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THE PROBABILISTIC NATURE OF MCSHANE'S IDENTITY: PLANAR TREE CODING OF SIMPLE LOOPS

FRANÇOIS LABOURIE AND SER PEOW TAN

With admiration, to our friend and mentor: Bill Goldman.

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1. INTRODUCTION

In his celebrated paper [10], Greg McShane showed that for a 1-cusped hyperbolic torus the following identity holds:

$$1 = \sum_{\gamma \in \mathcal{G}} \frac{1}{e^{\ell(\gamma)} + 1}, \quad (1)$$

where the sum runs over the set \mathcal{G} of oriented simple loops in the torus. Similarly for 1-cusped hyperbolic surfaces

$$1 = \sum_{p \in \mathcal{P}} \frac{1}{e^{\frac{\ell(\partial p)}{2}} + 1}. \quad (2)$$

where the sum runs now over the space \mathcal{P} of embedded 1-cusp pair of pants – or equivalently the space of oriented simple arcs from the cusp to itself.

This formula has had a lot of descendants and generalisations, to quasifuchsian spaces [14, 17, 16, 15, 4], to higher rank geometries [8], to surfaces with boundary components [11]. An extensive survey of geometric identities with similar features can be found in [3]. Quite notably, this identity has played a fundamental role in Mirzakhani acclaimed proof of the Kontsevich–Witten formula for intersection numbers [12].

Among these proofs stands a remarkable argument by Bowditch [2] proving McShane’s identity on the torus using Markoff triples and a combinatorial approach. In turn this interpretation has given rise to yet another higher rank construction [5] whose geometrical interpretation remains elusive.

The present paper grows out of the fascination of both authors for Bowditch’s proof and wishes to present a generalisation of Bowditch’s proof for higher genus surfaces, equally working in the context of higher geometries. More importantly, we want to present a new interpretation of McShane’s identity: essentially, we show that this formula has a probabilistic nature – as emphasized by the 1 on the lefthand side of the equations above – and we hope that in the future

this could lead to a better understanding of Mirzakhani's topological recursion.

Essentially our interpretation runs as follows. Let S be a topological surface with one marked point x_0 . As a rebranding of a Lemma by L. Mosher [13], we show that given any triangulation with x_0 as the only vertex, one can "code" any simple arc in S passing through x_0 by an edge of the original triangulation together with a word in the semigroup with two generators (Theorem 6.1.1) and similar results holds with infinite words and lamination (Theorem 6.9.1). We actually consider these words as embedded paths in some trivalent (except at the root) rooted planar tree \mathcal{T} .

Moreover, in that context the planar structure of the tree allows us to distinguish between *rational paths* – which end up coding rational laminations – and *irrational paths* – which interpret irrational laminations.

Now the geometry – a hyperbolic structure or more generally a cross ratio [8][6] – gives rise to a harmonic 1-form on \mathcal{F} and thus to a probability measure μ on the space of embedded paths.

Finally, using the Birman–Series Theorem [1] as in [10], one concludes that the measure of the set of irrational paths is zero, whereas the measure on the space of rational paths corresponds to the right-hand side of the equations above. Since the rational paths are countable and essentially labelled by simple paths through x_0 thanks to our coding theorem, McShane's identity is just expressing that the sum of the measure of rational paths is one.

Although, there are no new results in this article and in many ways the proofs are either trivial or well known, we hope that our presentation insisting on the probabilistic nature of McShane's identity will be helpful in the future.

The first section recalls harmonic measures on trees and how they define random paths in section 2. Then we specialize to planar trees in section 3 and describe the gap (the measure of rational paths) and error terms (the measure of irrational paths) of a general harmonic measure in Section 4. In Section 5, we relate the error term to the Lebesgue measure of a certain set on the circle. In section 6, we introduce a path coding geodesics once one chooses a triangulation. Then in section 7, we explain how hyperbolic structures and more generally cross ratios give rise to specific measures. In section 8, we show that the Birman–Series Theorem implies that the error term vanishes. Finally in Section 9, we indicate how one could generalize the above construction to surfaces with more boundary components.

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2. GREEN FORMULA FOR ROOTED TREES

We will recall in this section very standard material about harmonic analysis on graphs.

Let \mathcal{T} be a rooted tree whose root is denoted v_0 . Motivated by the applications to geodesics, we shall assume that all vertices –except for the root – has valence 3.

Every edge comes with an orientation: we say an edge is *positively oriented* or *positive* if it comes outward from the root, *negatively oriented* if it goes inward. We shall denote by $S(n)$ the sphere of combinatorial radius n centred at the root v_0 . If v is a vertex, let $|v|$ be the distance to the root, if e is a positive edge we define by $|e| := |v^+|$, where v^+ is the endpoint of e . Finally, if e is an edge we denote by \bar{e} the edge with the opposite orientation.

2.1. Harmonic forms. A function Φ on the set of edges of the tree is said to be a *1-form* if

$$\Phi(\bar{e}) = -\Phi(e),$$

moreover Φ is a *harmonic*, if for every vertex different from the root, if e_x is the incoming edge (from the root) and e_{\pm} the outgoing edges, we have

$$\Phi(e_x) = \Phi(e_+) + \Phi(e_-). \quad (3)$$

Since Φ is a function of the edges, we could consider it as a 1-form on the graph. Then the above equation just says that Φ considered as a 1-form is harmonic, everywhere except possibly at the root. Let us then define

$$\partial\Phi := \sum_{e \in S(1)} \Phi(e). \quad (4)$$

We have

Corollary 2.1.1. [GREEN FORMULA] *The following equality holds*

$$\sum_{e \in S(n)} \Phi(e) = \partial\Phi. \quad (5)$$

2.2. The harmonic measure. Assume now that Φ is positive on positive edges. Any vertex x in the tree other than the root is trivalent, let us define a probability measure μ_x on the edges having x as an extremities (two outgoing, one ingoing) in the following way

- the probability $\mu_x(e_x) = 0$ if e_x is the edge directed towards the root,
- Otherwise, let

$$\mu_x(e) = \frac{\Phi(e)}{\Phi(e_x)}.$$

This family of measures defines an inhomogeneous Markov process. Let \mathcal{P} be the set of infinite embedded paths in the tree starting from the root. Although we shall not need it we may identify \mathcal{P} with $\partial_\infty \mathcal{T}$. Let π_n be the map from \mathcal{P} to set of edges of \mathcal{T} which associates to a path its n step:

$$p^n := \pi_n(p) \in S(n).$$

We thus have the *harmonic measure*, see [9], μ_Φ on \mathcal{P} associated to this random process. By definition, this measure is such that

$$\pi_*^n(\mu_\Phi) = \Phi|_{S(n)}. \quad (6)$$

In particular

$$\mu_\Phi\{p\} = \Phi_\infty(p) \text{ and } \mu_\Phi(\mathcal{P}) = \partial\Phi, \quad (7)$$

where, given a path p , we define

$$\Phi_\infty(p) = \liminf_{n \rightarrow \infty} \Phi(\pi_n(p)).$$

Observe that since Φ is positive on positive edges – that is edges oriented away from the root – $\Phi(\pi_n(p))$ is decreasing as a function of n and thus Φ_∞ is actually the limit of $\Phi \circ \pi_n$.

By construction, if $e \in S(n)$, $\Phi(e)$ is the probability that a random path starting from the root arrives at e at the n step.

3. PLANAR TREE, COMPLEMENTARY REGIONS AND RATIONAL PATHS

A *planar tree* is a tree that can be embedded in the plane. Equivalently a planar structure on the tree is given by a cyclic order on the edges outgoing from each vertex. In particular,

- we obtain a cyclic order on $S(n)$,
- Given a positive edge e arriving in v , we define the edges $\mathcal{L}(e)$ and $\mathcal{R}(e)$ to be the outgoing edges from v so that $(\bar{e}, \mathcal{L}(e), \mathcal{R}(e))$ is positively oriented. In particular, the edges $\mathcal{L}(e)$ and $\mathcal{R}(e)$ are positive.

