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HIDA THEORY OVER SOME UNITARY SHIMURA VARIETIES WITHOUT ORDINARY LOCUS

RICCARDO BRASCA AND GIOVANNI ROSSO

Abstract. We develop Hida theory for Shimura varieties of type A without ordinary locus. In particular we show that the dimension of the space of ordinary forms is bounded independently of the weight and that there is a module of Λ-adic cuspidal ordinary forms which is of finite type over Λ, where Λ is a twisted Iwasawa algebra.

Introduction

In the historical paper [Hid86], Hida firstly constructed $\mathbb{Z}_p[\![T]\!]$-adic families of ordinary eigenforms. Since then a lot of work has been done in many directions, both relaxing the ordinary hypothesis and considering more general automorphic forms. The state-of-the-art result concerning families of ordinary forms is [Hid02], where the author shows that one can do Hida theory for all algebraic groups whose associated Shimura variety has a non-empty ordinary locus. Over the years Hida theory has had a lot of spectacular applications in number theory, ranging from proving cases of Mazur–Tate–Teitelbaum conjecture [GS93] to modularity lifting theorems [BLGGT14]. It would be hence extremely useful to have Hida theory in the greatest generality as possible; the aim of this paper is to complete this construction in the case of Shimura varieties of type A under the mild assumption that p is unramified in the Shimura datum.

Let us be more precise. In [Hid02] Hida axiomatised his approach to the study of ordinary families of p-adic modular forms. His machinery requires a certain set of conditions to be verified, such as:

- the rank of the ordinary part of the space of weight κ modular forms is bounded independently of κ;
- the existence of a Hasse invariant whose non-vanishing locus is the ordinary locus and which commutes with the U_p operators;
- a space of p-adic modular forms which contains as a dense subset all the classical modular forms, of all weights and all p^n-levels.

Of course he had in mind the standard notion of p-adic forms, *i.e.* forms defined over the ordinary locus, but in the axiomatic approach what one needs is just a huge space of p-adic modular forms where one can embed the classical forms. Usually one takes the space of functions on the Igusa tower, a space that parametrises trivializations of the relevant p-divisible group over the ordinary locus. Hida was able to show that all the assumptions of his theory are verified in this case and so he obtained the theory for general PEL type Shimura varieties with ordinary locus. In the paper at hand we want to generalise Hida theory to Shimura varieties of PEL type without ordinary locus using the same machinery.

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The first step is to replace the ordinary locus with the more general μ-ordinary locus, that is always dense in the (reduction modulo p of the) Shimura variety. The first two conditions can be checked thanks to the work on (reduced) μ-ordinary Hasse invariants [GN17, Her16a]. What we need after is an analogue of the Igusa tower to define the space of p-adic modular forms. Of course in general the universal μ-ordinary abelian scheme is not ordinary, but we have a good understanding of (the Dieudonné module of) his p-divisible group: it comes with a natural filtration whose graded pieces are generalised Lubin–Tate groups determined by the PEL data (more precisely by the signature at infinity of the unitary group). If the ordinary locus is not empty it is then equal to the μ-ordinary locus and this graded pieces are actually the connected-multiplicative and the étale part of the p-divisible group. To define the usual Igusa tower, one looks at the automorphisms of the graded pieces that moreover respect the polarisation, so in practice one takes the automorphisms of the connected component of the universal ordinary p-divisible group. We do a similar thing: we take for the Igusa tower the automorphisms of the graded pieces, but *we ignore the polarisation* and *we discard the étale part* (as, being étale, its module of invariant differentials is zero). This choice has two main effects:

- our Igusa tower can be extended to a fixed toroidal compactification of the Shimura variety. Note that this would not be possible if we had taken into account the polarisation, as the universal semiabelian scheme is not polarised. We view this possibility of extending the Igusa tower to the compactification as one of the main reasons to justify our choice;
- the Igusa tower seems to be 'too big', in the sense that its Galois group has too many p-adic characters (because we want to take these characters as the weights of our modular forms and we know the number of components a weight must have from the complex theory).

To fix this, our idea is to consider only characters that are *locally analytic* in a suitable sense. Roughly speaking, it means that the action of the character on the Lie algebra of $\mathbb{C}_p^*$ must have the same signature as the action on the Lie algebra of the Dieudonné module of the relevant graded pieces.

Let us be more explicit: we fix an unramified extension $\mathcal{O}$ of $\mathbb{Q}_p$ and a type $(d, \mathfrak{f})$ in the sense [Moo04]. The Dieudonné module M of the μ-ordinary Barsotti–Tate with $\mathcal{O}$-structure of type $(d, \mathfrak{f})$ is a free $\mathcal{O} \otimes_{\mathbb{Z}_p} \mathcal{O}$-module of rank h (where h depends on the type). It comes with a sub-$\mathcal{O}$-module Fil^1 that (and here is the big difference from the classical cases of Hida theory) is *not* a free $\mathcal{O} \otimes_{\mathbb{Z}_p} \mathcal{O}$-module, unless M is ordinary in the usual sense.

Suppose moreover that $h = 1$ (this is what we call a generalised Lubin–Tate). Classically, a p-adic weight would be a character of $\mathrm{Hom}_{\mathrm{cont}}(\mathcal{O}^*, \mathbb{C}_p^*)$. But, given the set of all $\sigma : \mathcal{O} \to \mathcal{O}_{\mathbb{C}_p}$, the action of $\mathcal{O}$ on Fil^1 is via a subset of these σ's. Our p-adic weights are defined as continuous characters of $\mathcal{O}^*$ which are σ-analytic for the σ's appearing in the action of $\mathcal{O}$ over Fil^1. (See Subsection 2.3 for the precise definition.)

We can then show that in this way we get characters (and hence weights) with the expected number of components. Note that in this case our weight space $\mathcal{W}$ and its Iwasawa algebra Λ will be a twisted version of the usual weight space and Iwasawa algebra.

Thanks to a variant of the Hodge–Tate map, we prove in Proposition 2.13 that classical modular forms can be interpreted as functions on the Igusa tower, see Section 2.4 for more details. We then define V^{cusp}, a subspace of the functions on the Igusa tower where classical cuspidal forms are dense, and we check that all the conditions we need hold, assuming that one has a reduced μ-ordinary Hasse

invariant. The main theorem, stating the existence of cuspidal Hida families and a control result, is the following:

Theorem. *Let G be a unitary group satisfying Assumption 1.3. We have constructed:*

(1) *An ordinary projector $e_G = e_G^2$ on V^{cusp} such that the Pontryagin dual of its ordinary part*
$$V^{\mathrm{cusp},*,\mathrm{ord}} := \mathrm{Hom}_{\mathbb{Z}_p}\left(V^{\mathrm{cusp},\mathrm{ord}}, \mathbb{Q}_p/\mathbb{Z}_p\right)$$
(which is naturally a Λ-module) is finite free over Λ°.

(2) *The Λ-module of cuspidal Hida families $\mathcal{M} := \mathrm{Hom}_\Lambda\left(V^{\mathrm{cusp},*,\mathrm{ord}}, \Lambda\right)$, which is of finite type over Λ.*

(3) *Given an arithmetic weight τ for Λ with dominant algebraic part $\kappa \in \mathcal{W}$ and finite order part $\varepsilon \in \mathrm{Hom}_{\mathrm{cont}}(\Lambda^*, \mathbb{C}_p^\times)$, let $\mathcal{P}_\tau$ be the corresponding prime ideal of Λ. Then*
$$\mathcal{M} \otimes /\mathcal{P}_\tau \xrightarrow{\sim} \varprojlim_m \varinjlim_l V_{m,l}^{\mathrm{cusp},\mathrm{ord}}[\tau],$$
and, if κ is very regular, combining it with (2.1) gives
$$\varinjlim_l \mathrm{S}\left(\mathcal{H} \cap \Gamma_1(p^l), \varepsilon\right)[1/p]^{\mathrm{ord}} \xrightarrow{\sim} (\mathcal{M} \otimes \Lambda/\mathcal{P}_\tau)[1/p].$$
Here the maps are equivariant under the action of the unramified Hecke algebra away from Np and the $\mathbb{U}_p$-operators.

Assumption 1.3 is simply to ensure that p 'does not ramify' in the Shimura datum.

A lot of people have recently been interested in the study of families of p-adic modular forms whenever the ordinary locus is empty. In [EM17] Eischen and Mantovan develop and study extensively a theory of p-adic modular forms in this setting, using a slightly different Igusa tower. They also prove many results that we have not covered, and among these we quote the problem of density of classical modular forms in the space of p-adic modular forms, the construction of differential operators in the spirit of Serre's θ operator, and the construction of explicit families of p-adic modular forms. In [Her17a] Hernandez constructs an eigenvariety for $\mathrm{U}(2,1)$ over a quadratic imaginary fields where p is inert and, using the results of [Her16b], his method generalises to all the cases considered in this paper.

We remark that there exist Shimura varieties of type D and PEL-type. In many case the ordinary locus is not empty and these cases have been dealt with in [Hid02] but there are a few cases when the ordinary locus is empty. The situation in this case is even simpler then the one in our paper, as the μ-ordinary p-divisible group is the product of a power of a multiplicative part and an étale part with a fixed generalised Lubin–Tate [Moo04, §3.2.3]. The same method of this paper easily generalises to this setting but for sake of brevity and exposition we shall not treat it.

Our approach to μ-ordinary p-divisible groups and, in general, a lot of our constructions are very 'linear algebra-style'. This is done on purpose as we are very confident that the methods of the paper can be generalised to all Shimura variety with a flat and surjective map to the stack of G-Zip, such as Hodge type Shimura varieties (in whose context one has the work of Goldring–Koskivirta [GK15] and Zhang [Zha17]). This will be the subject of a future paper.

The plan of the paper is the following. In Section 1 we fix some notations for Shimura varieties and Barsotti–Tate groups with $\mathcal{O}$-structure, in Section 2 we define an Igusa tower Ig and compare classical forms with functions on Ig. In Section 3 we

define two ordinary projectors: e_{GL}, of representation theoretic flavor and which will be used to prove the control theorem, and e_G, of geometric nature and that will serve to control the rank of ordinary parts. In Section 4 we put everything together to construct the module of cuspidal Hida families. Finally, in the appendix, we construct reduced mod p Hasse invariants, using a remark of Hernandez.

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1. Shimura datum and μ-ordinary locus

In this section we introduce the Shimura varieties we will work with. These will be Shimura varieties of PEL type, associated with certain unitary groups.

1.1. **PEL data.** Let F_0 be a totally real number field and let F be a totally imaginary quadratic extension of F (*i.e.* a CM field). Let d be $[F_0 : \mathbb{Q}]$ and let $\tau_1, \dots, \tau_d$ be the various embeddings $F_0 \hookrightarrow \mathbb{R}$. We choose once and for all a CM-type for F, *i.e.* we choose $\sigma_1, \dots, \sigma_d$ embeddings $F \hookrightarrow \mathbb{C}$ such that $\sigma_{i|F_0} = \tau_i$. In particular, $\mathrm{Hom}(F, \mathbb{C}) = \{\sigma_i, \bar{\sigma}_i\}_i$, where $\bar{\cdot}$ is the complex conjugation.

Let $\mathbb{V} \neq 0$ be an $\mathcal{O}_F$-lattice equipped with a nondegenerate bilinear pairing

$$\langle \cdot, \cdot \rangle \colon \mathbb{V} \times \mathbb{V} \to \mathbb{Z}\,.$$

We assume that $\langle \cdot, \cdot \rangle$ is symplectic with respect to $\mathcal{O}_F$, in the sense that it is alternating and moreover

$$\langle \gamma x, y \rangle = \langle x, \overline{\gamma} y \rangle$$

for all $\gamma \in \mathcal{O}_F$ and $x, y \in \mathbb{V}$. We set $V := \mathbb{V} \otimes_{\mathbb{Z}} \mathbb{Q}$. Note that the existence of the pairing $\langle \cdot, \cdot \rangle$ forces $\dim_{\mathbb{Q}}(V)$ to be even.

Remark 1.1. Usually, the above data always arise from an Hermitian form as follows (if $F_0 = \mathbb{Q}$ this is always the case). Recall that a Hermitian form

$$\Psi(\cdot, \cdot) \colon \mathbb{V} \times \mathbb{V} \to \mathcal{O}_F,$$

is an $\mathcal{O}_{F_0}$-bilinear form that is $\mathcal{O}_F$-linear in the first variable and such that

$$\Psi(y, x) = \overline{\Psi(y, x)}.$$

Suppose we are given such a Ψ that is nondegenerate. To obtain a pairing as above, we fix a totally negative element $\alpha \in \mathcal{O}_F$, so α is an algebraic integer such that $F = F_0(\sqrt{\alpha})$. Then, the $\mathcal{O}_{F_0}$-bilinear pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle \colon \mathbb{V} \times \mathbb{V} &\to \mathbb{Z} \\ (x, y) &\mapsto \mathrm{Tr}_{F/\mathbb{Q}}(\alpha \Psi(x, y)) \end{aligned}$$

is nondegenerate and symplectic with respect to $\mathcal{O}_F$.

Remark 1.2. More generally one can start with a simple algebra B over $\mathbb{Q}$, with center F and a positive involution $^*\colon B \to B$ such that $[F : F_0] = 2$, where F_0 is the subfield of F fixed by *. This the so called Case (A) of [Bra15]. (One need to use certain Morita equivalences in this case.) The above situation corresponds to $B = F$ and $\cdot^* = \bar{\cdot}$.

