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Abstract. For quickly decreasing potentials with one position variable, the first threshold zero is always a resonance of the Kramers-Fokker-Planck operator. In this article we study low-energy spectral properties of the operator and calculate large time asymptotics of solutions in terms of the Maxwellian.

1. Introduction

The Kramers equation is a special Fokker-Planck equation describing the Brownian motion in an external field. This equation was derived and used by H. A. Kramers [13] to describe kinetics of chemical reaction. Later on it turned out that it had more general applicability to different fields such as supersonic conductors, Josephson tunnelling junction and relaxation of dipoles ([19]). Mathematical analysis of the Kramers-Fokker-Planck (KFP, in short) equation is initially motivated by trend to equilibrium for confining potentials ([7, 9, 20]). Spectral problems of the KFP operator reveal to be quite interesting, because this operator is neither elliptic nor selfadjoint. After appropriate normalisation of physical constants and a change of unknowns, the KFP equation can be written into the form

$$\partial_t u(t;x,v) + Pu(t;x,v) = 0,\ (x,v) \in \mathbb{R}^n \times \mathbb{R}^n, t > 0, \tag{1.1}$$

with initial data

$$u(0;x,v) = u_0(x,v). \tag{1.2}$$

Here x and v represent respectively position and velocity of the particle, P is the KFP operator given by

$$P = -\Delta_v + \frac{1}{4}|v|^2 - \frac{n}{2} + v \cdot \nabla_x - \nabla V(x) \cdot \nabla_v, \tag{1.3}$$

where the potential $V(x)$ is supposed to be a real-valued C^1 function verifying

$$|V(x)| + \langle x \rangle |\nabla V(x)| \leq C \langle x \rangle^{-\rho}, \quad x \in \mathbb{R}^n, \tag{1.4}$$

for some $\rho \in \mathbb{R}$ and $\langle x \rangle = (1 + |x|^2)^{1/2}$. Let $\mathfrak{m}$ be the function defined by

$$\mathfrak{m}(x,v) = \frac{1}{(2\pi)^{\frac{n}{4}}} e^{-\frac{1}{2}(\frac{v^2}{2} + V(x))}. \tag{1.5}$$

Then $\mathfrak{M} = \mathfrak{m}^2$ is the Maxwellian ([19]) and $\mathfrak{m}$ verifies the stationary KFP equation

$$P\mathfrak{m} = 0 \quad \text{ on } \mathbb{R}^{2n}_{x,v}. \tag{1.6}$$

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From the point of view of spectral analysis, large time behavior of solutions of (1.1) is closely related to low-energy spectral properties of P. If $V(x) \geq C|x|$ for some constant $C > 0$ outside some compact, then $\mathfrak{m} \in L^2(\mathbb{R}^{2n}_{x,v})$ and zero is a discrete eigenvalue of P. This case has been studied by many authors. It is known that

$$u(t) - \langle u_0, \mathfrak{m}\rangle \mathfrak{m} = O(e^{-\sigma t}), \quad t \to +\infty, \tag{1.7}$$

in $L^2(\mathbb{R}^{2n})$, where $\sigma > 0$ can be evaluated in terms of spectral gap between zero eigenvalue and the real part of the other eigenvalues of P and $V(x)$ is normalized by

$$\int_{\mathbb{R}^n} e^{-V(x)} dx = 1.$$

See [7, 9, 10, 20] and references quoted therein. If $V(x)$ increases slowly: $V(x) \sim c\langle x\rangle^\beta$ for some constants $c > 0$ and $\beta \in]0, 1[$, then zero is an eigenvalue embedded in the essential spectrum of P and it is known that (1.7) still holds with the right-hand side replaced by $O(t^{-\infty})$ ([4, 5]) or more precisely by $O(e^{-at^{\frac{\beta}{2-\beta}}})$ for some $a > 0$ ([16]). For decreasing potentials ($\rho > 0$ in (1.4)), zero is no longer an eigenvalue of P. It is proved in [21] that for $n = 3$ and $\rho > 2$, one has

$$u(t) = \frac{1}{(4\pi t)^{\frac{3}{2}}} \langle u_0, \mathfrak{m}\rangle \mathfrak{m} + O(\frac{1}{t^{\frac{3}{2}+\epsilon}}), \quad t \to +\infty, \epsilon > 0, \tag{1.8}$$

in some weighted spaces. (1.8) shows that for rapidly decreasing potentials, space distribution of particles is still governed by the Maxwellian, but the density of distribution decreases in times in the same rate as for heat propagation. Time-decay estimates of local energies are also obtained in [21] for short-range potentials ($\rho > 1$) and in [16] for long-range potentials ($0 < \rho \leq 1$). See also [2, 8, 14, 15, 18] for other related works.

In this work we study one dimensional KFP equation with quickly decreasing potentials. It is known that for Schrödinger operators, low-energy spectral analysis in one and two dimensional cases is more difficult than higher dimensions and needs specific methods ([1, 3, 11]) because zero is already a threshold resonance of the Laplacian in dimension one and two. For the KFP operator with decreasing potentials, the notion of thresholds and threshold resonances is discussed in [21]. Although $\mathfrak{m}$ always verifies the stationary KFP equation $P\mathfrak{m} = 0$, a basic fact is that $\langle x\rangle^{-s}\mathfrak{m} \notin L^2(\mathbb{R}^{2n})$ if $n \geq 3$ and $1 < s < \frac{n}{2}$, while $\langle x\rangle^{-s}\mathfrak{m} \in L^2(\mathbb{R}^{2n})$ for any $s > 1$ if $n = 1, 2$. In language of threshold spectral analysis, this means that for $n \geq 3$, zero is not a resonance of P while for $n = 1, 2$, zero is a resonance of P with $\mathfrak{m}$ as a resonant state. This is the main difference between the present work and [21].

Set $P = P_0 + W$ where

$$P_0 = v \cdot \nabla_x - \Delta_v + \frac{1}{4}|v|^2 - \frac{n}{2} \text{ and } W = -\nabla_x V(x) \cdot \nabla_v. \tag{1.9}$$

P_0 and P are regarded as operators in $L^2(\mathbb{R}^{2n})$ with the maximal domain. They are then maximally accretive. Denote e^{-tP_0} and e^{-tP}, $t \geq 0$, the strongly continuous semigroups generated by $-P_0$ and $-P$, respectively. If $\rho > -1$, W is a relatively compact perturbation of the free KFP operator P_0: $W(P_0 + 1)^{-1}$ is a compact operator in $L^2(\mathbb{R}^{2n})$.

One can prove that

$$\sigma_{\rm ess}(P) = \sigma(P_0) = [0, +\infty[\tag{1.10}$$

and that non-zero complex eigenvalues of P have positive real parts and may accumulate only towards points in $[0, +\infty[$. It is unknown for decreasing potentials whether or not the complex eigenvalues does accumulate towards some point in $[0, +\infty[$.

The main result of this work is the following

Theorem 1.1. *Let $n = 1$ and $\rho > 4$. Then for any $s > \frac{5}{2}$, there exists some $\epsilon > 0$ such that*

$$e^{-tP} = \frac{1}{(4\pi t)^{\frac{1}{2}}} \left(\langle \cdot, \mathfrak{m} \rangle \mathfrak{m} + O(t^{-\epsilon}) \right), \quad t \to +\infty \tag{1.11}$$

as operators from $\mathcal{L}^{2,s}$ to $\mathcal{L}^{2,-s}$, where

$$\mathcal{L}^{2,r} = L^2(\mathbb{R}^2_{x,v}; \langle x \rangle^{2r} dxdv), \quad r \in \mathbb{R}.$$

To prove (1.11), the main task is to show that the resolvent $R(z) = (P - z)^{-1}$ has an asymptotics of the form

$$R(z) = \frac{i}{2\sqrt{z}} \langle \cdot, \mathfrak{m} \rangle \mathfrak{m} + O(|z|^{-\frac{1}{2}+\epsilon}) \tag{1.12}$$

as operators from $\mathcal{L}^{2,s}$ to $\mathcal{L}^{2,-s}$, for z near zero and $z \notin \mathbb{R}_+$. Although (1.12) and the decay assumption on the potential look the same as the resolvent asymptotics of one dimensional Schrödinger operators in the case where zero is a resonance but not an eigenvalue ([1, 3, 11]), its proof is quite different from the Schrödinger case. In fact, the known methods for the Schrödinger operator can not be applied to the KFP operator, mainly because the perturbation W is a first order differential operator. In this work we use the method of [21] to calculate the low energy asymptotic expansion for the free resolvent $R_0(z) = (P_0 - z)^{-1}$ of the form

$$R_0(z) = \frac{1}{\sqrt{z}} G_{-1} + G_0 + \sqrt{z} G_1 + \cdots \tag{1.13}$$

in appropriate spaces, where G_{-1} is an operator of rank one. By a careful analysis of the space $\mathcal{N}$ of resonant states of P defined by (4.2), we prove that $1 + G_0 W$ is invertible on $\mathcal{L}^{2,-s}$, $s > \frac{3}{2}$. (1.12) is derived from the equation

$$R(z) = D(z)(1 + M(z))^{-1} R_0(z) \tag{1.14}$$

for z near zero and $z \notin \mathbb{R}_+$, where

$$D(z) = (1 + R_1(z)W)^{-1} \text{ with } R_1(z) = R_0(z) - \frac{1}{\sqrt{z}} G_{-1}$$

and

$$M(z) = \frac{1}{\sqrt{z}} G_{-1} W D(z).$$

As in threshold spectral analysis for Schrödinger operators, a non-trivial problem here is to compute the value of some spectral constants involving the resonant state of P. Indeed, in most part of this work only the condition $\rho > 2$ is needed. The stronger assumption $\rho > 4$ is used to show that some number $m(z)$ is nonzero for z near 0 and $z \notin \mathbb{R}_+$ (see (5.24)), which allows to prove the invertibility of $1 + M(z)$ and to calculate

its inverse.

The organisation of this article is as follows. In Section 2, we recall some known results needed in this work. The low-energy asymptotics of the free resolvent in dimension one is calculated in Section 3. The threshold spectral analysis of P is carried out in Section 4. We prove in particular that zero resonance is simple and $1 + G_0W$ is invertible. The low-energy asymptotics of the full resolvent (1.12) is proved in Section 5, which implies in particular that if $\rho > 4$, zero is not an accumulation point of complex eigenvalues of P. Finally, Theorem 1.1 is deduced in Section 6 by using a high-energy resolvent estimate of [21] valid in all dimensions.