3.1. Complementary regions. Every edge e in the tree defines (injectively) a *complementary region* of the planar graph if we considered it properly embedded in the plane. This region, also denoted e , is bounded by the edges $\mathcal{L}^n \mathcal{R}(e)$ and $\mathcal{R}^n \mathcal{L}(e)$ for all non negative integers n and we define the paths

$$\partial^R e := \bigcup_{n \in \mathbb{N}} \mathcal{L}^n \mathcal{R}(e), \quad \partial^L e := \bigcup_{n \in \mathbb{N}} \mathcal{R}^n \mathcal{L}(e)$$

where the paths are completed by adding the necessary initial edges so that they start at v_0 (by abuse of notation we will usually consider this tail end of the path to be the path).

So far some complementary regions are missing from this construction, namely those innermost regions which are adjacent to the root v_0 . Let us label those as well. Let e^1, \dots, e^N are the edges stemming from the root v_0 in their cyclic order, let by definition $S(0) := \{(e^i, e^{i+1}) \mid i \in \{1, \dots\}\}$. Then any $f = (e^i, e^{i+1})$ defines also a *complementary region* bounded by the two paths $\mathcal{R}^n(e^i)$ and $\mathcal{L}^n(e^{i+1})$. By convention we write $\mathcal{L}f := e^i$ and $\mathcal{R}f := e^{i+1}$. Then, similarly

$$\partial^R f := \bigcup_{n \in \mathbb{N}} \mathcal{L}^n \mathcal{R}(f), \quad \partial^L f := \bigcup_{n \in \mathbb{N}} \mathcal{R}^n \mathcal{L}(f).$$

The set

$$\mathcal{F} := \bigsqcup_{n \in \mathbb{N}} S(n),$$

is the set of *complementary regions*.

3.2. Rational paths. The planar structure on the tree helps us to distinguish between "irrational" and "rational" paths: we saw that every complementary region e defines two paths $\partial^L e$ and $\partial^R e$ starting from v_0 . By definition we call these paths *rational* and any other paths *irrational*. Observe that the set \mathcal{Q} of rational paths is countable.

Remark that the set of paths \mathcal{P} inherits a lexicographic order from the order in $S(n)$. If p_0 and p_1 are two paths with $p_0 < p_1$ in the lexicographic order, we define

$$[p_0, p_1] := \{p \mid p_0 \leq p \leq p_1\}, \quad (8)$$

$$]p_0, p_1[:= \{p \mid p_0 < p < p_1\}. \quad (9)$$

We can observe the following fact:

Proposition 3.2.1. *Given two distinct paths p, q so that $]p, q[$ is empty, then there exists a complementary region so that $\{p, q\} = \{\partial^L e, \partial^R e\}$. In particular p, q are rational.*

4. GAP FUNCTION, ERROR TERMS AND THE GAP INEQUALITY

We now assume that we have a positive harmonic 1-form Φ defined on our tree. Given a complementary region e we define its *gap* as

$$\text{Gap}_\Phi(e) = \frac{1}{2} (\mu_\Phi \partial^L e) + \mu_\Phi(\partial^R e)$$

Our first result is the following:

Theorem 4.0.1. [GAP INEQUALITY] *We have*

$$\sum_{e \in \mathcal{F}} \text{Gap}_\Phi(e) \leq \frac{1}{2} \partial \Phi. \quad (10)$$

We will refer to

$$\text{Error}(\Phi) := \frac{1}{2} \partial \Phi - \sum_{e \in \mathcal{F}} \text{Gap}_\Phi(e) \quad (11)$$

as the error term of Φ . This gap inequality is almost a tautology and we shall give two immediate proofs. One emphasizes the probabilistic nature of the situation, the other uses the Green Formula and is very close to Bowditch's original idea.

4.1. Gap and error terms using the harmonic measure on the space of paths. We can express the gap and error terms using the harmonic measure

Proposition 4.1.1. *We have the equalities*

$$\text{Gap}(\Phi) = \frac{1}{2} \mu_\Phi(\mathcal{Q}), \quad \text{Error}(\Phi) = \frac{1}{2} \mu_\Phi(\mathcal{P} \setminus \mathcal{Q}).$$

Proof. Since $\mu_\Phi(\mathcal{P}) = \partial \Phi$, the second equality is a consequence of the first. To prove the first we have to notice that any rational path appears in the boundary of exactly one complementary region. Thus

$$\text{Gap}(\Phi) = \sum_{e \in \mathcal{F}} \text{Gap}_\Phi(e) = \frac{1}{2} \sum_{p \in \mathcal{Q}} \Phi_\infty(p) = \frac{1}{2} \mu_\Phi(\mathcal{Q}).$$

□

4.2. The gap inequality from the Green Formula. For any complementary region f and $n \in \mathbb{N}$ let

$$\text{Gap}_\Phi^n(f) := \frac{1}{2} (\Phi(\mathcal{L}^n \cdot \mathcal{R} \cdot f) + \Phi(\mathcal{R}^n \cdot \mathcal{L} \cdot f)).$$

Then obviously, from the positivity of Φ , we have

$$\text{Gap}_\Phi^n(f) \geq \text{Gap}_\Phi(f) := \liminf_{n \rightarrow \infty} \text{Gap}_\Phi^n(f).$$

We can now revisit the Green Formula by rearranging its term to get

Proposition 4.2.1. *We have*

$$\sum_{f \in \mathcal{S}(p) | p < n} \text{Gap}_\Phi^{n-p}(e) = \frac{1}{2} \partial\Phi, \quad (12)$$

The gap inequality follows from this formula since $\text{Gap}_\Phi(e) \leq \text{Gap}_\Phi^n(e)$.

Proof. For every n , we have the Green formula

$$\sum_{x \in \mathcal{S}(n)} \Phi(x) = \partial\Phi. \quad (13)$$

For all n , recall that

$$\text{Gap}_\Phi^n(e) = \frac{1}{2} (\Phi(\mathcal{L}^n \cdot \mathcal{R} \cdot e) + \Phi(\mathcal{R}^n \cdot \mathcal{L} \cdot e)). \quad (14)$$

Recall also that given an element z in $\mathcal{S}(n+1)$ there exist unique element z in $\mathcal{S}(p)$ with $p < n$ with $z \in \partial y$. or in other words $z = \mathcal{L}^{n-p} \cdot \mathcal{R} \cdot y$ or $z = \mathcal{R}^{n-p} \cdot \mathcal{L} \cdot y$. Then we have

$$\begin{aligned} \sum_{x \in \mathcal{S}(p) | p < n} \text{Gap}_\Phi^{n-p}(x) &= \frac{1}{2} \sum_{x \in \mathcal{S}(p) | p < n} (\Phi(\mathcal{L}^{n-p} \cdot \mathcal{R} \cdot x) + \Phi(\mathcal{R}^{n-p} \cdot \mathcal{L} \cdot x)) \\ &= \frac{1}{2} \sum_{y \in \mathcal{S}(n+1)} \Phi(y) = \frac{1}{2} \partial\Phi. \end{aligned} \quad (15)$$

□

5. THE ERROR TERM AND RANDOM VARIABLES ON \mathcal{P}

The terminology "rational" and "irrational" paths seems to suggest that the measure of rational paths ought to be zero. This is indeed the case when the probability is balanced: we have equal probability to take the left or right edge. However, we shall see on the contrary that in the geometric context which is underlying McShane's identity the measure of irrational paths is zero.

