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A hyperbolic approach for dissipative magnetohydrodynamics

Hubert Baty and Hiroaki Nishikawa

Abstract The hyperbolic method initially introduced by Nishikawa [1] for solving the diffusion equation is extended to a two-dimensional magnetohydrodynamic (MHD) model. The approach is based on reformulation of the dissipative terms in order to solve an equivalent first-order hyperbolic system. This enables the use of approximate Riemann solvers for handling dissipative and advective fluxes in the same way. The advantages of our method compared to traditional ones is illustrated by finding steady state solutions for magnetic reconnection process. In particular, the numerical solution is obtained with the same order of accuracy for the main and gradient variables over a wide range of magnetic Reynolds numbers, without any deterioration characteristic of more conventional schemes. Second, the convergence towards the steady state scales only linearly with the cell width h , giving thus a $O(1/h)$ acceleration factor. The improvement of the hyperbolic method and its extension to time-dependent MHD problems are presented. Finally, we discuss the importance of developing such new numerical methods for the sake of understanding the physical mechanisms underlying flares phenomena in plasmas.

1 Introduction

Most conventional magnetohydrodynamic (MHD) codes are based on methods using different Riemann-type solvers in order to handle discontinuities and shocks, as for example in the finite-volume based code AMRVAC [2]. Such methods are particularly adapted and efficient for purely hyperbolic equations. This is however not the case of the dissipative MHD model, where dissipative (resistive and/or viscous)

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terms play an important role. The dissipation is generally computed by a separated scheme (using a finite-difference source term approximation for example), making the comparison between the evaluation of the advective flux and the dissipative one not fully consistent. The reason for this separated treatment is due to different orders of spatial derivative of the main variable defining the fluxes, e.g. first and second order for the advective and dissipative flux respectively. An example of inconsistent behavior due to inconsistent construction of diffusive flux has been pointed out in the context of two-dimensionnal (2D) advection-diffusion problem. Indeed, a degradation of the order of accuracy has been observed for Reynolds numbers of order unity, when compared to pure advective/diffusive limits [3].

In MHD context, magnetic reconnection process involves large regions where the advection is dominant (i.e. where the local magnetic Reynolds number is much greater than one) and much smaller regions where the diffusion is dominant (i.e. where the local magnetic Reynolds number is much lower than one). As a consequence, a local deterioration of the spatial accuracy could propagate towards other regions and affect the solution in the whole domain. The solutions are also very sensitive to the detailed spatial profile of the resistivity parameter [4]. Thus, dissipative MHD requires a careful treatment of the dissipation terms and is clearly appealing for improved numerical schemes

We propose to cure the inconsistency by using the hyperbolic approach initially introduced for the diffusion equation [1], and later extended to other systems like Navier-Stokes one [5]. The basic idea is to transform the diffusive terms into hyperbolic ones by introducing diffusive fluxes as additional variables. These new variables that are gradients of the main variables, are also solutions of additional hyperbolic equations. Despite the common idea of using flux variables, this approach is different from the so-called mixed methods commonly employed in fluid mechanics. The plan of the paper is as follows. Section 2 presents the principles of hyperbolization and associated numerical procedure to find stationary solutions of a model equation. Section 3 is devoted to the extension of the hyperbolic method for our 2D MHD model. The advantages and performances of the method are illustrated in Sect.4, with a comparison of 2D stationary magnetic reconnection solutions obtained using our method with exact analytical solutions. Finally, we present ameliorations of our scheme and extensions (actually under development) to time-dependent problems in the last section before concluding.