Notation. For $r \geq 0$ and $s \in \mathbb{R}$, introduce the weighted Sobolev space

$$\mathcal{H}^{r,s} = \{u \in \mathcal{S}'(\mathbb{R}^{2n}); (1 - \Delta_v + |v|^2 + \langle D_x\rangle^{\frac{2}{3}})^{\frac{r}{2}}\langle x\rangle^s u \in L^2\}.$$

For $r < 0$, $\mathcal{H}^{r,s}$ is defined as the dual space of $\mathcal{H}^{-r,-s}$ with the dual product identified with the scalar product of L^2. The natural norm on $\mathcal{H}^{r,s}$ is denoted by $\|\cdot\|_{r,s}$. When no confusion is possible, we use $\|\cdot\|$ to denote the usual norm of $L^2(\mathbb{R}^{2n})$ or that bounded operators on L^2. Set $\mathcal{H}^r = \mathcal{H}^{r,0}$ and $\mathcal{L}^{2,s} = \mathcal{H}^{0,s}$. Denote $\mathcal{B}(r, s; r', s')$ the space of continuous linear operators from $\mathcal{H}^{r,s}$ to $\mathcal{H}^{r',s'}$. The weighted Sobolev spaces $\mathcal{H}^{r,s}$ are introduced in accordance with the sub-ellipticity of P_0: although P_0 does not map $\mathcal{H}^{1,s}$ to $\mathcal{H}^{-1,s}$, the sub-elliptic estimate of P_0 (Corollary 2.4) implies that $(P_0 + 1)^{-1} \in \mathcal{B}(-1, 0; 1, 0)$ and a commutator argument shows that $(P_0 + 1)^{-1} \in \mathcal{B}(-1, s; 1, s)$ for any $s \in \mathbb{R}$.

2. Preliminaries

In this Section we fix notation and state some known results which will be used in this work. Denote by P_0 the free KFP operator (with $\nabla V = 0$):

$$P_0 = v \cdot \nabla_x - \Delta_v + \frac{1}{4}|v|^2 - \frac{n}{2}, \quad x, v \in \mathbb{R}^n. \tag{2.1}$$

In terms of Fourier transform in x-variables, we have for $u \in D(P_0)$

$$\begin{aligned} (P_0u)(x, v) &= \mathcal{F}^{-1}_{x\to\xi}\hat{P}_0(\xi)\hat{u}(\xi, v), \quad \text{where} && (2.2)\\ \hat{P}_0(\xi) &= -\Delta_v + \frac{v^2}{4} - \frac{n}{2} + iv \cdot \xi, && (2.3)\\ \hat{u}(\xi, v) &= (\mathcal{F}_{x\to\xi}u)(\xi, v) \triangleq \int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(x, v)\, dx. && (2.4) \end{aligned}$$

Denote

$$D(\hat{P}_0) = \{f \in L^2(\mathbb{R}^{2n}_{\xi,v}); \hat{P}_0(\xi)f \in L^2(\mathbb{R}^{2n}_{\xi,v})\}. \tag{2.5}$$

Then $\hat{P}_0 \triangleq \mathcal{F}_{x\to\xi}P_0\mathcal{F}^{-1}_{x\to\xi}$ is a direct integral of the family of complex harmonic operators $\{\hat{P}_0(\xi); \xi \in \mathbb{R}^n\}$.

For fixed $\xi \in \mathbb{R}^n$, $\hat{P}_0(\xi)$ can be written as

$$\hat{P}_0(\xi) = -\Delta_v + \frac{1}{4}\sum_{j=1}^{n}(v_j + 2i\xi_j)^2 - \frac{n}{2} + |\xi|^2.$$

$\{\hat{P}_0(\xi), \xi \in \mathbb{R}^n\}$ is a holomorphic family of type A with constant domain $D = D(-\Delta_v + \frac{v^2}{4})$ in $L^2(\mathbb{R}^n_v)$. Its spectrum and eigenfunctions can be explicitly calculated. Let $F_j(s) = (-1)^j e^{\frac{s^2}{2}}\frac{d^j}{ds^j}e^{-\frac{s^2}{2}}$, $j \in \mathbb{N}$, be the Hermite polynomials and

$$\varphi_j(s) = (j!\sqrt{2\pi})^{-\frac{1}{2}} e^{-\frac{s^2}{4}} F_j(s)$$

the normalized Hermite functions. For $\xi \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}^n$, define

$$\psi_\alpha(v) = \prod_{j=1}^{n} \varphi_{\alpha_j}(v_j) \text{ and } \psi^\xi_\alpha(v) = \psi_\alpha(v + 2i\xi). \tag{2.6}$$

One can check ([21]) that the spectrum of $\hat{P}_0(\xi)$ is given by

$$\sigma(\hat{P}_0(\xi)) = \{l + \xi^2; l \in \mathbb{N}\}. \tag{2.7}$$

Each eigenvalue $l + \xi^2$ is semi-simple (i.e., its algebraic multiplicity and geometric multiplicity are equal) with multiplicity $m_l = \#\{\alpha \in \mathbb{N}^n; |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n = l\}$. The Riesz projection associated with the eigenvalue $l + \xi^2$ is given by

$$\Pi^\xi_l \phi = \sum_{\alpha, |\alpha| = l} \langle \phi, \psi^{-\xi}_\alpha \rangle \psi^\xi_\alpha, \quad \phi \in L^2. \tag{2.8}$$

The following result is useful to study the boundary values of the resolvent $R_0(z) = (P_0 - z)^{-1}$. Let $\hat{R}_0(z) = (\hat{P}_0 - z)^{-1}$ and $\hat{R}_0(z, \xi) = (\hat{P}_0(\xi) - z)^{-1}$ for $z \notin \mathbb{R}_+$. Then $R_0(z) = \mathcal{F}^{-1}_{x\to\xi}\hat{R}_0(z)\mathcal{F}_{x\to\xi}$.

Proposition 2.1. *Let $l \in \mathbb{N}$ and $l < a < l + 1$ be fixed. Take $\chi \geq 0$ and $\chi \in C^\infty_0(\mathbb{R}^n_\xi)$ with supp $\chi \subset \{\xi, |\xi| \leq a + 4\}$, $\chi(\xi) = 1$ when $|\xi| \leq a + 3$ and $0 \leq \chi(\xi) \leq 1$. Then one has*

$$\hat{R}_0(z, \xi) = \sum_{k=0}^{l} \chi(\xi)\frac{\Pi^\xi_k}{\xi^2 + k - z} + r_l(z, \xi), \tag{2.9}$$

for any $\xi \in \mathbb{R}^n$ and $z \in \mathbb{C}$ with $\mathrm{Re}\, z < a$ and $\mathrm{Im}\, z \neq 0$. Here $r_l(z, \xi)$ is holomorphic in z with $\mathrm{Re}\, z < a$ verifying the estimate

$$\sup_{\mathrm{Re}\, z < a, \xi \in \mathbb{R}^n} \|r_l(z, \xi)\|_{\mathcal{L}(L^2(\mathbb{R}^n_v))} < \infty. \tag{2.10}$$

See Proposition 2.7 of [21] for the proof. As a consequence of Proposition 2.1 and known results for the boundary values of the resolvent of $-\Delta_x$, we obtain the following

Corollary 2.2. *Let $n \geq 1$ and $R_0(z) = (P_0 - z)^{-1}$, $z \notin \mathbb{R}_+$.*

(a). With the notation of Proposition 2.1, one has

$$R_0(z) = \sum_{k=0}^{l} b^w_k(v, D_x, D_v)(-\Delta_x + k - z)^{-1} + r_l(z) \tag{2.11}$$

where $r_l(z)$ is $\mathcal{B}(L^2)$-valued holomorphic function for $\mathrm{Re}\, z < a$ *and $b_k^w(v, D_x, D_v)$ is the Weyl pseudo-differential operator with symbol $b_k(x, \xi, \eta)$ given by*

$$b_k(v,\xi,\eta) = \int_{\mathbb{R}^n} e^{-iv'\cdot\eta/2} \left(\sum_{|\alpha|=k} \chi(\xi)\psi_\alpha(v+v'+2i\xi)\psi_\alpha(v-v'+2i\xi) \right) dv'. \tag{2.12}$$

In particular,

$$b_0(v,\xi,\eta) = 2^{\frac{n}{2}} e^{-v^2-\eta^2+2iv\cdot\xi+2\xi^2}\chi(\xi). \tag{2.13}$$

(b). Let I be a compact interval of $\mathbb{R}$ *which does not contain any non negative integer. Then for any $s > \frac{1}{2}$, one has*

$$\sup_{\lambda\in I;\epsilon\in]0,1]} \|R_0(\lambda \pm i\epsilon)\|_{\mathcal{B}(-1,s;1,-s)} < \infty \tag{2.14}$$

The boundary values of the resolvent $R_0(\lambda\pm i0) = \lim_{\epsilon\to 0_+} R_0(\lambda\pm i\epsilon)$ exist in $\mathcal{B}(0, s; 0, -s)$ for $\lambda \in I$ and is Hölder-continuous in λ.

Seeing (2.11), it is natural to define $\mathbb{N}$ as set of thresholds of the KFP operator P ([21]). Note that an exponential upper-bound in λ for $R_0(\lambda\pm i\epsilon)$, $\epsilon > 0$ fixed, is obtained in [16] by method of harmonic analysis in Besov spaces.

For high energy resolvent estimate, we need the following result proved in Appendix A.2 of [18].

Theorem 2.3. *There exists some constant $C > 0$ such that*

$$\|(1 - \Delta_v + v^2 + |\xi|^{\frac{2}{3}} + |\lambda|^{\frac{1}{2}})(\hat{P}_0(\xi) + \frac{n}{2} + 1 - i\lambda)^{-1}\| \le C \tag{2.15}$$

uniformly in $\xi \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.

As consequence, we obtain a uniform sub-elliptic estimate for the free KFP operator.

Corollary 2.4. *One has*

$$|\lambda|\|u\|^2 + \|\Delta_v u\|^2 + \||v|^2 u\|^2 + \||D_x|^{\frac{2}{3}} u\|^2 \le C\|(P_0 + \frac{n+2}{2} - i\lambda)u\|^2, \tag{2.16}$$

for $u \in \mathcal{S}(\mathbb{R}^{2n}_{x,v})$ and $\lambda \in \mathbb{R}$. In addition, P_0 defined on $\mathcal{S}(\mathbb{R}^{2n}_{x,v})$ is essentially maximally accretive.

