0, L])$			
J	Δx	error in $\ell^\infty(0, T, \ell^2_\Delta(\mathbb{Z}))$ computed with $\tilde{E}_J$	numerical order
1600	$3.1250.10^{-2}$	$6.6322.10^{-3}$	
3200	$1.5625.10^{-2}$	$5.2115.10^{-3}$	0.34779
6400	$7.8125.10^{-3}$	$4.0950.10^{-3}$	0.34783
12800	$3.9063.10^{-3}$	$3.2699.10^{-3}$	0.32461
25600	$1.9531.10^{-3}$	$2.5937.10^{-3}$	0.33426
51200	$9.7656.10^{-4}$	$2.0449.10^{-3}$	0.34296

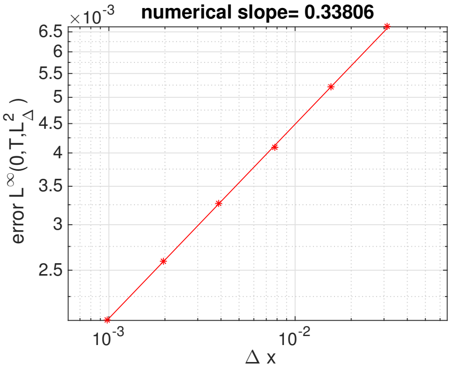

Figure 11: Experimental rate of convergence for $u_0 \in H^{2-}([0, L])$

$u_0 \in H^{3-}([0, L])$			
J	Δx	error in $\ell^\infty(0, T, \ell^2_\Delta(\mathbb{Z}))$ computed with $\tilde{E}_J$	numerical order
1600	$3.1250.10^{-2}$	$1.2295.10^{-2}$	
3200	$1.5625.10^{-2}$	$8.5386.10^{-3}$	0.52599
6400	$7.8125.10^{-3}$	$5.8964.10^{-3}$	0.53416
12800	$3.9063.10^{-3}$	$4.1913.10^{-3}$	0.49245
25600	$1.9531.10^{-3}$	$2.9576.10^{-3}$	0.50298
51200	$9.7656.10^{-4}$	$2.0778.10^{-3}$	0.50937

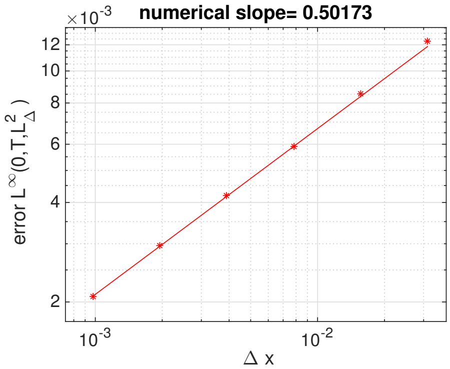

Figure 12: Experimental rate of convergence for $u_0 \in H^{3-}([0, L])$

We summarize the theoretical and numerical results in Figure 13. The blue line corresponds to the theoretical rate of convergence, the dashed line matches the conjectured rate and the red dots stand for the numerical rates of convergence. Both are intertwined, which validates the rate of convergence of $\frac{\min(s,6)}{6}$ with s the Sobolev regularity of the initial value.

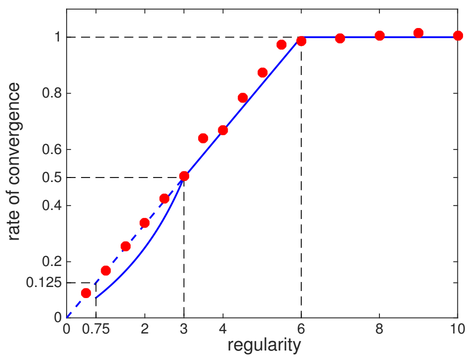

Figure 13: Rates of convergence according to the Sobolev regularity of u_0. – Rates proved in this paper (solid line) *versus* experimental rates (dots)

A Appendix : Consistency error

The consistency error is defined by (11). For simplicity, we note, in this appendix, $t^{n+1} = \inf(t^{n+1}, T)$ and $\Delta t = t^{n+1} - t^n$, which corresponds to the following expression, for $(n, j) \in [\![0, N]\!] \times \mathbb{Z}$

$$
\begin{aligned}
\epsilon_j^n = &\frac{1}{\Delta t^2 \Delta x} \int_{t^n}^{t^{n+1}} \int_{x_j}^{x_{j+1}} u(s+\Delta t, y) - u(s,y) dy ds \\
&+ \frac{1}{4\Delta x} \overbrace{\left[\left(\frac{1}{\Delta t \Delta x} \int_{t^n}^{t^{n+1}} \int_{x_j}^{x_{j+1}} u(s, y+\Delta x) dy ds \right)^2 - \left(\frac{1}{\Delta x \Delta t} \int_{t^n}^{t^{n+1}} \int_{x_j}^{x_{j+1}} u(s, y-\Delta x) dy ds \right)^2 \right]}^{=:(a)} \\
&+ \frac{1-\theta}{\Delta t \Delta x^4} \int_{t^n}^{t^{n+1}} \int_{x_j}^{x_{j+1}} u(s, y+2\Delta x) - 3u(s, y+\Delta x) + 3u(s,y) - u(s, y-\Delta x) dy ds \\
&+ \frac{\theta}{\Delta t \Delta x^4} \int_{t^{n+1}}^{t^{n+2}} \int_{x_j}^{x_{j+1}} u(s, y+2\Delta x) - 3u(s, y+\Delta x) + 3u(s,y) - u(s, y-\Delta x) dy ds \\
&- c \left(\frac{1}{2\Delta t \Delta x^2} \int_{t^n}^{t^{n+1}} \int_{x_j}^{x_{j+1}} u(s, y+\Delta x) - 2u(s,y) + u(s, y-\Delta x) dy ds \right).
\end{aligned}
\tag{74}
$$

Let us first deal with (a)-term. In order to simplify the following equations, we denote, for ν in $\mathbb{R}$

$$
K(\nu) := \left(\frac{1}{\Delta x \Delta t} \int_{x_j}^{x_{j+1}} \int_{t^n}^{t^{n+1}} u(s, y+\nu\Delta x) ds dy \right)^2.
$$

