Controllability of a 4 x 4 quadratic reaction-diffusion system

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Abstract

We consider a 4×4 nonlinear reaction-diffusion system posed on a smooth domain Ω of $\mathbb{R}^N$ ($N \geq 1$) with controls localized in some arbitrary nonempty open subset ω of the domain Ω. This system is a model for the evolution of concentrations in reversible chemical reactions. We prove the local exact controllability to stationary constant solutions of the underlying reaction-diffusion system for every $N \geq 1$ in any time $T > 0$. A specificity of this control system is the existence of some invariant quantities in the nonlinear dynamics. The proof is based on a linearization which uses return method and an adequate change of variables that creates crossed diffusion which will be used as coupling terms of second order. The controllability properties of the linearized system are deduced from Carleman estimates. A Kakutani's fixed-point argument enables to go back to the nonlinear parabolic system. Then, we prove a global controllability result in large time for $1 \leq N \leq 2$ thanks to our local controllabillity result together with a known theorem on the asymptotics of the free nonlinear reaction diffusion system.

Key words. Controllability to stationary states, parabolic system, nonlinear coupling, Carleman estimate, return method

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Contents

1 Introduction

Let $T > 0$, $N \in \mathbb{N}^*$, Ω be a bounded, connected, open subset of $\mathbb{R}^N$ of class C^2, $Q := (0,T) \times \Omega$ and ω a nonempty open subset of Ω.

1.1 Presentation of the nonlinear reaction-diffusion system

Let $(d_1, d_2, d_3, d_4) \in (0, +\infty)^4$. We are interested in the following reaction-diffusion system

$$\forall 1 \leq i \leq 4, \begin{cases} \partial_t u_i - d_i \Delta u_i = (-1)^i (u_1 u_3 - u_2 u_4) & \text{in } (0,T) \times \Omega, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega, \\ u_i(0,.) = u_{i,0} & \text{in } \Omega, \end{cases} \tag{1}$$

where n is the outer unit normal vector to $\partial\Omega$. This system is a model for the evolution of the concentration $u_i(.,.)$ in the reversible chemical reaction

$$U_1 + U_3 \rightleftharpoons U_2 + U_4,$$

by using the law of mass action, Fick's law and the fact that no substance crosses the boundary (Neumann conditions). For this quadratic system, global existence of weak solutions holds in any dimension.

Proposition 1.1. *[40, Proposition 5.12]*
Let $u_0 \in L^2(\Omega)^4$. Then, there exists a global weak solution (in the sense of the definition [40, Section 5, (5.12)]) to (1).

For dimensions $N = 1,\ 2$, it was proved that the solutions are bounded and therefore classical for bounded initial data (see [21], [34] and [36]). It was not known until recently whether they were bounded in higher dimension (see [40, Section 7, Problem 3] and references therein for more details). But, a very recent preprint [19] proves that these solutions are smooth.

1.2 The question

Let $(u_1^*, u_2^*, u_3^*, u_4^*) \in [0, +\infty)^4$ satisfying

$$u_1^* u_3^* = u_2^* u_4^*. \tag{2}$$

We will say that $(u_i^*)_{1 \leq i \leq 4}$ is a stationary constant solution of (1).

Remark 1.2. The nonnegative stationary solutions of (1) are constant (see Proposition 6.1 in Appendix). Thus, it is not restrective to assume that $(u_1^*, u_2^*, u_3^*, u_4^*) \in [0, +\infty)^4$.

The **question** we ask is the following : **Could one reach stationary constant solutions of** (1) **with localized controls in finite time?**

We introduce the notations:

$j \in \{1,2,3\}$ denotes the number of internal controls that we allow in the equations of (1),

$1_{i \leq j} := 1$ if $1 \leq i \leq j$ and 0 if $i > j$.

By symmetry of the system, we reduce our study to the case of controls entering in the first equations. Thus, we consider the following system

$$\forall 1 \leq i \leq 4, \begin{cases} \partial_t u_i - d_i \Delta u_i = (-1)^i (u_1 u_3 - u_2 u_4) + h_i 1_\omega 1_{i \leq j} & \text{in } (0,T) \times \Omega, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega, \\ u_i(0,.) = u_{i,0} & \text{in } \Omega. \end{cases} \tag{3}$$

We are interested in the L^∞-controllability properties of (3): For every $u_0 \in L^\infty(\Omega)^4$, does there exist $(h_i)_{1 \leq i \leq j} \in L^\infty(Q)^j$ such that the solution u of (3) satisfies

$$\forall i \in \{1,2,3,4\},\ u_i(T,.) = u_i^*? \tag{4}$$

1.3 Two partial answers

Our first main outcome is a **local controllability result in $L^\infty(\Omega)$ with controls in $L^\infty(Q)$ for** (3), i.e. we will show that for every $1 \leq j \leq 3$, there exists $\delta > 0$ such that for every $u_0 \in X_{j,(d_i),(u_i^*)}$ (a " natural " subspace of $L^\infty(\Omega)^4$, see Section 3.1), with $\|u_0 - u^*\|_{L^\infty(\Omega)^4} \leq \delta$, there exists $(h_i)_{1\leq i\leq j} \in L^\infty(Q)^j$ such that the solution u of (3) satisfies (4).

Our second main result is a **global controllability result in $L^\infty(\Omega)$ with controls in $L^\infty(Q)$ for** (3) **in large time and in small dimension**, i.e. we will prove that for every $1 \leq N \leq 2$, $1 \leq j \leq 3$, $u_0 \in X_{j,(d_i),(u_i^*)}$ which verifies a positivity condition (see (41)), there exist T^* sufficiently large and $(h_i)_{1\leq i\leq j} \in L^\infty((0,T^*)\times\Omega)^j$ such that the solution u of (3) (replace T with T^*) satisfies (4) (replace T with T^*).

The precise results are stated in Section 3 (see Theorem 3.2 and Theorem 3.6).

1.4 Bibliographical comments for the null-controllability of parabolic systems with localized controls

Now, we discuss the null-controllability of parabolic coupled parabolic systems. The following results will be useful for having a proof strategy of our two main results.

Remark 1.3. We choose to present parabolic systems with Dirichlet conditions because these results are more easy to find in the literature. However, all the following results can be adapted to the Neumann conditions.

1.4.1 Linear parabolic systems

The problem of null-controllability of the heat equation was solved independently by Lebeau, Robbiano in 1995 (see [38] or the survey [37]) and Fursikov, Imanuvilov in 1996 (see [30]) with Carleman estimates.

Theorem 1.4. *For every $u_0 \in L^2(\Omega)$, there exists $h \in L^2(Q)$ such that the solution u of*

$$\begin{cases} \partial_t u - \Delta u = h1_\omega & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,.) = u_0 & \text{in } \Omega, \end{cases}$$

satisfies $u(T,.) = 0$.

Then, null-controllability of linear parabolic systems was studied. A typical example is

$$\begin{cases} \partial_t u - D\Delta u = Au + Bh1_\omega & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,.) = u_0 & \text{in } \Omega, \end{cases} \tag{5}$$

where $u \in C([0,T];L^2(\Omega)^k)$ is the state, $h \in L^2(Q)^l$, $1 \leq l \leq k$, is the control, $D := diag(d_1,\dots,d_k)$ with $d_i \in (0,+\infty)$ is the *diffusion matrix*, $A \in \mathcal{M}_k(\mathbb{R})$ (matrix with k lines and k columns with entries in $\mathbb{R}$) is the *coupling matrix* and $B \in \mathcal{M}_{k,l}(\mathbb{R})$ (matrix with k lines and l columns with entries in $\mathbb{R}$) represents the *distribution of controls*.

Definition 1.5. System (5) is said to be null-controllable if for every $u_0 \in L^2(\Omega)^k$, there exists $h \in L^2(Q)^l$ such that the solution u of (5) satisfies $u(T,.) = 0$.

The triplet (D,A,B) plays an important role for null-controllabillity of (5) as the following theorem, proved by Ammar-Khodja, Benabdallah, Dupaix and Gonzalez-Burgos (which is a generalization of the well-known Kalman condition in finite dimension, see [14, Theorem 1.16]), shows us.

Theorem 1.6. *([5, Theorem 5.6])*
Let us denote by $(\lambda_m)_{m\geq 1}$ the sequence of positive eigenvalues of the unbounded operator $(-\Delta, H^2(\Omega)\cap H_0^1(\Omega))$ on $L^2(\Omega)$. Then, the following conditions are equivalent.

1. *System (5) is null-controllable.*
2. *For every $m \geq 1$, $rank((-\lambda_m D + A)|B) = k$, where*

$$((-\lambda_m D + A)|B) := \big(B, (-\lambda_m D + A)B, (-\lambda_m D + A)^2 B, \dots, (-\lambda_m D + A)^{k-1} B\big).$$

For example, let us consider the 2×2 toy-system,

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = a_{11} u_1 + a_{12} u_2 + h_1 1_\omega & \text{in } (0,T) \times \Omega, \\ \partial_t u_2 - d_2 \Delta u_2 = a_{21} u_1 + a_{22} u_2 & \text{in } (0,T) \times \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0,.) = u_0 & \text{in } \Omega, \end{cases} \tag{6}$$

where $a_{i,j} \in L^\infty(Q)$ for every $1 \leq i,j \leq 2$. We easily deduce from Theorem 1.6 the following proposition.

Proposition 1.7. *We assume $a_{ij} \in \mathbb{R}$ for every $1 \leq i,j \leq 2$. The following conditions are equivalent.*
1. *System (6) is null-controllable,*
2. *$a_{21} \neq 0$.*

Roughly speaking, u_1 can be driven to 0 thanks to the control h_1 and u_2 can be driven to 0 thanks to the *coupling term* $a_{21}u_1$. We have the following diagram

$$h_1 \overset{controls}{\rightsquigarrow} u_1 \overset{controls}{\rightsquigarrow} u_2.$$

We also have a more general result for the toy-model (6).

Proposition 1.8. *([5, Theorem 7.1])*
We assume that for every $1 \leq i,j \leq 2$, $a_{ij} \in L^\infty(Q)$ and there exist $t_1 < t_2 \in (0,T)$, a nonempty open subset $\omega_0 \subset \omega$ and $\varepsilon > 0$ such that for almost every $(t,x) \in (t_1,t_2) \times \omega_0$, $|a_{21}(t,x)| \geq \varepsilon$. Then, system (6) is null-controllable.

Roughly speaking, if the coupling term a_{21} lives somewhere in the control zone, then (u_1,u_2) can be driven to $(0,0)$. The case where $supp(a_{21}) \cap \omega = \emptyset$ is more difficult even if a_{21} depends only on the spatial variable: a minimal time of control can appear (see [6] and [7]).

In order to reduce the number of controls entering in the equations of a linear parabolic system, a good strategy is to transform the system into a *cascade system*. This type of system has been studied by Gonzalez-Burgos and de Teresa (see [32]). For example, let us consider the 3×3 toy system,

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = a_{11} u_1 + a_{12} u_2 + a_{13} u_3 + h_1 1_\omega & \text{in } (0,T) \times \Omega, \\ \partial_t u_2 - d_2 \Delta u_2 = a_{21} u_1 + a_{22} u_2 + a_{23} u_3 & \text{in } (0,T) \times \Omega, \\ \partial_t u_3 - d_3 \Delta u_3 = \quad a_{32} u_2 + a_{33} u_3 & \text{in } (0,T) \times \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0,.) = u_0 & \text{in } \Omega. \end{cases} \tag{7}$$

where for every $1 \leq i,j \leq 3$, $a_{ij} \in L^\infty(Q)$.

Proposition 1.9. *If there exist $t_1 < t_2 \in (0,T)$, a nonempty open subset $\omega_0 \subset \omega$ and $\varepsilon > 0$ such that for almost every $(t,x) \in (t_1,t_2) \times \omega_0$, $|a_{21}(t,x)| \geq \varepsilon$ and $|a_{32}(t,x)| \geq \varepsilon$, then system (7) is null-controllable.*

Roughly speaking, u_1 can be driven to 0 thanks to the control h_1, u_2 can be driven to 0 thanks to the coupling term $a_{21}u_1$ (which lives somewhere in the control zone) and u_3 can be driven to 0 thanks to the coupling term $a_{32}u_2$ (which lives somewhere in the control zone). Heuristically, we have the following diagram

$$h_1 \overset{controls}{\rightsquigarrow} u_1 \overset{controls}{\rightsquigarrow} u_2 \overset{controls}{\rightsquigarrow} u_3.$$

For more general results, see [4], [3], [2] and the survey [5, Sections 4, 5, 7].

We can also replace the coupling matrix A in the system (5) by a *differential operator of first order or second order*. In this case, there exist some similar results (see [31], [10] with a technical assumption on ω, [22], [23], [24]). For example, let us consider the particular case of the 2×2 system

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = g_{11}.\nabla u_1 + g_{12}.\nabla u_2 + a_{11} u_1 + a_{12} u_2 + h_1 1_\omega & \text{in } (0,T) \times \Omega, \\ \partial_t u_2 - d_2 \Delta u_2 = g_{21}.\nabla u_1 + g_{22}.\nabla u_2 + a_{21} u_1 + a_{22} u_2 & \text{in } (0,T) \times \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0,.) = u_0 & \text{in } \Omega, \end{cases} \tag{8}$$

where $a_{ij} \in \mathbb{R}$, $g_{ij} \in \mathbb{R}$ for every $1 \leq i,j \leq 2$. Then, system (8) is null-controllable if and only if $g_{21} \neq 0$ or $a_{21} \neq 0$. This result is due to Duprez and Lissy (see [23, Theorem 1]). It is proved by a *fictitious control method* and *algebraic solvability*, introduced for the first time by Coron and Lissy in [18]. However, the situation is much more complicated and is not well-understood in the case where a_{ij}, g_{ij} $(1 \leq i,j \leq 2)$ depend on the

spatial variable. One can see the surprising negative result of null-controllability: [24, Theorem 2]. When A is a differential operator of second order (take $A = \widetilde{A}\Delta + C$ with $(\widetilde{A}, C) \in \mathcal{M}_k(\mathbb{R})^2$ to simplify), the coupling matrix A disturbs the diagonal diffusion matrix D and creates a new "crossed" diffusion matrix: $\widetilde{D} = D - \widetilde{A}$. When $\widetilde{D}$ is not diagonalizable, there are few results (see [27] with a technical assumption on the dimension of the Jordan Blocks of $\widetilde{D}$ and [25, Theorem 5.1] when ω satisfies the geometric control condition (GCC)). Let us also keep in mind the following result which help to understand our analysis.

Proposition 1.10. *([35, Theorem 3], [27, Theorem 1.5])*
Let a_{11}, a_{12}, $d \in \mathbb{R}$. Let us consider the 2×2 toy system,

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = a_{11} u_1 + a_{12} u_2 + h_1 1_\omega & \text{in } (0,T) \times \Omega, \\ \partial_t u_2 - d_2 \Delta u_2 = d \Delta u_1 & \text{in } (0,T) \times \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0,.) = u_0 & \text{in } \Omega. \end{cases} \tag{9}$$

Then, the following conditions are equivalent.
1. *System (9) is null-controllable.*
2. *$d \neq 0$.*

Roughly speaking, u_1 can be driven to 0 thanks to the control h_1 and u_2 can be driven to 0 thanks to the *coupling term of second order* $d\Delta u_1$.

Remark 1.11. When it is possible, one can diagonalize the matrix $\widetilde{D} = \begin{pmatrix} d_1 & 0 \\ d & d_2 \end{pmatrix}$. Then, by a linear transformation together with Theorem 1.6, one can prove Proposition 1.10. However, in this paper, we choose the opposite strategy. We transform (3) into a system like (9) (with four equations). Indeed, such a system seems to be a *cascade system* with coupling terms of second order.

1.4.2 Nonlinear parabolic systems

Then, another challenging issue is the study of the null-controllability properties of semilinear parabolic systems. The usual strategy consists in *linearizing the system* around 0 and to deduce local controllability properties of the nonlinear system by controllability properties of the linearized system and a fixed-point argument.

For example, let us consider the 2×2 model system,

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = f_1(u_1, u_2) + h_1 1_\omega & \text{in } (0,T) \times \Omega, \\ \partial_t u_2 - d_2 \Delta u_2 = f_2(u_1, u_2) & \text{in } (0,T) \times \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0,.) = u_0 & \text{in } \Omega., \end{cases} \tag{10}$$

where f_1 and f_2 belong to $C^\infty(\mathbb{R}^2; \mathbb{R})$. Then, the following result is a consequence of Proposition 1.7.

Proposition 1.12. *Let us suppose that $\frac{\partial f_2}{\partial u_1}(0,0) \neq 0$. Then, there exists $\delta > 0$ such that for every $u_0 \in L^\infty(\Omega)^2$ which satisfies $\|u_0\|_{L^\infty(\Omega)^2} \leq \delta$, there exists $h_1 \in L^\infty(Q)$ such that the solution u of (10) verifies $u(T,.) = 0$.*

Remark 1.13. This result is well-known but it is difficult to find it in the literature (see [1, Theorem 6] with a restriction on the dimension $1 \leq N < 6$ and other function spaces or one can adapt the arguments given in [16] to get Proposition 1.12 for any $N \in \mathbb{N}^*$). For other results in this direction, see [44], [39], [33] and [12].

When f_2 does not satisfy the hypothesis of Proposition 1.12, another strategy consists in linearizing around a non trivial trajectory $(\overline{u_1}, \overline{u_2}, \overline{h_1})$ of the nonlinear system which goes from 0 to 0. This procedure is called the *return method* and was introduced by Coron in [13] (see [14, Chapter 6]). This method conjugated with Proposition 1.8 gives the following result.

Proposition 1.14. *We assume that there exist $t_1 < t_2 \in (0,T)$, a nonempty open subset $\omega_0 \subset \omega$ and $\varepsilon > 0$ such that $|\frac{\partial f_2}{\partial u_1}(\overline{u_1}, \overline{u_2})| \geq \varepsilon$ on $(t_1, t_2) \times \omega_0$. Then, there exists $\delta > 0$ such that for every $u_0 \in L^\infty(\Omega)^2$ which satisfies $\|u_0\|_{L^\infty(\Omega)^2} \leq \delta$, there exists $h_1 \in L^\infty(Q)$ such that the solution u of (10) verifies $u(T,.) = 0$.*

Proposition 1.14 is proved in [16] and used in [16] with $f_2(u_1, u_2) = u_1^3 + Ru_2$, where $R \in \mathbb{R}$, [15] and [17].

Finally, Chaves-Silva and Guerrero have studied the local controllability of the Keller-Segel system in which the nonlinearity involves derivative terms of order 2 (see [12]). Some ideas of [12] are exploited in our proof.

1.5 Proof strategy of the two main results

Let us return to the main question discussed in this paper (see Section 1.2) and the expected results as explained in Section 1.3.

The **local controllability result** is deduced from controllability properties of the linearized system around $(u_i^*)_{1\leq i\leq 4}$ of (3). This strategy presents two main difficulties.

For the case of 3 controls (see Section 4.1.1), if $(u_1^*, u_3^*, u_4^*) \neq (0,0,0)$, the linearized system is controllable and consequently the nonlinear result comes from an adaptation of Proposition 1.12. If $(u_1^*, u_3^*, u_4^*) = (0,0,0)$, the linearized system is not controllable. Then, we use the *return method* to overcome this problem and the nonlinear result comes from an adaptation of Proposition 1.14.

For the case of 2 controls and 1 control, there exist some invariant quantities in the nonlinear system and consequently in the linearized system, that prevent controllability from happening in the whole space $L^\infty(\Omega)^4$. Therefore, we restrict the initial data to a "natural" subspace of $L^\infty(\Omega)^4$ (see Section 3.1). A modified version (for Neumann conditions) of Theorem 1.6 cannot be applied to the linearized system of (3) because the rank condition is never satisfied (due to the invariant quantities). An *adequate change of variable* gets over this difficulty by creating *crossed-diffusion* and by using coupling matrices of second order (see Section 4.1.2 and Section 4.1.3). Then, we treat the controllability properties of the linearized system by adapting Proposition 1.9 and Proposition 1.10.

To summarize, we must require necessary conditions on the initial data. Consequently the local controllability result depends on: the coefficients $(d_i)_{1\leq i\leq 4}$ (i.e. the diffusion matrix), the state $(u_i^*)_{1\leq i\leq 4}$ (i.e. the coupling matrix of the linearized system of (3)), j (i.e. the number of controls that we put in the equations).

The **global controllability result** is a corollary of our local controllability result and a result by Desvillettes, Fellner and Pierre, Suzuki, Yamada, Zou concerning the asymptotics of the trajectory of (1) for $1 \leq N \leq 2$. Indeed, this known result claims that the solution $u(T,.)$ of (3) converges in $L^\infty(\Omega)^4$ to a particular positive stationary solution z of (1) when $T \to +\infty$ (see [21] or [42, Theorem 3] and [41, Theorem 3]). Then, the solution of (3) can be exactly driven to z by our first outcome. Finally, a connectedness-compactness argument enables to steer the solution of (3) from z to $(u_i^*)_{1\leq i\leq 4}$.

2 Properties of the nonlinear controlled system

2.1 Definitions and usual properties

In this part, we introduce the concept of *trajectory* of (3). This definition requires a well-posedness result (see Proposition 2.3).

First, we introduce some usual notations.

Let $k, l \in \mathbb{N}^*$, $\mathcal{A}$ an algebra. Then, $\mathcal{M}_k(\mathcal{A})$ (respectively $\mathcal{M}_{k,l}(\mathcal{A})$) denotes the algebra of matrices with k lines and k columns with entries in $\mathcal{A}$ (respectively the algebra of matrices with k lines and l columns with entries in $\mathcal{A}$).

For $k \in \mathbb{N}^*$ and $A \in \mathcal{M}_k(\mathbb{R})$, $Sp(M)$ denotes the set of complex eigenvalues of M,

$$Sp(M) := \{\lambda \in \mathbb{C} \ ; \ \exists X \in \mathbb{C}^k \setminus \{0\}, \ MX = \lambda X\}.$$

For $(a,b,c,d) \in \mathbb{R}^4$, we introduce

$$\forall i \in \mathbb{N}^*, \ f_i(a,b,c,d) := (-1)^i(ac - bd), \qquad f(a,b,c,d) = (f_i(a,b,c,d))_{1\leq i\leq 4}. \tag{11}$$

Definition 2.1. We introduce the space Y defined by

$$Y := L^2(0,T; H^1(\Omega)) \cap W^{1,2}(0,T; (H^1(\Omega))'). \tag{12}$$

Proposition 2.2. *From an easy adaptation of the proof of [26, Section 5.9.2, Theorem 3], we have*

$$Y \hookrightarrow C([0,T]; L^2(\Omega)). \tag{13}$$

Proposition 2.3. *Let $k \in \mathbb{N}^*$, $D \in \mathcal{M}_k(\mathbb{R})$ such that D is diagonalizable and $Sp(D) \subset (0,+\infty)$, $A \in \mathcal{M}_k(L^\infty(Q))$, $u_0 \in L^2(\Omega)^k$, $g \in L^2(Q)^k$. The following Cauchy problem admits a unique weak solution $u \in Y^k$*

$$\begin{cases} \partial_t u - D\Delta u = A(t,x)u + g & \text{in } (0,T)\times\Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,.) = u_0 & \text{in } \Omega. \end{cases}$$

This means that u is the unique function in Y^k that satisfies the variational fomulation

$$\forall w \in L^2(0,T;H^1(\Omega)^k),\ \int_0^T (\partial_t u, w)_{(H^1(\Omega)^k)',H^1(\Omega)^k)} + \int_Q D\nabla u.\nabla w = \int_Q (Au+g).w, \tag{14}$$

and

$$u(0,.) = u_0 \text{ in } L^2(\Omega)^k. \tag{15}$$

Moreover, there exists $C>0$ independent of u_0 and g such that

$$\|u\|_{Y^k} \leq C\left(\|u_0\|_{L^2(\Omega)^k} + \|g\|_{L^2(Q)^k}\right). \tag{16}$$

Finally, if $u_0 \in L^\infty(\Omega)^k$ and $g \in L^\infty(Q)^k$, then $u \in L^\infty(Q)^k$ and there exists $C>0$ independent of u_0 and g such that

$$\|u\|_{(Y\cap L^\infty(Q))^k} \leq C\left(\|u_0\|_{L^\infty(\Omega)^k} + \|g\|_{L^\infty(Q)^k}\right). \tag{17}$$

Remark 2.4. This proposition is more or less classical, but we could not find it as such in the literature and we give its proof in the Appendix (see Appendix A.1).

Definition 2.5. For $u_0 \in L^\infty(\Omega)^4$, $((u_i)_{1\leq i\leq 4},(h_i)_{1\leq i\leq j})$ is a trajectory of (3) if
1. $((u_i)_{1\leq i\leq 4},(h_i)_{1\leq i\leq j}) \in (Y\cap L^\infty(Q))^4 \times L^\infty(Q)^j$,
2. $(u_i)_{1\leq i\leq 4}$ is the (unique) solution of (3).

Moreover, $((u_i)_{1\leq i\leq 4},(h_i)_{1\leq i\leq j})$ is a trajectory of (3) reaching $(u_i^*)_{1\leq i\leq 4}$ if

$$\forall i \in \{1,\dots,4\},\ u_i(T,.) = u_i^*.$$

Remark 2.6. The concept of solution of (3) is the same as in Proposition 2.3 (take $D = diag(d_1,d_2,d_3,d_4)$, $A=0$ and $g = (g_i(u))_{1\leq i\leq 4}^T$ where $g_i(u) = f_i(u) + h_i 1_{i\leq j} 1_\omega$).

Remark 2.7. The uniqueness is a consequence of the following estimate.

Let $D = diag(d_1,d_2,d_3,d_4)$, $(h_i)_{1\leq i\leq j} \in L^\infty(Q)^j$, $u = (u_i)_{1\leq i\leq 4} \in (Y\cap L^\infty(Q))^4$, $\widetilde{u} = (\widetilde{u_i})_{1\leq i\leq 4} \in (Y\cap L^\infty(Q))^4$ be two solutions of (3), and $v = u - \widetilde{u}$. The function v satisfies (in the weak sense)

$$\begin{cases} \partial_t v - D\Delta v = f(u) - f(\widetilde{u}) & \text{in } (0,T)\times\Omega, \\ \frac{\partial v}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ v(0,.) = 0 & \text{in } \Omega. \end{cases} \tag{18}$$

By taking $w := v$ in the variational formulation of (18) (see also (14)) and by using the fact that the mapping $t \mapsto \|v(t)\|^2_{L^2(\Omega)^4}$ is absolutely continuous with $\frac{d}{dt}\|v(t)\|^2_{L^2(\Omega)^4} = 2(\partial_t v(t), v(t))_{(H^1(\Omega)^4)',H^1(\Omega)^4}$ for a.e. $0\leq t\leq T$ (see [26, Section 5.9.2, Theorem 3]), we find that

$$\frac{1}{2}\frac{d}{dt}\left(\|v\|^2_{L^2(\Omega)^4}\right) + \|D\nabla v\|^2_{L^2(\Omega)^4} = (f(u)-f(\widetilde{u}),v)_{L^2(\Omega)^4,L^2(\Omega)^4},\ \text{for a.e. } 0\leq t\leq T. \tag{19}$$

By using the facts that $(u,\widetilde{u}) \in L^\infty(Q)^4 \times L^\infty(Q)^4$, f is locally Lipschitz continuous on $\mathbb{R}^4$, we find the differential inequality

$$\frac{d}{dt}\left(\|v\|^2_{L^2(\Omega)^4}\right) \leq C\|v\|^2_{L^2(\Omega)^4},\ \text{for a.e. } 0\leq t\leq T. \tag{20}$$

Gronwall's lemma and the initial condition $v(0,.)=0$ prove that $v=0$ in $L^2(Q)^4$. Consequently, $u=\widetilde{u}$.

2.2 Invariant quantities of the nonlinear dynamics

In this section, we show that in the system (3), some invariant quantities exist. They impose some restrictions on the initial condition, for the controllability results.

2.2.1 Variation of the mass

Proposition 2.8. *Let $j\in\{1,2,3\}$, $u_0 \in L^\infty(\Omega)^4$, $((u_i)_{1\leq i\leq 4},(h_i)_{1\leq i\leq j})$ be a trajectory of (3). For every $1\leq i\leq 4$, the mapping $t\mapsto \int_\Omega u_i(t,x)dx$ is absolutely continuous with*

$$\frac{d}{dt}\int_\Omega u_i(t,x)dx = \int_\Omega \Big\{f_i(u_1(t,x),u_2(t,x),u_3(t,x),u_4(t,x)) + h_i(t,x)1_\omega(x)1_{i\leq j}\Big\}dx,\ \textit{for a.e. } 0\leq t\leq T. \tag{21}$$

Proof. We fix $1 \leq i \leq 4$. By using the fact that $u_i \in Y$ and from an easy adaptation of [26, Section 5.9.2, Theorem 3, (ii)], we deduce that the mapping $t \mapsto \int_\Omega u_i(t,x)dx$ is absolutely continuous and for a.e. $0 \leq t \leq T$,

$$\frac{d}{dt}\int_\Omega u_i(t,x)dx = (\partial_t u_i(t,.),1)_{(H^1(\Omega))',H^1(\Omega)}.$$

Then, by using that $((u_i)_{1\leq i\leq 4},(h_i)_{1\leq i\leq j})$ is the (unique) solution of (3) and by taking $w=1$ in (14), we find that for a.e. $0 \leq t \leq T$,

$$\begin{aligned}
&(\partial_t u_i(t,.),1)_{(H^1(\Omega))',H^1(\Omega)}\\
&= d_i(\nabla u_i(t,.),\nabla 1)_{L^2(\Omega),L^2(\Omega)} + \int_\Omega \Big\{ f_i(u_1(t,x),u_2(t,x),u_3(t,x),u_4(t,x)) + h_i(t,x)1_\omega(x)1_{i\leq j}\Big\}dx\\
&= \int_\Omega \Big\{ f_i(u_1(t,x),u_2(t,x),u_3(t,x),u_4(t,x)) + h_i(t,x)1_\omega(x)1_{i\leq j}\Big\}dx.
\end{aligned}$$

□

2.2.2 Case of 2 controls

Proposition 2.9. *Let $j=2$, $u_0 \in L^\infty(\Omega)^4$, $((u_i)_{1\leq i\leq 4},(h_i)_{1\leq i\leq 2})$ be a trajectory of (3) reaching $(u_i^*)_{1\leq i\leq 4}$. Then, we have*

$$\frac{1}{|\Omega|}\int_\Omega \Big(u_{3,0}(x)+u_{4,0}(x)\Big)dx = u_3^* + u_4^*, \tag{22}$$

$$\Big(d_3 = d_4\Big) \Rightarrow \Big(u_{3,0}+u_{4,0} = u_3^* + u_4^*\Big). \tag{23}$$

Proof. From (21), we have

$$\frac{d}{dt}\left(\int_\Omega (u_3(t,x)+u_4(t,x))dx\right) = 0 \text{ for a.e. } 0 \leq t \leq T.$$

Then, from Definition 2.5, (22) holds.