Let us indicate that the essential maximal accretivity of P_0 is discussed in [17]. Henceforth we still denote by P_0 its closure in L^2 with maximal domain $D(P_0) = \{u \in L^2(\mathbb{R}^{2n}_{x,v}); P_0 u \in L^2(\mathbb{R}^{2n}_{x,v})\}$. To determine the spectrum of P_0 which is unitarily equivalent with a direct integral of $\hat{P}_0(\xi)$, $\xi \in \mathbb{R}^n$, in addition to (2.7), one needs a resolvent estimate uniform with respect to $\xi \in \mathbb{R}^n$ proved in [21]: $\forall z \in \mathbb{C} \setminus \mathbb{R}_+$,

$$\sup_{\xi\in\mathbb{R}^n} \|(\hat{P}_0(\xi) - z)^{-1}\| \le C_z. \tag{2.17}$$

See [6] for the necessity of such uniform resolvent estimate in order to determine the spectrum of direct integral of a family of non-selfadjoint operators. (2.7) and (2.17) show that

$$\sigma(P_0) = \cup_{\xi\in\mathbb{R}^n}\sigma(\hat{P}_0(\xi)) = [0,+\infty[. \tag{2.18}$$

Under the condition (1.4) on V for some $\rho > -1$, $|\nabla V(x)| \to 0$ as $|x| \to +\infty$. By Corollary 2.4, $W = -\nabla V(x)\cdot\nabla_v$ is relatively compact with respect to P_0. It follows that

$$\sigma_{\rm ess}(P) = [0,+\infty[\tag{2.19}$$

and discrete spectrum of P is at most countable with possible accumulation points included in $[0,+\infty[$.

3. The free resolvent in dimension one

We use (2.11) with $l = 0$ to calculate the asymptotics of $R_0(z)$ near the first threshold zero.

Proposition 3.1. *Let $n = 1$. One has the following low-energy resolvent asymptotics for $R_0(z)$: for $s, s' > \frac{1}{2}$, there exists $\epsilon > 0$ such that*

$$R_0(z) = \frac{1}{\sqrt{z}}(G_{-1} + O(|z|^\epsilon)), \ \text{as } z \to 0, z \notin \mathbb{R}_+, \tag{3.1}$$

as operators in $\mathcal{B}(-1,s;1,-s')$. More generally, for any integer $N \geq 0$ and $s > N + \frac{3}{2}$, there exists $\epsilon > 0$

$$R_0(z) = \sum_{j=-1}^{N} z^{\frac{j}{2}}G_j + O(|z|^{\frac{N}{2}+\epsilon}), \ \text{as } z \to 0, z \notin \mathbb{R}_+, \tag{3.2}$$

as operators in $\mathcal{B}(-1,s;1,-s)$. Here the branch of $z^{\frac{1}{2}}$ is chosen such that its imaginary part is positive when $z \notin \mathbb{R}_+$ and $G_j \in \mathcal{B}(-1,s;1,-s)$ for $s > j+\frac{3}{2}$, $j \geq 0$. In particular,

$$G_{-1} = \frac{i}{2}\langle\cdot,\mathfrak{m}_0\rangle\mathfrak{m}_0 \tag{3.3}$$

$$G_0 = F_0 + F_1, \tag{3.4}$$

where

$$\mathfrak{m}_0(x,v) = 1\otimes\psi_0(v) \tag{3.5}$$

with $\psi_0(v) = \frac{1}{(2\pi)^{\frac{1}{4}}}e^{-\frac{v^2}{4}}$ the first eigenfunction of harmonic oscillator, F_0 is the operator with integral kernel

$$F_0(x,v;x',v') = -\frac{1}{2}\psi_0(v)\psi_0(v')|x-x'| \tag{3.6}$$

and $F_1 \in \mathcal{B}(-1,s;1,-s')$ for any $s, s' > \frac{1}{2}$.

Proof. For $z \notin \mathbb{R}_+$, (2.11) with $l = 0$ shows that

$$R_0(z) = b_0^w(v,D_x,D_v)(-\Delta_x - z)^{-1} + r_0(z), \tag{3.7}$$

with $r_0(z) \in \mathcal{B}(-1,0;1,0)$ holomorphic in z when Re $z < a$ for some $a \in]0,1[$. Here the cut-off $\chi(\xi)$ is chosen such that $\chi \in C_0^\infty(\mathbb{R}^n)$ and $\chi(\xi) = 1$ in a neighbourhood of $\{|\xi|^2 \leq a\}$. Therefore $r_0(z)$ admits a convergent expansion in powers of z for z near 0

$$r_0(z) = r_0(0) + zr_0'(0) + \cdots + z^n \frac{r_0^{(n)}(0)}{n!} + \cdots$$

in $\mathcal{B}(-1,0;1,0)$. It is sufficient to study the lower-energy expansion of $b_0^w(v, D_x, D_v)(-\Delta_x - z)^{-1}$.

Note that in one dimensional case, the integral kernel of the resolvent $(-\Delta_x - z)^{-1}$ is given by

$$\frac{i}{2\sqrt{z}} e^{i\sqrt{z}|x-y|}, z \notin \mathbb{R}_+, x, y \in \mathbb{R} \tag{3.8}$$

where the branch of $\sqrt{z}$ is chosen such that its imaginary part is positive for $z \notin \mathbb{R}_+$. The integral kernel of $b_0^w(v, D_x, D_v)(-\Delta_x - z)^{-1}$, $z \notin \mathbb{R}_+$, is given by

$$K(x, x'; v, v'; z) = \frac{i}{2\sqrt{z}} \int_{\mathbb{R}} e^{i\sqrt{z}|y-(x-x')|} \Phi(v, v', y)\, dy \tag{3.9}$$

with

$$\begin{aligned} \Phi(v, v', y) &= (2\pi)^{-\frac{3}{2}} e^{-\frac{1}{4}(v^2+v'^2)} \int_{\mathbb{R}} e^{i(y-v-v')\cdot\xi + 2\xi^2} \chi(\xi)\, d\xi \\ &= \psi_0(v)\psi_0(v')\Psi(y - v - v') \end{aligned} \tag{3.10}$$

where Ψ is the inverse Fourier transform of $e^{2\xi^2}\chi(\xi)$. Since $\chi \in C_0^\infty$, one has the following asymptotic expansion for $K(x, x'; v, v'; z)$: for any $\epsilon \in [0, 1]$ and $N \geq 0$

$$|K(x, x'; v, v'; z) - \sum_{j=-1}^{N} z^{\frac{j}{2}} K_j(x, x', v, v')| \leq C_{N,\epsilon} |z|^{\frac{N+\epsilon}{2}} |x - x'|^{N+1+\epsilon} e^{-\frac{1}{4}(v^2+v'^2)} \tag{3.11}$$

where

$$K_j(x, x'; v, v') = \frac{i^{j+2}}{2\,(j+1)!} \int_{\mathbb{R}} |y - (x - x')|^{j+1} \Phi(v, v', y) dy. \tag{3.12}$$

Remark that for $N \geq 0$, $s', s > N + \frac{1}{2}$ and $0 < \epsilon < \min\{s, s'\} - N - \frac{1}{2}$ and $\epsilon \in]0, \frac{1}{2}]$

$$\langle x \rangle^{-s} \langle x' \rangle^{-s'} |x - x'|^{N+\epsilon} e^{-\frac{1}{4}(v^2+v'^2)} \in L^2(\mathbb{R}^4).$$

We obtain the asymptotic expansion for $b_0^w(v, D_x, D_v)(-\Delta_x - z)^{-1}$ in powers of $z^{\frac{1}{2}}$ for z near 0 and $z \notin \mathbb{R}_+$.

$$b_0^w(v, D_x, D_v)(-\Delta_x - z)^{-1} = \sum_{j=-1}^{N} z^{\frac{j}{2}} K_j + O(|z|^{\frac{N}{2}+\epsilon}), \text{ as} \tag{3.13}$$

as operators in $\mathcal{B}(0, s'; 0, -s)$, $s', s > N + \frac{3}{2}$. By the sub-elliptic estimate of P_0, this expansion still holds in $\mathcal{B}(-1, s'; 1, -s)$. This proves (3.2) with G_k given by

$$G_{2j} = K_{2j} + \frac{r_0^{(j)}(0)}{j!}, \quad G_{2j-1} = K_{2j-1}, \quad j \geq 0. \tag{3.14}$$

To show (3.3) and (3.4), note that since $\chi(0) = 1$, one has

$$\int_{\mathbb{R}} \Phi(v, v', y)\, dy = \psi_0(v)\psi_0(v').$$

The first two terms in the expansion of $K(x, x'; v, v'; z)$ can be simplified as

$$\begin{aligned} K_{-1}(x, x', v, v') &= \frac{i}{2}\int_{\mathbb{R}} \Phi(v, v', y)dy = \frac{i}{2}\psi_0(v)\psi_0(v') && (3.15)\\ K_0(x, x', v, v') &= -\frac{1}{2}\int_{\mathbb{R}} \Phi(v, v', y)|y - (x - x')|dy && (3.16)\\ &= -\frac{1}{2}\psi_0(v)\psi_0(v')|x - x'| - \frac{1}{2}\int (|y - (x - x')| - |x - x'|)\Phi(v, v', y)dy. \end{aligned}$$

Therefore (3.3) is true and G_0 can be decomposed as: $G_0 = F_0 + F_1$ with F_0 defined by (3.6) and $F_1 = K_{0,1} + r_0(0)$, $K_{0,1}$ being the operator with the integral kernel

$$K_{0,1}(x, x', v, v') = -\frac{1}{2}\int_{\mathbb{R}} (|y - (x - x')| - |x - x'|)\Phi(v, v', y)dy,$$

which is a smooth function and

$$K_{0,1}(x, x', v, v') = O(\psi_0(v)\psi_0(v'))$$

for $|x - x'|$ large. Therefore $K_{0,1}$ is bounded in $B(-1, s; 1, -s')$ for any $s, s' > \frac{1}{2}$. This shows that $F_1 = K_{0,1} + r_0(0)$ has the same continuity property, which proves (3.4). □

Corollary 3.2. *Let $n = 1$ and e^{-tP_0}, $t \geq 0$, be the strongly continuous semigroup generated by $-P_0$. Then for any integer $N \geq 0$ and $s > 2N + \frac{1}{2}$, the following asymptotic expansion holds for some $\epsilon > 0$*

$$e^{-tP_0} = \sum_{k=0}^{N} t^{-\frac{2k+1}{2}}\beta_k G_{2k-1} + O(t^{-\frac{2N+1}{2}-\epsilon}), \quad t \to +\infty, \tag{3.17}$$

in $\mathcal{B}(0, s, 0, s)$. Here β_k is some non zero constant. In particular, the leading term $\beta_0 G_{-1}$ is a rank-one operator given by

$$\beta_0 G_{-1} = \frac{1}{(4\pi)^{\frac{1}{2}}}\langle \cdot, \mathfrak{m}_0\rangle \mathfrak{m}_0 : \mathcal{L}^{2,s} \to \mathcal{L}^{2,-s} \tag{3.18}$$

for any $s > \frac{1}{2}$.