2 The hyperbolic method

We consider a simple one-dimensionnal (1D) advection-diffusion equation for the real variable $\psi(x, t)$ (the real variable x denoting the space and the real positive one t is for time):

$$\frac{\partial \psi}{\partial t} + \frac{\partial(V\psi)}{\partial x} = \eta \frac{\partial^2 \psi}{\partial x^2} + E(x), \quad (1)$$

where the real variables $V(x)$ and $E(x)$ are given velocity and source terms, respectively. The dissipation is represented by the real coefficient η that is assumed to be uniform for simplicity. Following previously related works on the hyperbolic method, we consider the equivalent hyperbolic system:

$$\frac{\partial \psi}{\partial \tau} + \frac{\partial(V\psi)}{\partial x} - \frac{\partial(\eta p)}{\partial x} = E(x), \quad (2)$$

$$\frac{\partial p}{\partial \tau} - \frac{1}{T_r} \frac{\partial \psi}{\partial x} = -\frac{1}{T_r} p, \quad (3)$$

where p is an additional variable, and τ is a pseudo-time. The crucial remark is that, solving these two equations in the pseudo-steady state (i.e. for vanishing pseudo-time derivative terms) is equivalent of solving the original equation in the true steady state, as p is the x derivative of ψ in this limit. T_r is a relaxation time parameter defined as, $T_r = L_r^2/\eta$, where L_r is a length scale parameter. The hyperbolic method is different from classical relaxation methods, which result in stiff relaxation systems because of the requirement of a vanishing equivalent relaxation time [6]. In our model, T_r is not required to vanish because our hyperbolic model reduces to the advection-diffusion equation exactly in the steady state for any nonzero T_r . The choice of the L_r value, which also determines the T_r value, is mainly based on fast steady convergence requirement. For example, an optimal value of $L_r = 1/(2\pi)$ has been proposed in diffusion problems [1]. Different optimal formula for L_r depending on the typical problems addressed can be deduced, showing that L_r must be reduced as much as the Reynolds number is high [7].

2.1 Steady state: discretization and implementation

We re-write our system as:

$$\frac{\partial \mathbf{U}}{\partial \tau} + \frac{\partial \mathbf{F}}{\partial x} = \mathbf{S}, \quad (4)$$

where $\mathbf{U} = \begin{pmatrix} \psi \\ p \end{pmatrix}$, $\mathbf{F} = \begin{pmatrix} V\psi - \eta p \\ -\psi/T_r \end{pmatrix}$, and $\mathbf{S} = \begin{pmatrix} E \\ -p/T_r \end{pmatrix}$. The flux \mathbf{F} can be splitted

into two terms, an advective one \mathbf{F}^a and a dissipative one \mathbf{F}^d , with $\mathbf{F}^a = \begin{pmatrix} V\psi \\ 0 \end{pmatrix}$, and

$\mathbf{F}^d = \begin{pmatrix} -\eta p \\ -\psi/T_r \end{pmatrix}$. A finite-volume method is used with a spatial discretisation of the solution \mathbf{U} , namely \mathbf{U}_j , where \mathbf{U}_j is defined as the cell average of the solution of the j th spatial grid cell centered at x_j (see Appendix). Steady state solutions can be thus obtained by using a pseudo-time explicit iteration, $\mathbf{U}_j^{k+1} = \mathbf{U}_j^k - \Delta \tau \mathbf{Res}_j^k$, where k is the iteration counter, $\Delta \tau$ is the pseudo time step, and $-\mathbf{Res}_j^k$ is the residual which is required to vanish (up to a predefined given accuracy) when a steady state solution is reached.

The numerical fluxes are computed using upwind-type formula, and separating the Jacobian matrix, $\mathbf{A} = \frac{\partial \mathbf{F}}{\partial \mathbf{U}}$ into two parts, $|\mathbf{A}| \approx |\mathbf{A}^a| + |\mathbf{A}^d|$. We can thus take the maximum wave speeds to evaluate the Jacobians, The advective Jacobian $|\mathbf{A}^a|$ has a non zero eigenvalue that is equal to the local velocity V , while the dissipative Jacobian $|\mathbf{A}^d|$ has the following eigenvalues $\lambda = \pm \sqrt{\eta/T_r}$. There is an alternative way that consist in computing the eigenvalues of the full Jacobian in order to get an exactly unified scheme, but leading to more complicated implementation (especially for MHD equations). The explicit pseudo-time iteration to march towards the steady state is limited by a criterion,

$$\Delta \tau = CFL \times \text{Min} \left[\frac{h}{\sqrt{\eta/T_r} + a_n} \right], \quad (5)$$

where CFL is the Courant-Friedrichs-Lewy number less than or equal to one, and a_n is the local absolute velocity. One can also refer to the Appendix for more details.