Our choice of σ_i gives an isomorphism $F \otimes_{F_0,\tau_i} \mathbb{R} \cong \mathbb{C}$ and, thanks to the symplectic pairing $\langle\cdot,\cdot\rangle$, the $\mathbb{C}$-vector space $V \otimes_{F_0,\tau_i} \mathbb{R}$ inherits a skew Hermitian form. We denote with (a_i, b_i) its signature. (If we are in the case of Remark 1.1, these are the signatures of the Hermitian form Ψ.) Note that $a_i + b_i$ does not depend on i and it is equal to $n := \frac{\dim_{\mathbb{Q}}(V)}{2d}$. Up to renaming the embeddings of F_0 into $\mathbb{R}$ we may (and we actually do) assume that

$$a_1 \leq a_2 \leq \cdots \leq a_d \leq b_d \leq \cdots \leq b_2 \leq b_1.$$

Let G be the algebraic group scheme over $\mathbb{Z}$ given by the symplectic similitudes of $\mathbb{V}$, i.e for all ring R we have, functorially on R,

$$G(R) = \{(g,\lambda) \in \mathrm{GL}(\mathbb{V} \otimes_{\mathbb{Z}} R) \times R^* \text{ such that } \langle gx, gy\rangle = \lambda\langle x, y\rangle \text{ for all } x, y \in \mathbb{V}\}.$$

Note that λ is uniquely determined by g, so we will usually drop it from the notation. Also, we have a morphism

$$\begin{aligned} \nu\colon G &\to \mathbb{G}_{\mathrm{m}} \\ (g,\lambda) &\mapsto \lambda. \end{aligned}$$

We have that $G_{\mathbb{Q}}$ is a connected reductive algebraic group and by construction there is an isomorphism

$$G_{\mathbb{R}} \cong \mathrm{G}\left(\prod_{i=1}^{d} \mathrm{U}(a_i, b_i)_{\mathbb{R}}\right),$$

where $\mathrm{U}(a_i, b_i)_{\mathbb{R}}$ is the real unitary associated to the standard Hermitian form of signature (a_i, b_i). Here the notation $\mathrm{G}(\cdot)$ means that the similitude factors for the various embeddings $F_0 \hookrightarrow \mathbb{R}$ match.

We now fix a *polarisation* of $(\mathbb{V}, \langle\cdot,\cdot\rangle)$, i.e. an $\mathbb{R}$-algebra homomorphism

$$h\colon \mathbb{C} \to \mathrm{End}_{F\otimes_{\mathbb{Q}}\mathbb{R}}(V \otimes_{\mathbb{Q}} \mathbb{R})$$

such that, for all $z \in \mathbb{C}$ and all $x, y \in V \otimes_{\mathbb{Q}} \mathbb{R}$, we have:

- $\langle h(z)x, y,\rangle = \langle x, h(\overline{z}y)\rangle$;
- the pairing $(x, y) \mapsto \langle x, h(i)y\rangle$ is symmetric and positive definite.

This is the same as giving an homomorphism $\mathbb{S} \to G_{\mathbb{R}}$, where $\mathbb{S}$ is the Deligne torus, and it induces a complex structure on $V \otimes_{\mathbb{Q}} \mathbb{R}$. We then have a decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} \cong V_{\mathbb{C},1} \oplus V_{\mathbb{C},2},$$

where $h(z)$ acts on $V_{\mathbb{C},1}$ via multiplication by z and via multiplication by $\bar{z}$ on $V_{\mathbb{C},2}$. There is an isomorphism of $F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{C}^d$-modules

$$V_{\mathbb{C},1} \cong \prod_{i=1}^{d} \mathbb{C}^{a_i} \oplus \prod_{i=1}^{d} \overline{\mathbb{C}}^{b_i},$$

where $\mathbb{C}$ acts by multiplication on the first factor and by multiplication by the conjugate on the second factor. The reflex field E is by definition the field of definition of the isomorphism class of the complex F-representation $V_{\mathbb{C},1}$.

1.2. **The p-adic setting.** We let $p \neq 2$ be a fixed prime number. We fix once and for all embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $i_p\colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. We denote with $\mathcal{P}$ the corresponding prime ideal of $\mathcal{O}_E$ above p and we write $E_{\mathcal{P}}$ for the $\mathcal{P}$-adic completion of E.

Assumption 1.3. *From now on we assume that:*

- *p is unramified in F_0 and we denote by $\pi_1, \ldots, \pi_k$ the primes of $\mathcal{O}_{F_0}$ above p;*
- *the restriction of $\langle\cdot,\cdot\rangle$ to $\mathbb{V}_p$ gives a perfect pairing with values in $\mathbb{Z}_p$.*

We write d_i for the residue degree of π_i, so $d_1 + \cdots + d_k = d = [F_0 : \mathbb{Q}]$ and the completion of $\mathcal{O}_{F_0}$ at π_i is isomorphic to $\mathbb{Z}_{p^{d_i}}$. In particular we have

$$\mathcal{O}_{F_0,p} = \prod_{i=1}^{k} \mathbb{Z}_{p^{d_i}} .$$

Note that $\mathcal{O}_{F,p}$ depends on which primes above p in $\mathcal{O}_{F_0}$ are inert in $\mathcal{O}_F$: if π_i is split in $\mathcal{O}_F$ we will write $\pi_i = \pi_i^+ \pi_i^-$. Then we have

$$\mathcal{O}_{F,p} = \prod_{\pi_i \text{ split}} \left(\mathbb{Z}_{p^{d_i}} \times \mathbb{Z}_{p^{d_i}}\right) \times \prod_{\pi_i \text{ inert}} \mathbb{Z}_{p^{2d_i}},$$

where the decomposition corresponding to split primes is obtained accordingly to $\pi_i = \pi_i^+ \pi_i^-$ and we have fixed an identification $\mathbb{Z}_{p^{d_i}} = \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_{p^{d_i}}, \mathbb{Z}_p)$.

1.3. **Shimura varieties.** Let $\mathcal{H} \subset G(\mathbb{A}^p)$ be a compact open subgroup that we assume to be *sufficiently small*, fixed from now on. (To be precise we need $\mathcal{H}$ to be *neat* in the sense of [Lan13, Definition 1.4.1.8].) We are interested in the functor

$$X\colon \text{locally noetherian } \mathcal{O}_{E_{\mathcal{P}}} - \text{schemes} \to \mathbf{set}$$

that to S associates the isomorphism classes of the following data:

(1) an abelian scheme A/S (of dimension $nd = \frac{\dim_{\mathbb{Q}}(V)}{2}$);
(2) a polarisation $\lambda\colon A \to A^\vee$ of degree prime to p;
(3) an action $\iota\colon \mathcal{O}_F \to \mathrm{End}_S(A)$ on A/S such that $\lambda \circ \iota(\bar{x}) = \iota(x)^\vee \circ \lambda$ and $\det_{\mathrm{Lie}(A)}$ equals $\det_{V_{\mathbb{C},1}}$ as polynomials with coefficients in $\mathcal{O}_S$;
(4) a $\mathcal{H}$-level structure α in the sense of [Lan13, Definition 1.3.7.6].

We furthermore require the usual determinant condition of Kottwitz, see [Lan13, Definition 1.3.4.1]. In our particular case, this last condition can be formulated in a simple way: it is equivalent to assume that $\mathrm{Lie}(A)$ is isomorphic, Zariski locally on S, to the $\mathcal{O}_S \otimes_{\mathbb{Z}_p} \mathcal{O}_{F,p}$-module

$$\mathcal{O}_{F,p} = \mathcal{O}_S \otimes_{\mathbb{Z}_p} \left(\prod_{\pi_i \text{ split}} \left(\mathbb{Z}_{p^{d_i}}^{a_i} \oplus \mathbb{Z}_{p^{d_i}}^{b_i}\right) \times \prod_{\pi_i \text{ inert}} \mathbb{Z}_{p^{2d_i}}^{a_i+b_i} \right).$$

It is known (see for example [Lan13, Theorem 1.4.1.11 and Corollary 1.4.1.12]) that the functor X is representable by a quasi-projective scheme over $\mathrm{Spec}(\mathcal{O}_{E_{\mathcal{P}}})$, denoted again by X. Thanks to the unramifiedness assumption, X is smooth over $\mathrm{Spec}(\mathcal{O}_{E_{\mathcal{P}}})$. We have that $\dim(X) = d\sum_{i=1}^{d} a_i b_i$. We now let K be a finite extension of $\mathbb{Q}_p$, that we assume to be 'sufficiently big' (the meaning of this will change during the paper without any further comment). We base change X to $\mathcal{O}_K$, using the same notation.

1.4. **Shimura varieties of Iwahoric level.** Let $\mathcal{A}$ be the universal abelian scheme over X. Using the action of $\mathcal{O}_F$ on the p-divisible group associated to $\mathcal{A}$, we have a decomposition

$$\mathcal{A}[p^\infty] = \prod_{\pi_i \text{ split}} (\mathcal{A}[(\pi_i^+)^\infty] \oplus \mathcal{A}[(\pi_i^-)^\infty]) \times \prod_{\pi_i \text{ inert}} \mathcal{A}[\pi_i^\infty],$$

where $\mathcal{A}[(\pi_i^-)^\infty]$ is canonically identified with the Cartier dual of $\mathcal{A}[(\pi_i^+)^\infty]$. We have an action of $\mathbb{Z}_{p^{d_i}}$ on $\mathcal{A}[(\pi_i^+)^\infty]$ and on $\mathcal{A}[(\pi_i^-)^\infty]$, and are both Barsotti–Tate groups of height $d_i(a_i + b_i) = d_i \frac{\dim_{\mathbb{Q}}(V)}{2d}$. The first one has dimension $d_i a_i$ and the second one has dimension $d_i b_i$. If π_i is inert we have an action of $\mathbb{Z}_{p^{2d_i}}$ on $\mathcal{A}[(\pi_i)^\infty]$, that is an autodual p-divisible group of dimension $d_i(a_i + b_i)$.

Definition 1.4. Let $X_{\mathrm{Iw}} \to \mathrm{Spec}(\mathcal{O}_K)$ be the functor

$$X_{\mathrm{Iw}}\colon \text{locally noetherian } \mathcal{O}_K - \text{schemes} \to \mathbf{set}$$

that to S associates the isomorphism classes of the following data:

(1) a point $(A, \lambda, \iota, \alpha) \in X(S)$;
(2) for all i such that π_i is split, a filtration
$$0 = H_{i,0} \subset H_{i,1} \subset \cdots \subset H_{i,a_i+b_i} = A[\pi_i^+],$$
where each $H_{i,j}$ is a finite and flat subgroup of $A[\pi_i^+]$, stable under the action of $\mathbb{Z}_{p^{d_i}}$ and of height $d_i j$;
(3) for all i such that π_i is inert, a filtration
$$0 = H_{i,0} \subset H_{i,1} \subset \cdots \subset H_{i,a_i+b_i} = A[\pi_i],$$
where $H_{i,j}$ is a finite and flat subgroup of $A[\pi_i]$, stable under the action of $\mathbb{Z}_{p^{2d_i}}$ and of height $2d_i j$.

This functor is representable by a scheme, denoted again X_{Iw} and there is a proper morphism $X_{\mathrm{Iw}} \to X$.

Remark 1.5. Let $(A, \lambda, \iota, \alpha, (H_{i,\bullet})_i)$ be a point of X_{Iw} and let i be such that π_i is split. Taking the orthogonal with respect to the perfect pairing given by Cartier duality between $A[\pi_i^+]$ and $A[\pi_i^-]$, we have that $H_{i,\bullet}$ induces an analogous filtration on $A[\pi_i^-]$, so the choice of working with $A[\pi_i^+]$ is harmless.

1.5. **Compactifications.** We fix one and for all a smooth toroidal compactification X^{tor} of X. It comes with the universal semi-abelian scheme $\mathcal{G} \to X^{\mathrm{tor}}$ and admits a stratification $\bigsqcup_W X_W^{\mathrm{tor}}$, where W ranges among $\mathcal{H}$-equivalence classes of co-torsion-free sub-modules of $\mathbb{V}$. We denote by X^* the a minimal compactification and by π the proper map $\pi : X^{\mathrm{tor}} \to X^*$.

We let S be the μ-ordinary locus consisting of points whose associated p-divisible group is μ-ordinary in the sense of [Moo04]. We write S^* for the μ-ordinary locus of X^* and S^{tor} for the counter-image of S^* inside X^{tor}. We denote by Ha^μ a Hasse invariant for G. It is a modular form modulo p whose non-vanishing locus is the μ-ordinary locus. We shall sketch in the next section the idea behind its construction.

For a variety Y over $W(k)$, we denote by $Y_m := Y \times_{\mathrm{Spec}(W(k))} \mathrm{Spec}(W(k)/p^m)$ and by Y_∞ the corresponding formal scheme.

We conclude with three examples of algebraic group G (and hence of Shimura variety X) that in our opinion the reader should keep in mind.

Example 1.6. We suppose $F_0 = \mathbb{Q}$ and F a quadratic imaginary field where p is inert and that G at infinity is $\mathrm{U}(1,2)$.

Example 1.7. Generalising the previous example, we suppose F_0 is a totally real field of degree d and F a CM where p is inert, and that G at infinity is $\mathrm{U}(1,2d) \times \mathrm{U}(2,2d-1) \times \ldots \times \mathrm{U}(d,d+1)$.

Example 1.8. We suppose F_0 is a totally real field of degree 2 where p is inert and F a CM where p splits, and that G at infinity is $\mathrm{U}(1,4) \times \mathrm{U}(2,3)$.

1.6. **Dieudonné theory.** In this section we recall the theory of μ-ordinary p-divisible groups following [Moo04, Bij16]. We fix an unramified extension $\mathbb{Q}_{p^f}/\mathbb{Q}_p$ of degree f. We write $\mathcal{O} = \mathbb{Z}_{p^f}$ for its valuation ring, D for its Galois group and Fr for the Frobenius of D. Fix also a finite field k such that $W(k)$, that will be our base ring, contains $\mathcal{O}$. We denote by I the set of embeddings $\mathcal{O} \to W(k)$.