The proof of Corollary 3.2 uses Proposition 3.1 and a representation formula of the semigroup e^{-tP_0} as contour integral of the resolvent $R_0(z)$ in the right half-plane. See the proof of Theorem 1.1 for more details.

4. Threshold spectral properties

Assume that $V \in C^1(\mathbb{R}^n; \mathbb{R})$ and

$$|V(x)| + \langle x\rangle |\nabla V(x)| \leq C\langle x\rangle^{-\rho} \tag{4.1}$$

for some $\rho > 0$. Consider the null space of P defined by

$$\mathcal{N} = \{u; u \in \mathcal{H}^{1,-s}, \forall s > 1 \text{ and } Pu = 0\}. \tag{4.2}$$

Since zero is not an eigenvalue of P, $\mathcal{N}$ is the spaces of resonant states of P associated with zero resonance. See [21] for the definitions in general case. Remark that for $n = 1$, one can equally take $s > \frac{1}{2}$ in the above definition, instead of $s > 1$. But the condition $s > 1$ is necessary to define appropriately resonant states for $n = 2$. Clearly, $\mathfrak{m} \in \mathcal{N}$. We want to prove that in one dimensional case, one has: $\dim \mathcal{N} = 1$. In order to calculate the leading term of the resolvent expansion at threshold zero, we need also to calculate solutions of some integral equation.

Lemma 4.1. *Let $\rho > 0$ and $n = 1$. If $u \in \mathcal{H}^{1,-s}$ for some $s < \rho + \frac{1}{2}$ and satisfies the equation $Pu = 0$, then*

$$\langle Wu, \mathfrak{m}_0 \rangle = 0, \tag{4.3}$$

where

$$\mathfrak{m}_0(x, v) = 1 \otimes \psi_0(v).$$

Proof. Suppose for the moment $n \geq 1$. Since $u \in \mathcal{H}^{1,-s}$, one has $Wu \in \mathcal{H}^{0,\rho+1-s} \subset L^2$. Using the equation $Pu = 0$ and the ellipticity of P in velocity variables v, we deduce that $(-\Delta_v + v^2)u(x, \cdot) \in L^2(\mathbb{R}^n_v)$ a.e. in $x \in \mathbb{R}^n$. Taking scalar product of Pu with $\psi_0(v)$ in v-variables, one has

$$\langle (Pu)(x, \cdot), \psi_0 \rangle_v = 0, \quad \text{a. e.} \quad x \in \mathbb{R}^n.$$

Since ψ_0 is the first eigenfunction of the harmonic oscillator in v, one has also

$$\langle Pu, \psi_0 \rangle_v = \langle v \cdot \nabla_x u, \psi_0 \rangle_v - \langle \nabla_x V(x) \cdot \nabla_v u, \psi_0 \rangle_v$$

a. e. in $x \in \mathbb{R}^n$. These two relations imply that

$$2\nabla_x \cdot \langle \nabla_v u, \psi_0 \rangle_v + \nabla_x V(x) \cdot \langle \nabla_v u, \psi_0 \rangle_v = 0. \tag{4.4}$$

The above equation holds for $n \geq 1$. In the case $n = 1$, $\langle \nabla_v u, \psi_0 \rangle_v$ is a scalar function in x and the differential equation (4.4) determines $\langle \nabla_v u, \psi_0 \rangle_v$ up to some constant:

$$\langle \nabla_v u, \psi_0 \rangle_v = Ce^{-\frac{V(x)}{2}}, \text{ a. e. in } x \in \mathbb{R} \tag{4.5}$$

for some constant C. It is now clear that in one dimensional case, one has

$$\langle Wu, \mathfrak{m}_0 \rangle = -\int_{\mathbb{R}} V'(x) \langle \partial_v u, \psi_0 \rangle_v dx = -C \int_{\mathbb{R}} V'(x) e^{-\frac{V(x)}{2}} dx = 0, \tag{4.6}$$

because $V(x) \to 0$ as $|x| \to +\infty$. □

Lemma 4.1 is important in threshold spectral analysis of the KFP operator in dimension one. We believe that this result still holds when $n \geq 2$, but the last argument above does not hold if $n \geq 2$. In fact when $n \geq 2$, (4.4) only implies that the vector-valued function $\langle \nabla_v u, \psi_0 \rangle_v$ is of the form

$$\langle \nabla_v u, \psi_0 \rangle_v = e^{-\frac{V(x)}{2}} \overrightarrow{F}(x) \tag{4.7}$$

where $\overrightarrow{F} \in L^2(\mathbb{R}^n; \langle x \rangle^{-2s} dx)$ and $\nabla \cdot \overrightarrow{F} = 0$ in sense of distributions, which are not sufficient to conclude that $\langle Wu, \mathfrak{m}_0 \rangle = 0$.

From now on, assume that $\rho > 2$ and $n = 1$. Then by the sub-elliptic estimate for P_0, $G_0 W$ is a compact operator in $\mathcal{H}^{1,-s}$ for $\frac{3}{2} < s < \frac{\rho+1}{2}$. We want to study solutions of the integral equation

$$(1 + G_0 W)u = \beta \mathfrak{m}_0 \tag{4.8}$$

for $u \in \mathcal{H}^{1,-s}$ and $\beta \in \mathbb{C}$.

Lemma 4.2. *Let $\rho > 2$ and $u \in \mathcal{H}^{1,-s}$ for some $\frac{3}{2} < s < \frac{\rho+1}{2}$ such that $(1 + G_0 W)u = \beta \mathfrak{m}_0$ for some $\beta \in \mathbb{C}$. Then $Pu = 0$. In particular, one has: $\langle Wu, \mathfrak{m}_0 \rangle = 0$.*

Proof. One has seen that

$$R_0(z) = \frac{G_{-1}}{\sqrt{z}} + G_0 + o(1)$$

in $\mathcal{B}(-1, r; 1, -r)$ for any $r > \frac{3}{2}$. Therefore,

$$G_0 W u = \lim_{z \to 0, z \notin \mathbb{R}_+} (R_0(z) - \frac{G_{-1}}{\sqrt{z}}) W u$$

in $H^{1,-r}$. Since $P_0 G_{-1} = 0$ in $\mathcal{H}^{-1,r}$, one has for $\lambda < 0$

$$P_0 (R_0(\lambda) - \frac{G_{-1}}{\sqrt{\lambda}}) W u = W u + \lambda R_0(\lambda) W u.$$

The m-accretivity of P_0 implies

$$\|\lambda R_0(\lambda) W\| \leq 1, \quad \lambda < 0.$$

It follows that

$$\|\lambda R_0(\lambda) W u\| \leq \|W u\| \leq C \|u\|_{1,-s}, \quad \frac{3}{2} < s < \frac{\rho+1}{2},$$

uniformly in $\lambda < 0$. In addition, if $\frac{1}{2} < s' < \frac{\rho+1}{2}$, one has

$$\|\lambda R_0(\lambda) W u\|_{1,-s'} \leq \|\lambda R_0(\lambda)\|_{\mathcal{B}(0,s';0,-s')} \|W u\|_{0,s'} \leq C |\lambda|^{\frac{1}{2}} \|u\|_{1,-s}$$

for $\lambda < 0$. These two bounds show that

$$w - \lim_{\lambda \to 0_-} \lambda R_0(\lambda) W u = 0, \quad \text{in } L^2(\mathbb{R}^2). \tag{4.9}$$

Since $u = -G_0 W u + \beta \mathfrak{m}_0$ and $P_0 \mathfrak{m}_0 = 0$, the following equalities hold:

$$P_0 u = -w - \lim_{\lambda \to 0_-} P_0 (R_0(\lambda) - \frac{G_{-1}}{\sqrt{\lambda}}) W u = -W u$$

in sense of distributions. This proves that $Pu = 0$. In particular Lemma 4.1 shows that $\langle Wu, \mathfrak{m}_0 \rangle = 0$. □

Proposition 4.3. *Let $u \in \mathcal{H}^{1,-s}$ for some $\frac{3}{2} < s < \frac{\rho+1}{2}$ such that $(1 + G_0 W)u = \beta \mathfrak{m}_0$ for some $\beta \in \mathbb{C}$. Then one has*

$$u(x, v) = (\beta - C_1(x) - v C_2(x)) \psi_0(v) + r(x, v) \tag{4.10}$$

where $C_j \in L^\infty$ and $C_j' \in L^1$, $j = 1, 2$, and $(1 + v^2 - \partial_v^2) r \in L^2(\mathbb{R}^2_{x,v})$. In addition,

$$\lim_{x \to \pm\infty} C_1(x) = \pm d_1, \quad \lim_{x \to \pm\infty} C_2(x) = 0 \tag{4.11}$$

where

$$d_1 = -\frac{1}{4}\int\int_{\mathbb{R}^2}(x+\frac{v}{2})\psi_0(v)\nabla V(x)\nabla_v u(x,v)dxdv. \tag{4.12}$$

In particular, $u \in \mathcal{H}^{1,-s}$ *for any* $s > \frac{1}{2}$.

Proof. Recall that $G_0 = K_1 + r_0(0)$ where $r_0(0)$ is bounded from $\mathcal{H}^{-1}$ to $\mathcal{H}^1$ and K_1 is an operator of integral kernel

$$K_1(x,x';v,v') = -\frac{1}{2}\int_{\mathbb{R}}|y-(x-x')|\Phi(v,v';y)dy \tag{4.13}$$

with

$$\Phi(v,v',y) = \frac{1}{2}\psi_0(v)\psi_0(v')\Psi(y-v-v'),$$

Ψ being the inverse Fourier transform of $e^{2\xi^2}\chi(\xi)$. Let $u \in \mathcal{H}^{1,-s}$, $\frac{3}{2} < s < \frac{\rho+1}{2}$, such that $(1+G_0W)u = \beta\mathfrak{m}_0$. By Lemma 4.2,

$$\langle Wu, \mathfrak{m}_0\rangle = 0. \tag{4.14}$$

Set $w = K_1Wu$. Then $u + w - \beta\mathfrak{m}_0 = -r_0(0)Wu$ belongs to L^2. Let us study the asymptotic behavior of w as $|x| \to \infty$. Put

$$F(x',y,v,v') = \psi_0(v)\psi_0(v')\Psi(y-v-v')\nabla V(x')\nabla_v u(x',v').$$

Making use of the asymptotic expansion

$$|y-(x-x')| = |x-x'| - \frac{y(x-x')}{|x-x'|} + O(\frac{y^2}{|x-x'|})$$

for $|x-x'|$ large, one obtains that

$$\begin{aligned} w(x,v) &= \frac{1}{4}\int\int_{\mathbb{R}^3}|y-(x-x')|F(x',y,v,v')dydx'dv' \\ &\simeq \frac{1}{4}\int\int_{\mathbb{R}^3}(|x-x'| - \frac{y(x-x')}{|x-x'|})F(x',y,v,v')dydx'dv' \\ &= \frac{1}{4}\int\int_{\mathbb{R}^2}(|x-x'| - \frac{(v+v')(x-x')}{|x-x'|})\psi_0(v)\psi_0(v')\nabla V(x')\nabla_v u(x',v')dx'dv'. \end{aligned} \tag{4.15}$$

Here and in the following, "$\simeq$" means the equality modulo some term in $L^2(\mathbb{R}^2)$.