The ability of the above numerical scheme (using a second-order finite-volume discretization with an upwind flux) to obtain steady-state solutions is demonstrated, even when a quasi-singular layer tends to form [7]. It is also shown that the same order of accuracy for the main solution and gradient is attained for a wide range of magnetic Reynolds numbers, without any deterioration characteristic of more conventional schemes. Second, the convergence towards the steady state scales only linearly with the cell width h , giving thus a $O(1/h)$ acceleration factor. This is an expected advantage from using our CFL criterion over a conventional explicit scheme, where the CFL criterion is $\Delta t = CFL \times \text{Min} \left[\frac{h}{a_n}, \frac{h^2}{\eta} \right]$.

3 Hyperbolic method for the 2D MHD equations

We consider the 2D incompressible set of dissipative (viscous and resistive) MHD equations written in flux-vorticity scalar variables as follows,

$$\frac{\partial \psi}{\partial t} + \mathbf{V} \cdot \nabla \psi = \eta \nabla^2 \psi + E(x, y), \quad (6)$$

$$\frac{\partial \Omega}{\partial t} + \mathbf{V} \cdot \nabla \Omega = \nu \nabla^2 \Omega + \mathbf{B} \cdot \nabla J, \quad (7)$$

where the main variables are the magnetic flux function $\psi(x, y)$, and the scalar vorticity $\Omega(x, y)$. Both are functions of the 2D cartesian space coordinates (x, y) . The magnetic field vector \mathbf{B} is related to ψ via $\mathbf{B} = (\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x})$. The fluid velocity $\mathbf{V} = (V_x, V_y)$ is related to Ω via $\nabla \times \mathbf{V} = \Omega \mathbf{k}$ with \mathbf{k} is the unit vector perpendicular to the (x, y) plane. The scalar variable J is the current density (i.e. component of the current density perpendicular to the 2D plane) that can be deduced from the main variable trough $J = -\nabla^2 \psi$ as a consequence of Ampere's law, taking the magnetic

permeability constant equal to one in our units. The dissipative terms are the second derivatives terms involving the viscosity coefficient ν in Ω -equation, and the resistivity coefficient η in ψ -equation. For simplicity, these dissipative coefficients are assumed constant and uniform in the present work. The source term E represents a given electric field component perpendicular to the plane. Compared to a more conventional formulation using magnetic field and fluid velocity vectors as main variables, this set of equations has the advantage of ensuring the divergence free conditions on \mathbf{B} and \mathbf{V} . Two extra equations must however be introduced in order to deduce the fluid velocity components from the two main variables, that can be shown to be (using the incompressibility assumption $\nabla \mathbf{V} = 0$), namely

$$\nabla^2 V_x = -\frac{\partial \Omega}{\partial y}, \quad (8)$$

$$\nabla^2 V_y = \frac{\partial \Omega}{\partial x}. \quad (9)$$

3.1 Steady state: discretization and implementation

The hyperbolization of the ψ -equation follows,

$$\frac{\partial \psi}{\partial \tau} + \frac{\partial(V_x \psi)}{\partial x} + \frac{\partial(V_y \psi)}{\partial y} - \frac{\partial(\eta p_\psi)}{\partial x} - \frac{\partial(\eta q_\psi)}{\partial y} = E_\psi(x, y), \quad (10)$$

$$\frac{\partial p_\psi}{\partial \tau} - \frac{1}{T_\psi} \frac{\partial \psi}{\partial x} = -\frac{1}{T_\psi} p_\psi, \quad (11)$$

$$\frac{\partial q_\psi}{\partial \tau} - \frac{1}{T_\psi} \frac{\partial \psi}{\partial y} = -\frac{1}{T_\psi} q_\psi, \quad (12)$$

where we use the notation p_ψ and q_ψ for the x and y derivatives of ψ respectively in the true steady-state, and where E_ψ is the source term. We also introduce an associated relaxation time parameter, $T_\psi = L_r^2/\eta$. The equation for the vorticity function Ω can be also hyperbolized exactly in the same way (not shown here), where two associated gradient variables are now introduced, p_Ω and q_Ω . A second corresponding relaxation time is also introduced, $T_\Omega = L_r^2/\nu$, as a second dissipative coefficient ν (viscosity) is present for viscous plasmas.