For a fixed integer h we define M to be a free $W(k)$-module of rank fh with basis $(e_{\sigma,i})_{\sigma \in I, i=1,\ldots,h}$ and we also fix $\varepsilon : I^h \to \{0,1\}$ (this is equivalent to the map $\mathfrak{f} : I \to \{0,1,\ldots,h\}$ in [Moo04]). We define a Frobenius operator F and a Verschiebung V on M
$$F e_{\sigma,i} = p^{\varepsilon(\sigma,i)} e_{\mathrm{Fr}\sigma,i} \quad V e_{\sigma,i} = p^{1-\varepsilon(\sigma,i)} e_{\mathrm{Fr}^{-1}\sigma,i}.$$

Clearly $FV = VF = p$. For all i, we let $\mathcal{O}$ act on $e_{\sigma,i}$ via the σ-embedding. Hence M is a free $\mathcal{O}\otimes_{\mathbb{Z}_p} W(k)$-module of rank h. Via Dieudonné equivalence, this defines a Barsotti–Tate group with $\mathcal{O}$-action of ($\mathcal{O}$-)height h that shall be denote by BT_ε.

Take $h = 1$ and identify I with $\{1,\dots,f\}$; we shall write $\varepsilon^{(i)}$ for that map ε such that $\varepsilon(j) = 1$ if $j \geq i+1$ and 0 otherwise. For example, we have:

$$\begin{aligned}\mathrm{BT}_{(1,1,\dots,1)} &= \mu_{p^\infty}\otimes_{\mathbb{Z}_p}\mathcal{O};\\ \mathrm{BT}_{(0,\dots,0,1)} &= \mathcal{LT}_{\mathcal{O}}[p^\infty];\\ \mathrm{BT}_{(0,\dots,0)} &= K/\mathcal{O}\ (\text{or } \mathbb{Q}_p/\mathbb{Z}_p\otimes_{\mathbb{Z}_p}\mathcal{O}).\end{aligned}$$

If $\mathcal{D}$ denotes Cartier duality for p-divisible groups, note that $\mathcal{D}(\mathrm{BT}_\varepsilon) = \mathrm{BT}_{1-\varepsilon}$. In general, we shall denote by $\mathcal{LT}_i$ the BT with $\mathcal{O}$-structure $\mathrm{BT}_{\varepsilon^{(i)}}$ and call it generalised Lubin–Tate. Note that a generalised Lubin–Tate is rigid, in the sense that there is a unique deformation in characteristic zero [Moo04, Corollary 2.1.5].

We write $\mathrm{Fil}^1(M)$ for the $\mathcal{O}$-module spanned by the $e_{\sigma,i}$ for which $F(e_{\sigma,i}) \equiv 0 \bmod p$. (Note that it is not a free $\mathcal{O}\otimes_{\mathbb{Z}_p} W(k)$-module!)

For simplicity, we suppose now that p is inert in the totally real field F_0 and we write $\mathcal{O}$ for the p-adic completion of its ring of integers. We shall say that we are in case (U) if p is inert also in F and in case (L) if p splits. Recall that we fixed a unitary group with signature at infinity (a_i, b_i). We normalise the a_i's and b_i's so that $a_1 \leq a_2 \leq \dots \leq a_d \leq b_d \leq \dots \leq b_1$; this induces an isomorphism of I with $\{1,\dots,f\}$ that it will be fixed for the rest of the section. We also write σ_i for the embedding in $\mathrm{Hom}(\mathcal{O}_F\otimes\mathbb{Z}_p,\mathbb{C}_p)$ corresponding to a_i and $\bar{\sigma}_i$ for the one corresponding to b_i. We let $n = a_1 + b_1 = a_i + b_i$. For the rest of the subsection, let $\mathcal{A}^\mu$ be the μ-ordinary abelian variety over U, a fixed open subset of the ordinary locus *in characteristic p*.

In case (L), let $p = \pi^+\pi^-$. We have $\mathbb{Q}_{p^f} = F_{0,p}$, $f = d$ and, over $\overline{\mathbb{F}}_p$,

$$\mathcal{A}^\mu[p^\infty] \cong \mathcal{A}^\mu[(\pi^+)^\infty]\times\mathcal{A}^\mu[(\pi^-)^\infty] \cong \prod_{i=0}^{f}\mathcal{LT}_i^{a_{i+1}-a_i}\times\mathcal{D}\left(\prod_{i=0}^{f}\mathcal{LT}_i^{a_{i+1}-a_i}\right), \tag{1.1}$$

after defining $a_0 = 0$ and $a_{f+1} = n$. We shall denote the corresponding BT by BT_G and its Dieudonné module by M_G. We have $\dim(A[(\pi^+)^\infty]) = \sum_{i=1}^f a_i$ and $\dim(A[(\pi^-)^\infty]) = \sum_{i=1}^f b_i$ (which is the dimension of $\mathrm{Fil}^1(M_G)$).

Similarly in case (U) we have $\mathbb{Q}_{p^f} = F_p$, $2d = f$ and

$$\mathcal{A}^\mu[p^\infty] \cong \prod_{i=0}^{f}\mathcal{LT}_i^{a_{i+1}-a_i}, \tag{1.2}$$

where we have defined $a_{j+d} := b_{d-j+1}$. (Note that $a_{j+(d-j)+1} - a_{j+1+(d-j-1)+1} = b_j - b_{j+1} = a_{j+1} - a_j$ if $2 \leq j \leq d-1$. This ensure that $\mathcal{A}[p^\infty]$ is self-dual under Cartier duality.) We shall again denote the corresponding BT by BT_G and its Dieudonné module by M_G. As before we have $\dim(\mathcal{A}^\mu[p^\infty]) = dn$ (which is the dimension of $\mathrm{Fil}^1(M_G)$).

Remark 1.9. These explicit descriptions tell us immediately the condition for the ordinary locus to be non-empty in case (U): $a_1 = \dots = a_f$.

In both cases, we can identify M_G with $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}^\mu)$ and also $\mathrm{Fil}^1(M_G)$ with $\mathrm{Fil}^1\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}^\mu)$.

In the Appendix A we shall defined a refined Hasse invariant. In the case all the signatures are different, the invariants of the appendix coincide with the ones of [GN17, Her16a] and it is no harm for the reader to think about them when considering the Hasse invariant, as we need our reduced Hasse invariants only in

the proof of Theorem 4.4. Their Hasse invariant is the product of the partial Hasse invariants $\mathrm{Ha}^{\mu,\sigma_i}$ defined as the determinant of

$$\frac{\bigwedge^{a_i}\mathrm{Fr}^f}{p^{c_i}}:\bigwedge^{a_i}\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}^\mu)_{\sigma_i}/\mathrm{Fil}^1(\bigwedge^{a_i}\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}^\mu)_{\sigma_i})\to\bigwedge^{a_i}\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}^\mu)_{\sigma_i}/\mathrm{Fil}^1(\bigwedge^{a_i}\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}^\mu)_{\sigma_i}).$$

This also shows that their Hasse invariant Ha^μ is a modular form of parallel weight p^f-1 and that a power of it lifts to characteristic zero.

We quote from [Moo04, Proposition 2.1.9].

Proposition 1.10. *Let X be a deformation of BT_G to a local Artinian $W(k)$-algebra. Then, étale locally on the μ-ordinary locus, there is a unique filtration in BT with $\mathcal{O}$-structure X_i of X such that X_i lifts the slope filtration of BT_G (which is defined only étale locally). Moreover, we have a filtration $\{\mathcal{A}_i\}_{i=0}^{f+1}$ for $\mathcal{A}^\mu[p^\infty]$, where $\mathcal{A}_0=0$, $\mathcal{A}_{f+1}=\mathcal{A}^\mu[p^\infty]$, and $\mathcal{A}_1=\mathcal{A}[p^\infty]^{\mathrm{mult}}$ ($\cong(\mu_{p^\infty}\otimes\mathcal{O})^{a_1}$ in case (U) and $\cong(\mu_{p^\infty}\otimes\mathcal{O})^{a_1}\times(\mu_{p^\infty}\otimes\mathcal{O})^{b_1}$ in case (L)).*

2. Modular forms and the Igusa tower

2.1. **Classical modular forms.** Consider the group $P=\prod_{i=1}^d\mathrm{GL}_{a_i}\times\mathrm{GL}_{b_i}$. Its complex points coincide with the automorphisms of $(V_{\mathbb{C},1}\oplus V_{\mathbb{C},2})^{\mathbb{C}=\mathrm{Id}}$, where $\mathbb{C}=\mathrm{Id}$ means holomorphic part. Consider the semi-abelian variety $\mathcal{G}$ over X^{tor} and let us denote by e the unit section $e:X^{\mathrm{tor}}\to\mathcal{G}$. We have a locally free sheaf $\omega:=e^*\Omega^1_{\mathcal{G}/X^{\mathrm{tor}}}$.

Recall we had fixed a CM type $(\sigma_i,\bar\sigma_i)_{i=1}^d$ for (F_0,F); over F we have a decomposition

$$\omega\cong\bigoplus_{i=1}^d\omega_{\sigma_i}\oplus\omega_{\bar\sigma_i}$$

and we define $\mathcal{E}=\mathcal{E}^+\oplus\mathcal{E}^-$, where

$$\mathcal{E}^+\cong\bigoplus_i\underline{\mathrm{Isom}}(\mathcal{O}^{a_i}_{X^{\mathrm{tor}}},\omega_\sigma),$$
$$\mathcal{E}^-\cong\bigoplus_i\underline{\mathrm{Isom}}(\mathcal{O}^{b_i}_{X^{\mathrm{tor}}},\omega_{\bar\sigma_i}).$$

We have that $\xi:\mathcal{E}\to X^{\mathrm{tor}}$ is an (algebraic) P-torsor.

Let $\kappa=(k_{\sigma_i,1},\dots,k_{\sigma_i,a_i},k_{\bar\sigma_i,1},\dots,k_{\bar\sigma_i,b_i})_i$ be an algebraic weigh. We say that κ is:

- *dominant* if $k_{\sigma_i,1}\ge\dots\ge k_{\sigma_i,a_i}\ge k_{\bar\sigma_i,1}\ge\dots\ge k_{\bar\sigma_i,b_i}$ for all i;
- *regular* if $k_{\sigma_i,1}>\dots>k_{\sigma_i,a_i}>k_{\bar\sigma_i,1}>\dots>k_{\bar\sigma_i,b_i}$ for all i;
- *very regular* if $\kappa(w\alpha_pw\alpha_p^{-1})\in p\mathbb{Z}$ for all i, where $\alpha_p=(\alpha_{p,i})_i$ and $\alpha_{p,i}$ is the diagonal matrix with $1,p,\dots,p^{a_i-1},1,p,\dots,p^{b_i-1}$ on the diagonal. Here w ranges over the elements in the Weyl group of $\prod_i\mathrm{GL}_{a_i}\times\mathrm{GL}_{b_i}$.

The sheaf of weight κ automorphic forms is then

$$\omega^k=\xi_*\mathcal{O}_{\mathcal{E}}^{(\prod_i N_{a_i}\times N_{b_i})}[\kappa],$$

being $\prod_i N_{a_i}\times N_{b_i}$ the algebraic group of upper unipotent matrices in P. Locally for the Zariski topology, this sheaf is isomorphic to $R[\kappa]$, the algebraic representation of P of highest weight κ.

Definition 2.1. *For any $\mathcal{O}_K$-algebra R we define the space of weight k modular forms as*

$$\mathrm{M}_k(\mathcal{H},R):=\mathrm{H}^0(X^*{}_{/R},\pi_*\omega^\kappa)=\mathrm{H}^0(X^{\mathrm{tor}}{}_{/R},\omega^\kappa)$$

and the space of cusp forms

$$\mathrm{S}_k(\mathcal{H},R):=\mathrm{H}^0(X^*{}_{/R},\pi_*(\omega^\kappa(-D)))=\mathrm{H}^0(X^{\mathrm{tor}}{}_{/R},\omega^\kappa(-D)),$$

where D denotes the boundary of X^{tor}.

2.2. **The Igusa tower.** Recall that S is the μ-ordinary locus of our Shimura variety. Thanks to Proposition 1.10 we can define over S

$$\mathrm{Gr}^\bullet(\mathcal{A}^\mu[p^\infty]) = \prod_{i=1}^{f+1} \mathcal{A}_i/\mathcal{A}_{i-1}, \quad \mathrm{Gr}^\bullet(\mathcal{A}^\mu[p^\infty]^\circ) = \prod_{i=1}^{f} \mathcal{A}_i/\mathcal{A}_{i-1}.$$

The interest in removing the étale part lies in the fact that the connected part of $\mathcal{A}^\mu$ can be extended naturally to the whole S^{tor}. Indeed, let W be a cusp label for a component of the boundary of the toroidal compactification of rank r (hence $r \leq a_1$). Over X_W^{tor} we have the semi-abelian scheme

$$0 \to \mathbb{G}_m \otimes \mathcal{O} \otimes W \to \mathcal{G} \times_{X^{\mathrm{tor}}} X_W^{\mathrm{tor}} \to \mathcal{A}_W \to 0,$$

where $\mathcal{A}_W$ is the universal abelian variety associated with the Shimura datum for W. If we restrict to the ordinary locus and take the p^l-torsion we obtain

$$0 \to (\mu_{p^l} \otimes \mathcal{O})^r \to \mathcal{G} \times_{X^{\mathrm{tor}}} X_W^{\mathrm{tor}}[p^l] \to \mathcal{A}_W^\mu[p^l] \to 0$$

Using (1.1) or (1.2) we see that the difference between $A_W^\mu[p^l]$ and $A^\mu[p^l]$ is given by r copies of $\mu_{p^l} \otimes \mathcal{O}$ and $\mathbb{Z}/p^l\mathbb{Z} \otimes \mathcal{O}$. Hence, the $\mathrm{Gr}^\bullet(\mathcal{A}^{\mu,\circ})$ and $\mathrm{Gr}^\bullet(\mathcal{G}^{\mu,\circ})$ are the same, or better the first extends to the second.