A straightforward computation yields

$$
\begin{aligned}
K'(0) &= \frac{2}{\Delta x \Delta t^2} \int_{x_j}^{x_{j+1}} \int_{t^n}^{t^{n+1}} \int_{x_j}^{x_{j+1}} \int_{t^n}^{t^{n+1}} \partial_x u(\bar{s}, \bar{y}) u(s,y) d\bar{s} d\bar{y} ds dy \\
&= \frac{2}{\Delta x \Delta t^2} \int_{x_j}^{x_{j+1}} \int_{t^n}^{t^{n+1}} \int_{x_j}^{x_{j+1}} \int_{t^n}^{t^{n+1}} \left[\partial_x u(s,y) + \int_y^{\bar{y}} \partial_x^2 u(s,v) dv + \int_s^{\bar{s}} \partial_{xt} u(\tau, \bar{y}) d\tau \right] u(s,y) d\bar{s} d\bar{y} ds dy \\
&= \frac{2}{\Delta t} \int_{x_j}^{x_{j+1}} \int_{t^n}^{t^{n+1}} u(s,y) \partial_x u(s,y) ds dy + \frac{2}{\Delta t \Delta x} \int_{x_j}^{x_{j+1}} \int_{x_j}^{x_{j+1}} \int_{t^n}^{t^{n+1}} u(s,y) \int_y^{\bar{y}} \partial_x^2 u(s,v) dv ds d\bar{y} dy \\
&\qquad + \frac{2}{\Delta t^2 \Delta x} \int_{x_j}^{x_{j+1}} \int_{x_j}^{x_{j+1}} \int_{t^n}^{t^{n+1}} \int_{t^n}^{t^{n+1}} u(s,y) \int_s^{\bar{s}} \partial_{xt} u(\tau, \bar{y}) d\tau d\bar{s} ds d\bar{y} dy.
\end{aligned}
$$

Thus, the (a) term in Equation (74) rewrites

$$
(a) = K(1) - K(-1) = 2K'(0) + \int_0^1 K''(w)(1-w) dw + \int_0^1 K''(-w)(-1+w) dw,
$$

and after some Taylor expansions, it holds, since u satisfies (1a)

$$
\begin{aligned}
|\epsilon_j^n| \le \; & \frac{1}{\Delta x \Delta t}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}}\int_s^{s+\Delta t} |\partial_t^2 u(w,y)| dw ds dy + \frac{1}{\Delta t \Delta x^2}\int_{x_j}^{x_{j+1}}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}} |u(s,y)\int_y^{\bar{y}} \partial_x^2 u(s,v) dv| ds d\bar{y} dy \\
&+ \frac{1}{\Delta t^2 \Delta x^2}\int_{x_j}^{x_{j+1}}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\int_{t^n}^{t^{n+1}} |u(s,y)\int_s^{\bar{s}} \partial_{xt} u(\tau,\bar{y}) d\tau| d\bar{s} ds d\bar{y} dy + \frac{1}{4\Delta x}\int_0^1 |K''(w)|(1-w) dw \\
&+ \frac{1}{4\Delta x}\int_0^1 |K''(-w)|(1-w) dw + \frac{1-\theta}{\Delta t \Delta x}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}}\int_y^{y+2\Delta x} |\partial_x^4 u(s,z)| dz + \int_y^{y+\Delta x} |\partial_x^4 u(s,z)| dz \\
&+ \int_y^{y-\Delta x} |\partial_x^4 u(s,z)| dz dy ds + \frac{\theta}{\Delta t \Delta x}\int_{t^{n+1}}^{t^{n+2}}\int_{x_j}^{x_{j+1}}\int_y^{y+2\Delta x} |\partial_x^4 u(s,z)| dz + \int_y^{y+\Delta x} |\partial_x^4 u(s,z)| dz \\
&+ \int_y^{y-\Delta x} |\partial_x^4 u(s,z)| dz dy ds + \frac{c}{2\Delta x \Delta t}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}}\int_y^{y+\Delta x} |\partial_x^2 u(s,z)| dz + \int_y^{y-\Delta x} |\partial_x^2 u(s,z)| dz dy ds.
\end{aligned}
$$

Thus, once applying the Cauchy-Schwarz inequality, the previous inequality becomes

$$
\begin{aligned}
|\epsilon_j^n|^2 \le C \Bigg[& \frac{\Delta t}{\Delta x}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}} \left(\partial_t^2 u(w,y)\right)^2 dw dy + \frac{1}{\Delta t \Delta x}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}}\int_y^{\bar{y}} \left[u(s,y)\partial_x^2 u(s,v)\right]^2 dv ds dy d\bar{y} \\
&+ \frac{1}{\Delta t \Delta x^2}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\int_s^{\bar{s}} \left[u(s,y)\partial_{xt} u(\tau,\bar{y})\right]^2 d\tau d\bar{s} d\bar{y} ds dy + \frac{1}{8\Delta x^2}||K''||^2_{L^\infty([-1,1])} \\
&+ \frac{(1-\theta)^2\Delta x}{\Delta t}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}} \left(\partial_x^4 u(s,z)\right)^2 dz ds + \frac{\theta^2 \Delta x}{\Delta t}\int_{t^{n+1}}^{t^{n+2}}\int_{x_j}^{x_{j+1}} \left(\partial_x^4 u(s,z)\right)^2 dz ds \\
&+ \frac{c^2 \Delta x}{\Delta t}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}} \left(\partial_x^2 u(s,z)\right)^2 dz ds \Bigg],
\end{aligned}
$$

with C a constant.

However, one has

$$
\begin{aligned}
K''(\nu) = \; & \frac{2}{\Delta t^2}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}} \partial_x^2 u(s, y+\nu\Delta x) u(\bar{s}, \bar{y}+\nu\Delta x) d\bar{s} d\bar{y} ds dy \\
&+ \frac{2}{\Delta t^2}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}} \partial_x u(s, y+\nu\Delta x)\partial_x u(\bar{s}, \bar{y}+\nu\Delta x) d\bar{s} d\bar{y} ds dy,
\end{aligned}
$$

which gives, thanks to the Cauchy-Schwarz inequality

$$
\begin{aligned}
|K''(\nu)|^2 \le C \Bigg[& \frac{\Delta x^3}{\Delta t^2}\int_{t^n}^{t^{n+1}} ||u(\bar{s},.)||^2_{L_x^\infty}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}} \left(\partial_x^2 u(s, y+\nu\Delta x)\right)^2 ds dy d\bar{s} \\
&+ \left(\frac{2\Delta x}{\Delta t}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}} \left(\partial_x u(s, y+\nu\Delta x)\right)^2 ds dy\right)^2 \Bigg].
\end{aligned}
$$