The velocity components (V_x, V_y) are computed using the vorticity trough, $\nabla^2 V_x = -\frac{\partial \Omega}{\partial y}$, and $\nabla^2 V_y = \frac{\partial \Omega}{\partial x}$. The first Poisson equations can also be hyperbolized as,

$$\frac{\partial V_x}{\partial \tau} - \frac{\partial p_x}{\partial x} - \frac{\partial q_x}{\partial y} = \frac{\partial \Omega}{\partial y}, \quad (13)$$

$$\frac{\partial p_x}{\partial \tau} - \frac{1}{T_r} \frac{\partial V_x}{\partial x} = -\frac{1}{T_r} p_x, \quad (14)$$

$$\frac{\partial q_x}{\partial \tau} - \frac{1}{T_r} \frac{\partial V_x}{\partial y} = -\frac{1}{T_r} q_x, \quad (15)$$

for V_x variable, where p_x and q_x are the associated x and y gradient variables. In a similar way, the hyperbolization for V_y variable (not shown), lead to the introduction of the gradient variables (p_y, q_y) , that are the partial derivatives of V_x and V_y in the steady state, respectively. A third relaxation time T_r is thus defined as $L_r^2/1$.

The discretization follows the method presented in previous section for the 1D model equation (see also [7] for more details).

4 Results on steady-state magnetic reconnection

The hyperbolic method is tested for the MHD equations, in order to retrieve a particular solution for 2D steady-state magnetic reconnection process, namely the re-connective diffusion solution in an inviscid medium (i.e. for $\nu = 0$) [8]. The latter solution corresponds to the velocity and magnetic field profiles of the form:

$$\mathbf{V} = (-\alpha x, \alpha y - \frac{\beta}{\alpha} \frac{E_d}{\eta \mu} \text{Daw}(\mu x)), \quad (16)$$

and

$$\mathbf{B} = (\beta x, -\beta y + \frac{E_d}{\eta \mu} \text{Daw}(\mu x)), \quad (17)$$

respectively. $\text{Daw}(x)$ is the Dawson function given by, $\text{Daw}(x) = \int_0^x \exp(t^2 - x^2) dt$. E_d is the magnitude of a uniform electric field perpendicular to the (x, y) plane, and the three real constants (α, β, μ) are related via $\mu^2 = \frac{\alpha^2 - \beta^2}{2\eta\alpha}$ with $\beta < \alpha$. Typically, we use $E_d = 0.1$, $\alpha = 1$, and β is varied between 0 and unity. Exact Dirichlet boundary conditions are imposed.

The numerical results using the hyperbolic method are computed following the procedure described in detail elsewhere [7], and compared to the expected analytical solution. A second-order finite-volume discretisation on rectangular grids with up-wind flux is employed. First, the optimal L_r value for fastest convergence is shown to be dependent on the resistivity with L_r scaling approximately as $\eta^{1/2}$, as one can see in Fig.1 for $\beta = 0$ case. As an illustration, the typically obtained solutions for main and gradient variables are plotted in Fig. 2, and the L_1 errors on main and gradient variables are displayed in Fig. 3 for a few cases. As expected, the same second order of convergence is obtained for the main and gradient variables for a wide range of Lundquist numbers (i.e. resistivity values). We have also checked that, the convergence towards the steady state scales linearly with the cell width [7].