Moreover, u_3+u_4 satisfies

$$\begin{cases} \partial_t(u_3+u_4) - d_4\Delta(u_3+u_4) = (d_3-d_4)\Delta u_3 & \text{in } (0,T)\times\Omega,\\ \frac{\partial(u_3+u_4)}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases}$$

If $d_3=d_4$, then the backward uniqueness for the heat equation (which is a corollary of Lemma 2.11) proves that

$$\forall t\in[0,T],\ (u_3+u_4)(t,.) = (u_3+u_4)(T,.) = u_3^*+u_4^*. \tag{24}$$

This implies the necessary condition (23), stronger than (22), on the initial condition. □

2.2.3 Case of 1 control

Proposition 2.10. *Let $j=1$, $u_0 \in L^\infty(\Omega)^4$, $((u_i)_{1\leq i\leq 4},(h_i)_{1\leq i\leq 2})$ be a trajectory of (3) reaching $(u_i^*)_{1\leq i\leq 4}$. Then, we have*

$$\frac{1}{|\Omega|}\int_\Omega \Big(u_{2,0}(x)+u_{3,0}(x)\Big)dx = u_2^*+u_3^*,\ \frac{1}{|\Omega|}\int_\Omega \Big(u_{3,0}(x)+u_{4,0}(x)\Big)dx = u_3^*+u_4^*, \tag{25}$$

$$\Big(k\neq l\in\{2,3,4\},\ d_k=d_l\Big) \Rightarrow \Big(u_{k,0}-(-1)^{k-l}u_{l,0} = u_k^* - (-1)^{k-l}u_l^*\Big). \tag{26}$$

Proof. From (21), we have

$$\frac{d}{dt}\left(\frac{1}{|\Omega|}\int_\Omega (u_2(t,x)+u_3(t,x))dx\right) = 0,\ \frac{d}{dt}\left(\frac{1}{|\Omega|}\int_\Omega (u_3(t,x)+u_4(t,x))dx\right) = 0 \text{ for a.e. } 0\leq t\leq T.$$

Then, from Definition 2.5, (25) holds.

Moreover, if there exists $k\neq l\in\{2,3,4\}$ such that $d_k=d_l$, by using again the backward uniqueness for the heat equation, we get

$$\Big(k\neq l\in\{2,3,4\},\ d_k=d_l\Big) \Rightarrow \Big(\forall t\in[0,T],\ (u_k-(-1)^{k-l}u_l)(t,.) = (u_k-(-1)^{k-l}u_l)(T,.) = u_k^*-(-1)^{k-l}u_l^*\Big), \tag{27}$$

and in particular the necessary condition (26), stronger than (25), on the initial condition. □

2.3 More restrictive conditions on the initial condition when the target $(u_i^*)_{1\leq i\leq 4}$ vanishes

In the previous section, we have seen that there are invariant quantities in the dynamics of (3) which impose necessary conditions on the initial condition: (22), (25). Moreover, when some coefficients of diffusion d_i are equal, we have more invariant quantities in (3) which impose stronger necessary conditions on the initial condition: (23), (26).

2.3.1 The lemma of backward uniqueness

Lemma 2.11. ***Backward uniqueness***
Let $k \in \mathbb{N}^*$, $D = diag(d_1, \dots, d_k)$ where $d_i \in (0, +\infty)$, $C \in \mathcal{M}_k(L^\infty(Q))$, $\zeta_0 \in L^\infty(\Omega)^k$. Let $\zeta \in Y^k$ be the solution of

$$\begin{cases} \partial_t \zeta - D\Delta\zeta = C(t,x)\zeta & \text{in } (0,T)\times\Omega, \\ \frac{\partial \zeta}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \zeta(0,.) = \zeta_0 & \text{in } \Omega. \end{cases}$$

If $\zeta(T,.) = 0$, then for every $t \in [0,T]$, $\zeta(t,.) = 0$.

Proof. $\widetilde{\zeta}(t,x) = \exp(-t)\zeta(t,x) \in Y^k$ is the solution of the system

$$\begin{cases} \partial_t \widetilde{\zeta} - D\Delta\widetilde{\zeta} + I_k\widetilde{\zeta} = C(t,x)\widetilde{\zeta} & \text{in } (0,T)\times\Omega, \\ \frac{\partial \widetilde{\zeta}}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \zeta(0,.) = \widetilde{\zeta}_0 & \text{in } \Omega, \end{cases}$$

which verifies $\widetilde{\zeta}(T,.) = 0$.

Let us denote $A = -D\Delta + I_k$ which is a bounded linear operator from $H^1(\Omega)^k$ to $(H^1(\Omega)^k)'$. Indeed,

$$\forall (u,v) \in (H^1(\Omega)^k)^2, (Au)(v) = \sum_{i=1}^k d_i(\nabla u_i, \nabla v_i)_{L^2(\Omega),L^2(\Omega)} + \sum_{i=1}^k (u_i, v_i)_{L^2(\Omega),L^2(\Omega)},$$

$$\|Au\|_{(H^1(\Omega)^k)'} \leq \sqrt{\max(d_i)}\, \|u\|_{H^1(\Omega)^k}.$$

Then, A verifies the three hypotheses: (i), (ii) and (iii) of [9, Proposition II.1].

(i) is satisfied because A does not depend on t.

(ii) is a consequence of

$$\forall (u,v) \in (H^1(\Omega)^k)^2, (Au)(v) = (Av)(u).$$

(iii) is satisfied because

$$(Au,u) = \sum_{i=1}^k d_i(\nabla u_i, \nabla u_i)_{L^2(\Omega),L^2(\Omega)} + \sum_{i=1}^k (u_i,u_i)_{L^2(\Omega),L^2(\Omega)} \geq \min(\min_i(d_i), 1)\, \|u\|^2_{H^1(\Omega)^k}.$$

Let $B(t)$ be the family of operators in $L^2(0,T;\mathcal{L}(H^1(\Omega)^k, L^2(\Omega)^k))$ defined by

$$\forall u \in H^1(\Omega)^k,\ B(t)u(.) = C(t,.)u(.).$$

We have

$$\|B\|^2_{L^2(0,T;\mathcal{L}(H^1(\Omega)^k,L^2(\Omega)^k))} \leq \|C\|^2_{L^\infty(Q)^{k^2}}.$$

By applying [9, Theorem II.1], we get that for every $t \in [0,T]$, $\widetilde{\zeta}(t,.) = 0$. Then,

$$\forall t \in [0,T],\ \zeta(t,.) = 0.$$

□

2.3.2 Case of 2 controls

Proposition 2.12. *Let $j = 2$, $u_0 \in L^\infty(\Omega)^4$. If $((u_i)_{1\leq i\leq 4}, (h_i)_{1\leq i\leq 2})$ is a trajectory of (3) reaching $(u_i^*)_{1\leq i\leq 4}$, then we have*

$$\Big((u_3^*, u_4^*) = (0,0)\Big) \Rightarrow \Big((u_{3,0}, u_{4,0}) = (0,0)\Big). \tag{28}$$

Conversely, for every $u_0 \in L^\infty(\Omega)^4$ such that $(u_{3,0}, u_{4,0}) = (0,0)$, we can find $(h_i)_{1\leq i\leq 2} \in L^\infty(Q)^2$ such that the associated solution $(u_i)_{1\leq i\leq 4} \in L^\infty(Q)^4$ of (3) satisfies $(u_1,u_2,u_3,u_4)(T,.) = (u_1^*, u_2^*, 0, 0)$.

Proof. If $(u_3^*, u_4^*) = (0,0)$, it results from (3) that

$$\begin{cases} \partial_t u_3 - d_3\Delta u_3 = -u_1u_3 + u_2u_4 & \text{in } (0,T)\times\Omega, \\ \partial_t u_4 - d_4\Delta u_4 = u_1u_3 - u_2u_4 & \text{in } (0,T)\times\Omega, \\ \frac{\partial u_3}{\partial n} = \frac{\partial u_4}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{29}$$

By using the point 1 of Definition 2.5, we have

$$(u_1, u_2) \in L^\infty(Q)^2. \tag{30}$$

Then, from (29), (30), Definition 2.5: $(u_3, u_4)(T,.) = (0,0)$ and Lemma 2.11 with $k = 2$, $D = diag(d_3, d_4)$ and $C = \begin{pmatrix} -u_1 & u_2 \\ u_1 & -u_2 \end{pmatrix}$, we deduce that

$$\forall t \in [0,T],\ (u_3, u_4)(t,.) = (0,0),$$

and in particular (28).

Conversely, let $u_0 \in L^\infty(\Omega)^4$ be such that $(u_{3,0}, u_{4,0}) = (0,0)$. Then, (3) reduces to the following system

$$\begin{cases} \partial_t u_1 - d_1\Delta u_1 = h_1 1_\omega & \text{in } (0,T)\times\Omega, \\ \partial_t u_2 - d_2\Delta u_2 = h_2 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ (u_1, u_2)(0,.) = (u_{1,0}, u_{2,0}) & \text{in } \Omega. \end{cases} \tag{31}$$

The problem reduces to the null-controllability of two decoupled heat equations in $L^\infty(\Omega)$ with two localized control in $L^\infty(Q)$ which is a solved problem (see for example [28, Proposition 1]). Therefore, we can find $(h_i)_{1\le i\le 2} \in L^\infty(Q)^2$ such that the associated solution $(u_i)_{1\le i\le 4} \in L^\infty(Q)^4$ of (3) satisfies $(u_1, u_2, u_3, u_4)(T,.) = (u_1^*, u_2^*, 0, 0)$. □

Remark 2.13. Thanks to Proposition 2.12, we avoid the easy case $(u_3^*, u_4^*) = (0,0)$ for 2 controls in the sequel.

2.3.3 Case of 1 control

Proposition 2.14. *Let $j = 1$, $u_0 \in L^\infty(\Omega)^4$. If $((u_i)_{1\le i\le 4}, h_1)$ is a trajectory of (3) reaching $(u_i^*)_{1\le i\le 4}$, then we have*

$$\Big((u_3^*, u_2^*) = (0,0)\Big) \Rightarrow \Big((u_{2,0}, u_{3,0}, u_{4,0}) = (0,0,u_4^*)\Big), \tag{32}$$

$$\Big((u_3^*, u_4^*) = (0,0)\Big) \Rightarrow \Big((u_{2,0}, u_{3,0}, u_{4,0}) = (u_2^*,0,0)\Big). \tag{33}$$

Conversely, for every $u_0 \in L^\infty(\Omega)^4$ such that $u_{3,0} = 0$, we can find $h_1 \in L^\infty(Q)$ such that the associated solution $(u_i)_{1\le i\le 4} \in L^\infty(Q)^4$ of (3) satisfies $(u_1, u_2, u_3, u_4)(T,.) = (u_1^*, u_2^*, 0, u_4^*)$.

Proof. If $u_3^* = 0$, then from (2), $u_2^* = 0$ or $u_4^* = 0$. We assume that $(u_3^*, u_2^*) = (0,0)$ (the other case is similar). The backward uniqueness (i.e. Lemma 2.11) as in Section 2.3.2 leads to

$$\forall t \in [0,T],\ (u_3, u_2)(t,.) = (0,0).$$

Then, we deduce that

$$\begin{cases} \partial_t u_4 - d_4\Delta u_4 = 0 & \text{in } (0,T)\times\Omega, \\ \frac{\partial u_4}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{34}$$

The backward uniqueness for the heat equation applied to (34) proves that

$$\forall t \in [0,T],\ u_4(t,.) = u_4^*,$$

and in particular (32) and (33).

Conversely, let $u_0 \in L^\infty(\Omega)^4$ such that $u_{3,0} = 0$. Then, (3) reduces to the following system

$$\begin{cases} \partial_t u_1 - d_1\Delta u_1 = h_1 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial u_1}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ u_1(0,.) = u_{1,0} & \text{in } \Omega. \end{cases} \tag{35}$$

The problem reduces to the null-controllability of the heat equation in $L^\infty(\Omega)$ with a localized control in $L^\infty(Q)$ which is a solved problem (see for example [28, Proposition 1]). Therefore, we can find $h_1 \in L^\infty(Q)$ such that the associated solution $(u_i)_{1\le i\le 4} \in L^\infty(Q)^4$ of (3) satisfies $(u_1, u_2, u_3, u_4)(T,.) = (u_1^*, u_2^*, 0, u_4^*)$. □

Remark 2.15. Thanks to Proposition 2.14, we avoid the easy case $u_3^* = 0$ for 1 control in the sequel.

3 Main results

In this part, we present our two main results: a local controllability result and a large-time global controllabillity result for (3).

3.1 Local controllability under constraints

In Section 2.2 and Section 2.3, we have highlighted necessary conditions on initial conditions when $((u_i)_{1\leq i\leq 4},(h_i)_{1\leq i\leq j})$ is a trajectory reaching $(u_i^*)_{1\leq i\leq 4}$. They turn out to be sufficient for the existence of such trajectories at least for data close to $(u_i^*)_{1\leq i\leq 4}$. The goal of this subsection is to define subspaces of $L^\infty(\Omega)^4$ which take care of these conditions.

3.1.1 Case of 3 controls

We introduce

$$X_{3,(d_i),(u_i^*)} = L^\infty(\Omega)^4. \tag{36}$$

3.1.2 Case of 2 controls

The results of Section 2.2.2 and Section 2.3.2 are summed up in the following array.

	$(u_3^*,u_4^*)\neq(0,0)$
$d_3=d_4$	$u_{3,0}+u_{4,0}=u_3^*+u_4^*$
$d_3\neq d_4$	$\frac{1}{\lvert\Omega\rvert}\int_\Omega(u_{3,0}+u_{4,0})=u_3^*+u_4^*$

(37)

Then, we introduce

$$X_{2,(d_i),(u_i^*)} := \{u_0\in L^\infty(\Omega)^4\ ;\ u_0 \text{ satisfies the associated condition of (37)}\}. \tag{38}$$

For example, $X_{2,(1,2,3,4),(1,1,1,1)} = \{u_0\in L^\infty(\Omega)^4; \frac{1}{\lvert\Omega\rvert}\int_\Omega (u_{3,0}+u_{4,0})=2\}$.

3.1.3 Case of 1 control

The results of Section 2.2.3 and Section 2.3.3 are summed up in the following array.

	$u_3^*\neq 0$
$d_2=d_3=d_4$	$u_{2,0}+u_{3,0}=u_2^*+u_3^*,\ u_{3,0}+u_{4,0}=u_3^*+u_4^*$
$d_2\neq d_3,\ d_3=d_4$	$\frac{1}{\lvert\Omega\rvert}\int_\Omega(u_{2,0}+u_{3,0})=u_2^*+u_3^*,\ u_{3,0}+u_{4,0}=u_3^*+u_4^*$
$d_2=d_3, d_3\neq d_4$	$u_{2,0}+u_{3,0}=u_2^*+u_3^*,\ \frac{1}{\lvert\Omega\rvert}\int_\Omega(u_{3,0}+u_{4,0})=u_3^*+u_4^*$
$d_2=d_4,\ d_2\neq d_3$	$u_{2,0}-u_{4,0}=u_2^*-u_4^*,\ \frac{1}{\lvert\Omega\rvert}\int_\Omega(u_{3,0}+u_{4,0})=u_3^*+u_4^*$
$d_2\neq d_3,\ d_3\neq d_4,\ d_2\neq d_4$	$\frac{1}{\lvert\Omega\rvert}\int_\Omega(u_{2,0}+u_{3,0})=u_2^*+u_3^*,\ \frac{1}{\lvert\Omega\rvert}\int_\Omega(u_{3,0}+u_{4,0})=u_3^*+u_4^*$

(39)

Then, we introduce

$$X_{1,(d_i),(u_i^*)} := \{u_0\in L^\infty(\Omega)^4\ ;\ u_0 \text{ satisfies the associated condition of (39)}\}. \tag{40}$$

3.1.4 Local controllability result

Definition 3.1. Let $j\in\{1,2,3\}$, $(u_1^*,u_2^*,u_3^*,u_4^*)\in(\mathbb{R}^+)^4$ be such that (2) holds. The system (3) is **locally controllable to the state** $(u_i^*)_{1\leq i\leq 4}$ **in** $L^\infty(\Omega)^4$ **with controls in** $L^\infty(Q)^j$ if there exists $\delta>0$ such that for every $u_0\in X_{j,(d_i),(u_i^*)}$ which verifies $\|u_0-(u_i^*)_{1\leq i\leq 4}\|_{L^\infty(\Omega)^4}\leq\delta$, there exists $(h_i)_{1\leq i\leq j}\in L^\infty(Q)^j$ such that the solution $(u_i)_{1\leq i\leq 4}\in L^\infty(Q)^4$ to the Cauchy problem (3) satisfies

$$\forall i\in\{1,2,3,4\}, u_i(T,.)=u_i^*.$$

Theorem 3.2. *For every $j\in\{1,2,3\}$, for every $(u_1^*,u_2^*,u_3^*,u_4^*)\in(\mathbb{R}^+)^4$ which satisfies (2), the system (3) is* ***locally controllable to the state*** $(u_i^*)_{1\leq i\leq 4}$ ***in*** $L^\infty(\Omega)^4$ ***with controls in*** $L^\infty(Q)^j$.

Remark 3.3. The uniqueness of the solution $(u_i)_{1\leq i\leq 4}\in L^\infty(Q)^4$ is a consequence of Remark 2.7. The existence of the solution $(u_i)_{1\leq i\leq 4}\in L^\infty(Q)^4$ is a consequence of a good choice of controls $(h_i)_{1\leq i\leq j}\in L^\infty(Q)^j$ and more precisely of a fixed-point argument (see Section 4.5).

Remark 3.4. As we have said in the introduction, it was not known if L^∞ blow-up occurs or not in dimension $N>2$ for the free system (1) until recently (see [19]). Here, our strategy of control avoids blow-up and enables the solution to reach a stationary solution of (1).

Remark 3.5. In some particular cases (easy cases), this local controllability result can be improved in a global controllability result (see the case $(u_3^*, u_4^*) = (0,0)$ for 2 controls in Section 2.3.2 and the case $u_3^* = 0$ for 1 control in Section 2.3.3).

3.2 Large-time global controllability result

From Theorem 3.2, we establish a global controllability result in large time for $N = 1, 2$.

Theorem 3.6. *We assume that $N = 1$ or 2. Let $j \in \{1,2,3\}$ and $(u_i^*)_{1\leq i\leq 4} \in (\mathbb{R}^+)^4$ be such that (2) holds. Then, for every $u_0 \in X_{j,(d_i),(u_i^*)}$ satisfying*

$$\begin{array}{l} \forall 1 \leq i \leq 4,\ u_{i0} \geq 0, \\ \dfrac{1}{|\Omega|}\displaystyle\int_\Omega (u_{1,0}+u_{2,0}) > 0,\ \dfrac{1}{|\Omega|}\int_\Omega (u_{1,0}+u_{4,0}) > 0,\ \dfrac{1}{|\Omega|}\int_\Omega (u_{2,0}+u_{3,0}) > 0,\ \dfrac{1}{|\Omega|}\int_\Omega (u_{3,0}+u_{4,0}) > 0, \end{array} \tag{41}$$

there exists $T^* > 0$ (sufficiently large) and $(h_i)_{1\leq i\leq j} \in L^\infty((0,T^*)\times\Omega)^j$ such that the solution u of

$$\forall 1 \leq i \leq 4,\ \left\{ \begin{array}{ll} \partial_t u_i - d_i \Delta u_i = (-1)^i (u_1 u_3 - u_2 u_4) + h_i 1_\omega 1_{i\leq j} & \text{in } (0,T^*)\times\Omega, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } (0,T^*)\times\partial\Omega, \\ u_i(0,.) = u_{i,0} & \text{in } \Omega. \end{array} \right. \tag{42}$$

satisfies

$$u(T^*,.) = u^*. \tag{43}$$

Remark 3.7. The restriction on the dimension $N \in \{1,2\}$ is a consequence of the following property: the solution of the free system (1) converges in $L^\infty(\Omega)$ when $T \to +\infty$ to a particular stationary solution of (1) (see [21]). One can extend Theorem 3.6 to $N > 2$ if the convergence in $L^\infty(\Omega)$ (of the free system) holds. For $N > 2$, one only knows that a weak solution of the free system (1) converges in $L^1(\Omega)$ when $T \to +\infty$ to a particular stationary solution of (1) (see [42, Theorem 3]). But, for example, if we assume that *the diffusion coefficients d_i are close*, the weak solution of the free system (1) converges in $L^\infty(\Omega)$ when $T \to +\infty$ to a particular stationary solution of (1) (see [11, Proposition 1.3]).

Remark 3.8. The positivity assumption (41) is not restrictive. One can extend the result to nonnegative initial condition $u_0 \in X_{j,(d_i),(u_i^*)}$ (see [42, Section 5]).

4 Proof of Theorem 3.2: the local controllability to constant stationary states

The aim of this section is to prove Theorem 3.2. As usual, we study the properties of controllability of the **linearized system** around $(u_i^*)_{1\leq i\leq 4}$ of (3). First, we transform the problem by studying the null-controllability of a family of linear control systems (see Section 4.1). The **existence of controls in $L^2(Q)$** is a consequence of a duality method: the **Hilbert Uniqueness Method** introduced by Jacques-Louis Lions (see Section 4.3.1). It links the existence of controls in $L^2(Q)$ with an **observability inequality** for solution of the adjoint system. This type of inequalities is proved by **Carleman estimates** (see Section 4.3.2). In order to get more regular controls (in $L^p(Q)$ sense, $p \geq 2$), we use a sophistication of Hilbert Uniqueness Method called the **penalized Hilbert Uniqueness Method** introduced by Barbu (see Section 4.4.1). Indeed, this enables to have controls a bit better than $L^2(Q)$. Then, a **bootstrap method** gives controls in $L^\infty(Q)$ (see Section 4.4.2). A **fixed-point argument** concludes the proof (see Section 4.5).

Now, we develop a strategy in order to treat the cases of 1, 2 or 3 controls in a unified way.
We introduce the following notations

$$B_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ h^3 = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ 0 \end{pmatrix},\ B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},\ h^2 = \begin{pmatrix} h_1 \\ h_2 \\ 0 \\ 0 \end{pmatrix},\ B_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ h^1 = \begin{pmatrix} h_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{44}$$

Let $j \in \{1,2,3\}$, $(u_1^*, u_2^*, u_3^*, u_4^*) \in (\mathbb{R}^+)^4$ be such that (2) holds and $u_0 \in X_{j,(d_i),(u_i^*)}$ (see (36), (38) and (40)).

4.1 Linearization

We adopt the approach presented in Section 1.4.2.

4.1.1 3 controls, return method when $(u_1^*, u_3^*, u_4^*) = (0,0,0)$

We linearize (3) around $(u_i^*)_{1\leq i\leq 4}$ and we get the system,

$$\forall 1\leq i\leq 4, \begin{cases} \partial_t u_i - d_i\Delta u_i = (-1)^i(u_3^*u_1 - u_4^*u_2 + u_1^*u_3 - u_2^*u_4) + h_i 1_\omega 1_{i\leq 3} & \text{in } (0,T)\times\Omega, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ u_i(0,.) = u_{i,0} & \text{in } \Omega. \end{cases} \tag{45}$$

Roughly speaking, it is easy to control u_1, u_2, u_3 thanks to h_1, h_2, h_3. The main difficulty is to control u_4. Now, we present the heuristically way of controlling u_4.

4.1.1.1 First case: $(u_1^*, u_3^*, u_4^*) \neq (0,0,0)$. There is a coupling term in the fourth equation of (45) which enables to control u_4. For example, if $u_3^* \neq 0$, then u_1 controls u_4.

Remark 4.1. In this case, the linearized system (45) looks like the toy-model (6) and its controllability properties come from Proposition 1.7. Consequently, the local controllability of (3) can be proved as Proposition 1.12 for system (10).

4.1.1.2 Second case: $(u_1^*, u_3^*, u_4^*) = (0,0,0)$, return method. The fourth equation of (45) is decoupled from the other equations. In particular, if $u_4(0,.) \neq 0$, then $u_4(T,.) \neq 0$. Consequently, system (45) is not controllable. The idea is to linearize around a non trivial trajectory of (3) which comes from $(0, u_2^*, 0, 0)$ and goes to $(0, u_2^*, 0, 0)$ and which forces the appearance of a coupling term after linearization. It is the **return method**. Here, we take $\big((0, u_2^*, \overline{u_3}^\sharp, 0), (0,0,\overline{h_3}^\sharp)\big) := \big((0, u_2^*, g, 0), (0,0,\partial_t g - d_3\Delta g)\big)$ where g satisfies the following properties

$$g \in C^\infty(\overline{Q}),\ g \geq 0,\ g \neq 0,\ supp(g) \subset (0,T)\times\omega. \tag{46}$$

Then, if we linearize the system (3) around $\big((0, u_2^*, \overline{u_3}^\sharp, 0), (0,0,\overline{h_3}^\sharp)\big)$, then the fourth equation becomes

$$\partial_t u_4 - d_4\Delta u_4 = \overline{u_3}^\sharp u_1 - u_2^* u_4 \text{ in } (0,T)\times\Omega.$$

Roughly speaking, as $\overline{u_3}^\sharp \neq 0$ in the control zone, then u_1 controls u_4.

Remark 4.2. Here, the linearized system around the non trivial trajectory looks like the toy-model (6) and its controllability properties follow from Proposition 1.8. Consequently, the local controllability of (3) can be proved as Proposition 1.14 for (10).

4.1.1.3 Linearization in $L^\infty(Q)$ and null-controllability of a family of linear systems We define

$$\overline{u_3} := \begin{cases} u_3^* & \text{if } (u_1^*, u_3^*, u_4^*) \neq (0,0,0), \\ \overline{u_3}^\sharp & \text{if } (u_1^*, u_3^*, u_4^*) = (0,0,0), \end{cases} \quad \text{and } \overline{h_3} := \begin{cases} 0 & \text{if } (u_1^*, u_3^*, u_4^*) \neq (0,0,0), \\ \overline{h_3}^\sharp & \text{if } (u_1^*, u_3^*, u_4^*) = (0,0,0), \end{cases} \tag{47}$$

$$(\zeta, \widehat{h^3}) := (\zeta_1, \zeta_2, \zeta_3, \zeta_4, \widehat{h_1}, \widehat{h_2}, \widehat{h_3}) := (u_1 - u_1^*, u_2 - u_2^*, u_3 - \overline{u_3}, u_4 - u_4^*, h_1, h_2, h_3 - \overline{h_3}). \tag{48}$$

Then, (u, h^3) is a trajectory of (3) if and only if $(\zeta, \widehat{h^3})$ is a trajectory of the following system

$$\forall 1\leq i\leq 4,$$
$$\begin{cases} \partial_t\zeta_i - d_i\Delta\zeta_i = (-1)^i((\overline{u_3}+\zeta_3)\zeta_1 - (u_4^*+\zeta_4)\zeta_2 + u_1^*\zeta_3 - u_2^*\zeta_4) + \widehat{h_i} 1_\omega 1_{i\leq 3} & \text{in } (0,T)\times\Omega, \\ \frac{\partial\zeta_i}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \zeta_i(0,.) = u_{i,0} - u_i^* & \text{in } \Omega. \end{cases} \tag{49}$$

Then $(\zeta, \widehat{h^3})$ is a trajectory of

$$\begin{cases} \partial_t\zeta - D_3\Delta\zeta = G(\zeta)\zeta + B_3\widehat{h^3}1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial\zeta}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \zeta(0,.) = \zeta_0 & \text{in } \Omega, \end{cases} \tag{50}$$

where

$$D_3 := \begin{pmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{pmatrix}, \qquad G(\zeta) := \begin{pmatrix} -\overline{u_3}-\zeta_3 & u_4^*+\zeta_4 & -u_1^* & u_2^* \\ \overline{u_3}+\zeta_3 & -u_4^*-\zeta_4 & u_1^* & -u_2^* \\ -\overline{u_3}-\zeta_3 & u_4^*+\zeta_4 & -u_1^* & u_2^* \\ \overline{u_3}+\zeta_3 & -u_4^*-\zeta_4 & u_1^* & -u_2^* \end{pmatrix}. \tag{51}$$

Note that $G_{41}(0,0,0,0) = \overline{u_3}$. To simplify, we suppose the following fact: if $(u_1^*, u_3^*, u_4^*) \neq (0,0,0)$, then $u_3^* \neq 0$. Otherwise, we can easily adapt our proof strategy (see Remark 4.16). Then, from (46), there exist $t_1 < t_2 \in (0,T)$, a nonempty open subset $\omega_0 \subset\subset \omega$ and $M > 0$ such that

$$\forall (t,x) \in (t_1,t_2) \times \omega_0,\ G_{41}(0,0,0,0)(t,x) \geq 2/M,$$

$$\forall (k,l) \in \{1,\dots,4\}^2,\ \|G_{kl}(0,0,0,0)\|_{L^\infty(Q)} \leq M/2.$$

Consequently, we study the null-controllability of the linear systems

$$\begin{cases} \partial_t \zeta - D_3 \Delta \zeta = A\zeta + B_3 \widehat{h^3} 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial \zeta}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \zeta(0,.) = \zeta_0 & \text{in } \Omega, \end{cases} \tag{52}$$

where the matrix A verifies the following assumptions

$$\forall (t,x) \in (t_1,t_2) \times \omega_0,\ a_{41}(t,x) \geq 1/M, \tag{53}$$

$$\forall (k,l) \in \{1,\dots,4\}^2,\ \|a_{kl}\|_{L^\infty(Q)} \leq M. \tag{54}$$

Remark 4.3. To simplify the notations, we now denote $\widehat{h^3}$ by h^3.

4.1.2 2 controls, adequate change of variables

By Section 2.3.2, we can assume that $(u_3^*, u_4^*) \neq (0,0)$.