Therefore, it yields

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}} \Delta x|\epsilon_j^n|^2 \le C \Bigg[& \Delta t^2 \sup_{t\in[0,T]} ||\partial_t^2 u(t,.)||^2_{L_x^2} + \Delta x^2 \sup_{t\in[0,T]} ||u(t,.)||^2_{L_x^\infty} \sup_{t\in[0,T]} ||\partial_x^2 u(t,.)||^2_{L_x^2} \\
&+ \Delta t^2 \sup_{t\in[0,T]} ||u(t,.)||^2_{L_x^\infty} \sup_{t\in[0,T]} ||\partial_{xt} u(t,.)||^2_{L_x^2} + \Delta x^2 \sup_{t\in[0,T]} ||u(t,.)||^2_{L_x^\infty} \sup_{t\in[0,T]} ||\partial_x^2 u(t,.)||^2_{L_x^2} \\
&+ \Delta x^2 \sup_{t\in[0,T]} ||\partial_x u(t,.)||^2_{L_x^2} \sup_{t\in[0,T]} ||\partial_x u(t,.)||^2_{L_x^\infty} + \Delta x^2 \sup_{n\in[\![0,N]\!]} ||\partial_x^4 u||^2_{L_x^2} + \Delta x^2 \sup_{n\in[\![0,N]\!]} ||\partial_x^2 u||^2_{L_x^2} \Bigg].
\end{aligned}
\tag{75}
$$

Let us then compute $||\partial_t^2 u||_{L_x^2}$ in (75). Thanks to the KdV equation, the time derivative is equal to

$$\partial_t^2 u = 2u\left(\partial_x u\right)^2 + u^2\partial_x^2 u + 5\partial_x u\partial_x^3 u + 2u\partial_x^4 u + 3\left(\partial_x^2 u\right)^2 + \partial_x^6 u.$$

For the term $\partial_x u\partial_x^3 u$, we use then the relation, for all u and v in $H^{\alpha+\beta}(\mathbb{R})$

$$\left|\left|\partial_x^\alpha u\partial_x^\beta v\right|\right|_{L^2(\mathbb{R})} \leq C\left[||u||_{L^\infty(\mathbb{R})}||v||_{H^{\alpha+\beta}(\mathbb{R})} + ||v||_{L^\infty(\mathbb{R})}||u||_{H^{\alpha+\beta}(\mathbb{R})}\right]. \tag{76}$$

Hence

$$||\partial_t^2 u||_{L_x^2} \leq C\left[||u||_{L_x^\infty}||\partial_x u||_{L_x^4}^2 + ||u||_{L_x^\infty}^2||\partial_x^2 u||_{L_x^2} + ||u||_{L_x^\infty}||\partial_x^4 u||_{L_x^2} + ||u||_{L_x^\infty}||\partial_x^4 u||_{L_x^2} + ||\partial_x^2 u||_{L_x^4}^2 + ||\partial_x^6 u||_{L_x^2}\right].$$

For the term $||\partial_x u||_{L_x^4}$, we use an integration by parts and the Cauchy-Schwarz inequality to obtain

$$||\partial_x u||_{L_x^4}^4 = \int_{\mathbb{R}}\left(\partial_x u(x)\right)^3\partial_x u(x)dx = -\int_{\mathbb{R}} 3u(x)\partial_x^2 u(x)\left(\partial_x u(x)\right)^2 dx \leq 3\,||u||_{L_x^\infty}\left|\left|\partial_x^2 u\right|\right|_{L_x^2}||\partial_x u||_{L_x^4}^2.$$

We thus conclude $||\partial_x u||_{L_x^4}^2 \leq C\,||u||_{L_x^\infty}\left|\left|\partial_x^2 u\right|\right|_{L_x^2}$.

For the term $||\partial_x^2 u||_{L_x^4}^2$, we again use an integration by parts and the Cauchy-Schwarz inequality to write

$$||\partial_x^2 u||_{L_x^4}^4 = \int_{\mathbb{R}}\left(\partial_x^2 u(x)\right)^3\partial_x^2 u(x)dx = \int_{\mathbb{R}} -3\partial_x^3 u(x)\left(\partial_x^2 u(x)\right)^2\partial_x u(x)dx \leq 3||\partial_x^2 u||_{L_x^4}^2\sqrt{\int_{\mathbb{R}}\left(\partial_x^3 u(x)\right)^2\left(\partial_x u(dx)\right)^2 dx},$$

which implies thanks to Relation (76) $||\partial_x^2 u||_{L_x^4}^2 \leq C\,||u||_{L_x^\infty}\left|\left|\partial_x^4 u\right|\right|_{L_x^2}$.

For the $||\partial_{xt}u(t,\cdot)||_{L_x^2}$-term in (75), it holds

$$\begin{aligned}||\partial_{tx}u(t,\cdot)||_{L_x^2}^2 &= ||-\left(\partial_x u(t,\cdot)\right)^2 - u(t,\cdot)\partial_x^2 u(t,\cdot) - \partial_x^4 u(t,\cdot)||_{L_x^2}^2\\ &\leq C\left[||u(t,\cdot)||_{L_x^\infty}^2||\partial_x^2 u(t,\cdot)||_{L_x^2}^2 + ||\partial_x u(t,\cdot)||_{L_x^4}^4 + ||\partial_x^4 u(t,\cdot)||_{L_x^2}^2\right].\end{aligned}$$

To conclude, we obtain with (75)

$$\begin{aligned}||\epsilon^n||_{\ell^\infty([\![0,N]\!];\ell^2_\Delta(\mathbb{Z}))} \leq C&\Bigg[\Delta t\sup_{t\in[0,T]}\left(||u||_{L_x^\infty}^2||u||_{H_x^2} + ||u||_{L_x^\infty}||u||_{H_x^4} + ||u||_{H_x^6} + ||u||_{L_x^\infty}||u||_{H_x^2} + ||u||_{H_x^4}\right)\\ &+\Delta x\sup_{t\in[0,T]}\left(||u||_{L_x^\infty}||u||_{H_x^2} + ||\partial_x u||_{L_x^\infty}||u||_{H_x^1} + ||u||_{H_x^4} + ||u||_{H_x^2}\right)\Bigg],\end{aligned}$$

what can be simplified into

$$\begin{aligned}||\epsilon^n||_{\ell^\infty([\![0,N]\!];\ell^2_\Delta(\mathbb{Z}))} \leq C&\Bigg[\Delta t\sup_{t\in[0,T]}\left(||u||_{L_x^\infty}^2||u||_{H_x^2} + ||u||_{L_x^\infty}||u||_{H_x^4} + ||u||_{H_x^6}\right)\\ &+\Delta x\sup_{t\in[0,T]}\left(||u||_{L_x^\infty}||u||_{H_x^2} + ||\partial_x u||_{L_x^\infty}||u||_{H_x^1} + ||u||_{H_x^4}\right)\Bigg].\end{aligned}$$