Recall that since Ψ is the inverse Fourier transform of $e^{2\xi^2}\chi(\xi)$, one has

$$\int_{\mathbb{R}}\Psi(y)dy = 1, \quad \int_{\mathbb{R}}y\Psi(y)dy = 0$$

and that according to Lemma 4.1

$$\int_{\mathbb{R}^2}\psi_0(v')\nabla V(x')\nabla_v u(x',v')dx'dv' = -\langle Wu, \mathfrak{m}_0\rangle = 0. \tag{4.16}$$

The term related to $|x-x'|$ on the right-hand side of (4.15) is equal to

$$\begin{aligned}
&\frac{1}{4}\int\int_{\mathbb{R}^2}(|x-x'|)\psi_0(v)\psi_0(v')\nabla V(x')\nabla_v u(x',v')dx'dv' \\
&= \frac{1}{4}\left(\int_{-\infty}^{x}-\int_{x}^{+\infty}\right)(x-x')\psi_0(v)\nabla V(x')\langle\nabla_{v'}u(x',\cdot),\psi_0\rangle_{v'}dx' \qquad (4.17)
\end{aligned}$$

Applying (4.16), one has for $x\leq 0$

$$\begin{aligned}
&\left|x\left(\int_{-\infty}^{x}-\int_{x}^{+\infty}\right)\psi_0(v)\nabla V(x')\langle\nabla_{v'}u(x',\cdot),\psi_0\rangle_{v'}dx'\right| \\
&= \left|2x\int_{-\infty}^{x}\psi_0(v)\nabla V(x')\langle\nabla_{v'}u(x',\cdot),\psi_0\rangle_{v'}dx'\right| \\
&\leq C|x|\left\{\int_{-\infty}^{x}\langle x'\rangle^{-2(\rho+1-s)}dx'\right\}^{\frac{1}{2}}\psi_0(v)\|u\|_{\mathcal{H}^{1,-s}} \\
&\leq C'\langle x\rangle^{-\rho+s+\frac{1}{2}}\psi_0(v)\|u\|_{\mathcal{H}^{1,-s}} \qquad (4.18)
\end{aligned}$$

Since $\rho>2$ and $s<\frac{\rho+1}{2}$, this proves that the term

$$x\left(\int_{-\infty}^{x}-\int_{x}^{+\infty}\right)\nabla V(x')\langle\nabla_{v'}u(x',\cdot),\psi_0\rangle_{v'}dx'$$

is bounded for $x\leq 0$ and tends to 0 as $x\to-\infty$. The same conclusion also holds as $x\to+\infty$, using once more (4.16). In the same way one can check that

$$\left(\int_{-\infty}^{x}-\int_{x}^{+\infty}\right)x'\nabla V(x')\langle\nabla_{v'}u(x',\cdot),\psi_0\rangle_{v'}dx'$$

is bounded for $x\in\mathbb{R}$. The other terms in (4.15) can be studied in a similar way. Finally we obtain that

$$\begin{aligned}
w(x,v) &\simeq (C_1(x)+vC_2(x))\psi_0(v) \text{ where} \qquad (4.19)\\
C_1(x) &= \frac{1}{4}\int\int_{\mathbb{R}^2}(x-x'-\frac{v'}{2})\text{sgn }(x-x')\psi_0(v')\nabla V(x')\nabla_v u(x',v')dx'dv' \qquad (4.20)\\
C_2(x) &= -\frac{1}{8}\int\int_{\mathbb{R}^2}\text{sgn }(x-x')\psi_0(v')\nabla V(x')\nabla_v u(x',v')dx'dv'. \qquad (4.21)
\end{aligned}$$

It follows from Dominated Convergence Theorem that the limits

$$\lim_{x\to\pm\infty}C_j(x)=\pm d_j \qquad (4.22)$$

exist, where

$$\begin{aligned}
d_1 &= -\frac{1}{4}\int\int_{\mathbb{R}^2}(x'+\frac{v'}{2})\psi_0(v')\nabla V(x')\nabla_v u(x',v')dx'dv' \qquad (4.23)\\
d_2 &= -\frac{1}{8}\int\int_{\mathbb{R}^2}\psi_0(v')\nabla V(x')\nabla_v u(x',v')dx'dv'=0. \qquad (4.24)
\end{aligned}$$

This proves that

$$u\simeq\beta\mathfrak{m}_0-w\simeq(\beta-C_1(x)-vC_2(x))\psi_0(v)$$

modulo some terms in $L^2(\mathbb{R}^2)$. In particular, $u \in \mathcal{H}^{1,-s}$ for any $s > \frac{1}{2}$. Since $\rho > 2$, one can also check that $C_j'(x)$ belongs to $L^1(\mathbb{R})$, $j = 1, 2$. □

Theorem 4.4. *Assume $\rho > 2$. If $u \in \mathcal{H}^{1,-s}$, $\frac{3}{2} < s < \frac{\rho+1}{2}$, satisfies the equation $(1 + G_0 W)u = 0$, then $u = 0$.*

Proof. Let $\chi_1 \in C_0^\infty(\mathbb{R})$ be a cut-off with $\chi_1(\tau) = 1$ for $|\tau| \leq 1$ and $\chi_1(\tau) = 0$ for $|\tau| \geq 2$ and $0 \leq \chi_1(\tau) \leq 1$. Set $\chi_R(x) = \chi_1(\frac{x}{R})$ for $R \geq 1$ and $u_R(x, v) = \chi_R(x)u(x, v)$. Then one has

$$Pu_R = \frac{v}{R}\chi'(\frac{x}{R})u.$$

Taking the real part of the equality $\langle Pu_R, u_R\rangle = \langle \frac{v}{R}\chi'(\frac{x}{R})u, u_R\rangle$, one obtains

$$\int\int_{\mathbb{R}^2} |(\partial_v + \frac{v}{2})u(x, v)|^2 \chi_R(x)^2 \, dxdv = \langle \frac{v}{R}\chi'(\frac{x}{R})u, u_R\rangle. \tag{4.25}$$

According to Proposition 4.3, u can be decomposed as

$$u(x, v) = z(x, v) + r(x, v) \tag{4.26}$$

where $z(x, v) = -(C_1(x) + vC_2(x))\psi_0(v)$ and C_1, C_2 and r are given in Proposition 4.3. Since $\psi_0(v)$ is even in v, the term $\langle \frac{v}{R}\chi'(\frac{x}{R})z, \chi_R z\rangle$ is reduced to

$$\begin{aligned} &2\mathrm{Re}\, \langle \frac{v^2}{R}\chi'(\frac{x}{R})C_1\psi_0, \chi_R C_2\psi_0\rangle && (4.27)\\ &= \quad -\mathrm{Re} \int\int_{\mathbb{R}^2} v^2\psi_0(v)^2\chi_R(x)^2 \frac{d}{dx}(C_1(x)\overline{C}_2(x))dxdv && (4.28)\\ &\to \quad -\mathrm{Re} \int\int_{\mathbb{R}^2} v^2\psi_0(v)^2 \frac{d}{dx}(C_1(x)\overline{C}_2(x))dxdv = 0 && (4.29) \end{aligned}$$

as $R \to +\infty$, because $\frac{d}{dx}(C_1(x)\overline{C}_2(x))$ belongs to L^1 and $C_1(x)\overline{C}_2(x) \to 0$ as $|x| \to +\infty$. The term $|\langle \frac{v}{R}\chi'(\frac{x}{R})r, u_R\rangle|$ can be estimated by

$$|\langle \frac{v}{R}\chi'(\frac{x}{R})r, u_R\rangle| \leq CR^{-(1-s)}\|u\|_{L^{2,-s}}\|\langle v\rangle r\|_{L^2}$$

for $\frac{1}{2} < s < 1$. Similar estimate also holds for $|\langle \frac{v}{R}\chi'(\frac{x}{R})z, \chi_R r\rangle|$. Summing up, we proved that

$$\lim_{R\to+\infty} \langle \frac{v}{R}\chi'(\frac{x}{R})u, u_R\rangle = 0 \tag{4.30}$$

which implies that $(\partial_v + \frac{v}{2})u(x, v) = 0$ a.e. in x and v. Since $u \in \mathcal{H}^{1,-s}$ for any $s > \frac{1}{2}$ and $Pu = 0$, it follows that u is of the form $u(x, v) = D(x)e^{-\frac{v^2}{4}}$ for some $D \in L^{2,-s}(\mathbb{R})$ verifying the equation

$$D'(x) + \frac{1}{2}V'(x)D(x) = 0 \tag{4.31}$$

in sense of distributions on $\mathbb{R}$. It follows that $D(x) = \alpha e^{-\frac{V(x)}{2}}$ a.e. for some constant α. Hence

$$u(x, v) = \alpha e^{-\frac{v^2}{4} - \frac{V(x)}{2}}.$$

In particular, one has

$$\int_0^R \int_{\mathbb{R}_v} u(x,v)dvdx = \sqrt{\pi}\alpha R + O(1) \tag{4.32}$$

$$\int_{-R}^0 \int_{\mathbb{R}_v} u(x,v)dvdx = \sqrt{\pi}\alpha R + O(1) \tag{4.33}$$

as $R \to +\infty$. But according to Proposition 4.3, one has for some constant d_1

$$\int_0^R \int_{\mathbb{R}_v} u(x,v)dvdx = -\frac{d_1}{\sqrt{2}}R + o(R) \tag{4.34}$$

$$\int_{-R}^0 \int_{\mathbb{R}_v} u(x,v)dvdx = \frac{d_1}{\sqrt{2}}R + o(R). \tag{4.35}$$

as $R \to +\infty$. One concludes that $\alpha = d_1 = 0$. Therefore $u = 0$. □

Since G_0W is a compact operator on $\mathcal{H}^{1,-s}$, $\frac{3}{2} < s < \frac{\rho+1}{2}$, it follows from Theorem 4.4 that $1 + G_0W$ is invertible and

$$(1 + G_0W)^{-1} \in \mathcal{B}(1,-s;1,-s). \tag{4.36}$$

Theorem 4.5. *Let $\rho > 2$. One has:*

$$\mathcal{N} = \{u \in \mathcal{H}^{1,-s}; (1 + G_0W)u = \beta\mathfrak{m}_0 \text{ for some } \beta \in \mathbb{C}, \frac{3}{2} < s < \frac{\rho+1}{2}\}. \tag{4.37}$$

In particular, $\mathcal{N}$ is of dimension one and

$$(1 + G_0W)\mathfrak{m} = \mathfrak{m}_0 \tag{4.38}$$

Proof. To prove (4.37), it remains to prove the inclusion

$$\mathcal{N} \subset \left\{u \in \mathcal{H}^{1,-s}; (1 + G_0W)u = \beta\mathfrak{m}_0 \text{ for some } \beta \in \mathbb{C}, \frac{3}{2} < s < \frac{\rho+1}{2}\right\}. \tag{4.39}$$

The inclusion in the opposite sense is a consequence of Lemma 4.2 and Proposition 4.3.

