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# Homogenization of Periodic Structured Materials with Chiral Properties

Ouail Ouchetto, Badr Abou El Majd, Hassania Ouchetto, Brahim Essakhi and Saïd Zouhdi, Senior Member, IEEE

**Abstract**—This paper presents a new and efficient method to compute the quasi-static homogenized constitutive parameters of biperiodic chiral artificial material. In this method, the studied domain is reduced to an elementary cell with pseudo-periodic conditions on the lateral sides and the local electromagnetic properties are computed by using the finite element method. This chiral media is decomposed into two equivalent isotropic media and each one is treated separately. Indeed, the homogenized constitutive parameters of each isotropic media are expressed as a function of the macroscopic electromagnetic properties which are obtained by averaging the local electromagnetic properties. The homogenized constitutive parameters of the initial chiral material are expressed using the two equivalent isotropic materials. The validation of the numerical results is presented in the case of the lattices with 3D inclusions. In addition, the obtained results are compared to existing results in literature. Finally, the incidence wave angle impact on the homogenized parameters is studied.

**Index Terms**—Homogenization, metamaterials, homogenized constitutive parameters, finite element method (FEM), chirality.

## I. INTRODUCTION

IN the last decade there has been a growing interest in the investigation of artificial materials known as metamaterials. This is due to their exotic electromagnetic properties such as negative refraction, backward propagation, reverse Doppler effect, cloaking, diffraction-limit breaking imaging, reverse Vavilov-Cerenkov effect, etc [1]-[3]. These materials have shown a great potential in many applications like super-resolution lenses, slow light, data storage, optical switching, transformation optics, optical activity, circular dichroism, and so on [4]-[6]. They are made of large number of metallic or dielectric particles, periodically distributed, in a homogeneous host medium. Their electromagnetic properties depend on the shape, size, material composition and density of inclusions.

If the period of the studied composite material is small compared to the wavelength of the incident electromagnetic wave, this material can be seen as a homogeneous material. The aim of homogenization theory is to establish the macroscopic behaviour of such structure. This means that the heterogeneous structure is substituted by a homogeneous fictitious one (the homogenized structure). Within the electromagnetic community, homogenization of composite materials has a huge literature and proposed theories therein. It is to be noted that some

of the suggested theories, for predicting the effective properties, are based on analytical calculation such as Maxwell-Garnett, Bruggeman, Clausius-Mossotti mixing formulas, etc. [7]-[9]. However, other approaches are based on the principle of retrieval. Indeed, the homogenized constitutive parameters are obtained from the scattering properties of the medium by assuming the equivalence between a heterogeneous material and a uniform slab with unknown constitutive parameters [10]-[12]. The Floquet-Bloch method is a different approach of homogenization and is based on the resolution of an eigenvalue problem in unit cell [13]-[14]. It is to be underlined that the Floquet-Bloch waves were introduced by Floquet [15] and Bloch [16] and their properties are presented in [17] and [18]. The homogenized relative permittivity is given by the Hessian of the lowest eigenvalue at the Floquet-Bloch wave vector  $\mathbf{k} = 0$  representing the mismatch of the wavelength with the lattice. The Floquet-Bloch method is shown to be a good alternative to the classical homogenization method, when the contrast is modest. In the other side, an explicit expression of the effective permeability was defined by D. Felbacq et al. [19]. Their approach is a renormalization-based scheme that authorizes a completely general theory. In 2007, Silveirinha [20] proposed a source-driven approach to homogenization, leading to an integral equation that can be solved in closed form for the effective dielectric tensor. Recently, the effective constitutive parameters are defined from first-principles [21]. The proposed theory combines the approach of Floquet-based homogenization theories with the advantages of locality and general applicability of less accurate retrieval techniques.

Among mathematical proof based approaches is the multi-scale homogenization method. It consists of solving an elliptic problem in a unit cell with periodic boundary conditions [22]-[32]. The asymptotic analysis is called for in order to obtain an asymptotic expansion of the solution in terms of a small parameter which is the ratio of the two length-scales. The classical multi-scale homogenization technique are reconsidered by giving a new approach which based upon the periodic unfolding method [33]-[34]. The justification of the limiting homogeneous constitutive law is rigorously performed in the frequency and the time domains. The analysis and numerical study of this approach is presented for different cases [35]-[40]. For the multi-scale homogenization method, the studied domain is represented by an infinite triperiodic structure which is reduced to an elementary cell with triperiodic boundary conditions. This method can homogenize a structure with arbitrarily shaped inclusions and takes into account the effect of inclusions on the homogenized constitutive parameters. But it cannot evaluate directly the local fields in the elementary cell

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or in the whole studied structure. These fields can be evaluated only in the other bounded domain having different boundary conditions and the additional treatments are required. The motivation of this work is to present a new and efficient method to compute the homogenized constitutive parameters of the biperiodic chiral artificial structure having finite thickness.

This paper introduces a new homogenization method of the chiral periodic media. Compared to [41], this work has two novelties. The first one is a generalization and an extension of our previous work on lossy anisotropic periodic material to chiral periodic material. The second one is an improvement of the employed method by changing the boundary conditions used to evaluate the local fields. This change allows having a better precision in the chiral case. This technique is based on the reduction of the studied domain to an elementary cell with pseudo-biperiodic conditions on the lateral sides. The chiral elementary cell is decomposed into two isotropic equivalent cells, each cell is treated separately. Indeed, the computation of the local fields is performed by using the Finite Element Method (FEM) after choosing an appropriate boundary conditions on the different sides. The homogenized constitutive parameters of each cell are expressed as function of the macroscopic electromagnetic properties. These last are obtained from the local electromagnetic properties. Finally, the homogenized constitutive parameters of the initial chiral cell are expressed from those of the two equivalent isotropic cells. We note that the present method allows homogenizing the structures containing complex inclusions and also takes into account the effect of the geometry and the interaction of inclusion. In addition, it gives all electromagnetic properties as local and macroscopic fields, furthermore, it determines inductions in the initial studied domain. By comparison with Unfolding Finite Element Method (UFEM), the present method gives the direct information about the local electromagnetic field and it significantly reduces the computing time of both the constitutive effective parameters and the local fields.

This paper is organized as follows. The second section is devoted to define the studied domain and the adequate boundary conditions to compute the local properties. The third and the fourth section are dedicated to compute the local fields and to express the macroscopic properties respectively. The fifth section gives the expressions of the homogenized constitutive parameters of the chiral media. The sixth section presents the numerical validation of the present method and we compare the obtained results to those of the literature [42]-[44]. Finally, our conclusions are drawn in the last section.

