# Controllability of a $4 \times 4$ quadratic reaction-diffusion system 

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# Controllability of a $4 \times 4$ quadratic reaction-diffusion system 

Kévin Le Balc' $h^{1}$


#### Abstract

We consider a $4 \times 4$ nonlinear reaction-diffusion system posed on a smooth domain $\Omega$ of $\mathbb{R}^{N}$ ( $N \geq 1$ ) with controls localized in some arbitrary nonempty open subset $\omega$ of the domain $\Omega$. This system is a model for the evolution of concentrations in reversible chemical reactions. We prove the local exact controllability to stationary constant solutions of the underlying reaction-diffusion system for every $N \geq 1$ in any time $T>0$. A specificity of this control system is the existence of some invariant quantities in the nonlinear dynamics. The proof is based on a linearization which uses return method and an adequate change of variables that creates cross diffusion which will be used as coupling terms of second order. The controllability properties of the linearized system are deduced from Carleman estimates. A Kakutani's fixed-point argument enables to go back to the nonlinear parabolic system. Then, we prove a global controllability result in large time for $1 \leq N \leq 2$ thanks to our local controllabillity result together with a known theorem on the asymptotics of the free nonlinear reaction-diffusion system.


Keywords: Controllability to stationary states, parabolic system, nonlinear coupling, Carleman estimate, return method

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## 1. Introduction

Let $T>0, N \in \mathbb{N}^{*}, \Omega$ be a bounded, connected, open subset of $\mathbb{R}^{N}$ of class $C^{2}$ and let $\omega$ be a nonempty open subset of $\Omega$. The notation $Q:=(0, T) \times \Omega$ will be used throughout the paper.

### 1.1. Presentation of the nonlinear reaction-diffusion system

Let $\left(d_{1}, d_{2}, d_{3}, d_{4}\right) \in(0,+\infty)^{4}$. We are interested in the following reaction-diffusion system

$$
\forall 1 \leq i \leq 4, \begin{cases}\partial_{t} u_{i}-d_{i} \Delta u_{i}=(-1)^{i}\left(u_{1} u_{3}-u_{2} u_{4}\right) & \text { in }(0, T) \times \Omega,  \tag{1}\\ \frac{u_{i}}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega, \\ u_{i}(0, .)=u_{i, 0} & \text { in } \Omega,\end{cases}
$$

where $n$ is the outer unit normal vector to $\partial \Omega$. This system is a model for the evolution of the concentration $u_{i}(.,$.$) in the reversible chemical reaction$

$$
\begin{equation*}
U_{1}+U_{3} \rightleftharpoons U_{2}+U_{4} \tag{2}
\end{equation*}
$$

by using the law of mass action, Fick's law and the fact that no substance crosses the boundary (Neumann conditions). For this quadratic system, global existence of weak solutions holds in any dimension.

Proposition 1.1. [44, Proposition 5.12]
Let $u_{0} \in L^{2}(\Omega)^{4}, u_{0} \geq 0$. Then, there exists a global weak solution (in the sense of the definition [44, Section 5, (5.12)]) to (1).

For dimensions $N=1,2$, it was proved that the solutions are bounded and therefore classical for bounded initial data (see [21], [33] and [35]). It was not known until recently whether they were bounded in higher dimension (see [44, Section 7, Problem 3] and references therein for more details). But, two very recent preprints: [12] and [49] prove that these solutions are smooth.

### 1.2. The question

Let $\left(u_{1}^{*}, u_{2}^{*}, u_{3}^{*}, u_{4}^{*}\right) \in[0,+\infty)^{4}$ satisfying

$$
\begin{equation*}
u_{1}^{*} u_{3}^{*}=u_{2}^{*} u_{4}^{*} . \tag{3}
\end{equation*}
$$

We will say that $\left(u_{i}^{*}\right)_{1 \leq i \leq 4}$ is a stationary constant solution of (1).
Remark 1.2. The nonnegative stationary solutions of (1) are constant (see Proposition 6.1 in Appendix A). Thus, it is not restrictive to assume that $\left(u_{1}^{*}, u_{2}^{*}, u_{3}^{*}, u_{4}^{*}\right) \in[0,+\infty)^{4}$.

The question we ask is the following: Could one reach stationary constant solutions of (1) with localized controls in finite time? From a chemical viewpoint, we wonder whether one can act on the free reaction (2) by a localized external force to reach in finite time $T$ a particular steady state $\left(u_{i}^{*}\right)_{1 \leq i \leq 4}$. For instance, this force can be the addition or the removal of a chemical species in a specific location of the domain $\Omega$.

We introduce the notations:
$j \in\{1,2,3\}$ denotes the number of internal controls that we allow in the equations of (1),

$$
1_{i \leq j}:=1 \text { if } 1 \leq i \leq j \text { and } 0 \text { if } i>j .
$$

By symmetry of the system, we reduce our study to the case of controls entering in the first equations. Thus, we consider the following controlled system

$$
\forall 1 \leq i \leq 4, \begin{cases}\partial_{t} u_{i}-d_{i} \Delta u_{i}=(-1)^{i}\left(u_{1} u_{3}-u_{2} u_{4}\right)+h_{i} 1_{\omega} 1_{i \leq j} & \text { in }(0, T) \times \Omega,  \tag{4}\\ \frac{u_{i}}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega, \\ u_{i}(0, .)=u_{i, 0} & \text { in } \Omega .\end{cases}
$$

Here, $\left(u_{i}\right)_{1 \leq i \leq 4}(t,):. \Omega \rightarrow \mathbb{R}^{4}$ is the state to be controlled and $\left(h_{i}\right)_{1 \leq i \leq j}(t,):. \Omega \rightarrow \mathbb{R}^{j}$ is the control input supported in $\omega$. We are interested in the $L^{\infty}$-controllability properties of (4): For every $u_{0} \in L^{\infty}(\Omega)^{4}$, does there exist $\left(h_{i}\right)_{1 \leq i \leq j} \in L^{\infty}(Q)^{j}$ such that the solution $u$ of (4) satisfies

$$
\begin{equation*}
\forall i \in\{1,2,3,4\}, u_{i}(T, .)=u_{i}^{*} ? \tag{5}
\end{equation*}
$$

### 1.3. Two partial answers

Our first main outcome is a local controllability result in $L^{\infty}(\Omega)$ with controls in $L^{\infty}(Q)$ for (4), i.e. we will show that for every $1 \leq j \leq 3$, there exists $\delta>0$ such that for every $u_{0} \in X_{j,\left(d_{i}\right),\left(u_{i}^{*}\right)}$ (a " natural " subspace of $L^{\infty}(\Omega)^{4}$, see Section 3.1), with $\left\|u_{0}-u^{*}\right\|_{L^{\infty}(\Omega)^{4}} \leq \delta$, there exists $\left(h_{i}\right)_{1 \leq i \leq j} \in L^{\infty}(Q)^{j}$ such that the solution $u$ of (4) satisfies (5).

Our second main result is a global controllability result in $L^{\infty}(\Omega)$ with controls in $L^{\infty}(Q)$ for (4) in large time and in small dimension, i.e., we will prove that for every $1 \leq N \leq 2$, $1 \leq j \leq 3$, $u_{0} \in X_{j,\left(d_{i}\right),\left(u_{i}^{*}\right)}$ which verifies a positivity condition (see (42)), there exist $T^{*}$ sufficiently large and $\left(h_{i}\right)_{1 \leq i \leq j} \in L^{\infty}\left(\left(0, T^{*}\right) \times \Omega\right)^{j}$ such that the solution $u$ of (4) (replace $T$ with $T^{*}$ ) satisfies (5) (replace $T$ with $T^{*}$ ).

The precise results are stated in Section 3 (see Theorem 3.2 and Theorem 3.6).

### 1.4. Bibliographical comments for the null-controllability of parabolic systems with localized controls <br> Now, we discuss the null-controllability of parabolic coupled parabolic systems. The following results will be useful for having a proof strategy of our two main results.

Remark 1.3. We choose to present parabolic systems with Dirichlet conditions because these results are more easy to find in the literature. However, all the following results can be adapted to the Neumann conditions.

### 1.4.1. Linear parabolic systems

The problem of null-controllability of the heat equation was solved independently by Gilles Lebeau, Luc Robbiano in 1995 (see [39] or the survey [38]) and Andrei Fursikov, Oleg Imanuvilov in 1996 (see [29]) with Carleman estimates.

Theorem 1.4. [5, Corollary 2]
For every $u_{0} \in L^{2}(\Omega)$, there exists $h \in L^{2}(Q)$ such that the solution $u$ of

$$
\begin{cases}\partial_{t} u-\Delta u=h 1_{\omega} & \text { in }(0, T) \times \Omega, \\ u=0 & \text { on }(0, T) \times \partial \Omega, \\ u(0, .)=u_{0} & \text { in } \Omega,\end{cases}
$$

satisfies $u(T,)=$.0 .

Then, null-controllability of linear parabolic systems was studied. A typical example is

$$
\begin{cases}\partial_{t} u-D \Delta u=A u+B h 1_{\omega} & \text { in }(0, T) \times \Omega,  \tag{6}\\ u=0 & \text { on }(0, T) \times \partial \Omega, \\ u(0, .)=u_{0} & \text { in } \Omega,\end{cases}
$$

where $u \in C\left([0, T] ; L^{2}(\Omega)^{k}\right)$ is the state, $h \in L^{2}(Q)^{l}, 1 \leq l \leq k$, is the control, $D:=\operatorname{diag}\left(d_{1}, \ldots, d_{k}\right)$ with $d_{i} \in(0,+\infty)$ is the diffusion matrix, $A \in \mathcal{M}_{k}(\mathbb{R})$ (matrix with $k$ lines and $k$ columns with entries in $\mathbb{R}$ ) is the coupling matrix and $B \in \mathcal{M}_{k, l}(\mathbb{R}$ ) (matrix with $k$ lines and $l$ columns with entries in $\mathbb{R}$ ) represents the distribution of controls.
Definition 1.5. System (6) is said to be null-controllable if for every $u_{0} \in L^{2}(\Omega)^{k}$, there exists $h \in L^{2}(Q)^{l}$ such that the solution $u$ of (6) satisfies $u(T,)=$.0 .

The triplet ( $D, A, B$ ) plays an important role for null-controllabillity of (6) as the following theorem, proved by Farid Ammar-Khodja, Assia Benabdallah, Cédric Dupaix and Manuel Gonzalez-Burgos (which is a generalization of the well-known Kalman condition in finite dimension, see [15, Theorem 1.16]), shows us.

Theorem 1.6. [5, Theorem 5.6]
Let us denote by $\left(\lambda_{m}\right)_{m \geq 1}$ the sequence of positive eigenvalues of the unbounded operator $\left(-\Delta, H^{2}(\Omega) \cap H_{0}^{1}(\Omega)\right)$ on $L^{2}(\Omega)$. Then, the following conditions are equivalent.

1. System (6) is null-controllable.
2. For every $m \geq 1, \operatorname{rank}\left(\left(-\lambda_{m} D+A\right) \mid B\right)=k$, where

$$
\left(\left(-\lambda_{m} D+A\right) \mid B\right):=\left(B,\left(-\lambda_{m} D+A\right) B,\left(-\lambda_{m} D+A\right)^{2} B, \ldots,\left(-\lambda_{m} D+A\right)^{k-1} B\right) .
$$

For example, let us consider the $2 \times 2$ toy-system

$$
\begin{cases}\partial_{t} u_{1}-d_{1} \Delta u_{1}=a_{11} u_{1}+a_{12} u_{2}+h_{1} 1_{\omega} & \text { in }(0, T) \times \Omega,  \tag{7}\\ \partial_{t} u_{2}-d_{2} \Delta u_{2}=a_{21} u_{1}+a_{22} u_{2} & \text { in }(0, T) \times \Omega, \\ u=0 & \text { on }(0, T) \times \partial \Omega, \\ u(0, .)=u_{0} & \text { in } \Omega,\end{cases}
$$

where $a_{i, j} \in L^{\infty}(Q)$ for every $1 \leq i, j \leq 2$. We easily deduce from Theorem 1.6 the following proposition.

Proposition 1.7. We assume $a_{i j} \in \mathbb{R}$ for every $1 \leq i, j \leq 2$. The following conditions are equivalent.

1. System (7) is null-controllable.
2. $a_{21} \neq 0$.

Roughly speaking, $u_{1}$ can be driven to 0 thanks to the control $h_{1}$ and $u_{2}$ can be driven to 0 thanks to the coupling term $a_{21} u_{1}$. We have the following diagram

$$
h_{1} \stackrel{\text { controls }}{\rightsquigarrow \rightarrow} u_{1} \stackrel{\text { controls }}{\rightsquigarrow} u_{2} .
$$

We also have a more general result for the toy-model (7).
Proposition 1.8. [5, Theorem 7.1]
We assume that for every $1 \leq i, j \leq 2, a_{i j} \in L^{\infty}(Q)$ and there exist $t_{1}<t_{2} \in(0, T)$, a nonempty open subset $\omega_{0} \subset \omega$ and $\varepsilon>0$ such that for almost every $(t, x) \in\left(t_{1}, t_{2}\right) \times \omega_{0},\left|a_{21}(t, x)\right| \geq \varepsilon$. Then, system (7) is null-controllable.

Roughly speaking, if the coupling term $a_{21}$ lives somewhere in the control zone, then ( $u_{1}, u_{2}$ ) can be driven to $(0,0)$. The case where $\operatorname{supp}\left(a_{21}\right) \cap \omega=\emptyset$ is more difficult even if $a_{21}$ depends only on the spatial variable: a minimal time of control can appear (see [6] and [7]).

In order to reduce the number of controls entering in the equations of a linear parabolic system, a good strategy is to transform the system into a cascade system. This type of system has been studied by Manuel Gonzalez-Burgos and Luz de Teresa (see [31]). For example, let us consider the $3 \times 3$ toy system

$$
\begin{cases}\partial_{t} u_{1}-d_{1} \Delta u_{1}=a_{11} u_{1}+a_{12} u_{2}+a_{13} u_{3}+h_{1} 1_{\omega} & \text { in }(0, T) \times \Omega,  \tag{8}\\ \partial_{t} u_{2}-d_{2} \Delta u_{2}=a_{21} u_{1}+a_{22} u_{2}+a_{23} u_{3} & \text { in }(0, T) \times \Omega, \\ \partial_{t} u_{3}-d_{3} \Delta u_{3}= & a_{32} u_{2}+a_{33} u_{3} \\ u=0 & \text { in }(0, T) \times \Omega, \\ u(0, .)=u_{0} & \text { on }(0, T) \times \partial \Omega, \\ & \text { in } \Omega .\end{cases}
$$

where for every $1 \leq i, j \leq 3, a_{i j} \in L^{\infty}(Q)$.
Proposition 1.9. If there exist $t_{1}<t_{2} \in(0, T)$, a nonempty open subset $\omega_{0} \subset \omega$ and $\varepsilon>0$ such that for almost every $(t, x) \in\left(t_{1}, t_{2}\right) \times \omega_{0},\left|a_{21}(t, x)\right| \geq \varepsilon$ and $\left|a_{32}(t, x)\right| \geq \varepsilon$, then system (8) is null-controllable.

Roughly speaking, $u_{1}$ can be driven to 0 thanks to the control $h_{1}, u_{2}$ can be driven to 0 thanks to the coupling term $a_{21} u_{1}$ (which lives somewhere in the control zone) and $u_{3}$ can be driven to 0 thanks to the coupling term $a_{32} u_{2}$ (which lives somewhere in the control zone). Heuristically, we have the following diagram

$$
h_{1} \stackrel{\text { controls }}{\rightsquigarrow} u_{1} \stackrel{\text { controls }}{\rightsquigarrow>} u_{2} \stackrel{\text { controls }}{\rightsquigarrow} u_{3} \text {. }
$$

For more general results, see [4], [3], [2] and the survey [5, Sections 4, 5, 7].
We can also replace the coupling matrix $A$ in the system (6) by a differential operator of first order or second order. In this case, there exist some similar results (see [30], [10] with a technical assumption on $\omega$, [22], [23], [24]). For example, let us consider the particular case of the $2 \times 2$ system

$$
\begin{cases}\partial_{t} u_{1}-d_{1} \Delta u_{1}=g_{11} \cdot \nabla u_{1}+g_{12} \cdot \nabla u_{2}+a_{11} u_{1}+a_{12} u_{2}+h_{1} 1_{\omega} & \text { in }(0, T) \times \Omega,  \tag{9}\\ \partial_{t} u_{2}-d_{2} \Delta u_{2}=g_{21} \cdot \nabla u_{1}+g_{22} \cdot \nabla u_{2}+a_{21} u_{1}+a_{22} u_{2} & \text { in }(0, T) \times \Omega, \\ u=0 & \text { on }(0, T) \times \partial \Omega, \\ u(0, .)=u_{0} & \text { in } \Omega,\end{cases}
$$

where $a_{i j} \in \mathbb{R}, g_{i j} \in \mathbb{R}$ for every $1 \leq i, j \leq 2$. Then, system (9) is null-controllable if and only if $g_{21} \neq 0$ or $a_{21} \neq 0$. This result is due to Michel Duprez and Pierre Lissy (see [23, Theorem 1] and [50, Theorem 3.4] for a similar result). It is proved by a fictitious control method and algebraic solvability, introduced for the first time by Jean-Michel Coron in the context of stabilization of ordinary differential equations (see [14]). This type of method has also been used for NavierStokes equations by Jean-Michel Coron and Pierre Lissy in [19]. However, the situation is much more complicated and is not well-understood in the case where $a_{i j}, g_{i j}(1 \leq i, j \leq 2)$ depend on the spatial variable. One can see the surprising negative result of null-controllability: [24, Theorem 2]. When the matrix $A$ in (6) is a differential operator of second order (take $A=\widetilde{A} \Delta+C(t, x)$
with $(\widetilde{A}, C) \in \mathcal{M}_{k}(\mathbb{R}) \times L^{\infty}\left(Q ; \mathcal{M}_{k}(\mathbb{R})\right)$ to simplify), the coupling matrix $A$ disturbs the diagonal diffusion matrix $D$ and creates a new "cross" diffusion matrix: $\widetilde{D}=D-\widetilde{A}$. When $\widetilde{D}$ is not diagonalizable, there are few results (see [26] with a technical assumption on the dimension of the Jordan Blocks of $\widetilde{D}$ and the recent preprint [41, Section 3] when $C$ does not depend on time and space).

Let us also keep in mind the following result which help to understand our analysis.
Proposition 1.10. [34, Theorem 3], [26, Theorem 1.5]
Let $a_{11}, a_{12}, d \in \mathbb{R}$. Let us consider the $2 \times 2$ toy system

$$
\begin{cases}\partial_{t} u_{1}-d_{1} \Delta u_{1}=a_{11} u_{1}+a_{12} u_{2}+h_{1} 1_{\omega} & \text { in }(0, T) \times \Omega  \tag{10}\\ \partial_{t} u_{2}-d_{2} \Delta u_{2}=d \Delta u_{1} & \text { in }(0, T) \times \Omega \\ u=0 & \text { on }(0, T) \times \partial \Omega \\ u(0, .)=u_{0} & \text { in } \Omega\end{cases}
$$

Then, the following conditions are equivalent.

1. System (10) is null-controllable.
2. $d \neq 0$.

Roughly speaking, $u_{1}$ can be driven to 0 thanks to the control $h_{1}$ and $u_{2}$ can be driven to 0 thanks to the coupling term of second order $d \Delta u_{1}$.

Remark 1.11. When it is possible, one can diagonalize the matrix $\widetilde{D}=\left(\begin{array}{cc}d_{1} & 0 \\ d & d_{2}\end{array}\right)$. Then, by a linear transformation together with Theorem 1.6, one can prove Proposition 1.10. However, in this paper, we choose the opposite strategy. We transform (4) into a system like (10) (with four equations). Indeed, such a system seems to be a cascade system with coupling terms of second order.

### 1.4.2. Nonlinear parabolic systems

Then, another challenging issue is the study of the null-controllability properties of semilinear parabolic systems. The usual strategy consists in linearizing the system around 0 and to deduce local controllability properties of the nonlinear system by controllability properties of the linearized system and a fixed-point argument.

For example, let us consider the $2 \times 2$ model system

$$
\begin{cases}\partial_{t} u_{1}-d_{1} \Delta u_{1}=f_{1}\left(u_{1}, u_{2}\right)+h_{1} 1_{\omega} & \text { in }(0, T) \times \Omega  \tag{11}\\ \partial_{t} u_{2}-d_{2} \Delta u_{2}=f_{2}\left(u_{1}, u_{2}\right) & \text { in }(0, T) \times \Omega, \\ u=0 & \text { on }(0, T) \times \partial \Omega, \\ u(0, .)=u_{0} & \text { in } \Omega,\end{cases}
$$

where $f_{1}$ and $f_{2}$ belong to $C^{\infty}\left(\mathbb{R}^{2} ; \mathbb{R}\right)$. Then, the following result is a consequence of Proposition 1.7.

Proposition 1.12. Let us suppose that $\frac{\partial f_{2}}{\partial u_{1}}(0,0) \neq 0$. Then, there exists $\delta>0$ such that for every $u_{0} \in L^{\infty}(\Omega)^{2}$ which satisfies $\left\|u_{0}\right\|_{L^{\infty}(\Omega)^{2}} \leq \delta$, there exists $h_{1} \in L^{\infty}(Q)$ such that the solution $u$ of (11) verifies $u(T,)=$.0 .

Remark 1.13. This result is well-known but it is difficult to find it in the literature (see [1, Theorem 6] with a restriction on the dimension $1 \leq N<6$ and other function spaces or one can adapt the arguments given in [17] to get Proposition 1.12 for any $N \in \mathbb{N}^{*}$ ). For other results in this direction, see [51], [40], [32] and [13].

When $f_{2}$ does not satisfy the hypothesis of Proposition 1.12, another strategy consists in linearizing around a non trivial trajectory $\left(\overline{u_{1}}, \overline{u_{2}}, \overline{h_{1}}\right)$ of the nonlinear system which goes from 0 to 0 . This procedure is called the return method and was introduced by Jean-Michel Coron in [14] (see [15, Chapter 6]). This method conjugated with Proposition 1.8 gives the following result.

Proposition 1.14. We assume that there exist $t_{1}<t_{2} \in(0, T)$, a nonempty open subset $\omega_{0} \subset \omega$ and $\varepsilon>0$ such that $\left|\frac{\partial f_{2}}{\partial u_{1}}\left(\overline{u_{1}}, \overline{u_{2}}\right)\right| \geq \varepsilon$ on $\left(t_{1}, t_{2}\right) \times \omega_{0}$. Then, there exists $\delta>0$ such that for every $u_{0} \in L^{\infty}(\Omega)^{2}$ which satisfies $\left\|u_{0}\right\|_{L^{\infty}(\Omega)^{2}} \leq \delta$, there exists $h_{1} \in L^{\infty}(Q)$ such that the solution $u$ of (11) verifies $u(T,)=$.0 .

Proposition 1.14 is proved in [17] and used in [17] with $f_{2}\left(u_{1}, u_{2}\right)=u_{1}^{3}+R u_{2}$, where $R \in \mathbb{R}$, [16], [18] and [37].

Finally, Felipe Walison Chaves-Silva and Sergio Guerrero have studied the local controllability of the Keller-Segel system in which the nonlinearity involves derivative terms of order 2 (see [13]). Some ideas of [13] are exploited in our proof.

### 1.5. Proof strategy of the two main results

Let us return to the main question discussed in this paper (see Section 1.2) and the expected results as explained in Section 1.3.

The local controllability result is deduced from controllability properties of the linearized system around $\left(u_{i}^{*}\right)_{1 \leq i \leq 4}$ of (4). This strategy presents two main difficulties.

For the case of 3 controls (see Section 4.1.1), if $\left(u_{1}^{*}, u_{3}^{*}, u_{4}^{*}\right) \neq(0,0,0)$, the linearized system is controllable and consequently the nonlinear result comes from an adaptation of Proposition 1.12. If $\left(u_{1}^{*}, u_{3}^{*}, u_{4}^{*}\right)=(0,0,0)$, the linearized system is not controllable. Then, we use the return method to overcome this problem and the nonlinear result comes from an adaptation of Proposition 1.14.

For the case of 2 controls and 1 control, there exist some invariant quantities in the nonlinear system and consequently in the linearized system, that prevent controllability from happening in the whole space $L^{\infty}(\Omega)^{4}$. Therefore, we restrict the initial data to a "natural" subspace of $L^{\infty}(\Omega)^{4}$ (see Section 3.1). A modified version (for Neumann conditions) of Theorem 1.6 cannot be applied to the linearized system of (4) because the rank condition is never satisfied (due to the invariant quantities). An adequate change of variable gets over this difficulty by creating crossdiffusion and by using coupling matrices of second order (see Section 4.1.2 and Section 4.1.3). Then, we treat the controllability properties of the linearized system by adapting Proposition 1.9 and Proposition 1.10.

To summarize, we must require necessary conditions on the initial data. Consequently the local controllability result depends on: the coefficients $\left(d_{i}\right)_{1 \leq i \leq 4}$ (i.e. the diffusion matrix), the state $\left(u_{i}^{*}\right)_{1 \leq i \leq 4}$ (i.e. the coupling matrix of the linearized system of (4)), $j$ (i.e. the number of controls that we put in the equations).

The global controllability result is a corollary of our local controllability result and a result by Laurent Desvillettes, Klemens Fellner and Michel Pierre, Takashi Suzuki, Yoshio Yamada, Rong Zou concerning the asymptotics of the trajectory of (1) for $1 \leq N \leq 2$. Indeed, this known result claims that the solution $u(T,$.$) of (4) converges in L^{\infty}(\Omega)^{4}$ to a particular positive stationary solution $z$ of (1) when $T \rightarrow+\infty$ (see [21] or [46, Theorem 3] and [45, Theorem 3]). Then, the solution of (4) can be exactly driven to $z$ by our first outcome. Finally, a connectednesscompactness argument enables to steer the solution of (4) from $z$ to $\left(u_{i}^{*}\right)_{1 \leq i \leq 4}$.

## 2. Properties of the nonlinear controlled system

### 2.1. Definitions and usual properties

In this part, we introduce the concept of trajectory of (4). This definition requires a wellposedness result (see Proposition 2.3).

First, we introduce some usual notations.
Let $k, l \in \mathbb{N}^{*}, \mathcal{A}$ an algebra. Then, $\mathcal{M}_{k}(\mathcal{A})$ (respectively $\mathcal{M}_{k, l}(\mathcal{A})$ ) denotes the algebra of matrices with $k$ lines and $k$ columns with entries in $\mathcal{A}$ (respectively the algebra of matrices with $k$ lines and $l$ columns with entries in $\mathcal{A})$.

For $k \in \mathbb{N}^{*}$ and $A \in \mathcal{M}_{k}(\mathbb{R}), S p(M)$ denotes the set of complex eigenvalues of $M$,

$$
S p(M):=\left\{\lambda \in \mathbb{C} ; \exists X \in \mathbb{C}^{k} \backslash\{0\}, M X=\lambda X\right\}
$$

For $(a, b, c, d) \in \mathbb{R}^{4}$, we introduce

$$
\begin{equation*}
\forall i \in \mathbb{N}^{*}, f_{i}(a, b, c, d):=(-1)^{i}(a c-b d), \quad f(a, b, c, d)=\left(f_{i}(a, b, c, d)\right)_{1 \leq i \leq 4} . \tag{12}
\end{equation*}
$$

Definition 2.1. We introduce the space $Y$ defined by

$$
\begin{equation*}
Y:=L^{2}\left(0, T ; H^{1}(\Omega)\right) \cap H^{1}\left(0, T ;\left(H^{1}(\Omega)\right)^{\prime}\right) . \tag{13}
\end{equation*}
$$

Proposition 2.2. From an easy adaptation of the proof of [25, Section 5.9.2, Theorem 3], we have

$$
\begin{equation*}
Y \hookrightarrow C\left([0, T] ; L^{2}(\Omega)\right) . \tag{14}
\end{equation*}
$$

Proposition 2.3. Let $k \in \mathbb{N}^{*}, D \in \mathcal{M}_{k}(\mathbb{R})$ such that $D$ is diagonalizable and $S p(D) \subset(0,+\infty)$, $A \in \mathcal{M}_{k}\left(L^{\infty}(Q)\right), u_{0} \in L^{2}(\Omega)^{k}, g \in L^{2}(Q)^{k}$. The following Cauchy problem admits a unique weak solution $u \in Y^{k}$

$$
\begin{cases}\partial_{t} u-D \Delta u=A(t, x) u+g & \text { in }(0, T) \times \Omega, \\ \frac{\partial u}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega, \\ u(0, .)=u_{0} & \text { in } \Omega .\end{cases}
$$

This means that $u$ is the unique function in $Y^{k}$ that satisfies the variational fomulation

$$
\begin{equation*}
\forall w \in L^{2}\left(0, T ; H^{1}(\Omega)^{k}\right), \int_{0}^{T}\left(\partial_{t} u, w\right)_{\left.\left(H^{1}(\Omega)^{k}\right)^{\prime}, H^{1}(\Omega)^{k}\right)}+\int_{Q} D \nabla u \cdot \nabla w=\int_{Q}(A u+g) \cdot w, \tag{15}
\end{equation*}
$$

and

$$
\begin{equation*}
u(0, .)=u_{0} \text { in } L^{2}(\Omega)^{k} \tag{16}
\end{equation*}
$$

Moreover, there exists $C>0$ independent of $u_{0}$ and $g$ such that

$$
\begin{equation*}
\|u\|_{Y^{k}} \leq C\left(\left\|u_{0}\right\|_{L^{2}(\Omega)^{k}}+\|g\|_{L^{2}(Q)^{k}}\right) \tag{17}
\end{equation*}
$$

Finally, if $u_{0} \in L^{\infty}(\Omega)^{k}$ and $g \in L^{\infty}(Q)^{k}$, then $u \in L^{\infty}(Q)^{k}$ and there exists $C>0$ independent of $u_{0}$ and $g$ such that

$$
\begin{equation*}
\|u\|_{\left(Y \cap L^{\infty}(Q)\right)^{k}} \leq C\left(\left\|u_{0}\right\|_{L^{\infty}(\Omega)^{k}}+\|g\|_{L^{\infty}(Q)^{k}}\right) \tag{18}
\end{equation*}
$$

Remark 2.4. This proposition is more or less classical, but we could not find it as such in the literature and we give its proof in the Appendix (see Appendix A.1).

Definition 2.5. For $u_{0} \in L^{\infty}(\Omega)^{4},\left(\left(u_{i}\right)_{1 \leq i \leq 4},\left(h_{i}\right)_{1 \leq i \leq j}\right)$ is a trajectory of (4) if

1. $\left(\left(u_{i}\right)_{1 \leq i \leq 4},\left(h_{i}\right)_{1 \leq i \leq j}\right) \in\left(Y \cap L^{\infty}(Q)\right)^{4} \times L^{\infty}(Q)^{j}$,
2. $\left(u_{i}\right)_{1 \leq i \leq 4}$ is the (unique) solution of (4).

Moreover, $\left(\left(u_{i}\right)_{1 \leq i \leq 4},\left(h_{i}\right)_{1 \leq i \leq j}\right)$ is a trajectory of (4) reaching $\left(u_{i}^{*}\right)_{1 \leq i \leq 4}$ (in time $\left.T\right)$ if

$$
\forall i \in\{1, \ldots, 4\}, u_{i}(T, .)=u_{i}^{*}
$$

Remark 2.6. The concept of solution of (4) is the same as in Proposition 2.3 (take $D=$ $\operatorname{diag}\left(d_{1}, d_{2}, d_{3}, d_{4}\right), A=0$ and $g=\left(g_{i}(u)\right)_{1 \leq i \leq 4}^{T}$ where $\left.g_{i}(u)=f_{i}(u)+h_{i} 1_{i \leq j} 1_{\omega}\right)$.

Remark 2.7. The uniqueness is a consequence of the following estimate.
Let $D=\operatorname{diag}\left(d_{1}, d_{2}, d_{3}, d_{4}\right),\left(h_{i}\right)_{1 \leq i \leq j} \in L^{\infty}(Q)^{j}, u=\left(u_{i}\right)_{1 \leq i \leq 4} \in\left(Y \cap L^{\infty}(Q)\right)^{4}, \widetilde{u}=\left(\widetilde{u_{i}}\right)_{1 \leq i \leq 4} \in$ $\left(Y \cap L^{\infty}(Q)\right)^{4}$ be two solutions of (4), and $v=u-\widetilde{u}$. The function $v$ satisfies (in the weak sense)

$$
\begin{cases}\partial_{t} v-D \Delta v=f(u)-f(\widetilde{u}) & \text { in }(0, T) \times \Omega  \tag{19}\\ \frac{\partial v}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega \\ v(0, .)=0 & \text { in } \Omega\end{cases}
$$

By taking $w:=v$ in the variational formulation of (19) (see also (15)) and by using the fact that the mapping $t \mapsto\|v(t)\|_{L^{2}(\Omega)^{4}}^{2}$ is absolutely continuous with
$\frac{d}{d t}\|v(t)\|_{L^{2}(\Omega)^{4}}^{2}=2\left(\partial_{t} v(t), v(t)\right)_{\left(H^{1}(\Omega)^{4}\right)^{\prime}, H^{1}(\Omega)^{4}}$ for a.e. $0 \leq t \leq T$ (see [25, Section 5.9.2, Theorem $3]$ ), we find that

$$
\begin{equation*}
\frac{1}{2} \frac{d}{d t}\left(\|v\|_{L^{2}(\Omega)^{4}}^{2}\right)+\|D \nabla v\|_{L^{2}(\Omega)^{4}}^{2}=(f(u)-f(\widetilde{u}), v)_{L^{2}(\Omega)^{4}, L^{2}(\Omega)^{4}}, \text { for a.e. } 0 \leq t \leq T \tag{20}
\end{equation*}
$$

By using the facts that $(u, \widetilde{u}) \in L^{\infty}(Q)^{4} \times L^{\infty}(Q)^{4}, f$ is locally Lipschitz continuous on $\mathbb{R}^{4}$, we find the differential inequality

$$
\begin{equation*}
\frac{d}{d t}\left(\|v\|_{L^{2}(\Omega)^{4}}^{2}\right) \leq C\|v\|_{L^{2}(\Omega)^{4}}^{2}, \text { for a.e. } 0 \leq t \leq T \tag{21}
\end{equation*}
$$

Gronwall's lemma and the initial condition $v(0,)=$.0 prove that $v=0$ in $L^{2}(Q)^{4}$. Consequently, $u=\widetilde{u}$.

### 2.2. Invariant quantities of the nonlinear dynamics

In this section, we show that in the system (4), some invariant quantities exist. They impose some restrictions on the initial condition for the controllability results.

### 2.2.1. Variation of the mass

Proposition 2.8. Let $j \in\{1,2,3\}, u_{0} \in L^{\infty}(\Omega)^{4}$, $\left(\left(u_{i}\right)_{1 \leq i \leq 4},\left(h_{i}\right)_{1 \leq i \leq j}\right)$ be a trajectory of (4). For every $1 \leq i \leq 4$, the mapping $t \mapsto \int_{\Omega} u_{i}(t, x) d x$ is absolutely continuous with for a.e. $0 \leq t \leq T$,

$$
\begin{equation*}
\frac{d}{d t} \int_{\Omega} u_{i}(t, x) d x=\int_{\Omega}\left\{f_{i}\left(u_{1}(t, x), u_{2}(t, x), u_{3}(t, x), u_{4}(t, x)\right)+h_{i}(t, x) 1_{\omega}(x) 1_{i \leq j}\right\} d x \tag{22}
\end{equation*}
$$

Proof. We fix $1 \leq i \leq 4$. By using the fact that $u_{i} \in Y$ and from an easy adaptation of [25, Section 5.9.2, Theorem 3, (ii)], we deduce that the mapping $t \mapsto \int_{\Omega} u_{i}(t, x) d x$ is absolutely continuous and for a.e. $0 \leq t \leq T$,

$$
\frac{d}{d t} \int_{\Omega} u_{i}(t, x) d x=\left(\partial_{t} u_{i}(t, .), 1\right)_{\left(H^{1}(\Omega)\right)^{\prime}, H^{1}(\Omega)}
$$

Then, by using that $\left(\left(u_{i}\right)_{1 \leq i \leq 4},\left(h_{i}\right)_{1 \leq i \leq j}\right)$ is the (unique) solution of (4) and by taking $w=1$ in (15), we find that for a.e. $0 \leq t \leq T$,

$$
\begin{aligned}
& \left(\partial_{t} u_{i}(t, .), 1\right)_{\left(H^{1}(\Omega)\right)^{\prime}, H^{1}(\Omega)} \\
& =d_{i}\left(\nabla u_{i}(t, .), \nabla 1\right)_{L^{2}(\Omega), L^{2}(\Omega)}+\int_{\Omega}\left\{f_{i}\left(u_{1}(t, x), u_{2}(t, x), u_{3}(t, x), u_{4}(t, x)\right)+h_{i}(t, x) 1_{\omega}(x) 1_{i \leq j}\right\} d x \\
& =\int_{\Omega}\left\{f_{i}\left(u_{1}(t, x), u_{2}(t, x), u_{3}(t, x), u_{4}(t, x)\right)+h_{i}(t, x) 1_{\omega}(x) 1_{i \leq j}\right\} d x .
\end{aligned}
$$

### 2.2.2. Case of 2 controls

Proposition 2.9. Let $j=2, u_{0} \in L^{\infty}(\Omega)^{4}$, $\left(\left(u_{i}\right)_{1 \leq i \leq 4},\left(h_{i}\right)_{1 \leq i \leq 2}\right)$ be a trajectory of (4) reaching $\left(u_{i}^{*}\right)_{1 \leq i \leq 4}$ in time $T$. Then, we have

$$
\begin{gather*}
\frac{1}{|\Omega|} \int_{\Omega}\left(u_{3,0}(x)+u_{4,0}(x)\right) d x=u_{3}^{*}+u_{4}^{*}  \tag{23}\\
\left(d_{3}=d_{4}\right) \Rightarrow\left(u_{3,0}+u_{4,0}=u_{3}^{*}+u_{4}^{*}\right) . \tag{24}
\end{gather*}
$$

Proof. From (22), we have

$$
\frac{d}{d t}\left(\int_{\Omega}\left(u_{3}(t, x)+u_{4}(t, x)\right) d x\right)=0 \text { for a.e. } 0 \leq t \leq T
$$

Then, from Definition 2.5, (23) holds.
Moreover, $u_{3}+u_{4}$ satisfies

$$
\begin{cases}\partial_{t}\left(u_{3}+u_{4}\right)-d_{4} \Delta\left(u_{3}+u_{4}\right)=\left(d_{3}-d_{4}\right) \Delta u_{3} & \text { in }(0, T) \times \Omega, \\ \frac{\partial\left(u_{3}+u_{4}\right)}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega .\end{cases}
$$

If $d_{3}=d_{4}$, then the backward uniqueness for the heat equation (a corollary of Lemma 2.11) proves that

$$
\begin{equation*}
\forall t \in[0, T],\left(u_{3}+u_{4}\right)(t, .)=\left(u_{3}+u_{4}\right)(T, .)=u_{3}^{*}+u_{4}^{*} . \tag{25}
\end{equation*}
$$

This implies the necessary condition (24), stronger than (23), on the initial condition.
2.2.3. Case of 1 control

Proposition 2.10. Let $j=1, u_{0} \in L^{\infty}(\Omega)^{4},\left(\left(u_{i}\right)_{1 \leq i \leq 4},\left(h_{i}\right)_{1 \leq i \leq 2}\right)$ be a trajectory of (4) reaching $\left(u_{i}^{*}\right)_{1 \leq i \leq 4}$ in time $T$. Then, we have

$$
\begin{gather*}
\frac{1}{|\Omega|} \int_{\Omega}\left(u_{2,0}(x)+u_{3,0}(x)\right) d x=u_{2}^{*}+u_{3}^{*}, \frac{1}{|\Omega|} \int_{\Omega}\left(u_{3,0}(x)+u_{4,0}(x)\right) d x=u_{3}^{*}+u_{4}^{*}  \tag{26}\\
\left(k \neq l \in\{2,3,4\}, d_{k}=d_{l}\right) \Rightarrow\left(u_{k, 0}-(-1)^{k-l} u_{l, 0}=u_{k}^{*}-(-1)^{k-l} u_{l}^{*}\right) \tag{27}
\end{gather*}
$$

Proof. From (22), we have

$$
\frac{d}{d t}\left(\frac{1}{|\Omega|} \int_{\Omega}\left(u_{2}(t, x)+u_{3}(t, x)\right) d x\right)=0, \frac{d}{d t}\left(\frac{1}{|\Omega|} \int_{\Omega}\left(u_{3}(t, x)+u_{4}(t, x)\right) d x\right)=0 \text { for a.e. } 0 \leq t \leq T
$$

Then, from Definition 2.5, (26) holds.
Moreover, if there exists $k \neq l \in\{2,3,4\}$ such that $d_{k}=d_{l}$, by using again the backward uniqueness for the heat equation, we get

$$
\begin{align*}
& \left(k \neq l \in\{2,3,4\}, d_{k}=d_{l}\right) \\
& \Rightarrow\left(\forall t \in[0, T],\left(u_{k}-(-1)^{k-l} u_{l}\right)(t, .)=\left(u_{k}-(-1)^{k-l} u_{l}\right)(T, .)=u_{k}^{*}-(-1)^{k-l} u_{l}^{*}\right) \tag{28}
\end{align*}
$$

and in particular the necessary condition (27), stronger than (26), on the initial condition.