4.1.2.1 First case: $d_3 = d_4$.

From (24) and (38), system (3) reduces to

$$\forall 1 \leq i \leq 3,\ \begin{cases} \partial_t u_i - d_i \Delta u_i = (-1)^i (u_1 u_3 - u_2(u_3^* + u_4^* - u_3)) + h_i 1_\omega 1_{i\leq 2} & \text{in } (0,T)\times\Omega, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ u_i(0,.) = u_{i,0} & \text{in } \Omega. \end{cases} \tag{55}$$

We do not give the complete proof of Theorem 3.2 in this case because it is an easy adaptation of the study of the null-controllability of the linear systems (52) which satisfy (53), (54) (with three equations instead of four). Indeed, by linearization around $(u_i^*)_{1\leq i\leq 4}$ of (55), the equation satisfied by u_3 becomes

$$\partial_t u_3 - d_3 \Delta u_3 = -u_3^* u_1 + (u_3^* + u_4^*) u_2 - (u_1^* + u_2^*) u_3 \text{ in } (0,T)\times\Omega. \tag{56}$$

Then, there is a coupling term in (56) if and only if

$$(u_3^*, u_3^* + u_4^*) \neq (0,0) \text{ i.e. } (u_3^*, u_4^*) \neq (0,0). \tag{57}$$

4.1.2.2 Second case: $d_3 \neq d_4$.

We remark that

$$\boxed{(u_1,u_2,u_3,u_4)(T,.) = (u_1^*, u_2^*, u_3^*, u_4^*)}$$
$$\text{if and only if} \tag{58}$$
$$\boxed{(u_1,u_2,u_3,u_3+u_4)(T,.) = (u_1^*, u_2^*, u_3^*, u_3^* + u_4^*)}\,.$$

Therefore, we study the system satisfied by $(v_1,v_2,v_3,v_4) := (u_1,u_2,u_3,u_3+u_4)$,

$$\forall 1 \leq i \leq 3,\ \begin{cases} \partial_t v_i - d_i \Delta v_i = (-1)^i (v_1 v_3 - v_2(v_4 - v_3)) + h_i 1_\omega 1_{i\leq 2} & \text{in } (0,T)\times\Omega, \\ \partial_t v_4 - d_4 \Delta v_4 = (d_3 - d_4)\Delta v_3 & \text{in } (0,T)\times\Omega, \\ \frac{\partial v_i}{\partial n} = \frac{\partial v_4}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ (v_i, v_4)(0,.) = (u_{i,0}, u_{3,0} + u_{4,0}) & \text{in } \Omega. \end{cases} \tag{59}$$

Roughly speaking, v_4 can be controlled by v_3 thanks to the coupling term of second order $(d_3 - d_4)\Delta v_3$ in the second equation of (59) and v_3 can be controlled by v_1 or v_2 because the linearization of the first equation of (59) with $i = 3$, is

$$\partial_t v_3 - d_3 \Delta v_3 = -u_3^* v_1 + u_4^* v_2 - (u_1^* + u_2^*) v_3 + u_2^* v_4 \text{ in } (0,T)\times\Omega,$$

and $(u_3^*, u_4^*) \neq (0,0)$. Then, the proof of the controllability properties of the linearized-system of (59) follows the ideas of Proposition 1.9 and Proposition 1.10. The main difference is the nature of the coupling terms: one coupling term of second order $(d_3 - d_4)\Delta v_3$ and one coupling term of zero order $-u_3^* v_1$ if $u_3^* \neq 0$ or $u_4^* v_2$ if $u_4^* \neq 0$.

4.1.2.3 Linearization in $L^\infty(Q)$ and null-controllability of a family of linear systems when $d_3 \neq d_4$

We define

$$(\zeta, h^2) := (\zeta_1, \zeta_2, \zeta_3, \zeta_4, h_1, h_2) := (v_1 - u_1^*, v_2 - u_2^*, v_3 - u_3^*, v_4 - (u_3^* + u_4^*), h_1, h_2). \tag{60}$$

Then, (u, h^2) is a trajectory of (3) if and only if (ζ, h^2) is a trajectory of

$$\begin{cases} \partial_t \zeta - D_2 \Delta \zeta = G(\zeta)\zeta + B_2 h^2 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial \zeta}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \zeta(0,.) = \zeta_0 & \text{in } \Omega, \end{cases}$$

where

$$D_2 := \begin{pmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & (d_3 - d_4) & d_4 \end{pmatrix}, \qquad G(\zeta) := \begin{pmatrix} -(u_3^* + \zeta_3) & u_4^* + \zeta_4 - \zeta_3 & -u_1^* - u_2^* & u_2^* \\ u_3^* + \zeta_3 & -(u_4^* + \zeta_4 - \zeta_3) & u_1^* + u_2^* & -u_2^* \\ -(u_3^* + \zeta_3) & u_4^* + \zeta_4 - \zeta_3 & -u_1^* - u_2^* & u_2^* \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{61}$$

Note that $G_{31}(0,0,0,0) = -u_3^*$ and $G_{32}(0,0,0,0) = u_4^*$. Then, $(G_{31}(0,0,0,0), G_{32}(0,0,0,0)) \neq (0,0)$. To simplify, we suppose that $G_{31}(0,0,0,0) \neq 0$. The other case is similar. There exist $t_1 < t_2 \in (0,T)$, a nonempty open subset $\omega_0 \subset\subset \omega$ and $M > 0$ such that

$$\forall (t,x) \in (t_1,t_2)\times\omega_0,\ G_{31}(0,0,0,0)(t,x) \leq -2/M,$$

$$\forall (k,l) \in \{1,\dots,3\}\times\{1,\dots,3\},\ \|G_{kl}(0,0,0,0)\|_{L^\infty(Q)} \leq M/2,$$

$$G_{14} = -G_{24} = G_{34} = u_2^*,\ G_{41} = G_{42} = G_{43} = G_{44} = 0.$$

Consequently, we study the null-controllability of the linear systems

$$\begin{cases} \partial_t \zeta - D_2 \Delta \zeta = A\zeta + B_2 h^2 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial \zeta}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \zeta(0,.) = \zeta_0 & \text{in } \Omega, \end{cases} \tag{62}$$

where the matrix A verifies the following assumptions

$$\forall (t,x) \in (t_1,t_2)\times\omega_0,\ a_{31}(t,x) \leq -1/M, \tag{63}$$

$$\forall (k,l) \in \{1,\dots,3\}\times\{1,\dots,3\},\ \|a_{kl}\|_{L^\infty(Q)} \leq M, \tag{64}$$

$$a_{14} = -a_{24} = a_{34} = u_2^*, \tag{65}$$

$$a_{41} = a_{42} = a_{43} = a_{44} = 0. \tag{66}$$

Remark 4.4. Actually, we can show the null controllability of a bigger family of linear systems. Indeed, we can replace (65) by the more general assumption: a_{14}, a_{24}, $a_{34} \in \mathbb{R}$ because it does not change the proof of the null-controllability result of the linear systems like (62) (see Proposition 4.8). But, the more general case a_{14}, a_{24}, $a_{34} \in L^\infty(Q)$ is not handled by our proof of Proposition 4.8 (see Section 4.3.5 and in particular (135)).

Remark 4.5. The algebraic relation (66) is useful to prove the null-controllability result of the linear systems like (62) (see Proposition 4.8) because it creates the cascade form of (62). Indeed, the fourth and the third equation of (62) are

$$\partial_t \zeta_4 - d_4 \Delta \zeta_4 = (d_3 - d_4)\Delta\zeta_3 \text{ in } (0,T)\times\Omega, \text{ and } d_3 - d_4 \neq 0,$$

$$\partial_t \zeta_3 - d_3 \Delta \zeta_3 = a_{31}\zeta_1 + a_{32}\zeta_2 + a_{33}\zeta_3 + u_2^*\zeta_4 \text{ in } (0,T)\times\Omega, \text{ and } \forall (t,x) \in (t_1,t_2)\times\omega_0,\ a_{31}(t,x) \leq -1/M.$$

4.1.3 1 control, adequate change of variables

By Section 2.3.3, we can assume that $u_3^* \neq 0$.

4.1.3.1 First case: $\exists k \neq l \in \{2,3,4\},\ d_k = d_l$. We treat the case $d_2 = d_3$, $d_3 \neq d_4$. The other cases are similar. From (27) and (40), system (3) reduces to

$$\forall i \in \{1,2,4\},\ \begin{cases} \partial_t u_i - d_i \Delta u_i = (-1)^i (u_1(u_2^* + u_3^* - u_2) - u_2 u_4) + h_i 1_\omega 1_{i\leq 1} & \text{in } (0,T)\times\Omega, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ u_i(0,.) = u_{i,0} & \text{in } \Omega. \end{cases} \tag{67}$$

We remark that

$$\begin{aligned} &(u_1,u_2,u_4)(T,.) = (u_1^*,u_2^*,u_4^*) \\ &\text{if and only if} \\ &(u_1,u_2,u_2-u_4)(T,.) = (u_1^*,u_2^*,u_2^*-u_4^*). \end{aligned} \tag{68}$$

Therefore, we study the system satisfied by $(v_1,v_2,v_3) := (u_1,u_2,u_2-u_4)$,

$$\forall 1 \leq i \leq 2,\ \begin{cases} \partial_t v_i - d_i \Delta v_i = (-1)^i (v_1(u_2^* + u_3^* - v_2) - v_2(v_2 - v_3)) + h_i 1_\omega 1_{i\leq 1} & \text{in } (0,T)\times\Omega, \\ \partial_t v_3 - d_4 \Delta v_3 = (d_2 - d_4)\Delta v_2 & \text{in } (0,T)\times\Omega, \\ \frac{\partial v_i}{\partial n} = \frac{\partial v_3}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ (v_i(0,.), v_3(0,.)) = (u_{i,0}, u_{2,0} - u_{4,0}) & \text{in } \Omega. \end{cases} \tag{69}$$

We do not give the complete proof of Theorem 3.2 in this case because it is an easy adaptation of the study of the null-controllability of the linear systems (62) which satisfy (63), (64), (65) and (66) (with three equations instead of four). Indeed, v_3 can be controlled by v_2 thanks to the coupling term of second order $(d_2 - d_4)\Delta v_2$ in the second equation of (69) and v_2 can be controlled by v_1 because the linearization of the first equation of (69) with $i = 2$ is

$$\partial_t v_2 - d_2 \Delta v_2 = u_3^* v_1 + (-v_1^* - 2v_2^* + v_3^*) v_2 + u_2^* v_3 \text{ in } (0,T)\times\Omega,$$

where $(v_1^*, v_2^*, v_3^*) := (u_1^*, u_2^*, u_2^* - u_4^*)$ and $u_3^* \neq 0$.

4.1.3.2 Second case: $d_2 \neq d_3$, $d_3 \neq d_4$, $d_2 \neq d_4$. We introduce $\alpha \neq \beta$ such that

$$\alpha(d_2 - d_4) = \beta(d_3 - d_4) = 1, \text{ i.e. } \alpha = \frac{1}{d_2 - d_4} \text{ and } \beta = \frac{1}{d_3 - d_4}. \tag{70}$$

Then, we define $\gamma \neq 0$ by the algebraic relation

$$\alpha - \beta + \gamma = 0, \text{ i.e. } \gamma = \beta - \alpha. \tag{71}$$

We remark that

$$\boxed{(u_1,u_2,u_3,u_4)(T,.) = (u_1^*,u_2^*,u_3^*,u_4^*)}$$

if and only if

$$\boxed{(u_1,u_2,u_2+u_3,\alpha u_2 + \beta u_3 + \gamma u_4)(T,.) = (u_1^*,u_2^*,u_2^*+u_3^*,\alpha u_2^* + \beta u_3^* + \gamma u_4^*)}\ . \tag{72}$$

Therefore, we study the system satisfied by $(v_1,v_2,v_3,v_4) := (u_1,u_2,u_2+u_3,\alpha u_2 + \beta u_3 + \gamma u_4)$. We introduce the following notations

$$g_1(v_2,v_3,v_4) := \frac{\beta - \alpha}{\gamma} v_2 - \frac{\beta}{\gamma} v_3 + \frac{1}{\gamma} v_4 = u_4,\ g_2(v_2,v_3) := v_3 - v_2 = u_3. \tag{73}$$

We have

$$\forall 1 \leq i \leq 2, \begin{cases} \partial_t v_i - d_i \Delta v_i = (-1)^i \left(g_2(v_2,v_3) v_1 - g_1(v_2,v_3,v_4) v_2\right) + h_i 1_\omega 1_{i\leq 1} & \text{in } (0,T)\times\Omega, \\ \partial_t v_3 - d_3 \Delta v_3 = (d_2 - d_3)\Delta v_2 & \text{in } (0,T)\times\Omega, \\ \partial_t v_4 - d_4 \Delta v_4 = \Delta v_3 & \text{in } (0,T)\times\Omega, \\ \frac{\partial v_i}{\partial n} = \frac{\partial v_3}{\partial n} = \frac{\partial v_4}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ (v_i,v_3,v_4)(0,.) = (u_{i,0}, u_{2,0} + u_{3,0}, \alpha u_{2,0} + \beta u_{3,0} + \gamma u_{4,0}) & \text{in } \Omega. \end{cases} \tag{74}$$

Roughly speaking, v_4 can be controlled by v_3 thanks to the coupling term of second order Δv_3 in the third equation of (74) and v_3 can be controlled by v_2 thanks to the coupling term of second order $(d_2 - d_3)\Delta v_2$ in the second equation of (74) and v_2 can be controlled by v_1 because the linearization of the first equation of (74) with $i = 2$ is

$$\begin{aligned} \partial_t v_2 - d_2 \Delta v_2 &= g_2(v_2^*, v_3^*) v_1 - g_1(v_2^*, v_3^*, v_4^*) v_2 + v_1^* g_2(v_2, v_3) - v_2^* g_1(v_2, v_3, v_4) \\ &= u_3^* v_1 - g_1(v_2^*, v_3^*, v_4^*) v_2 + v_1^* g_2(v_2, v_3) - v_2^* g_1(v_2, v_3, v_4) \qquad \text{in } (0,T)\times\Omega, \end{aligned}$$

and $u_3^* \neq 0$. Then, the proof of the controllability properties of the linearized-system of (74) follows the ideas of Proposition 1.9 and Proposition 1.10. The main difference is the nature of the coupling terms: two coupling terms of second order Δv_3, $(d_2 - d_3)\Delta v_2$ and one coupling term of zero order $u_3^* v_1$.

4.1.3.3 Linearization in $L^\infty(Q)$ and null-controllability of a family of linear systems when $d_2 \neq d_3$, $d_2 \neq d_4$, $d_3 \neq d_4$ We define

$$(\zeta, h^1) := (\zeta_1, \zeta_2, \zeta_3, \zeta_4, h_1) := (v_1 - u_1^*, v_2 - u_2^*, v_3 - (u_2^* + u_3^*), v_4 - (\alpha u_2^* + \beta u_3^* + \gamma u_4^*), h_1). \tag{75}$$

Then, (u, h^1) is a trajectory of (3) if and only if (ζ, h^1) is a trajectory of

$$\begin{cases} \partial_t \zeta - D_1 \Delta \zeta = G(\zeta)\zeta + B_1 h^1 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial \zeta}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \zeta(0,.) = \zeta_0 & \text{in } \Omega, \end{cases}$$

where

$$D_1 := \begin{pmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & d_2 - d_3 & d_3 & 0 \\ 0 & 0 & 1 & d_4 \end{pmatrix}, \quad G(\zeta) := \begin{pmatrix} -(u_3^* + g_2(\zeta_2, \zeta_3)) & m_1 + g_1(\zeta_2, \zeta_3, \zeta_4) & -m_2 & m_3 \\ u_3^* + g_2(\zeta_2, \zeta_3) & -(m_1 + g_1(\zeta_2, \zeta_3, \zeta_4)) & m_2 & -m_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{76}$$

with $m_1 := u_1^* + u_2^* + u_4^*$, $m_2 := u_1^* + \frac{\beta}{\gamma} u_2^*$ and $m_3 = \frac{1}{\gamma} u_2^*$. Note that $G_{21}(0,0,0,0) = u_3^*$. There exist $t_1 < t_2 \in (0,T)$, a nonempty open subset $\omega_0 \subset\subset \omega$ and $M > 0$ such that

$$\forall (t,x) \in (t_1, t_2) \times \omega_0,\ G_{21}(0,0,0,0)(t,x) \geq 2/M,$$

$$\forall (k,l) \in \{1,2\} \times \{1,2\},\ \|G_{kl}(0,0,0,0)\|_{L^\infty(Q)} \leq M/2,$$

$$G_{13} = -G_{23} = -m_2,\ G_{14} = -G_{24} = m_3,\ G_{kl} = 0,\ 3 \leq k \leq 4,\ 1 \leq l \leq 4.$$

Consequently, we study the null-controllability of the linear systems

$$\begin{cases} \partial_t \zeta - D_1 \Delta \zeta = A\zeta + B_1 h^1 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial \zeta}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \zeta(0,.) = \zeta_0 & \text{in } \Omega, \end{cases} \tag{77}$$

where the matrix A verifies the following assumptions

$$\forall (t,x) \in (t_1, t_2) \times \omega_0,\ a_{21}(t,x) \geq 1/M, \tag{78}$$

$$\forall (k,l) \in \{1,2\} \times \{1,2\},\ \|a_{kl}\|_{L^\infty(Q)} \leq M, \tag{79}$$

$$a_{13} = -a_{23} = -m_2,\ a_{14} = -a_{24} = m_3, \tag{80}$$

$$a_{kl} = 0,\ 3 \leq k \leq 4,\ 1 \leq l \leq 4. \tag{81}$$

Remark 4.6. Actually, we can show the null controllability of a bigger family of linear systems. Indeed, we can replace (80) by the more general assumption: a_{13}, a_{23}, a_{14}, $a_{24} \in \mathbb{R}$ because it does not change the proof of the null-controllability result of the linear systems like (77) (see Proposition 4.8). But, the more general case a_{13}, a_{23}, a_{14}, $a_{24} \in L^\infty(Q)$ is not handled by our proof of Proposition 4.8 (see Section 4.3.7 and in particular (156) and (158)).

Remark 4.7. The algebraic relation (81) is useful to prove the null-controllability result of the linear systems like (77) (see Proposition 4.8) because it creates the cascade form of (77). Indeed, the fourth, the third and the second equation of (77) are

$$\partial_t \zeta_4 - d_4 \Delta \zeta_4 = \Delta \zeta_3 \text{ in } (0,T)\times\Omega,$$

$$\partial_t \zeta_3 - d_3 \Delta \zeta_3 = (d_2 - d_3)\Delta \zeta_2 \text{ in } (0,T)\times\Omega, \text{ and } (d_2 - d_3) \neq 0,$$

$$\partial_t \zeta_2 - d_2 \Delta \zeta_2 = a_{21}\zeta_1 + a_{22}\zeta_2 + m_2\zeta_3 - m_3\zeta_4 \text{ in } (0,T)\times\Omega, \text{ and } \forall (t,x) \in (t_1,t_2)\times\omega_0,\ a_{21}(t,x) \geq 1/M.$$

4.2 Null controllability in $L^2(\Omega)^4$ with controls in $L^\infty(Q)^j$ of a family of linear control systems

4.2.1 Main result of this subsection

We introduce the following notations,

$$\mathcal{E}_3 := \{A \in \mathcal{M}_4(L^\infty(Q)) \ ; \ A \text{ verifies the assumptions (53) and (54)}\}, \tag{82}$$

$$H_3 := L^2(\Omega)^4, \tag{83}$$

$$\mathcal{E}_2 := \{A \in \mathcal{M}_4(L^\infty(Q)) \ ; \ A \text{ verifies the assumptions (63), (64), (65) and (66)}\}, \tag{84}$$

$$H_2 := \left\{\zeta_0 \in L^2(\Omega)^4 \ ; \int_\Omega \zeta_{0,4} = 0\right\}, \tag{85}$$

$$\mathcal{E}_1 := \{A \in \mathcal{M}_4(L^\infty(Q)) \ ; \ A \text{ verifies the assumptions (78), (79), (80) and (81)}\}, \tag{86}$$

$$H_1 := \left\{\zeta_0 \in L^2(\Omega)^4 \ ; \int_\Omega \zeta_{0,3} = 0, \ \int_\Omega \zeta_{0,4} = 0\right\}. \tag{87}$$

The main result of this subsection is a null-controllability result in $L^2(\Omega)^4$ with controls in $L^\infty(Q)^j$ for families of linear control systems.

Proposition 4.8. *Let $j \in \{1,2,3\}$, D_j defined by (51), (61) or (76). There exists $C > 0$ such that, for every $A \in \mathcal{E}_j$ and $\zeta_0 = (\zeta_{0,1}, \zeta_{0,2}, \zeta_{0,3}, \zeta_{0,4}) \in H_j$, there exists $h^j \in L^\infty(Q)^j$ satisfying*

$$\left\|h^j\right\|_{L^\infty(Q)^j} \leq C \left\|\zeta_0\right\|_{L^2(\Omega)^4}, \tag{88}$$

such that the solution $\zeta \in Y^4$ to the Cauchy problem

$$\begin{cases} \partial_t \zeta - D_j \Delta \zeta = A\zeta + B_j h^j 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial \zeta}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \zeta(0,.) = \zeta_0 & \text{in } \Omega, \end{cases} \tag{89}$$

verifies

$$\zeta(T,.) = 0.$$

Remark 4.9. For every $1 \leq j \leq 3$, the diffusion matrices D_j defined by (51), (61) or (76) verify the assumption of Proposition 2.3 because they are similar to $diag(d_1, d_2, d_3, d_4)$.

4.2.2 Proof strategy of Proposition 4.8: Null controllability in $L^2(\Omega)^4$ with controls in $L^\infty(Q)^j$ of a family of linear control systems

- We let evolve the system without control in $(0,t_1)$ (take $h^j(t,.) = 0$ in $(0,t_1)$). From Proposition 2.2 and Proposition 2.3, we get the existence of $C > 0$ such that for every $A \in \mathcal{E}_j$, $\zeta_0 \in L^2(\Omega)^4$, the solution to the Cauchy problem satisfies
$$\left\|\zeta^*\right\|_{L^2(\Omega)^4} \leq C \left\|\zeta_0\right\|_{L^2(\Omega)^4},$$
where
$$\zeta^* = \zeta(t_1,.).$$
- Then, we find $h^j : (t_1,t_2)\times\Omega \to \mathbb{R}$ such that
$$\left\|h^j\right\|_{L^\infty((t_1,t_2)\times\Omega)^j} \leq C \left\|\zeta(t_1,.)\right\|_{L^2(\Omega)^4},$$
and the solution to the Cauchy problem
$$\begin{cases} \partial_t \zeta - D_j \Delta \zeta = A\zeta + B_j h^j 1_\omega & \text{in } (t_1,t_2)\times\Omega, \\ \frac{\partial \zeta}{\partial n} = 0 & \text{on } (t_1,t_2)\times\partial\Omega, \\ \zeta(t_1,.) = \zeta^* & \text{in } \Omega, \end{cases}$$
verifies
$$\zeta(t_2,.) = 0.$$
- Then, we set $h^j(t,.) = 0$ so that $h^j(t,.) = 0$ for $t \in (t_2,T)$.

This strategy gives

$$\zeta(T,.) = 0 \text{ and } \left\|h^j\right\|_{L^\infty((0,T)\times\omega)^j} \leq C \left\|\zeta_0\right\|_{L^2(\Omega)^4}.$$

To simplify, we now suppose

$$(t_1,t_2) = (0,T).$$

4.3 First step: Controls in $L^2(Q)^j$

The goal of this section is the proof of the following result.

Proposition 4.10. *Let $j \in \{1,2,3\}$. There exists $C>0$ such that, for every $A \in \mathcal{E}_j$ and for every $\zeta_0 \in H_j$, there exists a control $h^j \in L^2(Q)^j$ satisfying*

$$\left\|h^j\right\|_{L^2(Q)^j} \leq C \left\|\zeta_0\right\|_{L^2(\Omega)^4} \tag{90}$$

such that the solution $\zeta \in Y^4$ to the Cauchy problem (89) satisfies $\zeta(T,.)=0$.

The proof of Proposition 4.10 will be done in Section 4.3.3 for $j=3$, Section 4.3.5 for $j=2$, Section 4.3.7 for $j=1$. It requires technical preliminary results presented in Section 4.3.1, Section 4.3.2, Section 4.3.4, Section 4.3.6.

4.3.1 Hilbert Uniqueness Method

First, for $\Phi \in L^2(\Omega)$, $(\Phi)_\Omega$ denotes the mean value of Φ,

$$(\Phi)_\Omega := \frac{1}{|\Omega|}\int_\Omega \Phi,$$

and for $\Psi \in C([0,T];L^2(\Omega))$, $t \in [0,T]$, we introduce the notation

$$(\Psi)_\Omega(t) := \frac{1}{|\Omega|}\int_\Omega \Psi(t,x)dx.$$

By the HUM (Hilbert Uniqueness Method), the null-controllability result of Proposition 4.10 is equivalent to the following observability inequality: (92) (see [14, Theorem 2.44]).

Let $j \in \{1,2,3\}$, D_j defined by (51), (61) or (76). There exists $C>0$ such that, for every $A \in \mathcal{E}_j$ and $\varphi_T \in H_j$ (see (82), (83), (84), (85), (86), (87)) the solution φ of

$$\left\{\begin{array}{ll} -\partial_t\varphi - D_j^T\Delta\varphi = A^T\varphi & \text{in } (0,T)\times\Omega, \\ \frac{\partial\varphi}{\partial n}=0 & \text{on } (0,T)\times\partial\Omega, \\ \varphi(T,.)=\varphi_T & \text{in } \Omega, \end{array}\right. \tag{91}$$

verifies

$$\int_\Omega |\varphi(0,x)|^2dx \leq C\left(\sum_{i=1}^{j}\int\int_{(0,T)\times\omega}|\varphi_i(t,x)|^2dxdt\right). \tag{92}$$

It is easy to show that it is sufficient to prove the following observability inequalities.

There exists $C>0$ such that, for every $A \in \mathcal{E}_3$ and $\varphi_T \in L^2(\Omega)^4$, the solution φ of the adjoint system (91) verifies

$$\int_\Omega |\varphi(0,x)|^2dx \leq C\left(\sum_{i=1}^{3}\int\int_{(0,T)\times\omega}|\varphi_i(t,x)|^2dxdt\right). \tag{93}$$

There exists $C>0$ such that, for every $A \in \mathcal{E}_2$ and $\varphi_T \in L^2(\Omega)^4$, the solution φ of the adjoint system (91) verifies

$$\sum_{i=1}^{3}\left(\left\|\varphi_i(0,.)\right\|^2_{L^2(\Omega)}\right) + \left\|\varphi_4(0,.)-(\varphi_4)_\Omega(0)\right\|^2_{L^2(\Omega)} \leq C\left(\sum_{i=1}^{2}\int\int_{(0,T)\times\omega}|\varphi_i|^2dxdt\right). \tag{94}$$

There exists $C>0$ such that, for every $A \in \mathcal{E}_1$ and $\varphi_T \in L^2(\Omega)^4$, the solution φ of the adjoint system (91) verifies

$$\sum_{i=1}^{2}\left(\left\|\varphi_i(0,.)\right\|^2_{L^2(\Omega)}\right) + \sum_{i=3}^{4}\left(\left\|\varphi_i(0,.)-(\varphi_i)_\Omega(0)\right\|^2_{L^2(\Omega)}\right) \leq C\left(\int\int_{(0,T)\times\omega}|\varphi_1|^2dxdt\right). \tag{95}$$

4.3.2 Carleman estimates

We introduce several weight functions. Let $\omega'' \subset\subset \omega_0$ be a nonempty open subset and $\eta_0 \in C^2(\overline{\Omega})$ verifying

$$\forall x \in \Omega, \eta_0(x)>0,\ \eta_0=0 \text{ on } \partial\Omega,\ \forall x \in \overline{\Omega\setminus\omega''},\ |\nabla\eta_0(x)|>0.$$

The existence of such a function is proved in [14, Lemma 2.68]. Let $\lambda \geq 1$ a parameter. We remark that

$$1+f(\lambda) := 1+\exp(-\lambda\left\|\eta_0\right\|_\infty) < 2. \tag{96}$$

We define

$$\forall (t,x) \in (0,T)\times\Omega,\ \phi(t,x) := \frac{e^{\lambda\eta_0(x)}}{t(T-t)} > 0,\ \alpha(t,x) := \frac{e^{\lambda\eta_0(x)} - e^{2\lambda\|\eta_0\|_\infty}}{t(T-t)} < 0, \tag{97}$$

$$\forall t \in (0,T),\ \widehat{\alpha}(t) := \min_{x\in\overline{\Omega}} \alpha(t,x) = \frac{1-e^{2\lambda\|\eta_0\|_\infty}}{t(T-t)} < 0,\ \widehat{\phi}(t) := \min_{x\in\overline{\Omega}} \phi(t,x) = \frac{1}{t(T-t)} > 0. \tag{98}$$

Theorem 4.11. ***Carleman inequality***
Let $d \in (0,+\infty)$, ω' an open subset such that $\omega'' \subset\subset \omega' \subset\subset \omega_0$ and $\beta \in \mathbb{R}$. There exist $C = C(\Omega,\omega',\beta)$, $\lambda_0 = C(\Omega,\omega',\beta)$, $s_0 = s_0(\Omega,\omega',\beta)$ such that, for any $\lambda \geq \lambda_0$, $s \geq s_0(T+T^2)$, $\varphi_T \in L^2(\Omega)$ and $f \in L^2(Q)$, the solution φ to

$$\begin{cases} -\partial_t\varphi - d\Delta\varphi = f & \text{in } (0,T)\times\Omega, \\ \frac{\partial\varphi}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \varphi(T,.) = \varphi_T & \text{in } \Omega, \end{cases}$$

satisfies

$$\begin{aligned} &I(\beta,\lambda,s,\varphi) \\ &:= \lambda^4 \int_0^T\int_\Omega e^{2s\alpha}(s\phi)^{\beta+3}|\varphi|^2 dxdt + \lambda^2\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^{\beta+1}|\nabla\varphi|^2 dxdt + \int_0^T\int_\Omega e^{2s\alpha}(s\phi)^{\beta-1}\left(|\partial_t\varphi|^2 + |\Delta\varphi|^2\right) dxdt \\ &\leq C\left(\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^\beta |f|^2 dxdt + \int_0^T\int_{\omega'} \lambda^4 e^{2s\alpha}(s\phi)^{\beta+3}|\varphi|^2 dxdt\right). \end{aligned} \tag{99}$$

The original proof of this inequality can be found in [30, Lemma 1.2].

Remark 4.12. For a general introduction to global Carleman inequalities and their applications to the controllability of parabolic systems, one can see [29] (in particular, see [29, Lemma 1.3]). For Neumann conditions, one can see [28] and in particular [28, Lemma 1].

4.3.2.1 A parabolic regularity result in L^2 In the following, we consider initial conditions $\varphi_T \in C_0^\infty(\Omega)^4$ in order to improve the regularity of φ, solution of (91), and to allow some computations.