## II. STUDIED DOMAIN

As shown in Fig. 1, the artificial structure  $\Omega$  is modeled as a biperiodic array containing two constituents and obtained by suspending the identical chiral inclusions in a homogeneous chiral background. The periods of this structure are noted by  $\alpha$  and  $\beta$  along the axes  $(ox)$  and  $(oy)$  respectively, and the thickness is noted by  $d$ . The constitutive parameters, i.e., permittivity  $\epsilon$ , permeability  $\mu$ , and chirality  $\kappa$  of the structure, are biperiodic.

$$\gamma(\mathbf{u} + \alpha\mathbf{e}_1) = \gamma(\mathbf{u} + \beta\mathbf{e}_2) = \gamma(\mathbf{u}) \quad (1)$$

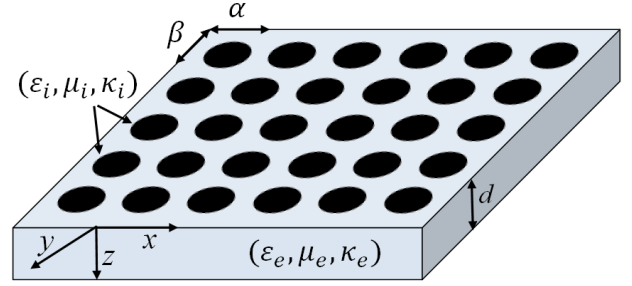


Fig. 1. Infinite biperiodic artificial material

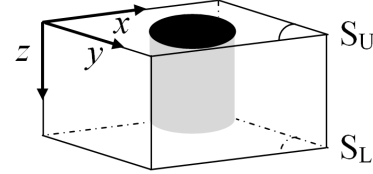


Fig. 2. Elementary cell.

where  $\gamma \in \{\epsilon, \mu, \kappa\}$ ,  $\mathbf{e}_1 = (1, 0, 0)$ ,  $\mathbf{e}_2 = (0, 1, 0)$  and  $\mathbf{e}_3 = (0, 0, 1)$  are the vectors of the canonical basis  $B$  for three-dimensional space, and  $\mathbf{u} = (x, y, z)$ .

The electromagnetic properties of the chiral medium, assuming time dependency is  $e^{j\omega t}$ , are described by the constitutive relations which are expressed as follow:

$$\begin{aligned} \mathbf{D} &= \epsilon\mathbf{E} - j\kappa\sqrt{\epsilon_0\mu_0}\mathbf{H} \\ \mathbf{B} &= \mu\mathbf{H} + j\kappa\sqrt{\epsilon_0\mu_0}\mathbf{E} \end{aligned} \quad (2)$$

where  $\mathbf{E}$  is the electric field,  $\mathbf{H}$  is the magnetic field,  $\mathbf{D}$  is the electric induction,  $\mathbf{B}$  is the magnetic displacement,  $\epsilon_0$  and  $\mu_0$  are the permittivity and the permeability of the vacuum.

The structure is illuminated by an incidence electromagnetic wave having an arbitrary incidence. This last is defined by the angles  $\theta_i$  and  $\phi_i$ , and the wavelength is considered large enough compared to the periods  $\alpha$  and  $\beta$ . For arbitrary incident plane wave, the direction of incidence is:

$$\vec{a}_i = \sin\theta_i \cos\phi_i \mathbf{e}_1 + \sin\theta_i \sin\phi_i \mathbf{e}_2 - \cos\theta_i \mathbf{e}_3 \quad (3)$$

The electric and the magnetic field depend on the electromagnetic properties of the material and verify the Helmholtz equations of chiral media. These equations are obtained by combining the Maxwell equations and the constitutive relations (2), and they are given by:

$$\begin{aligned} \nabla \times \mu^{-1} \nabla \times \mathbf{E} - 2\kappa \nabla \times \mathbf{E} + \omega^2 (\kappa^2 \epsilon_0 \mu_0 - \epsilon) \mathbf{E} &= -j\omega \mathbf{J} \\ \nabla \times \epsilon^{-1} \nabla \times \mathbf{H} + 2\kappa \nabla \times \mathbf{H} + \omega^2 (\kappa^2 \epsilon_0 \mu_0 - \mu) \mathbf{H} &= \\ (\nabla \times -\omega\kappa\sqrt{\epsilon_0\mu_0})\epsilon^{-1}\mathbf{J} \end{aligned} \quad (4)$$

where  $\nabla \times$  is the curl operator and  $\mathbf{J}$  is the excitation source.

We use the decomposition scheme which consists in transforming the chiral medium into their two equivalent isotropic media. Each one can be treated separately. This separation represents an important advantage. It reduces the complexity of the computation of the electromagnetic fields by removing the terms of  $\nabla \times \mathbf{E}$  and  $\nabla \times \mathbf{H}$  from (4). We note that

these two isotropic media are characterized by four equivalent permittivity/permeability parameters of  $\epsilon_{\pm}$  and  $\mu_{\pm}$  as follows:

$$\begin{aligned} \mathbf{D}_{\pm} &= \epsilon \mathbf{E}_{\pm} - j\kappa \sqrt{\epsilon_0 \mu_0} \mathbf{H}_{\pm} = \epsilon_{\pm} \mathbf{E}_{\pm} \\ \mathbf{B}_{\pm} &= j\kappa \sqrt{\epsilon_0 \mu_0} \mathbf{E}_{\pm} + \mu \mathbf{H}_{\pm} = \mu_{\pm} \mathbf{H}_{\pm} \end{aligned} \quad (5)$$

where the symbols  $+$  and  $-$  denote, respectively, the right-hand and the left-hand side circular polarized eigenwaves inside the chiral medium. We can easily verify that the respective equivalent permittivity  $\epsilon_{\pm}$  and permeability  $\mu_{\pm}$  of the eigenmodes should obey the following relation:

$$(\epsilon - \epsilon_{\pm})(\mu - \mu_{\pm}) = \kappa^2 \epsilon_0 \mu_0 \quad (6)$$

The permittivity  $\epsilon_{\pm}$  and the permeability  $\mu_{\pm}$  are expressed as function of  $\epsilon$ ,  $\mu$  and  $\kappa$  as follow:

$$\begin{aligned} \epsilon_{\pm} &= \epsilon \left( 1 \pm \kappa \sqrt{\frac{\epsilon_0 \mu_0}{\epsilon \mu}} \right) \\ \mu_{\pm} &= \mu \left( 1 \pm \kappa \sqrt{\frac{\epsilon_0 \mu_0}{\epsilon \mu}} \right) \end{aligned} \quad (7)$$

According to the Floquet's theorem, the local fields  $\mathbf{E}_{\pm}$  and  $\mathbf{H}_{\pm}$  are pseudo-periodic and perform the following relations:

$$\begin{aligned} \mathbf{f}(\mathbf{u} + \alpha \mathbf{e}_1) &= \mathbf{f}(\mathbf{u}) e^{-jk_{ox} \cdot \alpha} \\ \mathbf{f}(\mathbf{u} + \beta \mathbf{e}_2) &= \mathbf{f}(\mathbf{u}) e^{-jk_{oy} \cdot \beta} \end{aligned} \quad (8)$$

where  $k_{ox}$ ,  $k_{oy}$  and  $k_{oz}$  are the components of  $\mathbf{k}$  wave vector.