Let $u \in \mathcal{N}$ and $\lambda < 0$. Then $u \in \mathcal{H}^{1,-r}$ for $r > 1$ and r close to 1 and $P_0u = -Wu \in \mathcal{L}^{2,\rho+1-r}$. By Corollary 2.2, the resolvent $R_0(\lambda)$ can be decomposed as

$$R_0(\lambda) = b_0^w(v, D_x, D_v)(-\Delta_x - \lambda)^{-1} + r_0(\lambda) \tag{4.40}$$

where

$$b_0(v,\xi,\eta) = 2^{\frac{3}{2}} e^{-v^2-\eta^2+2iv\cdot\xi+2\xi^2}\chi(\xi)$$

with χ a smooth cut-off around 0 with compact support, and $r_0(\lambda)$ is uniformly bounded as operators in L^2 for $\lambda < a$ for some $a \in]0,1[$. One has

$$u + R_0(\lambda)Wu = -\lambda R_0(\lambda)u = -\lambda\left(b_0^w(v, D_x, D_v)(-\Delta_x - \lambda)^{-1} + r_0(\lambda)\right)u \tag{4.41}$$

for $\lambda < 0$. Recall the following estimate for $r_0(\lambda)$ (see (2.85) in [21]):

$$\|\langle x\rangle^{-s} r_0(\lambda)\langle x\rangle^s f\| \leq C(\|f\| + \|H_0 f\|) \tag{4.42}$$

for $f \in D(H_0)$, $\lambda < a$ and $s \in [0,2]$, where $H_0 = -\Delta_v + v^2 - \Delta_x$. It follows from (4.42) that

$$\lambda r_0(\lambda)u = O(|\lambda|), \quad \lambda < 0, \tag{4.43}$$

in $\mathcal{H}^{1,-r}$.

Let $\phi \in \mathcal{S}(\mathbb{R})$ such that $\int_{\mathbb{R}} \phi(x)dx = 1$. Then

$$\Pi = \langle \cdot, \phi \otimes \psi_0 \rangle \mathfrak{m}_0$$

is a projection on $\mathcal{H}^{1,-s}$ for any $s > \frac{1}{2}$ onto the linear span of $\mathfrak{m}_0$. Set $\Pi' = 1 - \Pi$. The term $\Pi' \lambda b_0^w(v, D_x, D_v)(-\Delta_x - \lambda)^{-1}u$ can be evaluated as follows. Making use of the inequality

$$|e^{-a} - e^{-b}| \leq |a - b|(e^{-a} + e^{-b}), \quad a, b \geq 0,$$

the quantity

$$\begin{aligned} &|\lambda \Pi' b_0^w(v, D_x, D_v)(-\Delta_x - \lambda)^{-1}u(x, v)| \\ &= \frac{\sqrt{|\lambda|}}{2} \left| \int_{\mathbb{R}^4} (e^{-\sqrt{|\lambda|}|y-(x-x')|} - e^{-\sqrt{|\lambda|}|y-(y'-x')|}) \phi(y') \Phi(v, v', y) u(x', v') \, dydy'dx'dv' \right| \end{aligned}$$

is bounded by

$$|\lambda| \int_{\mathbb{R}^4} |x - y'| (e^{-\sqrt{|\lambda|}|y-(x-x')|} + e^{-\sqrt{|\lambda|}|y-(y'-x')|}) |\phi(y') \Phi(v, v', y) u(x', v')| \, dydy'dx'dv'.$$

The integral involving the term $e^{-\sqrt{|\lambda|}|y-(x-x')|}$ can be evaluated as follows:

$$\begin{aligned} &|\lambda| \int_{\mathbb{R}^4} |x - y'| e^{-\sqrt{|\lambda|}|y-(x-x')|} |\phi(y') \Phi(v, v', y) u(x', v')| \, dydy'dx'dv' \\ &\leq C_1(1 + |x|)|\lambda| \int_{\mathbb{R}^3} e^{-\sqrt{|\lambda|}|y-(x-x')|} \left| \Phi(v, v', y) u(x', v') \right| \, dydx'dv' \\ &= C_2(1 + |x|)|\lambda| \int_{\mathbb{R}^3} e^{-\sqrt{|\lambda|}|y-(x-x')|} \left| \psi_0(v) \psi_0(v') \Psi(y - v - v') u(x', v') \right| \, dydx'dv' \\ &\leq C_3(1 + |x|)|\lambda| \|u\|_{\mathcal{L}^{2,-r}} \\ &\quad \times \left\{ \int_{\mathbb{R}^3} \left| \langle x' \rangle^r e^{-\sqrt{|\lambda|}|y-(x-x')|} \psi_0(v) \psi_0(v') \Psi(y - v - v') \right|^2 dydx'dv' \right\}^{\frac{1}{2}} \\ &\leq C_4(1 + |x|)^{1+r}|\lambda| \|u\|_{\mathcal{L}^{2,-r}} \left\{ \int_{\mathbb{R}} \left| \langle x' \rangle^r e^{-\sqrt{|\lambda|}|x'|} \right|^2 dx' \right\}^{\frac{1}{2}} \psi_0(v) \\ &\leq C_5(1 + |x|)^{1+r}|\lambda|^{\frac{3}{4} - \frac{r}{2}} \|u\|_{\mathcal{L}^{2,-r}} \psi_0(v) \end{aligned}$$

for some constants C_j. A similar upper-bound also holds for the integral involving the term $e^{-\sqrt{|\lambda|}|y-(y'-x')|}$. Putting them together, we obtain a point-wise upper-bound

$$|\lambda \left(\Pi' b_0^w(v, D_x, D_v)(-\Delta_x - \lambda)^{-1}u \right)(x, v)| \leq C(1 + |x|)^{1+r}|\lambda|^{\frac{3}{4} - \frac{r}{2}} \psi_0(v) \|u\|_{\mathcal{L}^{2,-r}} \quad (4.44)$$

This proves that for $1 < r < \frac{3}{2}$,

$$\lambda \Pi' b_0^w(v, D_x, D_v)(-\Delta_x - \lambda)^{-1}u \to 0, \text{ as } \lambda \to 0_- \quad (4.45)$$

in $\mathcal{L}^{2,-(\frac{3}{2}+r+\epsilon)}$, $\epsilon > 0$. Applying Π' to (4.41) and taking the limit $\lambda \to 0_-$, we get

$$\Pi'(1 + G_0 W)u = 0. \quad (4.46)$$

This means that there exists some constant $\beta \in \mathbb{C}$ such that $(1 + G_0W)u = \beta\mathfrak{m}_0$. The proof of (4.37) is complete.

Since $1 + G_0W$ is injective, one deduces from (4.37) that $\mathcal{N}$ is of dimension one. It is clear that $\mathfrak{m} \in \mathcal{N}$ and (4.37) implies that

$$(1 + G_0W)\mathfrak{m} = \beta\mathfrak{m}_0 \tag{4.47}$$

for some $\beta \in \mathbb{C}$. Proposition 4.3 applied to $\mathfrak{m}$ shows that $\mathfrak{m}$ has asymptotic behavior

$$\mathfrak{m}(x, v) = (\beta \mp d_1 + o(1))\psi_0(v), \quad x \to \pm\infty$$

with $d_1 \in \mathbb{C}$ given in Proposition 4.3. Comparing these relations with the trivial expansion of $\mathfrak{m}(x, v)$:

$$\mathfrak{m}(x, v) = (1 + O(\langle x\rangle^{-\rho}))\psi_0(v)$$

for $x \to \pm\infty$, one concludes that $\beta = 1$ and $d_1 = 0$, which prove (4.38). □

5. Low-energy expansion of the resolvent

Let $U_\delta = \{z; |z| < \delta, z \notin \mathbb{R}_+\}$, $\delta > 0$, and $\frac{3}{2} < s < \frac{\rho+1}{2}$. Recall that $(1 + G_0W)^{-1}$ exists and is bounded on $\mathcal{L}^{2,-s}$. Since

$$1 + R_0(z)W - \frac{1}{\sqrt{z}}G_{-1}W = 1 + G_0W + O(|z|^\epsilon) \tag{5.1}$$

in $\mathcal{L}^{2,-s}$ for $z \in U_\delta$, $1 + R_0(z)W - \frac{1}{\sqrt{z}}G_{-1}W$ is invertible for $z \in U_\delta$ if $\delta > 0$ is small enough. Denote

$$D(z) = \left(1 + R_0(z)W - \frac{1}{\sqrt{z}}G_{-1}W\right)^{-1}. \tag{5.2}$$

If $\rho > 2k + 2$, one has

$$D(z) = D_0 + \sum_{j=1}^{k} z^{\frac{j}{2}} D_j + O(|z|^{k+\epsilon}) \tag{5.3}$$

in $\mathcal{B}(1, -s; 1, -s)$ for $k + \frac{3}{2} < s < \frac{\rho+1}{2}$, where

$$D_0 = (1 + G_0W)^{-1} \tag{5.4}$$

$$D_1 = -D_0G_1WD_0 \tag{5.5}$$

$$D_2 = (D_0G_1W)^2D_0 - D_0G_2WD_0 \tag{5.6}$$

It follows that

$$(1 + R_0(z)W)^{-1} = D(z)(1 + M(z))^{-1} \tag{5.7}$$

where $M(z) = \frac{1}{\sqrt{z}}G_{-1}WD(z)$. $M(z)$ is an operator of rank one. In order to study the invertibility of $1 + M(z)$, consider the equation

$$(1 + M(z))u = f, \tag{5.8}$$

where $f \in \mathcal{L}^{2,-s}$ is given and $u = u(z)$ is to be determined. Take $\phi^*(x, v) = \chi(x)\psi_0(v)$ with $\chi \in \mathcal{S}(\mathbb{R})$ such that

$$\int_{\mathbb{R}} \chi(x)dx = 1.$$

Let $\Pi_0 = \langle \cdot, \phi^* \rangle \mathfrak{m}_0$. Then $\Pi_0^2 = \Pi_0$. Decompose f and u as $f = f_0 + f_1$ and $u = u_0 + u_1$ where $f_0 = \Pi_0 f$, $f_1 = (1 - \Pi_0) f$, and similarly for u. Equation (5.8) is equivalent with

$$u_1 = f_1 \text{ and} \tag{5.9}$$

$$C(z)(1 + \langle M(z)\mathfrak{m}_0, \phi^* \rangle) = \langle f, \phi^* \rangle - \langle M(z) f_1, \phi^* \rangle \tag{5.10}$$

where $C(z) = \langle u, \phi^* \rangle$ is some constant to be calculated. If $1 + \langle M(z)\mathfrak{m}_0, \phi^* \rangle \neq 0$ for $z \in U_\delta$, as we shall prove below, then $C(z)$ is uniquely determined by (5.10). Consequently, the equation $(1 + M(z))u = f$ has a unique solution given by

$$u = C(z)\mathfrak{m}_0 + f_1. \tag{5.11}$$

This will prove the invertibility of $1 + M(z)$ for $z \in U_\delta$.