Thus the consistency error is upper bounded by

$$||\epsilon^n||_{\ell^\infty([\![0,N]\!];\ell^2_\Delta(\mathbb{Z}))} \leq C\left\{\Delta t\sup_{t\in[0,T]}\left[\left(1 + ||u||_{L_x^\infty}^2\right)||u||_{H_x^6}\right] + \Delta x\sup_{t\in[0,T]}\left[\left(1 + ||u||_{L_x^\infty}\right)||u||_{H_x^4} + ||\partial_x u||_{L_x^\infty}||u||_{H_x^1}\right]\right\}.$$

B Appendix : Proofs of useful Lemmas of Section 4

Proof of Lemma 2. Identities (20) to (24b) are proved by developing the expressions. □

Proof of Lemma 3.
- Relation (25) is obtained by developing the sums.
- Relation (26) is proved by summing the square of Identity (22) with $b_j = \frac{a_j}{2}$.
- Relation (27) is obtained by developing the expression :

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \left[D_+D_-(a)_j\right]^2 &= \sum_{j\in\mathbb{Z}} \left[\frac{a_{j+1}-2a_j+a_{j-1}}{\Delta x^2}\right]^2 \\
&= \sum_{j\in\mathbb{Z}} \frac{a_{j+1}^2 - 4a_{j+1}a_j + 2a_{j+1}a_{j-1} + 4a_j^2 - 4a_ja_{j-1} + a_{j-1}^2}{\Delta x^4} \\
&= \sum_{j\in\mathbb{Z}} 4\frac{a_{j+1}^2 - 2a_{j+1}a_j + a_j^2}{\Delta x^4} - \frac{a_{j+1}^2 - 2a_{j-1}a_{j+1} + a_{j-1}^2}{\Delta x^4} \\
&= \sum_{j\in\mathbb{Z}} \frac{4}{\Delta x^2}\frac{a_{j+1}^2 - 2a_{j+1}a_j + a_j^2}{\Delta x^2} - \frac{4}{\Delta x^2}\frac{a_{j+1}^2 - 2a_{j-1}a_{j+1} + a_{j-1}^2}{4\Delta x^2}.
\end{aligned}$$

□

Proof of Lemma 4.
- Identities (29) and (30) derive from developing the sums.
- Relation (31) is a straightforward consequence of (24a).
- For (32), one has

$$\sum_{j\in\mathbb{Z}} D_+(a)_j\, a_ja_{j+1} = \sum_{j\in\mathbb{Z}} \frac{a_{j+1}^2a_j - a_j^2a_{j+1}}{\Delta x} = \sum_{j\in\mathbb{Z}} -\frac{a_{j+1}^3 - 3a_{j+1}^2a_j + 3a_{j+1}a_j^2 - a_j^3}{3\Delta x} = -\sum_{j\in\mathbb{Z}} \frac{\Delta x^2}{3}\left[\frac{a_{j+1}-a_j}{\Delta x}\right]^3.$$

- Likewise, Identity (33) is proved by the same way

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} D(a)_j\, a_{j-1}a_{j+1} = \sum_{j\in\mathbb{Z}} \frac{a_{j+1}^2a_{j-1} - a_{j+1}a_{j-1}^2}{2\Delta x} &= \sum_{j\in\mathbb{Z}} -\frac{4}{3}\frac{a_{j+1}^3 - 3a_{j+1}^2a_{j-1} + 3a_{j+1}a_{j-1}^2 - a_{j-1}^3}{8\Delta x} \\
&= \sum_{j\in\mathbb{Z}} -\frac{4\Delta x^2}{3}\left[\frac{a_{j+1}-a_{j-1}}{2\Delta x}\right]^3.
\end{aligned}$$

- The proof of (34) is based on a development of the expression of $D(ab)_j$ as explained below

$$\sum_{j\in\mathbb{Z}} a_jD(ab)_j = \sum_{j\in\mathbb{Z}} a_j\frac{a_{j+1}b_{j+1} - a_{j-1}b_{j-1}}{2\Delta x} = \sum_{j\in\mathbb{Z}} \frac{b_{j+1}a_{j+1}a_j - b_ja_{j+1}a_j}{2\Delta x} = \sum_{j\in\mathbb{Z}} \frac{b_{j+1}-b_j}{2\Delta x}a_{j+1}a_j.$$

- For Relation (35), we expand

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}} D_+D_-(a)_j\, D(ab)_j \\
&= \sum_{j\in\mathbb{Z}} \frac{(a_{j+1}-2a_j+a_{j-1})\left[a_{j+1}b_{j+1}\right] - (a_{j+1}-2a_j+a_{j-1})\left[a_{j-1}b_{j-1}\right]}{2\Delta x^3} \\
&= \sum_{j\in\mathbb{Z}} \frac{(a_{j+1})^2 b_{j+1} - 2a_ja_{j+1}b_{j+1} + a_{j-1}a_{j+1}b_{j+1} - a_{j+1}a_{j-1}b_{j-1} + 2a_ja_{j-1}b_{j-1} - (a_{j-1})^2 b_{j-1}}{2\Delta x^3}
\end{aligned}$$

$$= \sum_{j\in\mathbb{Z}} \frac{-2a_j a_{j+1} b_{j+1} + 2a_{j+1} a_j b_j}{2\Delta x^3} + \frac{a_{j-1} a_{j+1} b_{j+1} - a_{j+1} a_{j-1} b_{j-1}}{2\Delta x^3}$$
$$= \sum_{j\in\mathbb{Z}} -\frac{b_{j+1} - b_j}{\Delta x^3} a_{j+1} a_j + \sum_{j\in\mathbb{Z}} \frac{b_{j+1} - b_{j-1}}{2\Delta x^3} a_{j+1} a_{j-1}.$$

□

Proof of Corollary 2.

- Identity (36) is based on (34) with $b_j = \frac{a_j}{2}$ and Relation (32).
- Proving (37) corresponds to choose $b_j = \frac{a_j}{2}$ in Identity (35) and using Relations (32) and (33).