The decomposition of the chiral media into two fictional isotropic achiral materials and the Floquet's theorem allow transforming the study into two elementary cell, noted  $C_+$  and  $C_-$ , characterized, respectively, by  $(\epsilon_+, \mu_+)$  and  $(\epsilon_-, \mu_-)$ . Instead of resolving the Helmholtz equations (4) in  $C$ , we resolve the Helmholtz ones for the isotropic media in  $C_{\pm}$ :

$$\begin{aligned} \nabla \times \mu_{\pm}^{-1} \nabla \times \mathbf{E}_{\pm} - \omega^2 \epsilon_{\pm} \mathbf{E}_{\pm} &= j\omega \mathbf{J} \\ \nabla \times \epsilon_{\pm}^{-1} \nabla \times \mathbf{H}_{\pm} - \omega^2 \mu_{\pm} \mathbf{H}_{\pm} &= -j \nabla \times \epsilon_{\pm}^{-1} \mathbf{J} \end{aligned} \quad (9)$$

### III. COMPUTATION OF THE ELECTROMAGNETIC FIELD

In order to resolve the equations (9), we use the FEM and its ability to treat the domains containing complex shapes of inclusions. This method requires to impose the boundary conditions on the different sides of the unit cell  $C_{\pm}$ . Firstly, the upper surface  $S_U$  of unit cell is the excitation surface (see Fig. 2). Secondly, according to Floquet's theorem, we impose the pseudo-periodicity conditions on the lateral sides. Thirdly, it remains to determine the boundary conditions to impose on lower surface  $S_L$ . In [41], we added an infinite vacuum layer under the elementary cell, we introduced a fictitious boundary to delineate the studied extended cell on the one hand and to obtain a suitable approximation of the original problem, the Silver-Müller absorbing boundary conditions must be located sufficiently far from the obstacle on the other hand. We also imposed the pseudo periodicity conditions on the lateral sides of the vacuum layer. Choosing this boundary conditions has given good results in the lossy media case. However, in this configuration of the chiral media with a cascade of numerical results, the Silver-Müller absorbing conditions give just an average results. Then to improve the numerical results in the chiral media, we propose another configuration of the boundary conditions on the lower surface  $S_L$ . After removing the added vacuum layer, we impose the Neumann conditions

directly on the lower surface  $S_L$ . In fact, we can see the elementary cell as an open circuit.

The electric field solution of the first equation of (9) depends on the excitation source. To obtain this field according to  $TE$  and  $TM$  modes, we excite the upper surface  $S_U$  of the cell (see Fig. 2) by the normalized magnetic fields  $\mathbf{h}_{TE_{00}}$  and  $\mathbf{h}_{TM_{00}}$  respectively. They are the fundamental Floquet modes and their expressions are given as follows:

$$\begin{aligned} \mathbf{h}_{00}^{TE}(x, y) &= \frac{\exp(-jk_{0x}x - jk_{0y}y)}{\sqrt{\alpha\beta}} \times (\cos\varphi_i \mathbf{e}_1 + \sin\varphi_i \mathbf{e}_2) \\ \mathbf{h}_{00}^{TM}(x, y) &= \frac{\exp(-jk_{0x}x - jk_{0y}y)}{\sqrt{\alpha\beta}} \times (-\sin\varphi_i \mathbf{e}_1 + \cos\varphi_i \mathbf{e}_2) \end{aligned} \quad (10)$$

To compute the magnetic field according to the  $TE$  and  $TM$  modes, we use the same way. For this purpose, we excite the surface  $S_U$  by the normalized electric field  $\mathbf{e}_{TE_{00}}$  and  $\mathbf{e}_{TM_{00}}$  and we resolve the second equation of (9).

The weak formulation of the local field is obtained by multiplying the associate equation (9) by a test function and integrating over  $C_{\pm}$ . The elementary cell is decomposed to tetrahedral elements and we use the edge finite elements for the electric field.

$$[A_{\pm}](e_{\pm}) = (b) \quad (11)$$

where  $A_{\pm}$  is the matrix system of order  $n \times n$ ,  $(e_{\pm})$  is the unknown vector of order  $n$  which contains the degrees of freedom  $e_a$  on the edges of the mesh of  $C_{\pm}$ , and  $(b)$  is  $n \times 1$  column vector describing the known values of the field on the boundary conditions.

Building the matrix system is very important element in the computation of the local fields. For example, to compute  $\mathbf{E}_+$  and  $\mathbf{E}_-$ , we have to construct the associate matrices  $A_+$  and  $A_-$  where just the permittivity and the permeability which are different. For this, we propose optimizing the construction of these matrices. In fact, instead of constructing these matrices for the elementary cell, we construct the matrices associated to the inclusion  $C_i$  and the host media  $C_e$  by keeping the same edges' numbering of  $C$  for the edges of  $C_i$  and  $C_e$ . We note that the  $n \times n$  associated matrices are given as follows:

$$\begin{aligned} M_e(\gamma, \rho) &= \int_{C_e} \nabla \times \mathbf{w}_{\gamma}(\mathbf{u}) \cdot \nabla \times \mathbf{w}_{\rho}(\mathbf{u}) \mathbf{d}\mathbf{u} \\ M_i(\gamma, \rho) &= \int_{C_i} \nabla \times \mathbf{w}_{\gamma}(\mathbf{u}) \cdot \nabla \times \mathbf{w}_{\rho}(\mathbf{u}) \mathbf{d}\mathbf{u} \end{aligned} \quad (12)$$

and

$$\begin{aligned} N_e(\gamma, \rho) &= \int_{C_e} \mathbf{w}_{\gamma}(\mathbf{u}) \cdot \mathbf{w}_{\rho}(\mathbf{u}) \mathbf{d}\mathbf{u} \\ N_i(\gamma, \rho) &= \int_{C_i} \mathbf{w}_{\gamma}(\mathbf{u}) \cdot \mathbf{w}_{\rho}(\mathbf{u}) \mathbf{d}\mathbf{u} \end{aligned} \quad (13)$$

We use the tetrahedral mesh for the elementary cell and we represent the set of the tetrahedral elements of  $C_i$  and  $C_e$  by  $\mathcal{T}_i$  and  $\mathcal{T}_e$  respectively. We note  $\eta$  and  $\zeta$  two edges of a tetrahedron  $T$  in the mesh.

$$\begin{aligned} \forall \eta, \zeta \in T \subset \mathcal{T}_i &\Rightarrow M_e(\eta, \zeta) = N_e(\eta, \zeta) = 0 \\ \forall \eta, \zeta \in T \subset \mathcal{T}_e &\Rightarrow M_i(\eta, \zeta) = N_i(\eta, \zeta) = 0 \end{aligned} \quad (14)$$