Hence, we can define

$$\begin{aligned} \mathrm{Ig}_{m,l} &:= \mathrm{Isom}_{\mathcal{O}_F \otimes \mathbb{Z}_p\text{-BT}, S_m^{\mathrm{tor}}}(\mathrm{Gr}^\bullet(\mathcal{G}_m^{\mu,\circ}[p^l]), \mathrm{BT}^\circ_{G,m}[p^l]), \\ \mathrm{Ig} &:= \varinjlim_m \varprojlim_l \mathrm{Ig}_{m,l}. \end{aligned}$$

Note that we require the isomorphisms to respect the + and − parts in case (L) but do not require them to respect the polarisation condition (as $\mathcal{G}$ is no longer polarised).

Theorem 2.2. *The formal scheme* Ig *is a pro-étale Galois cover of S_∞^{tor} of Galois group $\prod_{i=0}^{f-1} \mathrm{GL}_{a_{i+1}-a_i}(\mathcal{O})$ in case (U) and $\prod_{i=0}^{f-1} \mathrm{GL}_{a_{i+1}-a_i}(\mathcal{O}) \times \prod_{i=1}^{f} \mathrm{GL}_{b_i-b_{i+1}}(\mathcal{O})$ in case (L).*

The Galois group that appears as automorphisms of the Igusa tower is the group of automorphisms of the (connected part of the) μ-ordinary BT. We denote by N_{Ig} the upper unipotent part of $\mathrm{Gal}(\mathrm{Ig}, S^{\mathrm{tor}})$ and by T_{Ig} the corresponding torus. We have that T_{Ig} is isomorphic to $(\mathcal{O}^*)^{a_f}$ in case (U) and $(\mathcal{O}^*)^{a_f} \times (\mathcal{O}^*)^{b_f}$ in case (L).

We can also push-forward Ig to the minimal compactification and we shall denote it by Ig^*. We shall write $\mathcal{I}^0$ and $\mathcal{I}^{*,0}$ for the sheaves of ideal of the boundary of the two Igusa tower. Note that $\pi_*\mathcal{O}_{\mathrm{Ig}} = \mathcal{O}_{\mathrm{Ig}^*}$ and $\pi_*\mathcal{I}^0 = \mathcal{I}^{*,0}$ for the same reasoning as [Lan13, Theorem 7.2.4.1]. (See also [Lan, Theorem 6.2.2.1].)

The attentive reader will have noticed that this group has too many characters. Taking G as in Example 1.6 with p inert in F, our Igusa tower has Galois group isomorphic to $\mathcal{O}^* \times \mathcal{O}^*$. The usual Hida theory would take for the weight space $\mathrm{Hom}_{\mathrm{cont},\mathbb{Z}_p}(\mathcal{O}^* \times \mathcal{O}^*, \mathbb{C}_p^*)$ while the classical weights are only three. Inspired by [Bra13], we define the weight space $\mathcal{W}$ to be $\mathrm{Hom}_{\mathrm{cont}}(\mathcal{O}^*, \mathbb{C}_p^*) \times \mathrm{Hom}_{\mathrm{cont},\mathcal{O}}(\mathcal{O}^*, \mathbb{C}_p^*)$ where $\mathrm{Hom}_{\mathrm{cont},\mathcal{O}}$ means that we consider continuous characters such that the induced morphism between Lie algebras is $\mathcal{O}$-linear. The reason is that the Fil^1 of the Dieudonné module of μ_{p^∞} is a two dimensional module (over $\mathcal{O}$) where $\mathcal{O}$ acts both via $\sigma : \mathcal{O} \to \mathcal{O}$ and $\bar{\sigma} : \mathcal{O} \to \mathcal{O}$, while on the Fil^1 of the Dieudonné module of $\mathcal{LT}_{\mathcal{O}}$ the action of $\mathcal{O}$ is only via σ. Every generalised $\mathcal{LT}_i$ will hence contribute to the weight space only according to the character of $\mathcal{O}$ acting on its Fil^1. Following this idea we now introduce our weight space.

2.3. **The weight space.** Our weight space will parametrise characters that are 'partially locally $\mathcal{O}$-analytic', in the sense that we want to control the induced morphism between Lie algebras. Let us be more precise.

Let A be a K-affinoid algebra and let $\chi \in \mathrm{Hom}_{\mathrm{cont}}(\mathcal{O}^*, A^*)$ be a continuous character. We have an induced morphism $d\chi\colon \mathrm{Lie}(\mathcal{O}^*) \to \mathrm{Lie}(A^*)$. This extends to a morphism, denoted with the same symbol,

$$d\chi\colon \mathrm{Lie}(\mathcal{O}^*) \otimes_{\mathbb{Z}_p} \mathcal{O} \to \mathrm{Lie}(A^*).$$

The Lie algebra of $\mathcal{O}^*$ and of A^* can be naturally identified with $\mathcal{O}$ and A, respectively. Moreover we can identify $\mathrm{Lie}(\mathcal{O}^*) \otimes_{\mathbb{Z}_p} \mathcal{O} = \mathcal{O} \otimes_{\mathbb{Z}_p} \mathcal{O}$ with $\prod \mathcal{O}$, where the product is over the embeddings of $\mathcal{O}$ in $\mathbb{C}_p$. In practice we get a morphism

$$d\chi\colon \prod_{\mathrm{Hom}(\mathcal{O},\mathbb{C}_p)} \mathcal{O} \to A.$$

Definition 2.3. Let J be a subset of $\mathrm{Hom}(\mathcal{O}, \mathbb{C}_p)$ and let $f\colon \mathrm{Lie}(\mathcal{O}^*) \to \mathrm{Lie}(A^*)$ be a $\mathbb{Z}_p$-linear morphism. We say that f is *J-linear* if the induced morphism $\mathcal{O} \otimes_{\mathbb{Z}_p} \mathcal{O} \to A$ factors through a map

$$\prod_J \mathcal{O} \to A.$$

We will write $\mathrm{Hom}_J(\mathrm{Lie}(\mathcal{O}^*), \mathrm{Lie}(A^*))$ for the set of J-linear morphism. Following [DI15, Définition 4.1], we say that $\chi \in \mathrm{Hom}_{\mathrm{cont}}(\mathcal{O}^*, A^*)$ is *J-analytic* if $d\chi$ is J-linear.

Let i be an integer between 0 and f. We define $J(\mathcal{LT}_i)$ to be the set of σ_j for $j \geq 1 + i$ and $J(\mathcal{D}(\mathcal{LT}_i))$ to be the set of σ_j for $j \leq i$. We also set

$$\mathrm{Hom}_{\mathrm{cont},\mathcal{LT}_i}(\mathcal{O}^*, A^*) := \{\chi \in \mathrm{Hom}_{\mathrm{cont}}(\mathcal{O}^*, A^*) \text{ such that } \chi \text{ is } J(\mathcal{LT}_i) - \text{analytic}\}.$$

The definition of $\mathrm{Hom}_{\mathrm{cont},\mathcal{D}(\mathcal{LT}_i)}(\mathcal{O}^*, A^*)$ is analogous. We call $\mathcal{W}_{\mathcal{LT}_i}$ the functor

$$A \mapsto \mathrm{Hom}_{\mathrm{cont},\mathcal{LT}_i}(\mathcal{O}^*, A^*).$$

By definition we have that $\mathcal{W}_{\mathcal{LT}_i}(A) \subseteq \mathrm{Hom}_{\mathrm{cont}}(\mathcal{O}^*, A^*)$ and $\mathrm{Hom}_{\mathrm{cont}}(\mathcal{O}^*, \cdot)$ is representable by the disjoint union of $p^f - 1$ open unit disks of dimension f (via $\mathcal{O}^* \cong \mu_{p^f-1} \times \mathcal{O} \cong \mu_{p^f-1} \times \mathbb{Z}_p^f$). Letting $\mathbf{D}_f$ be such a disk, by definition the diagram

$$\begin{tikzcd}
\mathcal{W}_{\mathcal{LT}_i}(A) \arrow[r, hook] \arrow[d, "d"] & \coprod \mathbf{D}_f(A) \arrow[d, "d"] \\
\mathrm{Hom}_{J(\mathcal{LT}_i)}(\mathrm{Lie}(\mathcal{O}^*), \mathrm{Lie}(A^*)) \arrow[r, hook] & \mathrm{Hom}(\mathrm{Lie}(\mathcal{O}^*), \mathrm{Lie}(A^*))
\end{tikzcd}$$

is cartesian. We can follow word by word the approach of [ST01, Sections 1 and 2] to show that $\mathcal{W}_{\mathcal{LT}_i}$ is representable by a closed analytic subvariety, denoted again $\mathcal{W}_{\mathcal{LT}_i}$, of $\coprod \mathbf{D}_f$. Similarly, we have the closed subvariety $\mathcal{W}_{\mathcal{D}(\mathcal{LT}_i)}$

Proposition 2.4. *We have that $\mathcal{W}_{\mathcal{LT}_0} = \coprod \mathbf{D}_f(A)$. In general, the base change to $\mathbb{C}_p$ of $\mathcal{W}_{\mathcal{LT}_i}$ is isomorphic to the disjoint union of $p^f - 1$ copies of an $(f-i)$-dimensional open unit disk. Similarly, $\mathcal{W}_{\mathcal{D}(\mathcal{LT}_f)} = \coprod \mathbf{D}_f(A)$ and the base change to $\mathbb{C}_p$ of $\mathcal{W}_{\mathcal{D}(\mathcal{LT}_i)}$ is isomorphic to the disjoint union of $p^f - 1$ copies of an i-dimensional open unit disk.*

Proof. The statement for $\mathcal{W}_{\mathcal{LT}_0}$ is obvious. For the general case we consider the Lubin-Tate group

$$\mathcal{G} := \prod_{j \geq i+1} \mathrm{BT}_{(0,\ldots,0,1,0,\ldots,0)},$$

where the 1 is in the j-position. (Note that this is *not* $\mathcal{LT}_i$ but the first filtered piece of the Dieudonné module of the two are isomorphic as $\mathcal{O} \otimes_{\mathbb{Z}_p} \mathcal{O}$-modules.) Using $\mathcal{G}$,

the proof is exactly the same as in [ST01, Section 3] (using $\mathcal{O}^* \cong \mu_{p^f-1} \times \mathcal{O}$). The proof for $\mathcal{W}_{\mathcal{D}(\mathcal{LT}_i)}$ is similar. □

Remark 2.5. If $i = f-1$ then, by [ST01, Section 3], we know that $\mathcal{W}_{\mathcal{LT}_i}$ is *not* isomorphic to a open unit disk, even after base change to any finite extension of K. We conjecture that the same hold for any $i \neq 0, f$.

Everything we said above for characters of $\mathcal{O}^*$ generalises immediately to characters of $(\mathcal{O}^*)^n$, where $n \geq 0$ is any integer. We obtain in this way the spaces $\mathcal{W}_{\mathcal{LT}_i^n}$ and $\mathcal{W}_{\mathcal{D}(\mathcal{LT}_i^n)}$. We are now ready to define the weight space we are interested in.

Definition 2.6. In case (L) then we define the weight space $\mathcal{W} = \mathcal{W}_G$ to be the rigid space given by

$$\prod_{i=0}^{f-1} \mathcal{W}_{\mathcal{LT}_i^{a_{i+1}-a_i}} \times \prod_{i=1}^{f} \mathcal{W}_{\mathcal{D}(\mathcal{LT}_i^{b_i-b_{i+1}})},$$

under the convention that $b_{f+1} = 0$.

In case (U) then we define the weight space $\mathcal{W} = \mathcal{W}_G$ to be the rigid space given by

$$\prod_{i=0}^{f-1} \mathcal{W}_{\mathcal{LT}_i^{a_{i+1}-a_i}}.$$

Remark 2.7. Note that $J(\mathcal{LT}_f) = \varnothing$ so $\mathrm{Hom}_{\mathrm{cont},\mathcal{LT}_f}(\mathcal{O}^*, \mathbb{C}_p^*) = 1$, the constant character. This gives another reason to discard the étale part in our Igusa tower: not only it has no Fil^1, *i.e.* no non-zero differentials and in particular no non-zero modular forms, but also the corresponding weight space is trivial.

The following lemma tells us the our weight space has the expected dimension.

Lemma 2.8. *We have that $\mathcal{W}$ is equidimensional of dimension nd.*

Proof. We can base change $\mathcal{W}$ to $\mathbb{C}_p$ and use Proposition 2.4. In case (L) we get

$$\dim(\mathcal{W}) = \sum_{i=0}^{f-1}(f-i)(a_{i+1}-a_i) + \sum_{i=1}^{f} i(b_i - b_{i+1}) =$$

$$\sum_{i=0}^{f-1}(f-i)(a_{i+1}-a_i) + \sum_{i=1}^{f} i(a_{i+1}-a_i) = f\sum_{i=0}^{f}(a_{i+1}-a_i) = f(a_{f+1}-a_0) = fn = nd.$$

In case (U) we get

$$\dim(\mathcal{W}) = \sum_{i=0}^{f-1}(f-i)(a_{i+1}-a_i) = fa_1 + (f-1)a_2 - (f-1)a_1 + \cdots + a_f - a_{f-1}$$

$$= a_1 + a_2 + \ldots + a_{f-1} + a_f = n(f/2) = nd.$$

□

In particular, these weights are the character of the intersection of $\mathrm{Gal}(\mathrm{Ig}/S^{\mathrm{tor}})$ with the $\mathcal{O}$-points of the parabolic P of [Moo04, §1.1.2] (which is also the parabolic P of Section 2.1). Explicitly, it consists of $\mathcal{O}$-linear maps $M_G \to M_G$ which preserve both the decomposition of BT_G into product of generalised Lubin–Tate groups and the $\mathcal{O}$-module $\mathrm{Fil}^1(M_G)$.