### 2.3. More restrictive conditions on the initial condition when the target $\left(u_{i}^{*}\right)_{1 \leq i \leq 4}$ vanishes

In the previous section, we have seen that there are invariant quantities in the dynamics of (4) which impose necessary conditions on the initial condition: (23), (26). Moreover, when some coefficients of diffusion $d_{i}$ are equal, we have more invariant quantities in (4) which impose stronger necessary conditions on the initial condition: (24), (27).

### 2.3.1. The lemma of backward uniqueness

## Lemma 2.11. Backward uniqueness

Let $k \in \mathbb{N}^{*}, D=\operatorname{diag}\left(d_{1}, \ldots, d_{k}\right)$ where $d_{i} \in(0,+\infty), C \in \mathcal{M}_{k}\left(L^{\infty}(Q)\right), \zeta_{0} \in L^{\infty}(\Omega)^{k}$. Let $\zeta \in Y^{k}$ be the solution of

$$
\begin{cases}\partial_{t} \zeta-D \Delta \zeta=C(t, x) \zeta & \text { in }(0, T) \times \Omega \\ \frac{\partial \zeta}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega \\ \zeta(0, .)=\zeta_{0} & \text { in } \Omega\end{cases}
$$

If $\zeta(T,)=$.0 , then for every $t \in[0, T], \zeta(t,)=$.0 .
Proof. $\widetilde{\zeta}(t, x)=\exp (-t) \zeta(t, x) \in Y^{k}$ is the solution of the system

$$
\begin{cases}\partial_{t} \widetilde{\zeta}-D \Delta \widetilde{\zeta}+I_{k} \widetilde{\zeta}=C(t, x) \widetilde{\zeta} & \text { in }(0, T) \times \Omega \\ \frac{\partial \widetilde{\zeta}}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega \\ \frac{\zeta}{\zeta}(0, .)=\widetilde{\zeta}_{0} & \text { in } \Omega,\end{cases}
$$

which verifies $\widetilde{\zeta}(T,)=$.0 .
Let us denote $A=-D \Delta+I_{k}$ which is a bounded linear operator from $H^{1}(\Omega)^{k}$ to $\left(H^{1}(\Omega)^{k}\right)^{\prime}$. Indeed,

$$
\forall(u, v) \in\left(H^{1}(\Omega)^{k}\right)^{2},(A u)(v)=\sum_{i=1}^{k} d_{i}\left(\nabla u_{i}, \nabla v_{i}\right)_{L^{2}(\Omega), L^{2}(\Omega)}+\sum_{i=1}^{k}\left(u_{i}, v_{i}\right)_{L^{2}(\Omega), L^{2}(\Omega)}
$$

$$
\|A u\|_{\left(H^{1}(\Omega)^{k}\right)^{\prime}} \leq \sqrt{1+\max \left(d_{i}\right)}\|u\|_{H^{1}(\Omega)^{k}}
$$

Then, $A$ verifies the three hypotheses: (i), (ii) and (iii) of [9, Proposition II.1].
(i) is satisfied because $A$ does not depend on $t$.
(ii) is a consequence of

$$
\forall(u, v) \in\left(H^{1}(\Omega)^{k}\right)^{2},(A u)(v)=(A v)(u)
$$

(iii) is satisfied because

$$
(A u, u)=\sum_{i=1}^{k} d_{i}\left(\nabla u_{i}, \nabla u_{i}\right)_{L^{2}(\Omega), L^{2}(\Omega)}+\sum_{i=1}^{k}\left(u_{i}, u_{i}\right)_{L^{2}(\Omega), L^{2}(\Omega)} \geq \min \left(\min _{i}\left(d_{i}\right), 1\right)\|u\|_{H^{1}(\Omega)^{k}}^{2} .
$$

Let $B(t)$ be the family of operators in $L^{2}\left(0, T ; \mathcal{L}\left(H^{1}(\Omega)^{k}, L^{2}(\Omega)^{k}\right)\right)$ defined by

$$
\forall u \in H^{1}(\Omega)^{k}, B(t) u(.)=C(t, .) u(.) .
$$

We have

$$
\|B\|_{L^{2}\left(0, T ; \mathcal{L}\left(H^{1}(\Omega)^{k}, L^{2}(\Omega)^{k}\right)\right)}^{2} \leq\|C\|_{L^{\infty}(Q)^{k^{2}}}^{2} .
$$

By applying [9, Theorem II.1], we get that for every $t \in[0, T], \widetilde{\zeta}(t,)=$.0 . Then,

$$
\forall t \in[0, T], \zeta(t, .)=0 .
$$

### 2.3.2. Case of 2 controls

Proposition 2.12. Let $j=2, u_{0} \in L^{\infty}(\Omega)^{4}$. If $\left(\left(u_{i}\right)_{1 \leq i \leq 4},\left(h_{i}\right)_{1 \leq i \leq 2}\right)$ is a trajectory of (4) reaching $\left(u_{i}^{*}\right)_{1 \leq i \leq 4}$ in time $T$, then we have

$$
\begin{equation*}
\left(\left(u_{3}^{*}, u_{4}^{*}\right)=(0,0)\right) \Rightarrow\left(\left(u_{3,0}, u_{4,0}\right)=(0,0)\right) . \tag{29}
\end{equation*}
$$

Conversely, for every $u_{0} \in L^{\infty}(\Omega)^{4}$ such that $\left(u_{3,0}, u_{4,0}\right)=(0,0)$, we can find $\left(h_{i}\right)_{1 \leq i \leq 2} \in L^{\infty}(Q)^{2}$ such that the associated solution $\left(u_{i}\right)_{1 \leq i \leq 4} \in L^{\infty}(Q)^{4}$ of (4) satisfies

$$
\left(u_{1}, u_{2}, u_{3}, u_{4}\right)(T, .)=\left(u_{1}^{*}, u_{2}^{*}, 0,0\right) .
$$

Proof. If $\left(u_{3}^{*}, u_{4}^{*}\right)=(0,0)$, it results from (4) that

$$
\begin{cases}\partial_{t} u_{3}-d_{3} \Delta u_{3}=-u_{1} u_{3}+u_{2} u_{4} & \text { in }(0, T) \times \Omega  \tag{30}\\ \partial_{t} u_{4}-d_{4} \Delta u_{4}=u_{1} u_{3}-u_{2} u_{4} & \text { in }(0, T) \times \Omega, \\ \frac{\partial u_{3}}{\partial n}=\frac{\partial u_{4}}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega\end{cases}
$$

By using the point 1 of Definition 2.5, we have

$$
\begin{equation*}
\left(u_{1}, u_{2}\right) \in L^{\infty}(Q)^{2} . \tag{31}
\end{equation*}
$$

Then, from (30), (31), Definition 2.5: $\left(u_{3}, u_{4}\right)(T,)=.(0,0)$ and Lemma 2.11 with $k=2, D=$ $\operatorname{diag}\left(d_{3}, d_{4}\right)$ and $C=\left(\begin{array}{cc}-u_{1} & u_{2} \\ u_{1} & -u_{2}\end{array}\right)$, we deduce that

$$
\forall t \in[0, T],\left(u_{3}, u_{4}\right)(t, .)=(0,0)
$$

and in particular (29).
Conversely, let $u_{0} \in L^{\infty}(\Omega)^{4}$ be such that $\left(u_{3,0}, u_{4,0}\right)=(0,0)$. Then, (4) reduces to the following system

$$
\begin{cases}\partial_{t} u_{1}-d_{1} \Delta u_{1}=h_{1} 1_{\omega} & \text { in }(0, T) \times \Omega,  \tag{32}\\ \partial_{t} u_{2}-d_{2} \Delta u_{2}=h_{2} 1_{\omega} & \text { in }(0, T) \times \Omega, \\ \frac{\partial u_{1}}{\partial n}=\frac{\partial u_{2}}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega, \\ \left(u_{1}, u_{2}\right)(0, .)=\left(u_{1,0}, u_{2,0}\right) & \text { in } \Omega\end{cases}
$$

The problem reduces to the null-controllability of two decoupled heat equations in $L^{\infty}(\Omega)$ with two localized control in $L^{\infty}(Q)$ which is a solved problem (see for example [27, Proposition 1]). Therefore, we can find $\left(h_{i}\right)_{1 \leq i \leq 2} \in L^{\infty}(Q)^{2}$ such that the associated solution $\left(u_{i}\right)_{1 \leq i \leq 4} \in L^{\infty}(Q)^{4}$ of (4) satisfies $\left(u_{1}, u_{2}, u_{3}, u_{4}\right)(T,)=.\left(u_{1}^{*}, u_{2}^{*}, 0,0\right)$.

Remark 2.13. Thanks to Proposition 2.12, we avoid the easy case $\left(u_{3}^{*}, u_{4}^{*}\right)=(0,0)$ for 2 controls in the sequel.

### 2.3.3. Case of 1 control

Proposition 2.14. Let $j=1, u_{0} \in L^{\infty}(\Omega)^{4}$. If $\left(\left(u_{i}\right)_{1 \leq i \leq 4}, h_{1}\right)$ is a trajectory of (4) reaching $\left(u_{i}^{*}\right)_{1 \leq i \leq 4}$ in time $T$, then we have

$$
\begin{align*}
& \left(\left(u_{3}^{*}, u_{2}^{*}\right)=(0,0)\right) \Rightarrow\left(\left(u_{2,0}, u_{3,0}, u_{4,0}\right)=\left(0,0, u_{4}^{*}\right)\right)  \tag{33}\\
& \left(\left(u_{3}^{*}, u_{4}^{*}\right)=(0,0)\right) \Rightarrow\left(\left(u_{2,0}, u_{3,0}, u_{4,0}\right)=\left(u_{2}^{*}, 0,0\right)\right) . \tag{34}
\end{align*}
$$

Conversely, for every $u_{0} \in L^{\infty}(\Omega)^{4}$ such that $u_{3,0}=0$, we can find $h_{1} \in L^{\infty}(Q)$ such that the associated solution $\left(u_{i}\right)_{1 \leq i \leq 4} \in L^{\infty}(Q)^{4}$ of (4) satisfies $\left(u_{1}, u_{2}, u_{3}, u_{4}\right)(T,)=.\left(u_{1}^{*}, u_{2}^{*}, 0, u_{4}^{*}\right)$.
Proof. If $u_{3}^{*}=0$, then from (3), $u_{2}^{*}=0$ or $u_{4}^{*}=0$. We assume that $\left(u_{3}^{*}, u_{2}^{*}\right)=(0,0)$ (the other case is similar). The backward uniqueness (i.e. Lemma 2.11) as in Section 2.3.2 leads to

$$
\forall t \in[0, T],\left(u_{3}, u_{2}\right)(t, .)=(0,0) .
$$

Then, we deduce that

$$
\begin{cases}\partial_{t} u_{4}-d_{4} \Delta u_{4}=0 & \text { in }(0, T) \times \Omega,  \tag{35}\\ \frac{\partial u_{4}}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega .\end{cases}
$$

The backward uniqueness for the heat equation applied to (35) proves that

$$
\forall t \in[0, T], u_{4}(t, .)=u_{4}^{*},
$$

and in particular (33) and (34).
Conversely, let $u_{0} \in L^{\infty}(\Omega)^{4}$ such that $u_{3,0}=0$. Then, (4) reduces to the following system

$$
\begin{cases}\partial_{t} u_{1}-d_{1} \Delta u_{1}=h_{1} 1_{\omega} & \text { in }(0, T) \times \Omega,  \tag{36}\\ \frac{\partial u_{1}}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega, \\ u_{1}(0, .)=u_{1,0} & \text { in } \Omega .\end{cases}
$$

The problem reduces to the null-controllability of the heat equation in $L^{\infty}(\Omega)$ with a localized control in $L^{\infty}(Q)$ which is a solved problem (see for example [27, Proposition 1]). Therefore, we can find $h_{1} \in L^{\infty}(Q)$ such that the associated solution $\left(u_{i}\right)_{1 \leq i \leq 4} \in L^{\infty}(Q)^{4}$ of (4) satisfies $\left(u_{1}, u_{2}, u_{3}, u_{4}\right)(T,)=.\left(u_{1}^{*}, u_{2}^{*}, 0, u_{4}^{*}\right)$.
Remark 2.15. Thanks to Proposition 2.14, we avoid the easy case $u_{3}^{*}=0$ for 1 control in the sequel.

## 3. Main results

In this part, we present our two main results: a local controllability result and a large-time global controllabillity result for (4).

### 3.1. Local controllability under constraints

In Section 2.2 and Section 2.3, we have highlighted necessary conditions on initial conditions when $\left(\left(u_{i}\right)_{1 \leq i \leq 4},\left(h_{i}\right)_{1 \leq i \leq j}\right)$ is a trajectory reaching $\left(u_{i}^{*}\right)_{1 \leq i \leq 4}$. They turn out to be sufficient for the existence of such trajectories at least for data close to $\left(u_{i}^{*}\right)_{1 \leq i \leq 4}$. The goal of this subsection is to define subspaces of $L^{\infty}(\Omega)^{4}$ which take care of these conditions.

### 3.1.1. Case of 3 controls

We introduce

$$
\begin{equation*}
X_{3,\left(d_{i}\right),\left(u_{i}^{*}\right)}=L^{\infty}(\Omega)^{4} . \tag{37}
\end{equation*}
$$

### 3.1.2. Case of 2 controls

The results of Section 2.2.2 and Section 2.3.2 are summed up in the following array.

$$
\begin{array}{|c|c|}
\hline & \left(u_{3}^{*}, u_{4}^{*}\right) \neq(0,0)  \tag{38}\\
\hline d_{3}=d_{4} & u_{3,0}+u_{4,0}=u_{3}^{*}+u_{4}^{*} \\
\hline d_{3} \neq d_{4} & \frac{1}{|\Omega|} \int_{\Omega}\left(u_{3,0}+u_{4,0}\right)=u_{3}^{*}+u_{4}^{*} \\
\hline
\end{array}
$$

Then, we introduce

$$
\begin{equation*}
X_{2,\left(d_{i}\right),\left(u_{i}^{*}\right)}:=\left\{u_{0} \in L^{\infty}(\Omega)^{4} ; u_{0} \text { satisfies the associated condition of (38) }\right\} . \tag{39}
\end{equation*}
$$

For example, $X_{2,(1,2,3,4),(1,1,1,1)}=\left\{u_{0} \in L^{\infty}(\Omega)^{4} ; \frac{1}{|\Omega|} \int_{\Omega}\left(u_{3,0}+u_{4,0}\right)=2\right\}$.

### 3.1.3. Case of 1 control

The results of Section 2.2.3 and Section 2.3.3 are summed up in the following array.

|  |  |
| :---: | :---: |
| $d_{2}=d_{3}=d_{4}$ | $u_{3,0}^{*} \neq 0$ |
| $d_{2} \neq d_{3}, d_{3}=d_{4}=u_{2}^{*}+u_{3}^{*}, u_{3,0}+u_{4,0}=u_{3}^{*}+u_{4}^{*}$ |  |
| $d_{2}=d_{3}, d_{3} \neq d_{4}$ | $\frac{1}{\|\Omega\|} \int_{\Omega}\left(u_{2,0}+u_{3,0}\right)=u_{2}^{*}+u_{3}^{*}, u_{3,0}+u_{4,0}=u_{3}^{*}+u_{4}^{*}$ |
| $d_{2}=d_{4}, d_{2} \neq d_{3}$ | $u_{2,0}+u_{3,0}=u_{2}^{*}+u_{3}^{*}, \frac{1}{\Omega \Omega} \int_{\Omega}\left(u_{3,0}+u_{4,0}\right)=u_{3}^{*}+u_{4}^{*}$ |
| $d_{2} \neq d_{3}, d_{3} \neq d_{4}, d_{2} \neq d_{4}$ | $\left.\frac{1}{\Omega,} \int_{\Omega, 0}-u_{4,0}=u_{2}^{*}-u_{4}^{*}, \frac{1}{\Omega \mid} \int_{\Omega, 0}+u_{3,0}\right)=u_{2}^{*}+u_{3}^{*}, \frac{1}{\Omega \Omega}\left(\int_{\Omega, 0}+u_{4,0}\right)=u_{3}^{*}+u_{4,0}^{*}$ |

Then, we introduce

$$
\begin{equation*}
X_{1,\left(d_{i}\right),\left(u_{i}^{*}\right)}:=\left\{u_{0} \in L^{\infty}(\Omega)^{4} ; u_{0} \text { satisfies the associated condition of (40) }\right\} . \tag{41}
\end{equation*}
$$

### 3.1.4. Local controllability result

Definition 3.1. Let $j \in\{1,2,3\},\left(u_{1}^{*}, u_{2}^{*}, u_{3}^{*}, u_{4}^{*}\right) \in\left(\mathbb{R}^{+}\right)^{4}$ be such that (3) holds. The system (4) is locally controllable to the state $\left(u_{i}^{*}\right)_{1 \leq i \leq 4}$ in $L^{\infty}(\Omega)^{4}$ with controls in $L^{\infty}(Q)^{j}$ if there exists $\delta>0$ such that for every $u_{0} \in X_{j,\left(d_{i}\right),\left(u_{i}^{*}\right)}$ (see (37), (39) and (41)) verifying $\left\|u_{0}-\left(u_{i}^{*}\right)_{1 \leq i \leq 4}\right\|_{L^{\infty}(\Omega)^{4}} \leq \delta$, there exists $\left(h_{i}\right)_{1 \leq i \leq j} \in L^{\infty}(Q)^{j}$ such that the solution $\left(u_{i}\right)_{1 \leq i \leq 4} \in L^{\infty}(Q)^{4}$ to the Cauchy problem (4) satisfies

$$
\forall i \in\{1,2,3,4\}, u_{i}(T, .)=u_{i}^{*} .
$$

Theorem 3.2. For every $j \in\{1,2,3\}$, for every $\left(u_{1}^{*}, u_{2}^{*}, u_{3}^{*}, u_{4}^{*}\right) \in\left(\mathbb{R}^{+}\right)^{4}$ which satisfies (3), the system (4) is locally controllable to the state $\left(u_{i}^{*}\right)_{1 \leq i \leq 4}$ in $L^{\infty}(\Omega)^{4}$ with controls in $L^{\infty}(Q)^{j}$.
Remark 3.3. The uniqueness of the solution $\left(u_{i}\right)_{1 \leq i \leq 4} \in L^{\infty}(Q)^{4}$ is a consequence of Remark 2.7. The existence of the solution $\left(u_{i}\right)_{1 \leq i \leq 4} \in L^{\infty}(Q)^{4}$ is a consequence of a good choice of controls $\left(h_{i}\right)_{1 \leq i \leq j} \in L^{\infty}(Q)^{j}$ and more precisely of a fixed-point argument (see Section 4.5).
Remark 3.4. As we have said in the introduction, it was not known if $L^{\infty}$ blow-up occurs or not in dimension $N>2$ for the free system (1) until recently (see [12]). Here, our strategy of control avoids blow-up and enables the solution to reach a stationary solution of (1).
Remark 3.5. In some particular cases (easy cases), this local controllability result can be improved in a global controllability result (see the case $\left(u_{3}^{*}, u_{4}^{*}\right)=(0,0)$ for 2 controls in Section 2.3.2 and the case $u_{3}^{*}=0$ for 1 control in Section 2.3.3).

### 3.2. Large-time global controllability result

From Theorem 3.2, we establish a global controllability result in large time for $N=1,2$.
Theorem 3.6. We assume that $N=1$ or 2 . Let $j \in\{1,2,3\}$ and $\left(u_{i}^{*}\right)_{1 \leq i \leq 4} \in\left(\mathbb{R}^{+}\right)^{4}$ be such that (3) holds. Then, for every $u_{0} \in X_{j,\left(d_{i}\right),\left(u_{i}^{*}\right)}$ satisfying

$$
\begin{align*}
& \forall 1 \leq i \leq 4, u_{i 0} \geq 0, \\
& \frac{1}{|\Omega|} \int_{\Omega}\left(u_{1,0}+u_{2,0}\right)>0, \frac{1}{|\Omega|} \int_{\Omega}\left(u_{1,0}+u_{4,0}\right)>0, \frac{1}{|\Omega|} \int_{\Omega}\left(u_{2,0}+u_{3,0}\right)>0, \frac{1}{|\Omega|} \int_{\Omega}\left(u_{3,0}+u_{4,0}\right)>0, \tag{42}
\end{align*}
$$

there exists $T^{*}>0\left(\right.$ sufficiently large) and $\left(h_{i}\right)_{1 \leq i \leq j} \in L^{\infty}\left(\left(0, T^{*}\right) \times \Omega\right)^{j}$ such that the solution $u$ of

$$
\forall 1 \leq i \leq 4, \begin{cases}\partial_{t} u_{i}-d_{i} \Delta u_{i}=(-1)^{i}\left(u_{1} u_{3}-u_{2} u_{4}\right)+h_{i} 1_{\omega} 1_{i \leq j} & \text { in }\left(0, T^{*}\right) \times \Omega,  \tag{43}\\ \frac{\partial u_{i}}{\partial n}=0 & \text { on }\left(0, T^{*}\right) \times \partial \Omega, \\ u_{i}(0, .)=u_{i, 0} & \text { in } \Omega,\end{cases}
$$

satisfies

$$
\begin{equation*}
u\left(T^{*}, .\right)=u^{*} . \tag{44}
\end{equation*}
$$

Remark 3.7. The restriction on the dimension $N \in\{1,2\}$ is a consequence of the following property: the solution of the free system (1) converges in $L^{\infty}(\Omega)$ when $T \rightarrow+\infty$ to a particular stationary solution of (1) (see [21]). One can extend Theorem 3.6 to $N>2$ if the convergence in $L^{\infty}(\Omega)$ (of the free system) holds. For $N>2$, one only knows that a weak solution of the free system (1) converges in $L^{1}(\Omega)$ when $T \rightarrow+\infty$ to a particular stationary solution of (1) (see [46, Theorem 3]). But, for example, if we assume that the diffusion coefficients $d_{i}$ are close, the weak solution of the free system (1) converges in $L^{\infty}(\Omega)$ when $T \rightarrow+\infty$ to a particular stationary solution of (1) (see [11, Proposition 1.3]).
Remark 3.8. The positivity assumption (42) is not restrictive. One can extend the result to nonnegative initial condition $u_{0} \in X_{j,\left(d_{i}\right),\left(u_{i}^{*}\right)}$ (see [46, Section 5]).

## 4. Proof of Theorem 3.2: the local controllability to constant stationary states

The aim of this section is to prove Theorem 3.2. As usual, we study the properties of controllability of the linearized system around $\left(u_{i}^{*}\right)_{1 \leq i \leq 4}$ of (4). First, we transform the problem by studying the null-controllability of a family of linear control systems (see Section 4.1). The existence of controls in $L^{2}(Q)$ is a consequence of a duality method: the Hilbert Uniqueness Method introduced by Jacques-Louis Lions (see Section 4.3.1). It links the existence of controls in $L^{2}(Q)$ with an observability inequality for solution of the adjoint system. This type of inequalities is proved by Carleman estimates (see Section 4.3.2). In order to get more regular controls (in $L^{p}(Q)$ sense, $p \geq 2$ ), we use a sophistication of Hilbert Uniqueness Method called the penalized Hilbert Uniqueness Method introduced by Viorel Barbu (see Section 4.4.1). Indeed, this enables to have controls a bit better than $L^{2}(Q)$. Then, a bootstrap method gives controls in $L^{\infty}(Q)$ (see Section 4.4.2). A fixed-point argument concludes the proof (see Section 4.5).

Now, we develop a strategy in order to treat the cases of 1,2 or 3 controls in a unified way. We introduce the following notations

$$
B_{3}=\left(\begin{array}{lll}
1 & 0 & 0  \tag{45}\\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{array}\right), h^{3}=\left(\begin{array}{l}
h_{1} \\
h_{2} \\
h_{3} \\
0
\end{array}\right), B_{2}=\left(\begin{array}{ll}
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0
\end{array}\right), h^{2}=\left(\begin{array}{c}
h_{1} \\
h_{2} \\
0 \\
0
\end{array}\right), B_{1}=\left(\begin{array}{l}
1 \\
0 \\
0 \\
0
\end{array}\right), h^{1}=\left(\begin{array}{c}
h_{1} \\
0 \\
0 \\
0
\end{array}\right) .
$$

Let $j \in\{1,2,3\},\left(u_{1}^{*}, u_{2}^{*}, u_{3}^{*}, u_{4}^{*}\right) \in\left(\mathbb{R}^{+}\right)^{4}$ be such that (3) holds and $u_{0} \in X_{j,\left(d_{i}\right),\left(u_{i}^{*}\right)}$ (see (37), (39) and (41)).

### 4.1. Linearization

We adopt the approach presented in Section 1.4.2.
4.1.1. 3 controls, return method when $\left(u_{1}^{*}, u_{3}^{*}, u_{4}^{*}\right)=(0,0,0)$

We linearize (4) around $\left(u_{i}^{*}\right)_{1 \leq i \leq 4}$ and we get the system: for every $1 \leq i \leq 4$,

$$
\begin{cases}\partial_{t} u_{i}-d_{i} \Delta u_{i}=(-1)^{i}\left(u_{3}^{*} u_{1}-u_{4}^{*} u_{2}+u_{1}^{*} u_{3}-u_{2}^{*} u_{4}\right)+h_{i} 1_{\omega} 1_{i \leq 3} & \text { in }(0, T) \times \Omega,  \tag{46}\\ \frac{\partial u_{i}}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega, \\ u_{i}(0, .)=u_{i, 0} & \text { in } \Omega\end{cases}
$$

Roughly speaking, it is easy to control $u_{1}, u_{2}, u_{3}$ thanks to $h_{1}, h_{2}, h_{3}$. The main difficulty is to control $u_{4}$. Now, we present the heuristic way of controlling $u_{4}$.
4.1.1.1. First case: $\left(u_{1}^{*}, u_{3}^{*}, u_{4}^{*}\right) \neq(0,0,0)$. There is a coupling term in the fourth equation of (46) which enables to control $u_{4}$. For example, if $u_{3}^{*} \neq 0$, then $u_{1}$ controls $u_{4}$.

Remark 4.1. In this case, the linearized system (46) looks like the toy-model (7) and its controllability properties come from Proposition 1.7. Consequently, the local controllability of (4) can be proved as in Proposition 1.12 for system (11).
4.1.1.2. Second case: $\left(u_{1}^{*}, u_{3}^{*}, u_{4}^{*}\right)=(0,0,0)$, return method. The fourth equation of (46) is decoupled from the other equations. In particular, if $u_{4}(0,) \neq$.0 , then $u_{4}(T,) \neq$.0 . Consequently, system (46) is not controllable. The idea is to linearize around a non trivial trajectory of (4) which comes from $\left(0, u_{2}^{*}, 0,0\right)$ and goes to $\left(0, u_{2}^{*}, 0,0\right)$ and which forces the appearance of a coupling term after linearization. It is the return method. Here, we take

$$
\left(\left(0, u_{2}^{*},{\overline{u_{3}}}^{\sharp}, 0\right),\left(0,0,{\overline{h_{3}}}^{\sharp}\right)\right):=\left(\left(0, u_{2}^{*}, g, 0\right),\left(0,0, \partial_{t} g-d_{3} \Delta g\right)\right),
$$

where $g$ satisfies the following properties

$$
\begin{equation*}
g \in C^{\infty}(\bar{Q}), g \geq 0, g \neq 0, \operatorname{supp}(g) \subset(0, T) \times \omega \tag{47}
\end{equation*}
$$

Then, if we linearize the system (4) around $\left(\left(0, u_{2}^{*},{\overline{u_{3}}}^{\sharp}, 0\right),\left(0,0,{\overline{h_{3}}}^{\sharp}\right)\right)$, then the fourth equation becomes

$$
\partial_{t} u_{4}-d_{4} \Delta u_{4}={\overline{u_{3}}}^{\sharp}(t, x) u_{1}-u_{2}^{*} u_{4} \text { in }(0, T) \times \Omega .
$$

Roughly speaking, as ${\overline{u_{3}}}^{\sharp} \neq 0$ in the control zone, then $u_{1}$ controls $u_{4}$.
Remark 4.2. Here, the linearized system around the non trivial trajectory looks like the toymodel (7) and its controllability properties follow from Proposition 1.8. Consequently, the local controllability of (4) can be proved as Proposition 1.14 for (11).
4.1.1.3. Linearization in $L^{\infty}(Q)$ and null-controllability of a family of linear systems. We define

$$
\begin{align*}
& \overline{u_{3}}:=\left\{\begin{array}{ll}
u_{3}^{*} & \text { if }\left(u_{1}^{*}, u_{3}^{*}, u_{4}^{*}\right) \neq(0,0,0), \\
\overline{u_{3}} & \text { if }\left(u_{1}^{*}, u_{3}^{*}, u_{4}^{*}\right)=(0,0,0),
\end{array} \text { and } \overline{h_{3}}:=\left\{\begin{array}{cc}
0 & \text { if }\left(u_{1}^{*}, u_{3}^{*}, u_{4}^{*}\right) \neq(0,0,0), \\
\overline{h_{3}} & \text { if }\left(u_{1}^{*}, u_{3}^{*}, u_{4}^{*}\right)=(0,0,0),
\end{array}\right.\right.  \tag{48}\\
& \left(\zeta, \widehat{h^{3}}\right):=\left(\zeta_{1}, \zeta_{2}, \zeta_{3}, \zeta_{4}, \widehat{h_{1}}, \widehat{h_{2}}, \widehat{h_{3}}\right):=\left(u_{1}-u_{1}^{*}, u_{2}-u_{2}^{*}, u_{3}-\overline{u_{3}}, u_{4}-u_{4}^{*}, h_{1}, h_{2}, h_{3}-\overline{h_{3}}\right) . \tag{49}
\end{align*}
$$

Thus, $\left(u, h^{3}\right)$ is a trajectory of (4) if and only if $\left(\zeta, \widehat{h^{3}}\right)$ is a trajectory of the following system

$$
\begin{align*}
& \forall 1 \leq i \leq 4, \\
& \begin{cases}\partial_{t} \zeta_{i}-d_{i} \Delta \zeta_{i} \\
\quad=(-1)^{i}\left(\left(\overline{u_{3}}+\zeta_{3}\right) \zeta_{1}-\left(u_{4}^{*}+\zeta_{4}\right) \zeta_{2}+u_{1}^{*} \zeta_{3}-u_{2}^{*} \zeta_{4}\right)+\widehat{h_{i}} 1_{\omega} 1_{i \leq 3} & \text { in }(0, T) \times \Omega \\
\frac{\partial_{\zeta}}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega \\
\zeta_{i}(0, .)=u_{i, 0}-u_{i}^{*} & \text { in } \Omega .\end{cases} \tag{50}
\end{align*}
$$

Then, $\left(\zeta, \widehat{h^{3}}\right)$ is a trajectory of

$$
\begin{cases}\partial_{t} \zeta-D_{3} \Delta \zeta=G(\zeta) \zeta+B_{3} \widehat{h^{3}} 1_{\omega} & \text { in }(0, T) \times \Omega  \tag{51}\\ \frac{\partial \zeta}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega \\ \zeta(0, .)=\zeta_{0} & \text { in } \Omega,\end{cases}
$$

where

$$
D_{3}:=\left(\begin{array}{cccc}
d_{1} & 0 & 0 & 0  \tag{52}\\
0 & d_{2} & 0 & 0 \\
0 & 0 & d_{3} & 0 \\
0 & 0 & 0 & d_{4}
\end{array}\right), \quad G(\zeta):=\left(\begin{array}{cccc}
-\overline{u_{3}}-\zeta_{3} & u_{4}^{*}+\zeta_{4} & -u_{1}^{*} & u_{2}^{*} \\
\overline{u_{3}}+\zeta_{3} & -u_{4}^{*}-\zeta_{4} & u_{1}^{*} & -u_{2}^{*} \\
-\overline{u_{3}}-\zeta_{3} & u_{4}^{*}+\zeta_{4} & -u_{1}^{*} & u_{2}^{*} \\
\overline{u_{3}}+\zeta_{3} & -u_{4}^{*}-\zeta_{4} & u_{1}^{*} & -u_{2}^{*}
\end{array}\right) .
$$

Note that $G_{41}(0,0,0,0)=\overline{u_{3}}$. To simplify, we suppose the following fact:
if $\left(u_{1}^{*}, u_{3}^{*}, u_{4}^{*}\right) \neq(0,0,0)$, then $u_{3}^{*} \neq 0$. Otherwise, we can easily adapt our proof strategy (see Remark 4.16). Then, from (47), there exist $t_{1}<t_{2} \in(0, T)$, a nonempty open subset $\omega_{0} \subset \subset$ and $M>0$ such that

$$
\begin{aligned}
& \forall(t, x) \in\left(t_{1}, t_{2}\right) \times \omega_{0}, G_{41}(0,0,0,0)(t, x) \geq 2 / M, \\
& \forall(k, l) \in\{1, \ldots, 4\}^{2},\left\|G_{k l}(0,0,0,0)\right\|_{L^{\infty}(Q)} \leq M / 2
\end{aligned}
$$

Consequently, we study the null-controllability of the linear systems

$$
\begin{cases}\partial_{t} \zeta-D_{3} \Delta \zeta=A \zeta+B_{3} \widehat{h^{3}} 1_{\omega} & \text { in }(0, T) \times \Omega  \tag{53}\\ \frac{\partial \zeta}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega \\ \zeta(0, .)=\zeta_{0} & \text { in } \Omega,\end{cases}
$$

where the matrix $A$ verifies the following assumptions

$$
\begin{align*}
& \forall(t, x) \in\left(t_{1}, t_{2}\right) \times \omega_{0}, a_{41}(t, x) \geq 1 / M  \tag{54}\\
& \forall(k, l) \in\{1, \ldots, 4\}^{2},\left\|a_{k l}\right\|_{L^{\infty}(Q)} \leq M . \tag{55}
\end{align*}
$$

Remark 4.3. To simplify the notations, we now denote $\widehat{h^{3}}$ by $h^{3}$.

### 4.1.2. 2 controls, adequate change of variables

By Section 2.3.2, we can assume that $\left(u_{3}^{*}, u_{4}^{*}\right) \neq(0,0)$.
4.1.2.1. First case: $d_{3}=d_{4}$. From (25) and (39), system (4) reduces to

$$
\begin{align*}
& \forall 1 \leq i \leq 3, \\
& \begin{cases}\partial_{t} u_{i}-d_{i} \Delta u_{i}=(-1)^{i}\left(u_{1} u_{3}-u_{2}\left(u_{3}^{*}+u_{4}^{*}-u_{3}\right)\right)+h_{i} 1_{\omega} 1_{i \leq 2} & \text { in }(0, T) \times \Omega, \\
\frac{\partial u_{i}}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega, \\
u_{i}(0, .)=u_{i, 0} & \text { in } \Omega .\end{cases} \tag{56}
\end{align*}
$$

We do not give the complete proof of Theorem 3.2 in this case because it is an easy adaptation of the study of the null-controllability of the linear systems (53) which satisfy (54), (55) (with three equations instead of four). Indeed, by linearization around $\left(u_{i}^{*}\right)_{1 \leq i \leq 4}$ of (56), the equation satisfied by $u_{3}$ becomes

$$
\begin{equation*}
\partial_{t} u_{3}-d_{3} \Delta u_{3}=-u_{3}^{*} u_{1}+\left(u_{3}^{*}+u_{4}^{*}\right) u_{2}-\left(u_{1}^{*}+u_{2}^{*}\right) u_{3} \text { in }(0, T) \times \Omega . \tag{57}
\end{equation*}
$$

Then, there is a coupling term in (57) if and only if

$$
\begin{equation*}
\left(u_{3}^{*}, u_{3}^{*}+u_{4}^{*}\right) \neq(0,0) \text { i.e. }\left(u_{3}^{*}, u_{4}^{*}\right) \neq(0,0) . \tag{58}
\end{equation*}
$$

4.1.2.2. Second case: $d_{3} \neq d_{4}$. We remark that

$$
\begin{equation*}
\left(u_{1}, u_{2}, u_{3}, u_{4}\right)(T, .)=\left(u_{1}^{*}, u_{2}^{*}, u_{3}^{*}, u_{4}^{*}\right) \tag{59}
\end{equation*}
$$

if and only if

$$
\left(u_{1}, u_{2}, u_{3}, u_{3}+u_{4}\right)(T, .)=\left(u_{1}^{*}, u_{2}^{*}, u_{3}^{*}, u_{3}^{*}+u_{4}^{*}\right) .
$$

Therefore, we study the system satisfied by $\left(v_{1}, v_{2}, v_{3}, v_{4}\right):=\left(u_{1}, u_{2}, u_{3}, u_{3}+u_{4}\right)$,

$$
\forall 1 \leq i \leq 3, \begin{cases}\partial_{t} v_{i}-d_{i} \Delta v_{i}=(-1)^{i}\left(v_{1} v_{3}-v_{2}\left(v_{4}-v_{3}\right)\right)+h_{i} 1_{\omega} 1_{i \leq 2} & \text { in }(0, T) \times \Omega,  \tag{60}\\ \partial_{t} v_{4}-d_{4} \Delta v_{4}=\left(d_{3}-d_{4}\right) \Delta v_{3} & \text { in }(0, T) \times \Omega, \\ \frac{\partial v_{i}}{\partial n} \frac{\partial v_{4}}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega, \\ \left(v_{i}, v_{4}\right)(0, .)=\left(u_{i, 0}, u_{3,0}+u_{4,0}\right) & \text { in } \Omega .\end{cases}
$$

Roughly speaking, $v_{4}$ can be controlled by $v_{3}$ thanks to the coupling term of second order $\left(d_{3}-d_{4}\right) \Delta v_{3}$ in the second equation of (60) and $v_{3}$ can be controlled by $v_{1}$ or $v_{2}$ because the linearization of the first equation of (60) with $i=3$ is

$$
\partial_{t} \nu_{3}-d_{3} \Delta v_{3}=-u_{3}^{*} \nu_{1}+u_{4}^{*} v_{2}-\left(u_{1}^{*}+u_{2}^{*}\right) v_{3}+u_{2}^{*} v_{4} \text { in }(0, T) \times \Omega,
$$

and $\left(u_{3}^{*}, u_{4}^{*}\right) \neq(0,0)$. Then, the proof of the controllability properties of the linearized-system of (60) follows the ideas of Proposition 1.9 and Proposition 1.10. The main difference is the nature of the coupling terms: one coupling term of second order $\left(d_{3}-d_{4}\right) \Delta v_{3}$ and one coupling term of zero order $-u_{3}^{*} \nu_{1}$ if $u_{3}^{*} \neq 0$ or $u_{4}^{*} v_{2}$ if $u_{4}^{*} \neq 0$.
4.1.2.3. Linearization in $L^{\infty}(Q)$ and null-controllability of a family of linear systems when $d_{3} \neq$ $d_{4}$. We define

$$
\begin{equation*}
\left(\zeta, h^{2}\right):=\left(\zeta_{1}, \zeta_{2}, \zeta_{3}, \zeta_{4}, h_{1}, h_{2}\right):=\left(v_{1}-u_{1}^{*}, v_{2}-u_{2}^{*}, v_{3}-u_{3}^{*}, v_{4}-\left(u_{3}^{*}+u_{4}^{*}\right), h_{1}, h_{2}\right) \tag{61}
\end{equation*}
$$

Then, $\left(u, h^{2}\right)$ is a trajectory of (4) if and only if $\left(\zeta, h^{2}\right)$ is a trajectory of

$$
\begin{cases}\partial_{t} \zeta-D_{2} \Delta \zeta=G(\zeta) \zeta+B_{2} h^{2} 1_{\omega} & \text { in }(0, T) \times \Omega \\ \frac{\partial \zeta}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega \\ \zeta(0, .)=\zeta_{0} & \text { in } \Omega\end{cases}
$$

where

$$
D_{2}:=\left(\begin{array}{cccc}
d_{1} & 0 & 0 & 0  \tag{62}\\
0 & d_{2} & 0 & 0 \\
0 & 0 & d_{3} & 0 \\
0 & 0 & \left(d_{3}-d_{4}\right) & d_{4}
\end{array}\right), \quad G(\zeta):=\left(\begin{array}{cccc}
-\left(u_{3}^{*}+\zeta_{3}\right) & u_{4}^{*}+\zeta_{4}-\zeta_{3} & -u_{1}^{*}-u_{2}^{*} & u_{2}^{*} \\
u_{3}^{*}+\zeta_{3} & -\left(u_{4}^{*}+\zeta_{4}-\zeta_{3}\right) & u_{1}^{*}+u_{2}^{*} & -u_{2}^{*} \\
-\left(u_{3}^{*}+\zeta_{3}\right) & u_{4}^{*}+\zeta_{4}-\zeta_{3} & -u_{1}^{*}-u_{2}^{*} & u_{2}^{*} \\
0 & 0 & 0 & 0
\end{array}\right) .
$$

Note that $G_{31}(0,0,0,0)=-u_{3}^{*}$ and $G_{32}(0,0,0,0)=u_{4}^{*}$. Then, $\left(G_{31}(0,0,0,0), G_{32}(0,0,0,0)\right) \neq$ $(0,0)$. To simplify, we suppose that $G_{31}(0,0,0,0) \neq 0$. The other case is similar. There exist $t_{1}<t_{2} \in(0, T)$, a nonempty open subset $\omega_{0} \subset \subset \omega$ and $M>0$ such that

$$
\begin{gathered}
\forall(t, x) \in\left(t_{1}, t_{2}\right) \times \omega_{0}, G_{31}(0,0,0,0)(t, x) \leq-2 / M, \\
\forall(k, l) \in\{1, \ldots, 3\} \times\{1, \ldots, 3\},\left\|G_{k l}(0,0,0,0)\right\|_{L^{\infty}(Q)} \leq M / 2, \\
G_{14}=-G_{24}=G_{34}=u_{2}^{*}, G_{41}=G_{42}=G_{43}=G_{44}=0 .
\end{gathered}
$$

Consequently, we study the null-controllability of the linear systems

$$
\begin{cases}\partial_{t} \zeta-D_{2} \Delta \zeta=A \zeta+B_{2} h^{2} 1_{\omega} & \text { in }(0, T) \times \Omega  \tag{63}\\ \frac{\partial \zeta}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega \\ \zeta(0, .)=\zeta_{0} & \text { in } \Omega\end{cases}
$$

where the matrix $A$ verifies the following assumptions

$$
\begin{gather*}
\forall(t, x) \in\left(t_{1}, t_{2}\right) \times \omega_{0}, a_{31}(t, x) \leq-1 / M,  \tag{64}\\
\forall(k, l) \in\{1, \ldots, 3\} \times\{1, \ldots, 3\},\left\|a_{k l}\right\|_{L^{\infty}(Q)} \leq M,  \tag{65}\\
a_{14}=-a_{24}=a_{34}=u_{2}^{*},  \tag{66}\\
a_{41}=a_{42}=a_{43}=a_{44}=0 . \tag{67}
\end{gather*}
$$

Remark 4.4. Actually, we can show the null controllability of a bigger family of linear systems. Indeed, we can replace (66) by the more general assumption: $a_{14}, a_{24}, a_{34} \in \mathbb{R}$ because it does not change the proof of the null-controllability result of the linear systems like (63) (see Proposition 4.8). But, the more general case $a_{14}, a_{24}, a_{34} \in L^{\infty}(Q)$ is not handled by our proof of Proposition 4.8 (see Section 4.3.5 and in particular (137)).