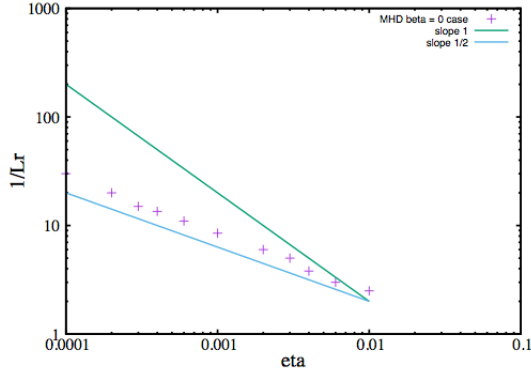


Fig. 1 Optimal $1/L_r$ value (in units of π) required for fastest numerical convergence towards steady-state, for 2D MHD reconnection with $\beta = 0$ at different η values. The η^{-1} and $\eta^{-1/2}$ dependences are indicated with plain line.

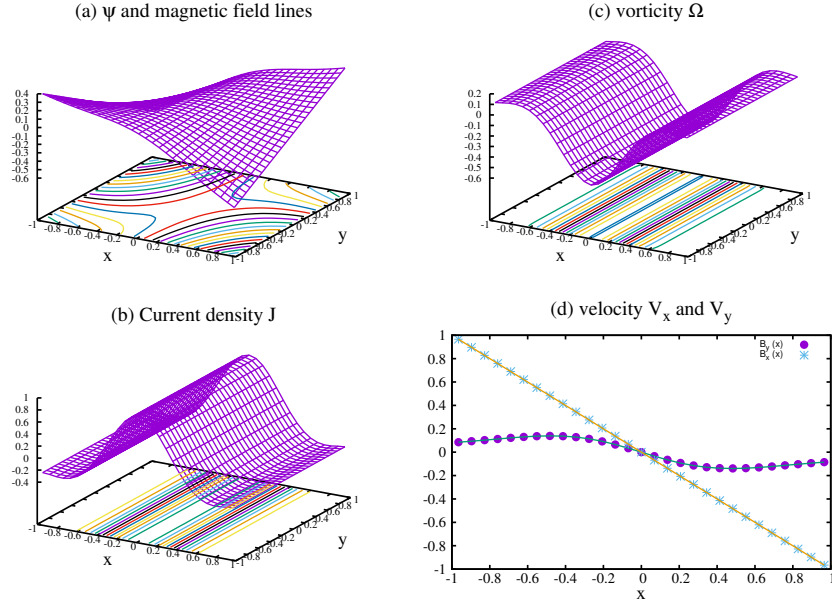


Fig. 2 Reconnection diffusion solution obtained for $\eta = 0.1$ and $\beta = 0.5$ using a moderate spatial resolution of 30×30 grid cells, covering the $[-1 : 1]^2$ domain. The flux variable $\psi(x, y)$ and iso-contours (in bottom plane) representing the magnetic field lines, the current density $J(x, y)$, the vorticity variable $\Omega(x, y)$, and velocity components $(V_x(x), V_y(x))$ at $y = 0$ are plotted in the different panels (a),(b),(c),(d), respectively.