Remark 2.9. For the reader who are unhappy about our weight space, we offer an alternative definition of the Igusa tower, more in the spirit of [EM17, Her17b] but that can still be extended to the boundary. Consider the filtration of the p-torsion of the universal μ-ordinary abelian variety over S_m given by

$$0 \subset \mathcal{A}[p^\infty]^{\mathrm{mult}} \subset \mathcal{A}[p^\infty]^\circ \subset \mathcal{A}[p^\infty],$$

where $\mathcal{A}^{\mathrm{mult}}$ resp. $\mathcal{A}^\circ$ denotes the multiplicative resp. connected part. The p-divisible group $\mathcal{A}[p^\infty]^\circ/\mathcal{A}[p^\infty]^{\mathrm{mult}}$ is naturally polarised. One could then define $\mathrm{Ig}_{m,l}$ as

$$\mathrm{Isom}_{\mathcal{O}-\mathrm{BT}}(\mathcal{A}[p^l]^{\mathrm{mult}}, \mathcal{O}\otimes\mu_{p^l}) \times \mathrm{Isom}_{\mathcal{O}-\mathrm{pol}-\mathrm{BT}}\left(\frac{\mathcal{A}[p^l]^\circ}{\mathcal{A}[p^l]^{\mathrm{mult}}}, \mathrm{BT}^*[p^l]\right),$$

where BT^* is the Barsotti–Tate with $\mathcal{O}$-structure isomorphic to $\mathcal{A}[p^\infty]^\circ/\mathcal{A}[p^\infty]^{\mathrm{mult}}$. These $\mathrm{Ig}_{m,l}$ extend to S_m^{tor}.

We end this subsection by defining our Iwasawa algebra Λ. Let us denote by $\mathcal{W}^\circ$ the connected component of $\mathcal{W}$ containing the trivial character. Let N be equal to $f(a_f+b_1)$ in case (L) and equal to fa_f in case (U). By the above discussion, $\mathcal{W}^\circ$ is a closed subvariety of $\mathbf{D}_N$, the N-dimensional open unit ball. Using Berthelot rigid fiber we have

$$\mathbf{D}_N = \mathrm{Spf}(\mathcal{O}_K[\![x_1,\dots,x_N]\!])^{\mathrm{rig}}.$$

By [ST01, Section 2] we have that

$$\mathcal{O}_{\mathcal{W}}(\mathcal{W}^\circ) = \mathcal{O}_{\mathbf{D}_N}(\mathbf{D}_N)/J,$$

where J is an ideal of $\mathcal{O}_{\mathbf{D}_N}(\mathbf{D}_N) \subset K[\![x_1,\dots,x_N]\!]$. We let I be the ideal of $\mathcal{O}_K[\![x_1,\dots,x_N]\!]$ defined by $I := J\cap\mathcal{O}_K[\![x_1,\dots,x_N]\!]$ and we define

$$\Lambda^\circ := \mathcal{O}_K[\![x_1,\dots,x_N]\!]/I.$$

In this way Λ° is $\mathcal{O}_K$-flat and by construction we have

$$\Lambda^{\circ\,\mathrm{rig}} = \mathcal{W}^\circ.$$

We shall denote by Λ the semi-local ring defined as the product over the connected components of $\mathcal{W}$ of copies of Λ°. We clearly have $\Lambda^{\mathrm{rig}} = \mathcal{W}$. By construction we have that elements of Λ are functions on $\mathcal{W}$ bounded by 1, but we do not know whether any such a function is an element of Λ. (This would be the case if, for example, Λ were a normal ring.)

2.4. **The Hodge–Tate map.** We define (p-divisible) p-adic modular forms as elements of $\tilde{V}$, where

$$\tilde{V} = (\varprojlim_m \varinjlim_l V_{m,l})^{N_{\mathrm{Ig}}},$$
$$V_{m,l} = \mathrm{H}^0(S_m^{\mathrm{tor}}, \mathcal{O}_{\mathrm{Ig}_{m,l}}) = \mathrm{H}^0(S_m^*, \mathcal{O}_{\mathrm{Ig}^*{}_{m,l}}).$$

Here N_{Ig} is the unipotent part of $\mathrm{Gal}(\mathrm{Ig}, S^{\mathrm{tor}})$. We have that T_{Ig} acts naturally on $\tilde{V}$. We want to interpret classical modular forms as p-adic modular forms.

Let U be an open subset of the μ-ordinary locus. Over U, given a trivialisation $\gamma\colon \mathcal{A}^\mu[p^\infty]^\circ \cong_U BT_G^\circ$ we can define a trivialisation of ω. The filtration of Proposition 1.10 gives a sequence of quotients $\{\omega_{\mathcal{A}_i} \twoheadrightarrow \omega_{\mathcal{A}_{i-1}}\}$. Over S these sheaves are locally free, hence the quotients all split and the splitting can be moreover chosen to be $\mathcal{O}$-equivariant. It can be extended to the toroidal compactification as $\mathcal{A}_{W,i}/\mathcal{A}_{W,0}$ are independent of W. Hence over the μ-ordinary locus we have an isomorphism

$$\omega = \bigoplus_{i=1}^f \omega_{\mathcal{A}_i/\mathcal{A}_{i-1}}.$$

Recall that $\mathcal{A}_{i+1}/\mathcal{A}_i \cong \mathcal{LT}_{i-1}^{a_i-a_{i-1}}$. As explained above in Subsection 1.6, we have fixed a basis $(e_{\sigma,i})$ of the Dieudonné module of $\mathcal{LT}_{i-1}^{a_i-a_{i-1}}$. This gives a basis $(e_{\sigma,j})$ of $\omega_{\mathcal{LT}_i^{a_i-a_{i-1}}}$. In particular $\{\gamma^*(e_{\sigma,j})\}$ is a basis for $\omega_{S^{\mathrm{tor}}\infty}$. Recall the $P = \prod_{i=1}^d \mathrm{GL}_{a_i}\times\mathrm{GL}_{b_i}$-torsor $\mathcal{E}$. Summing up, we have a map

$$\mathrm{HT}: \mathrm{Ig} \to \mathcal{E}\times S_\infty^{\mathrm{tor}}$$

sending $(\mathcal{A}_x^\mu, \gamma)$ to $(\mathcal{A}_x^\mu, \{\gamma^*(e_{\sigma,j})\})$.

Definition 2.10. Given a classical modular form f, viewed as a function of $\mathcal{E}$, we obtain, by restriction and composition with HT, the function $\mathrm{HT}^*(f)$ on Ig. Since by definition f is invariant under the action of $\prod_i N_{a_i} \times N_{b_i}$, we have that $\mathrm{HT}^*(f)$ is invariant under the action of N_{Ig}, *i.e.* we have

$$\mathrm{HT}^*(f) \in \tilde{V}.$$

Remark 2.11. The name Hodge–Tate could seem a bit out of turn: the map in [Pil12, Hid02] which compares classical and p-adic forms is induced by the Hodge–Tate map $\mathcal{A}^\vee \to \omega_{\mathcal{A}}$. In particular, the HT maps in *loc. cit.* would be ours composed with an involution.

Let's write down explicitly how the HT map sends $\mathrm{Gal}(\mathrm{Ig}/S^{\mathrm{tor}})$ into $\prod_{i=1}^d \mathrm{GL}_{a_i} \times \mathrm{GL}_{b_i}(\mathcal{O})$. In case (U), remember that $f = 2d$ and that we have written a_{j+d} for b_{d-j+1}. Let $(\gamma_i)_{i=0,\dots,f-1}$ be an element of $\prod_{i=0}^{f-1} \mathrm{GL}_{a_i - a_{i-1}}(\mathcal{O})$ we have explicitly

$$\mathrm{HT}((\gamma_i)_{i=0,\dots,f-1}) = \prod_{j=1}^{d} \Big([\sigma_j(\gamma_i)]_{i=0}^{j-1}, [\sigma_{f+1-j}(\gamma_i)]_{i=0}^{f-j} \Big),$$

where $[\sigma_j(\gamma_i)]_{i=0}^{j-1} = [\sigma_j(\gamma_0), \dots, \sigma_j(\gamma_{j-1})]$ is an element of $\prod_{i=0}^{j-1} \mathrm{GL}_{a_{i+1}-a_i}(\mathcal{O})$ embedded diagonally into $\mathrm{GL}_{a_j}(\mathcal{O})$.

In case (L) the description is similar; let $\gamma^+ = (\gamma_i^+)_{i=0,\dots,f-1}$ and $\gamma^- = (\gamma_i^-)_{i=1,\dots,f}$ be an element of $\prod_{i=0}^{f-1} \mathrm{GL}_{a_i - a_{i-1}}(\mathcal{O}) \times \prod_{i=1}^{f} \mathrm{GL}_{b_i - b_{i+1}}(\mathcal{O})$ we have explicitly

$$\mathrm{HT}(\gamma^+, \gamma^-) = \prod_{j=1}^{d} \Big([\sigma_j(\gamma_i^+)]_{i=0}^{j-1}, [\bar{\sigma}_j(\gamma_i^-)]_{i=j}^{f} \Big).$$

Example 2.12. When G is as in Example 1.7 with $f = 4$ we have

$$\begin{aligned}
&\mathrm{HT}(\gamma_0, \gamma_1, \gamma_2, \gamma_3) = \\
&= \left(\left(\sigma_1(\gamma_0), \begin{pmatrix} \sigma_4(\gamma_0) & & & \\ & \sigma_4(\gamma_1) & & \\ & & \sigma_4(\gamma_2) & \\ & & & \sigma_4(\gamma_3) \end{pmatrix} \right), \right. \\
&\left. \left(\begin{pmatrix} \sigma_2(\gamma_0) & \\ & \sigma_2(\gamma_1) \end{pmatrix}, \begin{pmatrix} \sigma_3(\gamma_0) & & \\ & \sigma_3(\gamma_1) & \\ & & \sigma_3(\gamma_2) \end{pmatrix} \right) \right)
\end{aligned}$$

Note that, without loss of generality, we can assume $\sigma_4 = \mathrm{Id}$. (This goes back to the choice of two fixed embeddings of $\overline{\mathbb{Q}}$ in $\mathbb{C}$ and $\mathbb{C}_p$). We hope that this also clarifies our choice of the weights in $\mathcal{W}$.

We define by Iw the Iwahori subgroup of G defined as the counter image of the Borel of $G(\mathcal{O}/p)$ (which is quasi-split as every algebraic group over a finite field) and by $\Gamma_1(p)$ the counter image of the corresponding unipotent. More generally, we denote by $\Gamma_0(p^l)$ (resp. $\Gamma_1(p^l)$) the subgroup of $G(\mathbb{Z}_p)$ whose elements, seen inside $G(\mathcal{O}_F)$ (which is $\cong \prod_{i=1}^{f/2} \mathrm{GL}_n(\mathcal{O}_F) \times \mathrm{GL}_n(\mathcal{O}_F)$ in case (U) and $\cong \prod_{i=1}^{f} \mathrm{GL}_n(\mathcal{O}_F) \times \mathrm{GL}_n(\mathcal{O}_F)$ in case (L)) reduce modulo p^l to a matrix in the Borel (resp. unipotent) subgroup of $P(\mathcal{O}_F/p^l)$.

Denote by $\mathrm{M}_\kappa\big(\mathcal{H} \cap \Gamma_1(p^l), \varepsilon, \mathcal{O}_K\big)$ the space of classical modular forms, defined over $\mathcal{O}_K$ (where K is our coefficient field), of weight κ, level $\mathcal{H} \times \Gamma_1(p^l)$ and transforming under the finite order character ε of $\Gamma_0(p^l)/\Gamma_1(p^l)$. We conclude with the following proposition:

Proposition 2.13. *We have a map*

$$\mathrm{M}_\kappa\left(\mathcal{H}\cap\Gamma_1(p^l),\varepsilon,\mathcal{O}_K\right)\otimes\mathcal{O}_K/p^m\to V_{m,l}^{N_{\mathrm{Ig}}}. \tag{2.1}$$

Proof. The Hodge–Tate map gives a natural map

$$\mathrm{M}_\kappa\left(\mathcal{H}\cap\Gamma_1(p^l),\varepsilon,\mathcal{O}_K\right)\otimes\mathcal{O}_K/p^m\to\mathrm{H}^0(\mathrm{Ig}_{m,l},\omega^\kappa)$$

given by restriction to the $\mathcal{O}/p^m$-points to the subgroup described above. Note that we have a linear map $l_{\mathrm{can}}:R[\kappa]\to\mathbb{A}^1$ which consists to evaluation at 1. This gives

$$l_{\mathrm{can}}:\mathrm{H}^0(\mathrm{Ig}_{m,l},\omega^\kappa)\to\mathrm{H}^0(\mathrm{Ig}_{m,l},\mathcal{O}_{\mathrm{Ig}_{m,l}})=V_{m,l}.$$

By the definition of $\Gamma_1(p^l)$ the image is invariant under the N_{Ig}. □

3. Hecke operators

As in [Hid02], we introduce two types of U_p's operators. Some representation-theoretic operators on the Igusa tower that shall be used to define an ordinary projector e_{GL}, and some geometric operators coming from the Shimura variety of Iwahori level for the projector e_G. We shall compare these two operators at the end of the section.