We develop in this section a framework to understand the error term as the measure on some set on the circle. More precisely, we describe an "increasing embedding" of \mathcal{P} in $\mathbb{R}/\partial\Phi\mathbb{Z}$. This will be closer to the original point of view of McShane's and will also us to define the error term as the Lebesgue measure of some set in the circle, which we call the Birman–Series phenomenon in Theorem 5.1.4.

5.1. Gap and error terms using random variables on \mathcal{P} . The Green Formula

$$\sum_{e \in S(n)} \Phi(e) = \partial\Phi,$$

as well as the circular order on $S(n)$ can be used to define a partition of $\mathbb{R}/\partial\Phi\mathbb{Z}$ in intervals labelled successively by edges in $S(n)$ and of length $\Phi(e)$. Such a partition is unique up to translation.

Using the extremities of this evolving partition rather than the length of the corresponding intervals, we express in Theorem 5.1.4 the values of the error term as the Lebesgue measure of some set \bar{X}_∞ .

In our future geometric application, the vanishing of the Lebesgue measure of this set – and consequently the vanishing of the error term – will be an application of the Coding Theorem 6.9.1 and the Birman–Series Theorem [1] which is a cornerstone of the proof of McShane's type identity as in [10] and [8].

Taking the mid points of each such interval gives rise to an increasing random variable Y_n on $S(n)$, which is unique up to translation when characterised by

$$Y_n(p_0) - Y_n(p_1) = \frac{1}{2}(\phi(p_0^n) + \phi(p_1^n)),$$

when $p_0 < p_1$ and there is no edge between p_0^n and p_1^n .

We may consider Y_n as a random variable on \mathcal{P} , and increasing n , obtain a variable X_∞ . Finally the goal of this section is to define the error term in terms of X_∞ .

5.1.1. The variables Y_n . Let us choose one “initial” complementary region f_0 whose boundary contains the root. Then the fact that the tree is planar gives a natural ordering of the edges of $S(n)$.

For every n , we construct a function Y_n on $S(n)$ with values in $\mathbb{R}/\partial\Phi\mathbb{Z}$, by

$$\begin{aligned} Y_n(e) &:= \frac{1}{2}\Phi(e) + \sum_{f \in S(n) | f < e} \Phi(f) \\ &= -\frac{1}{2}\Phi(e) + \sum_{f \in S(n) | f \leq e} \Phi(f). \end{aligned}$$

The following proposition is an immediate consequence of the definition of Y_n and the positivity of Φ .

Proposition 5.1.1. *For e and f in $S(n)$ with $e > f$, we have the inequalities*

$$Y_{n+1}(\mathcal{R} \cdot f) \leq Y_{n+1}(\mathcal{L} \cdot e) \leq Y_n(e) \leq Y_{n+1}(\mathcal{R} \cdot e), \quad (16)$$

$$|Y_{n+1}(g) - Y_n(e)| \leq \frac{1}{2}\Phi(h) \text{ for } \{g, h\} = \{\mathcal{R}(e), \mathcal{L}(e)\}, \quad (17)$$

5.1.2. *The variables X_n and X_∞ .* We then define for every path p in the set \mathcal{P} of infinite embedded paths starting from the root.

$$X_n(p) := Y_n(p^n). \quad (18)$$

Anticipating Proposition 5.1.3, we shall see that the variables X_n converges uniformly to a variable X_∞ .

Let us first prove the following Proposition:

Proposition 5.1.2. *Let p be a path. Let $\hat{p} = \{\hat{p}^n\}_{n \geq 2}$ be the sequence of edges so that $\hat{p}_n \in S(n)$ and $\{p^n, \hat{p}^n\} = \{\mathcal{R}(p^{n-1}), \mathcal{L}(p^{n-1})\}$. Then*

$$\sum_{n=1}^{\infty} \Phi(\hat{p}^n) < \infty,$$

and in particular

$$\lim_{n \rightarrow \infty} \Phi(\hat{p}^n) = 0.$$

Proof. Since $\Phi(p^p) + \Phi(\hat{p}^p) = \Phi(p^{p-1})$ it follows by induction that for all p ,

$$\Phi(p^p) + \sum_{n=2}^p \Phi(\hat{p}^n) = \Phi(p^{p-1}) + \sum_{n=2}^{p-1} \Phi(\hat{p}^n) = \Phi(p_1).$$

The result follows since Φ is positive. \square

The main observation of this section is

Proposition 5.1.3. *We have*

- (1) *The random variable X_n is increasing with respect to the lexicographic ordering on \mathcal{P} .*
 - (2) *The sequence X_n converges pointwise to a random variable X_∞ .*
- Moreover

$$X_\infty(p_1) - X_\infty(p_0) = \mu_\Phi([p_0, p_1]) - \frac{1}{2}(\Phi_\infty(p_0) + \Phi_\infty(p_1)), \quad (19)$$

$$= \mu_\Phi(]p_0, p_1]) + \frac{1}{2}(\Phi_\infty(p_0) + \Phi_\infty(p_1)). \quad (20)$$

Proof. The first item is a consequence of Proposition 5.1.1 Notice now, using the notation of Proposition 5.1.2, that Equation (17) implies that

$$|X_{n+1}(p) - X_n(p)| = \Phi(\hat{p}^n). \quad (21)$$

Thus by Proposition 5.1.2, X_n converges pointwise. Finally, let p_0 and p_1 be two paths with $p_0 < p_1$ in the lexicographic order. Let $[p_0, p_1] := \{p \mid p_0 \leq p \leq p_1\}$. Then for n large enough

$$X_n(p_1) - X_n(p_0) = \sum_{\substack{e \in \mathcal{S}(n) \\ p_0'' \leq e \leq p_1''}} \Phi(e) - \frac{1}{2} (\Phi(p_0'') + \Phi(p_1'')). \quad (22)$$

Thus

$$X_\infty(p_1) - X_\infty(p_0) = \mu_\Phi[p_0, p_1] - \frac{1}{2} (\Phi_\infty(p_0) + \Phi_\infty(p_1)).$$

This proves equation (19), equation (20) follows then from equation (7). \square

5.1.3. *The error term using X_∞ .* The set $X_\infty(\mathcal{P})$ might be not be closed, for instance when X_∞ is not continuous. Let us then define for every $p \in \mathcal{P}$,

$$\begin{aligned} X_\infty^L(p) &:= \sup_{q < p} (X_\infty(q)), \\ X_\infty^R(p) &:= \inf_{q > p} (X_\infty(q)), \\ \bar{X}_\infty &:= \overline{\bigcup_{p \in \mathcal{P} \setminus \mathcal{Q}} [X_\infty^R(p), X_\infty^L(p)]}. \end{aligned}$$

Then we can express the error term using the variable X_∞ :

Theorem 5.1.4. [BIRMAN–SERIES PHENOMENON]

$$\lambda(\bar{X}_\infty) = 2 \text{Error}(\Phi). \quad (23)$$

This will be a consequence of the following proposition:

Proposition 5.1.5. *Suppose that p is a limit of a strictly increasing sequence $\{p_n\}_{n \in \mathbb{N}}$, and q is limit of a strictly decreasing sequence $\{q_n\}_{n \in \mathbb{N}}$. Then*

$$X_\infty(p) - X_\infty^-(p) = \frac{1}{2} \Phi_\infty(p) \quad (24)$$

$$X_\infty^+(q) - X_\infty(q) = \frac{1}{2} \Phi_\infty(q) \quad (25)$$

$$(\mathbb{R} / \partial\Phi\mathbb{Z}) \setminus \bar{X}_\infty = \bigsqcup_{e \in \mathcal{F}}]X_\infty^L(\partial^L e), X_\infty^R(\partial^R e)[. \quad (26)$$

We first prove Theorem 5.1.4 from Proposition 5.1.5.