Definition 4.13. We define the following spaces of functions

$$H^2_{Ne}(\Omega) := \left\{ u \in H^2(\Omega)\ ;\ \frac{\partial u}{\partial n} = 0 \right\}, \qquad Y_2 := L^2(0,T;H^2_{Ne}(\Omega)) \cap H^1(0,T;L^2(\Omega)).$$

Proposition 4.14. *Let $k \in \mathbb{N}^*$, $D \in \mathcal{M}_k(\mathbb{R})$ such that $Sp(D) \subset (0,+\infty)$, $A \in \mathcal{M}_k(L^\infty(Q))$, $u_0 \in C_0^\infty(\Omega)^k$. From [20, Theorem 2.1], the following Cauchy problem admits a unique solution $u \in Y_2^k$*

$$\begin{cases} \partial_t u - D\Delta u = A(t,x)u & \text{in } (0,T)\times\Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,.) = u_0 & \text{in } \Omega. \end{cases}$$

4.3.2.2 A technical lemma for Carleman estimates By now, unless otherwise specified, we denote by C (respectively C_ε) various positive constants varying from line to line (respectively various positive constants varying from line to line and depending on the parameter ε). We insist on the fact that C and C_ε do not depend on λ and s, unless otherwise specified.

Lemma 4.15. *Let Φ, $\Psi \in Y_2$, $a \in L^\infty(Q)$, an open subset $\widetilde{\omega} \subset \omega_0$, $\Theta \in C^\infty(\overline{\Omega};[0,+\infty[)$ such that supp$(\Theta) \subset \widetilde{\omega}$*

and $r \in \mathbb{N}$. Then, for every $\varepsilon > 0$,

$$\forall (k,l) \in \mathbb{R}^2,\ k+l = 2r,\ \forall s \geq C,$$
$$\left| \int\int_{(0,T)\times\widetilde{\omega}} \Theta e^{2s\alpha} (s\phi)^r a \Phi \Psi \right| \leq \varepsilon \int\int_{(0,T)\times\Omega} e^{2s\alpha}(s\phi)^k |\Phi|^2 + C_\varepsilon \int\int_{(0,T)\times\widetilde{\omega}} e^{2s\alpha}(s\phi)^l |\Psi|^2, \tag{100}$$
$$\forall (k,l) \in \mathbb{R}^2,\ k+l = 2(r+2),\ \forall s \geq C,$$
$$\left| \int_0^T \int_{\widetilde{\omega}} \Theta e^{2s\alpha} (s\phi)^r \Phi \partial_t \Psi \right| \leq \varepsilon \left(\int_0^T \int_\Omega e^{2s\alpha}(s\phi)^k |\Phi|^2 + \int_0^T \int_\Omega e^{2s\alpha}(s\phi)^{k-4} |\partial_t \Phi|^2 \right) + C_\varepsilon \int_0^T \int_{\widetilde{\omega}} e^{2s\alpha}(s\phi)^l |\Psi|^2, \tag{101}$$
$$\forall (k,l) \in \mathbb{R}^2,\ k+l = 2(r+2),\ \forall s \geq C,$$
$$\left| \int_0^T \int_{\widetilde{\omega}} \Theta e^{2s\alpha} (s\phi)^r \Phi \Delta \Psi \right| \leq \varepsilon \left(\int_0^T \int_\Omega e^{2s\alpha}(s\phi)^k |\Phi|^2 + \int_0^T \int_\Omega e^{2s\alpha}(s\phi)^{k-2} |\nabla \Phi|^2 + \int_0^T \int_\Omega e^{2s\alpha}(s\phi)^{k-4} |\Delta \Phi|^2 \right) + C_\varepsilon \int_0^T \int_{\widetilde{\omega}} e^{2s\alpha}(s\phi)^l |\Psi|^2. \tag{102}$$
$$\forall (k,l) \in \mathbb{R}^2,\ k+l = 2r,\ \forall s \geq C,$$
$$\int_0^T \int_{\widetilde{\omega}} \Theta e^{2s\alpha} (s\phi)^r |\nabla \Phi|^2 \leq \varepsilon \left(\int_0^T \int_\Omega e^{2s\alpha}(s\phi)^k |\Delta \Phi|^2 + \int_0^T \int_\Omega e^{2s\alpha}(s\phi)^{k+2} |\nabla \Phi|^2 \right) + C_\varepsilon \int_0^T \int_{\widetilde{\omega}} e^{2s\alpha}(s\phi)^l |\Phi|^2. \tag{103}$$

Proof. The inequality (100) is an easy consequence of Young's inequality applied to

$$\left| \int\int_{(0,T)\times\widetilde{\omega}} \Theta e^{2s\alpha} (s\phi)^r a \Phi \Psi \right| \leq C \int\int_{(0,T)\times\widetilde{\omega}} \left(\sqrt{\varepsilon} e^{s\alpha} (s\phi)^{k/2} |\Phi| \right) \left(\frac{1}{\sqrt{\varepsilon}} \Theta e^{s\alpha} (s\phi)^{l/2} |\Psi| \right).$$

For (101), we integrate by parts with respect to the time variable

$$-\int_0^T \int_{\widetilde{\omega}} \Theta e^{2s\alpha} (s\phi)^r \Phi \partial_t \Psi = \int_0^T \int_{\widetilde{\omega}} \Theta e^{2s\alpha} (s\phi)^r \partial_t(\Phi) \Psi + \int_0^T \int_{\widetilde{\omega}} (\Theta e^{2s\alpha} (s\phi)^r)_t \Phi \Psi.$$

Moreover, by (97), we have $|(\Theta e^{2s\alpha}(s\phi)^r)_t| \leq C e^{2s\alpha} s^{r+1} \phi^{r+2} \leq e^{2s\alpha} s^{r+2} \phi^{r+2}$ for $s \geq C$. Then, we get (101) by applying Young's inequality to

$$\left| \int_0^T \int_{\widetilde{\omega}} \Theta e^{2s\alpha} (s\phi)^r \Phi \partial_t \Psi \right| \leq \int_0^T \int_{\widetilde{\omega}} \left(\sqrt{\varepsilon} e^{s\alpha} (s\phi)^{k/2-2} \partial_t \Phi \right) \left(\frac{1}{\sqrt{\varepsilon}} \Theta e^{s\alpha} (s\phi)^{l/2} \Psi \right) + \int_0^T \int_{\widetilde{\omega}} \left(\sqrt{\varepsilon} e^{s\alpha} (s\phi)^{k/2} \Phi \right) \left(\frac{1}{\sqrt{\varepsilon}} e^{s\alpha} (s\phi)^{l/2} \Psi \right).$$

For (102), by twice integrating by parts with respect to the spatial variable, we get

$$\int_0^T \int_{\widetilde{\omega}} \Theta e^{2s\alpha} (s\phi)^r \Phi \Delta \Psi = \int_0^T \int_{\widetilde{\omega}} \Delta(\Theta e^{2s\alpha} (s\phi)^r \Phi) \Psi.$$

Moreover, by (97), we have

$$|\Delta(\Theta e^{2s\alpha} (s\phi)^r \Phi| \leq C \left(e^{2s\alpha} (s\phi)^r |\Delta \Phi| + e^{2s\alpha} (s\phi)^{r+1} |\nabla \Phi| + e^{2s\alpha} (s\phi)^{r+2} |\Phi| \right).$$

Then, we deduce (102) by Young's inequality applied to

$$\left|\int_0^T\int_{\widetilde{\omega}}\Theta e^{2s\alpha}(s\phi)^r\Phi\Delta\Psi\right| \leq \int_0^T\int_{\widetilde{\omega}}\left(\sqrt{\varepsilon}e^{s\alpha}(s\phi)^{k/2-2}|\Delta\Phi|\right)\left(\frac{1}{\sqrt{\varepsilon}}e^{s\alpha}(s\phi)^{l/2}\Psi\right)$$
$$+\int_0^T\int_{\widetilde{\omega}}\left(\sqrt{\varepsilon}e^{s\alpha}(s\phi)^{k/2-1}|\nabla\Phi|\right)\left(\frac{1}{\sqrt{\varepsilon}}e^{s\alpha}(s\phi)^{l/2}\Psi\right)$$
$$+\int_0^T\int_{\widetilde{\omega}}\left(\sqrt{\varepsilon}e^{s\alpha}(s\phi)^{k/2}|\Phi|\right)\left(\frac{1}{\sqrt{\varepsilon}}e^{s\alpha}(s\phi)^{l/2}\Psi\right).$$

For (103), we integrate by parts with respect to the spatial variable,

$$\int_0^T\int_{\widetilde{\omega}}\Theta e^{2s\alpha}(s\phi)^r|\nabla\Phi|^2 = -\int_0^T\int_{\widetilde{\omega}}\Theta e^{2s\alpha}(s\phi)^r(\Delta\Phi)\Phi - \int_{\omega_0}\nabla(\Theta e^{2s\alpha}(s\phi)^r).(\nabla\Phi)\Phi.$$

By using $|\nabla(\Theta e^{2s\alpha}(s\phi)^r)| \leq Ce^{2s\alpha}(s\phi)^{r+1}$ which is a consequence of (97), we get (103) by Young's inequality. □

4.3.3 Proof with observation on three components: (93)

Proof. $\boxed{j=3}$

The proof is close to the proof of [16, Lemma 7].

Let $A \in \mathcal{E}_3$ (see (82)), $\varphi_T \in C_0^\infty(\Omega)^4$ (the general case comes from a density argument, see (117), Lemma 4.21 and Lemma 4.22), $\varphi \in Y_2^4$ be the solution of (91) (see Proposition 4.14) and ω_1 be an open subset such that $\omega'' \subset\subset \omega_1 \subset\subset \omega_0$. We have

$$\forall 1 \leq i \leq 4,\ \begin{cases} -\partial_t\varphi_i - d_i\Delta\varphi_i = a_{1i}\varphi_1 + a_{2i}\varphi_2 + a_{3i}\varphi_3 + a_{4i}\varphi_4 & \text{in } (0,T)\times\Omega, \\ \frac{\partial\varphi_i}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{104}$$

We apply (99) of Theorem 4.11 to each φ_i, $1 \leq i \leq 4$, with $\omega' = \omega_1$ and $\beta = 0$. Then, we sum (by using (54))

$$\forall \lambda, s \geq C,\ \sum_{i=1}^4 I(0,\lambda,s,\varphi_i) \leq C\left(\sum_{i=1}^4\left(\int_0^T\int_\Omega e^{2s\alpha}|\varphi_i|^2dxdt + \int_0^T\int_{\omega_1}\lambda^4e^{2s\alpha}(s\phi)^3|\varphi_i|^2dxdt\right)\right).$$

We fix $\lambda \geq C$ and we take s sufficiently large, then we can absorb the first right hand side term by the left hand side term. We get

$$\sum_{i=1}^4 I(0,\lambda,s,\varphi_i) \leq C\sum_{i=1}^4\int_0^T\int_{\omega_1}e^{2s\alpha}(s\phi)^3|\varphi_i|^2dxdt. \tag{105}$$

Now, λ, s are supposed to be fixed such that (105) holds and the constant C may depend on λ, s.

We have to get rid of the term $\int_0^T\int_{\omega_1}e^{2s\alpha}(s\phi)^3|\varphi_4|^2dxdt$ in order to prove the observability inequality (93). For this, we are going to use (53). So, we are going to estimate φ_4 by φ_i for every $1 \leq i \leq 3$ thanks to the first equation of (104) with $i = 1$.

Estimate of $\int_0^T\int_{\omega_1}e^{2s\alpha}(s\phi)^3|\varphi_4|^2dxdt$.

Let us introduce $\chi \in C^\infty(\overline{\Omega};[0,+\infty[)$, such that the support of χ is included in ω_0 and $\chi = 1$ in ω_1. We multiply the first equation of (104) with $i = 1$ by $\chi(x)e^{2s\alpha}(s\phi)^3\varphi_4$ and we integrate on $(0,T)\times\omega_0$, which leads to

$$\int_0^T\int_{\omega_1}e^{2s\alpha}(s\phi)^3|\varphi_4|^2dxdt \leq M\int_0^T\int_{\omega_1}e^{2s\alpha}(s\phi)^3a_{41}|\varphi_4|^2dxdt \text{ by (53)}$$
$$\leq M\int_0^T\int_{\omega_0}\chi(x)e^{2s\alpha}(s\phi)^3a_{41}|\varphi_4|^2dxdt$$
$$\int_0^T\int_{\omega_1}e^{2s\alpha}(s\phi)^3|\varphi_4|^2dxdt \leq M\int_0^T\int_{\omega_0}\chi(x)e^{2s\alpha}(s\phi)^3\varphi_4(-\partial_t\varphi_1 - d_1\Delta\varphi_1 - a_{11}\varphi_1 - a_{21}\varphi_2 - a_{31}\varphi_3)dxdt. \tag{106}$$

Remark 4.16. In Section 4.1.1, we suppose that if $(u_1^*, u_3^*, u_4^*) \neq (0,0,0)$, then $u_3^* \neq 0$. Consequently, we have (53). If, $u_1^* \neq 0$ (or respectively $u_4^* \neq 0$), we can easily adapt the preceding strategy. We can assume that

$$\forall (t,x) \in (t_1,t_2)\times\omega_0,\ a_{43}(t,x) \geq 1/M \text{ (or respectively } a_{42}(t,x) \leq -1/M),$$

and multiply the first equation of (104) with $i = 3$ (or respectively $i = 2$) by $\chi(x)e^{2s\alpha}(s\phi)^3\varphi_4$ (or $-\chi(x)e^{2s\alpha}(s\phi)^3\varphi_4$) and we integrate on $(0,T)\times\omega_0$.

Let $\varepsilon > 0$ which will be chosen small enough. Now, we want to estimate the right hand side term of (106) by $\sum_{i=1}^{3} \int_0^T \int_{\omega_1} e^{2s\alpha}(s\phi)^m |\varphi_i|^2 dxdt$ with $m \in \mathbb{N}$.

First, we treat the terms $\int_0^T \int_{\omega_0} \chi(x) e^{2s\alpha}(s\phi)^3 \varphi_4 a_{j1} \varphi_j dxdt$, for every $1 \leq j \leq 3$. By applying Lemma 4.15: (100) with $\Phi = \varphi_4$, $\Psi = \varphi_j$, $a = a_{j1}$ (recalling (54)), $\Theta = \chi$, $r = 3$ and $(k, l) = (3, 3)$, we have

$$\begin{aligned} &\left| \int \int_{(0,T)\times\omega_0} \chi(x) e^{2s\alpha}(s\phi)^3 \varphi_4 a_{j1}(t,x) \varphi_j dxdt \right| \\ &\leq \varepsilon \int \int_{(0,T)\times\Omega} e^{2s\alpha}(s\phi)^3 |\varphi_4|^2 dxdt + C_\varepsilon \int \int_{(0,T)\times\omega_0} e^{2s\alpha}(s\phi)^3 |\varphi_j|^2 dxdt. \end{aligned} \tag{107}$$

Then, we treat the term $-\int_0^T \int_{\omega_0} \chi(x) e^{2s\alpha}(s\phi)^3 \varphi_4 \partial_t \varphi_1 dxdt$. By applying Lemma 4.15: (101) with $\Phi = \varphi_4$, $\Psi = \varphi_1$, $a = 1$, $\Theta = \chi$, $r = 3$ and $(k, l) = (3, 7)$, we have

$$\begin{aligned} \left| \int_0^T \int_{\omega_0} \chi(x) e^{2s\alpha}(s\phi)^3 \varphi_4 \partial_t \varphi_1 dxdt \right| &\leq \varepsilon \left(\int_0^T \int_\Omega e^{2s\alpha}(s\phi)^3 |\varphi_4|^2 dxdt + \int_0^T \int_\Omega e^{2s\alpha}(s\phi)^{-1} |\partial_t \varphi_4|^2 dxdt \right) \\ &\quad + C_\varepsilon \int_0^T \int_{\omega_0} e^{2s\alpha}(s\phi)^7 |\varphi_1|^2 dxdt. \end{aligned} \tag{108}$$

Finally, the last term $-d_1 \int_0^T \int_{\omega_0} \chi(x) e^{2s\alpha}(s\phi)^3 \varphi_4 \Delta \varphi_1 dxdt$ is estimated as follows. By applying Lemma 4.15: (102) with $\Phi = \varphi_4$, $\Psi = \varphi_1$, $a = 1$, $\Theta = \chi$, $r = 3$ and $(k, l) = (3, 7)$, we have

$$\begin{aligned} &\left| d_1 \int_0^T \int_{\omega_0} \chi(x) e^{2s\alpha}(s\phi)^3 \varphi_4 \Delta \varphi_1 dxdt \right| \\ &\leq \varepsilon \left(\int_0^T \int_\Omega e^{2s\alpha}(s\phi)^{-1} |\Delta \varphi_4|^2 dxdt + \int_0^T \int_\Omega e^{2s\alpha}(s\phi) |\nabla \varphi_4|^2 dxdt + \int_0^T \int_\Omega e^{2s\alpha}(s\phi)^3 |\varphi_4|^2 dxdt \right) \\ &\quad + C_\varepsilon \int_0^T \int_{\omega_0} e^{2s\alpha}(s\phi)^7 |\varphi_1|^2 dxdt. \end{aligned} \tag{109}$$

Gathering (105), (106), (107), (108), (109), we get

$$\begin{aligned} &\sum_{i=1}^{4} I(0, \lambda, s, \varphi_i) \\ &\leq 3\varepsilon \left(\int_0^T \int_\Omega e^{2s\alpha}(s\phi)^3 |\varphi_4|^2 dxdt + \int_0^T \int_\Omega e^{2s\alpha}(s\phi) |\nabla \varphi_4|^2 dxdt + \int_0^T \int_\Omega e^{2s\alpha}(s\phi)^{-1} \left(|\partial_t \varphi_4|^2 + |\Delta \varphi_4|^2 \right) dxdt \right) \\ &\quad + C_\varepsilon \left(\sum_{i=1}^{3} \int_0^T \int_{\omega_0} e^{2s\alpha}(s\phi)^7 |\varphi_i|^2 dxdt \right). \end{aligned} \tag{110}$$

By taking ε small enough, we get

$$\sum_{i=1}^{4} I(0, \lambda, s, \varphi_i) \leq C_\varepsilon \left(\sum_{i=1}^{3} \int_0^T \int_{\omega_0} e^{2s\alpha}(s\phi)^7 |\varphi_i|^2 dxdt \right). \tag{111}$$

In particular, we deduce from (111) that

$$\sum_{i=1}^{4} \int_0^T \int_\Omega e^{2s\alpha}(s\phi)^3 |\varphi_i|^2 \leq C \left(\sum_{i=1}^{3} \int_0^T \int_{\omega_0} e^{2s\alpha}(s\phi)^7 |\varphi_i|^2 dxdt \right). \tag{112}$$

Then, by using the facts that

$$\min_{[T/4, 3T/4]\times\overline{\Omega}} e^{2s\alpha}(s\phi)^3 > 0, \tag{113}$$

and

$$e^{2s\alpha}(s\phi)^7 \in L^\infty((0,T)\times\Omega), \tag{114}$$

we get

$$\sum_{i=1}^{4} \int_{T/4}^{3T/4} \int_\Omega |\varphi_i|^2 dxdt \leq C \left(\sum_{i=1}^{3} \int_0^T \int_{\omega_0} |\varphi_i|^2 dxdt \right). \tag{115}$$

From the dissipation of the energy in time for (104) (see Lemma A.1 in the Appendix), we easily get

$$\|\varphi(0,.)\|^2_{L^2(\Omega)^4} \leq C\left(\sum_{i=1}^{4}\int_{T/4}^{3T/4}\int_{\Omega}|\varphi_i|^2dxdt\right). \tag{116}$$

Then, by using (115) and (116), we obtain

$$\|\varphi(0,.)\|^2_{L^2(\Omega)^4} \leq C\left(\sum_{i=1}^{3}\int_{0}^{T}\int_{\omega_0}|\varphi_i|^2dxdt\right). \tag{117}$$

This ends the proof of the observability inequality (93) because $\omega_0 \subset \omega$. □

Remark 4.17. Some stronger observability inequalities
We also have the following stronger inequality than (117) which can be proved from (112), (113) and (116). It will be used to find controls in $L^2_{wght}(Q) \subset L^2(Q)$ (see Section 4.4.1). We have

$$\|\varphi(0,.)\|^2_{L^2(\Omega)^4} \leq C\left(\sum_{i=1}^{3}\int_{0}^{T}\int_{\omega}e^{2s\alpha}(s\phi)^7|\varphi_i|^2dxdt\right). \tag{118}$$

Moreover, we also have an even stronger inequality (see (112)) than (117) and (118). It will be used to find controls in $L^\infty(Q)$ (see Section 4.4.2).

4.3.4 Density results

In this section, we show that we can assume that the data φ_T is regular i.e. $\varphi_T \in C_0^\infty(\Omega)^4$. Moreover, we also need some regularity on the coupling matrix A for the case $j = 1$. It's the purpose of Lemma 4.18.

Lemma 4.18. *Let $a \in L^\infty(Q)$. There exists $(a_k) \in (C_0^\infty(Q))^{\mathbb{N}}$ such that*

$$\|a_k\|_{L^\infty(Q)} \leq \|a\|_{L^\infty(Q)}, \tag{119}$$

$$a_k \underset{k\to+\infty}{\rightharpoonup^*} a \text{ in } L^\infty(Q). \tag{120}$$

Proof. Let $k \in \mathbb{N}^*$, $\alpha_k \in C_0^\infty((0,T);[0,1])$, $\alpha_k(t) = 1$ in $(1/k, T-1/k)$, $\beta_k \in C_0^\infty((\Omega);[0,1])$, $\beta_k(x) = 1$ in $\{x \in \Omega\ ;\ d(x,\partial\Omega) \geq 1/k\}$ and $\xi_k \in C_0^\infty(Q)$ be defined by $\xi_k(t,x) = \alpha_k(t)\beta_k(x)$. Let ρ_k be a mollifier sequence in Q such that $\int_Q \rho_k = 1$.

Then, it is easy to show that $a_k := \xi_k.(\rho_k \star a)$ satisfies the conclusion of the lemma. □

Remark 4.19. Actually, the previous lemma shows the density of $C_0^\infty(Q)$ in $L^\infty(Q)$ for the weak-star topology.

We also recall a particular case of the Aubin-Lions' lemma which is useful for the proof of Lemma 4.21.

Lemma 4.20. *[43, Section 8, Corollary 4]*
A bounded subset of Y (see Definition 2.1) is relatively compact in $L^2(Q)$.

Lemma 4.21. *Let $j \in \{1,2,3\}$, D_j defined by (51), (61) or (76), $A \in \mathcal{E}_j$ (see (82), (84) and (86)), $\varphi_T \in L^2(\Omega)^4$. We assume that*

$$\varphi_{T,k} \in C_0^\infty(\Omega)^4 \underset{k\to+\infty}{\to} \varphi_T \text{ in } L^2(\Omega)^4, \tag{121}$$

$$A_k \in \mathcal{M}_4(C_0^\infty(Q)) \underset{k\to+\infty}{\rightharpoonup^*} A \text{ in } L^\infty(Q)^{16}. \tag{122}$$

Then, the sequence of solutions $\varphi_k \in Y^4$ of

$$\begin{cases} -\partial_t\varphi_k - D_j^T\Delta\varphi_k = A_k^T\varphi_k & \text{in } (0,T)\times\Omega, \\ \frac{\partial\varphi_k}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \varphi_k(T,.) = \varphi_{T,k} & \text{in } \Omega, \end{cases} \tag{123}$$

weakly converges in Y^4 and strongly converges in $L^2(Q)^4$ to φ, the solution of (91).

Proof. First, recalling (121), we remark that $(\varphi_{T,k})_{k\in\mathbb{N}}$ is bounded in $L^2(\Omega)^4$. Secondly, recalling (122), we remark that (A_k) is bounded in $\mathcal{M}_4(L^\infty(Q))$. Then, from Proposition 2.3: (16), we get that $(\varphi_k)_{k\in\mathbb{N}}$ is bounded in Y^4. Then, up to a subsequence, we can suppose that there exists $\widetilde{\varphi}\in Y^4$ such that

$$\varphi_k \underset{k\to+\infty}{\rightharpoonup} \widetilde{\varphi} \text{ in } Y^4. \tag{124}$$

By Proposition 2.2, we can also suppose that

$$\varphi_k(T,.) \underset{k\to+\infty}{\rightharpoonup} \widetilde{\varphi}(T,.) \text{ in } L^2(\Omega)^4. \tag{125}$$

But, by (121), we deduce that

$$\varphi_k(T,.) = \varphi_{T,k} \underset{k\to+\infty}{\rightharpoonup} \varphi_T \text{ in } L^2(\Omega)^4. \tag{126}$$

Therefore, by (125) and (126), we get

$$\widetilde{\varphi}(T,.) = \varphi_T. \tag{127}$$

By Lemma 4.20, up to a subsequence, we can also assume that

$$\varphi_k \underset{k\to+\infty}{\to} \widetilde{\varphi} \text{ in } L^2(Q)^4. \tag{128}$$

Consequently, from (128) and (122), we have

$$A_k^T\varphi_k \underset{k\to+\infty}{\rightharpoonup} A^T\widetilde{\varphi} \text{ in } L^2(Q)^4. \tag{129}$$

By using (124), (129), (127) and by letting $k\to+\infty$ in (123), we have

$$\begin{cases} -\partial_t\widetilde{\varphi} - D_j^T\Delta\widetilde{\varphi} = A^T\widetilde{\varphi} & \text{in } (0,T)\times\Omega, \\ \frac{\partial\widetilde{\varphi}}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \widetilde{\varphi}(T,.) = \varphi_T & \text{in } \Omega. \end{cases} \tag{130}$$

By uniqueness in Proposition 2.3, we have $\widetilde{\varphi}=\varphi$. Then, $(\varphi_k)_{k\in\mathbb{N}}$ only has one limit-value: φ for the weak-convergence in Y^4 and for the strong convergence in $L^2(Q)^4$. The sequence $(\varphi_k)_{k\in\mathbb{N}}$ is relatively compact in Y equipped with the weak topology and $(\varphi_k)_{k\in\mathbb{N}}$ is relatively compact in $L^2(Q)^4$ equipped with the strong topology. Therefore,

$$\varphi_k \underset{k\to+\infty}{\rightharpoonup} \varphi \text{ in } Y^4,$$
$$\varphi_k \underset{k\to+\infty}{\to} \varphi \text{ in } L^2(Q)^4.$$

□

Lemma 4.22. *Let us suppose that $(\varphi_k)_{k\in\mathbb{N}}\in Y^{\mathbb{N}}$ weakly converges to φ in Y and strongly converges to φ in $L^2(Q)$. Then, we have*

$$\forall r\in\mathbb{N},\ \int_0^T\int_\Omega e^{2s\alpha}(s\phi)^r|\varphi_k|^2dxdt \underset{k\to+\infty}{\to} \int_0^T\int_\Omega e^{2s\alpha}(s\phi)^r|\varphi|^2dxdt,$$
$$\|\varphi(0,.)\|_{L^2(\Omega)} \le \liminf_{k\to+\infty}\|\varphi_k(0,.)\|_{L^2(\Omega)}.$$

Proof. The result is a consequence of the fact that $e^{2s\alpha}(s\phi)^r\in L^\infty(Q)$ and Proposition 2.2. □

4.3.5 Proof with observation on two components: (94)

4.3.5.1 Another parabolic regularity result For the cases $j=2$ (2 controls) and $j=1$ (1 control), the diffusion matrix is not diagonal (see (61) and (76)). It creates coupling terms of second order. Roughly speaking, we differentiate some equations of the adjoint system (91) in order to benefit from these coupling terms before applying Carleman estimates. The following lemma justifies this strategy.

Lemma 4.23. *Let $d\in(0,+\infty)$, $f\in L^2(0,T;H^2_{Ne}(\Omega))$ and $y_0\in C_0^\infty(\Omega)$. Let $y\in Y_2$ be the solution of*

$$\begin{cases} \partial_t y - d\Delta y = f & \text{in } (0,T)\times\Omega, \\ \frac{\partial y}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,.) = y_0 & \text{in } \Omega. \end{cases} \tag{131}$$

Then, $z := \Delta y\in Y_2$ is the solution of

$$\begin{cases} \partial_t z - d\Delta z = \Delta f & \text{in } (0,T)\times\Omega, \\ \frac{\partial z}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ z(0,.) = \Delta y_0 & \text{in } \Omega. \end{cases} \tag{132}$$

Proof. Let $\widetilde{z} \in Y_2$ be the solution of

$$\begin{cases} \partial_t \widetilde{z} - d\Delta\widetilde{z} = \Delta f & \text{in } (0,T)\times\Omega, \\ \frac{\partial \widetilde{z}}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \widetilde{z}(0,.) = \Delta y_0 & \text{in } \Omega. \end{cases} \tag{133}$$

By Proposition 2.2, we have $\widetilde{z} \in C([0,T];L^2(\Omega))$. Moreover, a.e. $t \in [0,T]$,

$$\frac{d}{dt}\int_\Omega \widetilde{z}(t,.) = d\int_\Omega \Delta\widetilde{z}(t,.) + \int_\Omega \Delta f(t,.) = 0.$$

Then, for every $t \in [0,T]$,

$$\int_\Omega \widetilde{z}(t,.) = \int_\Omega \widetilde{z}(0,.) = \int_\Omega \Delta y_0 = 0.$$

For every $t \in [0,T]$, let $\widetilde{y}(t,.)$ be the solution of

$$\begin{cases} \Delta\widetilde{y}(t,.) = \widetilde{z}(t,.) & \text{in } \Omega, \\ \frac{\partial \widetilde{y}(t,.)}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

By elliptic regularity, $\widetilde{y} \in C([0,T];H^2_{Ne}(\Omega)) \subset L^2(0,T;H^2_{Ne}(\Omega))$, $\partial_t\widetilde{y} \in L^2(0,T;H^2_{Ne}(\Omega)) \subset L^2(0,T;L^2(\Omega))$ since $\Delta\partial_t\widetilde{y} = \partial_t\widetilde{z}$. Moreover, $\widetilde{y}$ is the solution of (131) (by applying the operator Δ^{-1} to (133) and by using $\Delta^{-1}\partial_t\widetilde{z} = \partial_t\Delta^{-1}\widetilde{z}$). Then, by uniqueness, $\widetilde{y} = y$ and $\widetilde{z} = \Delta y$ is the solution of (132). □

4.3.5.2 Proof of the observability inequality: (94)

Proof. $\boxed{j=2}$

Let $A \in \mathcal{E}_2$ (see (84)), $\varphi_T \in C_0^\infty(\Omega)^4$ (the general case comes from a density argument, see (151), Lemma 4.21 and Lemma 4.22), $\varphi \in Y_2^4$ be the solution of (91) (see Proposition 4.14), ω_2 and ω_1 be two open subsets such that $\omega'' \subset\subset \omega_2 \subset\subset \omega_1 \subset\subset \omega_0$. Our goal is to prove (94).