Let us now study

$$m(z) = 1 + \langle M(z)\mathfrak{m}_0, \phi^* \rangle \tag{5.12}$$

for $z \in U_\delta$. Applying (5.3) with $k = 1$ (we need here the condition $\rho > 4$), one obtains

$$\langle M(z)\mathfrak{m}_0, \phi^* \rangle = \frac{i}{2\sqrt{z}} \langle W D(z)\mathfrak{m}_0, \mathfrak{m}_0 \rangle = \frac{i}{2\sqrt{z}} \left(\sigma_0 + \sqrt{z}\sigma_1 + O(|z|^{\frac{1}{2}+\epsilon}) \right) \tag{5.13}$$

where $\sigma_j = \langle W D_j \mathfrak{m}_0, \mathfrak{m}_0 \rangle$. By Theorem 4.5,

$$(1 + G_0 W)^{-1} \mathfrak{m}_0 = \mathfrak{m}. \tag{5.14}$$

Consequently

$$\sigma_0 = \langle W\mathfrak{m}, \mathfrak{m}_0 \rangle = 0 \tag{5.15}$$

and

$$\begin{aligned} \sigma_1 &= \langle (1 + G_0 W)^{-1} G_1 W (1 + G_0 W)^{-1} \mathfrak{m}_0, W\mathfrak{m}_0 \rangle \\ &= \langle G_1 W\mathfrak{m}, D_0^* W\mathfrak{m}_0 \rangle \end{aligned}$$

Let J be the symmetry in velocity variable defined by $J : g(x,v) \to (Jg)(x,v) = g(x,-v)$. Then $J^2 = 1$ and

$$JPJ = P^*, \quad JWJ = -W \quad \text{ and } JP_0J = P_0^*. \tag{5.16}$$

It follows that $(R_0(z)W)^* = JWR_0(\overline{z})J$, hence

$$(1 + G_0 W)^* = J(1 + WG_0)J. \tag{5.17}$$

We derive that

$$\begin{aligned} D_0^* W\mathfrak{m}_0 &= J(1 + WG_0)^{-1} JW\mathfrak{m}_0 \\ &= -J(1 + WG_0)^{-1} W\mathfrak{m}_0 \\ &= -JW(1 + G_0 W)^{-1}\mathfrak{m}_0 = -JW\mathfrak{m} = W\mathfrak{m}. \end{aligned}$$

This shows

$$\sigma_1 = \langle G_1 W\mathfrak{m}, W\mathfrak{m} \rangle. \tag{5.18}$$

Since $G_1 = \frac{1}{\sqrt{z}}(R_0(z) - \frac{1}{\sqrt{z}}G_{-1} - G_0) + O(|z|^\epsilon)$ in $\mathcal{B}(-1,s;1,-s)$, $s > \frac{5}{2}$, noticing that $G_{-1}W\mathfrak{m} = 0$, $(1+G_0W)\mathfrak{m} = \mathfrak{m}_0$, one obtains for $z = \lambda < 0$

$$\begin{aligned}\langle G_1 W\mathfrak{m}, W\mathfrak{m}\rangle &= -\frac{i}{\sqrt{|\lambda|}}\langle R_0(\lambda)W\mathfrak{m}, W\mathfrak{m}\rangle + O(|\lambda|^\epsilon) \quad (5.19)\\ &= i\sqrt{|\lambda|}\langle R_0(\lambda)\mathfrak{m}, W\mathfrak{m}\rangle + O(|\lambda|^\epsilon). \quad (5.20)\end{aligned}$$

Proposition 5.1. *Assume $\rho > 4$. One has*

$$\langle G_1 W\mathfrak{m}, W\mathfrak{m}\rangle = i\lim_{\lambda\to 0_-}\sqrt{|\lambda|}\langle R_0(\lambda)\mathfrak{m}, W\mathfrak{m}\rangle = 0. \quad (5.21)$$

Proof. Let $\lambda < 0$ and Π' be defined as in the proof of Theorem 4.5. Then $\langle R_0(\lambda)\mathfrak{m}, W\mathfrak{m}\rangle = \langle \Pi' R_0(\lambda)\mathfrak{m}, W\mathfrak{m}\rangle$, since $\langle \mathfrak{m}_0, W\mathfrak{m}\rangle = 0$. One has

$$R_0(\lambda)\mathfrak{m} = (b_0^w(v, D_x, D_v)(-\Delta_x - \lambda)^{-1} + r_0(\lambda))\mathfrak{m} \quad (5.22)$$

in $\mathcal{L}^{2,-r}$ for any $r > \frac{1}{2}$ and it follows from (4.42) that

$$\sqrt{|\lambda|}r_0(\lambda)\mathfrak{m} = O(\sqrt{|\lambda|}) \quad (5.23)$$

in $\mathcal{H}^{1,-r}$. Let us evaluate $\sqrt{|\lambda|}\Pi' b_0^w(v, D_x, D_v)(-\Delta_x - \lambda)^{-1}\mathfrak{m}$.

$$\begin{aligned}&\sqrt{|\lambda|}\Pi' b_0^w(v, D_x, D_v)(-\Delta_x - \lambda)^{-1}\mathfrak{m}(x,v)\\ &= \frac{i}{2}\int_{\mathbb{R}^4}(e^{-\sqrt{|\lambda|}|y-(x-x')|} - e^{-\sqrt{|\lambda|}|y-(y'-x')|})\phi(y')\Phi(v,v',y)\mathfrak{m}(x',v')\, dydy'dx'dv'\\ &= \frac{i}{2}\int_{\mathbb{R}^4}(e^{-\sqrt{|\lambda|}|y-(x-x')|} - e^{-\sqrt{|\lambda|}|y-(y'-x')|})\phi(y')\Phi(v,v',y)\mathfrak{m}_0(v')\, dydy'dx'dv'\\ &+ \frac{i}{2}\int_{\mathbb{R}^4}(e^{-\sqrt{|\lambda|}|y-(x-x')|} - e^{-\sqrt{|\lambda|}|y-(y'-x')|})\phi(y')\Phi(v,v',y)(\mathfrak{m}(x',v) - \mathfrak{m}_0(v'))\, dydy'dx'\\ &= \frac{i}{2}\int_{\mathbb{R}^4}(e^{-\sqrt{|\lambda|}|y-(x-x')|} - e^{-\sqrt{|\lambda|}|y-(y'-x')|})\phi(y')\Phi(v,v',y)(\mathfrak{m}(x',v) - \mathfrak{m}_0(v'))\, dydy'dx'\\ &= O(\sqrt{|\lambda|}|x|\psi_0(v))\end{aligned}$$

for $(x,v) \in \mathbb{R}^2$. The first term on the right-hand side of the second equality above vanishes by first integrating with respect to x' variable. In the last equality above, we used the upper bound

$$|e^{-\sqrt{|\lambda|}|y-(x-x')|} - e^{-\sqrt{|\lambda|}|y-(y'-x')|}| \le \sqrt{|\lambda|}|x-y'|\left(e^{-\sqrt{|\lambda|}|y-(x-x')|} + e^{-\sqrt{|\lambda|}|y-(y'-x')|}\right)$$

and the fact $\mathfrak{m} - \mathfrak{m}_0 = O(\langle x\rangle^{-\rho})\psi_0(v)$ to evaluate the integral. It follows that

$$\sqrt{|\lambda|}\langle \Pi' R_0(\lambda)\mathfrak{m}, W\mathfrak{m}\rangle = O(\sqrt{|\lambda|}), \quad \lambda \to 0_-$$

which finishes the proof of Proposition 5.1. □

Summing up, we proved that if $\rho > 4$, then $m(z) = 1 + \frac{i}{2\sqrt{z}}\langle WD(z)\mathfrak{m}_0, \mathfrak{m}_0\rangle$ verifies

$$m(z) = 1 + O(|z|^\epsilon), \quad \epsilon > 0, \quad (5.24)$$

for $z \in U_\delta$. Therefore $1 + M(z)$ is invertible for $z \in U_\delta$ with $\delta > 0$ small enough and the solution u to the equation $(1 + M(z))u = f$ is given by

$$\begin{aligned} u &= f_1 + \frac{1}{m(z)}(\langle f, \phi^* \rangle - \langle M(z) f_1, \phi^* \rangle)\varphi_0 \\ &= f - \langle f, \phi^* \rangle \mathfrak{m}_0 + \frac{1}{m(z)}(\langle f, \phi^* \rangle - \langle M(z)(f - \langle f, \phi^* \rangle \mathfrak{m}_0), \phi^* \rangle)\mathfrak{m}_0 \\ &= f - \frac{1}{m(z)} \langle M(z) f, \phi^* \rangle \mathfrak{m}_0. \end{aligned} \tag{5.25}$$

Taking notice that $\langle \mathfrak{m}_0, \phi^* \rangle = 1$, we proved the following

Proposition 5.2. *Let $\rho > 4$. Then $1 + M(z)$ is invertible in $\mathcal{B}(1, -s; -1, s)$, $s > \frac{3}{2}$, for $z \in U_\delta$. Its inverse is given by*

$$(1 + M(z))^{-1} = 1 - \frac{1}{m(z)\sqrt{z}} G_{-1} W D(z). \tag{5.26}$$

In addition, if $\rho > 2k + 2$ for some $k \geq 1$, one has

$$(1 + M(z))^{-1} = 1 - \frac{1}{m(z)\sqrt{z}} G_{-1} W \left(D_0 + \sum_{j=1}^{k} z^{\frac{j}{2}} D_j + O(|z|^{k+\epsilon}) \right) \tag{5.27}$$

in $\mathcal{B}(1, -s; 1, -s)$ for $k + \frac{3}{2} < s < \frac{\rho+1}{2}$, where D_j is given by (5.3).