The matrices  $M_i$ ,  $M_e$ ,  $N_i$  and  $N_e$  dependent only on the domain's mesh. They do not depend on constitutive parameters and on frequency. Since the constitutive parameters are constant in the inclusion and in the host media. Then, the

matrices  $A_+$  and  $A_-$  can be expressed as a function of these four matrices and they are given as follows:

$$\begin{aligned} A_+ &= \mu_{e,+}^{-1} M^e + \mu_{i,+}^{-1} M^i - \omega^2 (\epsilon_{e,+} N^e + \epsilon_{i,+} N^i) \\ A_- &= \mu_{e,-}^{-1} M^e + \mu_{i,-}^{-1} M^i - \omega^2 (\epsilon_{e,-} N^e + \epsilon_{i,-} N^i) \end{aligned} \quad (15)$$

where  $\epsilon_{i,+}$ ,  $\epsilon_{i,-}$ ,  $\mu_{i,+}$ ,  $\mu_{i,-}$  are the parameters of the inclusion, and  $\epsilon_{e,+}$ ,  $\epsilon_{e,-}$ ,  $\mu_{e,+}$ ,  $\mu_{e,-}$  are the parameters of the host media.

For the magnetic field  $\mathbf{H}_\pm$ , we use the same mesh with the facet elements. To optimize the construction of the matrices system  $B_+$  and  $B_-$  which associated to  $\mathbf{H}_+$  and  $\mathbf{H}_-$ , we employ the same method used for  $A_+$  and  $A_-$ .

#### IV. MACROSCOPIC ELECTROMAGNETIC PROPERTIES

The local electromagnetic response, which takes into account the different specificities of the structure on a microscopic scale, are encapsulated in the macroscopic electromagnetic response. This last is described by the macroscopic fields  $(\mathbf{E}_{\pm,m}, \mathbf{H}_{\pm,m})$  and the macroscopic inductions  $(\mathbf{D}_{\pm,m}, \mathbf{B}_{\pm,m})$ .

In the case of the isotropic and anisotropic heterogeneous media, the homogenized permittivity and permeability are independent respectively of the value of the permeability and the permittivity of the cell constituents. Then, for the computation of  $\epsilon_{\pm,h}$ , we take  $\mu_\pm = \mu_0$ . The same, for the computation of  $\mu_{\pm,h}$ , we take  $\epsilon_\pm = \epsilon_0$ . In addition, the resolution of the equations (9) allows obtaining the local fields  $(\mathbf{E}_\pm, \mathbf{H}_\pm)$ , and the use of the constitutive relationship (5) enables to evaluate the local inductions  $(\mathbf{D}_\pm, \mathbf{B}_\pm)$ .

We underline that the local fields are pseudo-periodic. Thus, the macroscopic fields  $(\mathbf{E}_{\pm,m}, \mathbf{H}_{\pm,m})$  can be obtained by averaging  $(\mathbf{E}_\pm, \mathbf{H}_\pm)$  over the  $C_\pm$  and their expressions are:

$$\mathbf{E}_{\pm,m} = \frac{1}{\alpha.\beta.d} \int_0^\alpha \int_0^\beta \int_0^d \mathbf{E}_\pm(\mathbf{u}) dx dy dz \quad (16)$$

$$\mathbf{H}_{\pm,m} = \frac{1}{\alpha.\beta.d} \int_0^\alpha \int_0^\beta \int_0^d \mathbf{H}_\pm(\mathbf{u}) dx dy dz \quad (17)$$

where  $\alpha.\beta.d$  represents the volume of  $C_\pm$ .

The permittivities  $\epsilon_\pm$  are biperiodic because the constitutive parameters  $\epsilon$ ,  $\mu$  and  $\kappa$  are biperiodic:

$$\begin{aligned} \epsilon_\pm(\mathbf{u} + \alpha\mathbf{e}_1) &= \\ \epsilon(\mathbf{u} + \alpha\mathbf{e}_1) \left( 1 \pm \kappa(\mathbf{u} + \alpha\mathbf{e}_1) \sqrt{\frac{\epsilon_0\mu_0}{\epsilon(\mathbf{u} + \alpha\mathbf{e}_1)\mu(\mathbf{u} + \alpha\mathbf{e}_1)}} \right) &= \\ \epsilon(\mathbf{u}) \left( 1 \pm \kappa(\mathbf{u}) \sqrt{\frac{\epsilon_0\mu_0}{\epsilon(\mathbf{u})\mu(\mathbf{u})}} \right) &= \epsilon_\pm(\mathbf{u}) \end{aligned} \quad (18)$$

$$\begin{aligned} \epsilon_\pm(\mathbf{u} + \beta\mathbf{e}_2) &= \\ \epsilon(\mathbf{u} + \beta\mathbf{e}_2) \left( 1 \pm \kappa(\mathbf{u} + \beta\mathbf{e}_2) \sqrt{\frac{\epsilon_0\mu_0}{\epsilon(\mathbf{u} + \beta\mathbf{e}_2)\mu(\mathbf{u} + \beta\mathbf{e}_2)}} \right) &= \\ \epsilon(\mathbf{u}) \left( 1 \pm \kappa(\mathbf{u}) \sqrt{\frac{\epsilon_0\mu_0}{\epsilon(\mathbf{u})\mu(\mathbf{u})}} \right) &= \epsilon_\pm(\mathbf{u}) \end{aligned} \quad (19)$$

Also the permeabilities  $\mu_\pm$  are biperiodic:

$$\mu_\pm(\mathbf{u} + \alpha\mathbf{e}_1) = \mu_\pm(\mathbf{u}) \quad (20)$$

$$\mu_\pm(\mathbf{u} + \beta\mathbf{e}_2) = \mu_\pm(\mathbf{u}) \quad (21)$$

The local inductions  $\mathbf{D}_\pm$  and  $\mathbf{B}_\pm$  are pseudo-periodic because the parameter  $\epsilon_\pm$  and  $\mu_\pm$  are periodic and the local fields  $\mathbf{E}_\pm$  and  $\mathbf{H}_\pm$  are pseudo-periodic:

$$\begin{aligned} \mathbf{D}_\pm(\mathbf{u} + \alpha\mathbf{e}_1) &= \epsilon_\pm(\mathbf{u} + \alpha\mathbf{e}_1) \mathbf{E}_\pm(\mathbf{u} + \alpha\mathbf{e}_1) \\ &= \epsilon_\pm(\mathbf{u}) \mathbf{E}_\pm(\mathbf{u}) e^{-jk_i\alpha} \\ &= \mathbf{D}_\pm(\mathbf{u}) e^{-jk_i\alpha} \end{aligned} \quad (22)$$

$$\begin{aligned} \mathbf{D}_\pm(\mathbf{u} + \beta\mathbf{e}_2) &= \epsilon_\pm(\mathbf{u} + \beta\mathbf{e}_2) \mathbf{E}_\pm(\mathbf{u} + \beta\mathbf{e}_2) \\ &= \epsilon_\pm(\mathbf{u}) \mathbf{E}_\pm(\mathbf{u}) e^{-jk_i\beta} \\ &= \mathbf{D}_\pm(\mathbf{u}) e^{-jk_i\beta} \end{aligned} \quad (23)$$

and

$$\mathbf{B}_\pm(\mathbf{u} + \alpha\mathbf{e}_1) = \mathbf{B}_\pm(\mathbf{u}) e^{-jk_i\alpha} \quad (24)$$

$$\mathbf{B}_\pm(\mathbf{u} + \beta\mathbf{e}_2) = \mathbf{B}_\pm(\mathbf{u}) e^{-jk_i\beta} \quad (25)$$