We have

$$\forall 1 \le i \le 2,\ \begin{cases} -\partial_t\varphi_i - d_i\Delta\varphi_i = a_{1i}\varphi_1 + a_{2i}\varphi_2 + a_{3i}\varphi_3 & \text{in } (0,T)\times\Omega, \\ -\partial_t\varphi_3 - d_3\Delta\varphi_3 = a_{13}\varphi_1 + a_{23}\varphi_2 + a_{33}\varphi_3 + (d_3-d_4)\Delta\varphi_4 & \text{in } (0,T)\times\Omega, \\ -\partial_t\varphi_4 - d_4\Delta\varphi_4 = u_2^*(\varphi_1 - \varphi_2 + \varphi_3) & \text{in } (0,T)\times\Omega, \\ \frac{\partial\varphi_i}{\partial n} = \frac{\partial\varphi_3}{\partial n} = \frac{\partial\varphi_4}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ (\varphi_i,\varphi_3,\varphi_4)(T,.) = (\varphi_{i,T},\varphi_{3,T},\varphi_{4,T}) & \text{in } \Omega. \end{cases} \tag{134}$$

From (134) and Lemma 4.23, we have

$$\begin{cases} -\partial_t(\Delta\varphi_4) - d_4\Delta(\Delta\varphi_4) = \Delta(u_2^*(\varphi_1 - \varphi_2 + \varphi_3)) & \text{in } (0,T)\times\Omega, \\ \frac{\partial\Delta\varphi_4}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \Delta\varphi_4(T,.) = \Delta\varphi_{4,T} & \text{in } \Omega. \end{cases} \tag{135}$$

We apply the Carleman inequality (99) for (135) with $\beta = 0$ and $\omega' = \omega_2$, for every $\lambda, s \ge C$,

$$I(0,\lambda,s,\Delta\varphi_4) \le C\left(\int_0^T\int_\Omega e^{2s\alpha}(|\Delta\varphi_1|^2 + |\Delta\varphi_2|^2| + |\Delta\varphi_3|^2)dxdt + \int_0^T\int_{\omega_2}\lambda^4 e^{2s\alpha}(s\phi)^3|\Delta\varphi_4|^2dxdt\right). \tag{136}$$

After this, we apply the Carleman inequality (99) for the first two equations of (134) with $\beta = 2$ and $\omega' = \omega_2$ to obtain (by (64)), for every $\lambda, s \ge C$,

$$\begin{aligned} I(2,\lambda,s,\varphi_1) + I(2,\lambda,s,\varphi_2) + I(2,\lambda,s,\varphi_3) \le{}& C\left(\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^2(|\varphi_1|^2 + |\varphi_2|^2| + |\varphi_3|^2 + |\Delta\varphi_4|^2)dxdt\right) \\ &+ C\left(\int_0^T\int_{\omega_2}\lambda^4 e^{2s\alpha}(s\phi)^5(|\varphi_1|^2 + |\varphi_2|^2| + |\varphi_3|^2)dxdt\right). \end{aligned} \tag{137}$$

We sum (136) and (137), for every $\lambda, s \ge C$,

$$\begin{aligned} &I(2,\lambda,s,\varphi_1) + I(2,\lambda,s,\varphi_2) + I(2,\lambda,s,\varphi_3) + I(0,\lambda,s,\Delta\varphi_4) \\ &\le C\left(\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^2(|\varphi_1|^2 + |\varphi_2|^2| + |\varphi_3|^2 + |\Delta\varphi_4|^2) + \int_0^T\int_{\omega_2}\lambda^4 e^{2s\alpha}(s\phi)^5(|\varphi_1|^2 + |\varphi_2|^2| + |\varphi_3|^2)\right) \\ &+ C\left(\int_0^T\int_\Omega e^{2s\alpha}(|\Delta\varphi_1|^2 + |\Delta\varphi_2|^2| + |\Delta\varphi_3|^2) + \int_0^T\int_{\omega_2}\lambda^4 e^{2s\alpha}(s\phi)^3|\Delta\varphi_4|^2\right). \end{aligned} \tag{138}$$

We fix $\lambda \geq C$ and we absorb the first and the third right-hand side terms of (138) by the left-hand side terms of (138), by taking s sufficiently large. Then,

$$I(2,\lambda,s,\varphi_1)+I(2,\lambda,s,\varphi_2)+I(2,\lambda,s,\varphi_3)+I(0,\lambda,s,\Delta\varphi_4)$$
$$\leq C\left(\int_0^T\int_{\omega_2} e^{2s\alpha}(s\phi)^5(|\varphi_1|^2+|\varphi_2|^2|+|\varphi_3|^2)dxdt+\int_0^T\int_{\omega_2} e^{2s\alpha}(s\phi)^3|\Delta\varphi_4|^2dxdt\right). \tag{139}$$

Now, λ, s are supposed to be fixed and the constant C may depend on λ, s.

Then, we have to get rid of $\int_0^T\int_{\omega_2} e^{2s\alpha}(s\phi)^3|\Delta\varphi_4|^2dxdt$ and $\int_0^T\int_{\omega_2} e^{2s\alpha}(s\phi)^5|\varphi_3|^2dxdt$. For the first term, we use the coupling term of second order $(d_3-d_4)\Delta$. For the second term, we use the coupling term of zero order thanks to property (63).

Estimate of $\int_0^T\int_{\omega_2} e^{2s\alpha}(s\phi)^3|\Delta\varphi_4|^2dxdt$.

Let us introduce $\chi_2 \in C^\infty(\overline{\Omega};[0,+\infty[)$, such that the support of χ_2 is included in ω_1 and $\chi_2=1$ in ω_2. We multiply the second equation of (134) by $sign(d_3-d_4)\chi_2(x)e^{2s\alpha}(s\phi)^3\Delta\varphi_4$ and we integrate on $(0,T)\times\omega_1$. As $d_3\neq d_4$, we have

$$\int_0^T\int_{\omega_2} e^{2s\alpha}(s\phi)^3|\Delta\varphi_4|^2dxdt \leq \int_0^T\int_{\omega_1}\chi_2(x)e^{2s\alpha}(s\phi)^3|\Delta\varphi_4|^2dxdt$$
$$\leq C\int_0^T\int_{\omega_1}\chi_2(x)e^{2s\alpha}(s\phi)^3\Delta\varphi_4(-\partial_t\varphi_3-d_3\Delta\varphi_3-a_{13}\varphi_1-a_{23}\varphi_2-a_{33}\varphi_3)dxdt. \tag{140}$$

Let $\varepsilon>0$ which will be chosen small enough. We estimate the right hand side of (140) in the same way as the one of (106):

- for terms involving $\Delta\varphi_4 a_{i3}\varphi_i$ with $1\leq i\leq 3$, we apply (100) with $\Phi=\Delta\varphi_4$, $\Psi=\varphi_i$, $a=a_{i3}\in L^\infty(Q)$, $1\leq i\leq 3$ (recalling (64)), $\Theta=\chi_2$ and $r=k=l=3$,
- for the term involving $\Delta\varphi_4\partial_t\varphi_3$, we apply (101) with $\Phi=\Delta\varphi_4$, $\Psi=\varphi_3$, $a=1$, $\Theta=\chi_2$ and $r=k=3$, $l=7$,
- for the term involving $\Delta\varphi_4\Delta\varphi_3$, we apply (102) with $\Phi=\Delta\varphi_4$, $\Psi=\varphi_3$, $a=d_3$, $\Theta=\chi_2$ and $r=k=3$, $l=7$.

From (139), (140), we get

$$I(2,\lambda,s,\varphi_1)+I(2,\lambda,s,\varphi_2)+I(2,\lambda,s,\varphi_3)+I(0,\lambda,s,\Delta\varphi_4)$$
$$\leq 3\varepsilon\left(\int_Q e^{2s\alpha}(s\phi)^3|\Delta\varphi_4|^2dxdt+\int_Q e^{2s\alpha}(s\phi)|\nabla\Delta\varphi_4|^2dxdt+\int_Q e^{2s\alpha}(s\phi)^{-1}\left(|\partial_t\Delta\varphi_4|^2+|\Delta\Delta\varphi_4|^2\right)dxdt\right)$$
$$+C_\varepsilon\left(\sum_{i=1}^3\int_0^T\int_{\omega_1} e^{2s\alpha}(s\phi)^7|\varphi_i|^2dxdt\right). \tag{141}$$

By taking ε small enough in (141), we get

$$I(2,s,\varphi_1)+I(2,s,\varphi_2)+I(2,s,\varphi_3)+I(0,s,\Delta\varphi_4)\leq C\left(\sum_{i=1}^3\int_0^T\int_{\omega_1} e^{2s\alpha}(s\phi)^7|\varphi_i|^2dxdt\right). \tag{142}$$

Estimate of $\int_0^T\int_{\omega_1} e^{2s\alpha}(s\phi)^7|\varphi_3|^2dxdt$.

Let us introduce $\chi_1\in C^\infty(\overline{\Omega};[0,+\infty[)$, such that the support of χ_1 is included in ω_0 and $\chi_1=1$ in ω_1. We multiply the first equation of the adjoint system (134) with $i=1$ by $-\chi_1(x)e^{2s\alpha}(s\phi)^7\varphi_3$ and we integrate on $(0,T)\times\omega_0$. By using (63), we have

$$\int_0^T\int_{\omega_1} e^{2s\alpha}(s\phi)^7|\varphi_3|^2dxdt\leq\int_0^T\int_{\omega_0}\chi_1(x)e^{2s\alpha}(s\phi)^7|\varphi_3|^2dxdt$$
$$\leq C\int_0^T\int_{\omega_0}\chi_1(x)e^{2s\alpha}(s\phi)^7\varphi_3(-\partial_t\varphi_1-d_1\Delta\varphi_1-a_{11}\varphi_1-a_{21}\varphi_2)dxdt. \tag{143}$$

Let $\varepsilon'>0$ which will be chosen small enough. We estimate the right hand side of (143) in the same way as the one of (106):

- for terms involving $\varphi_3 a_{i1}\varphi_i$ with $1\leq i\leq 2$, we apply (100) with $\Phi=\varphi_3$, $\Psi=\varphi_i$, $a=a_{i3}\in L^\infty(Q)$, $1\leq i\leq 2$ (recalling (64)), $\Theta=\chi_1$ and $r=7$, $k=5$, $l=9$,

- for the term involving $\varphi_3\partial_t\varphi_1$, we apply (101) with $\Phi=\varphi_3$, $\Psi=\varphi_1$, $a=1$, $\Theta=\chi_1$ and $r=7$, $k=5$, $l=13$,
- for the term involving $\varphi_3\Delta\varphi_1$, we apply (102) with $\Phi=\varphi_3$, $\Psi=\varphi_1$, $a=d_1$, $\Theta=\chi_1$ and $r=7$, $k=5$, $l=13$.

Then, we obtain

$$\int_0^T\int_{\omega_1} e^{2s\alpha}(s\phi)^7|\varphi_3|^2 \leq 3\varepsilon'\left(\int_0^T\int_\Omega e^{2s\alpha}((s\phi)^5|\varphi_3|^2+(s\phi)^3|\nabla\varphi_3|^2+(s\phi)(|\partial_t\varphi_3|^2+|\Delta\varphi_3|^2))\right) + C_{\varepsilon'}\left(\sum_{i=1}^2\int_0^T\int_{\omega_0} e^{2s\alpha}(s\phi)^{13}|\varphi_i|^2\right). \tag{144}$$

By using (142), (144) and by taking ε' sufficiently small, we get

$$I(2,\lambda,s,\varphi_1)+I(2,\lambda,s,\varphi_2)+I(2,\lambda,s,\varphi_3)+I(0,\lambda,s,\Delta\varphi_4) \leq C\left(\sum_{i=1}^2\int\int_{(0,T)\times\omega_0} e^{2s\alpha}(s\phi)^{13}|\varphi_i|^2\right). \tag{145}$$

Then, we deduce from (145) that we have

$$\sum_{i=1}^3\left(\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^5|\varphi_i|^2\right)+\int_0^T\int_\Omega e^{2s\widehat{\alpha}}(s\widehat{\phi})^3|\Delta\varphi_4|^2 \leq C\left(\sum_{i=1}^2\int\int_{(0,T)\times\omega_0} e^{2s\alpha}(s\phi)^{13}|\varphi_i|^2\right), \tag{146}$$

where $\widehat{\phi}$ and $\widehat{\alpha}$ are defined in (98). In particular, $\widehat{\phi}$ and $\widehat{\alpha}$ do not depend on the spatial variable x. In order to estimate φ_4 by $\Delta\varphi_4$, we use the classical lemma and the corollary that follow.

Lemma 4.24. ***Poincaré-Wirtinger inequality***
There exists $C=C(\Omega)$ such that

$$\forall u\in H^1(\Omega), \int_\Omega (u(x)-(u)_\Omega)^2\,dx \leq C\int_\Omega|\nabla u(x)|^2dx. \tag{147}$$

Corollary 4.25. *There exists $C=C(\Omega)$ such that*

$$\forall u\in H^2_{Ne}(\Omega):=\left\{u\in H^2(\Omega)\ ;\ \frac{\partial u}{\partial n}=0\right\},\ \int_\Omega|\nabla u(x)|^2dx\leq C\int_\Omega|\Delta u(x)|^2dx. \tag{148}$$

Proof. Let $u\in H^2_{Ne}(\Omega)$. We have by an integration by parts and by using (147),
$\int_\Omega|\nabla u|^2=-\int_\Omega(\Delta u)u=-\int_\Omega(\Delta u)(u-(u)_\Omega)\leq \|\Delta u\|_{L^2(\Omega)}\|u-(u)_\Omega\|_{L^2(\Omega}\leq C\|\Delta u\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega}$. □

Then, by applying the Poincaré-Wirtinger inequality (147) and (148) to φ_4, we deduce from (146) that

$$\sum_{i=1}^3\left(\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^5|\varphi_i|^2\right)+\int_0^T\int_\Omega e^{2s\widehat{\alpha}}(s\widehat{\phi})^3|\varphi_4-(\varphi_4)_\Omega|^2 \leq C\left(\sum_{i=1}^2\int\int_{(0,T)\times\omega_0} e^{2s\alpha}(s\phi)^{13}|\varphi_i|^2\right). \tag{149}$$

Now, from the dissipation in time of the energy of $(\varphi_1,\varphi_2,\varphi_3,\varphi_4-(\varphi_4)_\Omega)$ (see Lemma A.1 in the Appendix), we get

$$\sum_{i=1}^3\left(\|\varphi_i(0,.)\|^2_{L^2(\Omega)}\right)+\|\varphi_4(0,.)-(\varphi_4)_\Omega(0)\|^2_{L^2(\Omega)} \leq C\int_{T/4}^{3T/4}\left(\sum_{i=1}^3\left(\|\varphi_i(t,.)\|^2_{L^2(\Omega)}\right)+\|\varphi_4(t,.)-(\varphi_4)_\Omega(t)\|^2_{L^2(\Omega)}\right)dt. \tag{150}$$

Consequently, from (149), (150) and the same arguments given between (112) and (117), we easily deduce that

$$\sum_{i=1}^3\left(\|\varphi_i(0,.)\|^2_{L^2(\Omega)}\right)+\|\varphi_4(0,.)-(\varphi_4)_\Omega(0)\|^2_{L^2(\Omega)} \leq C\left(\sum_{i=1}^2\int\int_{(0,T)\times\omega} e^{2s\alpha}(s\phi)^{13}|\varphi_i|^2dxdt\right) \tag{151}$$

and consequently the observability inequality (94) because $e^{2s\alpha}(s\phi)^{13}$ is bounded.

This ends the proof of the observability inequality (94). □

4.3.6 Another Carleman inequality

Theorem 4.26. ***Carleman inequality***
Let $d \in (0,+\infty)$, ω' an open subset such that $\omega'' \subset\subset \omega' \subset\subset \omega_0$. There exist $C = C(\Omega,\omega')$, $\lambda_0 = \lambda_0(\Omega,\omega')$ such that, for every $\lambda \geq \lambda_0$, there exists $s_0 = s_0(\Omega,\omega',\lambda)$ such that, for any $s \geq s_0(T+T^2)$, any $\varphi_T \in L^2(\Omega)$ and any $f \in L^2(0,T;H^2_{Ne}(\Omega))$, the solution φ of

$$\left\{\begin{array}{ll} -\partial_t\varphi - d\Delta\varphi = \Delta f & \text{in } (0,T)\times\Omega, \\ \frac{\partial\varphi}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \varphi(T,.) = \varphi_T & \text{in } \Omega, \end{array}\right.$$

satisfies

$$\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^3|\varphi|^2dxdt \leq C\left(\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^4|f|^2dxdt + \int_0^T\int_{\omega'} e^{2s\alpha}(s\phi)^3|\varphi|^2dxdt\right). \tag{152}$$

The proof of this inequality can be found in [12, Lemma A.1] (see in particular that the equality [12, (A.3)] still holds for $f \in L^2(0,T;H^2_{Ne}(\Omega))$).

Remark 4.27. The estimate (152) is different from (99) because (99) gives us

$$\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^3|\varphi|^2dxdt \leq C\left(\int_0^T\int_\Omega e^{2s\alpha}|\Delta f|^2dxdt + \int_0^T\int_{\omega'} e^{2s\alpha}(s\phi)^3|\varphi|^2dxdt\right). \tag{153}$$

Therefore, (152) is useful when one wants an observation of φ in term of f (but not in term of Δf). Roughly, we remark that we have to *pay* this type of estimate with a weight $(s\phi)^4$ (see the first right hand side terms of (152) and (153)).

4.3.7 Proof with observation on one component: (95)

We have seen in Section 4.3.5 that parabolic regularity allows us to apply Δ to the third equation of (134) (see (135)) in order to benefit from the coupling term of second order $(d_3-d_4)\Delta\varphi_4$. The case $j=1$ needs more regularity because we have to benefit from **two** terms of coupling of second order. Therefore, we need to apply $\Delta\Delta$ (see (156)). There are two main difficulties. First, Proposition 4.14 only shows us that φ, the solution of (91) is in Y_2^4. However, we need: $\Delta\varphi \in Y_2^4$. That is why we regularize the coupling matrix $A \in \mathcal{E}_1$ (see Lemma 4.18). Secondly, we want an observation of $\Delta\Delta\varphi_4$ in term of $\Delta\varphi_1$, $\Delta\varphi_2$ (and not in term of $\Delta\Delta\varphi_1$, $\Delta\Delta\varphi_2$ because we do not have these terms in Carleman estimates applied to φ_1 and φ_2: see (160) and (161)). That's why we use Theorem 4.26.

Proof. $\boxed{j=1}$

Let $A \in \mathcal{M}_4(C_0^\infty(Q)) \cap \mathcal{E}_1$ (see (86)), $\varphi_T \in C_0^\infty(\Omega)^4$ (the general case comes from a density argument, see (183), Lemma 4.18, Lemma 4.21 and Lemma 4.22), $\varphi \in Y_2^4$ be the solution of (91) (see Proposition 4.14), ω_3, ω_2, ω_2' and ω_1 be four open subsets such that $\omega'' \subset\subset \omega_3 \subset\subset \omega_2 \subset\subset \omega_2' \subset\subset \omega_1 \subset\subset \omega_0$. Our goal is to prove (95).

We have

$$\left\{\begin{array}{ll} -\partial_t\varphi_1 - d_1\Delta\varphi_1 = a_{11}\varphi_1 + a_{21}\varphi_2 & \text{in } (0,T)\times\Omega, \\ -\partial_t\varphi_2 - d_2\Delta\varphi_2 = a_{12}\varphi_1 + a_{22}\varphi_2 + (d_2-d_3)\Delta\varphi_3 & \text{in } (0,T)\times\Omega, \\ -\partial_t\varphi_3 - d_3\Delta\varphi_3 = -m_2(\varphi_1-\varphi_2) + \Delta\varphi_4 & \text{in } (0,T)\times\Omega, \\ -\partial_t\varphi_4 - d_4\Delta\varphi_4 = m_3(\varphi_1-\varphi_2) & \text{in } (0,T)\times\Omega, \\ \frac{\partial\varphi}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \varphi(T,.) = \varphi_T & \text{in } \Omega. \end{array}\right. \tag{154}$$

First, by applying consecutively Lemma 4.23 to the fourth equation of (154), the third equation of (154), the second equation of (154), the first equation of (154), we get

$$\Delta\varphi \in L^2(0,T;H^2_{Ne}(\Omega))^4. \tag{155}$$

Consequently, we can apply $\Delta\Delta$ to the fourth equation of (154) by using (155) and Lemma 4.23,

$$\left\{\begin{array}{ll} -\partial_t(\Delta\Delta\varphi_4) - d_4\Delta(\Delta\Delta\varphi_4) = \Delta\Delta(m_3(\varphi_1-\varphi_2)) & \text{in } (0,T)\times\Omega, \\ \frac{\partial\Delta\Delta\varphi_4}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \Delta\Delta\varphi_4(T,.) = \Delta\Delta\varphi_{4,T} & \text{in } \Omega. \end{array}\right. \tag{156}$$

Then, we use the Carleman inequality (152) for (156) with $\omega' = \omega_3$ and $f = \Delta(m_3(\varphi_1 - \varphi_2)) \in L^2(0,T;H^2_{Ne}(\Omega))$, for every $\lambda, s \geq C$,

$$\int_0^T \int_\Omega e^{2s\alpha}(s\phi)^3 |\Delta\Delta\varphi_4|^2 \leq C\left(\int_0^T \int_\Omega e^{2s\alpha}(s\phi)^4 \left(|\Delta\varphi_1|^2 + |\Delta\varphi_2|^2\right) + \int_0^T \int_{\omega_3} e^{2s\alpha}(s\phi)^3 |\Delta\Delta\varphi_4|^2\right). \quad (157)$$

Remark 4.28. Here, we have to apply the Carleman estimate (152) instead of (99) in order to get in the right hand side of (157) only terms of order two (and not more) in φ_1, φ_2. Otherwise, we can't absorb the remaining terms thanks to Carleman estimates (99) applied to φ_1, φ_2.

Then, we apply Δ to the third equation of (154) thanks to (156) and Lemma 4.23, for every $\lambda, s \geq C$,

$$\begin{cases} -\partial_t(\Delta\varphi_3) - d_3\Delta(\Delta\varphi_3) = \Delta(-m_2(\varphi_1 - \varphi_2)) + \Delta\Delta\varphi_4 & \text{in } (0,T)\times\Omega, \\ \frac{\partial\Delta\varphi_3}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \Delta\varphi_3(T,.) = \Delta\varphi_{3,T} & \text{in } \Omega. \end{cases} \quad (158)$$

We use the Carleman inequality (99) with $\omega' = \omega_3$ and $\beta = 2$, for every $\lambda, s \geq C$,

$$I(2,\lambda,s,\Delta\varphi_3) \leq C\left(\int_0^T \int_\Omega e^{2s\alpha}(s\phi)^2(|\Delta\varphi_1|^2 + |\Delta\varphi_2|^2| + |\Delta\Delta\varphi_4|^2) + \int_0^T \int_{\omega_2} \lambda^4 e^{2s\alpha}(s\phi)^5 |\Delta\varphi_3|^2\right). \quad (159)$$

Then, we apply the Carleman inequality (99) with $\omega' = \omega_3$ and $\beta = 5$ to the second equation and the first equation of (154) (by (79)), for every $\lambda, s \geq C$,

$$\lambda I(5,\lambda,s,\varphi_2) \leq C\left(\int_0^T \int_\Omega \lambda e^{2s\alpha}(s\phi)^5(|\varphi_1|^2 + |\varphi_2|^2| + |\Delta\varphi_3|^2) + \int_0^T \int_{\omega_3} \lambda^5 e^{2s\alpha}(s\phi)^8 |\varphi_2|^2\right), \quad (160)$$

$$\lambda I(5,\lambda,s,\varphi_1) \leq C\left(\int_0^T \int_\Omega \lambda e^{2s\alpha}(s\phi)^5(|\varphi_1|^2 + |\varphi_2|^2|) + \int_0^T \int_{\omega_3} \lambda^5 e^{2s\alpha}(s\phi)^8 |\varphi_1|^2\right). \quad (161)$$

We sum (157), (159), (160), (161) and we take λ and s sufficiently large,

$$\begin{aligned} &\int_0^T \int_\Omega e^{2s\alpha}(s\phi)^3 |\Delta\Delta\varphi_4|^2 dxdt + I(2,\lambda,s,\Delta\varphi_3) + \lambda I(5,\lambda,s,\varphi_2) + \lambda I(5,\lambda,s,\varphi_1) \\ &\leq C\left(\int_0^T \int_{\omega_3} e^{2s\alpha}(s\phi)^3 |\Delta\Delta\varphi_4|^2 dxdt + \int_0^T \int_{\omega_3} \lambda^4 e^{2s\alpha}(s\phi)^5 |\Delta\varphi_3|^2 dxdt\right) \\ &+ C\left(\int_0^T \int_{\omega_3} \lambda^5 e^{2s\alpha}(s\phi)^8 |\varphi_2|^2 dxdt + \int_0^T \int_{\omega_3} \lambda^5 e^{2s\alpha}(s\phi)^8 |\varphi_1|^2 dxdt\right). \end{aligned} \quad (162)$$

Now, λ and s are supposed to be fixed. The constant C may depend on λ and s. We have

$$\begin{aligned} &\int_0^T \int_\Omega e^{2s\alpha}(s\phi)^3 |\Delta\Delta\varphi_4|^2 dxdt + I(2,\lambda,s,\Delta\varphi_3) + I(5,\lambda,s,\varphi_2) + I(5,\lambda,s,\varphi_1) \\ &\leq C\left(\int_0^T \int_{\omega_3} e^{2s\alpha}(s\phi)^3 |\Delta\Delta\varphi_4|^2 dxdt + \int_0^T \int_{\omega_3} e^{2s\alpha}(s\phi)^5 |\Delta\varphi_3|^2 dxdt\right) \\ &+ C\left(\int_0^T \int_{\omega_3} e^{2s\alpha}(s\phi)^8 |\varphi_2|^2 dxdt + \int_0^T \int_{\omega_3} e^{2s\alpha}(s\phi)^8 |\varphi_1|^2 dxdt\right). \end{aligned} \quad (163)$$

Remark 4.29. Here, we take advantage of the two parameters λ and s in Theorem 4.11. Indeed, if we forget λ, we would need to sum $\int_0^T \int_\Omega e^{2s\alpha}(s\phi)^3 |\Delta\Delta\varphi_4|^2 dxdt$, $I(4,s,\Delta\varphi_3)$, $I(6,s,\varphi_2)$ and $I(6,s,\varphi_1)$. Therefore, we would get in the right hand side $\int_0^T \int_\Omega e^{2s\alpha}(s\phi)^4 |\Delta\Delta\varphi_4|^2 dxdt$ which can't be absorbed by the left hand side.

Then ,we have to get rid of $\int_0^T \int_{\omega_3} e^{2s\alpha}(s\phi)^3 |\Delta\Delta\varphi_4|^2 dxdt$, $\int_0^T \int_{\omega_3} e^{2s\alpha}(s\phi)^5 |\Delta\varphi_3|^2 dxdt$ and $\int_0^T \int_{\omega_3} e^{2s\alpha}(s\phi)^8 |\varphi_2|^2 dxdt$. For the first term, we use the coupling term of fourth order $\Delta\Delta$. For the second term, we use the coupling term of second order $(d_2 - d_3)\Delta$. For the third term, we use the coupling term of zero order thanks to property (78).

Estimate of $\int_0^T \int_{\omega_3} e^{2s\alpha}(s\phi)^3 |\Delta\Delta\varphi_4|^2 dxdt$.

Let us introduce $\chi_3 \in C^\infty(\overline{\Omega};[0,+\infty[)$, such that the support of χ_3 is included in ω_2 and $\chi_3 = 1$ in ω_3. We multiply the first equation (158) by $(\chi_3(x))^2 e^{2s\alpha}(s\phi)^3 \Delta\Delta\varphi_4$ and we integrate on $(0,T)\times\omega_2$. We have

$$\begin{aligned}&\int_0^T\int_{\omega_2}(\chi_3(x))^2 e^{2s\alpha}(s\phi)^3|\Delta\Delta\varphi_4|^2 dxdt\\&\leq C\int_0^T\int_{\omega_2}(\chi_3(x))^2 e^{2s\alpha}(s\phi)^3\Delta\Delta\varphi_4(-\partial_t\Delta\varphi_3 - d_3\Delta\Delta\varphi_3 + m_2\Delta\varphi_1 - m_2\Delta\varphi_2)dxdt.\end{aligned}\tag{164}$$

Remark 4.30. One can see the presence of $(\chi_3(x))^2$ instead of $\chi_3(x)$ as before (see for example (106)). It is purely technical (see the proofs of Lemma 4.31 and Lemma 4.32).

Let $\varepsilon \in (0,1)$ which will be chosen small enough. First, for every $1 \leq i \leq 2$, by applying Lemma 4.15: (100) with $\Phi = \Delta\Delta\varphi_4$, $\Psi = \Delta\varphi_i$, $a = m_2$, $\Theta = (\chi_3)^2$, $r = 3$ and $(k,l) = (3,3)$, we have

$$\int_0^T\int_{\omega_2}\chi_3^2 e^{2s\alpha}(s\phi)^3(\Delta\Delta\varphi_4)m_2\Delta\varphi_i \leq \varepsilon\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^3|\Delta\Delta\varphi_4|^2 + C_\varepsilon\int_0^T\int_{\omega_2}\chi_3^2 e^{2s\alpha}(s\phi)^3|\Delta\varphi_i|^2.\tag{165}$$

But, the other terms in the right hand side of (164) i.e. $\int_0^T\int_{\omega_2}(\chi_3(x))^2 e^{2s\alpha}(s\phi)^3(\Delta\Delta\varphi_4)(\partial_t\Delta\varphi_3)dxdt$ and $\int_0^T\int_{\omega_2}(\chi_3(x))^2 e^{2s\alpha}(s\phi)^3(\Delta\Delta\varphi_4)(\Delta\Delta\varphi_3)dxdt$ can't be estimated as in Lemma 4.15 because we have not enough derivative terms in φ_4 in the left hand side of (163). In order to estimate these two terms, we follow the strategy developed in the proof of [12, Theorem 2.2] (see Appendix A.3 for the proof of the two following lemmas).