□

Proof of Lemma 5. Inequality (42) is based on Relation (23)

$$\sum_{j\in\mathbb{Z}} \left[D\left(ab\right)_j\right]^2 = \sum_{j\in\mathbb{Z}} \left[b_j D\left(a\right)_j + \frac{a_{j+1}}{2} D_+\left(b\right)_j + \frac{a_{j-1}}{2} D_-\left(b\right)_j\right]^2$$
$$= \sum_{j\in\mathbb{Z}} \left(b_j\right)^2 \left(D\left(a\right)_j\right)^2 + \sum_{j\in\mathbb{Z}} b_j a_{j+1} D\left(a\right)_j D_+\left(b\right)_j + \sum_{j\in\mathbb{Z}} b_j D\left(a\right)_j a_{j-1} D_-\left(b\right)_j$$
$$+ \sum_{j\in\mathbb{Z}} \left(\frac{a_{j+1}}{2} D_+\left(b\right)_j\right)^2 + \sum_{j\in\mathbb{Z}} \frac{a_{j+1} a_{j-1}}{2} D_+\left(b\right)_j D_-\left(b\right)_j + \sum_{j\in\mathbb{Z}} \left(\frac{a_{j-1}}{2} D_-\left(b\right)_j\right)^2.$$

Young inequality implies

$$\sum_{j\in\mathbb{Z}} \left(D\left(ab\right)_j\right)^2 \leq \sum_{j\in\mathbb{Z}} \left(b_j\right)^2 \left(D\left(a\right)_j\right)^2 + \sum_{j\in\mathbb{Z}} \frac{\left(b_j\right)^2}{2\Delta t} \left(a_{j+1}\right)^2 + \sum_{j\in\mathbb{Z}} \frac{\Delta t}{2} \left(D\left(a\right)_j\right)^2 \left(D_+\left(b\right)_j\right)^2 + \sum_{j\in\mathbb{Z}} \frac{\left(b_j\right)^2}{2\Delta t} \left(a_{j-1}\right)^2$$
$$+ \sum_{j\in\mathbb{Z}} \frac{\Delta t}{2} \left(D\left(a\right)_j\right)^2 \left(D_-\left(b\right)_j\right)^2 + \sum_{j\in\mathbb{Z}} \frac{3}{2} \left(\frac{a_{j+1}}{2} D_+\left(b\right)_j\right)^2 + \sum_{j\in\mathbb{Z}} \frac{3}{2} \left(\frac{a_{j-1}}{2} D_-\left(b\right)_j\right)^2.$$

□

Proof of Lemma 6. We use relation (23) to develop $D_+D_+D_-\left(a\right)_j D\left(ab\right)_j$ which gives

$$\sum_{j\in\mathbb{Z}} D_+D_+D_-\left(a\right)_j D\left(ab\right)_j = \sum_{j\in\mathbb{Z}} D_+D_+D_-\left(a\right)_j \left[b_j D\left(a\right)_j + \frac{a_{j+1}}{2} D_+\left(b\right)_j + \frac{a_{j-1}}{2} D_-\left(b\right)_j\right]$$
$$\leq \sum_{j\in\mathbb{Z}} b_j D_+D_+D_-\left(a\right)_j D\left(a\right)_j + \sum_{j\in\mathbb{Z}} \frac{\Delta t}{4} \left(D_+D_+D_-\left(a\right)_j\right)^2 \left|D_+\left(b\right)_j\right|$$
$$+ \sum_{j\in\mathbb{Z}} \frac{\left(a_{j+1}\right)^2}{4\Delta t} \left|D_+\left(b\right)_j\right| + \sum_{j\in\mathbb{Z}} \frac{\Delta t}{4} \left(D_+D_+D_-\left(a\right)_j\right)^2 \left|D_-\left(b\right)_j\right| + \sum_{j\in\mathbb{Z}} \frac{\left(a_{j-1}\right)^2}{4\Delta t} \left|D_-\left(b\right)_j\right|. \tag{77}$$

The conclusion comes from the following Lemma.

Lemma 12. *Let $\left(a_j\right)_{j\in\mathbb{Z}}$ and $\left(b_j\right)_{j\in\mathbb{Z}}$ be two sequences in $\ell^2_\Delta(\mathbb{Z})$, σ be in $\{0,1\}$ and ν be non negative. Then, it holds*

$$\sum_{j\in\mathbb{Z}} b_j D_+D_+D_-\left(a\right)_j D\left(a\right)_j \leq \sum_{j\in\mathbb{Z}} \frac{1}{2} \left\{\Delta x^\nu \left(\frac{|D_-(b)_j|^\sigma}{2} + \frac{|D_-(b)_j|^\sigma}{2}\right) - \frac{\Delta x}{2} D_- b_j\right\} \left(D_+D_-\left(a\right)_j\right)^2$$
$$+ \sum_{j\in\mathbb{Z}} \frac{1}{2\Delta x^\nu} \left|D_+\left(b\right)_j\right|^{2-\sigma} \left(D_+\left(a\right)_j\right)^2 - \sum_{j\in\mathbb{Z}} b_j \left(D_+D\left(a\right)_j\right)^2. \tag{78}$$

Proof of Lemma 12. By developing $D(a)_j$ and using the relation (29), it holds

$$\sum_{j\in\mathbb{Z}} b_j D_+D_+D_-(a)_j\, D(a)_j = \sum_{j\in\mathbb{Z}} \frac{b_j}{2} D_+D_+D_-(a)_j\, D_+(a)_j + \sum_{j\in\mathbb{Z}} \frac{b_j}{2} D_+D_+D_-(a)_j\, D_-(a)_j$$
$$= -\sum_{j\in\mathbb{Z}} D_+D_-(a)_j\, D_-\left(\frac{b_j}{2} D_+(a)_j\right) - \sum_{j\in\mathbb{Z}} D_+D_-(a)_j\, D_-\left(\frac{b_j}{2} D_-(a)_j\right).$$

We focus first on the term $-\sum_{j\in\mathbb{Z}} D_+D_-(a)_j\, D_-\left(\frac{b_j}{2} D_+(a)_j\right)$. Equality (21b) gives

$$-\sum_{j\in\mathbb{Z}} D_+D_-(a)_j\, D_-\left(\frac{b_j}{2} D_+(a)_j\right) = -\sum_{j\in\mathbb{Z}} D_+D_-(a)_j \left(\frac{D_-(b)_j}{2} D_-(a)_j + \frac{b_j}{2} D_+D_-(a)_j\right).$$