Then, the macroscopic inductions  $(\mathbf{D}_{\pm,m}, \mathbf{B}_{\pm,m})$  can be deduced by averaging the local inductions over the cell  $C_\pm$  and their expressions are giving by:

$$\begin{aligned} \mathbf{D}_{\pm,m} &= \frac{1}{\alpha.\beta.d} \int_0^\alpha \int_0^\beta \int_0^d \mathbf{D}_\pm(\mathbf{u}) dx dy dz \\ &= \frac{1}{\alpha.\beta.d} \int_0^\alpha \int_0^\beta \int_0^d \epsilon_\pm(\mathbf{u}) \mathbf{E}_\pm(\mathbf{u}) dx dy dz \end{aligned} \quad (26)$$

$$\begin{aligned} \mathbf{B}_{\pm,m} &= \frac{1}{\alpha.\beta.d} \int_0^\alpha \int_0^\beta \int_0^d \mathbf{B}_\pm(\mathbf{u}) dx dy dz \\ &= \frac{1}{\alpha.\beta.d} \int_0^\alpha \int_0^\beta \int_0^d \mu_\pm(\mathbf{u}) \mathbf{H}_\pm(\mathbf{u}) dx dy dz \end{aligned} \quad (27)$$

#### V. HOMOGENIZED CONSTITUTIVE PARAMETERS

The homogenization can be seen as a process of averaging of the Maxwell's equations, with rapidly varying coefficients, over the initial studied media. This process allows obtaining the macroscopic (or homogenized) version of Maxwell's equations with constant coefficients.

The computation of the volume integrals (16), (17), (26) and (27) permits to obtain the macroscopic fields  $(\mathbf{E}_{\pm,m}, \mathbf{H}_{\pm,m})$  and macroscopic inductions  $(\mathbf{D}_{\pm,m}, \mathbf{B}_{\pm,m})$ . Now, we assume that these macroscopic quantities satisfy the constitutive relationship  $(\mathbf{D}_{\pm,m} = \bar{\epsilon}_{\pm,h} \mathbf{E}_{\pm,m}$  and  $\mathbf{B}_{\pm,m} = \bar{\mu}_{\pm,h} \mathbf{H}_{\pm,m})$ . In the first time, we write the components of each macroscopic quantity as the sum of the real ( $Re$ ) and imaginary part ( $Im$ ). In the second time, we multiply the first equation by  $Re(\mathbf{E}_{\pm,m}^u) - jIm(\mathbf{E}_{\pm,m}^u)$  and the second one by  $Re(\mathbf{H}_{\pm,m}^u) - jIm(\mathbf{H}_{\pm,m}^u)$ , we obtain the expressions of the homogenized permittivity and permeability as diagonal matrices:

$$\begin{aligned} \bar{\epsilon}_{\pm,h}^{uu} &= \frac{\langle Re(\mathbf{D}_{\pm,m}^u), Re(\mathbf{E}_{\pm,m}^u) \rangle + \langle Im(\mathbf{D}_{\pm,m}^u), Im(\mathbf{E}_{\pm,m}^u) \rangle}{\|Re(\mathbf{E}_{\pm,m}^u)\|^2 + \|Im(\mathbf{E}_{\pm,m}^u)\|^2} \\ \bar{\mu}_{\pm,h}^{uu} &= \frac{\langle Re(\mathbf{B}_{\pm,m}^u), Re(\mathbf{H}_{\pm,m}^u) \rangle + \langle Im(\mathbf{B}_{\pm,m}^u), Im(\mathbf{H}_{\pm,m}^u) \rangle}{\|Re(\mathbf{H}_{\pm,m}^u)\|^2 + \|Im(\mathbf{H}_{\pm,m}^u)\|^2} \end{aligned} \quad (28)$$

where  $u \in \{x, y, z\}$ ,  $uu \in \{xx, yy, zz\}$ ,  $\langle \cdot, \cdot \rangle$  represents the scalar product and  $\|\cdot\|$  represents the Euclidean norm.

We have decomposed the initial inhomogeneous chiral cell  $C$  to two isotropic cells  $C_\pm$  ( $C_+$  and  $C_-$ ). Thus, we have proceeded to homogenize each one separately by computing the homogenized constitutive parameters  $\bar{\epsilon}_{\pm,h}$  and  $\bar{\mu}_{\pm,h}$ . From these parameters, we can express the homogenized constitutive parameters  $\bar{\epsilon}_h$ ,  $\bar{\mu}_h$  and  $\bar{\kappa}_h$  of the initial chiral cell  $C$ .

The homogenized cells of the chiral elementary cell  $C$  and their isotropic cells  $C_\pm$  are noted respectively by  $C_h$  and  $C_{\pm,h}$ . The homogenized cell  $C_h$  is a chiral media and it can be also regarded as the sum of two separate isotropic and homogeneous media. In addition, these two separate

homogenous isotropic media are nothing more than  $C_{\pm,h}$  which characterized by  $\bar{\epsilon}_{\pm,h}$  and  $\bar{\mu}_{\pm,h}$ . Then, these last can be expressed as a function of  $\bar{\epsilon}_h$ ,  $\bar{\mu}_h$  and  $\bar{\kappa}_h$ :

$$\begin{aligned}\bar{\epsilon}_{\pm,h}^{uu} &= \bar{\epsilon}_h^{uu} \left(1 \pm \bar{\kappa}_h^{uu} \sqrt{\frac{\epsilon_0 \mu_0}{\bar{\epsilon}_h^{uu} \bar{\mu}_h^{uu}}}\right) \\ \bar{\mu}_{\pm,h}^{uu} &= \bar{\mu}_h^{uu} \left(1 \pm \bar{\kappa}_h^{uu} \sqrt{\frac{\epsilon_0 \mu_0}{\bar{\epsilon}_h^{uu} \bar{\mu}_h^{uu}}}\right)\end{aligned}\quad (29)$$

where  $uu \in \{xx, yy, zz\}$ .