Proof. Since $] \partial^- e, \partial^+ e[= \emptyset$, we have that from the previous proposition and equation (20),

$$X_\infty(\partial^R e) - X_\infty(\partial^L e) = \frac{1}{2} (\Phi_\infty(\partial^L e) + \Phi_\infty(\partial^R e)) = \text{Gap}_\Phi(e).$$

Moreover from Equations (24) and (25), for $i = R$ or $i = L$

$$|X^i(\partial^i e) - X_\infty(\partial^i e)| = \frac{1}{2} \Phi_\infty(\partial^i e).$$

Thus combining the two preceding equations one gets that

$$\lambda([X_\infty^L(\partial^L e), X_\infty^R(\partial^R e)]) = X_\infty^R(\partial^R e) - X_\infty^L(\partial^L e) = 2 \text{Gap}_\Phi(e).$$

Thus from equation (26), we get

$$\begin{aligned} \partial\Phi &= \lambda(\overline{X}_\infty) + \sum_{e \in \mathcal{F}} \lambda([X_\infty^L(\partial^L e), X_\infty^R(\partial^R e)]) \\ &= \lambda(\overline{X}_\infty) + 2 \sum_{e \in \mathcal{F}} \text{Gap}_\Phi(e). \end{aligned}$$

Hence, $\lambda(\overline{X}_\infty(\mathcal{P})) = \partial\Phi - 2 \text{Gap}(\Phi) = 2 \text{Error}(\Phi)$. \square

Let us now prove Proposition 5.1.5

Proof. Given a path p and an infinite strictly increasing sequence $\{p_n\}_{n \in \mathbb{N}}$ converging to p , then $\{X_\infty(p_n)\}_{n \in \mathbb{N}}$ converges to $X_\infty^-(p)$. In particular, $\{X_\infty(p_n) - X_\infty(p_{n+1})\}_{n \in \mathbb{N}}$ converges to zero. Since by Equation (20),

$$X_\infty(p_{n+1}) - X_\infty(p_n) = \mu_\Phi([p_n, p_{n+1}[) + \frac{1}{2}(\Phi(p_n) + \Phi(p_{n+1})),$$

the sequence $\{\Phi_\infty(p_n)\}_{n \in \mathbb{N}}$ converges to zero. Then by Equation (20) again,

$$X_\infty(p) - X_\infty(p_n) = \mu_\Phi([p_n, p[) + \frac{1}{2}(\Phi(p_n) + \Phi(p)).$$

Thus taking the limit, one gets Equation (24). A symmetric argument yields Equation (25).

For the last statement, let $u \notin \overline{X}_\infty$. Since \overline{X}_∞ is closed, let $u \in]v, w[\subset \mathbb{R} / \partial\Phi \cdot \mathbb{Z} \setminus \overline{X}_\infty$ and let

$$\mathcal{P}^L = \{p \mid X_\infty^L(p) \leq v\}, \quad \mathcal{P}^R = \{p \mid X_\infty^R(p) \geq w\}.$$

Let now $p^R = \inf(p \in \mathcal{P}^R)$ and $p^L = \sup(p \in \mathcal{P}^L)$. By definition of u it follows that $X_\infty^L(p^L) \leq v$ and symmetrically that $X_\infty^R(p^R) \leq w$. Let $q \in]p^L, p^R[$. Then $X_\infty^L(q) \geq w > u$ and $X_\infty^R(q) \leq v < u$. This is impossible, hence $]p^L, p^R[= \emptyset$. It follows that $p^i = \partial^i e$ for some

complementary region e and $u \in]X_\infty^L(\partial^L e), X_\infty^R(\partial^R e)[$. Conversely, if $u \in]X_\infty^L(\partial^L e), X_\infty^R(\partial^R e)[$ then $u \notin \bar{X}_\infty$. \square

6. A ROOTED TREE CODING GEODESICS

This section is purely topological. Let S be a compact surface with one marked point x_0 and $\Sigma = S \setminus \{x_0\}$. Let $N = 6(1 - \chi(S))$. Let \mathcal{T} be the rooted trivalent tree, which has N edges at the root. As a corollary of our main result we will obtain the following result.

Theorem 6.0.1. *Let T be a triangulation of S whose only vertex is x_0 . There there exists a labelling of the edges of \mathcal{T} by simple curves passing through x_0 .*

Our actual results, Theorem 6.1.1 and 6.9.1 will be more precise and will allow us to describe simple infinite geodesics starting at x_0 as embedded paths in \mathcal{T} .

The main proposition is actually an elaboration on a Lemma by L. Mosher [13] as Saul Schleimer has explained to us and the proof is actually identical. Our point of view and purpose are nevertheless different.

6.1. A dynamical system on the set of triangulations. Let S be a closed oriented surface with one puncture x_0 and $\Sigma = S \setminus \{x_0\}$. An *ideal triangulation* of Σ is a triangulation of S , up to isotopy, whose only vertex is x_0 . Observe that the number N of oriented edges is $6(1 - \chi(S))$: the singular flat metric obtained by identifying the triangles with equilateral triangles of length 1, has total curvature $N\pi/3 - 2\pi$. Since this total curvature is $-2\pi\chi(S)$, we obtain that $N = 6(1 - \chi(S))$.

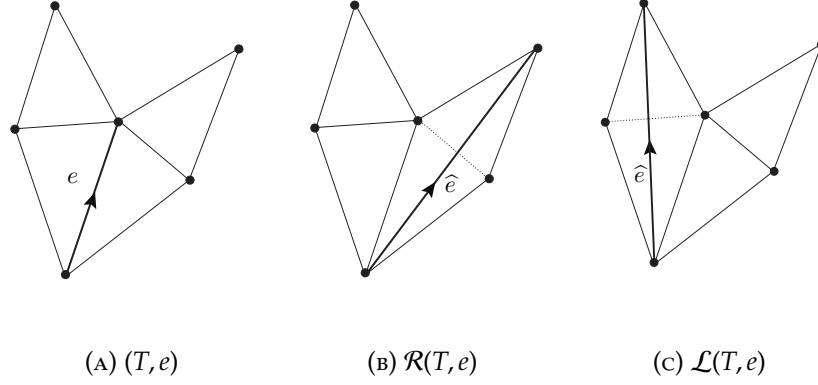
We will usually think of this triangulation as a triangulation by ideal triangles of the surface Σ equipped with a complete hyperbolic structure. Given a triangulation T and an oriented edge e , let \bar{e} be the same edge with the opposite orientation, and $s(e)$ the oriented edge whose origin is the end point of e such that e and $s(e)$ are both on the boundary of the triangle on the right of e .

Let \mathcal{S} be the space of pairs (T, e) where T is an ideal triangulation of S and e an oriented edge of T . We are going to describe three transformations \mathfrak{F} , \mathcal{R} and \mathcal{L} on \mathcal{S} .

First $\mathfrak{F}(T, e) = (\widehat{T}, \widehat{e})$ where \widehat{T} is the triangulation flipped at e , and \widehat{e} the new corresponding edge so that (e, \widehat{e}) is positively oriented.

Given (T, e) , define $\mathcal{R}(T, e) := \mathfrak{F}(T, s(e))$.

A similar definition holds for \mathcal{L} after changing the orientation of Σ . All together, these two transformations define an action of the free semigroup in two generators F_2^+ on \mathcal{S} .