Remark 4.5. The algebraic relation (67) is useful to prove the null-controllability result of the linear systems like (63) (see Proposition 4.8) because it creates the cascade form of (63). Indeed, the fourth and the third equation of (63) are

$$
\partial_{t} \zeta_{4}-d_{4} \Delta \zeta_{4}=\left(d_{3}-d_{4}\right) \Delta \zeta_{3} \text { in }(0, T) \times \Omega, \text { and } d_{3}-d_{4} \neq 0
$$

$\partial_{t} \zeta_{3}-d_{3} \Delta \zeta_{3}=a_{31} \zeta_{1}+a_{32} \zeta_{2}+a_{33} \zeta_{3}+u_{2}^{*} \zeta_{4}$ in $(0, T) \times \Omega$, and $\forall(t, x) \in\left(t_{1}, t_{2}\right) \times \omega_{0}, a_{31}(t, x) \leq-1 / M$.

### 4.1.3. 1 control, adequate change of variables

By Section 2.3.3, we can assume that $u_{3}^{*} \neq 0$.
4.1.3.1. First case: $\exists k \neq l \in\{2,3,4\}, d_{k}=d_{l}$. We treat the case $d_{2}=d_{3}, d_{3} \neq d_{4}$. The other cases are similar. From (28) and (41), system (4) reduces to

$$
\begin{align*}
& \forall i \in\{1,2,4\}, \\
& \begin{cases}\partial_{t} u_{i}-d_{i} \Delta u_{i}=(-1)^{i}\left(u_{1}\left(u_{2}^{*}+u_{3}^{*}-u_{2}\right)-u_{2} u_{4}\right)+h_{i} 1_{\omega} 1_{i \leq 1} & \text { in }(0, T) \times \Omega, \\
\frac{\partial u_{i}}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega, \\
u_{i}(0, .)=u_{i, 0} & \text { in } \Omega .\end{cases} \tag{68}
\end{align*}
$$

We remark that

$$
\begin{align*}
& \left(u_{1}, u_{2}, u_{4}\right)(T, .)=\left(u_{1}^{*}, u_{2}^{*}, u_{4}^{*}\right) \\
& \text { if and only if }  \tag{69}\\
& \left(u_{1}, u_{2}, u_{2}-u_{4}\right)(T, .)=\left(u_{1}^{*}, u_{2}^{*}, u_{2}^{*}-u_{4}^{*}\right) .
\end{align*}
$$

Therefore, we study the system satisfied by $\left(v_{1}, v_{2}, v_{3}\right):=\left(u_{1}, u_{2}, u_{2}-u_{4}\right)$,

$$
\begin{align*}
& \forall 1 \leq i \leq 2, \\
& \begin{cases}\partial_{t} v_{i}-d_{i} \Delta v_{i}=(-1)^{i}\left(v_{1}\left(u_{2}^{*}+u_{3}^{*}-v_{2}\right)-v_{2}\left(v_{2}-v_{3}\right)\right)+h_{i} 1_{\omega} 1_{i \leq 1} & \text { in }(0, T) \times \Omega, \\
\partial_{t} v_{3}-d_{4} \Delta v_{3}=\left(d_{2}-d_{4}\right) \Delta v_{2} & \text { in }(0, T) \times \Omega, \\
\frac{\partial v_{i}}{\partial n}=\frac{\partial v_{3}}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega, \\
\left(v_{i}(0, .), v_{3}(0, .)\right)=\left(u_{i, 0}, u_{2,0}-u_{4,0}\right) & \text { in } \Omega .\end{cases} \tag{70}
\end{align*}
$$

We do not give the complete proof of Theorem 3.2 in this case because it is an easy adaptation of the study of the null-controllability of the linear systems (63) which satisfy (64), (65), (66) and (67) (with three equations instead of four). Indeed, $v_{3}$ can be controlled by $v_{2}$ thanks to the coupling term of second order $\left(d_{2}-d_{4}\right) \Delta v_{2}$ in the second equation of (70) and $v_{2}$ can be controlled by $v_{1}$ because the linearization of the first equation of (70) with $i=2$ is

$$
\partial_{t} v_{2}-d_{2} \Delta v_{2}=u_{3}^{*} v_{1}+\left(-v_{1}^{*}-2 v_{2}^{*}+v_{3}^{*}\right) v_{2}+u_{2}^{*} v_{3} \text { in }(0, T) \times \Omega,
$$

where $\left(v_{1}^{*}, v_{2}^{*}, v_{3}^{*}\right):=\left(u_{1}^{*}, u_{2}^{*}, u_{2}^{*}-u_{4}^{*}\right)$ and $u_{3}^{*} \neq 0$.
4.1.3.2. Second case: $d_{2} \neq d_{3}, d_{3} \neq d_{4}, d_{2} \neq d_{4}$.. We introduce $\alpha \neq \beta$ such that

$$
\begin{equation*}
\alpha\left(d_{2}-d_{4}\right)=\beta\left(d_{3}-d_{4}\right)=1, \text { i.e. } \alpha=\frac{1}{d_{2}-d_{4}} \text { and } \beta=\frac{1}{d_{3}-d_{4}} . \tag{71}
\end{equation*}
$$

Then, we define $\gamma \neq 0$ by the algebraic relation

$$
\begin{equation*}
\alpha-\beta+\gamma=0 \text {, i.e. } \gamma=\beta-\alpha \text {. } \tag{72}
\end{equation*}
$$

We remark that

$$
\left(u_{1}, u_{2}, u_{3}, u_{4}\right)(T, .)=\left(u_{1}^{*}, u_{2}^{*}, u_{3}^{*}, u_{4}^{*}\right)
$$

if and only if

$$
\begin{equation*}
\left(u_{1}, u_{2}, u_{2}+u_{3}, \alpha u_{2}+\beta u_{3}+\gamma u_{4}\right)(T, .)=\left(u_{1}^{*}, u_{2}^{*}, u_{2}^{*}+u_{3}^{*}, \alpha u_{2}^{*}+\beta u_{3}^{*}+\gamma u_{4}^{*}\right) . \tag{73}
\end{equation*}
$$

Therefore, we study the system satisfied by $\left(v_{1}, v_{2}, v_{3}, v_{4}\right):=\left(u_{1}, u_{2}, u_{2}+u_{3}, \alpha u_{2}+\beta u_{3}+\gamma u_{4}\right)$. We introduce the following notations

$$
\begin{equation*}
g_{1}\left(v_{2}, v_{3}, v_{4}\right):=\frac{\beta-\alpha}{\gamma} v_{2}-\frac{\beta}{\gamma} v_{3}+\frac{1}{\gamma} v_{4}=u_{4}, g_{2}\left(v_{2}, v_{3}\right):=v_{3}-v_{2}=u_{3} \tag{74}
\end{equation*}
$$

We have

$$
\begin{align*}
& \forall 1 \leq i \leq 2, \\
& \begin{cases}\partial_{t} v_{i}-d_{i} \Delta v_{i}=(-1)^{i}\left(g_{2}\left(v_{2}, v_{3}\right) v_{1}-g_{1}\left(v_{2}, v_{3}, v_{4}\right) v_{2}\right)+h_{i} 1_{\omega} 1_{i \leq 1} & \text { in }(0, T) \times \Omega, \\
\partial_{t} v_{3}-d_{3} \Delta v_{3}=\left(d_{2}-d_{3}\right) \Delta v_{2} & \text { in }(0, T) \times \Omega, \\
\partial_{t} v_{4}-d_{4} \Delta v_{4}=\Delta v_{3} & \text { in }(0, T) \times \Omega, \\
\frac{\partial v_{i}}{\partial n}=\frac{\partial v_{3}}{\partial n}=\frac{\partial v_{4}}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega, \\
\left(v_{i}, v_{3}, v_{4}\right)(0, .)=\left(u_{i, 0}, u_{2,0}+u_{3,0}, \alpha u_{2,0}+\beta u_{3,0}+\gamma u_{4,0}\right) & \text { in } \Omega .\end{cases} \tag{75}
\end{align*}
$$

Roughly speaking, $v_{4}$ can be controlled by $v_{3}$ thanks to the coupling term of second order $\Delta v_{3}$ in the third equation of (75) and $v_{3}$ can be controlled by $v_{2}$ thanks to the coupling term of second order $\left(d_{2}-d_{3}\right) \Delta v_{2}$ in the second equation of (75) and $v_{2}$ can be controlled by $v_{1}$ because the linearization of the first equation of (75) with $i=2$ is

$$
\begin{array}{rlr}
\partial_{t} v_{2}-d_{2} \Delta v_{2} & =g_{2}\left(v_{2}^{*}, v_{3}^{*}\right) v_{1}-g_{1}\left(v_{2}^{*}, v_{3}^{*}, v_{4}^{*}\right) v_{2}+v_{1}^{*} g_{2}\left(v_{2}, v_{3}\right)-v_{2}^{*} g_{1}\left(v_{2}, v_{3}, v_{4}\right) \\
& =u_{3}^{*} v_{1}-g_{1}\left(v_{2}^{*}, v_{3}^{*}, v_{4}^{*}\right) v_{2}+v_{1}^{*} g_{2}\left(v_{2}, v_{3}\right)-v_{2}^{*} g_{1}\left(v_{2}, v_{3}, v_{4}\right) & \text { in }(0, T) \times \Omega,
\end{array}
$$

and $u_{3}^{*} \neq 0$. Then, the proof of the controllability properties of the linearized-system of (75) follows the ideas of Proposition 1.9 and Proposition 1.10. The main difference is the nature of the coupling terms: two coupling terms of second order $\Delta v_{3},\left(d_{2}-d_{3}\right) \Delta v_{2}$ and one coupling term of zero order $u_{3}^{*} \nu_{1}$.
4.1.3.3. Linearization in $L^{\infty}(Q)$ and null-controllability of a family of linear systems when $d_{2} \neq$ $d_{3}, d_{2} \neq d_{4}, d_{3} \neq d_{4}$. We define

$$
\begin{equation*}
\left(\zeta, h^{1}\right):=\left(\zeta_{1}, \zeta_{2}, \zeta_{3}, \zeta_{4}, h_{1}\right):=\left(v_{1}-u_{1}^{*}, v_{2}-u_{2}^{*}, v_{3}-\left(u_{2}^{*}+u_{3}^{*}\right), v_{4}-\left(\alpha u_{2}^{*}+\beta u_{3}^{*}+\gamma u_{4}^{*}\right), h_{1}\right) . \tag{76}
\end{equation*}
$$

Then, $\left(u, h^{1}\right)$ is a trajectory of (4) if and only if ( $\left.\zeta, h^{1}\right)$ is a trajectory of

$$
\begin{cases}\partial_{t} \zeta-D_{1} \Delta \zeta=G(\zeta) \zeta+B_{1} h^{1} 1_{\omega} & \text { in }(0, T) \times \Omega, \\ \frac{\partial \zeta}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega, \\ \zeta(0, .)=\zeta_{0} & \text { in } \Omega,\end{cases}
$$

where

$$
D_{1}:=\left(\begin{array}{cccc}
d_{1} & 0 & 0 & 0  \tag{77}\\
0 & d_{2} & 0 & 0 \\
0 & d_{2}-d_{3} & d_{3} & 0 \\
0 & 0 & 1 & d_{4}
\end{array}\right), \quad G(\zeta):=\left(\begin{array}{cccc}
-\left(u_{3}^{*}+g_{2}\left(\zeta_{2}, \zeta_{3}\right)\right) & m_{1}+g_{1}\left(\zeta_{2}, \zeta_{3}, \zeta_{4}\right) & -m_{2} & m_{3} \\
u_{3}^{*}+g_{2}\left(\zeta_{2}, \zeta_{3}\right) & -\left(m_{1}+g_{1}\left(\zeta_{2}, \zeta_{3}, \zeta_{4}\right)\right) & m_{2} & -m_{3} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)
$$

with $m_{1}:=u_{1}^{*}+u_{2}^{*}+u_{4}^{*}, m_{2}:=u_{1}^{*}+\frac{\beta}{\gamma} u_{2}^{*}$ and $m_{3}=\frac{1}{\gamma} u_{2}^{*}$. Note that $G_{21}(0,0,0,0)=u_{3}^{*}$. There exist $t_{1}<t_{2} \in(0, T)$, a nonempty open subset $\omega_{0} \subset \subset \omega$ and $M>0$ such that

$$
\begin{gathered}
\forall(t, x) \in\left(t_{1}, t_{2}\right) \times \omega_{0}, G_{21}(0,0,0,0)(t, x) \geq 2 / M, \\
\forall(k, l) \in\{1,2\} \times\{1,2\},\left\|G_{k l}(0,0,0,0)\right\|_{L^{\infty}(Q)} \leq M / 2, \\
G_{13}=-G_{23}=-m_{2}, G_{14}=-G_{24}=m_{3}, G_{k l}=0,3 \leq k \leq 4,1 \leq l \leq 4 .
\end{gathered}
$$

Consequently, we study the null-controllability of the linear systems

$$
\begin{cases}\partial_{t} \zeta-D_{1} \Delta \zeta=A \zeta+B_{1} h^{1} 1_{\omega} & \text { in }(0, T) \times \Omega  \tag{78}\\ \frac{\partial \zeta}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega \\ \zeta(0, .)=\zeta_{0} & \text { in } \Omega,\end{cases}
$$

where the matrix $A$ verifies the following assumptions

$$
\begin{gather*}
\forall(t, x) \in\left(t_{1}, t_{2}\right) \times \omega_{0}, a_{21}(t, x) \geq 1 / M,  \tag{79}\\
\forall(k, l) \in\{1,2\} \times\{1,2\},\left\|a_{k l}\right\|_{L^{\infty}(Q)} \leq M,  \tag{80}\\
a_{13}=-a_{23}=-m_{2}, a_{14}=-a_{24}=m_{3},  \tag{81}\\
a_{k l}=0,3 \leq k \leq 4,1 \leq l \leq 4 \tag{82}
\end{gather*}
$$

Remark 4.6. Actually, we can show the null controllability of a bigger family of linear systems. Indeed, we can replace (81) by the more general assumption: $a_{13}, a_{23}, a_{14}, a_{24} \in \mathbb{R}$ because it does not change the proof of the null-controllability result of the linear systems like (78) (see Proposition 4.8). But, the more general case $a_{13}, a_{23}, a_{14}, a_{24} \in L^{\infty}(Q)$ is not handled by our proof of Proposition 4.8 (see Section 4.3 .7 and in particular (158) and (160)).
Remark 4.7. The algebraic relation (82) is useful to prove the null-controllability result of the linear systems like (78) (see Proposition 4.8) because it creates the cascade form of (78). Indeed, the fourth, the third and the second equation of (78) are

$$
\begin{gathered}
\partial_{t} \zeta_{4}-d_{4} \Delta \zeta_{4}=\Delta \zeta_{3} \text { in }(0, T) \times \Omega, \\
\partial_{t} \zeta_{3}-d_{3} \Delta \zeta_{3}=\left(d_{2}-d_{3}\right) \Delta \zeta_{2} \text { in }(0, T) \times \Omega, \text { and }\left(d_{2}-d_{3}\right) \neq 0, \\
\partial_{t} \zeta_{2}-d_{2} \Delta \zeta_{2}=a_{21} \zeta_{1}+a_{22} \zeta_{2}+m_{2} \zeta_{3}-m_{3} \zeta_{4} \text { in }(0, T) \times \Omega, \text { and } \forall(t, x) \in\left(t_{1}, t_{2}\right) \times \omega_{0}, a_{21}(t, x) \geq 1 / M .
\end{gathered}
$$

4.2. Null controllability in $L^{2}(\Omega)^{4}$ with controls in $L^{\infty}(Q)^{j}$ of a family of linear control systems
4.2.1. Main result of this subsection

We introduce the following notations,

$$
\begin{align*}
& \mathcal{E}_{3}:=\left\{A \in \mathcal{M}_{4}\left(L^{\infty}(Q)\right) ; A \text { verifies the assumptions (54) and (55) }\right\},  \tag{83}\\
& H_{3}:=L^{2}(\Omega)^{4},  \tag{84}\\
& \mathcal{E}_{2}:=\left\{A \in \mathcal{M}_{4}\left(L^{\infty}(Q)\right) ; A \text { verifies the assumptions (64), (65), (66) and (67) }\right\},  \tag{85}\\
& H_{2}:=\left\{\zeta_{0} \in L^{2}(\Omega)^{4} ; \int_{\Omega} \zeta_{0,4}=0\right\},  \tag{86}\\
& \mathcal{E}_{1}:=\left\{A \in \mathcal{M}_{4}\left(L^{\infty}(Q)\right) ; A \text { verifies the assumptions (79), (80), (81) and (82) }\right\},  \tag{87}\\
& H_{1}:=\left\{\zeta_{0} \in L^{2}(\Omega)^{4} ; \int_{\Omega} \zeta_{0,3}=\int_{\Omega} \zeta_{0,4}=0\right\} . \tag{88}
\end{align*}
$$

The main result of this subsection is a null-controllability result in $L^{2}(\Omega)^{4}$ with controls in $L^{\infty}(Q)^{j}$ for families of linear control systems.

Proposition 4.8. Let $j \in\{1,2,3\}, D_{j}$ defined by (52), (62) or (77). There exists $C>0$ such that, for every $A \in \mathcal{E}_{j}$ and $\zeta_{0}=\left(\zeta_{0,1}, \zeta_{0,2}, \zeta_{0,3}, \zeta_{0,4}\right) \in H_{j}$, there exists $h^{j} \in L^{\infty}(Q)^{j}$ satisfying

$$
\begin{equation*}
\left\|h^{j}\right\|_{L^{\infty}(Q)^{j}} \leq C\left\|\zeta_{0}\right\|_{L^{2}(\Omega)^{4}}, \tag{89}
\end{equation*}
$$

such that the solution $\zeta \in Y^{4}$ to the Cauchy problem

$$
\begin{cases}\partial_{t} \zeta-D_{j} \Delta \zeta=A \zeta+B_{j} h^{j} 1_{\omega} & \text { in }(0, T) \times \Omega  \tag{90}\\ \frac{\partial \zeta}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega \\ \zeta(0, .)=\zeta_{0} & \text { in } \Omega\end{cases}
$$

verifies

$$
\zeta(T, .)=0 .
$$

Remark 4.9. For every $1 \leq j \leq 3$, the diffusion matrices $D_{j}$ defined by (52), (62) or (77) verify the assumption of Proposition 2.3 because they are similar to $\operatorname{diag}\left(d_{1}, d_{2}, d_{3}, d_{4}\right)$.
4.2.2. Proof strategy of Proposition 4.8: Null controllability in $L^{2}(\Omega)^{4}$ with controls in $L^{\infty}(Q)^{j}$ of a family of linear control systems

- We let evolve the system without control in $\left(0, t_{1}\right)$ (take $h^{j}(t,)=$.0 in $\left.\left(0, t_{1}\right)\right)$. From Proposition 2.2 and Proposition 2.3, we get the existence of $C>0$ such that for every $A \in \mathcal{E}_{j}, \zeta_{0} \in L^{2}(\Omega)^{4}$, the solution to the Cauchy problem satisfies

$$
\left\|\zeta^{*}\right\|_{L^{2}(\Omega)^{4}} \leq C\left\|\zeta_{0}\right\|_{L^{2}(\Omega)^{4}},
$$

where

$$
\zeta^{*}=\zeta\left(t_{1}, .\right)
$$

- Then, we find $h^{j}:\left(t_{1}, t_{2}\right) \times \Omega \rightarrow \mathbb{R}$ such that

$$
\left\|h^{j}\right\|_{L^{\infty}\left(\left(t_{1}, t_{2}\right) \times \Omega\right)^{j}} \leq C\left\|\zeta\left(t_{1}, .\right)\right\|_{L^{2}(\Omega)^{4}},
$$

and the solution to the Cauchy problem

$$
\begin{cases}\partial_{i} \zeta-D_{j} \Delta \zeta=A \zeta+B_{j} h^{j} 1_{\omega} & \text { in }\left(t_{1}, t_{2}\right) \times \Omega, \\ \frac{\partial \zeta}{\partial n}=0 & \text { on }\left(t_{1}, t_{2}\right) \times \partial \Omega, \\ \zeta\left(t_{1}, .\right)=\zeta^{*} & \text { in } \Omega,\end{cases}
$$

verifies

$$
\zeta\left(t_{2}, .\right)=0 .
$$

- Then, we set $h^{j}(t,)=$.0 so that $h^{j}(t,)=$.0 for $t \in\left(t_{2}, T\right)$.

This strategy gives

$$
\zeta(T, .)=0 \text { and }\left\|h^{j}\right\|_{L^{\infty}((0, T) \times \omega)^{j}} \leq C\left\|\zeta_{0}\right\|_{L^{2}(\Omega)^{4}} .
$$

To simplify, we now suppose

$$
\left(t_{1}, t_{2}\right)=(0, T)
$$

### 4.3. First step: Controls in $L^{2}(Q)^{j}$

The goal of this section is the proof of the following result.
Proposition 4.10. Let $j \in\{1,2,3\}$. There exists $C>0$ such that, for every $A \in \mathcal{E}_{j}$ and for every $\zeta_{0} \in H_{j}$, there exists a control $h^{j} \in L^{2}(Q)^{j}$ satisfying

$$
\begin{equation*}
\left\|h^{j}\right\|_{L^{2}(Q)^{j}} \leq C\left\|\zeta_{0}\right\|_{L^{2}(\Omega)^{4}} \tag{91}
\end{equation*}
$$

such that the solution $\zeta \in Y^{4}$ to the Cauchy problem (90) satisfies $\zeta(T,)=$.0 .
The proof of Proposition 4.10 will be done in Section 4.3 .3 for $j=3$, Section 4.3 .5 for $j=2$, Section 4.3 .7 for $j=1$. It requires technical preliminary results presented in Section 4.3.1, Section 4.3.2, Section 4.3.4, Section 4.3.6.

### 4.3.1. Hilbert Uniqueness Method

First, for $\Phi \in L^{2}(\Omega),(\Phi)_{\Omega}$ denotes the mean value of $\Phi$,

$$
(\Phi)_{\Omega}:=\frac{1}{|\Omega|} \int_{\Omega} \Phi
$$

and for $\Psi \in C\left([0, T] ; L^{2}(\Omega)\right), t \in[0, T]$, we introduce the notation

$$
(\Psi)_{\Omega}(t):=\frac{1}{|\Omega|} \int_{\Omega} \Psi(t, x) d x
$$

By the HUM (Hilbert Uniqueness Method), the null-controllability result of Proposition 4.10 is equivalent to the following observability inequality: (93) (see [15, Theorem 2.44]).

Let $j \in\{1,2,3\}, D_{j}$ defined by (52), (62) or (77). There exists $C>0$ such that, for every $A \in \mathcal{E}_{j}$ and $\varphi_{T} \in H_{j}$ (see (83), (84), (85), (86), (87), (88)) the solution $\varphi$ of

$$
\begin{cases}-\partial_{t} \varphi-D_{j}^{T} \Delta \varphi=A^{T} \varphi & \text { in }(0, T) \times \Omega  \tag{92}\\ \frac{\partial \varphi}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega \\ \varphi(T, .)=\varphi_{T} & \text { in } \Omega,\end{cases}
$$

verifies

$$
\begin{equation*}
\int_{\Omega}|\varphi(0, x)|^{2} d x \leq C\left(\sum_{i=1}^{j} \iint_{(0, T) \times \omega}\left|\varphi_{i}(t, x)\right|^{2} d x d t\right) \tag{93}
\end{equation*}
$$

It is easy to show that it is sufficient to prove the following observability inequalities.
There exists $C>0$ such that, for every $A \in \mathcal{E}_{3}$ and $\varphi_{T} \in L^{2}(\Omega)^{4}$, the solution $\varphi$ of the adjoint system (92) verifies

$$
\begin{equation*}
\int_{\Omega}|\varphi(0, x)|^{2} d x \leq C\left(\sum_{i=1}^{3} \iint_{(0, T) \times \omega}\left|\varphi_{i}(t, x)\right|^{2} d x d t\right) \tag{94}
\end{equation*}
$$

There exists $C>0$ such that, for every $A \in \mathcal{E}_{2}$ and $\varphi_{T} \in L^{2}(\Omega)^{4}$, the solution $\varphi$ of the adjoint system (92) verifies

$$
\begin{equation*}
\sum_{i=1}^{3}\left(\left\|\varphi_{i}(0, .)\right\|_{L^{2}(\Omega)}^{2}\right)+\left\|\varphi_{4}(0, .)-\left(\varphi_{4}\right)_{\Omega}(0)\right\|_{L^{2}(\Omega)}^{2} \leq C\left(\sum_{i=1}^{2} \iint_{(0, T) \times \omega}\left|\varphi_{i}\right|^{2} d x d t\right) \tag{95}
\end{equation*}
$$

There exists $C>0$ such that, for every $A \in \mathcal{E}_{1}$ and $\varphi_{T} \in L^{2}(\Omega)^{4}$, the solution $\varphi$ of the adjoint system (92) verifies

$$
\begin{equation*}
\sum_{i=1}^{2}\left(\left\|\varphi_{i}(0, .)\right\|_{L^{2}(\Omega)}^{2}\right)+\sum_{i=3}^{4}\left(\left\|\varphi_{i}(0, .)-\left(\varphi_{i}\right)_{\Omega}(0)\right\|_{L^{2}(\Omega)}^{2}\right) \leq C\left(\iint_{(0, T) \times \omega}\left|\varphi_{1}\right|^{2} d x d t\right) \tag{96}
\end{equation*}
$$

### 4.3.2. Carleman estimates

We introduce several weight functions. Let $\omega^{\prime \prime} \subset \subset \omega_{0}$ be a nonempty open subset and $\eta_{0} \in C^{2}(\bar{\Omega})$ verifying

$$
\forall x \in \Omega, \eta_{0}(x)>0, \eta_{0}=0 \text { on } \partial \Omega, \forall x \in \overline{\Omega \backslash \omega^{\prime \prime}},\left|\nabla \eta_{0}(x)\right|>0 .
$$

The existence of such a function is proved in [15, Lemma 2.68]. Let $\lambda \geq 1$ a parameter. We remark that

$$
\begin{equation*}
1+f(\lambda):=1+\exp \left(-\lambda\left\|\eta_{0}\right\|_{\infty}\right)<2 \tag{97}
\end{equation*}
$$

We define

$$
\begin{gather*}
\forall(t, x) \in(0, T) \times \Omega, \phi(t, x):=\frac{e^{\lambda \eta_{0}(x)}}{t(T-t)}>0, \alpha(t, x):=\frac{e^{\lambda \eta_{0}(x)}-e^{2 \lambda\left\|\eta_{0}\right\|_{\infty}}}{t(T-t)}<0,  \tag{98}\\
\forall t \in(0, T), \widehat{\alpha}(t):=\min _{x \bar{\Omega}} \alpha(t, x)=\frac{1-e^{2 \lambda\left\|\eta_{0}\right\|_{\infty}}}{t(T-t)}<0, \widehat{\phi}(t):=\min _{x \in \bar{\Omega}} \phi(t, x)=\frac{1}{t(T-t)}>0 . \tag{99}
\end{gather*}
$$

## Theorem 4.11. Carleman inequality

Let $d \in(0,+\infty), \omega^{\prime}$ an open subset such that $\omega^{\prime \prime} \subset \subset \omega^{\prime} \subset \subset \omega_{0}$ and $\beta \in \mathbb{R}$. There exist $C=C\left(\Omega, \omega^{\prime}, \beta\right), \lambda_{0}=C\left(\Omega, \omega^{\prime}, \beta\right), s_{0}=s_{0}\left(\Omega, \omega^{\prime}, \beta\right)$ such that, for any $\lambda \geq \lambda_{0}, s \geq s_{0}\left(T+T^{2}\right)$, $\varphi_{T} \in L^{2}(\Omega)$ and $f \in L^{2}(Q)$, the solution $\varphi$ to

$$
\begin{cases}-\partial_{t} \varphi-d \Delta \varphi=f & \text { in }(0, T) \times \Omega \\ \frac{\partial \varphi}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega \\ \varphi(T, .)=\varphi_{T} & \text { in } \Omega,\end{cases}
$$

satisfies

$$
\begin{align*}
I(\beta, \lambda, s, \varphi) & :=\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}\left(\lambda^{4}(s \phi)^{\beta+3}|\varphi|^{2}+\lambda^{2}(s \phi)^{\beta+1}|\nabla \varphi|^{2}+(s \phi)^{\beta-1}\left(\left|\partial_{t} \varphi\right|^{2}+|\Delta \varphi|^{2}\right)\right) d x d t \\
& \leq C\left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{\beta}|f|^{2} d x d t+\int_{0}^{T} \int_{\omega^{\prime}} \lambda^{4} e^{2 s \alpha}(s \phi)^{\beta+3}|\varphi|^{2} d x d t\right) \tag{100}
\end{align*}
$$

The original proof of this inequality can be found in [29, Lemma 1.2].
Remark 4.12. For a general introduction to global Carleman inequalities and their applications to the controllability of parabolic systems, one can see [28] (in particular, see [28, Lemma 1.3]). For Neumann conditions, one can see [27] and in particular [27, Lemma 1].
4.3.2.1. A parabolic regularity result in $L^{2}$. In the following, we consider initial conditions $\varphi_{T} \in$ $C_{0}^{\infty}(\Omega)^{4}$ in order to improve the regularity of $\varphi$, solution of (92), and to allow some computations.
Definition 4.13. We define the following spaces of functions

$$
H_{N e}^{2}(\Omega):=\left\{u \in H^{2}(\Omega) ; \frac{\partial u}{\partial n}=0\right\}, \quad Y_{2}:=L^{2}\left(0, T ; H_{N e}^{2}(\Omega)\right) \cap H^{1}\left(0, T ; L^{2}(\Omega)\right)
$$

Proposition 4.14. Let $k \in \mathbb{N}^{*}, D \in \mathcal{M}_{k}(\mathbb{R})$ such that $S p(D) \subset(0,+\infty), A \in \mathcal{M}_{k}\left(L^{\infty}(Q)\right)$, $u_{0} \in C_{0}^{\infty}(\Omega)^{k}$. From [20, Theorem 2.1], the following Cauchy problem admits a unique solution $u \in Y_{2}^{k}$

$$
\begin{cases}\partial_{t} u-D \Delta u=A(t, x) u & \text { in }(0, T) \times \Omega, \\ \frac{\partial u}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega, \\ u(0, .)=u_{0} & \text { in } \Omega .\end{cases}
$$

4.3.2.2. A technical lemma for Carleman estimates. By now, unless otherwise specified, we denote by $C$ (respectively $C_{\varepsilon}$ ) various positive constants varying from line to line (respectively various positive constants varying from line to line and depending on the parameter $\varepsilon$ ). We insist on the fact that $C$ and $C_{\varepsilon}$ do not depend on $\lambda$ and $s$, unless otherwise specified.
Lemma 4.15. Let $\Phi, \Psi \in Y_{2}, a \in L^{\infty}(Q)$, an open subset $\widetilde{\omega} \subset \omega_{0}, \Theta \in C^{\infty}(\bar{\Omega} ;[0,+\infty[)$ such that $\operatorname{supp}(\Theta) \subset \widetilde{\omega}$ and $r \in \mathbb{N}$. Then, for every $\varepsilon>0$,

$$
\begin{align*}
& \forall(k, l) \in \mathbb{R}^{2}, k+l=2 r, \forall s \geq C \\
& \left|\iint_{(0, T) \times \widetilde{\omega}} \Theta e^{2 s \alpha}(s \phi)^{r} a \Phi \Psi\right| \leq \varepsilon \iint_{(0, T) \times \Omega} e^{2 s \alpha}(s \phi)^{k}|\Phi|^{2}+C_{\varepsilon} \iint_{(0, T) \times \bar{\omega}} e^{2 s \alpha}(s \phi)^{l}|\Psi|^{2},  \tag{101}\\
& \forall(k, l) \in \mathbb{R}^{2}, k+l=2(r+2), \forall s \geq C \\
& \left|\int_{0}^{T} \int_{\widetilde{\omega}} \Theta e^{2 s \alpha}(s \phi)^{r} \Phi \partial_{t} \Psi\right| \leq \varepsilon\left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{k}|\Phi|^{2}+\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{k-4}\left|\partial_{t} \Phi\right|^{2}\right) \\
& +C_{\varepsilon} \int_{0}^{T} \int_{\widetilde{\omega}} e^{2 s \alpha}(s \phi)^{l}|\Psi|^{2} \tag{102}
\end{align*}
$$

$$
\begin{align*}
& \forall(k, l) \in \mathbb{R}^{2}, k+l=2(r+2), \forall s \geq C \\
& \left|\int_{0}^{T} \int_{\widetilde{\omega}} \Theta e^{2 s \alpha}(s \phi)^{r} \Phi \Delta \Psi\right| \\
& \leq \varepsilon\left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{k}|\Phi|^{2}+\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{k-2}|\nabla \Phi|^{2}+\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{k-4}|\Delta \Phi|^{2}\right) \\
& \quad+C_{\varepsilon} \int_{0}^{T} \int_{\widetilde{\omega}} e^{2 s \alpha}(s \phi)^{l}|\Psi|^{2} \tag{103}
\end{align*}
$$

$$
\begin{align*}
& \forall(k, l) \in \mathbb{R}^{2}, k+l=2 r, \forall s \geq C \\
& \begin{aligned}
\int_{0}^{T} \int_{\widetilde{\omega}} \Theta e^{2 s \alpha}(s \phi)^{r}|\nabla \Phi|^{2} \leq \varepsilon & \left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{k}|\Delta \Phi|^{2}+\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{k+2}|\nabla \Phi|^{2}\right) \\
& +C_{\varepsilon} \int_{0}^{T} \int_{\widetilde{\omega}} e^{2 s \alpha}(s \phi)^{l}|\Phi|^{2}
\end{aligned}
\end{align*}
$$

Proof. The inequality (101) is an easy consequence of Young's inequality applied to

$$
\left|\iint_{(0, T) \times \widetilde{\omega}} \Theta e^{2 s \alpha}(s \phi)^{r} a \Phi \Psi\right| \leq C \iint_{(0, T) \times \widetilde{\omega}}\left(\sqrt{\varepsilon} e^{s \alpha}(s \phi)^{k / 2}|\Phi|\right)\left(\frac{1}{\sqrt{\varepsilon}} \Theta e^{s \alpha}(s \phi)^{l / 2}|\Psi|\right)
$$

For (102), we integrate by parts with respect to the time variable

$$
-\int_{0}^{T} \int_{\widetilde{\omega}} \Theta e^{2 s \alpha}(s \phi)^{r} \Phi \partial_{t} \Psi=\int_{0}^{T} \int_{\widetilde{\omega}} \Theta e^{2 s \alpha}(s \phi)^{r} \partial_{t}(\Phi) \Psi+\int_{0}^{T} \int_{\widetilde{\omega}}\left(\Theta e^{2 s \alpha}(s \phi)^{r}\right)_{t} \Phi \Psi
$$

Moreover, by (98), we have $\left|\left(\Theta e^{2 s \alpha}(s \phi)^{r}\right)_{t}\right| \leq C e^{2 s \alpha} s^{r+1} \phi^{r+2} \leq e^{2 s \alpha} s^{r+2} \phi^{r+2}$ for $s \geq C$. Then, we get (102) by applying Young's inequality to

$$
\begin{aligned}
\left|\int_{0}^{T} \int_{\widetilde{\omega}} \Theta e^{2 s \alpha}(s \phi)^{r} \Phi \partial_{t} \Psi\right| & \leq \int_{0}^{T} \int_{\widetilde{\omega}}\left(\sqrt{\varepsilon} e^{s \alpha}(s \phi)^{k / 2-2} \partial_{t} \Phi\right)\left(\frac{1}{\sqrt{\varepsilon}} \Theta e^{s \alpha}(s \phi)^{l / 2} \Psi\right) \\
& +\int_{0}^{T} \int_{\widetilde{\omega}}\left(\sqrt{\varepsilon} e^{s \alpha}(s \phi)^{k / 2} \Phi\right)\left(\frac{1}{\sqrt{\varepsilon}} e^{s \alpha}(s \phi)^{l / 2} \Psi\right)
\end{aligned}
$$

For (103), by twice integrating by parts with respect to the spatial variable, we get

$$
\int_{0}^{T} \int_{\widetilde{\omega}} \Theta e^{2 s \alpha}(s \phi)^{r} \Phi \Delta \Psi=\int_{0}^{T} \int_{\widetilde{\omega}} \Delta\left(\Theta e^{2 s \alpha}(s \phi)^{r} \Phi\right) \Psi
$$

Moreover, by (98), we have

$$
\mid \Delta\left(\Theta e^{2 s \alpha}(s \phi)^{r} \Phi \mid \leq C\left(e^{2 s \alpha}(s \phi)^{r}|\Delta \Phi|+e^{2 s \alpha}(s \phi)^{r+1}|\nabla \Phi|+e^{2 s \alpha}(s \phi)^{r+2}|\Phi|\right)\right.
$$

Then, we deduce (103) by Young's inequality applied to

$$
\begin{aligned}
\left|\int_{0}^{T} \int_{\widetilde{\omega}} \Theta e^{2 s \alpha}(s \phi)^{r} \Phi \Delta \Psi\right| & \leq \int_{0}^{T} \int_{\widetilde{\omega}}\left(\sqrt{\varepsilon} e^{s \alpha}(s \phi)^{k / 2-2}|\Delta \Phi|\right)\left(\frac{1}{\sqrt{\varepsilon}} e^{s \alpha}(s \phi)^{l / 2} \Psi\right) \\
& +\int_{0}^{T} \int_{\widetilde{\omega}}\left(\sqrt{\varepsilon} e^{s \alpha}(s \phi)^{k / 2-1}|\nabla \Phi|\right)\left(\frac{1}{\sqrt{\varepsilon}} e^{s \alpha}(s \phi)^{l / 2} \Psi\right) \\
& +\int_{0}^{T} \int_{\widetilde{\omega}}\left(\sqrt{\varepsilon} e^{s \alpha}(s \phi)^{k / 2}|\Phi|\right)\left(\frac{1}{\sqrt{\varepsilon}} e^{s \alpha}(s \phi)^{l / 2} \Psi\right)
\end{aligned}
$$

For (104), we integrate by parts with respect to the spatial variable,

$$
\int_{0}^{T} \int_{\widetilde{\omega}} \Theta e^{2 s \alpha}(s \phi)^{r}|\nabla \Phi|^{2}=-\int_{0}^{T} \int_{\widetilde{\omega}} \Theta e^{2 s \alpha}(s \phi)^{r}(\Delta \Phi) \Phi-\int_{\omega_{0}} \nabla\left(\Theta e^{2 s \alpha}(s \phi)^{r}\right) \cdot(\nabla \Phi) \Phi .
$$

By using $\left|\nabla\left(\Theta e^{2 s \alpha}(s \phi)^{r}\right)\right| \leq C e^{2 s \alpha}(s \phi)^{r+1}$ which is a consequence of (98), we get (104) by Young's inequality. This concludes the proof of Lemma 4.15.