The known parameters  $\bar{\epsilon}_{\pm,h}$  and  $\bar{\mu}_{\pm,h}$  are expressed as a function of the unknown parameters  $\bar{\epsilon}_h$ ,  $\bar{\mu}_h$  and  $\bar{\kappa}_h$ . Then, by adding  $\bar{\epsilon}_{+,h}$  to  $\bar{\epsilon}_{-,h}$  and  $\bar{\mu}_{+,h}$  to  $\bar{\mu}_{-,h}$ , we eliminate the chirality term and we obtain the expressions of the  $\bar{\epsilon}_h$  and  $\bar{\mu}_h$ :

$$\begin{aligned}\bar{\epsilon}_h &= \frac{1}{2}(\bar{\epsilon}_{+,h} + \bar{\epsilon}_{-,h}) \\ \bar{\mu}_h &= \frac{1}{2}(\bar{\mu}_{+,h} + \bar{\mu}_{-,h})\end{aligned}\quad (30)$$

From the equations (29), the expressions of  $\bar{\kappa}_h$  are:

$$\begin{aligned}\bar{\kappa}_h^{uu} &= \pm \sqrt{\frac{\bar{\mu}_h^{uu}}{\epsilon_0 \mu_0 \bar{\epsilon}_h^{uu}} (\bar{\epsilon}_{\pm,h}^{uu} - \bar{\epsilon}_h^{uu})} \\ \bar{\kappa}_h^{uu} &= \pm \sqrt{\frac{\bar{\epsilon}_h^{uu}}{\epsilon_0 \mu_0 \bar{\mu}_h^{uu}} (\bar{\mu}_{\pm,h}^{uu} - \bar{\mu}_h^{uu})}\end{aligned}\quad (31)$$

We multiply the equations (31) term by term and we replace  $\bar{\epsilon}_h^{uu}$  and  $\bar{\mu}_h^{uu}$  by their expressions, we obtain:

$$\bar{\kappa}_h^{uu} = \sqrt{\frac{1}{4\epsilon_0 \mu_0} (\bar{\epsilon}_{+,h}^{uu} - \bar{\epsilon}_{-,h}^{uu}) (\bar{\mu}_{+,h}^{uu} - \bar{\mu}_{-,h}^{uu})} \quad (32)$$

The matrices  $\bar{\epsilon}_h$ ,  $\bar{\mu}_h$  and  $\bar{\kappa}_h$  are diagonal. Thus, we can express the matricial equation of the homogenized chirality:

$$\bar{\kappa}_h^2 = \frac{1}{4\epsilon_0 \mu_0} (\bar{\epsilon}_{+,h} - \bar{\epsilon}_{-,h}) (\bar{\mu}_{+,h} - \bar{\mu}_{-,h}) \quad (33)$$

## VI. NUMERICAL VALIDATION AND RESULTS

In this section, we present the numerical results of different types of structures. The computation of the parameters is done for an incidence wave defined by  $\theta_i = \pi/4$  and  $\phi_i = \pi/3$ , and the ratio of wavelength  $\lambda$  on period  $\alpha$  is  $\lambda/\alpha = 15$ . The obtained results are compared to Maxwell-Garnett (MG) formalism and UFEM [42]-[43]. We recall that the UFEM suppose that the studied structure is an infinite tri-periodic structure in the one hand; on the other hand, the homogenized parameters are computed in unit cell with tri-periodic conditions and they are expressed in terms of a sub-corrector.

### A. Lattice with circular cylinder inclusions

We consider an infinite biperiodic lattice of chiral identical inclusions suspended in free space. The inclusions are characterized by the relative permittivity  $\epsilon_{i,r} = 15$ , the relative permeability  $\mu_{i,r} = 15$  and the chirality  $\kappa_i = 7$ . The periods are  $\alpha = \beta = 1\text{cm}$  and the thickness is  $d = 1\text{cm}$ . We mention that the thickness of inclusions is also equal to  $1\text{cm}$ .

Fig. 3 plots the components  $\bar{\epsilon}_{h,r}^{xx}$  and  $\bar{\epsilon}_{h,r}^{yy}$  ( $\bar{\epsilon}_{h,r}^{xx} = \bar{\epsilon}_{h,r}^{yy}$ ) of the homogenized relative permittivity  $\bar{\epsilon}_{h,r}$  as a function of the volume fraction  $f$ . The present method and UFEM produce the same results for different values of  $f$  and a good agreement is shown throughout the curve. Indeed, the difference between these methods is less than 2%. For the large values of  $f$ , MG provides different values compared to other two methods. Indeed MG does not take into account the effect of the shape

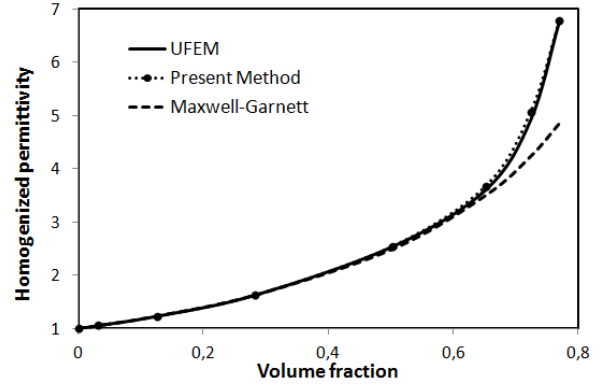


Fig. 3. Component  $\bar{\epsilon}_{h,r}^{xx}$  of  $\bar{\epsilon}_{h,r}$  of a lattice with circular cylinders.

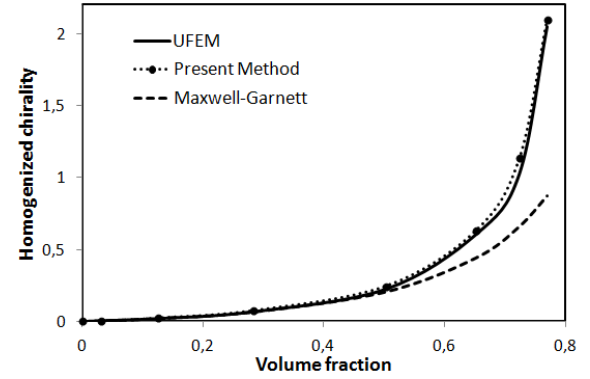


Fig. 4. Component  $\bar{\kappa}_h^{xx}$  of  $\bar{\kappa}_h$  of a lattice with circular cylinders.

of inclusions and the interactions between inclusions on the homogenized constitutive parameters.