FIGURE 1. Transformations \mathcal{R} and \mathcal{L}

Let also $L(x_0)$ be the space of simple loops, up to homotopy, passing through x_0 and π the natural projection from \mathcal{S} defined by $\pi(T, e) = e$. Our main result is the following

Theorem 6.1.1. [TOPOLOGICAL CODING] *Let T_0 be an ideal triangulation and f an oriented topological embedded loop on S through x_0 . Then, either f is an edge of T_0 or there exists a unique oriented edge e of T_0 and a unique element g in F_2^+ such that $\pi(g \cdot \mathfrak{F}(T_0, e)) = f$.*

In particular, the action of F_2^+ on \mathcal{S} is free. We will explain later a more precise version of this result.

We will also explain that this construction give rise to a coding of all embedded geodesics without self intersection issuing from x_0 in an auxiliary complete hyperbolic metric on Σ . A further property of this coding yields the Birman–Series Theorem.

We will also explain using this coding that every Hölder cross ratio on ∂_∞ gives by a harmonic measure construction a transverse measure with zero entropy on the set of simple infinite geodesics (emanating from a boundary component).

Then, generalising an idea from Bowditch in the punctured torus case [2], this construction will yield a new proof –or rather a new interpretation– of McShane identity for cross ratios [8].

6.2. An arboreal interpretation. For later use and in order to make the connection with the previous constructions, we present the results in terms of a planar tree.

Let T be a triangulation with n edges whose only vertex is x_0 . then let \mathcal{T} be the rooted trivalent (except at the root) planar tree with $2n$ root at the origin.

We first define a labelling of the edges and the complementary regions as follows:

- We label the initial edges by the pairs $\mathfrak{F}(T, e)$ using the cyclic order that comes from the cyclic order on the oriented simple curves $\pi(\mathfrak{F}(T, e))$.
- Next, label every other edge recursively so that if an edge a of the tree is labelled by (T', e') , the edge on the right of a is labelled by $\mathcal{R}((T', e'))$ and the edge on the left is labelled $\mathcal{L}((T', e'))$.
- We label the interior-most complementary regions f (those adjacent to the root v_0) by (T, e) , where $\mathcal{R}(T, e)$ is the label of the first edge of the tree on the right of f . It then follows that $\mathcal{L}(T, e)$ is the label of the first edge of the tree on the left of f .
- Finally, we label finally the other complementary regions by the label of the edge of the tree defining it.

Our theorem now can be rewritten as

Theorem 6.2.1. [LABELLING COMPLEMENTARY REGIONS] *Let T be a triangulation of S with n edges whose only vertex is x_0 . Let Ψ be the labelling of the complementary regions of \mathcal{T}_n by pairs (T', e') where e' is an oriented edge of the triangulation T' – described above. Then the map $f \mapsto \pi(\Psi(f))$ is a bijection from the space of complementary region to the space of simple curves passing through x_0 .*

6.3. Order preserving. In this section, we equip Σ with an auxiliary complete hyperbolic metric so that the triangles are realized as ideal hyperbolic triangles

Let $L(x_0)$ be the set of simple oriented geodesics based at x_0 . Observe that the orientation of Σ , as well as that of an auxiliary hyperbolic metric, gives a cyclic ordering on $L(x_0)$ eventually independent of the metric: we order the geodesics γ by the order on a small horosphere H at x_0 of the first intersection point in $H \cap \gamma$.

We now define a map Ψ from $E \times \mathbb{F}_2^+$ to $L(x_0)$ by

$$\Psi(e, g) := \pi(g \cdot \mathfrak{F}(T_0, e)). \quad (27)$$

We consider the cyclic ordering on E induced by $e \mapsto \Psi(e, 1)$ from the order on $L(x_0)$.

We fix a triangulation T . We define a cyclic order (by lexicographic ordering) on $E \times \mathbb{F}_2^+$. We now prove

Proposition 6.3.1. *The map Ψ from $E \times F_2^+$ to $L(x_0)$ is order preserving.*

As an immediate corollary, we get that the action of F_2^+ on \mathcal{S} is free.

Proof. We will prove the proposition by induction on the length of the word in F_2^+ . Since E has N -elements, the simple loops E and the simple loops of the form $\Psi(e, 1)$ divides the horocycle into $2N$ intervals so that elements of E and elements of $\Psi(E, 1)$ alternate. Furthermore, by the definition of the action of F_2^+ , we see that when g has word length one, $\pi(g \cdot \mathfrak{F}(T_0, e))$ lies in the interval to the left or right of the corresponding element $e \in E_{x_0}$ depending whether $g = \mathcal{R}$ or $g = \mathcal{L}$. The new $2N$ edges that are introduced alternate with the original $2N$ edges with respect to the cyclic ordering and again, by construction every pair of alternating edges (one of these being $\pi(g \cdot \mathfrak{F}(T_0, e))$) bound a triangle of the triangulation associated to $g \cdot \mathfrak{F}(T_0, e)$. Continuing inductively, we see that at any distance $d > 1$ from the root, we have $2^d N$ edges whose cyclic ordering agrees with that of $E \times F_2^+$ which completes the proof. \square

6.4. Reducing the complexity. We are going to define in this paragraph a *complexity invariant* associated to a triple (T, Δ, η) where

- T is a triangulation by geodesics arcs for a complete (auxiliary) hyperbolic metric of Σ ,
- η is an oriented simple geodesic starting at the puncture x_0 ,
- Δ is a *fundamental domain* for the surface with respect with the triangulation, that is a map from a triangulated disk D to S , which sends triangles to triangles, so that Δ restricted to the interior of D is injective with a dense image.

Our *complexity invariant* is a pair of integers

$$C(T, \Delta, \eta) = (n(T, \Delta, \eta), N(T, \Delta, \eta)), \quad (28)$$

defined in the following way. First if η is an edge of T , then

$$C(T, \Delta, \eta) = (1, 1). \quad (29)$$

If η is not an edge, let $(\eta_1, \eta_2, \dots, \eta_N)$ be the (ordered) connected components of $\Delta^{-1}(\eta)$. Then

- (1) $N(T, \Delta, \eta)$ is the number of connected components of $\Delta^{-1}(\eta)$, that is $N(T, \Delta, \eta) = N$,
- (2) $n(T, \Delta, \eta)$ is the number of triangles of D that η_1 intersects.

Our two main results are the following

Proposition 6.4.1. *Assume that $n(T, \Delta, \eta) = 1$ and $N(T, \Delta, \eta) > 2$, then there exists $\mathring{\Delta}$ so that*

$$N(T, \mathring{\Delta}, \eta) \leq N(T, \Delta, \eta) - 1, \quad (30)$$

$$2 \leq n(T, \mathring{\Delta}, \eta). \quad (31)$$

The second proposition is as follows.

Proposition 6.4.2. *Assume that $n(T, \Delta, \eta) > 1$ and let e be the first edge of T that η meets. Let $(\mathring{T}, \mathring{\delta}) := \mathfrak{F}(T, e)$. Then Δ is still a fundamental domain for \mathring{T} and furthermore,*

$$\begin{aligned} N(\mathring{T}, \Delta, \eta) &= N(T, \Delta, \eta), \\ n(\mathring{T}, \Delta, \eta) &= n(T, \Delta, \eta) - 1. \end{aligned} \quad (32)$$

6.5. Proof of proposition 6.4.1. By assumption there is a (closed) triangle δ in D such that $\eta_1 \subset \delta$. Let e_0 be the edge of δ opposite to the origin of η_1 , e_+ and e_- be the edges of δ on the left, respectively right of η_1 . Let also D^+ be the connected component of $D \setminus \delta$ whose closure contains e^+ and let us define similarly D^- (See figure 2).

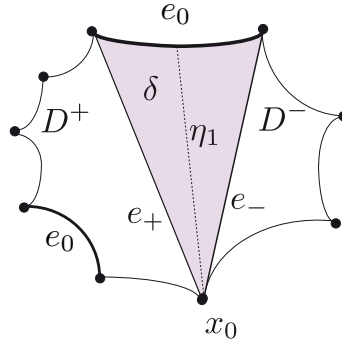


FIGURE 2. The subdomains of Δ

Now, we may as well assume that e_0 appears on the boundary of D^+ , the case where e_0 appears on the boundary of D^- is handled symmetrically. We remark that D^- could be empty.