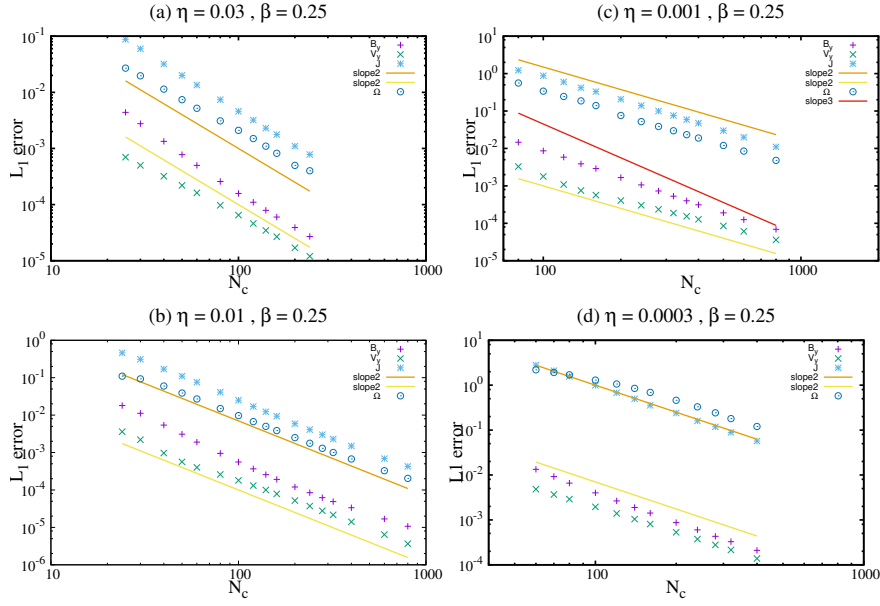


Fig. 3 Error convergence for the reconnective diffusion problem. L_1 error on $B_y, V_y, J,$ and Ω , as a function of the number of grid cells N_c in each direction. The linear (slope 1), parabolic (slope 2), and cubic (slope 3) convergence orders are also indicated with plain line. A non uniform cell spacing in x direction is used for $\eta = 0.0003$ case.

5 Extension to time dependent problems

The time dependent version of the hyperbolic method for our 1D model equation (Eq.1) can be written as,

$$\frac{\partial \psi}{\partial \tau} + \frac{\partial(V\psi)}{\partial x} - \frac{\partial(\eta p)}{\partial x} = E(x) - \frac{\partial \psi}{\partial t} \quad (18)$$

$$\frac{\partial p}{\partial \tau} - \frac{1}{T_r} \frac{\partial \psi}{\partial x} = -\frac{1}{T_r} p, \quad (19)$$

where an extra true time derivative term has been added on the right hand side of the main equation (as a source term). The idea is to evaluate this source term by using a backward difference formula, as for example,

$$\frac{\partial \psi}{\partial t} \approx \frac{3\psi - 4\psi^n + \psi^{n-1}}{2\Delta t}, \quad (20)$$

for a second order scheme (BDF2). Note that ψ^n and ψ^{n-1} are the values of ψ obtained at the two previous physical time steps t^n and t^{n-1} respectively. These schemes have been initially introduced and investigated for time-dependent advection-

diffusion equation [9]. The extension of our previous scheme (developed for steady state solutions) results now in a point-implicit pseudo-time iteration (with k index),

$$\left\{ \frac{\partial \psi_j}{\partial \tau} + \frac{3\psi_j}{2\Delta t} \right\}_{k+1} = \left\{ -\frac{1}{h} [\mathbf{F}_{j+1/2} - \mathbf{F}_{j-1/2}] + \frac{1}{h} \int_{I_j} S dx \right\}_k, \quad (21)$$

where h is the grid cell spacing, $\mathbf{F}_{j+1/2}$ and $\mathbf{F}_{j-1/2}$ represent the numerical upwind fluxes at the two cell interfaces of the j th- dual volume and S is a source term containing E and the remaining physical time derivative as $S = E + \frac{4\psi^n - \psi^{n-1}}{2\Delta t}$. The final formulation of the pseudo-time leads to,

$$\{\psi_j\}_{k+1} = \frac{1}{1 + \frac{3\Delta\tau}{2\Delta t}} \{\psi_j - \Delta\tau Res_j\}_k, \quad (22)$$

where $-Res_j$ is the residual hand side. Note that the second equation (for the gradient variable) is not changed as it does not contain such physical derivative term.

Our time-dependent BDF2 hyperbolic method has been tested on two simple problems, using the same second-order finite volume discretization described for steady-state case. First, the results for the 1D advection-diffusion of a Gaussian pulse (with a constant velocity) are illustrated in Fig. 4. For this moderate resolution, one can see that our scheme can well reproduce the advection as well as the small diffusion effect ($\eta = 10^{-3}$). Moreover, we have used two different slope limiters for the fluxes evaluation, namely the monotonized central (MC) limiter and the less numerically diffusive superbee limiter. The L_1 corresponding errors reported in Fig. 5 show that the second order expected for spatial and also time discretizations are effectively obtained. Secondly, the implementation of the method to solve a viscous Burgers equation (see Fig. 6) clearly demonstrate the necessity to implement a slope limiter in order to evitate non physical oscillations when a shock is forming (see case at $\eta = 10^{-4}$).