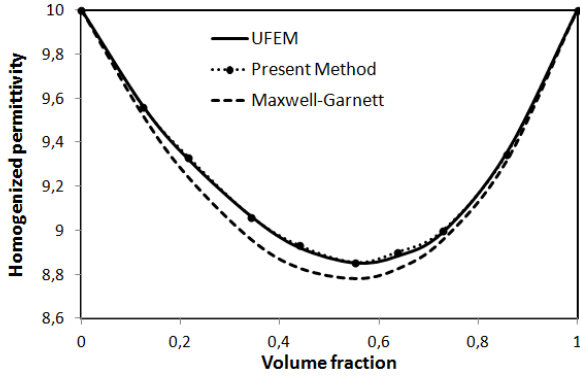
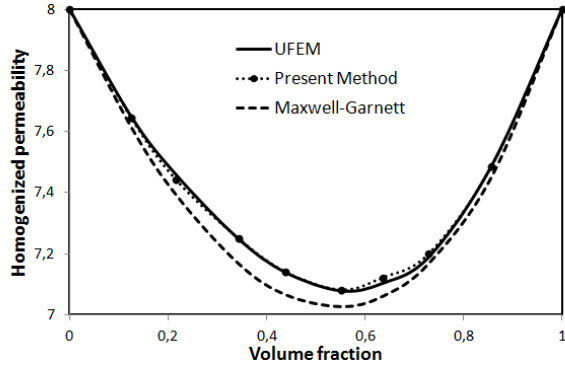
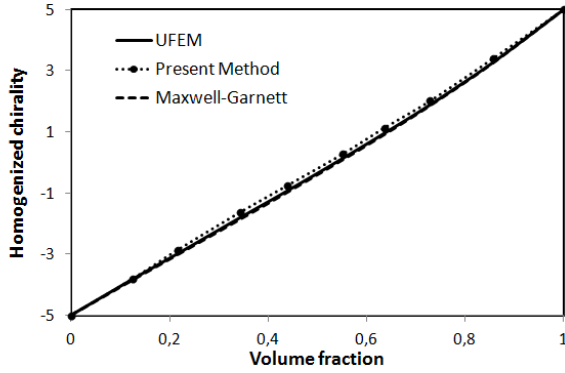
Fig. 4 represents the components  $\bar{\kappa}_h^{xx}$  and  $\bar{\kappa}_h^{yy}$  ( $\bar{\kappa}_h^{xx} = \bar{\kappa}_h^{yy}$ ) of the homogenized chirality  $\bar{\kappa}_h$ . We have the same behaviour, and the present method and UFEM give the same results. As we can see, from 0.5, the difference between the present method and MG increases as function of  $f$ .

### B. Lattice with cubic inclusions

We consider an infinite biperiodic lattice of identical chiral cubic inclusions characterized by the relative permittivity  $\epsilon_{i,r} = 10$ , the relative permeability  $\mu_{i,r} = 8$  and the chirality  $\kappa_i = 5$ . The host medium is also a chiral media and has the same relative permittivity  $\epsilon_{e,r} = 10$  and the same relative permeability  $\mu_{e,r} = 8$  but with an opposite chirality  $\kappa_e = -5$ . The periods and the thickness of this lattice are respectively  $\alpha = \beta = 1\text{cm}$  and  $d = 1\text{cm}$ .

Fig. 5 and Fig. 6 represent the component  $\bar{\epsilon}_{h,r}^{xx}$  and  $\bar{\mu}_{h,r}^{xx}$  of  $\bar{\epsilon}_{h,r}$  and  $\bar{\mu}_{h,r}$  respectively. We note that  $\bar{\epsilon}_{h,r}^{xx} = \bar{\epsilon}_{h,r}^{yy}$  and  $\bar{\mu}_{h,r}^{xx} = \bar{\mu}_{h,r}^{yy}$ . As we can observe,  $\bar{\epsilon}_{h,r}$  and  $\bar{\mu}_{h,r}$  are very close to UFEM's results. We underline that, although both  $\epsilon_i = \epsilon_e$  and  $\mu_i = \mu_e$ , the homogenized parameters  $\bar{\epsilon}_h$  and  $\bar{\mu}_h$  are not constant. This phenomena is due to the presence of the chirality. As in the first structure and for the same reasons, the results of the MG are different of those of the present method.

In Fig. 7, we present the component  $\bar{\kappa}_h^{xx}$  of  $\bar{\kappa}_h$ . As can be seen, the UFEM and the present method give the same


 Fig. 5. Component  $\bar{\epsilon}_{h,r}^{xx}$  of  $\bar{\epsilon}_{h,r}$  of a lattice with cubic inclusions.

 Fig. 6. Component  $\bar{\mu}_{h,r}^{xx}$  of  $\bar{\mu}_{h,r}$  of a lattice with cubic inclusions.

 Fig. 7. Component  $\bar{\kappa}_{h,r}^{xx}$  of  $\bar{\kappa}_{h,r}$  of a lattice with cubic inclusions.

results for different values of  $f$ . We note that the homogenized material is left-handed ( $\kappa_h < 0$ ) for the small value of  $f$  and is right-handed ( $\kappa_h > 0$ ) for the large value of  $f$ .

### C. Layered structure

We consider an infinite biperiodic lattice having different periods  $\alpha = 1.4cm$ ,  $\beta = 1cm$  and thickness  $d = 1cm$ . The unit cell is composed by two layers parallel to  $x$ -axis and perpendicular to  $y$ -axis. The thickness of these layers are  $d_1$  and  $d_2$  with  $d_1 + d_2 = \beta = 1cm$ . The first layer is characterized by the relative permittivity  $\epsilon_{1,r} = 5$ , the relative permeability  $\mu_{1,r} = 2$  and the chirality  $\kappa_1 = 0$ . The second one is characterized by the relative permittivity  $\epsilon_{2,r} = 20$ , the relative permeability  $\mu_{2,r} = 10$  and the chirality  $\kappa_2 = 5$ .

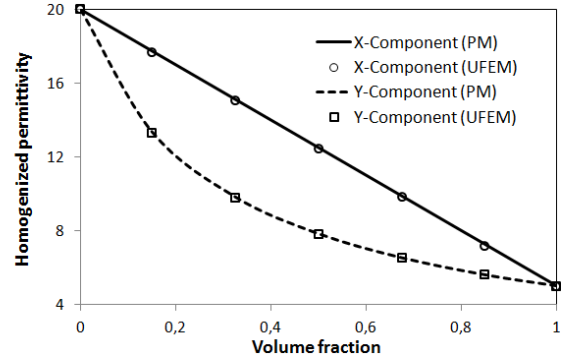
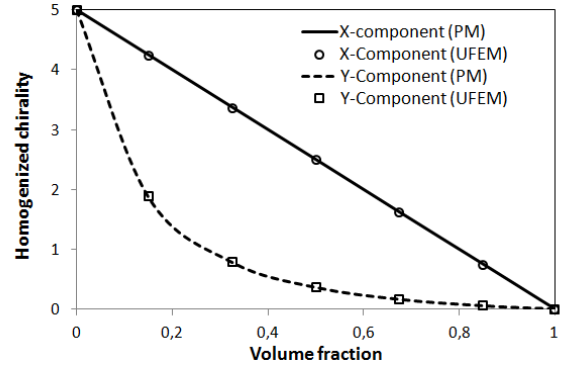

 Fig. 8. Components  $\bar{\epsilon}_{h,r}^{xx}$  and  $\bar{\epsilon}_{h,r}^{yy}$  of  $\bar{\epsilon}_{h,r}$  of a lattice with two layers.