Let then

$$D_0^- := D^- \cup \delta. \quad (33)$$

We now define a new triangulated disk and a new fundamental domain by

$$\mathring{D} := D_0^- \cup_{e_0} D^+, \quad (34)$$

where the notation means that we glue D_0^- and D^+ along e_0 . The fundamental domain is then constructed accordingly. Observe that D^+ is non empty and thus

$$n(T, \mathring{\Delta}, \eta) \geq 2. \quad (35)$$

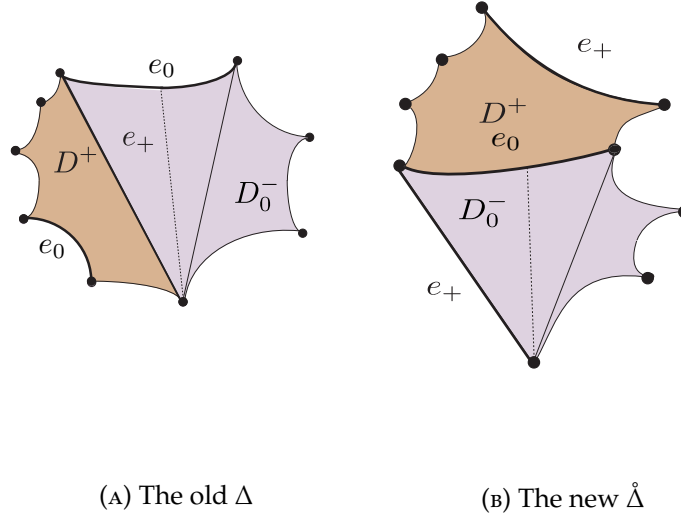


FIGURE 3. Mutating the fundamental domain

Our proposition thus reduces to the following assertion

$$N(T, \mathring{\Delta}, \eta) \leq N(T, \Delta, \eta) - 1, \quad (36)$$

which we now prove.

Let ϕ be a collection of edges of T and $i(\phi, \eta)$ be the cardinal of the intersection of η and ϕ . To make things precise, ϕ is a collection of curves in Σ (not in D) and thus there is no repetition of edges.

We now observe that if Δ is a fundamental domain and $\partial\Delta$ is the collection of edges of T of the boundary of Δ , then

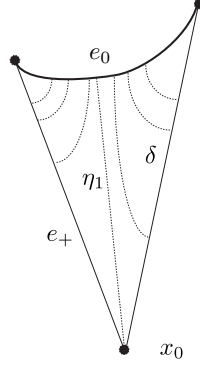
$$N(T, \Delta, \eta) = i(\partial\Delta, \eta) + 1.$$

Now by construction

$$\partial\mathring{\Delta} = (\partial\Delta \setminus \{e_0\}) \cup \{e_+\}.$$

Thus

$$N(T, \Delta, \eta) - N(T, \mathring{\Delta}, \eta) = i(e_0, \eta) - i(e_+, \eta).$$


 FIGURE 4. Intersection of η with e_0 and e_+

But since η_1 is embedded and goes from the vertex x_0 to e_0 , every η_i that intersects e_+ intersects e_0 as well (See figure 4). Furthermore, since η_1 intersects e_0 but not e_- it follows that

$$i(e_0, \eta) - i(e_+, \eta) = 1 + i(e_-, \eta) \geq 1. \quad (37)$$

This is what we wanted to prove.

6.6. Proof of proposition 6.4.2. By assumption, $n(T, \Delta, \eta) \geq 2$. Thus if δ_1 and δ_2 are the first triangles that η_1 encounters they are both in D . Thus, after flipping $\delta_1 \cup \delta_2$ along their common edge, we see that Δ is still a fundamental domain for \mathring{T} and thus obviously

$$N(\mathring{T}, \Delta, \eta) = N(T, \Delta, \eta). \quad (38)$$

Also, in the new triangulation of D , η_1 hits one less triangle, thus

$$n(\mathring{T}, \Delta, \eta) = n(T, \Delta, \eta) - 1, \quad (39)$$

This concludes the proof.

6.7. The coding. We now prove the coding theorem. The injectivity of the coding follows from Proposition 6.3.1. The surjectivity from the following

Proposition 6.7.1. *There exists constant A and B depending only on the topology of S so that, if f be any simple loop based at x_0 , if T_0 is any triangulation, then there exists e in T_0 , an element g of \mathbb{F}_2^+ such that*

$$\pi(g \cdot \mathfrak{F}(T_0, e)) = f, \quad (40)$$

and moreover, denoting by λ the word length of g , we have the rough inequality

$$\lambda \leq A \cdot i(f, T_0) + B \quad (41)$$

By definition, the length λ of the element g satisfying Equation (40) will be called the *tree length* of f and denoted $\lambda_{T_0}(f)$ where the subscript is omitted when no confusion is possible, similarly $i(f, T_0)$ will be called the *intersection length* of f .

Proof. We first want to prove the existence of g satisfying Equation (40).

Let η be a simple geodesic starting at the puncture x_0 . Let us choose a fundamental domain Δ for the initial triangulation T .

From the proof in the preceding paragraph, one sees that

$$\lambda(f) \leq \#\{T_0\} \cdot (N(T_0, \Delta, f) + 1). \quad (42)$$

Similarly one observes that

$$N(T_0, \Delta, f) - 1 = i(f, \partial\Delta) \leq i(f, T_0) \leq 2 \cdot \#\{T_0\} \cdot N(T_0, \Delta, f). \quad (43)$$

The result follows. \square

6.8. Spiraling. In this section, we equip Σ with an auxiliary complete hyperbolic metric of finite volume. Let $L(x_0)$ be the set of oriented simple geodesic arcs from x_0 to x_0 and \mathcal{P} be the set of embedded paths in the tree \mathcal{T} . Let also \mathcal{Q} be the set of rational paths as defined in Paragraph 3.2. Given a complementary region e , recall that we have two rational paths ∂^+e and ∂^-e .

Our goal is two prove that the simple arcs labelling $\pi_n(\partial^\pm e)$ both converges to laminations spiralling around a closed geodesic in a precise sense.

We first define the closed geodesic which the lamination spirals around. For $f \in L(x_0)$, define f^R to be the oriented closed geodesic in Σ homotopic to f in S on the right of f . Then f and f^R bounds a cylinder $Cyl^R(f)$ in Σ with one cusped geodesic boundary corresponding to f on the left, and one geodesic boundary corresponding to f^R on the right. Let f_∞^R be the unique geodesic lamination in $Cyl^R(f)$ starting from x_0 and spiralling around f^R , respecting the orientation.

We have a similar definition for f^L , $Cyl^L(f)$ and f_∞^L , replacing right by left in the above.

If now F is a complementary region, by an abuse of notation, we denote also by $\pi_n(\partial^i F)$ the closed geodesic arc from x_0 to x_0 representing the simple arc labelling the edge $\pi_n(\delta^i F)$ for $i = L$ or $i = R$. Our result is now

Proposition 6.8.1. *Let F be a complementary region labelled with the simple arc f . The sequence of geodesic arcs $\{\pi_n(\partial^R F)\}_{n \in \mathbb{N}}$ converges to f_∞^R and similarly $\{\pi_n(\partial^L F)\}_{n \in \mathbb{N}}$ converges to f_∞^L .*

Proof. The proof follows from (a) a simple observation about geodesics in the cylinder $Cyl^R(f)$ originating from x_0 , (b) Theorem 6.2.1 on topological coding, and (c) proposition 6.3.1 on order preserving.