Eventually, Young inequality provides

$$-\sum_{j\in\mathbb{Z}} D_+D_-(a)_j\, D_-\left(\frac{b_j}{2} D_+(a)_j\right) \le \sum_{j\in\mathbb{Z}} \frac{\Delta x^\nu |D_-b_j|^\sigma}{4} \left(D_+D_-(a)_j\right)^2 + \sum_{j\in\mathbb{Z}} \frac{|D_+(b)_j|^{2-\sigma}}{4\Delta x^\nu} \left(D_+(a)_j\right)^2$$
$$- \sum_{j\in\mathbb{Z}} \frac{b_j}{2} \left(D_+D_-(a)_j\right)^2.$$

For the term $-\sum_{j\in\mathbb{Z}} D_+D_-(a)_j\, D_-\left(\frac{b_j}{2} D_-(a)_j\right)$, one has thanks to Equality (21b),

$$-\sum_{j\in\mathbb{Z}} D_+D_-(a)_j\, D_-\left(\frac{b_j}{2} D_-(a)_j\right) = -\sum_{j\in\mathbb{Z}} D_+D_-(a)_j \left(\frac{D_-(b)_j}{2} D_-(a)_j + \frac{b_{j-1}}{2} D_-D_-(a)_j\right).$$

Hence, it holds (by Young inequality)

$$-\sum_{j\in\mathbb{Z}} D_+D_-(a)_j\, D_-\left(\frac{b_j}{2} D_-(a)_j\right)$$
$$\le \sum_{j\in\mathbb{Z}} \frac{\Delta x^\nu |D_-b_j|^\sigma}{4} \left(D_+D_-(a)_j\right)^2 + \sum_{j\in\mathbb{Z}} \frac{|D_+(b)_j|^{2-\sigma}}{4\Delta x^\nu} (D_+a_j)^2 - \sum_{j\in\mathbb{Z}} \frac{b_{j-1}}{2} D_+D_-(a)_j\, D_-D_-(a)_j$$
$$\le \sum_{j\in\mathbb{Z}} \frac{\Delta x^\nu |D_-b_j|^\sigma}{4} (D_+D_-a_j)^2 + \sum_{j\in\mathbb{Z}} \frac{|D_+(b)_j|^{2-\sigma}}{4\Delta x^\nu} (D_+a_j)^2$$
$$- \sum_{j\in\mathbb{Z}} b_{j-1} \left(\frac{D_+D_-a_j + D_-D_-a_j}{2}\right)^2 + \sum_{j\in\mathbb{Z}} \frac{b_{j-1}}{4} (D_+D_-a_j)^2 + \sum_{j\in\mathbb{Z}} \frac{b_{j-1}}{4} (D_-D_-a_j)^2$$
$$\le \sum_{j\in\mathbb{Z}} \left(\frac{\Delta x^\nu |D_-b_j|^\sigma}{4} + \frac{b_{j-1}+b_j}{4}\right) (D_+D_-a_j)^2 - \sum_{j\in\mathbb{Z}} b_j (D_+Da_j)^2 + \sum_{j\in\mathbb{Z}} \frac{|D_+(b)_j|^{2-\sigma}}{4\Delta x^\nu} \left(D_+(a)_j\right)^2.$$

By collecting the previous results, one has

$$\sum_{j\in\mathbb{Z}} b_j D_+D_+D_-(a)_j\, D(a)_j \le \sum_{j\in\mathbb{Z}} \left\{\frac{\Delta x^\nu |D_-b_j|^\sigma}{4} + \frac{\Delta x^\nu |D_-b_j|^\sigma}{4} + \frac{b_{j-1}-b_j}{4}\right\} (D_+D_-a_j)^2$$
$$+ \sum_{j\in\mathbb{Z}} \frac{|D_+(b)_j|^{2-\sigma}}{2\Delta x^\nu} \left(D_+(a)_j\right)^2 - \sum_{j\in\mathbb{Z}} b_j (D_+Da_j)^2.$$

Lemma 12 is then proved. □

Lemma 6 is a consequence, with $\nu = 0$.

□

Proof of Lemma 7. To prove Lemma 7, the following lemma is useful.

Lemma 13. *Let $(a_j)_{j\in\mathbb{Z}}$ be a sequence in $\ell^2_\Delta(\mathbb{Z})$, then one has*

$$||D_+a||_{\ell^4_\Delta} \leq \sqrt{3||a||_{\ell^\infty}||D_+D_-a||_{\ell^2_\Delta}}.$$

Proof of Lemma 13. We compute the following sum thanks to the discrete integration by parts (29)

$$\sum_{j\in\mathbb{Z}} (D_+a_j)^4 = \sum_{j\in\mathbb{Z}} (D_+a_j)^3 D_+a_j = -\sum_{j\in\mathbb{Z}} D_-\left[(D_+a_j)^3\right] a_j. \tag{79}$$

We use Relation (21b) to develop $D_-\left[(D_+a_j)^3\right]$ in (79) and $D_-\left[(D_+(a)_j)^2\right]$ in (80)

$$\begin{aligned}-\sum_{j\in\mathbb{Z}} D_-\left[(D_+a_j)^3\right] a_j &= -\sum_{j\in\mathbb{Z}} D_-\left[(D_+(a)_j)^2\right] D_-(a)_j a_j - \sum_{j\in\mathbb{Z}} (D_+a_j)^2 D_-D_+(a)_j a_j \qquad (80)\\ &= -\sum_{j\in\mathbb{Z}} D_+D_-(a)_j (D_-a_j)^2 a_j - \sum_{j\in\mathbb{Z}} D_+(a)_j D_+D_-(a)_j D_-(a)_j a_j - \sum_{j\in\mathbb{Z}} (D_+a_j)^2 D_+D_-(a)_j a_j.\end{aligned}$$

We recover $D(a)_j$ in the second sum, which gives

$$\begin{aligned}\sum_{j\in\mathbb{Z}} (D_+a_j)^4 &= -\sum_{j\in\mathbb{Z}} D_+D_-a_j (D_-a_j)^2 a_j - \sum_{j\in\mathbb{Z}} 2D_+D_-(a)_j a_j (D(a)_j)^2 + \sum_{j\in\mathbb{Z}} D_+D_-(a)_j a_j \frac{(D_+a_j)^2}{2}\\ &\quad + \sum_{j\in\mathbb{Z}} D_+D_-(a)_j a_j \frac{(D_-a_j)^2}{2} - \sum_{j\in\mathbb{Z}} (D_+a_j)^2 D_+D_-(a)_j a_j\\ &= -\sum_{j\in\mathbb{Z}} \frac{D_+D_-(a)_j a_j (D_-a_j)^2}{2} - 2\sum_{j\in\mathbb{Z}} D_+D_-(a)_j a_j (Da_j)^2 - \sum_{j\in\mathbb{Z}} D_+D_-(a)_j \frac{a_j (D_+a_j)^2}{2}.\end{aligned}$$