 Fig. 9. Components  $\bar{\kappa}_{h,r}^{xx}$  and  $\bar{\kappa}_{h,r}^{yy}$  of  $\bar{\kappa}_{h,r}$  of a lattice with two layers.

Fig. 8 represents the components  $\bar{\epsilon}_{h,r}^{xx}$  and  $\bar{\epsilon}_{h,r}^{yy}$  of  $\bar{\epsilon}_{h,r}$  as a function of  $f = d_1/(d_1 + d_2)$ , and Fig. 9 represents the components  $\bar{\kappa}_{h,r}^{xx}$  and  $\bar{\kappa}_{h,r}^{yy}$  of  $\bar{\kappa}_{h,r}$ . We note that  $f$  varies between 0 and 1 which represent  $d_1 = 0$  and  $d_2 = 0$  respectively. The obtained results are compared to UFEM and as we can see, the good agreement is obtained between these methods for different values of  $f$ .

### D. Lattice with hollow inclusions

Another type of infinite biperiodic lattice is studied in this subsection. The periods and the thickness of this structure are  $\alpha = \beta = 10mm$  and thickness is  $d = 12mm$ . The inclusions are a circular hollow cylinder having an inner diameter  $r_1 = 6mm$ , an outer diameter  $r_2 = 8mm$  and a height  $h = 2.5mm$ . The parameters of the inclusions are  $\epsilon_{i,r} = \mu_{i,r} = 13$  and  $\kappa_i = 6$ . The parameters of the host media are  $\epsilon_{e,r} = \mu_{i,r} = 5\eta$  and  $\kappa_e = 2\eta$  with  $\eta$  is a real.

Fig. 10 and Fig. 11 plot the components  $\bar{\epsilon}_{h,r}^{xx}$  and  $\bar{\kappa}_{h,r}^{xx}$  of  $\bar{\epsilon}_{h,r}$  and  $\bar{\kappa}_{h,r}$  respectively. We underline that  $\bar{\epsilon}_{h,r}^{xx} = \bar{\epsilon}_{h,r}^{yy}$  and  $\bar{\kappa}_{h,r}^{xx} = \bar{\kappa}_{h,r}^{yy}$ . The different components of the effective parameters are computed as function of  $\eta$  which varies between 1 and 10. As we can see from these figures, the present method and the UFEM produce the same results.

### E. Homogenized parameters as a function of incidence angle

The incidence angle is defined by  $\theta_i$  and  $\psi_i$  which are used in the expressions of the fundamental Floquet modes. These modes are used to determinate the local fields according  $TE$



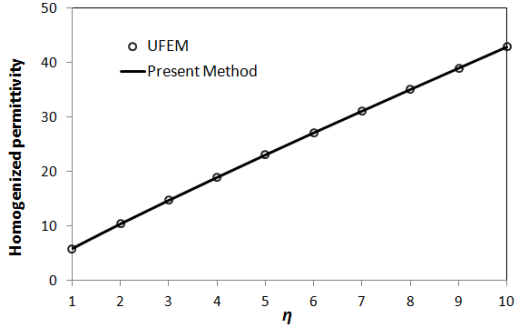
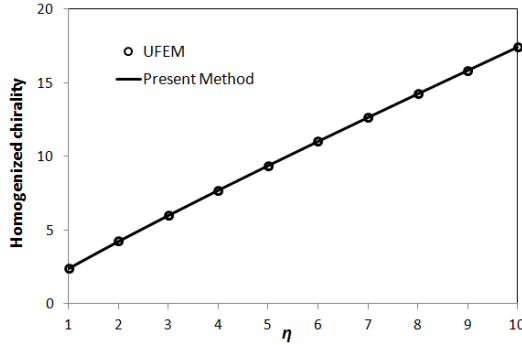

 Fig. 10. Component  $\bar{\epsilon}_{h,r}^{xx}$  of  $\bar{\epsilon}_{h,r}$  of a lattice with hollow inclusions.

 Fig. 11. Component  $\bar{\kappa}_h^{xx}$  of  $\bar{\kappa}_h$  of a lattice with hollow inclusions.

 TABLE I  
 $\bar{\epsilon}_{h,r}^{xx}$  FOR DIFFERENT VALUES OF  $\theta_i$  AND  $\phi_i$ .

	$\phi_i = 0$	$\phi_i = \pi/6$	$\phi_i = \pi/3$	$\phi_i = \pi/2$
$\theta_i = 0$	2.55	2.55	2.55	2.55
$\theta_i = \pi/6$	2.55	2.55	2.55	2.54
$\theta_i = \pi/3$	2.54	2.55	2.55	2.53

 TABLE II  
 $\bar{\kappa}_{h,r}^{xx}$  FOR DIFFERENT VALUES OF THE  $\theta_i$  AND  $\phi_i$ .

	$\phi_i = 0$	$\phi_i = \pi/6$	$\phi_i = \pi/3$	$\phi_i = \pi/2$
$\theta_i = 0$	0.241	0.241	0.240	0.240
$\theta_i = \pi/6$	0.243	0.243	0.242	0.242
$\theta_i = \pi/3$	0.248	0.248	0.247	0.245

and  $TM$  modes. In this subsection, we study the influence of the incidence angle on the homogenized parameters. For this, we retake the lattice with circular cylinder inclusions by keeping the same electromagnetic properties of inclusions and host media (see subsection VI. A). In addition, we take  $f = 0.502$  which corresponds to radius  $r = 0.4cm$ . Then, we compute  $(\bar{\epsilon}_h, \bar{\mu}_h, \bar{\kappa}_h)$  for different values of  $\theta_i$  and  $\psi_i$ .

Table 1 represents  $\bar{\epsilon}_{h,r}^{xx}$  for different values of  $\phi_i = 0, \pi/6, \pi/3, \pi/2$ , and of  $\theta_i = 0, \pi/6, \pi/3$ . Each line of this table gives the values of  $\bar{\epsilon}_{h,r}^{xx}$  at fixed  $\theta_i$  and each column gives the values of  $\bar{\epsilon}_{h,r}^{xx}$  at fixed  $\phi_i$ . As we can see, we obtain the same results and the relative error is less than 1%.

Table 2 shows the  $\bar{\kappa}_h^{xx}$  values for different values of  $\phi_i$  and  $\theta_i$ . As we can see, we have also the same results of  $\bar{\kappa}_h$ . The

numerical results, presented by Table 1 and Table 2, show that even when we change the values of the angle of incidence, the homogenized parameters remain constants. The obtained results confirm that the present method does not depend on the angle of incidence to compute the homogenized parameters.

## VII. CONCLUSION

In this paper, we have presented a new method to homogenize the biperiodic bi-isotropic artificial lattices. The accuracy and the effectiveness of the presented method has been shown by homogenizing an artificial structure with circular cylinder inclusions. The numerical results produced by the present method are compared with UFEM and the good agreement has been obtained.

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