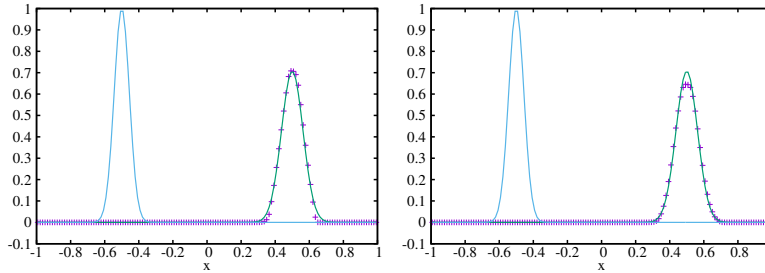


Fig. 4 Convection-diffusion of a 1D Gaussian pulse (initially centered in $x = -0.5$) using the point-implicit BDF2 scheme, obtained at a constant advection velocity $V = 1$ and a dissipation parameter $\eta = 10^{-3}$. The final integration time is $t = 1$ with a physical time step $\Delta t = 10^{-3}$. The number of grid cells is 140. Right panel corresponds to the use of a MC slope limiter, and left one to a superbee limiter. The exact solution (at $t = 1$) with plain line (centered in $x = 0.5$) is superposed to the numerical ones.

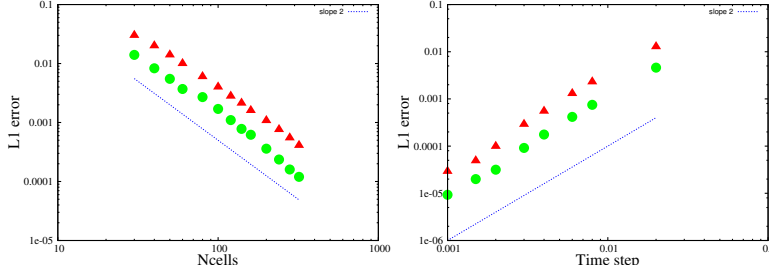


Fig. 5 L_1 error on the main variable ψ for the convection-diffusion of a 1D Gaussian pulse (see previous figure). Left panel is obtained for a physical time step $\Delta t = 10^{-3}$, and two different resistivity values $\eta = 10^{-2}$ and 10^{-4} as a function of the spatial resolution (e.g. the number of cells). Right panel is similar to the left panel, for a fixed number of cells (800) as a function of the physical time-step Δt . The second order of convergence is plotted on the two panels with plain line.

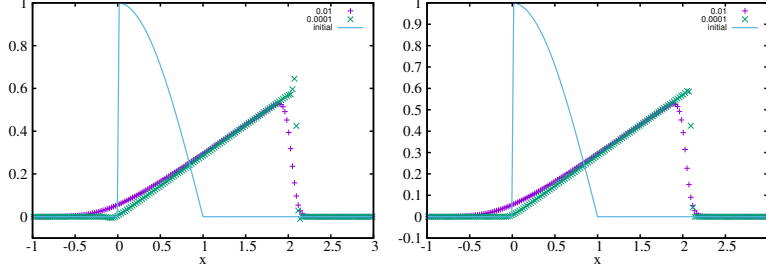


Fig. 6 Solution obtained for a time dependent Burger equation using two different resistivity values, $\eta = 10^{-2}$ and 10^{-4} . The final integration time is $t = 3.5$ for a physical time step $\Delta t = 10^{-4}$. The number of grid cells is 180. Left panel corresponds to a run without slope limiter, and right panel is for MC limiter. The initial ψ profile (at $t = 0$) is plotted using plain line.