Lemma 4.31. *We have*

$$\begin{aligned}&\int_0^T\int_{\omega_2}\chi_3^2 e^{2s\alpha}(s\phi)^3(\Delta\Delta\varphi_4)(\Delta\Delta\varphi_3)\\&\leq\varepsilon\left(\int_0^T\int_\Omega e^{2s\alpha}\Big\{(s\phi)^4(|\Delta\varphi_1|^2+|\Delta\varphi_2|^2)+(s\phi)|\Delta\Delta\varphi_3|^2+(s\phi)^3|\Delta\Delta\varphi_4|^2\Big\}\right)\\&+C_\varepsilon\left(\int_0^T\int_{\omega_2} e^{2s\alpha}\Big\{(s\phi)^{24}(|\varphi_1|^2+|\varphi_2|^2+|\Delta\varphi_3|^2)+(s\phi)^{22}(|\nabla\varphi_1|^2+|\nabla\varphi_2|^2+|\nabla\Delta\varphi_3|^2)\Big\}\right).\end{aligned}\tag{166}$$

Lemma 4.32. *We have*

$$\begin{aligned}&\int_0^T\int_{\omega_2}\chi_3^2 e^{2s\alpha}(s\phi)^3(\Delta\Delta\varphi_4)(\partial_t\Delta\varphi_3)\\&\leq\varepsilon\left(\int_0^T\int_\Omega e^{2s\alpha}\Big\{(s\phi)^4(|\Delta\varphi_1|^2+|\Delta\varphi_2|^2)+(s\phi)|\partial_t\Delta\varphi_3|^2+(s\phi)^3|\Delta\Delta\varphi_4|^2\Big\}\right)\\&+C_\varepsilon\left(\int_0^T\int_{\omega_2} e^{2s\alpha}\Big\{(s\phi)^{24}(|\varphi_1|^2+|\varphi_2|^2+|\Delta\varphi_3|^2)+(s\phi)^{22}(|\nabla\varphi_1|^2+|\nabla\varphi_2|^2+|\nabla\Delta\varphi_3|^2)\Big\}\right).\end{aligned}\tag{167}$$

Moreover, the proof of these two lemmas (see (309)) provides us another estimate which is useful to treat the right hand side of (165).

Lemma 4.33. *For every $1 \leq i \leq 2$, $\delta > 0$, we have*

$$\begin{aligned}&\int_0^T\int_{\omega_2} e^{2s\alpha}(s\phi)^3|\Delta\varphi_i|^2\\&\leq\delta\left(\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^4|\Delta\varphi_i|^2\right)+C_\delta\left(\int_0^T\int_{\omega_2} e^{2s\alpha}\Big\{(s\phi)^{24}(|\varphi_1|^2+|\varphi_2|^2+|\Delta\varphi_3|^2)+(s\phi)^{22}|\nabla\varphi_i|^2\Big\}\right).\end{aligned}\tag{168}$$

Gathering (165) and (168) with $\delta = \varepsilon/C_\varepsilon$, we find that for $1 \leq i \leq 2$,

$$\begin{aligned}&\int_0^T\int_{\omega_2}(\chi_3(x))^2 e^{2s\alpha}(s\phi)^3(\Delta\Delta\varphi_4)m_2\Delta\varphi_i dxdt\\&\leq\varepsilon\left(\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^3|\Delta\Delta\varphi_4|^2dxdt+\int_0^T\int_{\omega_2} e^{2s\alpha}(s\phi)^4|\Delta\varphi_i|^2dxdt\right)\\&+C_\varepsilon\left(\int_0^T\int_{\omega_2} e^{2s\alpha}(s\phi)^{24}(|\varphi_1|^2+|\varphi_2|^2+|\Delta\varphi_3|^2)dxdt+\int_0^T\int_{\omega_2} e^{2s\alpha}(s\phi)^{22}|\nabla\varphi_i|^2dxdt\right).\end{aligned}\tag{169}$$

From (164), (169), (166), (167), we get

$$\begin{aligned}
&\int_0^T \int_{\omega_2} (\chi_3(x))^2 e^{2s\alpha}(s\phi)^3 |\Delta\Delta\varphi_4|^2 dxdt \\
&\leq \varepsilon \left(\int_0^T \int_\Omega e^{2s\alpha} \Big\{ (s\phi)^4 (|\Delta\varphi_1|^2 + |\Delta\varphi_2|^2) + (s\phi)(|\partial_t \Delta\varphi_3|^2 + |\Delta\Delta\varphi_3|^2) + (s\phi)^3 |\Delta\Delta\varphi_4|^2 \Big\} dxdt \right) \\
&+ C_\varepsilon \left(\int_0^T \int_{\omega_2} e^{2s\alpha} \Big\{ (s\phi)^{24} (|\varphi_1|^2 + |\varphi_2|^2 + |\Delta\varphi_3|^2) + (s\phi)^{22} (|\nabla\varphi_1|^2 + |\nabla\varphi_2|^2 + |\nabla\Delta\varphi_3|^2) \Big\} dxdt \right). \qquad (170)
\end{aligned}$$

By using (163), (170) and by taking ε small enough, we have

$$\begin{aligned}
&\int_0^T \int_\Omega e^{2s\alpha}(s\phi)^3 |\Delta\Delta\varphi_4|^2 dxdt + I(2,\lambda,s,\Delta\varphi_3) + I(5,\lambda,s,\varphi_2) + I(5,\lambda,s,\varphi_1) \\
&\leq C \left(\int_0^T \int_{\omega_2} e^{2s\alpha} \Big\{ (s\phi)^{24} (|\varphi_1|^2 + |\varphi_2|^2 + |\Delta\varphi_3|^2) + (s\phi)^{22} (|\nabla\varphi_1|^2 + |\nabla\varphi_2|^2 + |\nabla\Delta\varphi_3|^2) \Big\} dxdt \right). \qquad (171)
\end{aligned}$$

Estimate of $\int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)^{22} |\nabla\Delta\varphi_3|^2 dxdt$.

Let us introduce $\widetilde{\chi_2} \in C^\infty(\overline{\Omega}; [0; +\infty[)$ such that $supp(\widetilde{\chi_2}) \subset \omega_2'$ and $\widetilde{\chi_2} = 1$ on ω_2. Then, by Lemma 4.15: (103) (with $\Phi = \Delta\varphi_3$, $\widetilde{\omega} = \omega_2$, $\Theta = \widetilde{\chi_2}$, $r = 22$ and $(k,l) = (1,43)$), for any $\varepsilon' > 0$, we have

$$\begin{aligned}
&\int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)^{22} |\nabla\Delta\varphi_3|^2 dxdt \\
&\leq \int_0^T \int_{\omega_2'} \widetilde{\chi_2} e^{2s\alpha}(s\phi)^{22} |\nabla\Delta\varphi_3|^2 dxdt \\
&\leq \varepsilon' \left(\int_0^T \int_\Omega e^{2s\alpha} \Big\{ (s\phi) |\Delta\Delta\varphi_3|^2 + (s\phi)^3 |\nabla\Delta\varphi_3|^2 \Big\} dxdt \right) + C_{\varepsilon'} \int_0^T \int_{\omega_2'} e^{2s\alpha}(s\phi)^{43} |\Delta\varphi_3|^2 dxdt. \qquad (172)
\end{aligned}$$

By taking ε' small enough and by using (171) and (172), we have

$$\begin{aligned}
&\int_0^T \int_\Omega e^{2s\alpha}(s\phi)^3 |\Delta\Delta\varphi_4|^2 dxdt + I(2,\lambda,s,\Delta\varphi_3) + I(5,\lambda,s,\varphi_2) + I(5,\lambda,s,\varphi_1) \\
&\leq C \left(\int_0^T \int_{\omega_2''} e^{2s\alpha}(s\phi)^{43} (|\varphi_1|^2 + |\varphi_2|^2 + |\Delta\varphi_3|^2) dxdt + \int_0^T \int_{\omega_2''} e^{2s\alpha}(s\phi)^{22} (|\nabla\varphi_1|^2 + |\nabla\varphi_2|^2) dxdt \right). \qquad (173)
\end{aligned}$$

Estimate of $\int_0^T \int_{\omega_2'} e^{2s\alpha}(s\phi)^{43} |\Delta\varphi_3|^2 dxdt$.

Let us introduce $\chi_2 \in C^\infty(\overline{\Omega}; [0, +\infty[)$, such that the support of χ_2 in included in ω_1 and $\chi_2 = 1$ in ω_2'. We multiply the second equation of (154) by $sign(d_2 - d_3)\chi_2(x) e^{2s\alpha}(s\phi)^{45} \Delta\varphi_3$ and we integrate on $(0,T) \times \omega_1$. As $d_2 \neq d_3$, we have

$$\begin{aligned}
&\int_0^T \int_{\omega_1} \chi_2(x) e^{2s\alpha}(s\phi)^{43} |\Delta\varphi_3|^2 dxdt \\
&\leq C \int_0^T \int_{\omega_1} \chi_2(x) e^{2s\alpha}(s\phi)^{43} \Delta\varphi_3 (-\partial_t\varphi_2 - d_2\Delta\varphi_2 - a_{12}\varphi_1 - a_{22}\varphi_2) dxdt. \qquad (174)
\end{aligned}$$

Let $\varepsilon'' > 0$ which will be chosen small enough. We estimate the right hand side of (174) in the same way as the one of (106):

- for terms involving $\Delta\varphi_3 a_{i2}\varphi_i$ with $1 \leq i \leq 2$, we apply (100) with $\Phi = \Delta\varphi_3$, $\Psi = \varphi_i$, $a = a_{i2} \in L^\infty(Q)$, $1 \leq i \leq 2$ (recalling (79)), $\Theta = \chi_2$ and $r = 43$, $k = 5$, $l = 81$,
- for the term involving $\Delta\varphi_3 \partial_t\varphi_2$, we apply (101) with $\Phi = \Delta\varphi_3$, $\Psi = \varphi_2$, $a = 1$, $\Theta = \chi_2$ and $r = 43$, $k = 5$, $l = 85$,
- for the term involving $\Delta\varphi_3\Delta\varphi_2$, we apply (102) with $\Phi = \Delta\varphi_3$, $\Psi = \varphi_2$, $a = d_2$, $\Theta = \chi_2$ and $r = 43$, $k = 5$, $l = 85$.

We get

$$\begin{aligned}
&\int_0^T\int_{\omega_1}\chi_2(x)e^{2s\alpha}(s\phi)^{43}|\Delta\varphi_3|^2dxdt\\
&\leq \varepsilon''\left(\int_0^T\int_\Omega e^{2s\alpha}\Big\{(s\phi)^5|\Delta\varphi_3|^2+(s\phi)^3|\nabla\Delta\varphi_3|^2+(s\phi)(|\partial_t\Delta\varphi_3|^2+|\Delta\Delta\varphi_3|^2\Big\}dxdt\right)\\
&\quad+C_{\varepsilon''}\int_0^T\int_{\omega_1}e^{2s\alpha}(s\phi)^{85}(|\varphi_1|^2+|\varphi_2|^2)dxdt. \qquad (175)
\end{aligned}$$

By taking ε'' sufficiently small, we get from (173), (175)

$$\begin{aligned}
&\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^3|\Delta\Delta\varphi_4|^2dxdt+I(2,\lambda,s,\Delta\varphi_3)+I(5,\lambda,s,\varphi_2)+I(5,\lambda,s,\varphi_1)\\
&\leq C\int_0^T\int_{\omega_1}e^{2s\alpha}(s\phi)^{85}(|\varphi_1|^2+|\varphi_2|^2)dxdt+\int_0^T\int_{\omega_2}e^{2s\alpha}(s\phi)^{22}(|\nabla\varphi_1|^2+|\nabla\varphi_2|^2)dxdt. \qquad (176)
\end{aligned}$$

Estimate of $\int_0^T\int_{\omega_2}e^{2s\alpha}(s\phi)^{22}|\nabla\varphi_i|^2dxdt$ for $1\leq i\leq 2$.

Applying Lemma 4.15: (103) (with $\Phi=\varphi_i$, $\widetilde{\omega}=\omega_1$, $\Theta=\chi_2$, $r=22$ and $(k,l)=(4,40)$), for any $\varepsilon'''>0$, we have

$$\begin{aligned}
&\int_0^T\int_{\omega_2}e^{2s\alpha}(s\phi)^{22}|\nabla\varphi_i|^2dxdt\\
&\leq\int_0^T\int_{\omega_1}\chi_2e^{2s\alpha}(s\phi)^{22}|\nabla\varphi_i|^2dxdt\\
&\leq\varepsilon'''\left(\int_0^T\int_\Omega e^{2s\alpha}\Big\{(s\phi)^4|\Delta\varphi_i|^2+(s\phi)^6|\nabla\varphi_i|^2\Big\}dxdt\right)+C_{\varepsilon'''}\int_0^T\int_{\omega_1}e^{2s\alpha}(s\phi)^{40}|\varphi_i|^2dxdt. \qquad (177)
\end{aligned}$$

By taking ε''' small enough and by using (176) and (177), we have

$$\begin{aligned}
&\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^3|\Delta\Delta\varphi_4|^2dxdt+I(2,\lambda,s,\Delta\varphi_3)+I(5,\lambda,s,\varphi_2)+I(5,\lambda,s,\varphi_1)\\
&\leq C\int_0^T\int_{\omega_1}e^{2s\alpha}(s\phi)^{85}(|\varphi_1|^2+|\varphi_2|^2)dxdt. \qquad (178)
\end{aligned}$$

Estimate of $\int_0^T\int_{\omega_1}e^{2s\alpha}(s\phi)^{85}|\varphi_2|^2dxdt$.

Let us introduce $\chi_1\in C^\infty(\overline{\Omega};[0,+\infty[)$, such that the support of χ_1 in included in ω_0 and $\chi_1=1$ in ω_1. We multiply the first equation of (91) by $\chi_1(x)e^{2s\alpha}(s\phi)^{85}\varphi_2$ and we integrate on $(0,T)\times\omega_0$. Recalling (78), we have

$$\begin{aligned}
&\int_0^T\int_{\omega_0}\chi_1(x)e^{2s\alpha}(s\phi)^{85}|\varphi_2|^2dxdt\\
&\leq C\int_0^T\int_{\omega_0}\chi_1(x)e^{2s\alpha}(s\phi)^{85}\varphi_2(-\partial_t\varphi_1-d_1\Delta\varphi_2-a_{11}\varphi_1)dxdt. \qquad (179)
\end{aligned}$$

We estimate the right hand side of (179) in the same way as the one of (106):

- for the term involving $\varphi_2a_{11}\varphi_1$, we apply (100) with $\Phi=\varphi_2$, $\Psi=\varphi_1$, $a=a_{11}\in L^\infty(Q)$ (recalling (79)), $\Theta=\chi_1$ and $r=85$, $k=8$, $l=162$,
- for the term involving $\varphi_2\partial_t\varphi_1$, we apply (101) with $\Phi=\varphi_2$, $\Psi=\varphi_1$, $a=1$, $\Theta=\chi_1$ and $r=85$, $k=8$, $l=166$,
- for the term involving $\varphi_2\Delta\varphi_1$, we apply (102) with $\Phi=\varphi_2$, $\Psi=\varphi_1$, $a=d_1$, $\Theta=\chi_1$ and $r=85$, $k=8$, $l=166$.

We get

$$\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^3|\Delta\Delta\varphi_4|^2+I(2,\lambda,s,\Delta\varphi_3)+I(5,\lambda,s,\varphi_2)+I(5,\lambda,s,\varphi_1)\leq C\int_0^T\int_{\omega_0}e^{2s\alpha}(s\phi)^{166}|\varphi_1|^2. \qquad (180)$$

Then, we can deduce from (98) and (180)

$$\sum_{i=1}^{2}\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^8|\varphi_i|^2+\int_0^T\int_\Omega e^{2s\widehat{\alpha}}\Big\{(s\widehat{\phi})^5|\Delta\varphi_3|^2+(s\widehat{\phi})^3|\Delta\Delta\varphi_4|^2\Big\}\leq C\int_0^T\int_\omega e^{2s\alpha}(s\phi)^{166}|\varphi_1|^2. \quad (181)$$

Now, we use Poincaré-Wirtinger inequality as in (149) to get

$$\begin{aligned}&\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^8(|\varphi_1|^2+|\varphi_2|^2)+e^{2s\widehat{\alpha}}\Big\{(s\widehat{\phi})^5|\varphi_3-(\varphi_3)_\Omega|^2+(s\widehat{\phi})^3|\varphi_4-(\varphi_4)_\Omega|^2\Big\}\\&\leq C\int_0^T\int_\omega e^{2s\alpha}(s\phi)^{166}|\varphi_1|^2. \quad (182)\end{aligned}$$

Now, from the dissipation of the energy of $(\varphi_1,\varphi_2,\varphi_3-(\varphi_3)_\Omega,\varphi_4-(\varphi_4)_\Omega)$ (see Lemma A.1 in Appendix A) and by using the same arguments as for 2 controls (see (150) and (151)), we easily get

$$\sum_{i=1}^{2}\|\varphi_i(0,.)\|^2_{L^2(\Omega)}+\sum_{i=3}^{4}\|\varphi_i(0,.)-(\varphi_i)(0,.)_\Omega\|^2_{L^2(\Omega)}\leq C\int_0^T\int_\omega e^{2s\alpha}(s\phi)^{166}|\varphi_1|^2dxdt, \quad (183)$$

and consequently the observability inequality (95).

This ends the proof of the observability inequality (95). □

4.4 Second step: Controls in $L^\infty(Q)^j$

4.4.1 Penalized Hilbert Uniqueness Method

The proof in this subsection follows ideas of [8] and [16, Section 3.1.2]. The goal is to get more regular controls in a some sense (see (202)) by considering a penalized problem.

Let $\varepsilon\in(0,1)$ and

$$M_3:=7,\ M_2:=13,\ M_1:=166.$$

We choose λ and s large enough such that (118), (151), (183) hold.

Let $j\in\{1,2,3\}$, $A\in\mathcal{E}_j$ (see (86), (84) and (82)), $\zeta_0\in H_j$ (see (87), (85), (83)). We define $J:L^2((0,T)\times\omega)^j\to\overline{\mathbb{R}^+}$,

$$\forall h^j\in L^2((0,T)\times\omega)^j,\ J(h^j)=\frac{1}{2}\int\int_{(0,T)\times\omega}e^{-2s\alpha}(s\phi)^{-M_j}|h^j|^2dxdt+\frac{1}{2\varepsilon}\|\zeta(T,.)\|^2_{L^2(\Omega)^4},$$

where $\zeta=(\zeta_1,\zeta_2,\zeta_3,\zeta_4)$ is the solution to the Cauchy problem (89) associated to the control h^j.

We condider the extremal problem

$$\inf_{h^j\in L^2((0,T)\times\omega)^j}J(h^j). \quad (184)$$

We have $e^{2s\alpha}(s\phi)^{M_j}\in L^\infty(Q)$. So, if $(e^{-2s\alpha}(s\phi)^{-M_j})^{1/2}h^j\in L^2((0,T)\times\omega)^j$ then $h^j\in L^2((0,T)\times\omega)^j$, otherwise $J(h^j)=+\infty$. Therefore,

$$\inf_{h^j\in L^2((0,T)\times\omega)^j}J(h^j)=\inf_{(e^{-2s\alpha}(s\phi)^{-M_j})^{1/2}h^j\in L^2((0,T)\times\omega)^j}J(h^j).$$

We introduce the notation $L^2_{wght}((0,T)\times\omega)^j$ for the set of functions h^j such that for every $1\leq i\leq j$, $(e^{-2s\alpha}(s\phi)^{-M_j})^{1/2}h_i\in L^2((0,T)\times\omega)$. $L^2_{wght}((0,T)\times\omega_0)^j$ is an Hilbert space equipped with the inner product $(h,k)=\sum_{i=1}^{j}\int\int_{(0,T)\times\omega_0}e^{-2s\alpha}(s\phi)^{-M_j}h_ik_idxdt$. We define

$$\forall h^j\in L^2_{wght}((0,T)\times\omega)^j,\ \widetilde{J}(h^j)=J(h^j).$$

$\widetilde{J}$ is a continuous, coercive, strictly convex functional on the Hilbert space $L^2_{wght}((0,T)\times\omega)^j$, then $\widetilde{J}$ has a unique minimum $h^{j,\varepsilon}$ with $(e^{-2s\alpha}(s\phi)^{-M_j})^{1/2}h^{j,\varepsilon}\in L^2((0,T)\times\omega)^j$. Let ζ^ε be the solution to the Cauchy problem (89) with control $h^{j,\varepsilon}$ and initial condition ζ_0.

The Euler-Lagrange equation gives

$$\forall h^j\in L^2_{wght}((0,T)\times\omega)^j,\ \sum_{i=1}^{j}\int\int_{(0,T)\times\omega}e^{-2s\alpha}(s\phi)^{-M_j}h_i^\varepsilon h_idxdt+\frac{1}{\varepsilon}\int_\Omega\zeta^\varepsilon(T,x).\zeta(T,x)dx=0, \quad (185)$$

where $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ is the solution to the Cauchy problem (89) associated to the control h^j and initial condition $\zeta_0 = 0$.

We introduce φ^ε the solution to the adjoint problem (91) with final condition $\varphi^\varepsilon(T,.) = -\frac{1}{\varepsilon}\zeta^\varepsilon(T,.)$. A duality argument between ζ and φ^ε gives

$$-\frac{1}{\varepsilon}\int_\Omega \zeta^\varepsilon(T,x).\zeta(T,x)dx = \sum_{i=1}^{j}\int\int_{(0,T)\times\omega_0} h_i\varphi_i^\varepsilon dxdt. \tag{186}$$

Then, we deduce from (185) and (186) that

$$\forall h^j \in L^2_{wght}((0,T)\times\omega)^j,\ \sum_{i=1}^{j}\int\int_{(0,T)\times\omega} e^{-2s\alpha}(s\phi)^{-M_j}h_i^\varepsilon h_i dxdt = \sum_{i=1}^{j}\int\int_{(0,T)\times\omega} h_i\varphi_i^\varepsilon dxdt.$$

Consequently,

$$\forall i \in \{1,\dots,j\},\ h_i^\varepsilon = e^{2s\alpha}(s\phi)^{M_j}\varphi_i^\varepsilon 1_\omega. \tag{187}$$

Another duality argument applied between ζ^ε and φ^ε together with (187) gives

$$-\frac{1}{\varepsilon}\left\|\zeta^\varepsilon(T,.)\right\|^2_{L^2(\Omega)^4} = \sum_{i=1}^{j}\int\int_{(0,T)\times\omega} e^{2s\alpha}(s\phi)^{M_j}|\varphi_i^\varepsilon|^2 dxdt + \int_\Omega \varphi^\varepsilon(0,x).\zeta_0(x)dx. \tag{188}$$

If $j=2$, we have $\int_\Omega \zeta_{0,4}(x)dx = 0$. Then,

$$\int_\Omega \varphi^\varepsilon(0,x).\zeta_0(x)dx = \sum_{i=1}^{3}\int_\Omega \varphi_i^\varepsilon(0,x)\zeta_{0,i}(x)dx + \int_\Omega(\varphi_4^\varepsilon(0,x) - (\varphi_4)_\Omega(0))\zeta_{0,4}(x)dx. \tag{189}$$

If $j=1$, we have $\int_\Omega \zeta_{0,3}(x)dx = 0$ and $\int_\Omega \zeta_{0,4}(x)dx = 0$. Then,

$$\begin{aligned}&\int_\Omega \varphi^\varepsilon(0,x).\zeta_0(x)dx\\ &= \sum_{i=1}^{2}\int_\Omega \varphi_i^\varepsilon(0,x)\zeta_{0,i}(x)dx + \int_\Omega(\varphi_3^\varepsilon(0,x) - (\varphi_3)_\Omega(0))\zeta_{0,4}(x)dx + \int_\Omega(\varphi_4^\varepsilon(0,x) - (\varphi_4)_\Omega(0))\zeta_{0,4}(x)dx.\end{aligned} \tag{190}$$

Then, from (118) for $j=3$, (151), (189) for $j=2$, (183), (190) for $j=1$ and (187), (188), we have

$$\frac{1}{\varepsilon}\left\|\zeta^\varepsilon(T,.)\right\|^2_{L^2(\Omega)^4} + \frac{1}{2}\left\|(e^{-2s\alpha}(s\phi)^{-M_j})^{1/2}h^{j,\varepsilon}\right\|^2_{L^2((0,T)\times\omega)^j} \le C\left\|\zeta_0\right\|^2_{L^2(\Omega)^4}. \tag{191}$$

In particular, from (191),

$$\zeta^\varepsilon(T,.) \underset{\varepsilon\to 0}{\to} 0 \text{ in } L^2(\Omega)^4, \tag{192}$$

and

$$\left\|B_j h^{j,\varepsilon}\right\|_{L^2(Q)^4} \le C. \tag{193}$$

Then, by using $A \in \mathcal{M}_4(L^\infty(Q))$ (see (86), (84) and (82)) and recalling (193), from Proposition 2.3 applied to (89), we deduce that

$$\left\|\zeta^\varepsilon\right\|_{Y^4} \le C. \tag{194}$$

So, from (194), up to a subsequence, we can suppose that there exists $\zeta \in Y$ such that

$$\zeta^\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} \zeta \text{ in } L^2(0,T;H^1(\Omega)^4), \tag{195}$$

$$\partial_t\zeta^\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} \partial_t\zeta \text{ in } L^2(0,T;(H^1(\Omega))'^4), \tag{196}$$

and from Proposition 2.2,

$$\zeta^\varepsilon(0,.) \underset{\varepsilon\to 0}{\rightharpoonup} \zeta(0,.) \text{ in } L^2(\Omega)^4,\ \zeta^\varepsilon(T,.) \underset{\varepsilon\to 0}{\rightharpoonup} \zeta(T,.) \text{ in } L^2(\Omega)^4. \tag{197}$$

Then, as we have $\zeta^\varepsilon(0,.) = \zeta_0$ and (192), we deduce that

$$\zeta(0,.) = \zeta_0, \text{ and } \zeta(T,.) = 0. \tag{198}$$

Moreover, from (191), up to a subsequence, we can suppose that there exists $h^j \in L^2_{wght}((0,T)\times\omega)^j$ such that

$$(h^{j,\varepsilon}) \underset{\varepsilon\to 0}{\rightharpoonup} h^j \text{ in } L^2_{wght}((0,T)\times\omega)^j, \tag{199}$$

and

$$\left\|(e^{-2s\alpha}(s\phi)^{-M_j})^{1/2}h^j\right\|^2_{L^2((0,T)\times\omega)^j} \leq \liminf_{\varepsilon\to 0}\left\|(e^{-2s\alpha}(s\phi)^{-M_j})^{1/2}h^{j,\varepsilon}\right\|^2_{L^2((0,T)\times\omega)^j} \leq C\left\|\zeta_0\right\|^2_{L^2(\Omega)^4}. \tag{200}$$

Then, from (195), (196), (199), we let $\varepsilon\to 0$ in the following equations

$$\begin{cases} \partial_t\zeta^\varepsilon - D\Delta\zeta^\varepsilon = A(t,x)\zeta^\varepsilon + B_j h^{j,\varepsilon}1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial\zeta^\varepsilon}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases}$$

and by using (198), we deduce

$$\begin{cases} \partial_t\zeta - D\Delta\zeta = A(t,x)\zeta + B_j h^j 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial\zeta}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ (\zeta(0,.),\zeta(T,.)) = (\zeta_0,0) & \text{in } \Omega. \end{cases} \tag{201}$$

Therefore, we have proved the existence of a control h^j such that $(e^{-2s\alpha}(s\phi)^{-M_j})^{1/2}h^j \in L^2((0,T)\times\omega)^j$ that drives the solution ζ of (89) to 0, and we have the estimate

$$\left\|(e^{-2s\alpha}(s\phi)^{-M_j})^{1/2}h^j\right\|^2_{L^2((0,T)\times\omega)^j} \leq C\left\|\zeta_0\right\|^2_{L^2(\Omega)^4}. \tag{202}$$

4.4.2 Bootstrap method

In the previous subsection, we proved the existence of a control h^j such that $(e^{-2s\alpha}(s\phi)^{-M_j})^{1/2}h^j \in L^2((0,T)\times\omega)^j$, i.e. a control h^j more regular than $L^2(Q)$. The key points are the link between $h^{j,\varepsilon}$ and φ^ε (i.e. (187)) and the weights of Carleman estimates. Now, we use an iterative process in order to find controls in $L^\infty(Q)^j$. We use the same key points together with parabolic regularity theorems. This section is inspired by [16, Section 3.1.2] and [44] (for the Neumann conditions). First, we are going to present the boostrap method for the case $j=3$ and after that, we explain the main differences for the case $j=2$ and $j=1$.