3.1. **Hecke operators on the Igusa tower.** We define GL-type Hecke operators following [Hid02, §2.6]. Let GL_m be the usual algebraic group, B_m the Borel of upper-triangular matrices, T_m its maximal torus and U_m the unipotent part. For $j=1,\dots,m$ we let α_j be the diagonal matrix $[1,\dots,1,p,\dots p]$ with j p's and we let t_j be the double coset $U_m(\mathbb{Z}_p)\alpha_jU_m(\mathbb{Z}_p)$. We can write as usual $t_j=\bigsqcup U_m(\mathbb{Z}_p)u$, where u varies in a set of representatives of the quotient $t_j/U_m(\mathbb{Z}_p)$. If f is a function on $\mathrm{GL}_m(\mathcal{O}/p^l\mathcal{O})$ invariant by the action of $B_m(\mathcal{O}/p^l\mathcal{O})$, we let t_j acts on f via the formula

$$f|t_j(\gamma)=\sum_u f(u.\gamma).$$

where $u.\gamma$ is defined as $\gamma^{-1}u^{-1}\gamma$ and the sum is independent of the choice of the set of representatives. If f is invariant only by $U_m(\mathcal{O}/p^l\mathcal{O})$ but is in the κ-eigenspace, for κ a character of $T_m(\mathcal{O}/p^l\mathcal{O})$, then $f|t_j$ is also well-defined and belongs to the same eigenspace for $T_m(\mathcal{O}/p^l\mathcal{O})$. We define $\mathbf{t}_m:=\prod_{j=1}^m t_j$. Note that $\prod_{j=1}^m\alpha_i$ contracts $U_m(\mathcal{O})$ into Id_m. Let A be a p-adically complete flat $\mathbb{Z}_p$-algebra, and let $K:=A[1/p]$. We define $R_A[\kappa]$ (resp. $F_K[\kappa]$) as the algebraic (resp. topological, in the sense of [Pil12, §3.1.2]) representation of $\mathrm{Res}^{\mathcal{O}}_{\mathbb{Z}_p}\mathrm{GL}_m$ (resp. of $\mathrm{GL}_m(\mathcal{O})$) of highest weight κ for $\mathrm{Res}^{\mathcal{O}}_{\mathbb{Z}_p}T$ (resp. $T_m(\mathcal{O})$). If κ is regular (resp. very regular) then $\mathbf{t}_m^{n!}$ converges to an operator e_m. The image $e_m(R_A[\kappa])$ (resp. $e_m(F_K[\kappa])$) is isomorphic, via the projection l_{can}, to the line generated by the highest weight vector (see [Pil12, Proposition 3.2, Proposition 3.3]).

We are ready to define the ordinary projector e_{GL}.

Definition 3.1. In case (U) we define

$$\mathbf{t}_{\mathrm{GL}}:=\prod_{i=0}^{f-1}\mathbf{t}_{a_{i+1}-a_i}.$$

and in case (L) we define

$$\mathbf{t}_{\mathrm{GL}}:=\prod_{i=0}^{f-1}\mathbf{t}_{a_{i+1}-a_i}\times\prod_{i=1}^{f}\mathbf{t}_{b_i-b_{i+1}}.$$

Since the Igusa tower is a cover of S_∞^{tor} with group $\prod_{i=0}^{f-1}\mathrm{GL}_{a_{i+1}-a_i}(\mathcal{O})$ in case (U) and $\prod_{i=0}^{f-1}\mathrm{GL}_{a_{i+1}-a_i}(\mathcal{O})\times\prod_{i=1}^{f}\mathrm{GL}_{b_i-b_{i+1}}(\mathcal{O})$ in case (L), we obtain an operator

$$\begin{aligned}\mathbf{t}_{\mathrm{GL}}\colon \tilde{V}&\to\tilde{V}\\ f(x,\gamma)&\mapsto f(x,\mathbf{t}_{\mathrm{GL}}(\gamma))\end{aligned}$$

Here x corresponds to a point of S_∞^{tor} and γ is a trivialisation as in the definition of the Igusa tower.

We have that $\mathbf{t}_{\mathrm{GL}}$ preserves $\tilde{V}[\kappa]$, for κ a character of T_{Ig}. Moreover, with the same reasoning as [Hid02, page 20], we see that it sends $V_{m,l+1}[\kappa]$ into $V_{m,l}[\kappa]$. In case (U), whenever it is well-defined (it is necessary to know that the module spanned by $\mathbf{t}_{\mathrm{GL}}^n(f)$ is of finite rank for all $f\in V_{m,l}$) we define

$$e_{\mathrm{GL}}:=\lim_{n_0,\dots,n_{f-1}}\prod_{i=0}^{f-1}\mathbf{t}_{a_{i+1}-a_i}^{n_i!}.$$

Similarly for case (L).

3.2. **Hecke operators on the Shimura variety.** In this section only we will work over the rigid fiber of our varieties, without changing the notation. We recall the definition of the correspondences C_i used to define the $\mathrm{U}_{p,i}$ operators from [Bij16, §2.3]. He defines correspondences over (the rigid fiber of) X_{Iw}.

In case (L), let $1\le i\le n-1$. The correspondence C_i parametrises points $(A,\lambda,\iota,\alpha,(H_{i,\bullet})_i)$ of X_{Iw} with in addition an $\mathcal{O}$-stable subgroup $L=L^+\oplus L^-$ for which $H_i\oplus L^+=A[\pi^+]$. By Remark 1.5 it is enough to specify L^+. We have two projections $p_1,p_2:C_i\to X_{\mathrm{Iw}}$ corresponding to forgetting L and taking the quotient by L. The universal isogeny $\mathcal{A}\to\mathcal{A}/L$ induces an isomorphism $p^*(\kappa):p_2^*\omega^\kappa\to p_2^*\omega^\kappa$. One defines $\tilde{\mathrm{U}}_{p,i}$ as the composition

$$\mathrm{H}^0(X_{\mathrm{Iw}},\omega^\kappa)\to\mathrm{H}^0(C_i,p_2^*\omega^\kappa)\xrightarrow{p^*(\kappa)}\mathrm{H}^0(C_i,p_1^*\omega^\kappa)\xrightarrow{\mathrm{Tr}(p_1)}\mathrm{H}^0(X_{\mathrm{Iw}},\omega^\kappa).$$

Define moreover:

$$\begin{aligned}N_i&=\sum_j\max(a_j-i,0)\kappa_{\sigma_j,a_j}+\max(j-a_j,0)\kappa_{\bar{\sigma}_j,b_j},\\ n_i&=\sum_j\min(a_j,j)\min(n-i,b_j).\end{aligned}$$

The number N_i comes from $p^*(\kappa)$ while n_i is an inseparability degree.

Definition 3.2. We define $\mathrm{U}_{p,i}:=p^{-N_i-n_i}\tilde{\mathrm{U}}_{p,i}$; it is well defined over $\mathcal{O}_K$ (see Remark 3.5).

In case (U) let $1\le i\le n/2$. If $i<n/2$ the correspondence C_i parametrises points $(A,\lambda,\iota,\alpha,(H_{i,\bullet})_i)$ of X_{Iw} with in addition an $\mathcal{O}$-stable subgroup L of $\mathcal{A}[p^2]$ such that $H_i\oplus L[p]=A[p]={H_i}^\perp\oplus pL$. If $i=n/2$ one considers L such that $H_i\oplus L=A[p]$.

Define:

$$\begin{aligned}N_i&=\sum_j\max(a_j-i,0)\kappa_{\sigma_j,a_j}+\max(b_j-i,b_j-a_j)\kappa_{\bar{\sigma}_j,b_j}\ \text{ if } i<n/2,\\ n_i&=n\sum_j\min(a_j,i)\ \text{ if } i<n/2,\\ N_i&=\sum_j\frac{b_j-a_j}{2}\kappa_{\bar{\sigma}_j,b_j}\ \text{ if } i=n/2,\\ n_i&=\frac{n}{2}\sum_j a_j\ \text{ if } i=n/2.\end{aligned}$$

The definitions of $\tilde{\mathrm{U}}_{p,i}$ and $\mathrm{U}_{p,i}$ are as before.

3.3. **A comparison.** We now want to extend the action of the operators $\mathrm{U}_{p,i}$ to the Igusa tower. The idea is that the filtration defined by Proposition 1.10 will give a section to X_{Iw}.

The first thing to do is to interpret these operators from a linear algebra perspective. Let $k = \overline{\mathbb{F}}_p$. We can identify the Dieudonné module of $\mathcal{A}^{\mu}_{/k}$ with the Tate module of $\mathcal{A}^{\mu}_{/\mathrm{Frac}(W(k))}$. We recall that the Tate module has rank $2dn$ over $\mathbb{Z}_p$, and that $f = d$ in case (L) and $f = 2d$ in case (U).

The Hecke operators in case (L) are the same as the ones of [Hid02, §6.5]. We let $\langle \overline{x}_1, \dots, \overline{x}_n \rangle$ be an $\mathcal{O}$-basis for $\mathcal{A}^{\mu}_{/\mathrm{Frac}(W(k))}[\pi^+]$ and $\langle \overline{x}_{n+1}, \dots, \overline{x}_{2n} \rangle$ be an $\mathcal{O}$-basis for $\mathcal{A}^{\mu}_{/\mathrm{Frac}(W(k))}[\pi^-]$. We say that the basis is adapted to L_i if L_i can be written as

$$L_i = \langle \overline{x}_{i+1}, \dots, \overline{x}_n, \overline{x}_{2n-i}, \dots, \overline{x}_{2n} \rangle.$$

The $\mathrm{U}_{p,i}$ operator can be interpreted as the double quotient operator $\mathrm{Iw}\beta_i\mathrm{Iw}$. Let us describe the corresponding matrix β_i: if $i > a_j$ the j-th component of β_i is $([p, \dots, p], [1, \dots, 1, p, \dots, p])$ where p appears a_j times in the first matrix and $i - a_j$ times in the second; if $a_j \geq i$ then it is $([1, \dots, 1, p, \dots, p], [1, \dots, 1])$ where p appears i times. The number n_i defined before coincides with the index defined by Hida, that he calculates explicitly in [Hid02, (6.12)].

In case (U), fix a basis $\langle x_1, \dots, x_n \rangle$ of the Tate module as $\mathcal{O}$-module adapted to L_i, and let $\langle \overline{x}_1, \dots, \overline{x}_n \rangle$ be the corresponding basis for $\mathcal{A}^{\mu}_{/\mathrm{Frac}(W(k))}[p^2]$. This means that $L_i \subset \mathcal{A}^{\mu}_{/\mathrm{Frac}(W(k))}[p^2]$ is given by

$$\langle p\overline{x}_{i+1}, \dots, p\overline{x}_{n-i}, \overline{x}_{n-i+1}, \dots, \overline{x}_n \rangle$$

if $i < n/2$. If $i = n/2$ then $L = \langle p\overline{x}_{n/2}, \dots, p\overline{x}_n \rangle \subset \mathcal{A}^{\mu}_{/\mathrm{Frac}(W(k))}[p]$. We can define for each L_i an adelic matrix β_i. If $i < n/2$ and $a_j \geq i$ then

$$\beta_i = [1, \dots, 1, p, \dots, p, p^2, \dots, p^2]$$

with i 1's and p^2's, and $n - 2i$ p's. At places where $i > a_j$ we have

$$\beta_i = [1, \dots, 1, p, \dots, p, p^2, \dots, p^2]$$

where 1 appears a_j times and p appears $n - 2a_j = b_j - a_j$ times. If $i = n/2$ then $\beta_i = [1, \dots, 1, p, \dots, p]$ with $n/2$ 1's and p's.

Remark 3.3. Note that the number n_i is the sum over all places of F_0 of the numbers defined in [Hid02, (6.10)]. This is not really clear at first sight so let's calculate them. Suppose $i \neq n/2$. Build a $a_j \times b_j$ table and put a $+1$ in the rows from $i+1$ to a_j (accounts for the multiplication by β_i), then add a -2 in the columns from $b_j - i + 1$ till b_j and a -1 in the column from $b_j - (a_j + b_j - 2i + i) = i - a_j$ till $b_j - i - 1$ (these account for the multiplication by β_i^{-1}).

If $a_j \leq i$ then there are no $+1$, -1 appears $a_j(b_j - i - 1 - (i - a_j - 1))) = a_j(n - 2i)$ times and -2 appears $a_j(b_j - (b_j - i + 1 - 1)) = a_j i$ times. Sum up the opposite to get $a_j(n - 2i) + 2a_j i = na_j$. Similar if $a_j > i$ or $i = n/2$.

Concerning the explanation for N_i, note that the matrix β_i is sent diagonally into

$$G(\mathcal{O} \otimes_{\mathbb{Z}_p} \mathcal{O}) \cong \prod_{j=1}^{d} (\mathrm{GL}_{a_j} \times \mathrm{GL}_{b_j}) \times (\mathrm{GL}_{b_j} \times \mathrm{GL}_{a_j})(\mathcal{O}) \tag{3.1}$$

and only the first factors of the two couples contribute (because they are the ones appearing in P). (So, for example in case (U), to justify the appearance of $\max(b_j - i, b_j - a_j)$, note that it is $b_j - i = (a_j - i) + (b_j - a_j)$ if $a_j \geq i$ and otherwise it is $b_j - a_j$, which is exactly the number of times that p appears in the two upper blocks $(\mathrm{GL}_{a_j} \times 1_{b_j}) \times (\mathrm{GL}_{b_j} \times 1_{a_j})$.)

In both case, one can define an operator $\mathbf{T}_i$ on $V_{m,l}[\kappa]$, similarly to [Hid02, page 36]. For each point $(\mathcal{A}_x, \gamma)$ where $\mathcal{A}_x$ is the μ-ordinary abelian variety corresponding to x in S_m and γ a class in $\mathrm{Gal}(\mathrm{Ig}_{m,l}/S_m^{\mathrm{tor}})/N_{\mathrm{Ig}}(\mathcal{O}/p^l)$ and for each isogeny $\xi : \mathcal{A}_x \to \mathcal{A}_x/L_x$ one defines a point $(\mathcal{A}_x/L_x, \gamma_{L_x})$ in $\mathrm{Ig}_{m,l}$.

Definition 3.4. We define the geometric operator $\mathbf{T}_i$ as follows:

$$\mathbf{T}_i f(x,\gamma) = p^{-n_i} \sum_{L_x \in p_2 p_1^{-1}(x)} f(\mathcal{A}_x/L_x, \gamma_{L_x}).$$

Remark 3.5. Note that this expression is well defined as the naive operators are divisible by n_i by [Hid02, Lemma 6.6].