Theorem 5.3. *Let $\rho > 4$. Then there exists some constant $\delta > 0$ such that if $s > \frac{5}{2}$*

$$R(z) = \frac{i}{2\sqrt{z}} \langle \cdot, \mathfrak{m} \rangle \mathfrak{m} + O(|z|^{-\frac{1}{2}+\epsilon}), \quad z \in U_\delta, \tag{5.28}$$

in $\mathcal{B}(-1, s; 1, -s)$ for some $\epsilon > 0$. In particular, P has no eigenvalues in U_δ. In addition, the boundary values $R(\lambda \pm i0)$ of $R(z)$ exist in $\mathcal{B}(-1, s; 1, -s)$, $s > \frac{3}{2}$, for $\lambda \in]0, \delta[$ and is Hölder continuous in $\lambda \in]0, \delta[$.

Proof. We see from the above calculation that $(1 + M(z))^{-1}$ admits an asymptotic expansion as $z \in U_\delta$ and $z \to 0$. The existence of the asymptotics of the resolvent $R(z)$ follows from the equation

$$R(z) = D(z)(1 + M(z))^{-1} R_0(z) = D(z) \left(1 - \frac{1}{m(z)\sqrt{z}} G_{-1} W D(z) \right) R_0(z). \tag{5.29}$$

Let us calculate its leading term.

$$\begin{aligned} &\left(1 - \frac{1}{m(z)\sqrt{z}} G_{-1} W D(z) \right) R_0(z) \\ &\equiv -\frac{1}{m(z)z} G_{-1} W D_0 G_{-1} + \frac{1}{\sqrt{z}} \left(G_{-1} - \frac{1}{m(z)} (G_{-1} W D_0 G_0 + G_{-1} W D_1 G_{-1}) \right). \end{aligned}$$

Here and in the following, "$\equiv$" means equality module some term which is of order $O(|z|^{-\frac{1}{2}+\epsilon})$ in $\mathcal{B}(-1, s; 1, -s)$, $s > \frac{5}{2}$. Recall that $G_{-1} = \frac{i}{2} \langle \cdot, \mathfrak{m}_0 \rangle \mathfrak{m}_0$, $D_0 = (1 + G_0 W)^{-1}$ and $(1 + G_0 W)^{-1} \mathfrak{m}_0 = \mathfrak{m}$. It follows that

$$G_{-1} W D_0 G_{-1} = \frac{i}{2} \langle W \mathfrak{m}, \mathfrak{m}_0 \rangle \langle \cdot, \mathfrak{m}_0 \rangle \mathfrak{m}_0 = 0. \tag{5.30}$$

Consequently

$$\begin{aligned} & D(z)\left(1-\frac{1}{m(z)\sqrt{z}}G_{-1}WD(z)\right)R_0(z) \\ \equiv\ & \frac{1}{\sqrt{z}}D_0\left(G_{-1}-\frac{1}{m(z)}(G_{-1}WD_0G_0+G_{-1}WD_1G_{-1})\right) \end{aligned}$$

Noticing that $m(z)=1+O(|z|^\epsilon)$, one obtains

$$\begin{aligned} R(z) &\equiv \frac{1}{\sqrt{z}}D_0G_{-1}(1-W(D_0G_0+D_1G_{-1})) \\ &= \frac{i}{2\sqrt{z}}\langle((1-W(D_0G_0+D_1G_{-1}))\cdot,\mathfrak{m}_0\rangle\mathfrak{m} \end{aligned} \tag{5.31}$$

Recall that $D_0^*W\mathfrak{m}_0=W\mathfrak{m}$ and $\langle G_1W\mathfrak{m},W\mathfrak{m}\rangle=0$ (see Proposition 5.1). One can simplify the leading term as follows:

$$\begin{aligned} & \langle(1-W(D_0G_0))\cdot,\mathfrak{m}_0\rangle \\ &= \langle\cdot,\mathfrak{m}_0\rangle+\langle\cdot,G_0^*D_0^*W\mathfrak{m}_0\rangle=\langle\cdot,\mathfrak{m}_0\rangle+\langle\cdot,G_0^*W\mathfrak{m}\rangle \\ &= \langle\cdot,\mathfrak{m}_0\rangle+\langle\cdot,JG_0JW\mathfrak{m}\rangle=\langle\cdot,\mathfrak{m}_0\rangle-\langle\cdot,G_0W\mathfrak{m}\rangle=\langle\cdot,\mathfrak{m}\rangle \end{aligned}$$

and

$$\begin{aligned} & \langle WD_1G_{-1}\cdot,\mathfrak{m}_0\rangle \\ &= -\langle WD_0G_1WD_0G_{-1}\cdot,\mathfrak{m}_0\rangle=-\frac{i}{2}\langle\cdot,\mathfrak{m}_0\rangle\langle WD_0G_1W\mathfrak{m},\mathfrak{m}_0\rangle \\ &= \frac{i}{2}\langle\cdot,\mathfrak{m}_0\rangle\langle G_1W\mathfrak{m},D_0^*W\mathfrak{m}_0\rangle=\frac{i}{2}\langle\cdot,\mathfrak{m}_0\rangle\langle G_1W\mathfrak{m},W\mathfrak{m}\rangle=0. \end{aligned}$$

This finishes the proof of (5.28). (5.28) implies that $R(z)$ has no poles in U_δ, hence P has no eigenvalues there. The last statement of Theorem 5.3 is a consequence of Corollary 2.2 (b) and (5.29), since the boundary values $D(\lambda\pm i0)$ exist in $\mathcal{B}(1,-s;1,-s)$, $s>\frac{3}{2}$, for $\lambda\in]0,\delta[$ and are continuous in λ. □

6. Large time asymptotics of solutions

The following high energy resolvent estimate is proved in [21].

Theorem 6.1. *Let $n\geq 1$ and assume (1.4) with $\rho\geq -1$. Then there exists $C>0$ such that $\sigma(P)\cap\{z;|\mathrm{Im}\ z|>C,\mathrm{Re}\ z\leq\frac{1}{C}|\mathrm{Im}\ z|^{\frac{1}{2}}\}=\emptyset$ and*

$$\|R(z)\|\leq\frac{C}{|z|^{\frac{1}{2}}}, \tag{6.1}$$

and

$$\|(1-\Delta_v+v^2)^{\frac{1}{2}}R(z)\|\leq\frac{C}{|z|^{\frac{1}{4}}}, \tag{6.2}$$

for $|\mathrm{Im}\ z|>C$ and $\mathrm{Re}\ z\leq\frac{1}{C}|\mathrm{Im}\ z|^{\frac{1}{2}}$.

Let $S(t) = e^{-tP}$, $t \geq 0$, be the one-parameter strongly continuous semigroup generated by $-P$. Then one can firstly represent $S(t)$ as

$$S(t)f = \frac{1}{2\pi i}\int_\gamma e^{-tz}R(z)f dz \tag{6.3}$$

for $f \in L^2(\mathbb{R}^2)$ and $t > 0$, where the contour γ is chosen such that

$$\gamma = \gamma_- \cup \gamma_0 \cup \gamma_+$$

with $\gamma_\pm = \{z; z = \pm iC + \lambda \pm iC\lambda^2, \lambda \geq 0\}$ and γ_0 is a curve in the left-half complexe plane joining $-iC$ and iC for some $C > 0$ sufficiently large, γ being oriented from $-i\infty$ to $+i\infty$.

Remark that under the condition (1.4) with $\rho > 0$, P has no eigenvalue on the imaginary axis ([9]). Making use of analytic deformation and Theorem 5.3, one obtains from (6.1) that

$$\langle S(t)f, g\rangle = \frac{1}{2\pi i}\int_\Gamma e^{-tz}\langle R(z)f, g\rangle dz, \quad t > 0, \tag{6.4}$$

for any $f, g \in \mathcal{L}^{2,s}$ with $s > \frac{5}{2}$. Here

$$\Gamma = \Gamma_- \cup \Gamma_0 \cup \Gamma_+$$

with

$$\Gamma_\pm = \{z; z = \delta + \lambda \pm i\delta^{-1}\lambda^2, \lambda \geq 0\}$$

for $\delta > 0$ small enough and

$$\Gamma_0 = \{z = \lambda + i0; \lambda \in [0, \delta]\} \cup \{z = \lambda - i0; \lambda \in [0, \delta]\}.$$

Γ is oriented from $-i\infty$ to $+i\infty$.

Proof of Theorem 1.1. By (6.4), one has for $f, g \in \mathcal{L}^{2,s}(\mathbb{R}^2)$ with $s > \frac{5}{2}$

$$\begin{aligned}\langle S(t)f, g\rangle &= \frac{1}{2\pi i}\left(\int_{\Gamma_0} + \int_{\Gamma_-} + \int_{\Gamma_+}\right) e^{-tz}\langle R(z)f, g\rangle\, dz \\ &\equiv I_1 + I_2 + I_3.\end{aligned}$$

For $\delta > 0$ appropriately small and fixed, it follows from Theorem 6.1 that there exist some constants $C, c > 0$ such that

$$|I_j| \leq Ce^{-ct}\|f\|\,\|g\|, \quad t > 0, \tag{6.5}$$

for $j = 2, 3$. Set

$$F_{-1} = \frac{i}{2}\langle \cdot, \mathfrak{m}\rangle \mathfrak{m}.$$

Applying Theorem 5.3, one has

$$\begin{aligned} I_1 &= \frac{1}{2\pi i}\int_0^\delta e^{-t\lambda}\langle (R(\lambda+i0)-R(\lambda-i0))f,g\rangle\, d\lambda \\ &= \frac{1}{\pi i}\int_0^\delta e^{-t\lambda}\lambda^{-\frac{1}{2}}\langle (F_{-1}+O(\lambda^\epsilon))f,g\rangle\, d\lambda \\ &= \frac{1}{\pi i}\int_0^{+\infty}\frac{1}{\sqrt{\lambda}}e^{-t\lambda}\langle F_{-1}f,g\rangle\, d\lambda + O(t^{-\frac{1}{2}-\epsilon})\|f\|_{0,s}\|g\|_{0,s} \\ &= \frac{1}{i\sqrt{\pi t}}\langle F_{-1}f,g\rangle + O(t^{-\frac{1}{2}-\epsilon}) \end{aligned} \tag{6.6}$$

as $t\to+\infty$ for some $\epsilon>0$. Using the formula for F_{-1}, we arrive at

$$S(t) = \frac{1}{(4\pi t)^{\frac{1}{2}}}\langle\cdot,\mathfrak{m}\rangle\mathfrak{m} + O(t^{-\frac{1}{2}-\epsilon}), t\to+\infty \tag{6.7}$$

as operators in $\mathcal{B}(0,s;0,-s)$ with $s>\frac{5}{2}$. Theorem 1.1 is proved. □

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