Multiplying by Δx and applying Cauchy-Schwarz inequality simplify the ℓ^4_Δ-norm

$$||D_+a||^4_{\ell^4_\Delta} \leq \frac{||a||_{\ell^\infty}}{2}||D_+D_-(a)||_{\ell^2_\Delta}||D_-a||^2_{\ell^4_\Delta} + 2||a||_{\ell^\infty}||D_+D_-(a)||_{\ell^2_\Delta}||Da||^2_{\ell^4_\Delta} + \frac{||a||_{\ell^\infty}}{2}||D_+D_-(a)||_{\ell^2_\Delta}||D_+a||^2_{\ell^4_\Delta}. \tag{81}$$

However, one has

$$\begin{aligned}||Da||^4_{\ell^4_\Delta} &= \sum_{j\in\mathbb{Z}} \Delta x \left(\frac{a_{j+1}-a_j+a_j-a_{j-1}}{2\Delta x}\right)^4\\ &= \sum_{j\in\mathbb{Z}} \Delta x \left(\frac{a_{j+1}-a_j+a_j-a_{j-1}}{2\Delta x}\right)^2 \left(\frac{a_{j+1}-a_j+a_j-a_{j-1}}{2\Delta x}\right)^2\\ &= \sum_{j\in\mathbb{Z}} \Delta x \left[2\left(\frac{a_{j+1}-a_j}{2\Delta x}\right)^2 + 2\left(\frac{a_j-a_{j-1}}{2\Delta x}\right)^2\right]\left[2\left(\frac{a_{j+1}-a_j}{2\Delta x}\right)^2 + 2\left(\frac{a_j-a_{j-1}}{2\Delta x}\right)^2\right]\\ &\leq \sum_{j\in\mathbb{Z}} \Delta x \frac{1}{4}\left(\frac{a_{j+1}-a_j}{\Delta x}\right)^4 + \frac{1}{2}\Delta x \left(\frac{a_{j+1}-a_j}{\Delta x}\right)^2\left(\frac{a_j-a_{j-1}}{\Delta x}\right)^2 + \frac{1}{4}\Delta x\left(\frac{a_j-a_{j-1}}{\Delta x}\right)^4\\ &\leq \sum_{j\in\mathbb{Z}} \Delta x \left(\frac{a_{j+1}-a_j}{\Delta x}\right)^4\\ &\leq ||D_+a||^4_{\ell^4_\Delta}.\end{aligned}$$

Equation (81) is changed as follows

$$||D_+a||_{\ell^4_\Delta}^4 \leq 3||a||_{\ell^\infty}||D_+D_-(a)||_{\ell^2_\Delta}||D_+a||_{\ell^4_\Delta}^2.$$

Then it holds

$$||D_+a||_{\ell^4_\Delta}^2 \leq 3||a||_{\ell^\infty}||D_+D_-(a)||_{\ell^2_\Delta}.$$

Lemma 13 is thus proved. $\square$

Remark 24. *Lemma 13 is also valid for the classical discrete ℓ^4-norm instead of ℓ^4_Δ-norm.*

To prove Lemma 7, we first develop the left hand side thanks to (23)

$$\sum_{j\in\mathbb{Z}} D_+D_+D_-\left(a\right)_j D\left(\frac{a^2}{2}\right)_j = \sum_{j\in\mathbb{Z}} D_+D_+D_-\left(a\right)_j \left[\frac{a_j}{2}D\left(a\right)_j + \frac{a_{j+1}}{4}D_+\left(a\right)_j + \frac{a_{j-1}}{4}D_-\left(a\right)_j\right].$$

• The first term $\sum_{j\in\mathbb{Z}}\frac{a_j}{2}D_+D_+D_-\left(a\right)_j D\left(a\right)_j$ is treated with Lemma 12 of Appendix B with $\nu = \frac{1}{2}-\gamma$ and $\sigma = 0$, which rewrites

$$\sum_{j\in\mathbb{Z}}\frac{a_j}{2}D_+D_+D_-(a)_jD(a)_j \leq \sum_{j\in\mathbb{Z}}\frac{1}{4}\left\{\Delta x^{\frac{1}{2}-\gamma} - \frac{\Delta x}{2}D_-a_j\right\}\left(D_+D_-a_j\right)^2 + \sum_{j\in\mathbb{Z}}\frac{1}{4\Delta x^{\frac{1}{2}-\gamma}}\left(D_+a_j\right)^4 - \sum_{j\in\mathbb{Z}}\frac{a_j}{2}\left(D_+Da_j\right)^2.$$

• For the second term, we integrate by parts thanks to (29) and (21b)

$$\begin{aligned}\sum_{j\in\mathbb{Z}} D_+D_+D_-\left(a\right)_j \frac{a_{j+1}}{4}D_+\left(a\right)_j &= \sum_{j\in\mathbb{Z}} -D_+D_-\left(a\right)_j D_-\left(\frac{a_{j+1}}{4}D_+\left(a\right)_j\right)\\ &= \sum_{j\in\mathbb{Z}} -D_+D_-\left(a\right)_j\left\{\frac{a_j}{4}D_+D_-\left(a\right)_j + \frac{\left(D_+\left(a\right)_j\right)^2}{4}\right\}.\end{aligned}$$

Young inequality completes the upper bound

$$\sum_{j\in\mathbb{Z}} D_+D_+D_-\left(a\right)_j\frac{a_{j+1}}{4}D_+\left(a\right)_j = \sum_{j\in\mathbb{Z}} -\left(D_+D_-(a)_j\right)^2\frac{a_j}{4} + \sum_{j\in\mathbb{Z}}\frac{1}{8}\left(D_+D_-(a)_j\right)^2\Delta x^{\frac{1}{2}-\gamma} + \sum_{j\in\mathbb{Z}}\frac{1}{8\Delta x^{\frac{1}{2}-\gamma}}\left(D_+(a)_j\right)^4.$$