4.4.2.1 Strong observability inequalities From (112) for the case $j=3$, (149) for the case $j=2$, (182) for the case $j=1$, (187) and (191), we deduce these inegalities which are useful for the bootstrap method:

$$(j=3) \Rightarrow \left(\sum_{i=1}^4\int_0^T\int_\Omega e^{2s\widehat{\alpha}}(s\widehat{\phi})^3|\varphi_i^\varepsilon|^2dxdt \leq C\left\|\zeta_0\right\|^2_{L^2(\Omega)^4}\right), \tag{203}$$

$$(j=2) \Rightarrow \left(\sum_{i=1}^3\int_0^T\int_\Omega e^{2s\widehat{\alpha}}(s\widehat{\phi})^3|\varphi_i^\varepsilon|^2dxdt + \int_0^T\int_\Omega e^{2s\widehat{\alpha}}(s\widehat{\phi})^3|\varphi_4^\varepsilon - (\varphi_4^\varepsilon)_\Omega|^2dxdt \leq C\left\|\zeta_0\right\|^2_{L^2(\Omega)^4}\right), \tag{204}$$

$$(j=1) \Rightarrow \left(\sum_{i=1}^2\int_0^T\int_\Omega e^{2s\widehat{\alpha}}(s\widehat{\phi})^3|\varphi_i^\varepsilon|^2dxdt + \sum_{i=3}^4\int_0^T\int_\Omega e^{2s\widehat{\alpha}}(s\widehat{\phi})^3|\varphi_i^\varepsilon - (\varphi_i^\varepsilon)_\Omega|^2dxdt \leq C\left\|\zeta_0\right\|^2_{L^2(\Omega)^4}\right). \tag{205}$$

4.4.2.1.1 Bootstrap Let $\delta>0$ which will be chosen sufficiently small and $(\delta_k)_{k\in\mathbb{N}} \in (\mathbb{R}^{+,*})^{\mathbb{N}}$ be a strictly increasing sequence such that $\delta_k \underset{k\to+\infty}{\to} \delta$. Let $(p_k)_{k\in\mathbb{N}}$ be the following sequence defined by induction

$$p_0 = 2,$$

$$p_{k+1} := \begin{cases} \frac{(N+2)p_k}{N+2-2p_k} & \text{if } p_k < \frac{N+2}{2}, \\ 2p_k & \text{if } p_k = \frac{N+2}{2}, \\ +\infty & \text{if } p_k > \frac{N+2}{2}. \end{cases}$$

Clearly, there exists $l\in\mathbb{N}$ such that

$$\forall k \geq l,\ p_k = +\infty. \tag{206}$$

Definition 4.34. We introduce the following spaces: for every $r\in[1,+\infty]$,

$$W^{2,r}_{Ne}(\Omega) := \left\{u\in W^{2,r}(\Omega)\ ;\ \frac{\partial u}{\partial n} = 0\right\}, \qquad Y_r = L^r(0,T;W^{2,r}_{Ne}(\Omega)) \cap W^{1,r}(0,T;L^r(\Omega)).$$

Definition 4.35. Let u be a function on Q. For $0 < \beta < 1$, we define

$$[u]_{\beta/2,\beta} = \sup_{(t,x),(t',x')\in Q,(t,x)\neq(t',x')} \frac{|u(t,x)-u(t',x')|}{(|t-t'|+|x-x'|^2)^{\beta/2}},$$

which is a semi-norm, and we denote by $C^{\beta/2,\beta}(\overline{Q})$ the set of all functions on Q such that $[u]_{\beta/2,\beta} < +\infty$, endowed with the norm

$$\|u\|_{\beta/2,\beta} = \left(\sup_{(t,x)\in Q} |u(t,x)|\right) + [u]_{\beta/2,\beta}.$$

Proposition 4.36. *Let $1 < p < +\infty$, $m \in \mathbb{N}^*$, $D \in \mathcal{M}_m(\mathbb{R})$ such that $Sp(D) \subset (0,+\infty)$, $A \in \mathcal{M}_m(L^\infty(Q))$, $f \in L^p(Q)^m$. From [20, Theorem 2.1], the following Cauchy problem admits a unique solution $u \in Y_p^m$*

$$\begin{cases} \partial_t u - D\Delta u = A(t,x)u + f & \text{in } (0,T)\times\Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,.) = 0 & \text{in } \Omega. \end{cases}$$

Moreover, there exists $C > 0$ independent of f such that

$$\|u\|_{Y_p^m} \leq C \|f\|_{L^p(Q)^k}.$$

Proposition 4.37. *([45, Theorem 1.4.1])*
Let $r \in [1,+\infty[$, we have

$$Y_r \hookrightarrow \begin{cases} L^{\frac{(N+2)r}{N+2-2r}}(Q) & \text{if } r < \frac{N+2}{2}, \\ L^{2r}(Q) & \text{if } r = \frac{N+2}{2}, \\ C^{\beta/2,\beta}(\overline{Q}) \hookrightarrow L^\infty(Q) \text{ with } 0 < \beta \leq 2 - \frac{N+2}{r} & \text{if } r > \frac{N+2}{2}. \end{cases}$$

$\boxed{j=3}$

In the following, C denotes various positive constants varying from one line to the other and does not depend of $\|\zeta_0\|_{L^2(\Omega}$.

We define for every $k \in \mathbb{N}$,

$$\psi^{\varepsilon,k} := e^{\widehat{\alpha}(s+\delta_k)}\varphi^\varepsilon.$$

For $k \in \mathbb{N}^*$, we have

$$\begin{cases} -\partial_t\psi^{\varepsilon,k} - D_3\Delta\psi^{\varepsilon,k} = A(t,x)\psi^{\varepsilon,k} + f_k & \text{in } (0,T)\times\Omega, \\ \frac{\partial\psi^{\varepsilon,k}}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \psi^{\varepsilon,k}(T,.) = 0 & \text{in } \Omega, \end{cases} \tag{207}$$

with

$$f_k(t,x) = -\partial_t(e^{\widehat{\alpha}(s+\delta_k)})\varphi^\varepsilon.$$

By using the fact that $(\delta_k)_{k\in\mathbb{N}}$ is strictly increasing, we easily have that

$$|f_k| \leq Ce^{\widehat{\alpha}(s+\delta_{k-1})}|\varphi^\varepsilon| = C|\psi^{\varepsilon,k-1}| \text{ in } (0,T)\times\Omega. \tag{208}$$

We show, by induction, that for every $k \in \mathbb{N}$, we have

$$\psi^{\varepsilon,k} \in L^{p_k}(Q)^4 \text{ and } \left\|\psi^{\varepsilon,k}\right\|_{L^{p_k}(Q)^4} \leq C\|\zeta_0\|_{L^2(\Omega)^4}. \tag{209}$$

The case $k = 0$ can be deduced from the fact that $\delta_0 > 0$ and (203).
Let $k \in \mathbb{N}^*$. We suppose that

$$\psi^{\varepsilon,k-1} \in L^{p_{k-1}}(Q)^4 \text{ and } \left\|\psi^{\varepsilon,k-1}\right\|_{L^{p_{k-1}}(Q)^4} \leq C\|\zeta_0\|_{L^2(\Omega)^4}. \tag{210}$$

Then, from (207), (208), (210) and from Proposition 4.36, we get

$$\psi^{\varepsilon,k} \in X^4_{p_{k-1}} \text{ and } \left\|\psi^{\varepsilon,k}\right\|_{X^4_{p_{k-1}}} \leq C\|\zeta_0\|_{L^2(\Omega)^4}. \tag{211}$$

Moreover, by Proposition 4.37, we have

$$\psi^{\varepsilon,k} \in L^{p_k}(Q)^4 \text{ and } \left\|\psi^{\varepsilon,k}\right\|_{L^{p_k}(Q)^4} \leq C\|\zeta_0\|_{L^2(\Omega)^4}.$$

This concludes the induction.

From (97) and (98), we remark that we have the following inequality

$$\alpha \leq \frac{\widehat{\alpha}}{1+f(\lambda)}, \tag{212}$$

because

$$(e^{\lambda\eta_0(x)} - e^{2\lambda\|\eta_0\|})(1+e^{-\lambda\|\eta_0\|}) = e^{\lambda\eta_0(x)} - e^{\lambda\|\eta_0\|} + 1 - e^{2\lambda\|\eta_0\|} \leq 1 - e^{2\lambda\|\eta_0\|}.$$

Moreover, from (96), we can pick $\delta > 0$ such that

$$2s - (1+f(\lambda))(s+\delta) = s(2-(1+f(\lambda))) - \delta(1+f(\lambda)) > 0. \tag{213}$$

Now, by applying consecutively (206), (187), (212), (213) and (209), we have for every $i \in \{1,\dots,3\}$,

$$\begin{aligned}
\|h_i^\varepsilon\|_{L^\infty(Q)^3} &= \|h_i^\varepsilon\|_{L^{p_l}(Q)^3} \qquad (214)\\
&= \left\|e^{2s\alpha}(s\phi)^7\varphi_i^\varepsilon\right\|_{L^{p_l}(Q)^3}\\
&\leq \left\|e^{2s\frac{\widehat{\alpha}}{1+f(\lambda)}}e^{-\widehat{\alpha}(s+\delta)}(s\phi)^7\right\|_{L^\infty(Q)}\left\|e^{\widehat{\alpha}(s+\delta)}\varphi_i^\varepsilon\right\|_{L^{p_l}(Q)}\\
&\leq C\left\|e^{\widehat{\alpha}(s+\delta)}\varphi_i^\varepsilon\right\|_{L^{p_l}(Q)}\\
&\leq C\left\|e^{\widehat{\alpha}(s+\delta_l)}\varphi_i^\varepsilon\right\|_{L^{p_l}(Q)} \quad (\delta_l \leq \delta \text{ and } \widehat{\alpha} < 0)\\
&\leq C\left\|\zeta_0\right\|_{L^2(\Omega)^4}. \qquad (215)
\end{aligned}$$

Therefore, from (215), we get

$$\|h_i^\varepsilon\|_{L^\infty(Q)^3} \leq C\left\|\zeta_0\right\|_{L^2(\Omega)^4}. \tag{216}$$

So, $(h^{3,\varepsilon})_\varepsilon$ is bounded in $L^\infty(Q)^3$, then up to a subsequence, we can suppose that there exists $h^3 \in L^\infty(Q)^3$ such that

$$h^{3,\varepsilon} \underset{\varepsilon\to 0}{\rightharpoonup}{}^* h^3 \text{ in } L^\infty(Q)^3, \tag{217}$$

and

$$\left\|h^3\right\|_{L^\infty(Q)^3} \leq C\left\|\zeta_0\right\|_{L^2(\Omega)^4}.$$

From (195), (196), (217), (198), we have

$$\begin{cases} \partial_t\zeta - D_3\Delta\zeta = A(t,x)\zeta + B_3h^3 1_\omega & \text{in } (0,T)\times\Omega,\\ \frac{\partial\zeta}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega,\\ (\zeta(0,.),\zeta(T,.)) = (\zeta_0,0) & \text{in } \Omega. \end{cases} \tag{218}$$

This ends the proof of Proposition 4.8 for the case $j=3$.

$\boxed{j=2}$

For every $k \in \mathbb{N}$, we introduce

$$\widetilde{\varphi^\varepsilon} := (\varphi_1^\varepsilon, \varphi_2^\varepsilon, \varphi_3^\varepsilon, \varphi_4^\varepsilon - (\varphi_4^\varepsilon)_\Omega)^T, \tag{219}$$

$$\psi^{\varepsilon,k} := e^{\widehat{\alpha}(s+\delta_k)}\widetilde{\varphi^{\varepsilon,k}}. \tag{220}$$

For $k \in \mathbb{N}^*$, we have

$$\begin{cases} -\partial_t\psi^{\varepsilon,k} - D_2\Delta\psi^{\varepsilon,k} = A(t,x)\psi^{\varepsilon,k} + f_k & \text{in } (0,T)\times\Omega,\\ \frac{\partial\psi^{\varepsilon,k}}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega,\\ \psi^{\varepsilon,k}(T,.) = 0 & \text{in } \Omega, \end{cases} \tag{221}$$

with

$$f_k(t,x) = -(e^{\widehat{\alpha}(s+\delta_k)})_t\widetilde{\varphi^{\varepsilon,k}} + \left(0,0,0,\left(u_2^*e^{\widehat{\alpha}(s+\delta_k)}\varphi_1^\varepsilon - u_2^*e^{\widehat{\alpha}(s+\delta_k)}\varphi_2^\varepsilon + u_2^*e^{\widehat{\alpha}(s+\delta_k)}\varphi_3^\varepsilon\right)_\Omega\right)^T,$$

because $A \in \mathcal{E}_2$ (see (84)). From the fact that $(\delta_k)_{k\in\mathbb{N}}$ is strictly increasing, we easily have

$$|f_k| \leq Ce^{\widehat{\alpha}(s+\delta_{k-1})}|\widetilde{\varphi^\varepsilon}| = C|\psi^{\varepsilon,k-1}| \text{ in } (0,T)\times\Omega. \tag{222}$$

Then, the strategy of bootstrap is exactly the same. The starting point comes from (204).

$\boxed{j=1}$

We apply the same strategy as for the case $j=2$. For every $k \in \mathbb{N}$, we introduce

$$\widetilde{\varphi^\varepsilon} := (\varphi_1^\varepsilon, \varphi_2^\varepsilon, \varphi_3^\varepsilon - (\varphi_3^\varepsilon)_\Omega, \varphi_4^\varepsilon - (\varphi_4^\varepsilon)_\Omega)^T, \tag{223}$$

$$\psi^{\varepsilon,k} := e^{\widehat{\alpha}(s+\delta_k)}\widetilde{\varphi^{\varepsilon,k}}. \tag{224}$$

The starting point comes from (205).

This ends the proof of Proposition 4.8.

4.5 Nonlinear problem

In order to prove Theorem 3.2, we use Proposition 4.8 together with a standard fixed-point argument.

4.5.1 Reduction to a fixed point problem

Let $j \in \{1,2,3\}$. We remark that $G : L^\infty(Q)^4 \to \mathcal{M}_4(L^\infty(Q))$ is continuous (see (51), (61) and (76)). Then, we get the existence of $\nu > 0$ small enough such that for every $z = (z_1, z_2, z_3, z_4) \in L^\infty(Q)^4$,

$$(\|z\|_{L^\infty(Q)^4} \leq \nu) \Rightarrow ((G(z_1, z_2, z_3, z_4)) \in \mathcal{E}_j), \tag{225}$$

where $\mathcal{E}_j$ are defined in (82), (84) and (86).

Let $\mathcal{Z}$ be the set of $z = (z_1, z_2, z_3, z_4) \in L^\infty(Q)^4$ such that $\|z\|_{L^\infty(Q)^4} \leq \nu$. From Proposition 4.8, we have proved that there exists $C_0 > 0$ such that for all $z = (z_1, z_2, z_3, z_4) \in \mathcal{Z}$ and for all $\zeta_0 \in L^\infty(Q)^4$, there exists a control $h^j \in L^\infty(Q)^j$ satisfying

$$\left\|h^j\right\|_{L^\infty(Q)^j} \leq C_0 \left\|\zeta_0\right\|_{L^2(\Omega)^4}, \tag{226}$$

such that the solution $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)^T \in (Y^4 \cap L^\infty(Q)^4)$ to the Cauchy problem

$$\begin{cases} \partial_t \zeta - D_j \Delta \zeta = G(z)\zeta + B_j h^j 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial \zeta}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \zeta(0,.) = \zeta_0 & \text{in } \Omega, \end{cases} \tag{227}$$

verifies

$$\zeta(T,.) = 0. \tag{228}$$

We fix $\zeta_0 \in L^\infty(Q)^4$.

We define $B : \mathcal{Z} \to L^\infty(Q)^4$ in the following way. For every $z = (z_1, z_2, z_3, z_4) \in \mathcal{Z}$, $B(z)$ is the set of $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4) \in L^\infty(Q)^4$ solution to the Cauchy problem (227), associated to a control $h^j \in L^\infty(Q)^j$ satisfying (226), and which verifies (228).

Our main result (i.e. Theorem 3.2) will be proved if we show that B has a fixed point (i.e. z is such that $z \in B(z)$).

We use the **Kakutani's fixed point theorem**.

Theorem 4.38. *Kakutani's fixed point theorem.*

1. *For every $z \in \mathcal{Z}$, $B(z)$ is a nonempty convex and closed subset of $L^\infty(Q)^4$.*
2. *There exists a convex compact set $K \subset \mathcal{Z}$ such that for every $z \in \mathcal{Z}, B(z) \subset K$.*
3. *B is upper semicontinuous in $L^\infty(Q)^4$, that is to say for all closed subset $\mathcal{A} \subset \mathcal{Z}$, $B^{-1}(\mathcal{A}) = \{z \in \mathcal{Z}; B(z) \cap \mathcal{A} \neq \emptyset\}$ is closed.*

Then, B has a fixed point.

4.5.2 Hypotheses of Kakutani's fixed point theorem

4.5.2.1 Proof of the point 1. Let $z \in \mathcal{Z}$.

$B(z)$ is nonempty because we have proved the existence of at least one control satisfying (226) that drives the solution to 0.

$B(z)$ is convex because the mapping $h \in L^\infty(Q)^j \mapsto \zeta \in L^\infty(Q)^4$, where ζ is the solution to the Cauchy problem (227), is affine and (226) is clearly verified by convex combinations of controls satisfying it.

$B(z)$ is closed. Indeed, let $(\zeta_k)_{k\in\mathbb{N}}$ be a sequence of $B(z)$ such that

$$\zeta_k \underset{k\to+\infty}{\to} \zeta \text{ in } L^\infty(Q)^4. \tag{229}$$

We introduce $(h_k^j)_{k\in\mathbb{N}}$ the sequence of controls associated to $(\zeta_k)_{k\in\mathbb{N}}$. In particular, for every $k\in\mathbb{N}$,

$$\left\|h_k^j\right\|_{L^\infty(Q)^j} \leq C_0 \left\|\zeta_0\right\|_{L^2(\Omega)^4}. \tag{230}$$

From (229) and (230), for every $k\in\mathbb{N}$,

$$\left\|G(z)\zeta_k + B_j h_k^j\right\|_{L^\infty(Q)^4} \leq C. \tag{231}$$

Then, from (231) and Proposition 2.3 applied to ζ_k which satisfies (227), we deduce that for every $k\in\mathbb{N}$,

$$\left\|\zeta_k\right\|_{(Y\cap L^\infty(Q))^4} \leq C. \tag{232}$$

So, from (232), up to a subsequence, we can suppose that there exists $\zeta\in Y$ such that

$$\zeta_k \underset{k\to+\infty}{\rightharpoonup} \zeta \text{ in } L^2(0,T;H^1(\Omega)^4), \tag{233}$$

$$\partial_t\zeta_k \underset{k\to+\infty}{\rightharpoonup} \partial_t\zeta \text{ in } L^2(0,T;(H^1(\Omega))'^4), \tag{234}$$

and, from Proposition 2.2,

$$\zeta_k(0,.) \underset{k\to+\infty}{\rightharpoonup} \zeta(0,.) \text{ in } L^2(\Omega)^4,\ \zeta_k(T,.) \underset{k\to+\infty}{\rightharpoonup} \zeta(T,.) \text{ in } L^2(\Omega)^4. \tag{235}$$

Then, as we have $\zeta_k(0,.)=\zeta_0$ and $\zeta_k(T,.)=0$ for every $k\in\mathbb{N}$, we deduce that

$$\zeta(0,.)=\zeta_0 \text{ and } \zeta(T,.)=0. \tag{236}$$

Moreover, from (230), up to a subsequence, we can suppose that there exists $h^j\in L^\infty(Q)^j$ such that

$$h_k^j \underset{k\to+\infty}{\rightharpoonup^*} h^j \text{ in } L^\infty(Q)^j, \tag{237}$$

and

$$\left\|h^j\right\|_{L^\infty(Q)^j} \leq \liminf_{k\to+\infty}\left\|h_k^j\right\|_{L^\infty(Q)^j} \leq C_0\left\|\zeta_0\right\|_{L^2(\Omega)^4}. \tag{238}$$

Then, from (233), (234), (235), (236) and (237), we let $k\to+\infty$ in the following equations (i.e. passing to the limit in the variational formulation (14))

$$\begin{cases} \partial_t\zeta_k - D_j\Delta\zeta_k = G(z)\zeta_k + B_j h_k^j 1_\omega & \text{in } (0,T)\times\Omega,\\ \frac{\partial\zeta_k}{\partial n}=0 & \text{on } (0,T)\times\partial\Omega,\\ (\zeta_k(0,.),\zeta_k(T,.)) = (\zeta_0,0) & \text{in } \Omega. \end{cases}$$

We deduce that

$$\begin{cases} \partial_t\zeta - D_j\Delta\zeta = G(z)\zeta + B_j h^j 1_\omega & \text{in } (0,T)\times\Omega,\\ \frac{\partial\zeta}{\partial n}=0 & \text{on } (0,T)\times\partial\Omega,\\ (\zeta(0,.),\zeta(T,.)) = (\zeta_0,0) & \text{in } \Omega. \end{cases} \tag{239}$$

Finally, from (239) and (238), we have $\zeta\in B(z)$.

4.5.2.2 Proof of the point 2.

Let $z\in\mathcal{Z}$.

By Proposition 2.3 and (226), we deduce that there exists $C_1>0$ such that

$$\forall z\in\mathcal{Z},\ \forall\zeta\in B(z),\ \left\|\zeta\right\|_{L^\infty(Q)^4} \leq C_1\left\|\zeta_0\right\|_{L^\infty(\Omega)^4}.$$

Now, we suppose that $\zeta_0\in L^\infty(\Omega)^4$ verifies

$$\left\|\zeta_0\right\|_{L^\infty(\Omega)^4} \leq \nu/C_1. \tag{240}$$

Then, we have

$$\forall z\in\mathcal{Z},\ B(z)\subset\mathcal{Z}. \tag{241}$$

Let $F\in L^\infty(Q)^4$ be the solution to the Cauchy problem

$$\begin{cases} \partial_t F - D_j\Delta F = 0 & \text{in } (0,T)\times\Omega,\\ \frac{\partial F}{\partial n}=0 & \text{on } (0,T)\times\partial\Omega,\\ F(0,.)=\zeta_0 & \text{in } \Omega. \end{cases} \tag{242}$$

Let $\zeta^* = \zeta - F$. Then, ζ^* is the solution to the following Cauchy problem

$$\begin{cases} \partial_t \zeta^* - D_j \Delta \zeta^* = G(z)\zeta + B_j h^j 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial \zeta^*}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \zeta^*(0,.) = 0 & \text{in } \Omega. \end{cases} \tag{243}$$

We can remark that there exists $C > 0$ such that

$$\left\| G(z)\zeta + B_j h^j 1_\omega \right\|_{L^\infty(Q)^4} \leq C. \tag{244}$$

From (244), Proposition 4.36 with $p = N+2$ applied to ζ^* (see (243)) and the Sobolev embedding theorem $Y_p \hookrightarrow C^{\beta/2,\beta}(\overline{Q})$ with $\beta > 0$ (see Proposition 4.37), we deduce that $\zeta^* \in C^0(\overline{Q})^4$ and there exists $C_2 > 0$ such that

$$\forall (t,x) \in \overline{Q}, \forall (t',x') \in \overline{Q},\ |\zeta^*(t,x) - \zeta^*(t',x')| \leq C_2(|t-t'|^{\beta/2} + |x-x'|^\beta). \tag{245}$$

Let K^* be the set of ζ^* such that (245) holds. Then, we have $(F + K^*) \cap \mathcal{Z}$ is a compact convex subset of $L^\infty(Q)^4$ by Ascoli's theorem and

$$\forall z \in \mathcal{Z}, B(z) \subset (F + K^*) \cap \mathcal{Z}.$$

Then, $K := (F + K^*) \cap \mathcal{Z}$ is a convex compact subset of $\mathcal{Z}$ such that the point 2 holds.

4.5.2.3 Proof of the point 3. Let A be a closed subset of $\mathcal{Z}$. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence of elements in $\mathcal{Z}$, $(\zeta_k)_{k\in\mathbb{N}}$ be a sequence of elements in $L^\infty(Q)^4$, and $z \in \mathcal{Z}$ be such that

$$z_k \underset{k\to+\infty}{\to} z \text{ in } L^\infty(Q)^4,$$

$$\forall k \in \mathbb{N},\ \zeta_k \in \mathcal{A},$$

$$\forall k \in \mathbb{N},\ \zeta_k \in B(z_k).$$

Let $(h_k^j)_{k\in\mathbb{N}}$ the sequence of controls associated to $(\zeta_k)_{k\in\mathbb{N}}$. As $\zeta_k \in B(z_k)$, we have

$$\forall k \in \mathbb{N},\ \left\| h_k^j \right\|_{L^\infty(Q)^j} \leq C_0 \left\| \zeta_0 \right\|_{L^2(\Omega)^4}.$$

By the point 2, we get that there exists a strictly increasing sequence $(k_l)_{l\in\mathbb{N}}$ of integers such that $\zeta_{k_l} \to \zeta$ in $L^\infty(Q)^4$ as $l \to +\infty$. As $\mathcal{A}$ is closed, we have $\zeta \in \mathcal{A}$, then it suffices to show that $\zeta \in B(z)$. The same arguments as in the point 1 give the result. This ends the proof of the point 3.

This concludes the proof of Theorem 3.2.

5 Proof of Theorem 3.6: the global controllability to constant stationary states

Proof. Let $N \in \{1,2\}$, $j = 3$ (we only prove the result for this case, the other cases are similar), $u_0 \in L^\infty(\Omega)^4$ satisfying the hypothesis (41), $(u_i^*)_{1\leq i\leq 4} \in (\mathbb{R}^+)^4$ satisfying (2).

From [42, Theorem 3] and [41, Theorem 3] (see also [21]), we deduce that the solution $u \in L^\infty((0,\infty)\times\Omega)^4$ of

$$\begin{cases} \partial_t u_i - d_i \Delta u_i = (-1)^i (u_1 u_3 - u_2 u_4) & \text{in } (0,\infty)\times\Omega, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ u_i(0,.) = u_{i,0} & \text{in } \Omega, \end{cases} \tag{246}$$

satisfies

$$\lim_{T\to+\infty} \|u(T,.) - z\|_{L^\infty(\Omega)^4} = 0, \tag{247}$$

where $z \in (\mathbb{R}^{+,*})^4$ is the unique nonnegative solution of

$$z_1 z_3 = z_2 z_4, \tag{248}$$

$$z_1 + z_2 = (u_{1,0})_\Omega + (u_{2,0})_\Omega,\ z_1 + z_4 = (u_{1,0})_\Omega + (u_{4,0})_\Omega, \tag{249}$$

$$z_3 + z_2 = (u_{3,0})_\Omega + (u_{2,0})_\Omega,\ z_3 + z_4 = (u_{3,0})_\Omega + (u_{4,0})_\Omega. \tag{250}$$

Case 1: $u_3^* \neq 0$. Let us define γ a path between z and $(u_i^*)_{1\leq i\leq 4}$,

$$\gamma : \left| \begin{array}{rcl} [0,1] & \longrightarrow & \{(v_1,v_2,v_3,v_4) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^{+,*} \times \mathbb{R}^+ \ ; \ v_1v_3 = v_2v_4\} \\ \theta & \longmapsto & \left(\frac{((1-\theta)z_2+\theta u_2^*)((1-\theta)z_4+\theta u_4^*)}{(1-\theta)z_3+\theta u_3^*}, (1-\theta)z_2+\theta u_2^*, (1-\theta)z_3+\theta u_3^*, (1-\theta)z_4+\theta u_4^* \right). \end{array} \right. \tag{251}$$

Let us define Φ in the following way,

$$\Phi : \left| \begin{array}{rcl} \Gamma := \{\gamma(\theta), \theta \in [0,1]\} & \longrightarrow & \mathbb{R}^{+,*} \\ (v_i) & \longmapsto & r_v, \end{array} \right. \tag{252}$$

where $r_v > 0$ is the radius of the ball of $L^\infty(\Omega)^4$ centered in $(v_i)_{1\leq i\leq 4}$ in which we have proved controllability to $(v_i)_{1\leq i\leq 4}$ (see Theorem 3.2). Precisely, r_v is given by (240). It is straightforward but tedious to see that

$$r := \inf \Phi > 0, \tag{253}$$

because there exists $\varepsilon > 0$ such that for every $\theta \in [0,1]$, $v_3 = (1-\theta)z_3 + \theta u_3^* \geq \varepsilon$. For more details, one can follow the dependence of the constant $r_v = \nu/C_1$ in function of the parameters $(v_i)_{1\leq i\leq 4}$ (see (240), (225), (226), Proposition 4.8 for the definition of the constant C_0, (51), (53), (54) and Section 4.3.3 for the dependence of this constant C_0 in term of $(v_i)_{1\leq i\leq 4}$).

By (247), there exists $T_1 > 0$ such that $\|u(T_1,.) - z\|_{L^\infty(\Omega)^4} < r$, where u is the solution of (246). By (252) and (253), there exists $h^{3,1} \in L^\infty((T_1, T_1+T)\times\Omega)^3$ such that the solution u^1 of (3), with $(0,T) = (T_1, T_1+T)$ and $u^1(T_1,.) = u(T_1,.)$, satisfies $u^1(T_1+T,.) = z$.

The mapping γ is continuous on the compact set $[0,1]$, so γ is uniformly continuous on $[0,1]$ by Heine's theorem. Consequently, there exists $\eta > 0$ such that for every $\theta_1, \theta_2 \in [0,1]$, verifying $|\theta_1 - \theta_2| \leq \eta$, $\|\gamma(\theta_1) - \gamma(\theta_2)\|_\infty < r$. Moreover, there exists $m \in \mathbb{N}^*$ sufficiently large such that $m\eta \leq 1 < (m+1)\eta$. Therefore, let us define $\theta_k = k\eta$ for $k \in \{0,\dots,m\}$ and $\theta_{m+1} = 1$. Then, we have

$$\Gamma \subset \bigcup_{i=1}^{m+1} B(\gamma(\theta_i), r). \tag{254}$$

We remark that we have $\gamma(\theta_0) = z$, $\gamma(\theta_{m+1}) = u^*$ and $\|\gamma(\theta_i) - \gamma(\theta_{i+1})\|_\infty < r$ for every $i \in \{1,\dots,m\}$ by definition of η.

We have $\|z - \gamma(\theta_1)\|_\infty = \|\gamma(\theta_0) - \gamma(\theta_1)\|_\infty < r$. Then, by (252) and (253), there exists $h^{3,2} \in L^\infty((T_1 + T, T_1 + 2T)\times\Omega)^3$ such that the solution u^2 of (3), with $(0,T) = (T_1+T, T_1+2T)$ and $u^1(T_1+T,.) = z$, satisfies $u^1(T_1+2T,.) = \gamma(\theta_1)$.

By repeating m times this strategy, we get the existence of a control $h^3 \in L^\infty((0, T_1 + (m+2)T)\times\Omega)$ so that $h^3(t,.) = 0$ for $t \in (0,T_1)$, $h^3(t,.) = h^{3,1}(t,.)$ for $t \in (T_1, T_1+T)$, ... , $h^3(t,.) = h^{3,m+2}(t,.)$ for $t \in (T_1 + (m+1)T, T_1 + (m+2)T)$, such the solution u of

$$\forall 1 \leq i \leq 4, \ \left\{ \begin{array}{ll} \partial_t u_i - d_i\Delta u_i = (-1)^i(u_1u_3 - u_2u_4) + h_i^3 1_\omega & \text{in } (0, T_1+(m+2)T)\times\Omega, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } (0, T_1+(m+2)T)\times\partial\Omega, \\ u_i(0,.) = u_{i,0} & \text{in } \Omega, \end{array} \right. \tag{255}$$

satisfies $u(T_1 + (m+2)T,.) = u^*$.

Case 2: $u_3^* = 0$. From (2), we have $u_2^* = 0$ or $u_4^* = 0$. We can assume that $u_2^* = 0$. The other case is similar. By Theorem 3.2, we know that there exists $\widehat{r} > 0$ such that $\forall \widetilde{u}^* \in B(u^*, \widehat{r})_{L^\infty(\Omega)^4}$, we can find a control $h^3 \in L^\infty((0,T)\times\Omega)^3$ that enables to go from $\widetilde{u}^*$ to u^*. Consequently, we choose β such that $0 < \beta < \widehat{r}/2$ and $\frac{\beta(u_4^*+\widehat{r}/2)}{u_1^*+\widehat{r}/2} < \widehat{r}/2$ and we set $\widetilde{u}^* := (u_1^* + \widehat{r}/2, \beta, \frac{\beta(u_4^*+\widehat{r}/2)}{u_1^*+\widehat{r}/2}, u_4^* + \widehat{r}/2) \in B(u^*, \widehat{r})$. We remark that $\widetilde{u}^*$ satisfies (2) and $\widetilde{u_3}^* \neq 0$. Then, from the first case of the proof, we can find a control which drives z to $\widetilde{u}^*$. Next, we can find a control which drives $\widetilde{u}^*$ to u^*. □

6 Comments, perspectives and open problems

6.1 ω_i instead of ω

An interesting open problem could be the generalization of Theorem 3.2 to the system

$$\left\{ \begin{array}{ll} \partial_t u_i - d_i\Delta u_i = (-1)^i(u_1u_3 - u_2u_4) + h_i 1_{\omega_i} 1_{i\leq j} & \text{in } (0,T)\times\Omega, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ u_i(0,.) = u_{i,0} & \text{in } \Omega, \end{array} \right. \tag{256}$$

where for every $i \in \{1, \dots, j\}$, ω_i are nonempty open subsets such that $\omega_i \subset \Omega$ and $\bigcap_{i=1}^{j} \omega_i = \emptyset$ (otherwise, the generalization is straightforward).