### 4.3.3. Proof with observation on three components: (94)

Proof. $j=3$
The proof is close to the proof of [17, Lemma 7].
Let $A \in \mathcal{E}_{3}$ (see (83)), $\varphi_{T} \in C_{0}^{\infty}(\Omega)^{4}$ (the general case comes from a density argument, see (119), Lemma 4.21 and Lemma 4.22), $\varphi \in Y_{2}^{4}$ be the solution of (92) (see Proposition 4.14) and $\omega_{1}$ be an open subset such that $\omega^{\prime \prime} \subset \subset \omega_{1} \subset \subset \omega_{0}$. We have

$$
\forall 1 \leq i \leq 4, \begin{cases}-\partial_{t} \varphi_{i}-d_{i} \Delta \varphi_{i}=a_{1 i} \varphi_{1}+a_{2 i} \varphi_{2}+a_{3 i} \varphi_{3}+a_{4 i} \varphi_{4} & \text { in }(0, T) \times \Omega  \tag{105}\\ \frac{\partial \varphi_{i}}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega\end{cases}
$$

We apply (100) of Theorem 4.11 to each $\varphi_{i}, 1 \leq i \leq 4$, with $\omega^{\prime}=\omega_{1}$ and $\beta=0$. Then, we sum (by using (55)): for every $\lambda \geq C$,

$$
\begin{equation*}
\sum_{i=1}^{4} I\left(0, \lambda, s, \varphi_{i}\right) \leq C\left(\sum_{i=1}^{4}\left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}\left|\varphi_{i}\right|^{2} d x d t+\int_{0}^{T} \int_{\omega_{1}} \lambda^{4} e^{2 s \alpha}(s \phi)^{3}\left|\varphi_{i}\right|^{2} d x d t\right)\right) \tag{106}
\end{equation*}
$$

We fix $\lambda \geq C$ and we take $s$ sufficiently large, then we can absorb the first right hand side term by the left hand side term of (106). We get

$$
\begin{equation*}
\sum_{i=1}^{4} I\left(0, \lambda, s, \varphi_{i}\right) \leq C \sum_{i=1}^{4} \int_{0}^{T} \int_{\omega_{1}} e^{2 s \alpha}(s \phi)^{3}\left|\varphi_{i}\right|^{2} d x d t \tag{107}
\end{equation*}
$$

Now, $\lambda, s$ are supposed to be fixed such that (107) holds and the constant $C$ may depend on $\lambda, s$.

We have to get rid of the term $\int_{0}^{T} \int_{\omega_{1}} e^{2 s \alpha}(s \phi)^{3}\left|\varphi_{4}\right|^{2} d x d t$ in order to prove the observability inequality (94). For this, we are going to use (54). So, we are going to estimate $\varphi_{4}$ by $\varphi_{i}$ for every $1 \leq i \leq 3$ thanks to the first equation of (105) with $i=1$.
$\underline{\text { Estimate of } \int_{0}^{T} \int_{\omega_{1}} e^{2 s \alpha}(s \phi)^{3}\left|\varphi_{4}\right|^{2} d x d t}$.
Let us introduce $\chi \in C^{\infty}\left(\bar{\Omega} ;\left[0,+\infty[)\right.\right.$, such that the support of $\chi$ is included in $\omega_{0}$ and $\chi=1$ in $\omega_{1}$. We multiply the first equation of (105) with $i=1$ by $\chi(x) e^{2 s \alpha}(s \phi)^{3} \varphi_{4}$ and we integrate on
$(0, T) \times \omega_{0}$, which leads to

$$
\begin{align*}
& \int_{0}^{T} \int_{\omega_{1}} e^{2 s \alpha}(s \phi)^{3}\left|\varphi_{4}\right|^{2} d x d t \\
& \leq M \int_{0}^{T} \int_{\omega_{1}} e^{2 s \alpha}(s \phi)^{3} a_{41}\left|\varphi_{4}\right|^{2} d x d t \text { by } \\
& \leq M \int_{0}^{T} \int_{\omega_{0}} \chi(x) e^{2 s \alpha}(s \phi)^{3} a_{41}\left|\varphi_{4}\right|^{2} d x d t \\
& \leq M \int_{0}^{T} \int_{\omega_{0}} \chi(x) e^{2 s \alpha}(s \phi)^{3} \varphi_{4}\left(-\partial_{t} \varphi_{1}-d_{1} \Delta \varphi_{1}-a_{11} \varphi_{1}-a_{21} \varphi_{2}-a_{31} \varphi_{3}\right) d x d t \tag{108}
\end{align*}
$$

Remark 4.16. In Section 4.1.1, we suppose that if $\left(u_{1}^{*}, u_{3}^{*}, u_{4}^{*}\right) \neq(0,0,0)$, then $u_{3}^{*} \neq 0$. Consequently, we have (54). If, $u_{1}^{*} \neq 0$ (or respectively $u_{4}^{*} \neq 0$ ), we can easily adapt the preceding strategy. We can assume that

$$
\left.\forall(t, x) \in\left(t_{1}, t_{2}\right) \times \omega_{0}, a_{43}(t, x) \geq 1 / M \text { (or respectively } a_{42}(t, x) \leq-1 / M\right)
$$

and multiply the first equation of (105) with $i=3$ (or respectively $i=2$ ) by $\chi(x) e^{2 s \alpha}(s \phi)^{3} \varphi_{4}$ (or $\left.-\chi(x) e^{2 s \alpha}(s \phi)^{3} \varphi_{4}\right)$ and we integrate on $(0, T) \times \omega_{0}$.

Let $\varepsilon>0$ which will be chosen small enough. Now, we want to estimate the right hand side term of (108) by $\sum_{i=1}^{3} \int_{0}^{T} \int_{\omega_{1}} e^{2 s \alpha}(s \phi)^{m}\left|\varphi_{i}\right|^{2} d x d t$ with $m \in \mathbb{N}$.

First, we treat the terms $\int_{0}^{T} \int_{\omega_{0}} \chi(x) e^{2 s \alpha}(s \phi)^{3} \varphi_{4} a_{j 1} \varphi_{j} d x d t$, for every $1 \leq j \leq 3$. By applying Lemma 4.15: (101) with $\Phi=\varphi_{4}, \stackrel{\Psi}{\omega_{0}}=\varphi_{j}, a=a_{j 1}$ (recalling (55)), $\Theta=\chi, r=3$ and $(k, l)=(3,3)$, we have

$$
\begin{align*}
& \left|\iint_{(0, T) \times \omega_{0}} \chi(x) e^{2 s \alpha}(s \phi)^{3} \varphi_{4} a_{j 1}(t, x) \varphi_{j} d x d t\right| \\
& \leq \varepsilon \iint_{(0, T) \times \Omega} e^{2 s \alpha}(s \phi)^{3}\left|\varphi_{4}\right|^{2} d x d t+C_{\varepsilon} \iint_{(0, T) \times \omega_{0}} e^{2 s \alpha}(s \phi)^{3}\left|\varphi_{j}\right|^{2} d x d t \tag{109}
\end{align*}
$$

Then, we treat the term $-\int_{0}^{T} \int_{\omega_{0}} \chi(x) e^{2 s \alpha}(s \phi)^{3} \varphi_{4} \partial_{t} \varphi_{1} d x d t$. By applying Lemma 4.15: (102) with $\Phi=\varphi_{4}, \Psi=\varphi_{1}, a=1, \Theta \stackrel{\chi}{=} \chi, r=3$ and $(k, l)=(3,7)$, we have

$$
\begin{align*}
\left|\int_{0}^{T} \int_{\omega_{0}} \chi e^{2 s \alpha}(s \phi)^{3} \varphi_{4} \partial_{t} \varphi_{1}\right| & \leq \varepsilon\left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}\left\{(s \phi)^{3}\left|\varphi_{4}\right|^{2}+(s \phi)^{-1}\left|\partial_{t} \varphi_{4}\right|^{2}\right\}\right) \\
& +C_{\varepsilon} \int_{0}^{T} \int_{\omega_{0}} e^{2 s \alpha}(s \phi)^{7}\left|\varphi_{1}\right|^{2} \tag{110}
\end{align*}
$$

Finally, the last term $-d_{1} \int_{0}^{T} \int_{\omega_{0}} \chi(x) e^{2 s \alpha}(s \phi)^{3} \varphi_{4} \Delta \varphi_{1} d x d t$ is estimated as follows. By applying Lemma 4.15: (103) with $\Phi=\varphi_{4}, \Psi=\varphi_{1}, a=1, \Theta=\chi, r=3$ and $(k, l)=(3,7)$, we have

$$
\begin{align*}
\left|d_{1} \int_{0}^{T} \int_{\omega_{0}} \chi e^{2 s \alpha}(s \phi)^{3} \varphi_{4} \Delta \varphi_{1}\right| & \leq \varepsilon\left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}\left\{(s \phi)^{-1}\left|\Delta \varphi_{4}\right|^{2}+(s \phi)\left|\nabla \varphi_{4}\right|^{2}+(s \phi)^{3}\left|\varphi_{4}\right|^{2}\right\}\right) \\
& +C_{\varepsilon} \int_{0}^{T} \int_{\omega_{0}} e^{2 s \alpha}(s \phi)^{7}\left|\varphi_{1}\right|^{2} \tag{111}
\end{align*}
$$

Gathering (107), (108), (109), (110), (111), we get

$$
\begin{align*}
\sum_{i=1}^{4} I\left(0, \lambda, s, \varphi_{i}\right) & \leq 3 \varepsilon\left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}\left\{(s \phi)^{3}\left|\varphi_{4}\right|^{2}+(s \phi)\left|\nabla \varphi_{4}\right|^{2}+(s \phi)^{-1}\left(\left|\partial_{t} \varphi_{4}\right|^{2}+\left|\Delta \varphi_{4}\right|^{2}\right)\right\}\right) \\
& +C_{\varepsilon}\left(\sum_{i=1}^{3} \int_{0}^{T} \int_{\omega_{0}} e^{2 s \alpha}(s \phi)^{7}\left|\varphi_{i}\right|^{2} d x d t\right) \tag{112}
\end{align*}
$$

By taking $\varepsilon$ small enough, we get

$$
\begin{equation*}
\sum_{i=1}^{4} I\left(0, \lambda, s, \varphi_{i}\right) \leq C_{\varepsilon}\left(\sum_{i=1}^{3} \int_{0}^{T} \int_{\omega_{0}} e^{2 s \alpha}(s \phi)^{7}\left|\varphi_{i}\right|^{2} d x d t\right) \tag{113}
\end{equation*}
$$

In particular, we deduce from (113) that

$$
\begin{equation*}
\sum_{i=1}^{4} \int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{3}\left|\varphi_{i}\right|^{2} \leq C\left(\sum_{i=1}^{3} \int_{0}^{T} \int_{\omega_{0}} e^{2 s \alpha}(s \phi)^{7}\left|\varphi_{i}\right|^{2} d x d t\right) \tag{114}
\end{equation*}
$$

Then, by using the facts that

$$
\begin{equation*}
\min _{[T / 4,3 T / 4] \times \bar{\Omega}} e^{2 s \alpha}(s \phi)^{3}>0, \tag{115}
\end{equation*}
$$

and

$$
\begin{equation*}
e^{2 s \alpha}(s \phi)^{7} \in L^{\infty}((0, T) \times \Omega) \tag{116}
\end{equation*}
$$

we get

$$
\begin{equation*}
\sum_{i=1}^{4} \int_{T / 4}^{3 T / 4} \int_{\Omega}\left|\varphi_{i}\right|^{2} d x d t \leq C\left(\sum_{i=1}^{3} \int_{0}^{T} \int_{\omega_{0}}\left|\varphi_{i}\right|^{2} d x d t\right) \tag{117}
\end{equation*}
$$

From the dissipation of the energy in time for (105) (see Lemma A. 1 in the Appendix), we easily get

$$
\begin{equation*}
\|\varphi(0, .)\|_{L^{2}(\Omega)^{4}}^{2} \leq C\left(\sum_{i=1}^{4} \int_{T / 4}^{3 T / 4} \int_{\Omega}\left|\varphi_{i}\right|^{2} d x d t\right) \tag{118}
\end{equation*}
$$

Then, by using (117) and (118), we obtain

$$
\begin{equation*}
\|\varphi(0, .)\|_{L^{2}(\Omega)^{4}}^{2} \leq C\left(\sum_{i=1}^{3} \int_{0}^{T} \int_{\omega_{0}}\left|\varphi_{i}\right|^{2} d x d t\right) \tag{119}
\end{equation*}
$$

This ends the proof of the observability inequality (94) because $\omega_{0} \subset \omega$.

## Remark 4.17. Some stronger observability inequalities

We also have the following stronger inequality than (119) which can be proved from (114), (115) and (118). It will be used to find controls in $L_{w g h t}^{2}(Q) \subset L^{2}(Q)$ (see Section 4.4.1). We have

$$
\begin{equation*}
\|\varphi(0, .)\|_{L^{2}(\Omega)^{4}}^{2} \leq C\left(\sum_{i=1}^{3} \int_{0}^{T} \int_{\omega} e^{2 s \alpha}(s \phi)^{7}\left|\varphi_{i}\right|^{2} d x d t\right) \tag{120}
\end{equation*}
$$

Moreover, we also have an even stronger inequality (see (114)) than (119) and (120). It will be used to find controls in $L^{\infty}(Q)$ (see Section 4.4.2).

### 4.3.4. Density results

In this section, we show that we can assume that the data $\varphi_{T}$ is regular i.e. $\varphi_{T} \in C_{0}^{\infty}(\Omega)^{4}$. Moreover, we also need some regularity on the coupling matrix $A$ for the case $j=1$. It's the purpose of Lemma 4.18.

Lemma 4.18. Let $a \in L^{\infty}(Q)$. There exists $\left(a_{k}\right) \in\left(C_{0}^{\infty}(Q)\right)^{\mathbb{N}}$ such that

$$
\begin{gather*}
\left\|a_{k}\right\|_{L^{\infty}(Q)} \leq\|a\|_{L^{\infty}(Q)},  \tag{121}\\
a_{k} \underset{k \rightarrow+\infty}{ }{ }^{*} a \text { in } L^{\infty}(Q) . \tag{122}
\end{gather*}
$$

Proof. Let $k \in \mathbb{N}^{*}, \alpha_{k} \in C_{0}^{\infty}((0, T) ;[0,1]), \alpha_{k}(t)=1$ in $(1 / k, T-1 / k), \beta_{k} \in C_{0}^{\infty}((\Omega) ;[0,1])$, $\beta_{k}(x)=1$ in $\{x \in \Omega ; d(x, \partial \Omega) \geq 1 / k\}$ and $\xi_{k} \in C_{0}^{\infty}(Q)$ be defined by $\xi_{k}(t, x)=\alpha_{k}(t) \beta_{k}(x)$. Let $\rho_{k}$ be a mollifier sequence in $Q$ such that $\int_{Q} \rho_{k}=1$.

Then, it is easy to show that $a_{k}:=\xi_{k} \cdot\left(\rho_{k} * a\right)$ satisfies the conclusion of Lemma 4.18.
Remark 4.19. Actually, the previous lemma shows the density of $C_{0}^{\infty}(Q)$ in $L^{\infty}(Q)$ for the weakstar topology.

We also recall a particular case of the Aubin-Lions' lemma which is useful for the proof of Lemma 4.21.

Lemma 4.20. [48, Section 8, Corollary 4]
A bounded subset of $Y$ (see Definition 2.1) is relatively compact in $L^{2}(Q)$.
Lemma 4.21. Let $j \in\{1,2,3\}, D_{j}$ defined by (52), (62) or (77), $A \in \mathcal{E}_{j}$ (see (83), (85) and (87)), $\varphi_{T} \in L^{2}(\Omega)^{4}$. We assume that

$$
\begin{gather*}
\varphi_{T, k} \in C_{0}^{\infty}(\Omega)^{4} \underset{k \rightarrow+\infty}{\rightarrow} \varphi_{T} \text { in } L^{2}(\Omega)^{4}  \tag{123}\\
A_{k} \in \mathcal{M}_{4}\left(C_{0}^{\infty}(Q)\right) \underset{k \rightarrow+\infty}{\rightharpoonup^{*}} A \text { in } L^{\infty}(Q)^{16} \tag{124}
\end{gather*}
$$

Then, the sequence of solutions $\varphi_{k} \in Y^{4}$ of

$$
\begin{cases}-\partial_{t} \varphi_{k}-D_{j}^{T} \Delta \varphi_{k}=A_{k}^{T} \varphi_{k} & \text { in }(0, T) \times \Omega  \tag{125}\\ \frac{\partial \varphi_{k}}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega \\ \varphi_{k}(T, .)=\varphi_{T, k} & \text { in } \Omega,\end{cases}
$$

weakly converges in $Y^{4}$ and strongly converges in $L^{2}(Q)^{4}$ to $\varphi$, the solution of (92).
Proof. First, recalling (123), we remark that $\left(\varphi_{T, k}\right)_{k \in \mathbb{N}}$ is bounded in $L^{2}(\Omega)^{4}$. Secondly, recalling (124), we remark that $\left(A_{k}\right)$ is bounded in $\mathcal{M}_{4}\left(L^{\infty}(Q)\right.$ ). Then, from Proposition 2.3: (17), we get that $\left(\varphi_{k}\right)_{k \in \mathbb{N}}$ is bounded in $Y^{4}$. Then, up to a subsequence, we can suppose that there exists $\widetilde{\varphi} \in Y^{4}$ such that

$$
\begin{equation*}
\varphi_{k} \underset{k \rightarrow+\infty}{\rightharpoonup} \widetilde{\varphi} \text { in } Y^{4} \tag{126}
\end{equation*}
$$

By Proposition 2.2, we can also suppose that

$$
\begin{equation*}
\varphi_{k}(T, .) \underset{k \rightarrow+\infty}{\rightharpoonup} \widetilde{\varphi}(T, .) \text { in } L^{2}(\Omega)^{4} \tag{127}
\end{equation*}
$$

But, by (123), we deduce that

$$
\begin{equation*}
\varphi_{k}(T, .)=\varphi_{T, k} \underset{k \rightarrow+\infty}{\rightharpoonup} \varphi_{T} \text { in } L^{2}(\Omega)^{4} \tag{128}
\end{equation*}
$$

Therefore, by (127) and (128), we get

$$
\begin{equation*}
\widetilde{\varphi}(T, .)=\varphi_{T} \tag{129}
\end{equation*}
$$

By Lemma 4.20, up to a subsequence, we can also assume that

$$
\begin{equation*}
\varphi_{k} \underset{k \rightarrow+\infty}{\rightarrow} \widetilde{\varphi} \text { in } L^{2}(Q)^{4} \tag{130}
\end{equation*}
$$

Consequently, from (130) and (124), we have

$$
\begin{equation*}
A_{k}^{T} \varphi_{k} \underset{k \rightarrow+\infty}{\rightharpoonup} A^{T} \widetilde{\varphi} \text { in } L^{2}(Q)^{4} \tag{131}
\end{equation*}
$$

By using (126), (131), (129) and by letting $k \rightarrow+\infty$ in (125), we have

$$
\begin{cases}-\partial_{t} \widetilde{\varphi}-D_{j}^{T} \Delta \widetilde{\varphi}=A^{T} \widetilde{\varphi} & \text { in }(0, T) \times \Omega  \tag{132}\\ \frac{\partial \widetilde{\varphi}}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega \\ \bar{\varphi}(T, .)=\varphi_{T} & \text { in } \Omega\end{cases}
$$

By uniqueness in Proposition 2.3, we have $\widetilde{\varphi}=\varphi$. Then, $\left(\varphi_{k}\right)_{k \in \mathbb{N}}$ only has one limit-value: $\varphi$ for the weak-convergence in $Y^{4}$ and for the strong convergence in $L^{2}(Q)^{4}$. The sequence $\left(\varphi_{k}\right)_{k \in \mathbb{N}}$ is relatively compact in $Y$ equipped with the weak topology and $\left(\varphi_{k}\right)_{k \in \mathbb{N}}$ is relatively compact in $L^{2}(Q)^{4}$ equipped with the strong topology. Therefore,

$$
\begin{gathered}
\varphi_{k} \underset{k \rightarrow+\infty}{\rightarrow} \varphi \text { in } Y^{4}, \\
\varphi_{k} \underset{k \rightarrow+\infty}{\rightarrow} \varphi \text { in } L^{2}(Q)^{4} .
\end{gathered}
$$

This concludes the proof of Lemma 4.21.
Lemma 4.22. Let us suppose that $\left(\varphi_{k}\right)_{k \in \mathbb{N}} \in Y^{\mathbb{N}}$ weakly converges to $\varphi$ in $Y$ and strongly converges to $\varphi$ in $L^{2}(Q)$. Then, we have

$$
\begin{gathered}
\forall r \in \mathbb{N}, \int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{r}\left|\varphi_{k}\right|^{2} d x d t \underset{k \rightarrow+\infty}{\rightarrow} \int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{r}|\varphi|^{2} d x d t, \\
\|\varphi(0, .)\|_{L^{2}(\Omega)} \leq \liminf _{k \rightarrow+\infty}\left\|\varphi_{k}(0, .)\right\|_{L^{2}(\Omega)}
\end{gathered}
$$

Proof. The result is a consequence of the fact that $e^{2 s \alpha}(s \phi)^{r} \in L^{\infty}(Q)$ and Proposition 2.2.

### 4.3.5. Proof with observation on two components: (95)

4.3.5.1. Another parabolic regularity result. For the cases $j=2$ (2 controls) and $j=1$ ( 1 control), the diffusion matrix is not diagonal (see (62) and (77)). It creates coupling terms of second order. Roughly speaking, we differentiate some equations of the adjoint system (92) in order to benefit from these coupling terms before applying Carleman estimates. The following lemma justifies this strategy.

Lemma 4.23. Let $d \in(0,+\infty), f \in L^{2}\left(0, T ; H_{N e}^{2}(\Omega)\right)$ and $y_{0} \in C_{0}^{\infty}(\Omega)$. Let $y \in Y_{2}$ be the solution of

$$
\begin{cases}\partial_{t} y-d \Delta y=f & \text { in }(0, T) \times \Omega  \tag{133}\\ \frac{\partial y}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega \\ y(0, .)=y_{0} & \text { in } \Omega\end{cases}
$$

Then, $z:=\Delta y \in Y_{2}$ is the solution of

$$
\begin{cases}\partial_{t} z-d \Delta z=\Delta f & \text { in }(0, T) \times \Omega,  \tag{134}\\ \frac{\partial z}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega, \\ z(0, .)=\Delta y_{0} & \text { in } \Omega\end{cases}
$$

Proof. Let $\widetilde{z} \in Y_{2}$ be the solution of

$$
\begin{cases}\partial_{t} \widetilde{z}-d \Delta \widetilde{z}=\Delta f & \text { in }(0, T) \times \Omega,  \tag{135}\\ \frac{\partial z}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega, \\ \frac{\partial n}{z}(0, .)=\Delta y_{0} & \text { in } \Omega .\end{cases}
$$

By Proposition 2.2, we have $\widetilde{z} \in C\left([0, T] ; L^{2}(\Omega)\right)$. Moreover, a.e. $t \in[0, T]$,

$$
\frac{d}{d t} \int_{\Omega} \widetilde{z}(t, .)=d \int_{\Omega} \Delta \widetilde{z}(t, .)+\int_{\Omega} \Delta f(t, .)=0
$$

Then, for every $t \in[0, T]$,

$$
\int_{\Omega} \widetilde{z}(t, .)=\int_{\Omega} \widetilde{z}(0, .)=\int_{\Omega} \Delta y_{0}=0 .
$$

For every $t \in[0, T]$, let $\widetilde{y}(t$, . ) be the solution of

$$
\begin{cases}\Delta \widetilde{y}(t, .)=\widetilde{z}(t, .) & \text { in } \Omega, \\ \frac{\partial \bar{y}(t,)}{\partial n}=0 & \text { on } \partial \Omega .\end{cases}
$$

By elliptic regularity, $\widetilde{y} \in C\left([0, T] ; H_{N e}^{2}(\Omega)\right) \subset L^{2}\left(0, T ; H_{N e}^{2}(\Omega)\right)$, $\partial_{t} \widetilde{y} \in L^{2}\left(0, T ; H_{N e}^{2}(\Omega)\right) \subset$ $L^{2}\left(0, T ; L^{2}(\Omega)\right)$ since $\Delta \partial_{t} \widetilde{y}=\partial_{t} \widetilde{z}$. Moreover, $\widetilde{y}$ is the solution of (133) (by applying the operator $\Delta^{-1}$ to (135) and by using $\Delta^{-1} \partial_{t} \bar{z}=\partial_{t} \Delta^{-1} \bar{z}$ ). Then, by uniqueness, $\widetilde{y}=y$ and $\widetilde{z}=\Delta y$ is the solution of (134).
4.3.5.2. Proof of the observability inequality: (95).

Proof. $j=2$
Let $A \in \mathcal{E}_{2}$ (see (85)), $\varphi_{T} \in C_{0}^{\infty}(\Omega)^{4}$ (the general case comes from a density argument, see (153), Lemma 4.21 and Lemma 4.22), $\varphi \in Y_{2}^{4}$ be the solution of (92) (see Proposition 4.14), $\omega_{2}$ and $\omega_{1}$ be two open subsets such that $\omega^{\prime \prime} \subset \subset \omega_{2} \subset \subset \omega_{1} \subset \subset \omega_{0}$. Our goal is to prove (95).

We have: for every $1 \leq i \leq 2$,

$$
\begin{cases}-\partial_{t} \varphi_{i}-d_{i} \Delta \varphi_{i}=a_{1 i} \varphi_{1}+a_{2 i} \varphi_{2}+a_{3 i} \varphi_{3} & \text { in }(0, T) \times \Omega  \tag{136}\\ -\partial_{t} \varphi_{3}-d_{3} \Delta \varphi_{3}=a_{13} \varphi_{1}+a_{23} \varphi_{2}+a_{33} \varphi_{3}+\left(d_{3}-d_{4}\right) \Delta \varphi_{4} & \text { in }(0, T) \times \Omega \\ -\partial_{t} \varphi_{4}-d_{4} \Delta \varphi_{4}=u_{2}^{*}\left(\varphi_{1}-\varphi_{2}+\varphi_{3}\right) & \text { in }(0, T) \times \Omega \\ \frac{\partial \varphi_{i}}{\partial n}=\frac{\partial \varphi_{3}}{\partial n}=\frac{\partial \varphi_{4}}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega \\ \left(\varphi_{i}, \varphi_{3}, \varphi_{4}\right)(T, .)=\left(\varphi_{i, T}, \varphi_{3, T}, \varphi_{4, T}\right) & \text { in } \Omega\end{cases}
$$

From (136) and Lemma 4.23, we have

$$
\begin{cases}-\partial_{t}\left(\Delta \varphi_{4}\right)-d_{4} \Delta\left(\Delta \varphi_{4}\right)=\Delta\left(u_{2}^{*}\left(\varphi_{1}-\varphi_{2}+\varphi_{3}\right)\right) & \text { in }(0, T) \times \Omega,  \tag{137}\\ \frac{\partial \Delta \varphi_{4}}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega, \\ \Delta \varphi_{4}(T, .)=\Delta \varphi_{4, T} & \text { in } \Omega .\end{cases}
$$

We apply the Carleman inequality (100) for (137) with $\beta=0$ and $\omega^{\prime}=\omega_{2}$, for every $\lambda, s \geq C$,

$$
\begin{equation*}
I\left(0, \lambda, s, \Delta \varphi_{4}\right) \leq C\left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}\left(\left|\Delta \varphi_{1}\right|^{2}+\left|\Delta \varphi_{2}\right|^{2}\left|+\left|\Delta \varphi_{3}\right|^{2}\right)+\int_{0}^{T} \int_{\omega_{2}} \lambda^{4} e^{2 s \alpha}(s \phi)^{3}\left|\Delta \varphi_{4}\right|^{2}\right)\right. \tag{138}
\end{equation*}
$$

After this, we apply the Carleman inequality (100) for the first two equations of (136) with $\beta=2$ and $\omega^{\prime}=\omega_{2}$ to obtain (by (65)), for every $\lambda, s \geq C$,

$$
\begin{align*}
\sum_{i=1}^{3} I\left(2, \lambda, s, \varphi_{i}\right) \leq & C\left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{2}\left(\left|\varphi_{1}\right|^{2}+\left|\varphi_{2}\right|^{2}\left|+\left|\varphi_{3}\right|^{2}+\left|\Delta \varphi_{4}\right|^{2}\right)\right)\right. \\
& +C\left(\int_{0}^{T} \int_{\omega_{2}} \lambda^{4} e^{2 s \alpha}(s \phi)^{5}\left(\left|\varphi_{1}\right|^{2}+\left|\varphi_{2}\right|^{2}\left|+\left|\varphi_{3}\right|^{2}\right)\right)\right. \tag{139}
\end{align*}
$$

We sum (138) and (139), for every $\lambda, s \geq C$,

$$
\begin{align*}
& \sum_{i=1}^{3} I\left(2, \lambda, s, \varphi_{i}\right)+I\left(0, \lambda, s, \Delta \varphi_{4}\right) \\
& \leq C\left(\int _ { 0 } ^ { T } \int _ { \Omega } e ^ { 2 s \alpha } \left((s \phi)^{2}\left(\left|\varphi_{1}\right|^{2}+\left|\varphi_{2}\right|^{2}\left|+\left|\varphi_{3}\right|^{2}+\left|\Delta \varphi_{4}\right|^{2}\right)+\left|\Delta \varphi_{1}\right|^{2}+\left|\Delta \varphi_{2}\right|^{2}\left|+\left|\Delta \varphi_{3}\right|^{2}\right)\right)\right.\right. \\
& +C\left(\int_{0}^{T} \int_{\omega_{2}} \lambda^{4} e^{2 s \alpha}\left((s \phi)^{5}\left(\left|\varphi_{1}\right|^{2}+\left|\varphi_{2}\right|^{2}\left|+\left|\varphi_{3}\right|^{2}\right)+(s \phi)^{3}\left|\Delta \varphi_{4}\right|^{2}\right)\right) .\right. \tag{140}
\end{align*}
$$

We fix $\lambda \geq C$ and we absorb the first right-hand side term of (140) by the left-hand side terms of (140), by taking $s$ sufficiently large. Then,

$$
\begin{align*}
& \sum_{i=1}^{3} I\left(2, \lambda, s, \varphi_{i}\right)+I\left(0, \lambda, s, \Delta \varphi_{4}\right) \\
& \leq C\left(\int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}\left((s \phi)^{5}\left(\left|\varphi_{1}\right|^{2}+\left|\varphi_{2}\right|^{2}\left|+\left|\varphi_{3}\right|^{2}\right)+(s \phi)^{3}\left|\Delta \varphi_{4}\right|^{2}\right)\right)\right. \tag{141}
\end{align*}
$$

Now, $\lambda, s$ are supposed to be fixed and the constant $C$ may depend on $\lambda, s$.
Then, we have to get rid of $\int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}(s \phi)^{3}\left|\Delta \varphi_{4}\right|^{2} d x d t$ and $\int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}(s \phi)^{5}\left|\varphi_{3}\right|^{2} d x d t$. For the first term, we use the coupling term of second order $\left(d_{3}-d_{4}\right) \Delta$. For the second term, we use the coupling term of zero order thanks to property (64).

Estimate of $\int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}(s \phi)^{3}\left|\Delta \varphi_{4}\right|^{2} d x d t$.
Let us introduce $\chi_{2} \in C^{\infty}\left(\bar{\Omega} ;\left[0,+\infty[)\right.\right.$, such that the support of $\chi_{2}$ is included in $\omega_{1}$ and $\chi_{2}=1$ in $\omega_{2}$. We multiply the second equation of (136) by $\operatorname{sign}\left(d_{3}-d_{4}\right) \chi_{2}(x) e^{2 s \alpha}(s \phi)^{3} \Delta \varphi_{4}$ and
we integrate on $(0, T) \times \omega_{1}$. As $d_{3} \neq d_{4}$, we have

$$
\begin{align*}
& \int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}(s \phi)^{3}\left|\Delta \varphi_{4}\right|^{2} d x d t \leq \int_{0}^{T} \int_{\omega_{1}} \chi_{2}(x) e^{2 s \alpha}(s \phi)^{3}\left|\Delta \varphi_{4}\right|^{2} d x d t \\
& \leq C \int_{0}^{T} \int_{\omega_{1}} \chi_{2}(x) e^{2 s \alpha}(s \phi)^{3} \Delta \varphi_{4}\left(-\partial_{t} \varphi_{3}-d_{3} \Delta \varphi_{3}-a_{13} \varphi_{1}-a_{23} \varphi_{2}-a_{33} \varphi_{3}\right) d x d t \tag{142}
\end{align*}
$$

Let $\varepsilon>0$ which will be chosen small enough. We estimate the right hand side of (142) in the same way as the one of (108):

- for terms involving $\Delta \varphi_{4} a_{i 3} \varphi_{i}$ with $1 \leq i \leq 3$, we apply (101) with $\Phi=\Delta \varphi_{4}, \Psi=\varphi_{i}$, $a=a_{i 3} \in L^{\infty}(Q), 1 \leq i \leq 3$ (recalling (65)), $\Theta=\chi_{2}$ and $r=k=l=3$,
- for the term involving $\Delta \varphi_{4} \partial_{t} \varphi_{3}$, we apply (102) with $\Phi=\Delta \varphi_{4}, \Psi=\varphi_{3}, a=1, \Theta=\chi_{2}$ and $r=k=3, l=7$,
- for the term involving $\Delta \varphi_{4} \Delta \varphi_{3}$, we apply (103) with $\Phi=\Delta \varphi_{4}, \Psi=\varphi_{3}, a=d_{3}, \Theta=\chi_{2}$ and $r=k=3, l=7$.
From (141), (142), we get

$$
\begin{align*}
& \sum_{i=1}^{3} I\left(2, \lambda, s, \varphi_{i}\right)+I\left(0, \lambda, s, \Delta \varphi_{4}\right) \\
& \leq 3 \varepsilon\left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}\left\{(s \phi)^{3}\left|\Delta \varphi_{4}\right|^{2}+(s \phi)\left|\nabla \Delta \varphi_{4}\right|^{2}+(s \phi)^{-1}\left(\left|\partial_{t} \Delta \varphi_{4}\right|^{2}+\left|\Delta \Delta \varphi_{4}\right|^{2}\right)\right\}\right) \\
& +C_{\varepsilon}\left(\sum_{i=1}^{3} \int_{0}^{T} \int_{\omega_{1}} e^{2 s \alpha}(s \phi)^{7}\left|\varphi_{i}\right|^{2}\right) \tag{143}
\end{align*}
$$

By taking $\varepsilon$ small enough in (143), we get

$$
\begin{equation*}
\sum_{i=1}^{3} I\left(2, \lambda, s, \varphi_{i}\right)+I\left(0, s, \Delta \varphi_{4}\right) \leq C\left(\sum_{i=1}^{3} \int_{0}^{T} \int_{\omega_{1}} e^{2 s \alpha}(s \phi)^{7}\left|\varphi_{i}\right|^{2} d x d t\right) \tag{144}
\end{equation*}
$$

$\underline{\text { Estimate of } \int_{0}^{T} \int_{\omega_{1}} e^{2 s \alpha}(s \phi)^{7}\left|\varphi_{3}\right|^{2} d x d t .}$
Let us introduce $\chi_{1} \in C^{\infty}\left(\bar{\Omega} ;\left[0,+\infty[)\right.\right.$, such that the support of $\chi_{1}$ is included in $\omega_{0}$ and $\chi_{1}=1$ in $\omega_{1}$. We multiply the first equation of the adjoint system (136) with $i=1$ by $-\chi_{1}(x) e^{2 s \alpha}(s \phi)^{7} \varphi_{3}$ and we integrate on $(0, T) \times \omega_{0}$. By using (64), we have

$$
\begin{align*}
& \int_{0}^{T} \int_{\omega_{1}} e^{2 s \alpha}(s \phi)^{7}\left|\varphi_{3}\right|^{2} d x d t \leq \int_{0}^{T} \int_{\omega_{0}} \chi_{1}(x) e^{2 s \alpha}(s \phi)^{7}\left|\varphi_{3}\right|^{2} d x d t \\
& \leq C \int_{0}^{T} \int_{\omega_{0}} \chi_{1}(x) e^{2 s \alpha}(s \phi)^{7} \varphi_{3}\left(-\partial_{t} \varphi_{1}-d_{1} \Delta \varphi_{1}-a_{11} \varphi_{1}-a_{21} \varphi_{2}\right) d x d t \tag{145}
\end{align*}
$$

Let $\varepsilon^{\prime}>0$ which will be chosen small enough. We estimate the right hand side of (145) in the same way as the one of (108):

- for terms involving $\varphi_{3} a_{i 1} \varphi_{i}$ with $1 \leq i \leq 2$, we apply (101) with $\Phi=\varphi_{3}, \Psi=\varphi_{i}$, $a=a_{i 3} \in L^{\infty}(Q), 1 \leq i \leq 2$ (recalling (65)), $\Theta=\chi_{1}$ and $r=7, k=5, l=9$,
- for the term involving $\varphi_{3} \partial_{t} \varphi_{1}$, we apply (102) with $\Phi=\varphi_{3}, \Psi=\varphi_{1}, a=1, \Theta=\chi_{1}$ and $r=7, k=5, l=13$,
- for the term involving $\varphi_{3} \Delta \varphi_{1}$, we apply (103) with $\Phi=\varphi_{3}, \Psi=\varphi_{1}, a=d_{1}, \Theta=\chi_{1}$ and $r=7, k=5, l=13$.
Then, we obtain

$$
\begin{align*}
\int_{0}^{T} \int_{\omega_{1}} e^{2 s \alpha}(s \phi)^{7}\left|\varphi_{3}\right|^{2} \leq & 3 \varepsilon^{\prime}\left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}\left\{(s \phi)^{5}\left|\varphi_{3}\right|^{2}+(s \phi)^{3}\left|\nabla \varphi_{3}\right|^{2}+(s \phi)\left(\left|\partial_{t} \varphi_{3}\right|^{2}+\left|\Delta \varphi_{3}\right|^{2}\right)\right\}\right) \\
& +C_{\varepsilon^{\prime}}\left(\sum_{i=1}^{2} \int_{0}^{T} \int_{\omega_{0}} e^{2 s \alpha}(s \phi)^{13}\left|\varphi_{i}\right|^{2}\right) \tag{146}
\end{align*}
$$

By using (144), (146) and by taking $\varepsilon^{\prime}$ sufficiently small, we get

$$
\begin{equation*}
\sum_{i=1}^{3} I\left(2, \lambda, s, \varphi_{i}\right)+I\left(0, \lambda, s, \Delta \varphi_{4}\right) \leq C\left(\sum_{i=1}^{2} \iint_{(0, T) \times \omega_{0}} e^{2 s \alpha}(s \phi)^{13}\left|\varphi_{i}\right|^{2}\right) \tag{147}
\end{equation*}
$$

Then, we deduce from (147) that we have

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega} \sum_{i=1}^{3} e^{2 s \alpha}(s \phi)^{5}\left|\varphi_{i}\right|^{2}+e^{2 s \widehat{\alpha}}(\widehat{\phi \phi})^{3}\left|\Delta \varphi_{4}\right|^{2} \leq C\left(\sum_{i=1}^{2} \iint_{(0, T) \times \omega_{0}} e^{2 s \alpha}(s \phi)^{13}\left|\varphi_{i}\right|^{2}\right), \tag{148}
\end{equation*}
$$

where $\widehat{\phi}$ and $\widehat{\alpha}$ are defined in (99). In particular, $\widehat{\phi}$ and $\widehat{\alpha}$ do not depend on the spatial variable $x$. In order to estimate $\varphi_{4}$ by $\Delta \varphi_{4}$, we use the classical lemma and the corollary that follow.