• For the third term, Relation (29) together with (21a) gives

$$\begin{aligned}&\sum_{j\in\mathbb{Z}} D_+D_+D_-\left(a\right)_j\frac{a_{j-1}}{4}D_-\left(a\right)_j = \sum_{j\in\mathbb{Z}} -D_+D_+\left(a\right)_j D_+\left(\frac{a_{j-1}}{4}D_-\left(a\right)_j\right)\\ &= \sum_{j\in\mathbb{Z}} -D_+D_+\left(a\right)_j\left\{\frac{a_j}{4}D_+D_-\left(a\right)_j + \frac{D_+\left(a\right)_{j-1}}{4}D_-\left(a\right)_j\right\}\\ &= \sum_{j\in\mathbb{Z}} -\frac{a_j}{2}\left(\frac{D_+D_+\left(a\right)_j + D_+D_-\left(a\right)_j}{2}\right)^2 + \sum_{j\in\mathbb{Z}}\frac{a_j}{8}\left(D_+D_+\left(a\right)_j\right)^2 + \sum_{j\in\mathbb{Z}}\frac{a_j}{8}\left(D_+D_-\left(a\right)_j\right)^2 - \sum_{j\in\mathbb{Z}} D_+D_+\left(a\right)_j\frac{\left(D_-\left(a\right)_j\right)^2}{4}\\ &\leq \sum_{j\in\mathbb{Z}} -\frac{a_j}{2}\left(D_+D(a)_j\right)^2 + \sum_{j\in\mathbb{Z}}\frac{a_j + a_{j-1}}{8}\left(D_+D_-(a)_j\right)^2 + \sum_{j\in\mathbb{Z}}\frac{\Delta x^{\frac{1}{2}-\gamma}}{8}\left(D_+D_-(a)_j\right)^2 + \sum_{j\in\mathbb{Z}}\frac{1}{8\Delta x^{\frac{1}{2}-\gamma}}\left(D_+(a)_j\right)^4.\end{aligned}$$

Gathering all these results yields

$$\begin{aligned}\sum_{j\in\mathbb{Z}} D_+D_+D_-(a)_jD\left(\frac{a^2}{2}\right)_j \leq \sum_{j\in\mathbb{Z}}\left\{\frac{\Delta x^{\frac{1}{2}-\gamma}}{2} - \frac{\Delta x}{8}D_-(a)_j + \frac{a_{j-1}-a_j}{8}\right\}\left(D_+D_-(a)_j\right)^2 + \sum_{j\in\mathbb{Z}}\frac{1}{2\Delta x^{\frac{1}{2}-\gamma}}\left(D_+a_j\right)^4\\ - \sum_{j\in\mathbb{Z}} a_j\left(D_+D(a)_j\right)^2.\end{aligned}$$

Lemma 13 concludes this proof

$$\sum_{j\in\mathbb{Z}} D_+D_+D_-(a)_jD\left(\frac{a^2}{2}\right)_j \leq \sum_{j\in\mathbb{Z}} \left\{\frac{\Delta x^{\frac{1}{2}-\gamma}+||a||_{\ell^\infty}+9\frac{||a||_{\ell^\infty}^2}{\Delta x^{\frac{1}{2}-\gamma}}}{2}\right\}\left(D_+D_-(a)_j\right)^2+\sum_{j\in\mathbb{Z}}||a||_{\ell^\infty}\left(D_+D(a)_j\right)^2.$$

□

Proof of Lemma 8. For Identity (44), the right hand side yields

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\left[D(a)_j\right]^2\frac{a_{j+1}b_{j+1}+a_{j-1}b_{j-1}}{2}-\sum_{j\in\mathbb{Z}}D(b)_j\left[D(a)_j\right]^3\frac{4\Delta x^2}{3}-\sum_{j\in\mathbb{Z}}\frac{1}{3}DD(b)_j\,a_j^3\\
&=\sum_{j\in\mathbb{Z}}\frac{a_{j+1}^3b_{j+1}+a_{j+1}^2a_{j-1}b_{j-1}-2a_{j+1}^2a_{j-1}b_{j+1}-2a_{j+1}a_{j-1}^2b_{j-1}+a_{j-1}^2a_{j+1}b_{j+1}+a_{j-1}^3b_{j-1}}{8\Delta x^2}\\
&-\sum_{j\in\mathbb{Z}}\left(\frac{b_{j+1}-b_{j-1}}{2\Delta x}\right)\frac{a_{j+1}^3-3a_{j+1}^2a_{j-1}+3a_{j+1}a_{j-1}^2-a_{j-1}^3}{8\Delta x^3}\left(\frac{4\Delta x^2}{3}\right)-\sum_{j\in\mathbb{Z}}\frac{1}{3}\frac{b_{j+2}-2b_j+b_{j-2}}{4\Delta x^2}a_j^3.
\end{aligned}$$

Adding the first two sums gives

$$\sum_{j\in\mathbb{Z}}\frac{a_{j+1}^3b_{j+1}-3a_{j+1}^2a_{j-1}b_{j-1}-3a_{j-1}^2a_{j+1}b_{j+1}+a_{j-1}^3b_{j-1}+2a_{j-1}^3b_{j+1}+2b_{j-1}a_{j+1}^3}{24\Delta x^2}.$$

The sum yields then

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\left[D(a)_j\right]^2\frac{a_{j+1}b_{j+1}+a_{j-1}b_{j-1}}{2}-\sum_{j\in\mathbb{Z}}D(b)_j\left[D(a)_j\right]^3\frac{4\Delta x^2}{3}-\sum_{j\in\mathbb{Z}}\frac{1}{3}DD(b)_j\,a_j^3\\
&=\sum_{j\in\mathbb{Z}}\frac{a_{j+1}^3b_{j+1}-3a_{j+1}^2a_{j-1}b_{j-1}-3a_{j-1}^2a_{j+1}b_{j+1}+a_{j-1}^3b_{j-1}+2a_{j-1}^3b_{j+1}+2b_{j-1}a_{j+1}^3}{24\Delta x^2}\\
&\qquad-\sum_{j\in\mathbb{Z}}\frac{1}{3}\frac{b_{j+1}a_{j-1}^3-2b_ja_j^3+b_{j-1}a_{j+1}^3}{4\Delta x^2}\\
&=\sum_{j\in\mathbb{Z}}\frac{a_{j+1}^3b_{j+1}-a_{j+1}^2a_{j-1}b_{j-1}-a_{j-1}^2a_{j+1}b_{j+1}+a_{j-1}^3b_{j-1}}{8\Delta x^2}\\
&=\sum_{j\in\mathbb{Z}}\left(\frac{a_{j+1}b_{j+1}-a_{j-1}b_{j-1}}{2\Delta x}\right)\left(\frac{a_{j+1}^2-a_{j-1}^2}{4\Delta x}\right)\\
&=\sum_{j\in\mathbb{Z}}D(ab)_j\,D\left(\frac{a^2}{2}\right)_j.
\end{aligned}$$

□

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