6 Conclusion

Our hyperbolic scheme is shown to reproduce stationary solutions for 2D MHD reconnection problems, with the expected same convergence order for main and gradients variables. The extension for accurate time-dependent reconnection is actually under development. Preliminary results obtained using the BDF2 scheme are promising. In order to speed up the convergence procedure iteration at each time step, the scheme presented in this work would probably require some improvements like higher order spatial reconstruction and/or an implicit pseudo-time solver [10]. There is actually a great deal of interest on the subject, as the reconnection mechanism is considered to give an efficient energy mechanism on a fast time-scale that is

nearly independent of the resistivity. This is of considerable importance to explain flares occurring in many astrophysical plasmas. For example, the numerical investigation of the role of plasmoids in magnetic reconnection is complicated, as conventional codes lack some convergence properties due to the stochastic feature of the associated reconnection mechanism in such very high Lundquist plasmas [11]. The existence of fast regime of reconnection is also very dependent on the spatial variation of the resistivity [4]. We have checked that the hyperbolic method also works when a spatially localized profile is prescribed for the resistivity. The use of our hyperbolic scheme would be thus particularly promising in this context. Finally, we mention that another interesting hyperbolic approach for Navier-Stokes equations has been proposed very recently [12].

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Appendix

We consider a Riemann problem for the j th cell where the fluxes are evaluated at the two cell interfaces $x_{j-1/2}$ and $x_{j+1/2}$ respectively. The semi-discrete discretization of the system (Equ. 5) with our cell-centered scheme over the dual volume $I_j = [x_{j-1/2}, x_{j+1/2}]$ is thus:

$$\frac{\partial \mathbf{U}_j}{\partial \tau} = -\frac{1}{h} [\mathbf{F}_{j+1/2} - \mathbf{F}_{j-1/2}] + \frac{1}{h} \int_{I_j} \mathbf{S} dx, \quad (23)$$

where h is the grid cell volume (not necessarily uniform). The numerical flux is computed using the upwind formula:

$$\mathbf{F}_{j+1/2} = \frac{1}{2} (\mathbf{F}_L + \mathbf{F}_R) - \frac{1}{2} |\mathbf{A}| (\mathbf{U}_R - \mathbf{U}_L), \quad (24)$$

where the subscripts L and R stand for the left and right sides of the cell interface situated at $x_{j+1/2}$ respectively. The first term is computed from an average value of the two fluxes $\mathbf{F}_L = \mathbf{F}(\mathbf{U}_L)$ and $\mathbf{F}_R = \mathbf{F}(\mathbf{U}_R)$. We can therefore construct a numerical scheme by using advective/dissipative numerical fluxes based on upwind-type formulation, and taking the maximum wave speeds to evaluate the Jacobians,

$$\mathbf{F}_{j+1/2}^a = \begin{pmatrix} \frac{1}{2} [(V\psi)_L + (V\psi)_R] - \frac{a_n}{2} [\psi_R - \psi_L] \\ 0 \end{pmatrix}, \quad (25)$$

$$\mathbf{F}_{j+1/2}^d = \begin{pmatrix} \frac{1}{2} [(-\eta p)_L + (-\eta p)_R] - \frac{1}{2} \sqrt{\eta/T_r} [\psi_R - \psi_L] \\ \frac{1}{2} [(-\psi/T_r)_L + (-\psi/T_r)_R] - \frac{1}{2} \sqrt{\eta/T_r} [p_R - p_L] \end{pmatrix}, \quad (26)$$

where $a_n = |V|$. Finally, the source term is computed using a simple point integration approximation,

$$\int_{I_j} \mathbf{S} dx \approx \mathbf{S}_j h = \begin{pmatrix} hE_j \\ -hp_j/T_r \end{pmatrix}. \quad (27)$$

We use a second-order accuracy method, where the nodal gradients are evaluated by a linear least-squares (LSQ) method. The Left and right states are thus evaluated by a linear extrapolation from the cell centers. More explicitly we use,

$$\psi_L = \psi_j + \frac{1}{2}h\nabla\psi_j, \quad \psi_R = \psi_k - \frac{1}{2}h\nabla\psi_k, \quad (28)$$

$$p_L = p_j + \frac{1}{2}h\nabla p_j, \quad p_R = p_k - \frac{1}{2}h\nabla p_k, \quad (29)$$

where $\nabla\psi_j$ is the gradient of ψ computed by the LSQ method at node j , and similarly for ∇p_j (we have $k = j + 1$).

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