## Lemma 4.24. Poincaré-Wirtinger inequality

There exists $C=C(\Omega)$ such that

$$
\begin{equation*}
\forall u \in H^{1}(\Omega), \int_{\Omega}\left(u(x)-(u)_{\Omega}\right)^{2} d x \leq C \int_{\Omega}|\nabla u(x)|^{2} d x \tag{149}
\end{equation*}
$$

Corollary 4.25. There exists $C=C(\Omega)$ such that

$$
\begin{equation*}
\forall u \in H_{N e}^{2}(\Omega):=\left\{u \in H^{2}(\Omega) ; \frac{\partial u}{\partial n}=0\right\}, \int_{\Omega}|\nabla u(x)|^{2} d x \leq C \int_{\Omega}|\Delta u(x)|^{2} d x \tag{150}
\end{equation*}
$$

Proof. Let $u \in H_{N e}^{2}(\Omega)$ satisfying $\|\nabla u\|_{L^{2}(\Omega} \neq 0$. Otherwise, the inequality (150) is trivial. We have by an integration by parts and by using (149),

$$
\begin{aligned}
\int_{\Omega}|\nabla u|^{2}=-\int_{\Omega}(\Delta u) u=-\int_{\Omega}(\Delta u)\left(u-(u)_{\Omega}\right) & \leq\|\Delta u\|_{L^{2}(\Omega)}\left\|u-(u)_{\Omega}\right\|_{L^{2}(\Omega} \\
& \leq C\|\Delta u\|_{L^{2}(\Omega)}\|\nabla u\|_{L^{2}(\Omega}
\end{aligned}
$$

We conclude the proof of Corollary 4.25 by simplifying by $\|\nabla u\|_{L^{2}(\Omega}$.
Then, by applying the Poincaré-Wirtinger inequality (149) and (150) to $\varphi_{4}$, we deduce from (148) that

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega} \sum_{i=1}^{3} e^{2 s \alpha}(s \phi)^{5}\left|\varphi_{i}\right|^{2}+e^{2 s \widehat{\alpha}}(\widehat{s \phi})^{3}\left|\varphi_{4}-\left(\varphi_{4}\right)_{\Omega}\right|^{2} \leq C\left(\sum_{i=1}^{2} \iint_{(0, T) \times \omega_{0}} e^{2 s \alpha}(s \phi)^{13}\left|\varphi_{i}\right|^{2}\right) \tag{151}
\end{equation*}
$$

Now, from the dissipation in time of the energy of $\left(\varphi_{1}, \varphi_{2}, \varphi_{3}, \varphi_{4}-\left(\varphi_{4}\right)_{\Omega}\right)$ (see Lemma A. 1 in the Appendix), we get

$$
\begin{align*}
& \sum_{i=1}^{3}\left(\left\|\varphi_{i}(0, .)\right\|_{L^{2}(\Omega)}^{2}\right)+\left\|\varphi_{4}(0, .)-\left(\varphi_{4}\right)_{\Omega}(0)\right\|_{L^{2}(\Omega)}^{2} \\
& \leq C \int_{T / 4}^{3 T / 4}\left(\sum_{i=1}^{3}\left(\left\|\varphi_{i}(t, .)\right\|_{L^{2}(\Omega)}^{2}\right)+\left\|\varphi_{4}(t, .)-\left(\varphi_{4}\right)_{\Omega}(t)\right\|_{L^{2}(\Omega)}^{2}\right) d t . \tag{152}
\end{align*}
$$

Consequently, from (151), (152) and the same arguments given between (114) and (119), we easily deduce that

$$
\begin{equation*}
\sum_{i=1}^{3}\left(\left\|\varphi_{i}(0, .)\right\|_{L^{2}(\Omega)}^{2}\right)+\left\|\varphi_{4}(0, .)-\left(\varphi_{4}\right)_{\Omega}(0)\right\|_{L^{2}(\Omega)}^{2} \leq C\left(\sum_{i=1}^{2} \iint_{(0, T) \times \omega} e^{2 s \alpha}(s \phi)^{13}\left|\varphi_{i}\right|^{2} d x d t\right) \tag{153}
\end{equation*}
$$

and consequently the observability inequality (95) because $e^{2 s \alpha}(s \phi)^{13}$ is bounded.
This ends the proof of the observability inequality (95).

### 4.3.6. Another Carleman inequality

## Theorem 4.26. Carleman inequality

Let $d \in(0,+\infty)$, $\omega^{\prime}$ an open subset such that $\omega^{\prime \prime} \subset \subset \omega^{\prime} \subset \subset \omega_{0}$. There exist $C=C\left(\Omega, \omega^{\prime}\right)$, $\lambda_{0}=\lambda_{0}\left(\Omega, \omega^{\prime}\right)$ such that, for every $\lambda \geq \lambda_{0}$, there exists $s_{0}=s_{0}\left(\Omega, \omega^{\prime}, \lambda\right)$ such that, for any $s \geq s_{0}\left(T+T^{2}\right)$, any $\varphi_{T} \in L^{2}(\Omega)$ and any $f \in L^{2}\left(0, T ; H_{N e}^{2}(\Omega)\right)$, the solution $\varphi$ of

$$
\begin{cases}-\partial_{t} \varphi-d \Delta \varphi=\Delta f & \text { in }(0, T) \times \Omega, \\ \frac{\partial \varphi}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega, \\ \varphi(T, .)=\varphi_{T} & \text { in } \Omega,\end{cases}
$$

satisfies

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{3}|\varphi|^{2} d x d t \leq C\left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{4}|f|^{2} d x d t+\int_{0}^{T} \int_{\omega^{\prime}} e^{2 s \alpha}(s \phi)^{3}|\varphi|^{2} d x d t\right) \tag{154}
\end{equation*}
$$

The proof of this inequality can be found in [13, Lemma A.1] (see in particular that the equality $[13,(\mathrm{~A} .3)]$ still holds for $\left.f \in L^{2}\left(0, T ; H_{N e}^{2}(\Omega)\right)\right)$.

Remark 4.27. The estimate (154) is different from (100) because (100) gives us

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{3}|\varphi|^{2} d x d t \leq C\left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}|\Delta f|^{2} d x d t+\int_{0}^{T} \int_{\omega^{\prime}} e^{2 s \alpha}(s \phi)^{3}|\varphi|^{2} d x d t\right) \tag{155}
\end{equation*}
$$

Therefore, (154) is useful when one wants an observation of $\varphi$ in term of $f$ (but not in term of $\Delta f$ ). Roughly, we remark that we have to pay this type of estimate with a weight $(s \phi)^{4}$ (see the first right hand side terms of (154) and (155)).

### 4.3.7. Proof with observation on one component: (96)

We have seen in Section 4.3.5 that parabolic regularity allows us to apply $\Delta$ to the third equation of (136) (see (137)) in order to benefit from the coupling term of second order $\left(d_{3}-\right.$ $\left.d_{4}\right) \Delta \varphi_{4}$. The case $j=1$ requires more regularity because we have to benefit from two terms of coupling of second order. Therefore, we need to apply $\Delta \Delta$ (see (158)). There are two main difficulties. First, Proposition 4.14 only shows us that $\varphi$, the solution of (92) is in $Y_{2}^{4}$. However, we need: $\Delta \varphi \in Y_{2}^{4}$. That is why we regularize the coupling matrix $A \in \mathcal{E}_{1}$ (see Lemma 4.18). Secondly, we want an observation of $\Delta \Delta \varphi_{4}$ in term of $\Delta \varphi_{1}, \Delta \varphi_{2}$ (and not in term of $\Delta \Delta \varphi_{1}, \Delta \Delta \varphi_{2}$ because we do not have these terms in Carleman estimates applied to $\varphi_{1}$ and $\varphi_{2}$ : see (162) and (163)). That is why we use Theorem 4.26.

Proof. $j=1$
Let $A \in \mathcal{M}_{4}\left(C_{0}^{\infty}(Q)\right) \cap \mathcal{E}_{1}$ (see (87)), $\varphi_{T} \in C_{0}^{\infty}(\Omega)^{4}$ (the general case comes from a density argument, see (185), Lemma 4.18, Lemma 4.21 and Lemma 4.22), $\varphi \in Y_{2}^{4}$ be the solution of (92) (see Proposition 4.14), $\omega_{3}, \omega_{2}, \omega_{2}^{\prime}$ and $\omega_{1}$ be four open subsets such that $\omega^{\prime \prime} \subset \subset \omega_{3} \subset \subset \omega_{2} \subset \subset$ $\omega_{2}^{\prime} \subset \subset \omega_{1} \subset \subset \omega_{0}$. Our goal is to prove (96).

We have

$$
\begin{cases}-\partial_{t} \varphi_{1}-d_{1} \Delta \varphi_{1}=a_{11} \varphi_{1}+a_{21} \varphi_{2} & \text { in }(0, T) \times \Omega,  \tag{156}\\ -\partial_{t} \varphi_{2}-d_{2} \Delta \varphi_{2}=a_{12} \varphi_{1}+a_{22} \varphi_{2}+\left(d_{2}-d_{3}\right) \Delta \varphi_{3} & \text { in }(0, T) \times \Omega, \\ -\partial_{t} \varphi_{3}-d_{3} \Delta \varphi_{3}=-m_{2}\left(\varphi_{1}-\varphi_{2}\right)+\Delta \varphi_{4} & \text { in }(0, T) \times \Omega, \\ -\partial_{t} \varphi_{4}-d_{4} \Delta \varphi_{4}=m_{3}\left(\varphi_{1}-\varphi_{2}\right) & \text { in }(0, T) \times \Omega, \\ \frac{\partial \varphi}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega, \\ \varphi(T, .)=\varphi_{T} & \text { in } \Omega .\end{cases}
$$

First, by using the regularity: $\varphi \in Y_{2}^{4}$ and by applying consecutively Lemma 4.23 to the fourth equation of (156), the third equation of (156), the second equation of (156), the first equation of (156), we get

$$
\begin{equation*}
\Delta \varphi \in L^{2}\left(0, T ; H_{N e}^{2}(\Omega)\right)^{4} \tag{157}
\end{equation*}
$$

Consequently, we can apply $\Delta \Delta$ to the fourth equation of (156) by using (157) and Lemma 4.23,

$$
\begin{cases}-\partial_{\partial}\left(\Delta \Delta \varphi_{4}\right)-d_{4} \Delta\left(\Delta \Delta \varphi_{4}\right)=\Delta \Delta\left(m_{3}\left(\varphi_{1}-\varphi_{2}\right)\right) & \text { in }(0, T) \times \Omega,  \tag{158}\\ \frac{\partial \Delta \Delta \varphi_{4}}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega, \\ \Delta \Delta \varphi_{4}(T, .)=\Delta \Delta \varphi_{4, T} & \text { in } \Omega .\end{cases}
$$

Then, we use the Carleman inequality (154) for (158) with $\omega^{\prime}=\omega_{3}$ and $f=\Delta\left(m_{3}\left(\varphi_{1}-\varphi_{2}\right)\right) \in$ $L^{2}\left(0, T ; H_{N e}^{2}(\Omega)\right)$, for every $\lambda, s \geq C$,

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{3}\left|\Delta \Delta \varphi_{4}\right|^{2} \leq C\left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{4}\left(\left|\Delta \varphi_{1}\right|^{2}+\left|\Delta \varphi_{2}\right|^{2}\right)+\int_{0}^{T} \int_{\omega_{3}} e^{2 s \alpha}(s \phi)^{3}\left|\Delta \Delta \varphi_{4}\right|^{2}\right) \tag{159}
\end{equation*}
$$

Remark 4.28. Here, we have to apply the Carleman estimate (154) instead of (100) in order to get in the right hand side of (159) only terms of order two (and not more) in $\varphi_{1}, \varphi_{2}$. Otherwise, we cannot absorb the remaining terms thanks to Carleman estimates (100) applied to $\varphi_{1}, \varphi_{2}$.

Then, we apply $\Delta$ to the third equation of (156) thanks to (158) and Lemma 4.23, for every $\lambda, s \geq C$,

$$
\begin{cases}-\partial_{t}\left(\Delta \varphi_{3}\right)-d_{3} \Delta\left(\Delta \varphi_{3}\right)=\Delta\left(-m_{2}\left(\varphi_{1}-\varphi_{2}\right)\right)+\Delta \Delta \varphi_{4} & \text { in }(0, T) \times \Omega,  \tag{160}\\ \frac{\partial \Delta \varphi_{3}}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega, \\ \Delta \varphi_{3}(T, .)=\Delta \varphi_{3, T} & \text { in } \Omega .\end{cases}
$$

We use the Carleman inequality (100) with $\omega^{\prime}=\omega_{3}$ and $\beta=2$, for every $\lambda, s \geq C$,

$$
\begin{equation*}
I\left(2, \lambda, s, \Delta \varphi_{3}\right) \leq C\left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{2}\left(\left|\Delta \varphi_{1}\right|^{2}+\left|\Delta \varphi_{2}\right|^{2}\left|+\left|\Delta \Delta \varphi_{4}\right|^{2}\right)+\int_{0}^{T} \int_{\omega_{2}} \lambda^{4} e^{2 s \alpha}(s \phi)^{5}\left|\Delta \varphi_{3}\right|^{2}\right) .\right. \tag{161}
\end{equation*}
$$

Then, we apply the Carleman inequality (100) with $\omega^{\prime}=\omega_{3}$ and $\beta=5$ to the second equation and the first equation of (156) (by (80)), for every $\lambda, s \geq C$,

$$
\begin{align*}
& \lambda I\left(5, \lambda, s, \varphi_{2}\right) \leq C\left(\int_{0}^{T} \int_{\Omega} \lambda e^{2 s \alpha}(s \phi)^{5}\left(\left|\varphi_{1}\right|^{2}+\left|\varphi_{2}\right|^{2}\left|+\left|\Delta \varphi_{3}\right|^{2}\right)+\int_{0}^{T} \int_{\omega_{3}} \lambda^{5} e^{2 s \alpha}(s \phi)^{8}\left|\varphi_{2}\right|^{2}\right),\right.  \tag{162}\\
& \lambda I\left(5, \lambda, s, \varphi_{1}\right) \leq C\left(\int_{0}^{T} \int_{\Omega} \lambda e^{2 s \alpha}(s \phi)^{5}\left(\left|\varphi_{1}\right|^{2}+\left|\varphi_{2}\right|^{2} \mid\right)+\int_{0}^{T} \int_{\omega_{3}} \lambda^{5} e^{2 s \alpha}(s \phi)^{8}\left|\varphi_{1}\right|^{2}\right) . \tag{163}
\end{align*}
$$

We sum (159), (161), (162), (163) and we take $\lambda$ and $s$ sufficiently large,

$$
\begin{align*}
& \int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{3}\left|\Delta \Delta \varphi_{4}\right|^{2} d x d t+I\left(2, \lambda, s, \Delta \varphi_{3}\right)+\lambda I\left(5, \lambda, s, \varphi_{2}\right)+\lambda I\left(5, \lambda, s, \varphi_{1}\right) \\
& \leq C\left(\int_{0}^{T} \int_{\omega_{3}} e^{2 s \alpha}(s \phi)^{3}\left|\Delta \Delta \varphi_{4}\right|^{2} d x d t+\int_{0}^{T} \int_{\omega_{3}} \lambda^{4} e^{2 s \alpha}(s \phi)^{5}\left|\Delta \varphi_{3}\right|^{2} d x d t\right) \\
& +C\left(\int_{0}^{T} \int_{\omega_{3}} \lambda^{5} e^{2 s \alpha}(s \phi)^{8}\left|\varphi_{2}\right|^{2} d x d t+\int_{0}^{T} \int_{\omega_{3}} \lambda^{5} e^{2 s \alpha}(s \phi)^{8}\left|\varphi_{1}\right|^{2} d x d t\right) . \tag{164}
\end{align*}
$$

Now, $\lambda$ and $s$ are supposed to be fixed. The constant $C$ may depend on $\lambda$ and $s$. We have

$$
\begin{align*}
& \int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{3}\left|\Delta \Delta \varphi_{4}\right|^{2} d x d t+I\left(2, \lambda, s, \Delta \varphi_{3}\right)+I\left(5, \lambda, s, \varphi_{2}\right)+I\left(5, \lambda, s, \varphi_{1}\right) \\
& \leq C\left(\int_{0}^{T} \int_{\omega_{3}} e^{2 s \alpha}(s \phi)^{3}\left|\Delta \Delta \varphi_{4}\right|^{2} d x d t+\int_{0}^{T} \int_{\omega_{3}} e^{2 s \alpha}(s \phi)^{5}\left|\Delta \varphi_{3}\right|^{2} d x d t\right) \\
& +C\left(\int_{0}^{T} \int_{\omega_{3}} e^{2 s \alpha}(s \phi)^{8}\left|\varphi_{2}\right|^{2} d x d t+\int_{0}^{T} \int_{\omega_{3}} e^{2 s \alpha}(s \phi)^{8}\left|\varphi_{1}\right|^{2} d x d t\right) . \tag{165}
\end{align*}
$$

Remark 4.29. Here, we take advantage of the two parameters $\lambda$ and $s$ in Theorem 4.11. Indeed, if we forget $\lambda$, we would need to sum $\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{3}\left|\Delta \Delta \varphi_{4}\right|^{2} d x d t, I\left(4, s, \Delta \varphi_{3}\right), I\left(6, s, \varphi_{2}\right)$ and $I\left(6, s, \varphi_{1}\right)$. Therefore, we would get in the right hand side $\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{4}\left|\Delta \Delta \varphi_{4}\right|^{2} d x d t$ which cannot be absorbed by the left hand side.

Then , we have to get rid of $\int_{0}^{T} \int_{\omega_{3}} e^{2 s \alpha}(s \phi)^{3}\left|\Delta \Delta \varphi_{4}\right|^{2} d x d t, \int_{0}^{T} \int_{\omega_{3}} e^{2 s \alpha}(s \phi)^{5}\left|\Delta \varphi_{3}\right|^{2} d x d t$ and $\int_{0}^{T} \int_{\omega_{3}} e^{2 s \alpha}(s \phi)^{8}\left|\varphi_{2}\right|^{2} d x d t$. For the first term, we use the coupling term of fourth order $\Delta \Delta$. For the second term, we use the coupling term of second order $\left(d_{2}-d_{3}\right) \Delta$. For the third term, we use the coupling term of zero order thanks to property (79).

Estimate of $\int_{0}^{T} \int_{\omega_{3}} e^{2 s \alpha}(s \phi)^{3}\left|\Delta \Delta \varphi_{4}\right|^{2} d x d t$.
Let us introduce $\chi_{3} \in C^{\infty}\left(\bar{\Omega} ;\left[0,+\infty[)\right.\right.$, such that the support of $\chi_{3}$ is included in $\omega_{2}$ and $\chi_{3}=1$ in $\omega_{3}$. We multiply the first equation (160) by $\left(\chi_{3}(x)\right)^{2} e^{2 s \alpha}(s \phi)^{3} \Delta \Delta \varphi_{4}$ and we integrate on
$(0, T) \times \omega_{2}$. We have

$$
\begin{align*}
& \int_{0}^{T} \int_{\omega_{2}}\left(\chi_{3}(x)\right)^{2} e^{2 s \alpha}(s \phi)^{3}\left|\Delta \Delta \varphi_{4}\right|^{2} d x d t \\
& \leq C \int_{0}^{T} \int_{\omega_{2}}\left(\chi_{3}(x)\right)^{2} e^{2 s \alpha}(s \phi)^{3} \Delta \Delta \varphi_{4}\left(-\partial_{t} \Delta \varphi_{3}-d_{3} \Delta \Delta \varphi_{3}+m_{2} \Delta \varphi_{1}-m_{2} \Delta \varphi_{2}\right) d x d t \tag{166}
\end{align*}
$$

Remark 4.30. One can see the presence of $\left(\chi_{3}(x)\right)^{2}$ instead of $\chi_{3}(x)$ as before (see for example (108)). It is purely technical (see the proofs of Lemma 4.31 and Lemma 4.32).

Let $\varepsilon \in(0,1)$ which will be chosen small enough. First, for every $1 \leq i \leq 2$, by applying Lemma 4.15: (101) with $\Phi=\Delta \Delta \varphi_{4}, \Psi=\Delta \varphi_{i}, a=m_{2}, \Theta=\left(\chi_{3}\right)^{2}, r=3$ and $(k, l)=(3,3)$, we have

$$
\begin{equation*}
\int_{0}^{T} \int_{\omega_{2}} \chi_{3}^{2} e^{2 s \alpha}(s \phi)^{3}\left(\Delta \Delta \varphi_{4}\right) m_{2} \Delta \varphi_{i} \leq \varepsilon \int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{3}\left|\Delta \Delta \varphi_{4}\right|^{2}+C_{\varepsilon} \int_{0}^{T} \int_{\omega_{2}} \chi_{3}^{2} e^{2 s \alpha}(s \phi)^{3}\left|\Delta \varphi_{i}\right|^{2} \tag{167}
\end{equation*}
$$

But, the other terms in the right hand side of (166) i.e. $\int_{0}^{T} \int_{\omega_{2}}\left(\chi_{3}(x)\right)^{2} e^{2 s \alpha}(s \phi)^{3}\left(\Delta \Delta \varphi_{4}\right)\left(\partial_{t} \Delta \varphi_{3}\right) d x d t$ and $\int_{0}^{T} \int_{\omega_{2}}\left(\chi_{3}(x)\right)^{2} e^{2 s \alpha}(s \phi)^{3}\left(\Delta \Delta \varphi_{4}\right)\left(\Delta \Delta \varphi_{3}\right) d x d t$ cannot be estimated as in Lemma 4.15 because we have not enough derivative terms in $\varphi_{4}$ in the left hand side of (165). In order to estimate these two terms, we follow the strategy developed in the proof of [13, Theorem 2.2] (see Appendix A. 3 for the proof of the two following lemmas).
Lemma 4.31. We have

$$
\begin{align*}
& \int_{0}^{T} \int_{\omega_{2}} \chi_{3}^{2} e^{2 s \alpha}(s \phi)^{3}\left(\Delta \Delta \varphi_{4}\right)\left(\Delta \Delta \varphi_{3}\right) \\
& \leq \varepsilon\left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}\left\{(s \phi)^{4}\left(\left|\Delta \varphi_{1}\right|^{2}+\left|\Delta \varphi_{2}\right|^{2}\right)+(s \phi)\left|\Delta \Delta \varphi_{3}\right|^{2}+(s \phi)^{3}\left|\Delta \Delta \varphi_{4}\right|^{2}\right\}\right) \\
& +C_{\varepsilon}\left(\int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}\left\{(s \phi)^{24}\left(\left|\varphi_{1}\right|^{2}+\left|\varphi_{2}\right|^{2}+\left|\Delta \varphi_{3}\right|^{2}\right)+(s \phi)^{22}\left(\left|\nabla \varphi_{1}\right|^{2}+\left|\nabla \varphi_{2}\right|^{2}+\left|\nabla \Delta \varphi_{3}\right|^{2}\right)\right\}\right) \tag{168}
\end{align*}
$$

Lemma 4.32. We have

$$
\begin{align*}
& \int_{0}^{T} \int_{\omega_{2}} \chi_{3}^{2} e^{2 s \alpha}(s \phi)^{3}\left(\Delta \Delta \varphi_{4}\right)\left(\partial_{t} \Delta \varphi_{3}\right) \\
& \leq \varepsilon\left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}\left\{(s \phi)^{4}\left(\left|\Delta \varphi_{1}\right|^{2}+\left|\Delta \varphi_{2}\right|^{2}\right)+(s \phi)\left|\partial_{t} \Delta \varphi_{3}\right|^{2}+(s \phi)^{3}\left|\Delta \Delta \varphi_{4}\right|^{2}\right\}\right) \\
& +C_{\varepsilon}\left(\int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}\left\{(s \phi)^{24}\left(\left|\varphi_{1}\right|^{2}+\left|\varphi_{2}\right|^{2}+\left|\Delta \varphi_{3}\right|^{2}\right)+(s \phi)^{22}\left(\left|\nabla \varphi_{1}\right|^{2}+\left|\nabla \varphi_{2}\right|^{2}+\left|\nabla \Delta \varphi_{3}\right|^{2}\right)\right\}\right) \tag{169}
\end{align*}
$$

Moreover, the proof of these two lemmas (see (A.43)) provides us another estimate which is useful to treat the right hand side of (167).

Lemma 4.33. For every $1 \leq i \leq 2, \delta>0$, we have

$$
\begin{align*}
& \int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}(s \phi)^{3}\left|\Delta \varphi_{i}\right|^{2} \\
& \leq \delta\left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{4}\left|\Delta \varphi_{i}\right|^{2}\right) \\
& +C_{\delta}\left(\int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}\left\{(s \phi)^{24}\left(\left|\varphi_{1}\right|^{2}+\left|\varphi_{2}\right|^{2}+\left|\Delta \varphi_{3}\right|^{2}\right)+(s \phi)^{22}\left|\nabla \varphi_{i}\right|^{2}\right\}\right) \tag{170}
\end{align*}
$$

Gathering (167) and (170) with $\delta=\varepsilon / C_{\varepsilon}$, we find that for $1 \leq i \leq 2$,

$$
\begin{align*}
& \int_{0}^{T} \int_{\omega_{2}}\left(\chi_{3}(x)\right)^{2} e^{2 s \alpha}(s \phi)^{3}\left(\Delta \Delta \varphi_{4}\right) m_{2} \Delta \varphi_{i} d x d t \\
& \leq \varepsilon\left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{3}\left|\Delta \Delta \varphi_{4}\right|^{2}+\int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}(s \phi)^{4}\left|\Delta \varphi_{i}\right|^{2}\right) \\
& +C_{\varepsilon}\left(\int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}(s \phi)^{24}\left(\left|\varphi_{1}\right|^{2}+\left|\varphi_{2}\right|^{2}+\left|\Delta \varphi_{3}\right|^{2}\right)+\int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}(s \phi)^{22}\left|\nabla \varphi_{i}\right|^{2}\right) \tag{171}
\end{align*}
$$

From (166), (171), (168), (169), we get

$$
\begin{align*}
& \int_{0}^{T} \int_{\omega_{2}}\left(\chi_{3}(x)\right)^{2} e^{2 s \alpha}(s \phi)^{3}\left|\Delta \Delta \varphi_{4}\right|^{2} d x d t \\
& \leq \varepsilon\left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}\left\{(s \phi)^{4}\left(\left|\Delta \varphi_{1}\right|^{2}+\left|\Delta \varphi_{2}\right|^{2}\right)+(s \phi)\left(\left|\partial_{t} \Delta \varphi_{3}\right|^{2}+\left|\Delta \Delta \varphi_{3}\right|^{2}\right)+(s \phi)^{3}\left|\Delta \Delta \varphi_{4}\right|^{2}\right\}\right) \\
& +C_{\varepsilon}\left(\int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}\left\{(s \phi)^{24}\left(\left|\varphi_{1}\right|^{2}+\left|\varphi_{2}\right|^{2}+\left|\Delta \varphi_{3}\right|^{2}\right)+(s \phi)^{22}\left(\left|\nabla \varphi_{1}\right|^{2}+\left|\nabla \varphi_{2}\right|^{2}+\left|\nabla \Delta \varphi_{3}\right|^{2}\right)\right\}\right) . \tag{172}
\end{align*}
$$

By using (165), (172) and by taking $\varepsilon$ small enough, we have

$$
\begin{align*}
& \int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{3}\left|\Delta \Delta \varphi_{4}\right|^{2} d x d t+I\left(2, \lambda, s, \Delta \varphi_{3}\right)+I\left(5, \lambda, s, \varphi_{2}\right)+I\left(5, \lambda, s, \varphi_{1}\right) \\
& \leq C\left(\int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}\left\{(s \phi)^{24}\left(\left|\varphi_{1}\right|^{2}+\left|\varphi_{2}\right|^{2}+\left|\Delta \varphi_{3}\right|^{2}\right)+(s \phi)^{22}\left(\left|\nabla \varphi_{1}\right|^{2}+\left|\nabla \varphi_{2}\right|^{2}+\left|\nabla \Delta \varphi_{3}\right|^{2}\right)\right\}\right) \tag{173}
\end{align*}
$$

$\underline{\text { Estimate of } \int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}(s \phi)^{22}\left|\nabla \Delta \varphi_{3}\right|^{2} d x d t}$.
Let us introduce $\widetilde{\chi_{2}} \in C^{\infty}\left(\bar{\Omega} ;\left[0 ;+\infty[)\right.\right.$ such that $\operatorname{supp}\left(\widetilde{\chi_{2}}\right) \subset \omega_{2}^{\prime}$ and $\widetilde{\chi_{2}}=1$ on $\omega_{2}$. Then, by Lemma 4.15: (104) (with $\Phi=\Delta \varphi_{3}, \widetilde{\omega}=\omega_{2}, \Theta=\widetilde{\chi_{2}}, r=22$ and $(k, l)=(1,43)$ ), for any $\varepsilon^{\prime}>0$, we have

$$
\begin{align*}
& \int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}(s \phi)^{22}\left|\nabla \Delta \varphi_{3}\right|^{2} \\
& \leq \int_{0}^{T} \int_{\omega_{2}^{\prime}} \widetilde{\chi_{2}} e^{2 s \alpha}(s \phi)^{22}\left|\nabla \Delta \varphi_{3}\right|^{2} \\
& \leq \varepsilon^{\prime}\left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}\left\{(s \phi)\left|\Delta \Delta \varphi_{3}\right|^{2}+(s \phi)^{3}\left|\nabla \Delta \varphi_{3}\right|^{2}\right\}\right)+C_{\varepsilon^{\prime}} \int_{0}^{T} \int_{\omega_{2}^{\prime}} e^{2 s \alpha}(s \phi)^{43}\left|\Delta \varphi_{3}\right|^{2} \tag{174}
\end{align*}
$$

By taking $\varepsilon^{\prime}$ small enough and by using (173) and (174), we have

$$
\begin{align*}
& \int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{3}\left|\Delta \Delta \varphi_{4}\right|^{2} d x d t+I\left(2, \lambda, s, \Delta \varphi_{3}\right)+I\left(5, \lambda, s, \varphi_{2}\right)+I\left(5, \lambda, s, \varphi_{1}\right) \\
& \leq C\left(\int_{0}^{T} \int_{\omega_{2}^{\prime \prime}} e^{2 s \alpha}(s \phi)^{43}\left(\left|\varphi_{1}\right|^{2}+\left|\varphi_{2}\right|^{2}+\left|\Delta \varphi_{3}\right|^{2}\right)+\int_{0}^{T} \int_{\omega_{2}^{\prime \prime}} e^{2 s \alpha}(s \phi)^{22}\left(\left|\nabla \varphi_{1}\right|^{2}+\left|\nabla \varphi_{2}\right|^{2}\right)\right) \tag{175}
\end{align*}
$$

$\underline{\text { Estimate of } \int_{0}^{T} \int_{\omega_{2}^{\prime}} e^{2 s \alpha}(s \phi)^{43}\left|\Delta \varphi_{3}\right|^{2} d x d t .}$
Let us introduce $\chi_{2} \in C^{\infty}\left(\bar{\Omega} ;\left[0,+\infty[)\right.\right.$, such that the support of $\chi_{2}$ in included in $\omega_{1}$ and $\chi_{2}=1$ in $\omega_{2}^{\prime}$. We multiply the second equation of (156) by $\operatorname{sign}\left(d_{2}-d_{3}\right) \chi_{2}(x) e^{2 s \alpha}(s \phi)^{45} \Delta \varphi_{3}$ and we integrate on $(0, T) \times \omega_{1}$. As $d_{2} \neq d_{3}$, we have

$$
\begin{align*}
& \int_{0}^{T} \int_{\omega_{1}} \chi_{2}(x) e^{2 s \alpha}(s \phi)^{43}\left|\Delta \varphi_{3}\right|^{2} d x d t \\
& \leq C \int_{0}^{T} \int_{\omega_{1}} \chi_{2}(x) e^{2 s \alpha}(s \phi)^{43} \Delta \varphi_{3}\left(-\partial_{t} \varphi_{2}-d_{2} \Delta \varphi_{2}-a_{12} \varphi_{1}-a_{22} \varphi_{2}\right) d x d t \tag{176}
\end{align*}
$$

Let $\varepsilon^{\prime \prime}>0$ which will be chosen small enough. We estimate the right hand side of (176) in the same way as the one of (108):

- for terms involving $\Delta \varphi_{3} a_{i 2} \varphi_{i}$ with $1 \leq i \leq 2$, we apply (101) with $\Phi=\Delta \varphi_{3}, \Psi=\varphi_{i}$, $a=a_{i 2} \in L^{\infty}(Q), 1 \leq i \leq 2$ (recalling (80)), $\Theta=\chi_{2}$ and $r=43, k=5, l=81$,
- for the term involving $\Delta \varphi_{3} \partial_{t} \varphi_{2}$, we apply (102) with $\Phi=\Delta \varphi_{3}, \Psi=\varphi_{2}, a=1, \Theta=\chi_{2}$ and $r=43, k=5, l=85$,
- for the term involving $\Delta \varphi_{3} \Delta \varphi_{2}$, we apply (103) with $\Phi=\Delta \varphi_{3}, \Psi=\varphi_{2}, a=d_{2}, \Theta=\chi_{2}$ and $r=43, k=5, l=85$.
We get

$$
\begin{align*}
& \int_{0}^{T} \int_{\omega_{1}} \chi_{2} e^{2 s \alpha}(s \phi)^{43}\left|\Delta \varphi_{3}\right|^{2} \\
& \leq \varepsilon^{\prime \prime}\left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}\left\{(s \phi)^{5}\left|\Delta \varphi_{3}\right|^{2}+(s \phi)^{3}\left|\nabla \Delta \varphi_{3}\right|^{2}+(s \phi)\left(\left|\partial_{t} \Delta \varphi_{3}\right|^{2}+\left|\Delta \Delta \varphi_{3}\right|^{2}\right\}\right)\right. \\
& \quad+C_{\varepsilon^{\prime \prime}} \int_{0}^{T} \int_{\omega_{1}} e^{2 s \alpha}(s \phi)^{85}\left(\left|\varphi_{1}\right|^{2}+\left|\varphi_{2}\right|^{2}\right) . \tag{177}
\end{align*}
$$

By taking $\varepsilon^{\prime \prime}$ sufficiently small, we get from (175), (177)

$$
\begin{align*}
& \int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{3}\left|\Delta \Delta \varphi_{4}\right|^{2} d x d t+I\left(2, \lambda, s, \Delta \varphi_{3}\right)+I\left(5, \lambda, s, \varphi_{2}\right)+I\left(5, \lambda, s, \varphi_{1}\right) \\
& \leq C \int_{0}^{T} \int_{\omega_{1}} e^{2 s \alpha}(s \phi)^{85}\left(\left|\varphi_{1}\right|^{2}+\left|\varphi_{2}\right|^{2}\right)+\int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}(s \phi)^{22}\left(\left|\nabla \varphi_{1}\right|^{2}+\left|\nabla \varphi_{2}\right|^{2}\right) \tag{178}
\end{align*}
$$

$\underline{\text { Estimate of } \int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}(s \phi)^{22}\left|\nabla \varphi_{i}\right|^{2} d x d t \text { for } 1 \leq i \leq 2 .}$

Applying Lemma 4.15: (104) (with $\Phi=\varphi_{i}, \widetilde{\omega}=\omega_{1}, \Theta=\chi_{2}, r=22$ and $(k, l)=(4,40)$ ), for any $\varepsilon^{\prime \prime \prime}>0$, we have

$$
\begin{align*}
& \int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}(s \phi)^{22}\left|\nabla \varphi_{i}\right|^{2} d x d t \\
& \leq \int_{0}^{T} \int_{\omega_{1}} \chi_{2} e^{2 s \alpha}(s \phi)^{22}\left|\nabla \varphi_{i}\right|^{2} d x d t \\
& \leq \varepsilon^{\prime \prime \prime}\left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}\left\{(s \phi)^{4}\left|\Delta \varphi_{i}\right|^{2}+(s \phi)^{6}\left|\nabla \varphi_{i}\right|^{2}\right\} d x d t\right)+C_{\varepsilon^{\prime \prime \prime}} \int_{0}^{T} \int_{\omega_{1}} e^{2 s \alpha}(s \phi)^{40}\left|\varphi_{i}\right|^{2} d x d t \tag{179}
\end{align*}
$$

By taking $\varepsilon^{\prime \prime \prime}$ small enough and by using (178) and (179), we have

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{3}\left|\Delta \Delta \varphi_{4}\right|^{2}+I\left(2, \lambda, s, \Delta \varphi_{3}\right)+\sum_{i=1}^{2} I\left(5, \lambda, s, \varphi_{i}\right) \leq C \int_{0}^{T} \int_{\omega_{1}} e^{2 s \alpha}(s \phi)^{85}\left(\left|\varphi_{1}\right|^{2}+\left|\varphi_{2}\right|^{2}\right) . \tag{180}
\end{equation*}
$$

Estimate of $\int_{0}^{T} \int_{\omega_{1}} e^{2 s \alpha}(s \phi)^{85}\left|\varphi_{2}\right|^{2} d x d t$.
Let us introduce $\chi_{1} \in C^{\infty}\left(\bar{\Omega} ;\left[0,+\infty[)\right.\right.$, such that the support of $\chi_{1}$ in included in $\omega_{0}$ and $\chi_{1}=1$ in $\omega_{1}$. We multiply the first equation of (92) by $\chi_{1}(x) e^{2 s \alpha}(s \phi)^{85} \varphi_{2}$ and we integrate on $(0, T) \times \omega_{0}$. Recalling (79), we have

$$
\begin{align*}
& \int_{0}^{T} \int_{\omega_{0}} \chi_{1}(x) e^{2 s \alpha}(s \phi)^{85}\left|\varphi_{2}\right|^{2} d x d t \\
& \leq C \int_{0}^{T} \int_{\omega_{0}} \chi_{1}(x) e^{2 s \alpha}(s \phi)^{85} \varphi_{2}\left(-\partial_{t} \varphi_{1}-d_{1} \Delta \varphi_{2}-a_{11} \varphi_{1}\right) d x d t \tag{181}
\end{align*}
$$

We estimate the right hand side of (181) in the same way as the one of (108):

- for the term involving $\varphi_{2} a_{11} \varphi_{1}$, we apply (101) with $\Phi=\varphi_{2}, \Psi=\varphi_{1}, a=a_{11} \in L^{\infty}(Q)$ (recalling (80)), $\Theta=\chi_{1}$ and $r=85, k=8, l=162$,
- for the term involving $\varphi_{2} \partial_{t} \varphi_{1}$, we apply (102) with $\Phi=\varphi_{2}, \Psi=\varphi_{1}, a=1, \Theta=\chi_{1}$ and $r=85, k=8, l=166$,
- for the term involving $\varphi_{2} \Delta \varphi_{1}$, we apply (103) with $\Phi=\varphi_{2}, \Psi=\varphi_{1}, a=d_{1}, \Theta=\chi_{1}$ and $r=85, k=8, l=166$.
We get

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{3}\left|\Delta \Delta \varphi_{4}\right|^{2}+I\left(2, \lambda, s, \Delta \varphi_{3}\right)+\sum_{i=1}^{2} I\left(5, \lambda, s, \varphi_{i}\right) \leq C \int_{0}^{T} \int_{\omega_{0}} e^{2 s \alpha}(s \phi)^{166}\left|\varphi_{1}\right|^{2} \tag{182}
\end{equation*}
$$

Then, we can deduce from (99) and (182)

$$
\begin{equation*}
\sum_{i=1}^{2} \int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{8}\left|\varphi_{i}\right|^{2}+\int_{0}^{T} \int_{\Omega} e^{2 s \widehat{\alpha}}\left\{(\widehat{s \phi})^{5}\left|\Delta \varphi_{3}\right|^{2}+(\widehat{s \phi})^{3}\left|\Delta \Delta \varphi_{4}\right|^{2}\right\} \leq C \int_{0}^{T} \int_{\omega} e^{2 s \alpha}(s \phi)^{166}\left|\varphi_{1}\right|^{2} \tag{183}
\end{equation*}
$$

Now, we use Poincaré-Wirtinger inequality as in (151) to get

$$
\begin{align*}
& \int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{8}\left(\left|\varphi_{1}\right|^{2}+\left|\varphi_{2}\right|^{2}\right)+e^{2 s \widehat{\alpha}}\left\{(s \widehat{\phi})^{5}\left|\varphi_{3}-\left(\varphi_{3}\right)_{\Omega}\right|^{2}+(s \widehat{\phi})^{3}\left|\varphi_{4}-\left(\varphi_{4}\right)_{\Omega}\right|^{2}\right\} \\
& \leq C \int_{0}^{T} \int_{\omega} e^{2 s \alpha}(s \phi)^{166}\left|\varphi_{1}\right|^{2} . \tag{184}
\end{align*}
$$

Now, from the dissipation of the energy of $\left(\varphi_{1}, \varphi_{2}, \varphi_{3}-\left(\varphi_{3}\right)_{\Omega}, \varphi_{4}-\left(\varphi_{4}\right)_{\Omega}\right)$ (see Lemma A. 1 in Appendix A) and by using the same arguments as for 2 controls (see (152) and (153)), we easily get

$$
\begin{equation*}
\sum_{i=1}^{2}\left\|\varphi_{i}(0, .)\right\|_{L^{2}(\Omega)}^{2}+\sum_{i=3}^{4}\left\|\varphi_{i}(0, .)-\left(\varphi_{i}\right)(0, .)_{\Omega}\right\|_{L^{2}(\Omega)}^{2} \leq C \int_{0}^{T} \int_{\omega} e^{2 s \alpha}(s \phi)^{166}\left|\varphi_{1}\right|^{2} d x d t \tag{185}
\end{equation*}
$$

and consequently the observability inequality (96).
This ends the proof of the observability inequality (96).

### 4.4. Second step: Controls in $L^{\infty}(Q)^{j}$

### 4.4.1. Penalized Hilbert Uniqueness Method

The proof in this subsection follows ideas of [8] and [17, Section 3.1.2]. The goal is to get more regular controls in some sense (see (203)) by considering a penalized problem.