We define $\mathbf{T} := \prod_{i=1}^{n/2} \mathbf{T}_i$. Assuming it converges, we can hence define $e_G = \lim_n \mathbf{T}^{n!}$. From the matrix description and the decomposition in (3.1) it is clear that the projector e_{GL} is contained in e_G. It is clear that we have the relation $\mathbf{T}_i = \mathrm{U}_{p,i}$ (after composition with (2.1)) hence the operators of the previous section are well defined.

To conclude, we make the following observation.

Lemma 3.6. *Let $\mathcal{A}^\mu$ be the μ-ordinary universal abelian variety over $\overline{\mathbb{F}}_p$. For every L_i we have $\mathrm{Ha}^\mu(\mathcal{A}^\mu) = \mathrm{Ha}^\mu(\mathcal{A}^\mu/L_i)$. In particular*

$$\mathrm{U}_{p,i}(\mathrm{Ha}^\mu f) = \mathrm{Ha}^\mu \mathrm{U}_{p,i}(f)$$

and moreover we can normalise Ha^μ so that $\mathrm{Ha}^\mu \equiv 1$ on S^{tor}.

Proof. Using the Dieudonné functor, we see that the quotient $\mathcal{A}^\mu \to \mathcal{A}^\mu/L_i$ corresponds in case (U) to the sub-lattice

$$M' := \langle x_1, \dots, x_i, px_{i+1}, \dots, px_{n-i}, p^2 x_{n-i+1}, \dots, p^2 x_n \rangle$$

of $M := \langle x_1, \dots, x_n \rangle$. By the definition of the Hasse invariant, we are calculating the determinant of $\bigwedge^{a_i} \mathrm{Fr}^f$, suitably normalised. Moreover the filtration on M' is induced by the one of M, hence the determinant is the same. This tells us also that Ha^μ is constant on S^{tor}.

The same holds for case (L). □

3.4. **Prime-to-p Hecke operators.** We define now the Hecke algebra. Let N be a prime-to-p integer containing all prime numbers which are norms of prime ramified in F or for which G is not quasi-split and let $\mathfrak{l}$ be a prime ideal of F_0 above $l \nmid pN$, we define

$$\mathbb{T}_{G,\mathfrak{l}} = \mathbb{Z}_p[G(\mathcal{O}_{F_{0,\mathfrak{l}}}) \smallsetminus G(F_{0,\mathfrak{l}})/G(\mathcal{O}_{F_{0,\mathfrak{l}}})].$$

If $\mathfrak{l}$ is split in F, then $G(F_{0,\mathfrak{l}}) \cong \mathrm{GL}_{a+b}(F_{0,\mathfrak{l}}) \times \mathbb{G}_m(\mathbb{Q}_l)$ and $\mathbb{T}_{G,\mathfrak{l}}$ is generated by the same matrices. If $\mathfrak{l}$ is inert, $G(F_{0,\mathfrak{l}})$ is contained in $\mathrm{GL}_{a+b}(F_{\mathfrak{l}})$ and generated by the same diagonal matrices.

The abstract Hecke algebra of prime-to-Np level is then

$$\mathbb{T}_G^{(Np)} = \bigotimes_{\mathfrak{l} \nmid Np}' \mathbb{T}_{G,\mathfrak{l}}.$$

It acts naturally on the space of p-adic modular forms.

4. Hida theory

In this section we do not suppose that p is inert in F_0. The p-torsion of $\mathcal{A}^\mu$ decompose accordingly to the decomposition of p in F_0, and for each π_i dividing p we can define an Igusa tower Ig_{π_i}, a weight space $\mathcal{W}_{\pi_i}$ etc. as in Section 2, and we denote by

$$\mathrm{Ig} = \mathrm{Ig}_{\pi_1} \times_{S^{\mathrm{tor}}} \mathrm{Ig}_{\pi_2} \times \cdots \times_{S^{\mathrm{tor}}} \mathrm{Ig}_{\pi_k},$$

and the same for $\mathcal{W}$.

We also have the two projectors e_{G,π_i} and e_{GL,π_i}. We set

$$e_G = \prod_{\pi_i|p \text{ in } F_0} e_{G,\pi_i}, \quad e_{\mathrm{GL}} = \prod_{\pi_i|p \text{ in } F_0} e_{\mathrm{GL},\pi_i}.$$

We are almost ready to develop Hida theory but first we need some propositions. Recall that D is the divisor of $X^{\mathrm{tor}} \smallsetminus X$ and $\pi : X^{\mathrm{tor}} \to X^*$.

Proposition 4.1. *We have a short exact sequence:*

$$0 \to \mathrm{H}^0(S^*, \pi_*(\omega^\kappa(-D))) \overset{\times p^m}{\to} \mathrm{H}^0(S^*, \pi_*(\omega^\kappa(-D)) \to \mathrm{H}^0(S^*_m, \pi_*(\omega^\kappa(-D))) \to 0.$$

Proof. We use the q-expansion as described in [BR17, Proposition 1.3] at a point $x \in X^*$:

$$\prod_h \mathrm{H}^0(\mathcal{Z}, \mathcal{L}(h) \otimes \omega^k(-D))^{\Gamma(h)},$$

where h runs over the element of maximal rank in the abelian part of the unipotent for G, $\mathcal{Z}$ is isogenous to a power of a universal abelian variety whose dimension depends on x and $\Gamma(h)$ is the stabiliser of h inside $\mathcal{H}$. As we have chosen a neat level for X and h has maximal rank, then all stabilisers $\Gamma(h)$ are trivial. We are left to use the exact sequence of sheaves

$$0 \to \mathcal{L}(h) \otimes \omega^k(-D) \overset{\times p^m}{\to} L(h) \otimes \omega^k(-D) \to \mathcal{L}(h) \otimes \omega^k(-D)_m \to 0.$$

Using Mumford vanishing theorem [Mum70, §III.16] for the H^1 of $\mathcal{L}(h)$ (remember that h has maximal rank) we can conclude. □

Remark 4.2. An alternative proof can be made, using the fact that $R^1\pi_*\mathcal{O}_{X^{\mathrm{tor}}(-D)} = 0$ and that S^* is affine.

The proposition can fail whenever one consider non-cuspidal forms. This is because there are some restrictive conditions on the weight to have non-cuspidal forms [BR17, Theorem 1.12] in characteristic zero, while the same is not true modulo p (imagine a weight which is not parallel but it becomes modulo p). Nevertheless, one can hope to develop Hida theory after restricting to weights for which there is no *a priori* obstruction to the existence of Eisenstein series. The interested reader is referred to [LR17] where this strategy has been exploited to construct non-cuspidal Hida families of Siegel modular forms.

Proposition 4.3. *If κ is big enough, then for all $f \in \mathrm{H}^0(X_{\mathrm{Iw}}, \omega^\kappa)$ we have*

$$\prod_i \mathrm{U}_{p,i} f(x) \equiv 0 \bmod p$$

for all $x \in X_{\mathrm{Iw}} \smallsetminus S$.

Proof. We mimic the proof for the Siegel case, as given in [Pil12, Proposition A.5]. Suppose not all the a_j's are equal (otherwise we have a non-empty ordinary locus). Let NP be the μ-ordinary Newton polygon and NP_j the Newton polygon which is above N only at the point from a_j to $a_{j'}$ (where $a_j = \cdots = a_{j'}$) (like in [Her16a, Figure 7], but we admit that more consecutive slopes could be the same).

Let $\mathcal{N}_j$ be the corresponding Newton stratum and x any point in it. The corresponding p-divisible group $\mathcal{A}_x[p^\infty]$ admits a filtration $\{\mathcal{A}_l\}$ for $l = 0, \ldots, j-1, j'+1, f+1$.

As the Newton polygon NP_j is above NP, the slopes on $H := \mathcal{A}_{j'+1}/\mathcal{A}_{j-1}$ are bigger then the ones of the $j'+1$ filtered piece of $\mathcal{A}^\mu$. We need to understand (a bit) what is the matrix M of first filtered piece of the Dieudonné module of $\mathcal{A}_x/L$ w.r.t the first filtration of the Dieudonné module of $\mathcal{A}_x$, for a subgroup that appear in the correspondence C_i for a certain i such that $L \cap H \neq 0$. In this case the matrix

M is gonna differ, at places σ_k with $j \leq k \leq j'$ from the k-th component of β_i, the matrix defined in Section 3.3 at least in one entry, where a 1 is substituted by a p^ε, with $0 < \varepsilon < 1$ (this number is related to the σ_k degree of H, in the sense of Fargues [Far11]).

A similar calculation to the one that led to the choice of the number N_i in [Bij16, Definition 2.4, 2.9] tells us that the image of the map $p^*(\kappa)$ is gonna be contained inside $p^{M_i(x)} e^* \Omega^1 \mathcal{A}_x / L / W(k)$ where

$$M_i(x) = N_i + n_i + \sum_k \varepsilon_k \, \kappa_{\sigma_k, a_k},$$

with at least one $\varepsilon_k > 0$. If $\kappa_{\sigma_k, a_k} > 1/\varepsilon_k$ then $M_i - N_i - n_i \geq 1$, *i.e.* $p \mid U_{p,i} f(x)$. As the reunion of $\mathcal{N}_j$ over the $j's$ is dense in $X \smallsetminus S$ [Ham15, Theorem 1.1] we can conclude. □

We conclude with the following very important theorem.

Theorem 4.4. *We have that*

$$\dim(\mathrm{H}^0((X^{\mathrm{tor}}_{\mathrm{Iw}})_1, \omega^\kappa(-D))^{\mathrm{ord}}) \text{ and } \mathrm{rk}(\mathrm{H}^0(X^{\mathrm{tor}}_{\mathrm{Iw}}, \omega^\kappa(-D))^{\mathrm{ord}})$$

are bounded independently of κ.

Proof. First note that if f is ordinary then $\mathrm{Ha}^\mu f$ is ordinary too by Lemma 3.6. If the weight is big enough then by Proposition 4.3 $\mathrm{U}_p f$ vanishes outside the ordinary locus, and so does f; thank to Theorem A.2 we can choose Ha^μ so that it has simple zeros and hence we can divide f by Ha^μ. Multiplication by powers of the Hasse invariant induces

$$\mathrm{H}^0((X^{\mathrm{tor}}_{\mathrm{Iw}})_1, \omega_1^\kappa)^{\mathrm{ord}} \xrightarrow{\cong} \mathrm{H}^0((X^{\mathrm{tor}}_{Iw})_1, \omega_1^{\kappa'})^{\mathrm{ord}}.$$

The same proof tells us that

$$\begin{aligned} \mathrm{H}^0(S^{\mathrm{tor}}_1, \omega^\kappa(-D))^{\mathrm{ord}} &\xrightarrow{\cong} \bigcup_r \frac{\mathrm{H}^0(X^{\mathrm{tor}}_{\mathrm{Iw}}, \omega^{\kappa' + rw(\mathrm{Ha}^\mu)}(-D)) \otimes \mathcal{O}/p}{\mathrm{Ha}^{\mu r}}^{\mathrm{ord}} \\ &\xrightarrow{\cong} \mathrm{H}^0((X^{\mathrm{tor}}_{\mathrm{Iw}})_1, \omega_1^\kappa(-D))^{\mathrm{ord}}. \end{aligned}$$

where we used Proposition 4.1, *i.e.* the surjectivity of the reduction modulo p for cusp forms. This implies that the space of ordinary modular forms modulo p of fixed weight κ is bounded, and the above isomorphism between different weights implies that the bound is independent of κ. Using again Proposition 4.1 one sees that the same holds for $\mathrm{H}^0(X^{\mathrm{tor}}_{\mathrm{Iw}}, \omega^\kappa(-D))^{\mathrm{ord}}$. □

Remark 4.5. Note that this is the only point where we need that the Hasse invariant has simple zeroes.

We now define a space of p-adic modular forms we can control:

$$V^{\mathrm{cusp}} := \varinjlim_{m,l,\kappa} \mathrm{H}^0(X^{\mathrm{tor}}_m(p^l), \omega^\kappa(-D)),$$

where the limit is seen inside $\tilde{V}$ via the map (2.1). By slight abuse of notation, we shall write $V^{\mathrm{cusp}}_{m,l}$ for $\varinjlim_\kappa \mathrm{H}^0(X^{\mathrm{tor}}_m(p^l), \omega^\kappa(-D))$.

Proposition 4.6. *The ordinary projector* e_{GL} *and* e_G *are well-defined on* V^{cusp}.

Proof. The only thing to check is that for every element f of V^{cusp} the span of $\mathbf{T}_i^n f$ is of finite rank. But by definition f is the reduction modulo p^m of a classical forms of weight κ and level $\Gamma_1(p^l)$ for which this is clear. This show that e_G converges and the same is true for e_{GL} (because it is contained in e_G). □

We recall that Λ is the semi-local Iwasawa algebra we associated with $\mathcal{W}$ and Λ° the local ring we associated with $\mathcal{W}^\circ$. We say that a character τ is arithmetic if it can be written as $\varepsilon\kappa$ with $\kappa \in \mathcal{W}(\mathbb{C}_p)$ algebraic and a finite order part $\varepsilon \in \mathrm{Hom}_{\mathrm{cont}}(\Lambda^*, \mathbb{C}_p^\times)^{\mathrm{tors}}$.

We can now apply Hida machinery and we get the main theorem:

Theorem 4.7. *We have constructed:*

(1) *An ordinary projector $e_G = e_G^2$ on V^{cusp} such that the Pontryagin dual of its ordinary part*

$$V^{\mathrm{cusp},*,\mathrm{ord}} := \mathrm{Hom}_{\mathbb{Z}_p}\left(V^{\mathrm{cusp},\mathrm{ord}}, \mathbb{Q}_p/\mathbb{Z}_p\right)$$

(which is naturally a Λ-module) is finite free over Λ°.