First observe that every simple geodesic arc in $L(x_0)$ whose beginning lies in $Cyl^R(f)$ must intersect f^R . In particular, the simple arcs $d_n := \pi_n(\partial^R F)$ all intersect f^R and f_∞^R is a left bound for d_n with respect to the cyclic ordering. Next, the cyclic ordering implies that d_n are monotone to the left, so $\lim_{n \rightarrow \infty} d_n$ exists. Let D be the positive Dehn twist about f^R , then $\lim_{n \rightarrow \infty} D^n(d_0) = f_\infty^R$. If $\lim_{n \rightarrow \infty} d_n = l \neq f_\infty^R$, then there exists m such that $D^m(d_0)$ is between l and f_∞^R . But the order preserving property now implies that for k sufficiently large, d_k is between f_∞^R and l which is a contradiction. Hence, $\lim_{n \rightarrow \infty} d_n = f_\infty^R$. \square

6.9. Coding non self intersecting geodesics. The previous proposition extends in the following way. Let $L_\infty(x_0)$ be the set of non properly embedded simple arcs from x_0 . The proof of this Theorem follows immediately from original arguments by McShane's in [10].

Theorem 6.9.1. [CODING SIMPLE ARCS AND LAMINATIONS] *The map Ψ extends in a unique way to a bijection Ψ_∞ from \mathcal{P} to $L_\infty(x_0)$, so that if $p \in \mathcal{P}$ then*

$$\lim_{n \rightarrow \infty} \Psi(p^n) = \Psi_\infty(p).$$

Proof. We just proved that Ψ extend to Ψ_∞ for rational paths, Ψ_∞ – restricted to irrational path – is monotone. Let us now consider an irrational path p . Let us define

$$\begin{aligned} \Psi_\infty^-(p) &= \sup\{\Psi_\infty(q) \mid q \in \mathcal{Q}, q < p\}, \\ \Psi_\infty^+(p) &= \inf\{\Psi_\infty(q) \mid q \in \mathcal{Q}, q > p\}. \end{aligned}$$

If w is a properly embedded simple geodesic arc from x_0 to x_0 , then either $\partial^R w < p$ or $\partial^L w > p$ for the lexicographic order on paths coming from the coding. Since $\Psi_\infty(\partial^L w) < \Psi(w) < \Psi_\infty(\partial^R w)$, it follows that either $\Psi(w) < \Psi_\infty^-(p)$ or $\Psi_\infty^+(p) < \Psi(w)$. Now by McShane's gap Lemma [10], if we have two distinct non properly embedded simple arcs starting at x_0 , then there is always a simple properly embedded arc in the sector between them. Thus $\Psi_\infty^-(p) = \Psi_\infty^+(p)$. This concludes the proof. \square

7. CROSS RATIO AND A HARMONIC FORM

We now explain in this section how the geometry – be it hyperbolic, or that corresponding to a higher rank representation of the surface group – has a counterpart in our planar tree picture as a harmonic 1-form in the sense of the first section.

7.1. Cross ratio. We now consider the boundary at infinity $\partial_\infty \pi_1(\Sigma)$ of $\pi_1(\Sigma)$. Every element α in $\pi_1(\Sigma)$ has two fixed points on $\partial_\infty \pi_1(\Sigma)$: the attracting fixed point α^+ , and the repelling fixed point α^- .

We consider a Hölder equivariant cross ratio \mathbf{b} on $\partial_\infty \pi_1(\Sigma)$ (See [8, 7, 6] for definitions).

Recall the the length of an element γ is then

$$\ell(\gamma) := \log(\mathbf{b}(\gamma^-, \gamma^+, y, \gamma \cdot y)), \quad (44)$$

where y is any element of $\partial_\infty \pi_1(\Sigma)$ not fixed by γ .

A triangle Δ in a triangulation T , gives rise to three peripheral elements α, β and γ of $\pi_1(\Sigma)$, so that the triple (α, β, γ) is well defined up to global conjugation.

Similarly an edge e of the triangulation T gives rise to four peripheral elements that we denote by

$$\delta(T, e), \alpha_1(T, e), \gamma(T, e), \alpha(T, e), \quad (45)$$

which corresponds to the vertices of the lozenge defined the the two triangles bounded by e , labelled using the cyclic ordering and starting at the initial vertex of e .

7.2. A divergence free vector field. Then we define

$$\Phi(T, e) := \log \mathbf{b}(\delta^+(T, e), \delta^-(T, e), \alpha_0^+(T, e), \alpha_1^+(T, e)), \quad (46)$$

where δ is the initial vertex of the edge, α_0 and α_1 are the vertices of Δ_0 and Δ_1 opposite to e . Then we have

Proposition 7.2.1. [DIVERGENCE FREE] *The vector field Φ is divergence free, moreover*

$$\partial\Phi = \ell(\partial S). \quad (47)$$

This proposition is a immediate consequence of the multiplicative cocycle properties of cross ratio.

7.3. A special case: hyperbolic surface with one cusp. Instead of explaining in detail the not so exciting proof of the previous example, we instead give another related example of the construction of the harmonic 1-form in the case of an hyperbolic surface with cusp which is perhaps more illuminating.

Let us consider an embedded horosphere H around the cusp. Every triangulation T defines an ideal hyperbolic triangulation of the surface. Then every oriented edge e of T defines a point e_H in H , namely the (first) intersection of H with the geodesic associated to e . Let then K be the collection of these points, and given $e_H \in K$, let $e^+H \in K$ be the point immediately on the right and e^-_H be the point immediately on the left (for some auxiliary orientation of H). Let finally $I(e)$, be the interval on H with extremities e^\pm_H and containing e_H . Define

$$\Phi(T, e) = \frac{1}{2} \frac{\text{length}(I(e))}{\text{length}(H)}.$$

An elementary geometric construction shows you that

Proposition 7.3.1. *The form Φ is harmonic:*

$$\Phi(\mathcal{R}(T, e)) + \Phi(\mathcal{L}(T, e)) = \Phi(T, e).$$

Moreover $\partial\Phi = 1$.

7.4. Gaps and pairs of pants. Let us move back again to the case of a general cross ratio and make the connection with [8].

Recall that a pair of pants is a regular homotopy class of immersion of the plane minus two points – that we identify with $S^1 \times]-1, 1[\setminus \{1, 0\}$ for later use – in S . A pair of pants is alternatively described by a triple (α, β, γ) of elements of $\pi_1(\Sigma)$, defined up to conjugation so that $\alpha \cdot \gamma \cdot \beta = 1$. The set of pairs of pants is then $\pi_1(\Sigma)^3 / \pi_1(\Sigma)$ where the latter action is by conjugation. A pair of pants is embedded if it can be represented by an embedding.

Then, following [8]¹.

Definition 7.4.1. *Let \mathbf{b} be a cross ratio. The gap of the pair of pants $P = (\alpha, \beta, \gamma)$ with respect to the cross ratio \mathbf{b} is*

$$\text{Gap}_{\mathbf{b}}(P) := \log(\mathbf{b}(\alpha^+, \alpha^-, \gamma^-, \beta^+)). \tag{48}$$

Observe that a simple loop ε passing through x_0 define a pair of pants $P(\varepsilon)$, associated to the embedding of $S^1 \times]-1, 1[\setminus \{1, 0\}$ corresponding to the identification of a tubular neighbourhood of ε with $S^1 \times]-1, 1[$.

¹We use a different convention for cross ratio

We saw that if T is a triangulation, then ε labels exactly one complementary region $f(\varepsilon)$ of the tree \mathcal{T} . In particular, one can associate a gap to the harmonic function associated to \mathbf{b} .

The two notions of gap coincide.