Let $\varepsilon \in(0,1)$ and

$$
M_{3}:=7, M_{2}:=13, M_{1}:=166
$$

We choose $\lambda$ and $s$ large enough such that (120), (153), (185) hold.
Let $j \in\{1,2,3\}, A \in \mathcal{E}_{j}$ (see (87), (85) and (83)), $\zeta_{0} \in H_{j}$ (see (88), (86), (84)). We introduce the notation $L_{w g h t}^{2}((0, T) \times \omega)^{j}$ for the set of functions $h^{j}$ such that for every $1 \leq i \leq j$, $\left(e^{-2 s \alpha}(s \phi)^{-M_{j}}\right)^{1 / 2} h_{i} \in L^{2}((0, T) \times \omega)$. The set $L_{w g h t}^{2}\left((0, T) \times \omega_{0}\right)^{j}$ is an Hilbert space equipped with the inner product $(h, k)=\sum_{i=1}^{j} \iint_{(0, T) \times \omega_{0}} e^{-2 s \alpha}(s \phi)^{-M_{j}} h_{i} k_{i} d x d t$. We define

$$
\forall h^{j} \in L_{w g h t}^{2}((0, T) \times \omega)^{j}, J\left(h^{j}\right):=\frac{1}{2} \iint_{(0, T) \times \omega} e^{-2 s \alpha}(s \phi)^{-M_{j}}\left|h^{j}\right|^{2} d x d t+\frac{1}{2 \varepsilon}\|\zeta(T, .)\|_{L^{2}(\Omega)^{4}}^{2},
$$

where $\zeta=\left(\zeta_{1}, \zeta_{2}, \zeta_{3}, \zeta_{4}\right)$ is the solution to the Cauchy problem (90) associated to the control $h^{j}$.
The mapping $J$ is a continuous, coercive, strictly convex functional on the Hilbert space $L_{w g h t}^{2}((0, T) \times \omega)^{j}$, then $J$ has a unique minimum $h^{j, \varepsilon}$ with $\left(e^{-2 s \alpha}(s \phi)^{-M_{j}}\right)^{1 / 2} h^{j, \varepsilon} \in L^{2}((0, T) \times \omega)^{j}$. Let $\zeta^{\varepsilon}$ be the solution to the Cauchy problem (90) with control $h^{j, \varepsilon}$ and initial condition $\zeta_{0}$.

The Euler-Lagrange equation gives

$$
\begin{equation*}
\forall h^{j} \in L_{w g h t}^{2}((0, T) \times \omega)^{j}, \sum_{i=1}^{j} \iint_{(0, T) \times \omega} e^{-2 s \alpha}(s \phi)^{-M_{j}} h_{i}^{\varepsilon} h_{i}+\frac{1}{\varepsilon} \int_{\Omega} \zeta^{\varepsilon}(T, .) \zeta \zeta(T, .)=0, \tag{186}
\end{equation*}
$$

where $\zeta=\left(\zeta_{1}, \zeta_{2}, \zeta_{3}, \zeta_{4}\right)$ is the solution to the Cauchy problem (90) associated to the control $h^{j}$ and initial condition $\zeta_{0}=0$.

We introduce $\varphi^{\varepsilon}$ the solution to the adjoint problem (92) with final condition $\varphi^{\varepsilon}(T,)=$. $-\frac{1}{\varepsilon} \zeta^{\varepsilon}(T,$.$) . A duality argument between \zeta$ and $\varphi^{\varepsilon}$ gives

$$
\begin{equation*}
-\frac{1}{\varepsilon} \int_{\Omega} \zeta^{\varepsilon}(T, x) . \zeta(T, x) d x=\sum_{i=1}^{j} \iint_{(0, T) \times \omega_{0}} h_{i} \varphi_{i}^{\varepsilon} d x d t \tag{187}
\end{equation*}
$$

Then, we deduce from (186) and (187) that

$$
\forall h^{j} \in L_{w g h t}^{2}((0, T) \times \omega)^{j}, \sum_{i=1}^{j} \iint_{(0, T) \times \omega} e^{-2 s \alpha}(s \phi)^{-M_{j}} h_{i}^{\varepsilon} h_{i} d x d t=\sum_{i=1}^{j} \iint_{(0, T) \times \omega} h_{i} \varphi_{i}^{\varepsilon} d x d t
$$

Consequently,

$$
\begin{equation*}
\forall i \in\{1, \ldots, j\}, h_{i}^{\varepsilon}=e^{2 s \alpha}(s \phi)^{M_{j}} \varphi_{i}^{\varepsilon} 1_{\omega} . \tag{188}
\end{equation*}
$$

Another duality argument applied between $\zeta^{\varepsilon}$ and $\varphi^{\varepsilon}$ together with (188) gives

$$
\begin{equation*}
-\frac{1}{\varepsilon}\left\|\zeta^{\varepsilon}(T, .)\right\|_{L^{2}(\Omega)^{4}}^{2}=\sum_{i=1}^{j} \iint_{(0, T) \times \omega} e^{2 s \alpha}(s \phi)^{M_{j}}\left|\varphi_{i}^{\varepsilon}\right|^{2} d x d t+\int_{\Omega} \varphi^{\varepsilon}(0, x) \cdot \zeta_{0}(x) d x \tag{189}
\end{equation*}
$$

If $j=2$, we have $\int_{\Omega} \zeta_{0,4}(x) d x=0$. Then,

$$
\begin{equation*}
\int_{\Omega} \varphi^{\varepsilon}(0, x) \cdot \zeta_{0}(x) d x=\sum_{i=1}^{3} \int_{\Omega} \varphi_{i}^{\varepsilon}(0, x) \zeta_{0, i}(x) d x+\int_{\Omega}\left(\varphi_{4}^{\varepsilon}(0, x)-\left(\varphi_{4}\right)_{\Omega}(0)\right) \zeta_{0,4}(x) d x \tag{190}
\end{equation*}
$$

If $j=1$, we have $\int_{\Omega} \zeta_{0,3}(x) d x=0$ and $\int_{\Omega} \zeta_{0,4}(x) d x=0$. Then,

$$
\begin{equation*}
\int_{\Omega} \varphi^{\varepsilon}(0, x) \cdot \zeta_{0}(x) d x=\sum_{i=1}^{2} \int_{\Omega} \varphi_{i}^{\varepsilon}(0, .) \zeta_{0, i}(.)+\sum_{i=3}^{4} \int_{\Omega}\left(\varphi_{i}^{\varepsilon}(0, .)-\left(\varphi_{i}\right)_{\Omega}(0)\right) \zeta_{0, i}(.) . \tag{191}
\end{equation*}
$$

Then, from (120) for $j=3$, (153), (190) for $j=2$, (185), (191) for $j=1$ and (188), (189), we have

$$
\begin{equation*}
\frac{1}{\varepsilon}\left\|\zeta^{\varepsilon}(T, .)\right\|_{L^{2}(\Omega)^{4}}^{2}+\frac{1}{2}\left\|\left(e^{-2 s \alpha}(s \phi)^{-M_{j}}\right)^{1 / 2} h^{j, \varepsilon}\right\|_{L^{2}((0, T) \times \omega)^{j}}^{2} \leq C\left\|\zeta_{0}\right\|_{L^{2}(\Omega)^{4}}^{2} \tag{192}
\end{equation*}
$$

In particular, from (192),

$$
\begin{equation*}
\zeta^{\varepsilon}(T, .) \underset{\varepsilon \rightarrow 0}{\rightarrow} 0 \text { in } L^{2}(\Omega)^{4} \tag{193}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|B_{j} h^{j, \varepsilon}\right\|_{L^{2}(Q)^{j}} \leq C . \tag{194}
\end{equation*}
$$

Then, by using $A \in \mathcal{M}_{4}\left(L^{\infty}(Q)\right)$ (see (87), (85) and (83)) and recalling (194), from Proposition 2.3 applied to (90), we deduce that

$$
\begin{equation*}
\left\|\zeta^{\varepsilon}\right\|_{Y^{4}} \leq C . \tag{195}
\end{equation*}
$$

So, from (195), up to a subsequence, we can suppose that there exists $\zeta \in Y^{4}$ such that

$$
\begin{gather*}
\zeta^{\varepsilon} \underset{\varepsilon \rightarrow 0}{\rightharpoonup} \zeta \text { in } L^{2}\left(0, T ; H^{1}(\Omega)^{4}\right),  \tag{196}\\
\partial_{t} \zeta^{\varepsilon} \underset{\varepsilon \rightarrow 0}{\rightharpoonup} \partial_{t} \zeta \text { in } L^{2}\left(0, T ;\left(H^{1}(\Omega)\right)^{\prime 4}\right), \tag{197}
\end{gather*}
$$

and from Proposition 2.2,

$$
\begin{equation*}
\zeta^{\varepsilon}(0, .) \underset{\varepsilon \rightarrow 0}{\rightharpoonup} \zeta(0, .) \text { in } L^{2}(\Omega)^{4}, \zeta^{\varepsilon}(T, .) \underset{\varepsilon \rightarrow 0}{\rightharpoonup} \zeta(T, .) \text { in } L^{2}(\Omega)^{4} \tag{198}
\end{equation*}
$$

Then, as we have $\zeta^{\varepsilon}(0,)=.\zeta_{0}$ and (193), we deduce that

$$
\begin{equation*}
\zeta(0, .)=\zeta_{0}, \text { and } \zeta(T, .)=0 \tag{199}
\end{equation*}
$$

Moreover, from (192), up to a subsequence, we can suppose that there exists $h^{j} \in L_{w g h t}^{2}((0, T) \times$ $\omega)^{j}$ such that

$$
\begin{equation*}
\left(h^{j, \varepsilon}\right) \underset{\varepsilon \rightarrow 0}{\stackrel{\rightharpoonup}{x}} h^{j} \text { in } \underset{46}{L_{w g h t}^{2}}((0, T) \times \omega)^{j}, \tag{200}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|\left(e^{-2 s \alpha}(s \phi)^{-M_{j}}\right)^{1 / 2} h^{j}\right\|_{L^{2}((0, T) \times \omega)^{j}}^{2} \leq \lim _{\varepsilon \rightarrow 0} \inf \|\left(e^{-2 s \alpha}(s \phi)^{\left.-M_{j}\right)^{1 / 2} h^{j, \varepsilon}\left\|_{L^{2}((0, T) \times \omega)^{j}}^{2} \leq C\right\| \zeta_{0} \|_{L^{2}(\Omega)^{4}}^{2} . . . . ~ . ~}\right. \tag{201}
\end{equation*}
$$

Then, from (196), (197), (200), we let $\varepsilon \rightarrow 0$ in the following equations

$$
\begin{cases}\partial_{t} \zeta^{\varepsilon}-D \Delta \zeta^{\varepsilon}=A(t, x) \zeta^{\varepsilon}+B_{j} h^{j, \varepsilon} 1_{\omega} & \text { in }(0, T) \times \Omega \\ \frac{\partial \zeta^{\varepsilon}}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega\end{cases}
$$

and by using (199), we deduce

$$
\begin{cases}\partial_{t} \zeta-D \Delta \zeta=A(t, x) \zeta+B_{j} h^{j} 1_{\omega} & \text { in }(0, T) \times \Omega  \tag{202}\\ \frac{\partial \zeta}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega \\ (\zeta(0, .), \zeta(T, .))=\left(\zeta_{0}, 0\right) & \text { in } \Omega\end{cases}
$$

Therefore, we have proved the existence of a control $h^{j}$ such that $\left(e^{-2 s \alpha}(s \phi)^{-M_{j}}\right)^{1 / 2} h^{j} \in L^{2}((0, T) \times$ $\omega)^{j}$ that drives the solution $\zeta$ of (90) to 0 , and we have the estimate

$$
\begin{equation*}
\left\|\left(e^{-2 s \alpha}(s \phi)^{-M_{j}}\right)^{1 / 2} h^{j}\right\|_{L^{2}((0, T) \times \omega)^{j}}^{2} \leq C\left\|\zeta_{0}\right\|_{L^{2}(\Omega)^{4}}^{2} . \tag{203}
\end{equation*}
$$

### 4.4.2. Bootstrap method

In the previous subsection, we proved the existence of a control $h^{j} \in L_{w g h t}^{2}((0, T) \times \omega)^{j}$ i.e. a control $h^{j}$ more regular than $L^{2}(Q)$. The key points are the link between $h^{j, \varepsilon}$ and $\varphi^{\varepsilon}$ (i.e. (188)) and the weights of Carleman estimates. Now, we use an iterative process in order to find controls in $L^{\infty}(Q)^{j}$. We use the same key points together with parabolic regularity theorems. This section is inspired by [17, Section 3.1.2] and [51] (for the Neumann conditions). First, we are going to present the boostrap method for the case $j=3$ and after that, we explain the main differences for the case $j=2$ and $j=1$.
4.4.2.1. Strong observability inequalities. From (114) for the case $j=3$, (151) for the case $j=2$, (184) for the case $j=1$, (188) and (192), we deduce these inegalities which are useful for the bootstrap method:

$$
\begin{gather*}
(j=3) \Rightarrow\left(\sum_{i=1}^{4} \int_{0}^{T} \int_{\Omega} e^{2 \widehat{\alpha}}(\widehat{s \phi})^{3}\left|\varphi_{i}^{\varepsilon}\right|^{2} d x d t \leq C\left\|\zeta_{0}\right\|_{L^{2}(\Omega)^{4}}^{2}\right)  \tag{204}\\
(j=2) \Rightarrow\left(\int_{0}^{T} \int_{\Omega} \sum_{i=1}^{3} e^{2 s \widehat{\alpha}}(\widehat{s \phi})^{3}\left|\varphi_{i}^{\varepsilon}\right|^{2}+e^{2 s \widehat{\alpha}}(\widehat{s \phi})^{3}\left|\varphi_{4}^{\varepsilon}-\left(\varphi_{4}^{\varepsilon}\right)_{\Omega}\right|^{2} \leq C\left\|\zeta_{0}\right\|_{L^{2}(\Omega)^{4}}^{2}\right)  \tag{205}\\
(j=1) \Rightarrow\left(\int_{0}^{T} \int_{\Omega} \sum_{i=1}^{2} e^{2 s \widehat{\alpha}}(\widehat{s \phi})^{3}\left|\varphi_{i}^{\varepsilon}\right|^{2}+\sum_{i=3}^{4} e^{2 \widehat{\alpha}}(\widehat{\phi \phi})^{3}\left|\varphi_{i}^{\varepsilon}-\left(\varphi_{i}^{\varepsilon}\right)_{\Omega}\right|^{2} \leq C\left\|\zeta_{0}\right\|_{L^{2}(\Omega)^{4}}^{2}\right) . \tag{206}
\end{gather*}
$$

4.4.2.2. Bootstrap. Let $\delta>0$ which will be chosen sufficiently small and $\left(\delta_{k}\right)_{k \in \mathbb{N}} \in\left(\mathbb{R}^{+, *}\right)^{\mathbb{N}}$ be a strictly increasing sequence such that $\delta_{k} \underset{k \rightarrow+\infty}{\rightarrow} \delta$. Let $\left(p_{k}\right)_{k \in \mathbb{N}}$ be the following sequence defined by induction

$$
p_{0}=2
$$

$$
p_{k+1}:=\left\{\begin{array}{cl}
\frac{(N+2) p_{k}}{N+2-2 p_{k}} & \text { if } p_{k}<\frac{N+2}{2} \\
2 p_{k} & \text { if } p_{k}=\frac{N+2}{2} \\
+\infty & \text { if } p_{k}>\frac{N+2}{2} .
\end{array}\right.
$$

Clearly, we have that

$$
\begin{equation*}
\exists l \in \mathbb{N}, \forall k \geq l, p_{k}=+\infty . \tag{207}
\end{equation*}
$$

Definition 4.34. We introduce the following spaces: for every $r \in[1,+\infty]$,

$$
W_{N e}^{2, r}(\Omega):=\left\{u \in W^{2, r}(\Omega) ; \frac{\partial u}{\partial n}=0\right\}, \quad Y_{r}=L^{r}\left(0, T ; W_{N e}^{2, r}(\Omega)\right) \cap W^{1, r}\left(0, T ; L^{r}(\Omega)\right) .
$$

Definition 4.35. Let $u$ be a function on $Q$. For $0<\beta<1$, we define

$$
[u]_{\beta / 2, \beta}=\sup _{(t, x),\left(t^{\prime}, x^{\prime}\right) \in Q,(t, x) \neq\left(t^{\prime}, x^{\prime}\right)} \frac{\left|u(t, x)-u\left(t^{\prime}, x^{\prime}\right)\right|}{\left(\left|t-t^{\prime}\right|+\left|x-x^{\prime}\right|^{2}\right)^{\beta / 2}},
$$

which is a semi-norm, and we denote by $C^{\beta / 2, \beta}(\bar{Q})$ the set of all functions on $Q$ such that $[u]_{\beta / 2, \beta}<$ $+\infty$, endowed with the norm

$$
\|u\|_{\beta / 2, \beta}=\left(\sup _{(t, x) \in Q}|u(t, x)|\right)+[u]_{\beta / 2, \beta} .
$$

Proposition 4.36. Let $1<p<+\infty, m \in \mathbb{N}^{*}, D \in \mathcal{M}_{m}(\mathbb{R})$ such that $S p(D) \subset(0,+\infty)$, $A \in$ $\mathcal{M}_{m}\left(L^{\infty}(Q)\right), f \in L^{p}(Q)^{m}$. From [20, Theorem 2.1], the following Cauchy problem admits a unique solution $u \in Y_{p}^{m}$

$$
\begin{cases}\partial_{t} u-D \Delta u=A(t, x) u+f & \text { in }(0, T) \times \Omega, \\ \frac{\partial u}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega, \\ u(0, .)=0 & \text { in } \Omega .\end{cases}
$$

Moreover, there exists $C>0$ independent of $f$ such that

$$
\|u\|_{Y_{p}^{m}} \leq C\|f\|_{L^{p}(Q)^{k}}
$$

Proposition 4.37. [52, Theorem 1.4.1]
Let $r \in[1,+\infty[$, we have

$$
Y_{r} \hookrightarrow\left\{\begin{array}{cl}
L^{\frac{(N+2) r}{N+2-2 r}}(Q) & \text { if } r<\frac{N+2}{N_{2}^{2}}, \\
L^{2 r}(Q) & \text { if } r=\frac{N+2}{2}, \\
C^{\beta / 2, \beta}(\bar{Q}) \hookrightarrow L^{\infty}(Q) \text { with } 0<\beta \leq 2-\frac{N+2}{r} & \text { if } r>\frac{N+2}{2} .
\end{array}\right.
$$

$j=3$
In the following, $C$ denotes various positive constants varying from one line to the other and does not depend of $\left\|\zeta_{0}\right\|_{L^{2}(\Omega}$.

We define for every $k \in \mathbb{N}$,

$$
\begin{equation*}
\psi^{\varepsilon, k}:=e^{\widetilde{\alpha}\left(s+\delta_{k}\right)} \varphi^{\varepsilon} \tag{208}
\end{equation*}
$$

For $k \in \mathbb{N}^{*}$, by using (208) and the adjoint system (92) satisfied by $\varphi^{\varepsilon}$, we have

$$
\begin{cases}-\partial_{t} \psi^{\varepsilon, k}-D_{3} \Delta \psi^{\varepsilon, k}=A(t, x) \psi^{\varepsilon, k}+f_{k} & \text { in }(0, T) \times \Omega  \tag{209}\\ \frac{\partial \psi^{t, k}}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega \\ \psi^{\varepsilon, k}(T, .)=0 & \text { in } \Omega,\end{cases}
$$

with

$$
f_{k}(t, x)=-\partial_{t}\left(e^{\widetilde{\alpha}\left(s+\delta_{k}\right)}\right) \varphi^{\varepsilon} .
$$

By using the fact that $\left(\delta_{k}\right)_{k \in \mathbb{N}}$ is strictly increasing, we easily have that

$$
\begin{equation*}
\left|f_{k}\right| \leq C e^{\widetilde{\alpha}\left(s+\delta_{k-1}\right)}\left|\varphi^{\varepsilon}\right|=C\left|\psi^{\varepsilon, k-1}\right| \text { in }(0, T) \times \Omega . \tag{210}
\end{equation*}
$$

We show, by induction, that for every $0 \leq k \leq l$ (see (207)), we have

$$
\begin{equation*}
\psi^{\varepsilon, k} \in L^{p_{k}}(Q)^{4} \text { and }\left\|\psi^{\varepsilon, k}\right\|_{L^{p_{k}}(Q)^{4}} \leq C\left\|\zeta_{0}\right\|_{L^{2}(\Omega)^{4}} . \tag{211}
\end{equation*}
$$

The case $k=0$ can be deduced from the fact that $\delta_{0}>0$ and the strong observability inequality (204).

Let $1 \leq k \leq l$. We suppose that

$$
\begin{equation*}
\psi^{\varepsilon, k-1} \in L^{p_{k-1}}(Q)^{4} \text { and }\left\|\psi^{\varepsilon, k-1}\right\|_{L^{p_{k-1}}(Q)^{4}} \leq C\left\|\zeta_{0}\right\|_{L^{2}(\Omega)^{4}} \tag{212}
\end{equation*}
$$

Then, from (209), (210), (212) and from the maximal regularity theorem: Proposition 4.36, we get

$$
\begin{equation*}
\psi^{\varepsilon, k} \in X_{p_{k-1}}^{4} \text { and }\left\|\psi^{\varepsilon, k}\right\|_{X_{p k-1}^{4}} \leq C\left\|\zeta_{0}\right\|_{L^{2}(\Omega)^{4}} . \tag{213}
\end{equation*}
$$

Moreover, by the Sobolev embedding Proposition 4.37, we have

$$
\psi^{\varepsilon, k} \in L^{p_{k}}(Q)^{4} \text { and }\left\|\psi^{\varepsilon, k}\right\|_{L^{p_{k}}(Q)^{4}} \leq C\left\|\zeta_{0}\right\|_{L^{2}(\Omega)^{4}} .
$$

This concludes the induction.
From (98) and (99), we remark that we have the following inequality

$$
\begin{equation*}
\alpha \leq \frac{\widehat{\alpha}}{1+f(\lambda)} \tag{214}
\end{equation*}
$$

because

$$
\left(e^{\lambda \eta_{0}(x)}-e^{2 \lambda\left\|\eta_{0}\right\|_{\infty}}\right)\left(1+e^{-\lambda\left\|\eta_{0}\right\|_{\infty}}\right)=e^{\lambda \eta_{0}(x)}-e^{\lambda\left\|\eta_{0}\right\|_{\infty}}+1-e^{2 \lambda\left\|\eta_{0}\right\|_{\infty}} \leq 1-e^{2 \lambda\left\|\eta_{0}\right\|_{\infty}} .
$$

Moreover, from (97), we can pick $\delta>0$ such that

$$
\begin{equation*}
2 s-(1+f(\lambda))(s+\delta)=s(2-(1+f(\lambda)))-\delta(1+f(\lambda))>0 . \tag{215}
\end{equation*}
$$

Now, by applying consecutively (207), (188), (214), (215) and (211), we have for every $i \in$ $\{1, \ldots, 3\}$,

$$
\begin{align*}
\left\|h_{i}^{\varepsilon}\right\|_{L^{\infty}(Q)}=\left\|h_{i}^{\varepsilon}\right\|_{L^{p_{l}}(Q)} & =\left\|e^{2 s \alpha}(s \phi)^{7} \varphi_{i}^{\varepsilon}\right\|_{L^{p_{l}}(Q)} \\
& \leq\left\|e^{\widehat{\alpha}\left(\frac{2 s}{1+f(x)}-(s+\delta)\right.}(s \phi)^{7}\right\|_{L^{\infty}(Q)}\left\|e^{\widehat{\alpha}(s+\delta)} \varphi_{i}^{\varepsilon}\right\|_{L^{p_{l}}(Q)} \\
& \leq C\left\|e^{\widehat{\alpha}(s+\delta)} \varphi_{i}^{\varepsilon}\right\|_{L^{p_{l}}(Q)} \\
& \leq C\left\|e^{\widehat{\alpha}\left(s+\delta_{l}\right)} \varphi_{i}^{\varepsilon}\right\|_{L^{p_{l}(Q)}}\left(\delta_{l} \leq \delta \text { and } \widehat{\alpha}<0\right) \\
& \leq C\left\|\zeta_{0}\right\|_{L^{2}(\Omega)^{4}} \tag{216}
\end{align*}
$$

Therefore, from (216), we get

$$
\begin{equation*}
\left\|h_{i}^{\varepsilon}\right\|_{L^{\infty}(Q)} \leq C\left\|\xi_{0}\right\|_{L^{2}(\Omega)^{4}} . \tag{217}
\end{equation*}
$$

So, $\left(h^{3, \varepsilon}\right)_{\varepsilon}$ is bounded in $L^{\infty}(Q)^{3}$, then up to a subsequence, we can suppose that there exists $h^{3} \in L^{\infty}(Q)^{3}$ such that

$$
\begin{equation*}
h^{3, \varepsilon} \underset{\varepsilon \rightarrow 0}{\rightarrow}{ }^{*} h^{3} \text { in } L^{\infty}(Q)^{3}, \tag{218}
\end{equation*}
$$

and

$$
\left\|h^{3}\right\|_{L^{\infty}(Q)^{3}} \leq C\left\|\zeta_{0}\right\|_{L^{2}(\Omega)^{4}} .
$$

From (196), (197), (218), (199), we have

$$
\begin{cases}\partial_{t} \zeta-D_{3} \Delta \zeta=A(t, x) \zeta+B_{3} h^{3} 1_{\omega} & \text { in }(0, T) \times \Omega,  \tag{219}\\ \frac{\partial \zeta}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega, \\ (\zeta(0, .), \zeta(T, .))=\left(\zeta_{0}, 0\right) & \text { in } \Omega .\end{cases}
$$

This ends the proof of Proposition 4.8 for the case $j=3$.

$$
j=2
$$

For every $k \in \mathbb{N}$, we introduce

$$
\begin{align*}
\widetilde{\varphi^{\varepsilon}} & :=\left(\varphi_{1}^{\varepsilon}, \varphi_{2}^{\varepsilon}, \varphi_{3}^{\varepsilon}, \varphi_{4}^{\varepsilon}-\left(\varphi_{4}^{\varepsilon}\right)_{\Omega}\right)^{T},  \tag{220}\\
\psi^{\varepsilon, k} & :=e^{\widetilde{\alpha}\left(s+\delta_{k}\right)} \widehat{\varphi^{\varepsilon, k}} . \tag{221}
\end{align*}
$$

For $k \in \mathbb{N}^{*}$, we have

$$
\begin{cases}-\partial_{1} \psi^{\varepsilon, k}-D_{2} \Delta \psi^{\varepsilon, k}=A(t, x) \psi^{\varepsilon, k}+f_{k} & \text { in }(0, T) \times \Omega  \tag{222}\\ \frac{\partial \psi^{\varepsilon, k}}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega \\ \psi^{\varepsilon, k}(T, .)=0 & \text { in } \Omega,\end{cases}
$$

with

$$
f_{k}(t, x)=-\left(e^{\widetilde{\alpha}\left(s+\delta_{k}\right)}\right)_{t} \widetilde{\varphi^{\varepsilon, k}}+\left(0,0,0,\left(u_{2}^{*} e^{\widetilde{\alpha}\left(s+\delta_{k}\right)} \varphi_{1}^{\varepsilon}-u_{2}^{*} e^{\widetilde{\alpha}\left(s+\delta_{k}\right)} \varphi_{2}^{\varepsilon}+u_{2}^{*} e^{\widetilde{\alpha}\left(s+\delta_{k}\right)} \varphi_{3}^{\varepsilon}\right)_{\Omega}\right)^{T},
$$

because $A \in \mathcal{E}_{2}$ (see (85)). From the fact that $\left(\delta_{k}\right)_{k \in \mathbb{N}}$ is strictly increasing, we easily have

$$
\begin{equation*}
\left|f_{k}\right| \leq C e^{\widetilde{\alpha}\left(s+\delta_{k-1}\right)}\left|\widetilde{\varphi^{\varepsilon}}\right|=C\left|\psi^{\varepsilon, k-1}\right| \text { in }(0, T) \times \Omega . \tag{223}
\end{equation*}
$$

Then, the strategy of bootstrap is exactly the same. The starting point comes from the strong observability inequality (205).

$$
j=1
$$

We apply the same strategy as for the case $j=2$. For every $k \in \mathbb{N}$, we introduce

$$
\begin{align*}
\widetilde{\varphi^{\varepsilon}} & :=\left(\varphi_{1}^{\varepsilon}, \varphi_{2}^{\varepsilon}, \varphi_{3}^{\varepsilon}-\left(\varphi_{3}^{\varepsilon}\right)_{\Omega}, \varphi_{4}^{\varepsilon}-\left(\varphi_{4}^{\varepsilon}\right)_{\Omega}\right)^{T},  \tag{224}\\
\psi^{\varepsilon, k} & :=e^{\widetilde{\alpha}\left(s+\delta_{k}\right)} \widetilde{\varphi^{\varepsilon, k}} . \tag{225}
\end{align*}
$$

The starting point comes from the strong observability inequality (206).
This ends the proof of Proposition 4.8.

### 4.5. Nonlinear problem

In order to prove Theorem 3.2, we use Proposition 4.8 together with a standard fixed-point argument.

### 4.5.1. Reduction to a fixed point problem

Let $j \in\{1,2,3\}$. We remark that $G: L^{\infty}(Q)^{4} \rightarrow \mathcal{M}_{4}\left(L^{\infty}(Q)\right)$ is continuous (see (52), (62) and (77)). Then, we get the existence of $v>0$ small enough such that for every $z=\left(z_{1}, z_{2}, z_{3}, z_{4}\right) \in$ $L^{\infty}(Q)^{4}$,

$$
\begin{equation*}
\left(\|z\|_{L^{\infty}(Q)^{4}} \leq v\right) \Rightarrow\left(\left(G\left(z_{1}, z_{2}, z_{3}, z_{4}\right)\right) \in \mathcal{E}_{j}\right), \tag{226}
\end{equation*}
$$

where $\mathcal{E}_{j}$ are defined in (83), (85) and (87).
Let $\mathcal{Z}$ be the set of $z=\left(z_{1}, z_{2}, z_{3}, z_{4}\right) \in L^{\infty}(Q)^{4}$ such that $\|z\|_{L^{\infty}(Q)^{4}} \leq v$. From Proposition 4.8, we have proved that there exists $C_{0}>0$ such that for all $z=\left(z_{1}, z_{2}, z_{3}, z_{4}\right) \in \mathcal{Z}$ and for all $\zeta_{0} \in L^{\infty}(Q)^{4}$, there exists a control $h^{j} \in L^{\infty}(Q)^{j}$ satisfying

$$
\begin{equation*}
\left\|h^{j}\right\|_{L^{\infty}(Q)^{j}} \leq C_{0}\left\|\zeta_{0}\right\|_{L^{2}(\Omega)^{4}}, \tag{227}
\end{equation*}
$$

such that the solution $\zeta=\left(\zeta_{1}, \zeta_{2}, \zeta_{3}, \zeta_{4}\right)^{T} \in\left(Y^{4} \cap L^{\infty}(Q)^{4}\right)$ to the Cauchy problem

$$
\begin{cases}\partial_{t} \zeta-D_{j} \Delta \zeta=G(z) \zeta+B_{j} h^{j} 1_{\omega} & \text { in }(0, T) \times \Omega  \tag{228}\\ \frac{\partial \zeta}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega \\ \zeta(0, .)=\zeta_{0} & \text { in } \Omega\end{cases}
$$

verifies

$$
\begin{equation*}
\zeta(T, .)=0 . \tag{229}
\end{equation*}
$$

We fix $\zeta_{0} \in L^{\infty}(Q)^{4}$.
We define $B: \mathcal{Z} \rightarrow L^{\infty}(Q)^{4}$ in the following way. For every $z=\left(z_{1}, z_{2}, z_{3}, z_{4}\right) \in \mathcal{Z}, B(z)$ is the set of $\zeta=\left(\zeta_{1}, \zeta_{2}, \zeta_{3}, \zeta_{4}\right) \in L^{\infty}(Q)^{4}$ solution to the Cauchy problem (228), associated to a control $h^{j} \in L^{\infty}(Q)^{j}$ satisfying (227), and which verifies (229).

Our main result (i.e. Theorem 3.2) will be proved if we show that $B$ has a fixed point (i.e. $z$ is such that $z \in B(z)$ ).

We use the Kakutani's fixed point theorem.
Theorem 4.38. Kakutani's fixed point theorem.

1. For every $z \in \mathcal{Z}, B(z)$ is a nonempty convex and closed subset of $L^{\infty}(Q)^{4}$.
2. There exists a convex compact set $K \subset \mathcal{Z}$ such that for every $z \in \mathcal{Z}, B(z) \subset K$.
3. $B$ is upper semicontinuous in $L^{\infty}(Q)^{4}$, that is to say for all closed subset $\mathcal{A} \subset \mathcal{Z}, B^{-1}(\mathcal{A})=$ $\{z \in \mathcal{Z} ; B(z) \cap \mathcal{A} \neq \emptyset\}$ is closed.
Then, $B$ has a fixed point.

### 4.5.2. Hypotheses of Kakutani's fixed point theorem

### 4.5.2.1. Proof of the point 1 . Let $z \in \mathcal{Z}$.

$B(z)$ is nonempty because we have proved the existence of at least one control satisfying (227) that drives the solution to 0 .
$B(z)$ is convex because the mapping $h \in L^{\infty}(Q)^{j} \mapsto \zeta \in L^{\infty}(Q)^{4}$, where $\zeta$ is the solution to the

Cauchy problem (228), is affine and (227) is clearly verified by convex combinations of controls satisfying it.
$B(z)$ is closed. Indeed, let $\left(\zeta_{k}\right)_{k \in \mathbb{N}}$ be a sequence of $B(z)$ such that

$$
\begin{equation*}
\zeta_{k} \underset{k \rightarrow+\infty}{ } \zeta \text { in } L^{\infty}(Q)^{4} \tag{230}
\end{equation*}
$$

We introduce $\left(h_{k}^{j}\right)_{k \in \mathbb{N}}$ the sequence of controls associated to $\left(\zeta_{k}\right)_{k \in \mathbb{N}}$. In particular, for every $k \in \mathbb{N}$,

$$
\begin{equation*}
\left\|h_{k}^{j}\right\|_{L^{\infty}(Q)^{j}} \leq C_{0}\left\|\zeta_{0}\right\|_{L^{2}(\Omega)^{4}} \tag{231}
\end{equation*}
$$

From (230) and (231), for every $k \in \mathbb{N}$,

$$
\begin{equation*}
\left\|G(z) \zeta_{k}+B_{j} h_{k}^{j}\right\|_{L^{\infty}(Q)^{4}} \leq C . \tag{232}
\end{equation*}
$$

Then, from (232) and Proposition 2.3 applied to $\zeta_{k}$ which satisfies (228), we deduce that for every $k \in \mathbb{N}$,

$$
\begin{equation*}
\left\|\zeta_{k}\right\|_{\left(Y \cap L^{\infty}(Q)\right)^{4}} \leq C . \tag{233}
\end{equation*}
$$

So, from (233), up to a subsequence, we can suppose that there exists $\zeta \in Y^{4}$ such that

$$
\begin{gather*}
\zeta_{k} \underset{k \rightarrow+\infty}{\rightharpoonup} \zeta \text { in } L^{2}\left(0, T ; H^{1}(\Omega)^{4}\right),  \tag{234}\\
\partial_{t} \zeta_{k} \underset{k \rightarrow+\infty}{\rightharpoonup} \partial_{t} \zeta \text { in } L^{2}\left(0, T ;\left(H^{1}(\Omega)\right)^{4}\right), \tag{235}
\end{gather*}
$$

and, from Proposition 2.2,

$$
\begin{equation*}
\zeta_{k}(0, .) \underset{k \rightarrow+\infty}{\rightharpoonup} \zeta(0, .) \text { in } L^{2}(\Omega)^{4}, \zeta_{k}(T, .) \underset{k \rightarrow+\infty}{\rightharpoonup} \zeta(T, .) \text { in } L^{2}(\Omega)^{4} . \tag{236}
\end{equation*}
$$

Then, as we have $\zeta_{k}(0,)=.\zeta_{0}$ and $\zeta_{k}(T,)=$.0 for every $k \in \mathbb{N}$, we deduce that

$$
\begin{equation*}
\zeta(0, .)=\zeta_{0} \text { and } \zeta(T, .)=0 . \tag{237}
\end{equation*}
$$

Moreover, from (231), up to a subsequence, we can suppose that there exists $h^{j} \in L^{\infty}(Q)^{j}$ such that

$$
\begin{equation*}
h_{k}^{j} \underset{k \rightarrow+\infty}{*} h^{j} \text { in } L^{\infty}(Q)^{j} \tag{238}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|h^{j}\right\|_{L^{\infty}(Q)^{j}} \leq \liminf _{k \rightarrow+\infty}\left\|h_{k}^{j}\right\|_{L^{\infty}(Q)^{j}} \leq C_{0}\left\|\zeta_{0}\right\|_{L^{2}(\Omega)^{4}} \tag{239}
\end{equation*}
$$

Then, from (234), (235), (236), (237) and (238), we let $k \rightarrow+\infty$ in the following equations (i.e. passing to the limit in the variational formulation (15))

$$
\begin{cases}\partial_{t} \zeta_{k}-D_{j} \Delta \zeta_{k}=G(z) \zeta_{k}+B_{j} h_{k}^{j} 1_{\omega} & \text { in }(0, T) \times \Omega, \\ \frac{\partial \zeta_{k}}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega, \\ \left(\zeta_{k}(0, .), \zeta_{k}(T, .)\right)=\left(\zeta_{0}, 0\right) & \text { in } \Omega\end{cases}
$$

We deduce that

$$
\begin{cases}\partial_{t} \zeta-D_{j} \Delta \zeta=G(z) \zeta+B_{j} h^{j} 1_{\omega} & \text { in }(0, T) \times \Omega  \tag{240}\\ \frac{\partial \zeta}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega \\ (\zeta(0, .), \zeta(T, .))=\left(\zeta_{0}, 0\right) & \text { in } \Omega\end{cases}
$$

Finally, from (240) and (239), we have $\zeta \in B(z)$.

### 4.5.2.2. Proof of the point 2. Let $z \in \mathcal{Z}$.

By Proposition 2.3 and (227), we deduce that there exists $C_{1}>0$ such that

$$
\forall z \in \mathcal{Z}, \forall \zeta \in B(z),\|\zeta\|_{L^{\infty}(Q)^{4}} \leq C_{1}\left\|\zeta_{0}\right\|_{L^{\infty}(\Omega)^{4}}
$$

Now, we suppose that $\zeta_{0} \in L^{\infty}(\Omega)^{4}$ verifies

$$
\begin{equation*}
\left\|\zeta_{0}\right\|_{L^{\infty}(\Omega)^{4}} \leq v / C_{1} . \tag{241}
\end{equation*}
$$

Then, we have

$$
\begin{equation*}
\forall z \in \mathcal{Z}, B(z) \subset \mathcal{Z} \tag{242}
\end{equation*}
$$

Let $F \in L^{\infty}(Q)^{4}$ be the solution to the Cauchy problem

$$
\begin{cases}\partial_{t} F-D_{j} \Delta F=0 & \text { in }(0, T) \times \Omega,  \tag{243}\\ \frac{\partial F}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega, \\ F(0, .)=\zeta_{0} & \text { in } \Omega .\end{cases}
$$

Let $\zeta^{*}=\zeta-F$, where $\zeta \in B(z)$ with $z \in \mathcal{Z}$. We also denote by $h^{j}$ the control associated to $\zeta$. Then, $\zeta^{*}$ is the solution to

$$
\begin{cases}\partial_{t} \zeta^{*}-D_{j} \Delta \zeta^{*}=G(z) \zeta+B_{j} h^{j} 1_{\omega} & \text { in }(0, T) \times \Omega  \tag{244}\\ \frac{\partial \zeta^{*}}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega \\ \zeta^{*}(0, .)=0 & \text { in } \Omega .\end{cases}
$$

From (226), (242) and (227), we can remark that there exists $C>0$ such that

$$
\begin{equation*}
\left\|G(z) \zeta+B_{j} h^{j} 1_{\omega}\right\|_{L^{\infty}(Q)^{4}} \leq C . \tag{245}
\end{equation*}
$$

From (245), Proposition 4.36 with $p=N+2$ applied to $\zeta^{*}$ (see (244)) and the Sobolev embedding theorem $Y_{p} \hookrightarrow C^{\beta / 2, \beta}(\bar{Q})$ with $\beta>0$ (see Proposition 4.37), we deduce that $\zeta^{*} \in C^{0}(\bar{Q})^{4}$ and there exists $C_{2}>0$ such that

$$
\begin{equation*}
\forall(t, x) \in \bar{Q}, \forall\left(t^{\prime}, x^{\prime}\right) \in \bar{Q},\left|\zeta^{*}(t, x)-\zeta^{*}\left(t^{\prime}, x^{\prime}\right)\right| \leq C_{2}\left(\left|t-t^{\prime}\right|^{\beta / 2}+\left|x-x^{\prime}\right|^{\beta}\right) \tag{246}
\end{equation*}
$$

Let $K^{*}$ be the set of $\zeta^{*}$ such that (246) holds. Then, we have $\left(F+K^{*}\right) \cap \mathcal{Z}$ is a compact convex subset of $L^{\infty}(Q)^{4}$ by Ascoli's theorem and

$$
\forall z \in \mathcal{Z}, B(z) \subset\left(F+K^{*}\right) \cap \mathcal{Z} .
$$

Then, $K:=\left(F+K^{*}\right) \cap \mathcal{Z}$ is a convex compact subset of $\mathcal{Z}$ such that the point 2 holds.
4.5.2.3. Proof of the point 3. Let $A$ be a closed subset of $\mathcal{Z}$. Let $\left(z_{k}\right)_{k \in \mathbb{N}}$ be a sequence of elements in $\mathcal{Z},\left(\zeta_{k}\right)_{k \in \mathbb{N}}$ be a sequence of elements in $L^{\infty}(Q)^{4}$, and $z \in \mathcal{Z}$ be such that

$$
\begin{gathered}
z_{k} \underset{k \rightarrow+\infty}{\rightarrow} z \text { in } L^{\infty}(Q)^{4}, \\
\forall k \in \mathbb{N}, \zeta_{k} \in \mathcal{A}, \\
\forall k \in \mathbb{N}, \zeta_{k} \in B\left(z_{k}\right) .
\end{gathered}
$$

Let $\left(h_{k}^{j}\right)_{k \in \mathbb{N}}$ the sequence of controls associated to $\left(\zeta_{k}\right)_{k \in \mathbb{N}}$. As $\zeta_{k} \in B\left(z_{k}\right)$, we have

$$
\forall k \in \mathbb{N},\left\|h_{k}^{j}\right\|_{L^{\infty}(Q)^{j}} \leq C_{0}\left\|\zeta_{0}\right\|_{L^{2}(\Omega)^{4}}
$$

By the point 2 , we get that there exists a strictly increasing sequence $\left(k_{l}\right)_{l \in \mathbb{N}}$ of integers such that $\zeta_{k_{l}} \rightarrow \zeta$ in $L^{\infty}(Q)^{4}$ as $l \rightarrow+\infty$. As $\mathcal{A}$ is closed, we have $\zeta \in \mathcal{A}$, then it suffices to show that $\zeta \in B(z)$. The same arguments as in the point 1 give the result. This ends the proof of the point 3 .