(2) *The Λ-module of Hida families $\mathcal{M} := \mathrm{Hom}_\Lambda\left(V^{\mathrm{cusp},*,\mathrm{ord}}, \Lambda\right)$, which is of finite type over Λ.*

(3) *Given an arithmetic weight τ with dominant algebraic part $\kappa \in \mathcal{W}$ and finite order part $\varepsilon \in \mathrm{Hom}_{\mathrm{cont}}(\Lambda^*, \mathbb{C}_p^\times)$, let $\mathcal{P}_\tau$ be the corresponding prime ideal of Λ. Then*

$$\mathcal{M} \otimes /\mathcal{P}_\tau \xrightarrow{\sim} \varprojlim_m \varinjlim_l V_{m,l}^{\mathrm{cusp},\mathrm{ord}}[\tau],$$

and, if κ is very regular, combining it with (2.1) *gives*

$$\varinjlim_l \mathrm{S}\left(\mathcal{H} \cap \Gamma_1(p^l), \varepsilon\right)[1/p]^{\mathrm{ord}} \xrightarrow{\sim} (\mathcal{M} \otimes \Lambda/\mathcal{P}_\tau)[1/p].$$

Here the maps are equivariant under the action of the unramified Hecke algebra away from Np and the $\mathbb{U}_p$-operators.

Proof. We need to check that all the hypotheses of [Hid02] are satisfied. (Hyp1) is verified by Proposition 4.1 and (Hyp2) is satisfied by the, maybe artificial, definition of V^{cusp} and (Hyp1) (note that it holds trivially for $\tilde{V}$ because these are just section of the structural sheaf); hence we can define the projectors e_G and e_{GL}. Moreover (C) holds by Lemma 3.6 and as already seen this implies that (F) holds (see Theorem 4.4). As Λ is a product of local rings we can use (F) and Nakayama's Lemma to see that $V^{\mathrm{cusp},*,\mathrm{ord}}$ is of finite type, and moreover free over Λ° [Pil12, Théorème]. So we have Hida theory, *i.e.* the first two points of the theorem.

What is left is to show that the last map is an isomorphism, *i.e.* the control theorem. We have the diagram

$$\begin{tikzcd} R[\kappa] \arrow[r] \arrow[d, "l_{\mathrm{can}}"] & F[\kappa] \arrow[d, "l_{\mathrm{can}}"] \\ W(k)[\kappa] \arrow[r] & W(k)[\kappa], \end{tikzcd}$$

where $R[\kappa]$ is the algebraic representation corresponding to ω^κ, $F[\kappa]$ is the topological representation obtained restricting $R[\kappa]$ to $\mathrm{Gal}(\mathrm{Ig}/S^{\mathrm{tor}})$, and $W(k)[\kappa]$ is a one dimensional module where T_{Ig} acts via κ. Now, as already said [Pil12, Proposition 3.3], whenever κ is very regular the projector e_{GL} makes the map l_{can} into an isomorphism, and the same on the left (in which case it is enough for the weight to be dominant). Hence the map on the top is also an isomorphism when restricted to the ordinary part. But this was the map used to define (2.1), so this map restricted to the ordinary part is an isomorphism. □

Appendix A. Reduced μ-ordinary Hasse invariants

In [Her16a, Proposition 9.17] Hernandez shows that in certain cases (namely, when the ordinary locus is not empty or when the a_i are all distinct) his Hasse invariants are reduced. This is not always the case as in Remark 9.23 of *loc. cit.* he gives examples when they are not reduced. Let us recall the example: suppose that there is only one prime above p in F and suppose that $a_j = a_{j+1}$. Then we have

$$\mathrm{Fr}^1 : M_{\sigma_j} \xrightarrow{\sim} M_{\sigma_{j+1}},$$
$$\mathrm{Fr}^{f-1} : M_{\sigma_{j+1}} \xrightarrow{\sim} M_{\sigma_j}.$$

If we denote by ξ (resp. η) the determinant of $\bigwedge^{a_j} \mathrm{Fr}^{\alpha}/p^{c_j}$ (resp. $\bigwedge^{a_j} \mathrm{Fr}^{f-\alpha}/p^{c_j}$) modulo Fil^1, then

$$\mathrm{Ha}^{\mu,\sigma_j} = \xi \otimes \eta^p, \quad \mathrm{Ha}^{\mu,\sigma_{j+1}} = \eta \otimes \xi^{p^{f-1}}.$$

The idea is to consider separately ξ and η (or more of such if more consecutive a_j's are equal) to have partial Hasse invariants *à la* Goren [Gor01].

We suppose for an instant that p is inert in F_0 and let $\mathcal{O}$ be F_p in case (U) and $F_{0,p}$ in case (L). We denote by f the degree of $\mathcal{O}$. If we are in case (U), recall that we defined $a_{f/2+i} = b_{f/2+1-i}$.

For all $1 \le j \le f$, let $\alpha \ge 0$ be such that $a_j = \ldots = a_{j+\alpha}$, we define

$$\mathrm{Ha}^{\mu,\sigma_{j+\beta}} := \frac{\bigwedge^{d_j} \mathrm{Fr}}{p^{c_\beta}} : \frac{\bigwedge^{d_j} M_{\sigma_{j+\beta}}}{\mathrm{Fil}^1(\bigwedge^{d_j} M_{\sigma_{j+\beta}})} \to \frac{\bigwedge^{d_j} M_{\sigma_{j+\beta+1}}}{\mathrm{Fil}^1(\bigwedge^{d_j} M_{\sigma_{j+\beta+1}})} \text{ for } 0 \le \beta \le \alpha - 1;$$
$$\mathrm{Ha}^{\mu,\sigma_{j+\alpha}} := \frac{\bigwedge^{d_j} \mathrm{Fr}^{f-\alpha}}{p^{c_\beta}} : \frac{\bigwedge^{d_j} M_{\sigma_{j+\alpha}}}{\mathrm{Fil}^1(\bigwedge^{d_j} M_{\sigma_{j+\alpha}})} \to \frac{\bigwedge^{d_j} M_{\sigma_j}}{\mathrm{Fil}^1(\bigwedge^{d_j} M_{\sigma_j})}$$

for $d_j = \dim(\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}^\mu)_{\sigma_j}/\mathrm{Fil}^1(\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}^\mu)_{\sigma_j})) = n - a_j$ and $c_j = \sum_{k=1}^{j-1} k(a_{k+1} - a_k)$.

Note that M is isomorphic to its Cartier dual, hence we get $M \cong \mathcal{D}(M)^{\mathrm{Fr}^{f/2}}$ (meaning the underling $\mathcal{O}$-module is the same but the action of σ on one side corresponds to the action of $\bar{\sigma}$ on the other).

Because of this we get

$$\frac{\bigwedge^{d_j} M_{\sigma_{j+\beta}}}{\mathrm{Fil}^1(\bigwedge^{d_j} M_{\sigma_{j+\beta}})} \cong \frac{\bigwedge^{d_j} \mathcal{D}(M)^{\mathrm{Fr}^{f/2}}_{\sigma_{j+\beta}}}{\mathrm{Fil}^1(\bigwedge^{d_j} M_{\sigma_{j+\beta}})}$$
$$\cong \frac{\bigwedge^{d_j} \mathcal{D}(M)_{\sigma_{f-(j+\beta)+1}}}{\mathrm{Fil}^1(\bigwedge^{d_j} M_{\sigma_{f-(j+\beta)+1}})}$$

and also

$$\mathrm{Fil}^1 M_{\sigma_i} \cong M_{\sigma_{f-i+1}}/\mathrm{Fil}^1 M_{\sigma_{f-i+1}}$$

This implies $\mathrm{Ha}^{\mu,\sigma_{j+\beta}} = \mathrm{Ha}^{\mu,\bar{\sigma}_{j+\beta}}$.

In case (L) instead we split $M = M^+ \oplus M^-$ and we repeat the same definitions with M^+ in place of M, *i.e.* we consider only the Dieudonné module of $\mathcal{A}^\mu[(\pi^+)^\infty]$.

Definition A.1. For all primes π_i we define $\mathrm{Ha}^\mu_{\pi_i} = \prod_{i=1}^d \mathrm{Ha}^{\mu,\sigma_i}$ and in general

$$\mathrm{Ha}^\mu := \prod_{i=1}^k \mathrm{Ha}^\mu_{\pi_i}.$$

Theorem A.2. *The divisor $V(\mathrm{Ha}^\mu)$ in S_1^* is reduced.*

Proof. It is enough to show that the same holds for S_1. Moreover, as everything splits according to the action of $\mathcal{O}_F \otimes \mathbb{Z}_p$, it is not harmful to work with a BT-with $\mathcal{O}$-structure, for $\mathcal{O}$ any of $\mathcal{O}_{F,\pi_i}$ (if π_i is inert in F/F_0) or $\mathcal{O}_{F_0,\pi_i}$ (if π_i splits).

As in Proposition 4.3 we consider a Newton stratum $\mathcal{N}_j$ of abelian varieties whose Newton polygon NP_j coincides with the μ-ordinary Newton polygon NP except at the breaking point a_j (and the dual point b_j). First of all notice that if $a_i \neq a_j$ then $\mathrm{Ha}^{\mu,\sigma_i}$ doesn't distinguish between NP_j and NP. So we have to analyse only $\mathrm{Ha}^{\mu,\sigma_i}$ for $j \leq i \leq j+\alpha$ with $\alpha \geq 0$ and $a_j = \ldots = a_{j+\alpha}$. Only for this theorem, we let $H := \mathcal{A}_{j+\alpha+1}/\mathcal{A}_{j-1}$, for $\{\mathcal{A}_k\}$ the filtration for the p-torsion of the universal abelian variety over $\mathcal{N}_j$. It is a BT with $\mathcal{O}$-structure of $\mathcal{O}$-height 2. We follow very closely the proof of [Her16a, Proposition 9.17]: we need to study the matrices of the Frobenius on the Dieudonné module M_H of H. We let $\{e_{1,i}, e_{2,i}\}_{i=1,\ldots,f}$ be a basis of M_H such that $\mathcal{O}$ acts via σ_i on $\{e_{1,i}, e_{2,i}\}$. We have

$$\mathrm{rk}_{\mathcal{O}}\mathrm{Fil}^1 M_{H,\sigma_i} = \begin{cases} 0 & \text{if } i < j \\ 1 & \text{if } j \leq i \leq j+\alpha \\ 2 & \text{if } i > j+\alpha \end{cases}.$$

By our choice of NP_j we know that the two slopes of Fr^f on $\{e_{1,i}, e_{2,i}\}$ (seen as a Fr^f crystal of rank two) are the same and that the dimension of H is $\alpha+1$. The Dieudonné–Manin classification of Dieudonné modules tells us that H is isogenous over $\mathbb{Q}_p^{\mathrm{ur}}$ to

$$M_{2,s} = \mathbb{Q}_p^{\mathrm{ur}}[\Phi]/(\Phi^2 - p^s),$$

if $s = 2(f-j-\alpha)+\alpha+1$ is odd, and

$$M_{1,s/2} \oplus M_{1,s/2} = \mathbb{Q}_p^{\mathrm{ur}}[\Phi]/(\Phi - p^{s/2}) \oplus \mathbb{Q}_p^{\mathrm{ur}}[\Phi]/(\Phi - p^{s/2}),$$

if s is even. Here Φ is Fr^f.

We are interested in the matrix of the isogeny $\mathrm{Fr} : M_{\sigma_j} \to M_{\sigma_{j+1}}$. It can only be $\begin{pmatrix} 0 & p \\ 1 & 0 \end{pmatrix}$ otherwise the slopes would be distinct.

We know that the deformation space of H has dimension $\alpha+1$ [Moo04, §2.1.4] and thanks to [Moo04, Proposition 2.1.8] the matrix of Fr on the the universal BT group with $\mathcal{O}$-structure of type as H is

$$\begin{pmatrix} X_1 & p \\ 1 & 0 \end{pmatrix}$$

and, similarly, for all $\beta \leq \alpha - 1$ the universal matrix for $\mathrm{Fr} : M_{\sigma_{j+\beta}} \to M_{\sigma_{j+\beta+1}}$ is

$$\begin{pmatrix} X_{\beta+1} & p \\ 1 & 0 \end{pmatrix}.$$

In particular if $\beta \neq \alpha$ we have

$$\begin{array}{rccc} \mathrm{Fr}: & M_{H,\sigma_{j+\beta}}/\mathrm{Fil}^1(M_{H,\sigma_{j+\beta}}) & \to & M_{H,\sigma_{j+\beta+1}}/\mathrm{Fil}^1(M_{H,\sigma_{j+\beta+1}}) \\ & e_{1,j+\beta} & \mapsto & (X_{\beta+1}e_{1,j+\beta+1} + e_{2,j+\beta+1})/e_{2,j+\beta+1} = X_{\beta+1}e_{1,j+\beta+1}. \end{array}$$

So these $\mathrm{Ha}^{\mu,\sigma_{j+\beta}}$ are non-zero on the tangent space at a point in $\mathcal{N}_j$, and their images are all independent. For $\mathrm{Ha}^{\mu,\sigma_{j+\alpha}}$ a similar calculation (here we have to take a power of Fr and divide by a power of p, and note that the remaining M_{H,σ_i} do not contribute to the tangent space as they are "rigid") gives that on the tangent space $\mathrm{Ha}^{\mu,\sigma_{j+\alpha}}$ is $X_{\alpha+1}$. In particular the product is a non-zero linear form on the tangent space. □

Remark A.3. Note that Lemma 3.6 is not affected if one uses this new Hasse invariant as the quotient of the lattice of $\mathcal{A}$ by the one of $\mathcal{A}/L_i$ is free over $\mathcal{O}/p \otimes_{\mathbb{F}_p} \mathcal{O}/p$.

Remark A.4. The Hasse invariant just defined is a product of modular forms of non-parallel weight $(0,\dots,-1,p,0,\dots)$ or $(0,\dots,p^{f-\alpha},0,\dots,0,-1,0,\dots)$ and it is non-cuspidal modulo p so in general one can not expect to lift (not even a power of it) to characteristic zero.

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