Proposition 7.4.2. *Let Φ be the associated divergence free vector field to a cross ratio \mathbf{b} as in Equation (46). Then if f is a complementary region,*

$$\text{Gap}_\Phi(f(\varepsilon)) = \text{Gap}_\mathbf{b}(P(\varepsilon)). \quad (49)$$

Proof. This follows from Proposition 6.8.1. Indeed, by construction, if f is a complementary region in \mathcal{T} an induction shows that

$$G_\Phi^n(f) = \mathbf{b}(\alpha^+, \alpha^-, \gamma_n^-, \beta_n^+),$$

where γ_n is the simple arc associated to the edge $\mathcal{R} \cdot \mathcal{L}^{n-1} \cdot \mathcal{R}f$, and β_n is the simple arc associated to the edge $\mathcal{L} \cdot \mathcal{R}^{n-1} \cdot \mathcal{L}f$. Then by Proposition 6.8.1, γ_n converges to $\gamma = \partial^{\mathcal{R}}f$ and β_n converges to $\beta = \partial^{\mathcal{L}}f$. \square

Similarly in the case of hyperbolic surfaces with cusp one can get explicit formulas for the gaps in terms of lengths of simple curves bounding the pair of pants as in [10], [11] and [8].

8. THE ERROR TERM AND THE BIRMAN–SERIES THEOREM

Our Gap Formula for planar trees now reads, since every complementary region is labelled by a simple loop passing through x_0 , as

$$\ell(\partial S) = \text{Error}(\Phi) + \sum_{\varepsilon \in L(x_0)} \text{Gap}_\mathbf{b}(P(\varepsilon)),$$

where $L(x_0)$ is the set of simple loops up to homotopy passing through x_0 . To recover the McShane's identity, we need to prove that the error term vanishes.

As in McShane's original proof and most of the subsequent proofs – except notably the proof by Bowditch in [2] – the vanishing of the error term relies on Birman–Series Theorem [1]: the closure of the reunion of the space of simple closed geodesics has Hausdorff dimension 1.

We will sketch this proof this only in the context of hyperbolic surface with one cusp and use the apparatus developed in paragraph 5.1.

In this context $\mathbb{R}/\partial\phi\mathbb{Z}$ is identified with the horosphere H centered at x_0 . Then if e is a simple loop passing through x_0 corresponding to an edge in $S(n)$ of the tree \mathcal{T} , $Y_n(e)$ is the intersection of e with the horosphere centered at x_0 – normalized so that it has length one. If \mathbf{p} is an irrational path corresponding to an infinite simple geodesic γ

initiating at the cusp, we obtain that $X_\infty^-(p) = X_\infty^+(p)$ which corresponds to the intersection of γ with the horosphere H .

Then $\overline{X_\infty(\mathcal{P} \setminus \mathcal{Q})}$ is included in the closure of the intersection of the horosphere H with the union of simple geodesics initiating in x_0 . Thus, according to the Birman–Series Theorem, $\lambda(\overline{X_\infty(\mathcal{P} \setminus \mathcal{Q})}) = 0$. Hence, by Theorem 5.1.4, the error term vanishes.

9. SURFACES WITH MORE THAN ONE BOUNDARY

If S has more than one puncture, x_0, \dots, x_{n-1} , $n \geq 2$, we may still define $\Sigma = S \setminus \{x_0, \dots, x_{n-1}\}$ and a triangulation of Σ such that the set of vertices is $\{x_0, \dots, x_{n-1}\}$. Again it is useful to endow S with an auxiliary complete hyperbolic structure so that x_0, \dots, x_{n-1} correspond to cusps. We choose a distinguished cusp, say x_0 , and this time, we let $L(x_0)$ be the set of oriented simple geodesic arcs starting from x_0 and terminating in another (not necessarily distinct) cusp. We can now define \mathcal{S} to be the space of pairs (T, e) where T is an ideal triangulation of S and e is an oriented edge of T which is in $L(x_0)$.

The transformations \mathfrak{F} , \mathcal{L} and \mathcal{R} are defined as before, with some slight modifications. To start with, note that \mathfrak{F} is defined on the pairs (T, e) , where e is an edge of T but it is not necessary that $e \in L(x_0)$ (e may originate from a different cusp). On the other hand, the transformations \mathcal{L} and \mathcal{R} may not be defined on $(T, e) \in \mathcal{S}$ for certain non-proper triangulations T as we shall see later. Nonetheless, where \mathcal{L} or \mathcal{R} is defined on (T, e) , then $\mathcal{L}(T, e) \in \mathcal{S}$ and $\mathcal{R}(T, e) \in \mathcal{S}$. We now form the rooted tree as in the previous case, with the following modifications:

- The root v_0 of the tree is an ideal triangulation T_0 of S with vertices in the set x_0, \dots, x_{n-1} .
- The neighbors of v_0 are the pairs $(\mathring{T}, \mathring{e}) \in \mathcal{S}$ such that $(\mathring{T}, \mathring{e}) = \mathfrak{F}(T_0, e)$ for some oriented edge e of T_0 which is not necessarily in $L(x_0)$ (note however that we do require that $\mathring{e} \in L(x_0)$). In particular, it is possible for v_0 to be of valence 1 which occurs when only one edge of T_0 (counted twice) is in $L(x_0)$. By construction, all the neighbors of v_0 are labeled by some element $(T, e) \in \mathcal{S}$ and e is adjacent to two distinct triangles of T .
- For a vertex v labeled by (T, e) at distance $n \geq 1$ from v_0 , $\mathcal{R}(T, e)$ and $\mathcal{L}(T, e)$ are defined as before, provided that the corresponding flip moves are possible. Each admissible move results in a vertex at distance $n + 1$ from v_0 adjacent to v .

- In the case of a vertex (T, e) for which $\mathcal{R}(T, e)$ is not defined, we call the vertex *right blocked*. Similarly, if $\mathcal{L}(T, e)$ is not defined, we call the vertex *left blocked*.
- A vertex (T, e) is right blocked if $s^2(e) = \overline{s(e)}$ or equivalently, if e bounds a punctured disk on the (see figure 5). An analogous statement holds for left blocked vertices. If S is not a thrice punctured sphere, then a vertex $v \neq v_0$ cannot be both left blocked and right blocked. Hence the vertices of the rooted tree apart from the root have valence two or three.

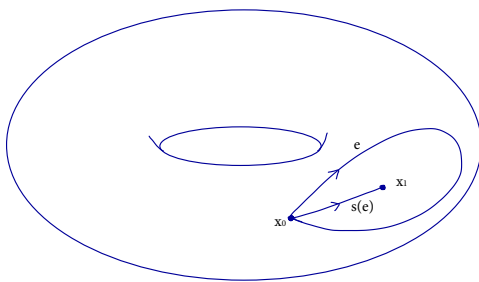


FIGURE 5. An example of (T, e) which is right-blocked. Here S is a torus with two punctures x_0, x_1 and only the edges e and $s(e)$ of the triangulation T are shown.

With this set-up, we have the following generalization of the topological coding theorem, with essentially the same proof:

Theorem 9.0.1. [TOPOLOGICAL CODING FOR SURFACES WITH MORE THAN ONE BOUNDARY] *Let T_0 be an ideal triangulation and f an oriented topological embedded arc on S starting from x_0 and ending in $\{x_0, \dots, x_{n-1}\}$. Then, either f is an edge of T_0 or there exists a unique oriented edge e of T_0 and a unique element g in \mathbb{F}_2^+ such that $\pi(g \cdot \mathfrak{F}(T_0, e)) = f$.*

Similarly, the order preserving property holds.

Remark: In the computation of the gap functions and the McShane identity, the “end gaps” correspond to the vertices which are left or right blocked.

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