## This concludes the proof of Theorem 3.2.

## 5. Proof of Theorem 3.6: the global controllability to constant stationary states

Proof. Let $N \in\{1,2\}, j=3$ (we only prove the result for this case, the other cases are similar), $u_{0} \in L^{\infty}(\Omega)^{4}$ satisfying the hypothesis (42), $\left(u_{i}^{*}\right)_{1 \leq i \leq 4} \in\left(\mathbb{R}^{+}\right)^{4}$ satisfying (3).

From [46, Theorem 3] and [45, Theorem 3] (see also [21]), we deduce that the solution $u \in L^{\infty}((0, \infty) \times \Omega)^{4}$ of

$$
\forall 1 \leq i \leq 4, \begin{cases}\partial_{t} u_{i}-d_{i} \Delta u_{i}=(-1)^{i}\left(u_{1} u_{3}-u_{2} u_{4}\right) & \text { in }(0, \infty) \times \Omega,  \tag{247}\\ \frac{\partial u_{i}}{\partial n}=0 & \text { on }(0, \infty) \times \partial \Omega, \\ u_{i}(0, .)=u_{i, 0} & \text { in } \Omega,\end{cases}
$$

satisfies

$$
\begin{equation*}
\lim _{T \rightarrow+\infty}\|u(T, .)-z\|_{L^{\infty}(\Omega)^{4}}=0, \tag{248}
\end{equation*}
$$

where $z \in\left(\mathbb{R}^{+, *}\right)^{4}$ is the unique nonnegative solution of

$$
\begin{align*}
& z_{1} z_{3}=z_{2} z_{4}  \tag{249}\\
& z_{1}+z_{2}=\left(u_{1,0}\right)_{\Omega}+\left(u_{2,0}\right)_{\Omega}, z_{1}+z_{4}=\left(u_{1,0}\right)_{\Omega}+\left(u_{4,0}\right)_{\Omega}  \tag{250}\\
& z_{3}+z_{2}=\left(u_{3,0}\right)_{\Omega}+\left(u_{2,0}\right)_{\Omega}, z_{3}+z_{4}=\left(u_{3,0}\right)_{\Omega}+\left(u_{4,0}\right)_{\Omega} . \tag{251}
\end{align*}
$$

Case 1: $u_{3}^{*} \neq 0$. Let us define a path $\gamma$ between $z$ and $\left(u_{i}^{*}\right)_{1 \leq i \leq 4}$,

$$
\begin{align*}
\gamma: \mid 0,1] & \longrightarrow\left\{\left(v_{1}, v_{2}, v_{3}, v_{4}\right) \in \mathbb{R}^{+} \times \mathbb{R}^{+} \times \mathbb{R}^{+, *} \times \mathbb{R}^{+} ; v_{1} v_{3}=v_{2} v_{4}\right\} \\
\theta & \longmapsto\left(\frac{\left((1-\theta) z_{2}+\theta u_{2}^{*}\right)\left((1-\theta) z_{4}+\theta u_{4}^{*}\right)}{(1-\theta) z_{3}+\theta u_{3}^{*}},(1-\theta) z_{2}+\theta u_{2}^{*},(1-\theta) z_{3}+\theta u_{3}^{*},(1-\theta) z_{4}+\theta u_{4}^{*}\right) . \tag{252}
\end{align*}
$$

Let us define $\Phi$ in the following way,

$$
\Phi: \left\lvert\, \begin{align*}
& \Gamma:=\{\gamma(\theta), \theta \in[0,1]\} \longrightarrow  \tag{253}\\
& \mathbb{R}^{+, *} \\
&\left(v_{i}\right) \longmapsto
\end{align*} r_{v}\right.,
$$

where $r_{v}>0$ is the radius of the ball of $L^{\infty}(\Omega)^{4}$ centered in $\left(v_{i}\right)_{1 \leq i \leq 4}$ in which we have proved controllability to $\left(v_{i}\right)_{1 \leq i \leq 4}$ (see Theorem 3.2). Precisely, $r_{v}$ is given by (241). It is straightforward but tedious to see that

$$
\begin{equation*}
r:=\inf \Phi>0 \tag{254}
\end{equation*}
$$

because there exists $\varepsilon>0$ such that for every $\theta \in[0,1], v_{3}=(1-\theta) z_{3}+\theta u_{3}^{*} \geq \varepsilon$. For more details, one can follow the dependence of the constant $r_{v}=v / C_{1}$ in function of the parameters
$\left(v_{i}\right)_{1 \leq i \leq 4}$ (see (241), (226), (227), Proposition 4.8 for the definition of the constant $C_{0}$, (52), (54), (55) and Section 4.3.3 for the dependence of this constant $C_{0}$ in term of $\left.\left(v_{i}\right)_{1 \leq i \leq 4}\right)$.

By (248), there exists $T_{1}>0$ such that $\left\|u\left(T_{1}, .\right)-z\right\|_{L^{\infty}(\Omega)^{4}}<r$, where $u$ is the solution of (247). By (253) and (254), there exists $h^{3,1} \in L^{\infty}\left(\left(T_{1}, T_{1}+T\right) \times \Omega\right)^{3}$ such that the solution $u^{1}$ of (4), with $(0, T)=\left(T_{1}, T_{1}+T\right)$ and $u^{1}\left(T_{1},.\right)=u\left(T_{1},.\right)$, satisfies $u^{1}\left(T_{1}+T,.\right)=z$.

The mapping $\gamma$ is continuous on the compact set $[0,1]$, so $\gamma$ is uniformly continuous on $[0,1]$ by Heine's theorem. Consequently, there exists $\eta>0$ such that for every $\theta_{1}, \theta_{2} \in[0,1]$, verifying $\left|\theta_{1}-\theta_{2}\right| \leq \eta,\left\|\gamma\left(\theta_{1}\right)-\gamma\left(\theta_{2}\right)\right\|_{\infty}<r$. Moreover, there exists $m \in \mathbb{N}^{*}$ sufficiently large such that $m \eta \leq 1<(m+1) \eta$. Therefore, let us define $\theta_{k}=k \eta$ for $k \in\{0, \ldots, m\}$ and $\theta_{m+1}=1$. Then, we have

$$
\begin{equation*}
\Gamma \subset \bigcup_{i=0}^{m+1} B\left(\gamma\left(\theta_{i}\right), r\right) \tag{255}
\end{equation*}
$$

We remark that we have $\gamma\left(\theta_{0}\right)=z, \gamma\left(\theta_{m+1}\right)=u^{*}$ and $\left\|\gamma\left(\theta_{i}\right)-\gamma\left(\theta_{i+1}\right)\right\|_{\infty}<r$ for every $i \in$ $\{1, \ldots, m\}$ by definition of $\eta$.

We have $\left\|z-\gamma\left(\theta_{1}\right)\right\|_{\infty}=\left\|\gamma\left(\theta_{0}\right)-\gamma\left(\theta_{1}\right)\right\|_{\infty}<r$. Then, by (253) and (254), there exists $h^{3,2} \in$ $L^{\infty}\left(\left(T_{1}+T, T_{1}+2 T\right) \times \Omega\right)^{3}$ such that the solution $u^{2}$ of (4), with $(0, T)=\left(T_{1}+T, T_{1}+2 T\right)$ and $u^{1}\left(T_{1}+T,.\right)=z$, satisfies $u^{1}\left(T_{1}+2 T,.\right)=\gamma\left(\theta_{1}\right)$.

By repeating $m$ times this strategy, we get the existence of a control $h^{3} \in L^{\infty}\left(\left(0, T_{1}+(m+\right.\right.$ 2) $T) \times \Omega$ ) so that $h^{3}(t,)=$.0 for $t \in\left(0, T_{1}\right), h^{3}(t,)=.h^{3,1}(t,$.$) for t \in\left(T_{1}, T_{1}+T\right), \ldots$, $h^{3}(t,)=.h^{3, m+2}(t,$.$) for t \in\left(T_{1}+(m+1) T, T_{1}+(m+2) T\right)$, such the solution $u$ of

$$
\forall 1 \leq i \leq 4, \begin{cases}\partial_{t} u_{i}-d_{i} \Delta u_{i}=(-1)^{i}\left(u_{1} u_{3}-u_{2} u_{4}\right)+h_{i}^{3} 1_{\omega} & \text { in }\left(0, T_{1}+(m+2) T\right) \times \Omega  \tag{256}\\ \frac{\partial u_{i}}{\partial n}=0 & \text { on }\left(0, T_{1}+(m+2) T\right) \times \partial \Omega \\ u_{i}(0, .)=u_{i, 0} & \text { in } \Omega\end{cases}
$$

satisfies $u\left(T_{1}+(m+2) T,.\right)=u^{*}$.
Case 2: $u_{3}^{*}=0$. From (3), we have $u_{2}^{*}=0$ or $u_{4}^{*}=0$. We can assume that $u_{2}^{*}=0$. The other case is similar. By Theorem 3.2, we know that there exists $\widehat{r}>0$ such that for every $\widetilde{u}^{*} \in B\left(u^{*}, \widetilde{r}_{L^{\infty}(\Omega)^{4}}\right.$, we can find a control $h^{3} \in L^{\infty}((0, T) \times \Omega)^{3}$ that enables to go from $\widetilde{u}^{*}$ to $u^{*}$. Consequently, we choose $\beta$ such that $0<\beta<\widehat{r} / 2$ and $\frac{\beta\left(u_{4}^{*}+\widehat{r} / 2\right)}{u_{1}^{*}+\widetilde{r} / 2}<\widehat{r} / 2$ and we set $\widetilde{u}^{*}:=$ $\left(u_{1}^{*}+\widehat{r} / 2, \beta, \frac{\beta\left(u_{4}^{*}+\overparen{r} / 2\right)}{u_{1}^{*}+\widetilde{r} / 2}, u_{4}^{*}+\widehat{r} / 2\right) \in B\left(u^{*}, \widehat{r}\right)$. We remark that $\widetilde{u}^{*}$ satisfies ( 3 ) and ${\widetilde{u_{3}}}^{*} \neq 0$. Then, from the first case of the proof, we can find a control which drives $z$ to $\widetilde{u}^{*}$. Next, we can find a control which drives $\widetilde{u}^{*}$ to $u^{*}$.

## 6. Comments, perspectives and open problems

6.1. $\omega_{i}$ instead of $\omega$

An interesting open problem could be the generalization of Theorem 3.2 to the system

$$
\forall 1 \leq i \leq 4, \begin{cases}\partial_{t} u_{i}-d_{i} \Delta u_{i}=(-1)^{i}\left(u_{1} u_{3}-u_{2} u_{4}\right)+h_{i} 1_{\omega_{i}} 1_{i \leq j} & \text { in }(0, T) \times \Omega,  \tag{257}\\ \frac{\partial u_{i}}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega, \\ u_{i}(0, .)=u_{i, 0} & \text { in } \Omega,\end{cases}
$$

where for every $i \in\{1, \ldots, j\}, \omega_{i}$ are nonempty open subsets such that $\omega_{i} \subset \Omega$ and $\bigcap_{i=1}^{j} \omega_{i}=\emptyset$ (otherwise, the generalization is straightforward).

### 6.2. Stationary solutions

We only have considered nonnegative stationary constant solutions of (1). It is not restrictive because of the following proposition.

Proposition 6.1. Let $\left(u_{i}\right)_{1 \leq i \leq 4} \in C^{2}(\bar{\Omega})^{4}$ be a nonnegative solution of

$$
\forall 1 \leq i \leq 4, \begin{cases}-d_{i} \Delta u_{i}=(-1)^{i}\left(u_{1} u_{3}-u_{2} u_{4}\right) & \text { in } \Omega  \tag{258}\\ \frac{\partial u_{i}}{\partial n}=0 & \text { on } \partial \Omega\end{cases}
$$

Then, for every $1 \leq i \leq 4, u_{i}$ is constant.
Proof. Let $\varepsilon>0$. For every $i \in\{1, \ldots, 4\}$, let us denote $u_{i}^{\varepsilon}=u_{i}+\varepsilon$ and $w_{i}^{\varepsilon}=u_{i}^{\varepsilon}\left(\log u_{i}^{\varepsilon}-1\right)+1$. Note that $w_{i}^{\varepsilon} \geq 0$ for every $i \in\{1, \ldots, 4\}$. We have

$$
\begin{equation*}
\forall 1 \leq i \leq 4, \nabla w_{i}^{\varepsilon}=\log \left(u_{i}^{\varepsilon}\right) \nabla u_{i}^{\varepsilon}, \Delta w_{i}^{\varepsilon}=\log \left(u_{i}^{\varepsilon}\right) \Delta u_{i}^{\varepsilon}+\frac{\left|\nabla u_{i}^{\varepsilon}\right|^{2}}{u_{i}^{\varepsilon}} . \tag{259}
\end{equation*}
$$

Then, from (258) and (259), we have that for every $1 \leq i \leq 4$,

$$
\begin{cases}-d_{i} \Delta w_{i}^{\varepsilon}+d_{i} \frac{\left|\nabla u_{i}^{\varepsilon}\right|^{2}}{u_{i}^{\varepsilon}}=(-1)^{i} \log \left(u_{i}^{\varepsilon}\right)\left(u_{1}^{\varepsilon} u_{3}^{\varepsilon}-u_{2}^{\varepsilon} u_{4}^{\varepsilon}-\varepsilon\left(u_{1}+u_{3}-u_{2}-u_{4}\right)\right) & \text { in } \Omega,  \tag{260}\\ \frac{\partial w_{i}^{\varepsilon}}{\partial n}=0 & \text { on } \partial \Omega .\end{cases}
$$

We add the four equations of (260) and we integrate on $\Omega$. We get

$$
\begin{align*}
0 & +\int_{\Omega} \sum_{i=1}^{4} d_{i} \frac{\left|\nabla u_{i}^{\varepsilon}\right|^{2}}{u_{i}^{\varepsilon}} \\
= & -\left(\int_{\Omega}\left(\log \left(u_{1}^{\varepsilon} u_{3}^{\varepsilon}\right)-\log \left(u_{2}^{\varepsilon} u_{4}^{\varepsilon}\right)\right)\left(u_{1}^{\varepsilon} u_{3}^{\varepsilon}-u_{2}^{\varepsilon} u_{4}^{\varepsilon}\right)\right) \\
& +\varepsilon\left(\int_{\Omega}\left(\log \left(u_{1}^{\varepsilon} u_{3}^{\varepsilon}\right)-\log \left(u_{2}^{\varepsilon} u_{4}^{\varepsilon}\right)\right)\left(u_{1}+u_{3}-u_{2}-u_{4}\right)\right) \\
\leq & \varepsilon\left(\int_{\Omega}\left(\log \left(u_{1}^{\varepsilon} u_{3}^{\varepsilon}\right)-\log \left(u_{2}^{\varepsilon} u_{4}^{\varepsilon}\right)\right)\left(u_{1}+u_{3}-u_{2}-u_{4}\right)\right) . \tag{261}
\end{align*}
$$

Moreover,

$$
\begin{equation*}
\forall 1 \leq i \leq 4, \quad \int_{\Omega} d_{i} \frac{\left|\nabla u_{i}^{\varepsilon}\right|^{2}}{u_{i}^{\varepsilon}}=\int_{\Omega} 4 d_{i}\left|\nabla \sqrt{u_{i}^{\varepsilon}}\right|^{2} . \tag{262}
\end{equation*}
$$

Consequently, from (261), (262) and by taking $\varepsilon$ sufficiently small, for every $1 \leq i \leq 4$,

$$
\begin{aligned}
\int_{\Omega} 4 d_{i}\left|\nabla \sqrt{u_{i}^{\varepsilon}}\right|^{2} & \leq \varepsilon\left(\int_{\Omega}\left(\log \left(u_{1}^{\varepsilon} u_{3}^{\varepsilon}\right)-\log \left(u_{2}^{\varepsilon} u_{4}^{\varepsilon}\right)\right)\left(u_{1}+u_{3}-u_{2}-u_{4}\right)\right) \\
& \leq \varepsilon\left(\int_{\Omega}\left|\log \left(\varepsilon^{4}\right)\right|\left|u_{1}+u_{3}-u_{2}-u_{4}\right|\right) .
\end{aligned}
$$

Then, by letting $\varepsilon \rightarrow 0$, we get that

$$
\forall 1 \leq i \leq 4, \quad \int_{\Omega} 4 d_{i}\left|\nabla \sqrt{u_{i}}\right|^{2}=0 .
$$

Consequently, for every $1 \leq i \leq 4, u_{i}$ is constant.
We can also remark that there exist non constant solutions of (258). For example, in the case of $\left(d_{1}, d_{2}, d_{3}, d_{4}\right)=(1,1,1,1),\left(u_{1}^{*}, u_{2}^{*}, u_{3}^{*}, u_{4}^{*}\right)=\left(\varphi_{\lambda},-\varphi_{\lambda}, \varphi_{\lambda}-\lambda,-\varphi_{\lambda}\right)$, where $\lambda>0$ and $\varphi_{\lambda}$ are respectively an eigenvalue and a corresponding eigenfunction of the unbounded operator $\left(-\Delta, H_{N e}^{2}(\Omega)\right)$ (see Definition 4.13), is a solution of (258). The result of Theorem 3.2 is still valid for non constant stationary solutions under a natural condition of sign of $\left(u_{1}^{*}, u_{2}^{*}, u_{3}^{*}, u_{4}^{*}\right)$ on a nonempty open subset $\omega_{0} \subset \omega$ (see (54), (64), (79) after linearization). There is only one nontrivial thing to verify. For the proof of the observability inequalities (95) and (96), the application of $\Delta$ to some equations does not create "bad" terms. A good meaning to be convinced is to look at the inequality (138) which becomes

$$
\begin{equation*}
I\left(0, \lambda, s, \Delta \varphi_{4}\right) \leq C\left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}\left\{\sum_{i=1}^{3}\left|\Delta \varphi_{i}\right|^{2}+\left|\nabla \varphi_{i}\right|^{2}\left|+\left|\varphi_{i}\right|^{2}\right\}+\int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}(s \phi)^{3}\left|\Delta \varphi_{4}\right|^{2}\right)\right. \tag{263}
\end{equation*}
$$

It is clear that $\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}\left(\sum_{i=1}^{3}\left|\Delta \varphi_{i}\right|^{2}+\left|\nabla \varphi_{i}\right|^{2}\left|+\left|\varphi_{i}\right|^{2}\right)\right.$ can be absorbed by the left hand side of (140) by taking $s$ sufficiently large.

### 6.3. Nonnegative solutions and nonnegative controls

In the spirit of the works [43] and [47] and in order to make the model more realistic, an interesting open problem could be: for nonnegative initial conditions $\left(u_{i, 0}\right)_{1 \leq i \leq 4}$, and nonnegative stationary state $\left(u_{i}^{*}\right)_{1 \leq i \leq 4}$, does there exit a control $\left(h_{i}\right)_{1 \leq i \leq j}$ such that the solution $\left(u_{i}\right)_{1 \leq i \leq 4}$ of (4) remains nonnegative and satisfies (5)?

### 6.4. Constraints on the initial condition for the controllability of the linearized system

The goal of this section is to show that the linear transformation we do before linearization (see (59) and (73)), seems to be essential. Indeed, this adequate change of variable leads to control all possible initial conditions (see the necessary conditions on the initial conditions due to invariant quantities of the nonlinear dynamics: Section 2.2). One could think about [3, Theorem 5.3] which gives sufficient conditions of controllability when the rank condition of Theorem 1.6 is not verified. But it reduces the space of initial condition once more and it becomes "artificial" in our case.

The linearized-system of (4) around $\left(u_{i}^{*}\right)_{1 \leq i \leq 4}$ is

$$
\begin{cases}\partial_{t} u-D \Delta u=A u+B_{j} h^{j} 1_{\omega} & \text { in }(0, T) \times \Omega,  \tag{264}\\ \frac{\partial u}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega \\ u(0, .)=u_{0} & \text { in } \Omega,\end{cases}
$$

where

$$
u=\left(u_{1}, u_{2}, u_{3}, u_{4}\right)^{T}, D=\operatorname{diag}\left(d_{1}, d_{2}, d_{3}, d_{4}\right), A=\left(\begin{array}{cccc}
-u_{3}^{*} & u_{4}^{*} & -u_{1}^{*} & u_{2}^{*}  \tag{265}\\
u_{3}^{*} & -u_{4}^{*} & u_{1}^{*} & -u_{2}^{*} \\
-u_{3}^{*} & u_{4}^{*} & -u_{1}^{*} & u_{2}^{*} \\
u_{3}^{*} & -u_{4}^{*} & u_{1}^{*} & -u_{2}^{*}
\end{array}\right) \text {, }
$$

and $B_{j}, h^{j}$ are defined in (45).
Definition 6.2. The system (264) is $\left(u_{i}^{*}\right)_{1 \leq i \leq 4}$-controllable if for every $u_{0} \in L^{2}(\Omega)^{4}$, there exists $h^{j} \in L^{2}(Q)^{j}$ such that the solution $u$ of (264) satisfies $u(T,)=.u^{*}$.

We would also use [3, Theorem 1] in order to deduce the necessary and sufficient condition of controllability to $\left(u_{i}^{*}\right)_{1 \leq i \leq 4}$ for (264). First, let us denote by $\left(\lambda_{k}\right)_{k \in \mathbb{N}}$ the increasing sequence of the eigenvalues of the unbounded operator $\left(-\Delta, H_{N e}^{2}(\Omega)\right)$ (see Definition 4.13 for the definition of $H_{N e}^{2}(\Omega)$ ). In particular, $\lambda_{0}=0$.
Theorem 6.3. The system (264) is $\left(u_{i}^{*}\right)_{1 \leq i \leq 4}$-controllable if and only if

$$
\begin{equation*}
\forall k \in \mathbb{N}, \operatorname{rank}\left(-\lambda_{k} D+A \mid B_{j}\right)=4, \tag{266}
\end{equation*}
$$

where

$$
\left(\left(-\lambda_{k} D+A\right) \mid B_{j}\right):=\left(B_{j},\left(-\lambda_{k} D+A\right) B_{j},\left(-\lambda_{k} D+A\right)^{2} B_{j},\left(-\lambda_{k} D+A\right)^{3} B_{j}\right)
$$

For $j=3$, we can check that for every $k \in \mathbb{N}, \operatorname{rank}\left(-\lambda_{k} D+A \mid B_{3}\right)=4$ if and only if $\left(u_{1}^{*}, u_{3}^{*}, u_{4}^{*}\right) \neq(0,0,0)$. It is consistent with Section 4.1.1.1.

For $j=2$ and $d_{3} \neq d_{4}$, we can check that $\operatorname{rank}\left(\lambda_{0}+A \mid B_{2}\right)<4$, then $(264)$ is not $\left(u_{i}^{*}\right)_{1 \leq i \leq 4^{-}}$ controllable. It is consistent with the hypothesis we have to make for the initial condition i.e. (23). But, we can deduce from [3, Theorem 5.3] that (264) is $\left(u_{i}^{*}\right)_{1 \leq i \leq 4}$-controllable for initial conditions verifying

$$
\begin{equation*}
\forall i \in\{1, \ldots, 4\}, \frac{1}{|\Omega|} \int_{\Omega} u_{i, 0}(x)=u_{i}^{*} \tag{267}
\end{equation*}
$$

The condition (267) is a more restrictive hypothesis than (23). It is only a sufficient condition. Actually, we have found a necessary and sufficient condition on the initial data for $\left(u_{i}^{*}\right)_{1 \leq i \leq 4^{-}}$ controllability.

Proposition 6.4. Let $j=2, d_{3} \neq d_{4}$.
For every $u_{0} \in L^{2}(\Omega)^{4}$ such that $\frac{1}{|\Omega|} \int_{\Omega}\left(u_{3,0}+u_{4,0}\right)=u_{3}^{*}+u_{4}^{*}$, there exists $h^{2} \in L^{2}(Q)^{2}$ such that the solution $u$ of (264) satisfies $u(T,)=.u^{*}$.
If $u_{0} \in L^{2}(\Omega)^{4}$ does not satisfy $\frac{1}{|\Omega|} \int_{\Omega}\left(u_{3,0}+u_{4,0}\right)=u_{3}^{*}+u_{4}^{*}$, for every $h^{2} \in L^{2}(Q)^{2}$, the solution $u$ of (264) does not satisfy $u(T,)=.u^{*}$.

Proof. The necessary condition of controllability is a consequence of

$$
\text { a.e. } t \in[0, T], \frac{d}{d t}\left(\int_{\Omega}\left(u_{3}(t, x)+u_{4}(t, x)\right) d x\right)=0 \text {. }
$$

The sufficient condition of controllability is a consequence of the adequate change of variable $\left(v_{1}, v_{2}, v_{3}, v_{4}\right):=\left(u_{1}, u_{2}, u_{3}, u_{3}+u_{4}\right)$ and the proof of the observability inequality (95).

Remark 6.5. We chose to state our previous result in the particular case $j=2$ and $d_{3} \neq d_{4}$ for simplicity but one can generalize this proposition to other cases.

An interesting open problem could consist in trying to find precisely the initial conditions that can be controlled for systems of the form (264) when (266) is not satisfied. This will lead to a better understanding of the controllability properties of a large class of nonlinear reactiondiffusion systems.

### 6.5. More general nonlinear reaction-diffusion systems

Let $k \in \mathbb{N}^{*},\left(\alpha_{1}, \ldots, \alpha_{k}\right) \in(\mathbb{N})^{n},\left(\beta_{1}, \ldots, \beta_{k}\right) \in(\mathbb{N})^{k}$ such that for every $1 \leq i \leq k, \alpha_{i} \neq \beta_{i}$, $\left(d_{1}, \ldots, d_{k}\right) \in(0,+\infty)^{k}$ and $J \subset\{1, \ldots, k\}$. We consider the following nonlinear controlled reaction-diffusion system:

$$
\forall 1 \leq i \leq k, \begin{cases}\partial_{t} u_{i}-d_{i} \Delta u_{i}= &  \tag{268}\\ \quad\left(\beta_{i}-\alpha_{i}\right)\left(\prod_{k=1}^{n} u_{k}^{\alpha_{k}}-\prod_{k=1}^{n} u_{k}^{\beta_{k}}\right)+h_{i} 1_{\omega} 1_{i \in J} & \text { in }(0, T) \times \Omega, \\ \frac{\partial u_{i}}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega, \\ u_{i}(0, .)=u_{i, 0} & \text { in } \Omega .\end{cases}
$$

The article [36] by the author treats the local-controllability of (268) around nonnegative (constant) stationary states by using the same kind of change of variables as in (59) and (73). Nevertheless, the proof of observability inequalities for the linearized system cannot follow the same strategy as performed in Section 4.3.7. Indeed, if we apply Carleman estimates to each equation of the adjoint system, it leads to some global terms in the right hand side of the inequality that cannot be absorbed by the left hand side. Thus, as in [26, Hypothesis 3], a similar technical obstruction appears. Inspired by the recent work of Pierre Lissy and Enrique Zuazua (see [41, Section 3]), who obtained sharp results for the null-controllability of non-diagonalizable systems of parabolic equations, the author proves the null-controllability of the linearized system. Then, the source term method introduced by Yuning Liu, Takéo Takahashi, Marius Tucsnak (see [42]) enables to go back to the nonlinear reaction-diffusion system.

## Appendix A. Appendix

## Appendix A.1. $L^{\infty}$-estimate for parabolic systems

We give the proof of Proposition 2.3.
Proof. By using the fact that $D$ is diagonalizable and $S p(D) \subset(0,+\infty)$, we only have to prove the result when $D=\operatorname{diag}\left(d_{1}, \ldots, d_{k}\right)$ with $d_{i} \in(0,+\infty)$.

The first point of the proof i.e. the existence and the uniqueness of the weak solution $u \in Y^{k}$ is based on Galerkin approximations and energy estimates. One can easily adapt the arguments given in [25, Section 7.1.2] to the Neumann cases.

The second point of the proof i.e. the $L^{\infty}$ estimate is based on Stampacchia's method. We introduce

$$
\begin{equation*}
l(t)=\left(l_{1}(t), \ldots, l_{k}(t)\right)^{T}:=l_{0} \exp (t M)(1, \ldots, 1)^{T}=: L(t)(1, \ldots, 1)^{T} \in \mathbb{R}^{k} \tag{A.1}
\end{equation*}
$$

for every $t \in[0, T]$ and $l_{0}, M \in(0,+\infty)$ which will be chosen later. By (15), we have

$$
\begin{align*}
& \forall w \in L^{2}\left(0, T ; H^{1}(\Omega)^{k}\right), \\
& \int_{0}^{T}\left(\partial_{t} u, w\right)_{\left.\left(H^{1}(\Omega)^{k}\right)^{\prime}, H^{1}(\Omega)^{k}\right)}-\int_{Q}\left(\operatorname{sign}(u) l^{\prime}\right) \cdot w+\int_{Q} D \nabla u \cdot \nabla w=\int_{Q}(A u+g) \cdot w-\int_{Q}\left(\operatorname{sign}(u) l^{\prime}\right) \cdot w, \tag{A.2}
\end{align*}
$$

where $\operatorname{sign}(u) l^{\prime}=\left(\operatorname{sign}\left(u_{1}\right) l_{1}^{\prime}, \ldots, \operatorname{sign}\left(u_{k}\right) l_{k}^{\prime}\right)^{T}$. We fix $t \in[0, T]$ and we apply (A.2) with $w$ defined by

$$
\begin{aligned}
\forall(\tau, x) \in[0, T] \times \Omega, w(\tau, x) & :=\operatorname{sign}(u)(|u|(t, x)-l(t))^{+} 1_{[0, t]}(\tau) \\
& :=\left(\operatorname{sign}\left(u_{1}\right)\left(\left|u_{1}\right|(t, x)-l_{1}(t)\right)^{+}, \ldots,\left(\operatorname{sign}\left(u_{k}\right)\left(\left|u_{k}\right|(t, x)-l(t)\right)^{+}\right)^{T} 1_{[0, t]}(\tau) .\right.
\end{aligned}
$$

We get

$$
\begin{align*}
& \int_{0}^{t} \frac{1}{2} \frac{d}{d \tau} \int_{\Omega} \sum_{i=0}^{k}\left(\left(\left|u_{i}\right|(\tau, x)-l_{i}(\tau)\right)^{+}\right)^{2} d x d \tau+\int_{0}^{t} \int_{\Omega} \sum_{i=0}^{k} d_{i} \nabla u_{i} \cdot \nabla u_{i} 1_{\left|u_{i}\right| \geq l_{i}} \\
& =\int_{0}^{t} \int_{\Omega} \sum_{i=0}^{k}\left(\sum_{j=0}^{k} a_{i j} u_{j}+g_{i}-\operatorname{sign}\left(u_{i}\right) l_{i}^{\prime}\right) \operatorname{sign}\left(u_{i}\right)\left(\left|u_{i}\right|-l_{i}\right)^{+} . \tag{A.3}
\end{align*}
$$

We remark that

$$
-\operatorname{sign}\left(u_{i}\right) l_{i}^{\prime} \operatorname{sign}\left(u_{i}\right)\left(\left|u_{i}\right|-l_{i}\right)^{+}=-l_{i}^{\prime}\left(\left|u_{i}\right|-l_{i}\right)^{+} .
$$

Moreover, we have

$$
\begin{align*}
\left(\sum_{j=0}^{k} a_{i j} u_{j}+g_{i}-\operatorname{sign}\left(u_{i}\right) l_{i}^{\prime}\right) \operatorname{sign}\left(u_{i}\right)\left(\left|u_{i}\right|-l_{i}\right)^{+} & \leq\left(\sum_{j=0}^{k}\left|a_{i j}\right|\left|u_{j}\right|+\left|g_{i}\right|-l_{i}^{\prime}\right)\left(\left|u_{i}\right|-l_{i}\right)^{+} \\
& \leq\left(\sum_{j=0}^{k}\left|a_{i j}\right|\left(\left|u_{j}\right|-l_{j}\right)^{+}+A_{i}\right)\left(\left|u_{i}\right|-l_{i}\right)^{+}, \tag{A.4}
\end{align*}
$$

where $A_{i}:=\sum_{j=0}^{k} l_{j}\left|a_{i j}\right|+g_{i}-l_{i}^{\prime}=L \sum_{j=0}^{k}\left|a_{i j}\right|+g_{i}-M L$ (see (A.1)). We choose $l_{0}, M \in(0,+\infty)$ such that

$$
\begin{equation*}
M \geq \max _{i}\left\{\left\|\sum_{j=0}^{k}\left|a_{i j}\right|\right\|_{\infty}+1\right\}, l_{0}=\max _{i}\left\{\left\|u_{0 i}\right\|_{\infty}+\left\|g_{i}\right\|_{\infty}\right\} . \tag{A.5}
\end{equation*}
$$

Then, we find

$$
\begin{equation*}
A_{i} \leq L(M-1)+l_{0}-M L \leq L(M-1)+L-M L \leq 0 . \tag{A.6}
\end{equation*}
$$

By using $l_{0} \geq \max _{i}\left\|u_{0 i}\right\|_{\infty}, \int_{0}^{t} \int_{\Omega} \sum_{i=0}^{k} d_{i} \nabla u_{i} . \nabla u_{i} 1_{\left|u_{i}\right| \geq l_{i}} \geq 0$, (A.4), (A.6), together with (A.3), we have that for every $t \in[0, T]$,

$$
\begin{equation*}
\int_{\Omega} \sum_{i=0}^{k}\left(\left(\left|u_{i}\right|(t, x)-l_{i}(t)\right)^{+}\right)^{2} d x \leq 2 \int_{0}^{t} \int_{\Omega} \sum_{i=0}^{k} \sum_{j=0}^{k}\left|a_{i j}\right|\left(\left|u_{j}\right|-l_{j}\right)^{+}\left(\left|u_{i}\right|-l_{i}\right)^{+} d x d \tau . \tag{A.7}
\end{equation*}
$$

Cauchy-Schwartz inequality applied to the right hand side term of (A.7) gives

$$
\begin{equation*}
\forall t \in[0, T], \int_{\Omega} \sum_{i=0}^{k}\left(\left(\left|u_{i}\right|(t, x)-l_{i}(t)\right)^{+}\right)^{2} d x \leq C \int_{0}^{t} \int_{\Omega} \sum_{i=0}^{k}\left(\left(\left|u_{i}\right|(\tau, x)-l_{i}(\tau)\right)^{+}\right)^{2} d x d \tau \tag{A.8}
\end{equation*}
$$

where $C:=2 k \max _{i, j}\left\|a_{i j}\right\|_{\infty}$. Gronwall's lemma applied to (A.8) gives

$$
\begin{equation*}
\forall i \in\{1, \ldots, k\}, \forall t \in[0, T],\left|u_{i}(t)\right| \leq l_{i}(t)=l_{0} \exp (t M) \tag{A.9}
\end{equation*}
$$

Therefore, from (A.9), we deduce (18) with our choice of $l_{0}$ (see (A.5)).

## Appendix A.2. Dissipation of the energy for crossed-diffusion parabolic systems

The goal of this section is to give a sketch of the proof of the dissipation of the energy (in time) for some parabolic systems.

Lemma A.1. Let $j \in\{1,2,3\}, D_{j}$ defined by (52), (62), (77), $A \in \mathcal{E}_{j}$ (see (83), (85) and (87)), $\varphi_{T} \in L^{2}(\Omega)^{4}$ and $\varphi$ be the solution of the following Cauchy problem

$$
\begin{cases}-\varphi_{t}-D_{j}^{T} \Delta \varphi=A^{T} \varphi & \text { in }(0, T) \times \Omega \\ \frac{\partial \varphi}{\partial n}=0 & \text { on }(0, T) \times \partial \Omega \\ \varphi(T, .)=\varphi_{T} & \text { in } \Omega\end{cases}
$$

Then, there exists $C>0$ such that for every $\left(t_{1}, t_{2}\right) \in[0, T]^{2}, t_{1}<t_{2}$,

$$
\begin{align*}
& \sum_{i=1}^{j+1}\left\|\varphi_{i}\left(t_{1}, .\right)\right\|_{L^{2}(\Omega)}^{2}+\sum_{i=j+2}^{4}\left\|\varphi_{i}\left(t_{1}, .\right)-\left(\varphi_{i}\right)_{\Omega}\left(t_{1}\right)\right\|_{L^{2}(\Omega)}^{2} \\
& \left.\leq C\left(\sum_{i=1}^{j+1}\left\|\varphi_{i}\left(t_{2}, .\right)\right\|_{L^{2}(\Omega)}^{2}+\sum_{i=j+2}^{4} \| \varphi_{i}\left(t_{2}, .\right)-\left(\varphi_{i}\right)\right)_{\Omega}\left(t_{2}\right) \|_{L^{2}(\Omega)}^{2}\right) \tag{A.10}
\end{align*}
$$

Proof. By using the fact that $D_{j}$ is diagonalizable, we only have to prove the result when $D$ is diagonal. First, we introduce $\psi=\left(\varphi_{1}, \ldots, \varphi_{j+1}, \varphi_{j+2}-\left(\varphi_{j+2}\right)_{\Omega}(),. \ldots, \varphi_{4}-\left(\varphi_{4}\right)_{\Omega}().\right)$. We look for the parabolic system satisfied by $\psi$. Then, we multiply the variational formulation (see (15)) by $w(t, x)=\psi(t, x) 1_{\left[t_{1}, t_{2}\right]}(t)$. By Young inequalities, we find a differential inequality as follows

$$
\text { a.e. } t \in\left[t_{1}, t_{2}\right], \frac{d}{d t}\|\psi(t)\|_{L^{2}(\Omega)}^{2} \leq C\|\psi(t)\|_{L^{2}(\Omega)}^{2}
$$

Then, we use Gronwall's lemma to deduce (A.10).
Appendix A.3. Technical estimates for the observability inequality in the case of 1 control
The goal of this section is to prove Lemma 4.31 and Lemma 4.32. We use the same notations as in Section 4.3.7. We recall that $s$ is supposed to be fixed and the constants $C$ may depend on $s$.

First, we recall two classical facts on the heat equation for Dirichlet conditions: a wellposedness result and a regularity result.

## Appendix A.3.1. General lemmas

Proposition A.2. Let $d \in(0,+\infty)$, $u_{0} \in L^{2}(\Omega), g \in L^{2}(Q)$. From [25, Section 7.1, Theorem 3 and Theorem 4], the following Cauchy problem admits a unique weak solution $u \in Z:=$ $L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right) \cap W^{1,2}\left(0, T ; H^{-1}(\Omega)\right)$

$$
\begin{cases}\partial_{t} u-d \Delta u=g & \text { in }(0, T) \times \Omega, \\ u=0 & \text { on }(0, T) \times \partial \Omega \\ u(0, .)=u_{0} & \text { in } \Omega\end{cases}
$$

This means that $u$ is the unique function in $Z$ that satisfies the variational fomulation

$$
\begin{equation*}
\forall w \in L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right), \int_{0}^{T}\left(\partial_{t} u, w\right)_{H^{-1}(\Omega), H_{0}^{1}(\Omega)}+\int_{Q} d \nabla u \cdot \nabla w=\int_{Q} g w, \tag{A.11}
\end{equation*}
$$

and

$$
\begin{equation*}
u(0, .)=u_{0} \text { in } L^{2}(\Omega) . \tag{A.12}
\end{equation*}
$$

Moreover, there exists $C>0$ independent of $u_{0}$ and $g$ such that

$$
\|u\|_{Z} \leq C\left(\left\|u_{0}\right\|_{L^{2}(\Omega)}+\|g\|_{L^{2}(Q)}\right) .
$$

Proposition A.3. Let $d \in(0,+\infty), g \in L^{2}(Q)$, $u_{0} \in C_{0}^{\infty}(\Omega)$. From Proposition A.2, the following Cauchy problem admits a unique weak solution $u \in Z$

$$
\begin{cases}\partial_{t} u-d \Delta u=g & \text { in }(0, T) \times \Omega \\ u=0 & \text { on }(0, T) \times \partial \Omega \\ u(0, .)=u_{0} & \text { in } \Omega\end{cases}
$$

Moreover, from [25, Section 7.1, Theorem 5], $u \in Z_{2}:=L^{2}\left(0, T, H^{2}(\Omega) \cap H_{0}^{1}(\Omega)\right) \cap W^{1,2}\left(0, T ; L^{2}(\Omega)\right)$ and if $u_{0}=0$, then there exists $C>0$ independent of $g$ such that

$$
\|u\|_{Z_{2}} \leq C\|g\|_{L^{2}(Q)} .
$$

The following lemma is inspired by the proof of [13, Theorem 2.2].
Lemma A.4. Let $d \in(0,+\infty), f \in Y_{2}$ (see Definition 4.13), $\Phi_{T} \in C_{0}^{\infty}(\Omega), \widetilde{\omega}$ be an open subset such that $\widetilde{\omega} \subset \subset \omega_{0}, \chi \in C^{\infty}(\bar{\Omega} ;[0,+\infty[)$ such that supp $(\chi) \subset \widetilde{\omega},(r, k) \in \mathbb{R} \times[1,+\infty)$, $\Theta=\chi e^{s \alpha}(s \phi)^{r}$. Let $\Phi \in Z_{2}$ (see Proposition A.3) be the solution of

$$
\begin{cases}-\partial_{t} \Phi-d \Delta \Phi=\Delta f & \text { in }(0, T) \times \Omega  \tag{A.13}\\ \Phi=0 & \text { on }(0, T) \times \partial \Omega \\ \Phi(T, .)=\Phi_{T} & \text { in } \Omega\end{cases}
$$

We decompose

$$
\begin{equation*}
\Theta \Phi=\eta+\psi, \tag{A.14}
\end{equation*}
$$

where $\eta \in Z_{2}$ and $\psi \in Z_{2}$ satisfy

$$
\begin{gather*}
\begin{cases}-\partial_{t} \eta-d \Delta \eta=\Theta \Delta f & \text { in }(0, T) \times \Omega, \\
\eta=0 & \text { on }(0, T) \times \partial \Omega, \\
\eta(T, .)=0 & \text { in } \Omega,\end{cases}  \tag{A.15}\\
\left\{\begin{array}{lll}
-\partial_{t} \psi-d \Delta \psi=-\left(\partial_{t} \Theta\right) \Phi-2 d \nabla \Theta . \nabla \Phi-d(\Delta \Theta) \Phi & \text { in }(0, T) \times \Omega, \\
\psi=0 & \text { on }(0, T) \times \partial \Omega, \\
\psi(T, .)=0 & \text { in } \Omega .
\end{array}\right. \tag{A.16}
\end{gather*}
$$

Then, there exist $\widetilde{\chi} \in C^{\infty}(\bar{\Omega} ;[0,+\infty[)$ such that $\operatorname{supp}(\widetilde{\chi}) \subset \subset \widetilde{\omega}, \widetilde{\chi}=1$ on $\operatorname{supp}(\chi)$ and $C>0$ such that

$$
\begin{gather*}
\|\eta\|_{L^{2}(Q)}^{2} \leq C \int_{0}^{T} \int_{\widetilde{\omega}} \widetilde{\chi}^{2} e^{2 s \alpha}(s \phi)^{2(r+2)}|f|^{2},  \tag{A.17}\\
\left\|\frac{\psi}{(s \phi)^{k}}\right\|_{L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right)}^{2}+\left\|\left(\frac{\psi}{(s \phi)^{k}}\right)\right\|_{t} \|_{L^{2}\left(0, T ; H^{-1}(\Omega)\right)}^{2} \\
\leq C\left(\|\eta\|_{L^{2}(Q)}^{2}+\int_{0}^{T} \int_{62} e^{2 s \alpha}(s \phi)^{2(r+2-k)}|\Phi|^{2}\right) . \tag{A.18}
\end{gather*}
$$

Proof. Let $\Gamma \in L^{2}(Q)$ and let $z \in Z_{2}$ be the solution of

$$
\begin{cases}\partial_{t} z-d \Delta z=\Gamma & \text { in }(0, T) \times \Omega  \tag{A.19}\\ z=0 & \text { on }(0, T) \times \partial \Omega \\ z(0, .)=0 & \text { in } \Omega\end{cases}
$$

By Proposition A.3, we have

$$
\begin{equation*}
\|z\|_{L^{2}\left(0, T ; H^{2}(\Omega)\right)}^{2} \leq C\|\Gamma\|_{L^{2}(Q)}^{2} \tag{A.20}
\end{equation*}
$$

A duality argument between (A.15) and (A.19) gives

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega} \eta \Gamma d x d t=\int_{0}^{T} \int_{\Omega} \Theta \Delta(f) z d x d t \tag{A.21}
\end{equation*}
$$

We integrate by parts with respect to the spatial variable,

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega} \Theta \Delta(f) z d x d t=\int_{0}^{T} \int_{\Omega} f \Delta(\Theta z) d x d t \tag{A.22}
\end{equation*}
$$

There exists $\widetilde{\chi} \in C^{\infty}(\bar{\Omega} ;[0,+\infty[)$ such that $\operatorname{supp} \widetilde{\chi}) \subset \subset \widetilde{\omega}, \widetilde{\chi}=1$ on $\operatorname{supp}(\chi)$ and

$$
\begin{equation*}
\forall i \in\{1,2\},\left|D_{x}^{i} \Theta\right| \leq C \widetilde{\chi}(s \phi)^{r+i} e^{s \alpha} \text { in }(0, T) \times \Omega \tag{A.23}
\end{equation*}
$$

Therefore, from (A.20) and (A.23), we can deduce that

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega} f \Delta(\Theta z) d x d t \leq \frac{1}{2}\|\Gamma\|_{L^{2}(Q)}^{2}+C \int_{0}^{T} \int_{\Omega} \widetilde{\chi}^{2} e^{2 s \alpha}(s \phi)^{2(r+2)}|f|^{2} d x d t \tag{A.24}
\end{equation*}
$$

By using (A.21), (A.22), (A.24) and by taking $\Gamma=\eta$, we deduce (A.17).
We introduce

$$
\begin{equation*}
\rho=(s \phi)^{-k} \tag{A.25}
\end{equation*}
$$

Then, we have

$$
\begin{cases}-\partial_{t}(\rho \psi)-d \Delta(\rho \psi)=\rho\left(-\left(\partial_{t} \Theta\right) \Phi-2 d \nabla \Theta . \nabla \Phi-d(\Delta \Theta) \Phi\right) &  \tag{A.26}\\ & -\left(\partial_{t} \rho\right) \psi-2 d \nabla \rho \cdot \nabla \psi-d(\Delta \rho) \psi \\ \rho \psi=0 & \text { in }(0, T) \times \Omega \\ \rho \psi(T, .)=0 & \text { on }(0, T) \times \partial \Omega \\ \text { in } \Omega\end{cases}
$$

We estimate the source term of (A.26). We have by definition of $\Theta$, the fact that $k \geq 1$, (A.14), (A.25) and the embedding $L^{2}(\Omega) \hookrightarrow H^{-1}(\Omega)$, the following estimates

$$
\begin{equation*}
\left\|\rho \partial_{t}(\Theta) \Phi\right\|_{L^{2}(Q)}^{2} \leq C \int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{2(r+2-k)}|\Phi|^{2} \tag{A.27}
\end{equation*}
$$

$$
\begin{align*}
\|\rho \nabla \Theta . \nabla \Phi\|_{L^{2}\left(0, T ; H^{-1}(\Omega)\right)}^{2} & =\|\nabla \cdot(\rho \Phi \nabla \Theta)-(\rho(\Delta \Theta) \Phi)-(\nabla \rho \cdot \nabla \Theta) \Phi\|_{L^{2}\left(0, T ; H^{-1}(\Omega)\right)}^{2} \\
& \leq C\left(\|\rho \Phi \nabla \Theta\|_{L^{2}(Q)}^{2}+\|\rho(\Delta \Theta) \Phi\|_{L^{2}(Q)}^{2}+\|(\nabla \rho \cdot \nabla \Theta) \Phi\|_{L^{2}(Q)}^{2}\right) \\
& \leq C \int_{0}^{T} \int_{\Omega} e^{2 s \alpha}\left((s \phi)^{2(r+1-k)}+(s \phi)^{2(r+2-k)}+(s \phi)^{2(r+1-k)}\right)|\Phi|^{2} \\
& \leq C \int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{2(r+2-k)}|\Phi|^{2}, \tag{A.28}
\end{align*}
$$

$$
\begin{align*}
&\left\|\left(\partial_{t} \rho\right) \psi\right\|_{L^{2}(Q)}^{2}=\left\|\left(\partial_{t} \rho\right)(\Theta \Phi-\eta)\right\|_{L^{2}(Q)}^{2} \\
& \leq C\left(\int_{0}^{T} \int_{\Omega}(s \phi)^{2(-k+1)}|\eta|^{2}+\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{2(r+1-k)}|\Phi|^{2}\right) \\
& \leq C\left(\int_{0}^{T} \int_{\Omega}|\eta|^{2}+\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{2(r+2-k)}|\Phi|^{2}\right),  \tag{A.29}\\
&\|\nabla \rho \cdot \nabla \psi\|_{L^{2}\left(0, T ; H^{-1}(\Omega)\right)}^{2}=\|\nabla \cdot(\psi \nabla \rho)-\psi \Delta \rho\|_{L^{2}\left(0, T ; H^{-1}(\Omega)\right)}^{2} \\
&=\|\nabla \cdot((\Theta \Phi-\eta) \nabla \rho)-(\Theta \Phi-\eta) \Delta \rho\|_{L^{2}\left(0, T ; H^{-1}(\Omega)\right)}^{2} \\
& \leq C\left(\int_{0}^{T} \int_{\Omega}(s \phi)^{-2 k}|\eta|^{2}+\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{2(r-k)}|\Phi|^{2}\right) . \\
& \leq C\left(\int_{0}^{T} \int_{\Omega}|\eta|^{2}+\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{2(r+2-k)}|\Phi|^{2}\right) \tag{A.30}
\end{align*}
$$

By using (A.26), (A.27), (A.28), (A.29), (A.30) and Proposition A.2, we deduce (A.18).
Corollary A.5. We take the same notations as in Lemma A.4 and $g \in Y_{2}$. Then, for every $\delta>0$,

$$
\begin{align*}
& \int_{0}^{T} \int_{\widetilde{\omega}} \chi e^{s \alpha} \psi \Delta g \\
& \leq \delta\left(\int_{0}^{T} \int_{\widetilde{\omega}} \widetilde{\chi}^{2} e^{2 s \alpha}(s \phi)^{2(r+2)}|f|^{2}+\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{2(r+2-k)}|\Phi|^{2}\right) \\
& \quad+C_{\delta} \int_{0}^{T} \int_{\widetilde{\omega}} e^{2 s \alpha}(s \phi)^{2(k+1)}|\nabla g|^{2},  \tag{A.31}\\
& \int_{0}^{T} \int_{\widetilde{\omega}} \chi e^{s \alpha} \psi \partial_{t} g \\
& \leq \delta\left(\int_{0}^{T} \int_{\widetilde{\omega}} \widetilde{\chi}^{2} e^{2 s \alpha}(s \phi)^{2(r+2)}|f|^{2}+\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{2(r+2-k)}|\Phi|^{2}\right)  \tag{A.32}\\
& \quad+C_{\delta}\left(\int_{0}^{T} \int_{\widetilde{\omega}} e^{2 s \alpha}(s \phi)^{2(k+2)}|g|^{2}+\int_{0}^{T} \int_{\widetilde{\omega}} e^{2 s \alpha}(s \phi)^{2 k}|\nabla g|^{2}\right) \tag{A.33}
\end{align*}
$$

Proof. We integrate by parts with respect to the spatial variable and we use (A.18), (A.17),

$$
\begin{aligned}
& \int_{0}^{T} \int_{\widetilde{\omega}} \chi e^{s \alpha} \psi \Delta g \\
& =-\int_{0}^{T} \int_{\widetilde{\omega}} \frac{\psi}{(s \phi)^{k}} \nabla\left(\chi e^{s \alpha}(s \phi)^{k}\right) \cdot \nabla g-\int_{0}^{T} \int_{\widetilde{\omega}} \chi e^{s \alpha}(s \phi)^{k} \nabla\left(\frac{\psi}{(s \phi)^{k}}\right) \cdot \nabla g \\
& \leq \delta\left\|\frac{\psi}{(s \phi)^{k}}\right\|_{L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right)}^{2}+C_{\delta} \int_{0}^{T} \int_{\widetilde{\omega}} e^{2 s \alpha}(s \phi)^{2(k+1)}|\nabla g|^{2} \\
& \leq \delta\left(\|\eta\|_{L^{2}(Q)}^{2}+\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{2(r+2-k)}|\Phi|^{2}\right)+C_{\delta} \int_{0}^{T} \int_{\widetilde{\omega}} e^{2 s \alpha}(s \phi)^{2(k+1)}|\nabla g|^{2} \\
& \leq \delta\left(\int_{0}^{T} \int_{\widetilde{\omega}} \widetilde{\chi}^{2} e^{2 s \alpha}(s \phi)^{2(r+2)}|f|^{2}+\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{2(r+2-k)}|\Phi|^{2}\right) \\
& \quad+C_{\delta} \int_{0}^{T} \int_{\widetilde{\omega}} e^{2 s \alpha}(s \phi)^{2(k+1)}|\nabla g|^{2} .
\end{aligned}
$$

We integrate by parts with respect to the time variable and we use (A.18), (A.17),

$$
\begin{aligned}
& \int_{0}^{T} \int_{\widetilde{\omega}} \chi e^{s \alpha} \psi \partial_{t} g \\
& = \\
& -\left\langle\left(\frac{\psi}{(s \phi)^{k}}\right)_{t}, \chi e^{s \alpha}(s \phi)^{k} g\right\rangle_{L^{2}\left(0, T ; H^{-1}(\Omega)\right), L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right)} \\
& -\int_{0}^{T} \int_{\widetilde{\omega}} \frac{\psi}{(s \phi)^{k}} \chi \partial_{t}\left(e^{s \alpha}(s \phi)^{k}\right) g \\
& \leq \delta\left\|\left(\frac{\psi}{(s \phi)^{k}}\right)\right\|_{t}^{2}\left\|_{L^{2}\left(0, T ; H^{-1}(\Omega)\right)}+C_{\delta}\right\| \chi e^{s \alpha}(s \phi)^{k} g \|_{L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right)}^{2} \\
& +\delta\left\|\left(\frac{\psi}{(s \phi)^{k}}\right)\right\|_{L^{2}\left(\left(0, T ; L^{2}(\Omega)\right)\right.}^{2}+C_{\delta} \int_{0}^{T} \int_{\widetilde{\omega}}\left|\partial_{t}\left(e^{s \alpha}(s \phi)^{k}\right)\right|^{2}|g|^{2} \\
& \leq \delta\left(\|\eta\|_{L^{2}(Q)}^{2}+\int_{0}^{T} \int_{\Omega}^{2 s \alpha}(s \phi)^{2(r+2-k)}|\Phi|^{2}\right) \\
& +C_{\delta}\left(\int_{0}^{T} \int_{\widetilde{\omega}} e^{2 s \alpha}(s \phi)^{2(k+2)}|g|^{2}+\int_{0}^{T} \int_{\widetilde{\omega}} e^{2 s \alpha}(s \phi)^{2 k}|\nabla g|^{2}\right) \\
& \leq \delta\left(\int_{0}^{T} \int_{\widetilde{\omega}} \widetilde{\chi}^{2} e^{2 s \alpha}(s \phi)^{2(r+2)}|f|^{2}+\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{2(r+2-k)}|\Phi|^{2}\right) \\
& \quad+C_{\delta}\left(\int_{0}^{T} \int_{\widetilde{\omega}} e^{2 s \alpha}(s \phi)^{2(k+2)}|g|^{2}+\int_{0}^{T} \int_{\widetilde{\omega}} e^{2 s \alpha}(s \phi)^{2 k}|\nabla g|^{2}\right) .
\end{aligned}
$$

Appendix A.3.2. Proof of technical lemmas: Lemma 4.31 and Lemma 4.32
Let $\varepsilon \in(0,1)$. We introduce

$$
\begin{gather*}
\theta=\chi_{3} e^{s \alpha}(s \phi)^{3} .  \tag{A.34}\\
65
\end{gather*}
$$

The function $\theta \Delta \Delta \varphi_{4}$ satisfies the following parabolic system (see (158)),

$$
\begin{cases}-\partial_{t}\left(\theta \Delta \Delta \varphi_{4}\right)-d_{4} \Delta\left(\theta \Delta \Delta \varphi_{4}\right) & \\ \quad=\theta \Delta \Delta\left(m_{3}\left(\varphi_{1}-\varphi_{2}\right)\right)-\partial_{t} \theta \Delta \Delta \varphi_{4}-2 d_{4} \nabla \theta \cdot \nabla\left(\Delta \Delta \varphi_{4}\right)-d_{4} \Delta \theta \Delta \Delta \varphi_{4} & \text { in }(0, T) \times \Omega, \\ \theta \Delta \Delta \varphi_{4}=0 & \text { on }(0, T) \times \partial \Omega,  \tag{A.35}\\ \theta \Delta \Delta \varphi_{4}(T, .)=0 & \text { in } \Omega .\end{cases}
$$

We decompose

$$
\begin{equation*}
\theta \Delta \Delta \varphi_{4}=\eta+\psi, \tag{A.36}
\end{equation*}
$$

where $\eta$ and $\psi$ solve, respectively,

$$
\begin{gather*}
\begin{cases}-\partial_{t} \eta-d_{4} \Delta \eta=\theta \Delta \Delta\left(m_{3}\left(\varphi_{1}-\varphi_{2}\right)\right) & \text { in }(0, T) \times \Omega, \\
\eta=0 & \text { on }(0, T) \times \partial \Omega, \\
\eta(T, .)=0 & \text { in } \Omega,\end{cases}  \tag{A.37}\\
\begin{cases}-\partial_{t} \psi-d_{4} \Delta \psi=-\partial_{t} \theta \Delta \Delta \varphi_{4}-2 d_{4} \nabla \theta \cdot \nabla\left(\Delta \Delta \varphi_{4}\right)-d_{4} \Delta \theta \Delta \Delta \varphi_{4} & \text { in }(0, T) \times \Omega, \\
\psi=0 & \text { on }(0, T) \times \partial \Omega, \\
\psi(T, .)=0 & \text { in } \Omega .\end{cases} \tag{A.38}
\end{gather*}
$$

Appendix A.3.2.1. Proof of Lemma 4.31. We have

$$
\begin{equation*}
\int_{0}^{T} \int_{\omega_{2}}\left(\chi_{3}(x)\right)^{2} e^{2 s \alpha}(s \phi)^{3}\left(\Delta \Delta \varphi_{4}\right)\left(\Delta \Delta \varphi_{3}\right) d x d t=\int_{0}^{T} \int_{\omega_{2}} \chi_{3}(x) e^{s \alpha}(\eta+\psi)\left(\Delta \Delta \varphi_{3}\right) d x d t \tag{A.39}
\end{equation*}
$$

The first term in the right-hand side of (A.39) can be estimated as follows,

$$
\begin{align*}
\int_{0}^{T} \int_{\omega_{2}} \chi_{3}(x) e^{s \alpha} \eta\left(\Delta \Delta \varphi_{3}\right) d x d t & \leq \varepsilon \int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)\left|\Delta \Delta \varphi_{3}\right|^{2}+C_{\varepsilon} \int_{0}^{T} \int_{\omega_{2}}\left(\chi_{3}(x)\right)^{2}(s \phi)^{-1} \eta^{2} \\
& \leq \varepsilon \int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)\left|\Delta \Delta \varphi_{3}\right|^{2}+C_{\varepsilon} \int_{0}^{T} \int_{\Omega} \eta^{2} \tag{A.40}
\end{align*}
$$

Lemma A.6. For every $\delta>0$,

$$
\begin{align*}
\int_{0}^{T} \int_{\Omega}|\eta|^{2} d x d t \leq & \delta \int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{4}\left(\left|\Delta \varphi_{1}\right|^{2}+\left|\Delta \varphi_{2}\right|^{2}\right) \\
& +C_{\delta} \int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}\left\{(s \phi)^{24}\left(\left|\varphi_{1}\right|^{2}+\left|\varphi_{2}\right|^{2}+\left|\Delta \varphi_{3}\right|^{2}\right)+(s \phi)^{22}\left(\left|\nabla \varphi_{1}\right|^{2}+\left|\nabla \varphi_{2}\right|^{2}\right)\right\} \tag{A.41}
\end{align*}
$$

Proof. The idea of the proof is to apply two times Lemma A. 4 because the source term of (A.37) is $\theta \Delta \Delta(\ldots)$.

Step 1: We apply Lemma A.4: (A.17) with $d=d_{4}, f=m_{3} \Delta\left(\varphi_{1}-\varphi_{2}\right), \Phi_{T}=\Delta \Delta \varphi_{4, T}$, $\widetilde{\omega}=\omega_{2}, \underline{\chi}=\chi_{3}, r=3, \Theta=\theta, \Phi=\Delta \Delta \varphi_{4}$ and the decomposition (A.36). Then, there exists $\widetilde{\chi_{3}} \in C^{\infty}\left(\bar{\Omega} ;\left[0,+\infty[)\right.\right.$ such that $\operatorname{supp}\left(\widetilde{\chi_{3}}\right) \subset \subset \omega_{2}, \widetilde{\chi_{3}}=1$ on $\operatorname{supp}\left(\chi_{3}\right)$ and

$$
\begin{equation*}
\|\eta\|_{L^{2}(Q)}^{2} \leq C \int_{0}^{T} \int_{\omega_{2}}\left(\widetilde{\chi_{3}}\right)^{2} e^{2 s \alpha}(s \phi)^{10}\left(\left|\Delta \varphi_{1}\right|^{2}+\left|\Delta \varphi_{2}\right|^{2}\right) d x d t \tag{A.42}
\end{equation*}
$$

Remark A.7. This estimate is not sufficient because we can not absorb the right hand side term of (A.42) by the left hand side term of (165).

Step 2: Now, our aim is to prove that for every $i \in\{1,2\}, \delta>0$, we have

$$
\begin{align*}
& \int_{0}^{T} \int_{\omega_{2}}\left(\widetilde{\chi_{3}}\right)^{2} e^{2 s \alpha}(s \phi)^{10}\left|\Delta \varphi_{i}\right|^{2} d x d t \\
& \leq \delta\left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{4}\left|\Delta \varphi_{i}\right|^{2} d x d t\right) \\
& +C_{\delta}\left(\int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}(s \phi)^{24}\left(\left|\varphi_{1}\right|^{2}+\left|\varphi_{2}\right|^{2}+\left|\Delta \varphi_{3}\right|^{2}\right) d x d t+\int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}(s \phi)^{22}\left|\nabla \varphi_{i}\right|^{2} d x d t\right) \tag{A.43}
\end{align*}
$$

Remark A.8. This previous estimate is also useful for the proof of the observability inequality with one component (see (170)).

First, we remark that

$$
\int_{0}^{T} \int_{\omega_{2}}\left(\widetilde{\chi_{3}}\right)^{2} e^{2 s \alpha}(s \phi)^{10}\left|\Delta \varphi_{i}\right|^{2}=\int_{0}^{T} \int_{\omega_{2}} \widetilde{\chi_{3}} e^{s \alpha} \widetilde{\theta} \Delta \varphi_{i} \Delta \varphi_{i}
$$

with

$$
\begin{equation*}
\widetilde{\theta}=\widetilde{\chi_{3}} e^{s \alpha}(s \phi)^{10} \tag{A.44}
\end{equation*}
$$

Moreover, $\widetilde{\theta} \Delta \varphi_{i}$ satisfies the following parabolic system (see (156) and Lemma 4.23),

$$
\begin{cases}\left.-\partial_{t}\left(\widetilde{\theta} \Delta \varphi_{i}\right)-d_{i} \Delta \widetilde{\theta} \Delta \varphi_{i}\right) &  \tag{A.45}\\ \quad=\widetilde{\theta} \Delta\left(a_{1 i} \varphi_{1}+a_{2 i} \varphi_{2}+\delta_{i 2}\left(d_{2}-d_{3}\right) \Delta \varphi_{3}\right) & \\ \quad-\widetilde{\partial}_{t} \widetilde{\theta} \Delta \varphi_{i}-2 d_{i} \nabla \widetilde{\nabla} \cdot \nabla\left(\Delta \varphi_{i}\right)-d_{i} \Delta \widetilde{\theta} \Delta \varphi_{i} & \text { in }(0, T) \times \Omega \\ \widetilde{\theta} \Delta \varphi_{i}=0 & \text { on }(0, T) \times \partial \Omega \\ \widetilde{\theta} \Delta \varphi_{i}(T, .)=0 & \text { in } \Omega .\end{cases}
$$

We decompose

$$
\begin{equation*}
\widetilde{\theta} \Delta \varphi_{i}=\widetilde{\eta}_{i}+\widetilde{\psi}_{i}, \tag{A.46}
\end{equation*}
$$

where $\widetilde{\eta}_{i}$ and $\widetilde{\psi}_{i}$ solve, respectively,

$$
\begin{align*}
& \begin{cases}-\partial_{t} \widetilde{\eta}_{i}-d_{i} \Delta \widetilde{\eta}_{i}=\widetilde{\theta} \Delta\left(a_{1 i} \varphi_{1}+a_{2 i} \varphi_{2}+\delta_{i 2}\left(d_{2}-d_{3}\right) \Delta \varphi_{3}\right) & \text { in }(0, T) \times \Omega, \\
\widetilde{\eta}_{i}=0 & \text { on }(0, T) \times \partial \Omega, \\
\widetilde{\eta}_{i}(T, .)=0 & \text { in } \Omega,\end{cases}  \tag{A.47}\\
& \begin{cases}-\partial_{t} \widetilde{\psi}_{i}-d_{i} \Delta \widetilde{\psi_{i}}=-\partial_{t} \widetilde{\theta} \Delta \varphi_{i}-2 d_{i} \nabla \widetilde{\theta} \cdot \nabla\left(\Delta \varphi_{i}\right)-d_{i} \Delta \widetilde{\theta} \Delta \varphi_{i} & \text { in }(0, T) \times \Omega, \\
\widetilde{\psi_{i}}=0 & \text { on }(0, T) \times \partial \Omega, \\
\widetilde{\psi_{i}}(T, .)=0 & \text { in } \Omega .\end{cases} \tag{A.48}
\end{align*}
$$

We have

$$
\begin{equation*}
\int_{0}^{T} \int_{\omega_{2}}\left(\widetilde{\chi_{3}}\right)^{2} e^{2 s \alpha}(s \phi)^{10}\left|\Delta \varphi_{i}\right|^{2} d x d t=\int_{0}^{T} \int_{\omega_{2}} \widetilde{\chi_{3}} e^{s \alpha}\left(\widetilde{\eta_{i}}+\widetilde{\psi_{i}}\right)\left(\Delta \varphi_{i}\right) d x d t \tag{A.49}
\end{equation*}
$$

The first term in the right-hand side of (A.49) can be estimated as follows,

$$
\begin{equation*}
\int_{0}^{T} \int_{\omega_{2}}{\widetilde{\chi_{3}}}^{s \alpha} \widetilde{\eta}_{i}\left(\Delta \varphi_{i}\right) d x d t \leq \delta \int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{4}\left|\Delta \varphi_{i}\right|^{2} d x d t+C_{\delta} \int_{0}^{T} \int_{\Omega} \widetilde{\eta}_{i}^{2} d x d t \tag{A.50}
\end{equation*}
$$

Then, we apply Lemma A.4: (A.17) with $d=d_{i}, f=a_{1 i} \varphi_{1}+a_{2 i} \varphi_{2}+\delta_{i 2}\left(d_{2}-d_{3}\right) \Delta \varphi_{3} \in Y_{2}$ (because $\left.A \in \mathcal{M}_{4}\left(C_{0}^{\infty}(Q)\right)\right), \Phi_{T}=\Delta \varphi_{i, T}, \widetilde{\omega}=\omega_{2}, \chi=\widetilde{\chi_{3}}, r=10, \Theta=\widetilde{\theta}, \Phi=\Delta \varphi_{i}$ and the decomposition (A.46). There exists $\chi_{3}{ }^{\sharp} \in C^{\infty}\left(\bar{\Omega} ;\left[0,+\infty[)\right.\right.$ such that $\operatorname{supp}\left(\chi_{3}{ }^{\sharp}\right) \subset \subset \omega_{2}$ and $C$ which depends on $\|A\|_{L^{\infty}(Q)}$

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega}\left|\widetilde{\eta}_{i}\right|^{2} d x d t \leq C \int_{0}^{T} \int_{\omega_{2}}\left(\chi_{3}^{\sharp}\right)^{2} e^{2 s \alpha}(s \phi)^{24}\left(\left|\varphi_{1}\right|^{2}+\left|\varphi_{2}\right|^{2}+\left|\Delta \varphi_{3}\right|^{2}\right) d x d t . \tag{A.51}
\end{equation*}
$$

Then, (A.50) and (A.51) give

$$
\begin{align*}
& \int_{0}^{T} \int_{\omega_{2}} \widetilde{\chi}_{3} e^{s \alpha} \widetilde{\eta}_{i}\left(\Delta \varphi_{i}\right) d x d t \\
& \leq \delta \int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{4}\left|\Delta \varphi_{i}\right|^{2} d x d t+C_{\delta} \int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}(s \phi)^{24}\left(\left|\varphi_{1}\right|^{2}+\left|\varphi_{2}\right|^{2}+\left|\Delta \varphi_{3}\right|^{2}\right) d x d t \tag{A.52}
\end{align*}
$$

For the second term in the right-hand side of (A.49), we use Corollary A.5: (A.31) with $d=d_{i}$, $f=a_{1 i} \varphi_{1}+a_{2 i} \varphi_{2}+\delta_{i 2}\left(d_{2}-d_{3}\right) \Delta \varphi_{3} \in Y_{2}, \Phi_{T}=\Delta \varphi_{i, T}, \widetilde{\omega}=\omega_{2}, \chi=\widetilde{\chi_{3}},(r, k)=(10,10), \Theta=\widetilde{\theta}$, $\Phi=\Delta \varphi_{i}$ and the decomposition (A.46)). Then, we have

$$
\begin{align*}
& \int_{0}^{T} \int_{\omega_{2}} \widetilde{\chi}_{3} e^{s \alpha} \widetilde{\psi}_{i} \Delta \varphi_{i} \\
& \leq \delta\left(\int_{0}^{T} \int_{\omega_{2}}\left(\chi_{3}^{\sharp}\right)^{2} e^{2 s \alpha}(s \phi)^{24}\left(\left|\varphi_{1}\right|^{2}+\left|\varphi_{2}\right|^{2}+\left|\Delta \varphi_{3}\right|^{2}\right)+\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{4}\left|\Delta \varphi_{i}\right|^{2}\right) \\
& \quad+C_{\delta} \int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}(s \phi)^{22}\left|\nabla \varphi_{i}\right|^{2} . \tag{A.53}
\end{align*}
$$

Gathering (A.49), (A.52) and (A.53), we have (A.43).
The estimates (A.42) and (A.43) give (A.41).
End of the proof of Lemma 4.31: Applying Lemma A. 6 with $\delta=\varepsilon / C_{\varepsilon}$, we find

$$
\begin{align*}
& \int_{0}^{T} \int_{\Omega}|\eta|^{2} d x d t \\
& \leq \frac{\varepsilon}{C_{\varepsilon}} \int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{4}\left(\left|\Delta \varphi_{1}\right|^{2}+\left|\Delta \varphi_{2}\right|^{2}\right) d x d t \\
& \quad+C_{\varepsilon}^{\prime} \int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}\left\{(s \phi)^{24}\left(\left|\varphi_{1}\right|^{2}+\left|\varphi_{2}\right|^{2}+\left|\Delta \varphi_{3}\right|^{2}\right)+(s \phi)^{22}\left(\left|\nabla \varphi_{1}\right|^{2}+\left|\nabla \varphi_{2}\right|^{2}\right)\right\} d x d t \tag{A.54}
\end{align*}
$$

Then, we put (A.54) in (A.40) to get

$$
\begin{align*}
& \int_{0}^{T} \int_{\omega_{2}} \chi_{3}(x) e^{s \alpha} \eta\left(\Delta \Delta \varphi_{3}\right) \\
& \leq \varepsilon\left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)\left|\Delta \Delta \varphi_{3}\right|^{2}+\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{4}\left(\left|\Delta \varphi_{1}\right|^{2}+\left|\Delta \varphi_{2}\right|^{2}\right)\right) \\
& +C_{\varepsilon}\left(\int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}(s \phi)^{24}\left(\left|\varphi_{1}\right|^{2}+\left|\varphi_{2}\right|^{2}+\left|\Delta \varphi_{3}\right|^{2}\right)+\int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}(s \phi)^{22}\left(\left|\nabla \varphi_{1}\right|^{2}+\left|\nabla \varphi_{2}\right|^{2}\right)\right) \tag{A.55}
\end{align*}
$$

Lemma A.9. For every $\delta>0$,

$$
\begin{align*}
& \int_{0}^{T} \int_{\omega_{2}} \chi_{3} e^{s \alpha} \psi\left(\Delta \Delta \varphi_{3}\right) d x d t \\
& \leq \delta\left(\int_{0}^{T} \int_{\omega_{2}}\left(\widetilde{\chi_{3}}\right)^{2} e^{2 s \alpha}(s \phi)^{10}\left(\left|\Delta \varphi_{1}\right|^{2}+\left|\Delta \varphi_{2}\right|^{2}\right)+\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{3}\left|\Delta \Delta \varphi_{4}\right|^{2}\right) \\
& \quad+C_{\delta} \int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}(s \phi)^{9}\left|\nabla \Delta \varphi_{3}\right|^{2} \tag{A.56}
\end{align*}
$$

Proof. We apply Corollary A.5: (A.31) with $d=d_{4}, f=m_{3} \Delta\left(\varphi_{1}-\varphi_{2}\right), \Phi_{T}=\Delta \Delta \varphi_{4, T}, \widetilde{\omega}=\omega_{2}$, $\chi=\chi_{3},(r, k)=(3,7 / 2), \Theta=\theta, \Phi=\Delta \Delta \varphi_{4}$, the decomposition (A.36) and $g=\Delta \varphi_{3}$.

Applying Lemma A. 9 with $\delta=\varepsilon$, we find

$$
\begin{align*}
& \int_{0}^{T} \int_{\omega_{2}} \chi_{3} e^{s \alpha} \psi\left(\Delta \Delta \varphi_{3}\right) d x d t \\
& \leq \varepsilon\left(\int_{0}^{T} \int_{\omega_{2}}\left(\widetilde{\chi_{3}}\right)^{2} e^{2 s \alpha}(s \phi)^{10}\left(\left|\Delta \varphi_{1}\right|^{2}+\left|\Delta \varphi_{2}\right|^{2}\right)+\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{3}\left|\Delta \Delta \varphi_{4}\right|^{2}\right) \\
& \quad+C_{\varepsilon} \int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}(s \phi)^{9}\left|\nabla \Delta \varphi_{3}\right|^{2} \tag{A.57}
\end{align*}
$$

Then, we put (A.43) with $\delta=\varepsilon$ in (A.57) to get

$$
\begin{align*}
& \int_{0}^{T} \int_{\omega_{2}} \chi_{3} e^{s \alpha} \psi\left(\Delta \Delta \varphi_{3}\right) \\
& \leq \varepsilon\left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}\left\{(s \phi)^{4}\left(\left|\Delta \varphi_{1}\right|^{2}+\left|\Delta \varphi_{2}\right|^{2}\right)+(s \phi)^{3}\left|\Delta \Delta \varphi_{4}\right|^{2}\right\}\right) \\
& +C_{\varepsilon}\left(\int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}\left\{(s \phi)^{24}\left(\left|\varphi_{1}\right|^{2}+\left|\varphi_{2}\right|^{2}+\left|\Delta \varphi_{3}\right|^{2}\right)+(s \phi)^{22}\left(\left|\nabla \varphi_{1}\right|^{2}+\left|\nabla \varphi_{2}\right|^{2}+\left|\nabla \Delta \varphi_{3}\right|^{2}\right)\right\}\right) \tag{A.58}
\end{align*}
$$

Therefore, recalling (A.39), (A.55), (A.58), we get (168) and consequently Lemma 4.31.

Appendix A.3.2.2. Proof of Lemma 4.32. We have by (A.34) and (A.36)

$$
\begin{equation*}
\int_{0}^{T} \int_{\omega_{2}}\left(\chi_{3}(x)\right)^{2} e^{2 s \alpha}(s \phi)^{3}\left(\Delta \Delta \varphi_{4}\right)\left(\partial_{t} \Delta \varphi_{3}\right) d x d t=\int_{0}^{T} \int_{\omega_{2}} \chi_{3}(x) e^{s \alpha}(\eta+\psi) \partial_{t}\left(\Delta \varphi_{3}\right) d x d t \tag{A.59}
\end{equation*}
$$

We easily have by Young's inequality

$$
\begin{equation*}
\int_{0}^{T} \int_{\omega_{2}} \chi_{3}(x) e^{s \alpha} \eta \partial_{t}\left(\Delta \varphi_{3}\right) d x d t \leq \varepsilon \int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}(s \phi)\left|\partial_{t}\left(\Delta \varphi_{3}\right)\right|^{2} d x d t+C_{\varepsilon} \int_{0}^{T} \int_{\Omega}|\eta|^{2} d x d t \tag{A.60}
\end{equation*}
$$

By using Lemma A. 6 with $\delta=\varepsilon / C_{\varepsilon}$, we can deduce from (A.60) that

$$
\begin{align*}
& \int_{0}^{T} \int_{\omega_{2}} \chi_{3}(x) e^{s \alpha} \eta\left(\partial_{t} \Delta \varphi_{3}\right) d x d t \\
& \leq \varepsilon\left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)\left|\partial_{t} \Delta \varphi_{3}\right|^{2}+\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{4}\left(\left|\Delta \varphi_{1}\right|^{2}+\left|\Delta \varphi_{2}\right|^{2}\right)\right) \\
& +C_{\varepsilon}\left(\int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}(s \phi)^{24}\left(\left|\varphi_{1}\right|^{2}+\left|\varphi_{2}\right|^{2}+\left|\Delta \varphi_{3}\right|^{2}\right)+\int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}(s \phi)^{22}\left(\left|\nabla \varphi_{1}\right|^{2}+\left|\nabla \varphi_{2}\right|^{2}\right)\right) \tag{A.61}
\end{align*}
$$

Then, we estimate the other term in the right hand side of (A.59).
Lemma A.10. For every $\delta>0$,

$$
\begin{align*}
& \int_{0}^{T} \int_{\omega_{2}} \chi_{3} e^{s \alpha} \psi \partial_{t} \Delta \varphi_{3} \\
& \leq \delta\left(\int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}(s \phi)^{10}\left(\left|\Delta \varphi_{1}\right|^{2}+\left|\Delta \varphi_{2}\right|^{2}\right)+\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}(s \phi)^{3}\left|\Delta \Delta \varphi_{4}\right|^{2}\right) \\
& \quad+C_{\delta}\left(\int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}(s \phi)^{11}\left|\Delta \varphi_{3}\right|^{2}+\int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}(s \phi)^{7}\left|\nabla \Delta \varphi_{3}\right|^{2}\right) \tag{A.62}
\end{align*}
$$

Proof. We apply Corollary A.5: (A.33) with $d=d_{4}, f=\Delta\left(\varphi_{1}-\varphi_{2}\right), \Phi_{T}=\Delta \Delta \varphi_{4, T}, \widetilde{\omega}=\omega_{2}$, $\chi=\chi_{3},(r, k)=(3,7 / 2), \Theta=\theta, \Phi=\Delta \Delta \varphi_{4}$, the decomposition (A.36) and $g=\Delta \varphi_{3}$.

Then, we put (A.43) with $\delta=\varepsilon$ in (A.62) to get

$$
\begin{align*}
& \int_{0}^{T} \int_{\omega_{2}} \chi_{3} e^{s \alpha} \psi\left(\partial_{t} \Delta \varphi_{3}\right) \\
& \leq \varepsilon\left(\int_{0}^{T} \int_{\Omega} e^{2 s \alpha}\left\{(s \phi)^{4}\left(\left|\Delta \varphi_{1}\right|^{2}+\left|\Delta \varphi_{2}\right|^{2}\right)+(s \phi)^{3}\left|\Delta \Delta \varphi_{4}\right|^{2}\right\}\right) \\
& +C_{\varepsilon}\left(\int_{0}^{T} \int_{\omega_{2}} e^{2 s \alpha}\left\{(s \phi)^{24}\left(\left|\varphi_{1}\right|^{2}+\left|\varphi_{2}\right|^{2}+\left|\Delta \varphi_{3}\right|^{2}\right)+(s \phi)^{22}\left(\left|\nabla \varphi_{1}\right|^{2}+\left|\nabla \varphi_{2}\right|^{2}+\left|\nabla \Delta \varphi_{3}\right|^{2}\right)\right\}\right) \tag{A.63}
\end{align*}
$$

Recalling (A.59), (A.61), (A.63), we get (169) and consequently Lemma 4.32.
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