6.2 Stationary solutions

We only have considered nonnegative stationary **constant** solutions of (1). It is not restrictive because of the following proposition.

Proposition 6.1. *Let $(u_i) \in C^2(\overline{\Omega})^4$ be a nonnegative solution of*

$$\begin{cases} -d_i \Delta u_i = (-1)^i (u_1 u_3 - u_2 u_4) & \text{in } \Omega, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{257}$$

Then, for every $1 \leq i \leq 4$, u_i is constant.

Proof. Let $\varepsilon > 0$. For every $i \in \{1, \dots, 4\}$, let us denote $u_i^\varepsilon = u_i + \varepsilon$ and $w_i^\varepsilon = u_i^\varepsilon(\log u_i^\varepsilon - 1) + 1$. Note that $w_i^\varepsilon \geq 0$ for every $i \in \{1, \dots, 4\}$. We have

$$\forall i,\ \nabla w_i^\varepsilon = \log(u_i^\varepsilon) \nabla u_i^\varepsilon,\ \Delta w_i^\varepsilon = \log(u_i^\varepsilon) \Delta u_i^\varepsilon + \frac{|\nabla u_i^\varepsilon|^2}{u_i^\varepsilon}. \tag{258}$$

Then, from (257) and (258), we have

$$\forall i,\ \begin{cases} -d_i \Delta w_i^\varepsilon + d_i \frac{|\nabla u_i^\varepsilon|^2}{u_i^\varepsilon} = (-1)^i \log(u_i^\varepsilon)(u_1^\varepsilon u_3^\varepsilon - u_2^\varepsilon u_4^\varepsilon - \varepsilon(u_1 + u_3 - u_2 - u_4)) & \text{in } \Omega, \\ \frac{\partial w_i^\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{259}$$

We add the four equations of (259) and we integrate on Ω. We get

$$\begin{aligned} &0 + \int_\Omega \sum_{i=1}^{4} d_i \frac{|\nabla u_i^\varepsilon|^2}{u_i^\varepsilon} \\ &= -\left(\int_\Omega (\log(u_1^\varepsilon u_3^\varepsilon) - \log(u_2^\varepsilon u_4^\varepsilon))(u_1^\varepsilon u_3^\varepsilon - u_2^\varepsilon u_4^\varepsilon) \right) + \varepsilon \left(\int_\Omega (\log(u_1^\varepsilon u_3^\varepsilon) - \log(u_2^\varepsilon u_4^\varepsilon))(u_1 + u_3 - u_2 - u_4) \right) \\ &\leq \varepsilon \left(\int_\Omega (\log(u_1^\varepsilon u_3^\varepsilon) - \log(u_2^\varepsilon u_4^\varepsilon))(u_1 + u_3 - u_2 - u_4) \right). \end{aligned} \tag{260}$$

Moreover,

$$\int_\Omega d_i \frac{|\nabla u_i^\varepsilon|^2}{u_i^\varepsilon} = \int_\Omega 4 d_i |\nabla \sqrt{u_i^\varepsilon}|^2. \tag{261}$$

Consequently, from (260), (261) and by taking ε sufficiently small,

$$\begin{aligned} \int_\Omega 4 d_i |\nabla \sqrt{u_i^\varepsilon}|^2 &\leq \varepsilon \left(\int_\Omega (\log(u_1^\varepsilon u_3^\varepsilon) - \log(u_2^\varepsilon u_4^\varepsilon))(u_1 + u_3 - u_2 - u_4) \right) \\ &\leq \varepsilon \left(\int_\Omega |\log(\varepsilon^4)||u_1 + u_3 - u_2 - u_4| \right). \end{aligned}$$

Then, by letting $\varepsilon \to 0$, we get that

$$\forall i, \int_\Omega 4 d_i |\nabla \sqrt{u_i}|^2 = 0.$$

Consequently, for every $1 \leq i \leq 4$, u_i is constant. □

We can also remark that there exist non constant solutions of (257). For example, in the case of $(d_1, d_2, d_3, d_4) = (1, 1, 1, 1)$, $(u_1^*, u_2^*, u_3^*, u_4^*) = (\varphi_\lambda, -\varphi_\lambda, \varphi_\lambda - \lambda, -\varphi_\lambda)$, where $\lambda > 0$ and φ_λ are respectively an eigenvalue and a corresponding eigenfunction of the unbounded operator $(-\Delta, H^2_{Ne}(\Omega))$ (see Definition 4.13), is a solution of (257). The result of Theorem 3.2 is still valid for non constant stationary solutions under a natural condition of sign of $(u_1^*, u_2^*, u_3^*, u_4^*)$ on ω (see (53), (63), (78) after linearization). There is only one nontrivial thing to verify. For the proof of the observability inequalities (94) and (95), the application of Δ to some equations does not create "bad" terms. A good meaning to be convinced is to look at the inequality (136) which becomes

$$I(0, \lambda, s, \Delta\varphi_4) \leq C \left(\int_0^T \int_\Omega e^{2s\alpha} \left\{ \sum_{i=1}^{3} |\Delta\varphi_i|^2 + |\nabla\varphi_i|^2| + |\varphi_i|^2 \right\} + \int_0^T \int_{\omega_2} e^{2s\alpha} (s\phi)^3 |\Delta\varphi_4|^2 \right). \tag{262}$$

It is clear that $\int_0^T \int_\Omega e^{2s\alpha} \left(\sum_{i=1}^{3} |\Delta\varphi_i|^2 + |\nabla\varphi_i|^2| + |\varphi_i|^2 \right)$ can be absorbed by the left hand side of (138) by taking s sufficiently large.

6.3 Constraints on the initial condition for the controllability of the linearized system

The goal of this section is to show that the linear transformation we do before linearization (see (58) and (72)), seems to be essential. Indeed, this adequate change of variable leads to control all possible initial conditions (see the necessary conditions on the initial conditions due to invariant quantities of the nonlinear dynamics: Section 2.2). One could think about [3, Theorem 5.3] which gives sufficient conditions of controllability when the rank condition of Theorem 1.6 is not verified. But it reduces the space of initial condition once more and it becomes "artificial" in our case.

The linearized-system of (3) around $(u_i^*)_{1\leq i\leq 4}$ is

$$\begin{cases} \partial_t u - D\Delta u = Au + B_j h^j 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,.) = u_0 & \text{in } \Omega, \end{cases} \tag{263}$$

where

$$u = (u_1,u_2,u_3,u_4)^T,\ D = diag(d_1,d_2,d_3,d_4),\ A = \begin{pmatrix} -u_3^* & u_4^* & -u_1^* & u_2^* \\ u_3^* & -u_4^* & u_1^* & -u_2^* \\ -u_3^* & u_4^* & -u_1^* & u_2^* \\ u_3^* & -u_4^* & u_1^* & -u_2^* \end{pmatrix}, \tag{264}$$

and B_j, h^j are defined in (44).

Definition 6.2. The system (263) is $(u_i^*)_{1\leq i\leq 4}$-controllable if for every $u_0 \in L^2(\Omega)^4$, there exists $h^j \in L^2(Q)^j$ such that the solution u of (263) satisfies $u(T,.) = u^*$.

We would also use [3, Theorem 1] in order to deduce the necessary and sufficient condition of controllability to $(u_i^*)_{1\leq i\leq 4}$ for (263). First, let us denote by $(\lambda_k)_{k\in\mathbb{N}}$ the increasing sequence of the eigenvalues of the unbounded operator $(-\Delta, H^2_{Ne}(\Omega))$ (see Definition 4.13 for the definition of $H^2_{Ne}(\Omega)$). In particular, $\lambda_0 = 0$.

Theorem 6.3. *The system* (263) *is* $(u_i^*)_{1\leq i\leq 4}$*-controllable if and only*

$$\forall k\in\mathbb{N},\ rank(-\lambda_k D + A|B_j) = 4, \tag{265}$$

where

$$((-\lambda_k D + A)|B_j) := \big(B_j, (-\lambda_k D + A)B_j, (-\lambda_k D + A)^2 B_j, (-\lambda_k D + A)^3 B_j\big).$$

For $j = 3$, we can check that for every $k\in\mathbb{N}$, $rank(-\lambda_k D + A|B_3) = 4$ if and only if $(u_1^*, u_3^*, u_4^*) \neq (0,0,0)$. It is consistent with Section 4.1.1.1.

For $j = 2$ and $d_3 \neq d_4$, we can check that $rank(\lambda_0 + A|B_2) < 4$, then (263) is not $(u_i^*)_{1\leq i\leq 4}$-controllable. It is consistent with the hypothesis we have to make for the initial condition i.e. (22). But, we can deduce from [3, Theorem 5.3] that (263) is $(u_i^*)_{1\leq i\leq 4}$-controllable for initial conditions verifying

$$\forall i\in\{1,\dots,4\},\ \frac{1}{|\Omega|}\int_\Omega u_{i,0}(x) = u_i^*. \tag{266}$$

The condition (266) is a more restrictive hypothesis than (22). It is only a sufficient condition. Actually, we have found a necessary and sufficient condition on the initial data for $(u_i^*)_{1\leq i\leq 4}$-controllability.

Proposition 6.4. *Let $j = 2$, $d_3 \neq d_4$.*
For every $u_0 \in L^2(\Omega)^4$ such that $\frac{1}{|\Omega|}\int_\Omega (u_{3,0} + u_{4,0}) = u_3^* + u_4^*$, there exists $h^2 \in L^2(Q)^2$ such that the solution u of (263) *satisfies $u(T,.) = u^*$.*
If $u_0 \in L^2(\Omega)^4$ does not satisfy $\frac{1}{|\Omega|}\int_\Omega (u_{3,0} + u_{4,0}) = u_3^* + u_4^*$, for every $h^2 \in L^2(Q)^2$, the solution u of (263) *does not satisfy $u(T,.) = u^*$.*

Proof. The necessary condition of controllability is a consequence of

$$\text{a.e. } t\in[0,T],\ \frac{d}{dt}\left(\int_\Omega (u_3(t,x) + u_4(t,x))dx\right) = 0.$$

The sufficient condition of controllability is a consequence of the adequate change of variable $(v_1,v_2,v_3,v_4) := (u_1,u_2,u_3,u_3+u_4)$ and the proof of the observability inequality (94). □

Remark 6.5. We chose to state our previous result in the particular case $j = 2$ and $d_3 \neq d_4$ for simplicity but one can generalize this proposition to other cases.

As far as we know, an interesting open problem could consist in finding a **necessary** and **sufficient** condition for the controllabillity of general linear parabolic systems of the form (263) when (265) is not satisfied.

A Appendix

A.1 L^∞-estimate for parabolic systems

We give the proof of Proposition 2.3.

Proof. By using the fact that D is diagonalizable and $Sp(D) \subset (0,+\infty)$, we only have to prove the result when $D = diag(d_1,\dots,d_k)$ with $d_i \in (0,+\infty)$.

The first point of the proof i.e. the existence and the uniqueness of the weak solution $u \in Y^k$ is based on Galerkin approximations and energy estimates. One can easily adapt the arguments given in [26, Section 7.1.2] to the Neumann cases.

The second point of the proof i.e. the L^∞ estimate is based on Stampacchia's method. We introduce

$$l(t) = (l_1(t),\dots,l_k(t))^T := l_0 \exp(tM)(1,\dots,1)^T =: L(t)(1,\dots,1)^T \in \mathbb{R}^k, \tag{267}$$

for every $t \in [0,T]$ and $l_0, M \in (0,+\infty)$ which will be chosen later. By (14), we have

$$\begin{aligned}&\forall w \in L^2(0,T;H^1(\Omega)^k),\\ &\int_0^T (\partial_t u, w)_{(H^1(\Omega)^k)',H^1(\Omega)^k)} - \int_Q (sign(u)l').w + \int_Q D\nabla u.\nabla w = \int_Q (Au+g).w - \int_Q (sign(u)l').w,\end{aligned} \tag{268}$$

where $sign(u)l' = (sign(u_1)l_1',\dots,sign(u_k)l_k')^T$. We fix $t \in [0,T]$ and we apply (268) with w defined by

$$\begin{aligned}\forall (\tau,x) \in [0,T]\times\Omega, w(\tau,x) &:= sign(u)(|u|(t,x) - l(t))^+ 1_{[0,t]}(\tau)\\ &:= \left(sign(u_1)(|u_1|(t,x)-l_1(t))^+,\dots,(sign(u_k)(|u_k|(t,x)-l(t))^+\right)^T 1_{[0,t]}(\tau).\end{aligned}$$

We get

$$\begin{aligned}&\int_0^t \frac{1}{2}\frac{d}{d\tau}\int_\Omega \sum_{i=0}^k \left((|u_i|(\tau,x)-l_i(\tau))^+\right)^2 dxd\tau + \int_0^t\int_\Omega \sum_{i=0}^k d_i\nabla u_i.\nabla u_i 1_{|u_i|\geq l_i}\\ &= \int_0^t\int_\Omega \sum_{i=0}^k \left(\sum_{j=0}^k a_{ij}u_j + g_i - sign(u_i)l_i'\right) sign(u_i)(|u_i|-l_i)^+.\end{aligned} \tag{269}$$

We remark that

$$-sign(u_i)l_i' sign(u_i)(|u_i|-l_i)^+ = -l_i'(|u_i|-l_i)^+.$$

Moreover, we have

$$\begin{aligned}\left(\sum_{j=0}^k a_{ij}u_j + g_i - sign(u_i)l_i'\right) sign(u_i)(|u_i|-l_i)^+ &\leq \left(\sum_{j=0}^k |a_{ij}||u_j| + |g_i| - l_i'\right)(|u_i|-l_i)^+\\ &\leq \left(\sum_{j=0}^k |a_{ij}|(|u_j|-l_j)^+ + A_i\right)(|u_i|-l_i)^+,\end{aligned} \tag{270}$$

where $A_i := \sum_{j=0}^k l_j|a_{ij}| + g_i - l_i' = L\sum_{j=0}^k |a_{ij}| + g_i - ML$ (see (267)). We choose $l_0, M \in (0,+\infty)$ such that

$$M \geq \max_i \left\{ \left\| \sum_{j=0}^k |a_{ij}| \right\|_\infty + 1 \right\},\ l_0 = \max_i \left\{ \|u_{0i}\|_\infty + \|g_i\|_\infty \right\}. \tag{271}$$

Then, we find

$$A_i \leq L(M-1) + l_0 - ML \leq L(M-1) + L - ML \leq 0. \tag{272}$$

By using $l_0 \geq \max_i \|u_{0i}\|_\infty$, $\int_0^t\int_\Omega \sum_{i=0}^k d_i\nabla u_i.\nabla u_i 1_{|u_i|\geq l_i} \geq 0$, (270), (272), together with (269), we have

$$\forall t \in [0,T],\ \int_\Omega \sum_{i=0}^k \left((|u_i|(\tau,x)-l_i(\tau))^+\right)^2 dxd\tau \leq 2\int_0^t\int_\Omega \sum_{i=0}^k\sum_{j=0}^k |a_{ij}|(|u_j|-l_j)^+(|u_i|-l_i)^+ dxd\tau. \tag{273}$$

Cauchy-Schwartz inequality applied to the right hand side term of (273) gives

$$\forall t \in [0,T],\ \int_\Omega \sum_{i=0}^{k} \Big((|u_i|(\tau,x) - l_i(\tau))^+ \Big)^2 dx d\tau \leq C \int_0^t \int_\Omega \sum_{i=0}^{k} \Big((|u_i|(\tau,x) - l_i(\tau))^+ \Big)^2 dx d\tau, \tag{274}$$

where $C := 2k \max_{i,j} \|a_{ij}\|_\infty$. Gronwall's lemma applied to (274) gives

$$\forall i \in \{1,\dots,k\},\ \forall t \in [0,T],\ |u_i(t)| \leq l_i(t) = l_0 \exp(tM). \tag{275}$$

Therefore, from (275), we deduce (17) with our choice of l_0 (see (271)). □

A.2 Dissipation of the energy for crossed-diffusion parabolic systems

The goal of this section is to give a sketch of the proof of the dissipation of the energy (in time) for some parabolic systems.

Lemma A.1. *Let $j \in \{1,2,3\}$, D_j defined by (51), (61), (76), $A \in \mathcal{E}_j$ (see (82), (84) and (86)), $\varphi_T \in L^2(\Omega)^4$ and φ be the solution of the following Cauchy problem*

$$\begin{cases} -\varphi_t - D_j^T \Delta \varphi = A^T \varphi & \text{in } (0,T)\times\Omega, \\ \frac{\partial \varphi}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \varphi(T,.) = \varphi_T & \text{in } \Omega. \end{cases}$$

Then, there exists $C > 0$ such that for every $(t_1,t_2) \in [0,T]^2, t_1 < t_2$,

$$\begin{aligned} &\sum_{i=1}^{j+1} \|\varphi_i(t_1,.)\|^2_{L^2(\Omega)} + \sum_{i=j+2}^{4} \|\varphi_i(t_1,.) - (\varphi_i)_\Omega(t_1)\|^2_{L^2(\Omega)} \\ &\leq C \left(\sum_{i=1}^{j+1} \|\varphi_i(t_2,.)\|^2_{L^2(\Omega)} + \sum_{i=j+2}^{4} \|\varphi_i(t_2,.) - (\varphi_i)_\Omega(t_2)\|^2_{L^2(\Omega)} \right). \end{aligned} \tag{276}$$

Proof. By using the fact that D_j is diagonalizable, we only have to prove the result when D is diagonal. First, we introduce $\psi = \Big(\varphi_1,\dots,\varphi_{j+1},\varphi_{j+2} - (\varphi_{j+2})_\Omega(.),\dots,\varphi_4 - (\varphi_4)_\Omega(.)\Big)$. We look for the parabolic system satisfied by ψ. Then, we multiply the variational formulation (see (14)) by $w(t,x) = \psi(t,x)1_{[t_1,t_2]}(t)$. By Young inequalities, we find a differential inequality as follows

$$\text{a.e. } t \in [t_1,t_2],\ \frac{d}{dt} \|\psi(t)\|^2_{L^2(\Omega)} \leq C \|\psi(t)\|^2_{L^2(\Omega)}.$$

Then, we use Gronwall's lemma. □

A.3 Technical estimates for the observability inequality in the case of 1 control

The goal of this section is to prove Lemma 4.31 and Lemma 4.32. We use the same notations as in Section 4.3.7. We recall that s is supposed to be fixed and the constants C may depend on s.

First, we recall two classical facts on the heat equation for Dirichlet conditions: a well-posedness result and a regularity result.

A.3.1 General lemmas

Proposition A.2. *Let $d \in (0,+\infty)$, $u_0 \in L^2(\Omega)$, $g \in L^2(Q)$. From [26, Section 7.1, Theorem 3 and Theorem 4], the following Cauchy problem admits a unique weak solution $u \in Z := L^2(0,T;H^1_0(\Omega)) \cap W^{1,2}(0,T;H^{-1}(\Omega))$*

$$\begin{cases} \partial_t u - d\Delta u = g & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,.) = u_0 & \text{in } \Omega. \end{cases}$$

This means that u is the unique function in Z that satisfies the variational fomulation

$$\forall w \in L^2(0,T;H^1_0(\Omega)),\ \int_0^T (\partial_t u, w)_{H^{-1}(\Omega),H^1_0(\Omega)} + \int_Q d\nabla u.\nabla w = \int_Q gw, \tag{277}$$

and

$$u(0,.) = u_0 \text{ in } L^2(\Omega). \tag{278}$$

Moreover, there exists $C > 0$ independent of u_0 and g such that

$$\|u\|_Z \leq C \left(\|u_0\|_{L^2(\Omega)} + \|g\|_{L^2(Q)} \right).$$

Proposition A.3. *Let $d \in (0, +\infty)$, $g \in L^2(Q)$, $u_0 \in C_0^\infty(\Omega)$. From Proposition A.2, the following Cauchy problem admits a unique weak solution $u \in Z$*

$$\begin{cases} \partial_t u - d\Delta u = g & \text{in } (0,T) \times \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0,.) = u_0 & \text{in } \Omega. \end{cases}$$

Moreover, from [26, Section 7.1, Theorem 5], $u \in Z_2 := L^2(0,T, H^2(\Omega) \cap H_0^1(\Omega)) \cap W^{1,2}(0,T; L^2(\Omega))$ and if $u_0 = 0$, then there exists $C > 0$ independent of g such that

$$\|u\|_{Z_2} \leq C \|g\|_{L^2(Q)}.$$

The following lemma is inspired by the proof of [12, Theorem 2.2].

Lemma A.4. *Let $d \in (0, +\infty)$, $f \in Y_2$ (see Definition 4.13), $\Phi_T \in C_0^\infty(\Omega)$, $\widetilde{\omega}$ be an open subset such that $\widetilde{\omega} \subset\subset \omega_0$, $\chi \in C^\infty(\overline{\Omega}; [0, +\infty[)$ such that $supp(\chi) \subset\subset \widetilde{\omega}$, $(r,k) \in \mathbb{R} \times [1, +\infty)$, $\Theta = \chi e^{s\alpha}(s\phi)^r$. Let $\Phi \in Z_2$ (see Proposition A.3) be the solution of*

$$\begin{cases} -\partial_t \Phi - d\Delta\Phi = \Delta f & \text{in } (0,T) \times \Omega, \\ \Phi = 0 & \text{on } (0,T) \times \partial\Omega, \\ \Phi(T,.) = \Phi_T & \text{in } \Omega. \end{cases} \tag{279}$$

We decompose

$$\Theta\Phi = \eta + \psi, \tag{280}$$

where $\eta \in Z_2$ and $\psi \in Z_2$ satisfy

$$\begin{cases} -\partial_t \eta - d\Delta\eta = \Theta\Delta f & \text{in } (0,T) \times \Omega, \\ \eta = 0 & \text{on } (0,T) \times \partial\Omega, \\ \eta(T,.) = 0 & \text{in } \Omega, \end{cases} \tag{281}$$

$$\begin{cases} -\partial_t \psi - d\Delta\psi = -(\partial_t\Theta)\Phi - 2d\nabla\Theta.\nabla\Phi - d(\Delta\Theta)\Phi & \text{in } (0,T) \times \Omega, \\ \psi = 0 & \text{on } (0,T) \times \partial\Omega, \\ \psi(T,.) = 0 & \text{in } \Omega. \end{cases} \tag{282}$$

Then, there exist $\widetilde{\chi} \in C^\infty(\overline{\Omega}; [0, +\infty[)$ such that $supp(\widetilde{\chi}) \subset\subset \widetilde{\omega}$, $\widetilde{\chi} = 1$ on $supp(\chi)$ and $C > 0$ such that

$$\|\eta\|^2_{L^2(Q)} \leq C \int_0^T \int_{\widetilde{\omega}} \widetilde{\chi}^2 e^{2s\alpha}(s\phi)^{2(r+2)} |f|^2, \tag{283}$$

$$\begin{aligned} &\left\| \frac{\psi}{(s\phi)^k} \right\|^2_{L^2(0,T;H_0^1(\Omega))} + \left\| \left(\frac{\psi}{(s\phi)^k} \right)_t \right\|^2_{L^2(0,T;H^{-1}(\Omega))} \\ &\leq C \left(\|\eta\|^2_{L^2(Q)} + \int_0^T \int_\Omega e^{2s\alpha}(s\phi)^{2(r+2-k)} |\Phi|^2 \right). \end{aligned} \tag{284}$$

Proof. Let $\Gamma \in L^2(Q)$ and let $z \in Z_2$ be the solution of

$$\begin{cases} \partial_t z - d\Delta z = \Gamma & \text{in } (0,T) \times \Omega, \\ z = 0 & \text{on } (0,T) \times \partial\Omega, \\ z(0,.) = 0 & \text{in } \Omega. \end{cases} \tag{285}$$

By Proposition A.3, we have

$$\|z\|^2_{L^2(0,T;H^2(\Omega))} \leq C \|\Gamma\|^2_{L^2(Q)}. \tag{286}$$

A duality argument between (281) and (285) gives

$$\int_0^T \int_\Omega \eta\Gamma dxdt = \int_0^T \int_\Omega \Theta\Delta(f) z dxdt. \tag{287}$$

We integrate by parts with respect to the spatial variable,

$$\int_0^T \int_\Omega \Theta \Delta(f) z dx dt = \int_0^T \int_\Omega f \Delta(\Theta z) dx dt. \tag{288}$$

There exists $\widetilde{\chi} \in C^\infty(\overline{\Omega}; [0, +\infty[)$ such that $supp(\widetilde{\chi}) \subset\subset \widetilde{\omega}$, $\widetilde{\chi} = 1$ on $supp(\chi)$ and

$$\forall i \in \{1, 2\},\ |D_x^i \Theta| \leq C \widetilde{\chi} (s\phi)^{r+i} e^{s\alpha} \text{ in } (0, T) \times \Omega. \tag{289}$$

Therefore, from (286) and (289), we can deduce that

$$\int_0^T \int_\Omega f \Delta(\Theta z) dx dt \leq \frac{1}{2} \|\Gamma\|^2_{L^2(Q)} + C \int_0^T \int_\Omega \widetilde{\chi}^2 e^{2s\alpha} (s\phi)^{2(r+2)} |f|^2 dx dt. \tag{290}$$

By using (287), (288), (290) and by taking $\Gamma = \eta$, we deduce (283).

We introduce

$$\rho = (s\phi)^{-k}. \tag{291}$$

Then, we have

$$\begin{cases} -\partial_t(\rho\psi) - d\Delta(\rho\psi) = \rho(-(\partial_t\Theta)\Phi - 2d\nabla\Theta.\nabla\Phi - d(\Delta\Theta)\Phi) \\ \qquad\qquad -(\partial_t\rho)\psi - 2d\nabla\rho.\nabla\psi - d(\Delta\rho)\psi & \text{in } (0, T) \times \Omega, \\ \rho\psi = 0 & \text{on } (0, T) \times \partial\Omega, \\ \rho\psi(T, .) = 0 & \text{in } \Omega. \end{cases} \tag{292}$$

We estimate the source term of (292). We have by definition of Θ, the fact that $k \geq 1$, (280), (291) and $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$,

$$\|\rho\partial_t(\Theta)\Phi\|^2_{L^2(Q)} \leq C \int_0^T \int_\Omega e^{2s\alpha} (s\phi)^{2(r+2-k)} |\Phi|^2, \tag{293}$$

$$\begin{aligned} &\|\rho\nabla\Theta.\nabla\Phi\|^2_{L^2(0,T;H^{-1}(\Omega))} = \|\nabla.(\rho\Phi\nabla\Theta) - (\rho(\Delta\Theta)\Phi) - (\nabla\rho.\nabla\Theta)\Phi\|^2_{L^2(0,T;H^{-1}(\Omega))} \\ &\leq C \left(\|\rho\Phi\nabla\Theta\|^2_{L^2(Q)} + \|\rho(\Delta\Theta)\Phi\|^2_{L^2(Q)} + \|(\nabla\rho.\nabla\Theta)\Phi\|^2_{L^2(Q)} \right) \\ &\leq C \int_0^T \int_\Omega e^{2s\alpha} \left((s\phi)^{2(r+1-k)} + (s\phi)^{2(r+2-k)} + (s\phi)^{2(r+1-k)} \right) |\Phi|^2 \\ &\leq C \int_0^T \int_\Omega e^{2s\alpha} (s\phi)^{2(r+2-k)} |\Phi|^2, \end{aligned} \tag{294}$$

$$\begin{aligned} &\|(\partial_t\rho)\psi\|^2_{L^2(Q)} = \|(\partial_t\rho)(\Theta\Phi - \eta)\|^2_{L^2(Q)} \leq C \left(\int_0^T \int_\Omega (s\phi)^{2(-k+1)} |\eta|^2 dx dt + \int_0^T \int_\Omega e^{2s\alpha} (s\phi)^{2(r+1-k)} |\Phi|^2 \right) \\ &\leq C \left(\int_0^T \int_\Omega |\eta|^2 dx dt + \int_0^T \int_\Omega e^{2s\alpha} (s\phi)^{2(r+2-k)} |\Phi|^2 \right), \end{aligned} \tag{295}$$

$$\begin{aligned} &\|\nabla\rho.\nabla\psi\|^2_{L^2(0,T;H^{-1}(\Omega))} = \|\nabla.(\psi\nabla\rho) - \psi\Delta\rho\|^2_{L^2(0,T;H^{-1}(\Omega))} = \|\nabla.((\Theta\Phi - \eta)\nabla\rho) - (\Theta\Phi - \eta)\Delta\rho\|^2_{L^2(0,T;H^{-1}(\Omega))} \\ &\leq C \left(\int_0^T \int_\Omega (s\phi)^{-2k} |\eta|^2 dx dt + \int_0^T \int_\Omega e^{2s\alpha} (s\phi)^{2(r-k)} |\Phi|^2 \right). \\ &\leq C \left(\int_0^T \int_\Omega |\eta|^2 dx dt + \int_0^T \int_\Omega e^{2s\alpha} (s\phi)^{2(r+2-k)} |\Phi|^2 \right). \end{aligned} \tag{296}$$

By using (292), (293), (294), (295), (296) and Proposition A.2, we deduce (284). □

Corollary A.5. *We take the same notations as in Lemma A.4 and $g \in Y_2$. Then, for every $\delta > 0$,*

$$\begin{aligned} &\int_0^T \int_{\widetilde{\omega}} \chi e^{s\alpha} \psi \Delta g \\ &\leq \delta \left(\int_0^T \int_{\widetilde{\omega}} \widetilde{\chi}^2 e^{2s\alpha} (s\phi)^{2(r+2)} |f|^2 + \int_0^T \int_\Omega e^{2s\alpha} (s\phi)^{2(r+2-k)} |\Phi|^2 \right) \\ &\quad + C_\delta \int_0^T \int_{\widetilde{\omega}} e^{2s\alpha} (s\phi)^{2(k+1)} |\nabla g|^2, \end{aligned} \tag{297}$$

$$\int_0^T \int_{\widetilde{\omega}} \chi e^{s\alpha} \psi \partial_t g$$
$$\leq \delta \left(\int_0^T \int_{\widetilde{\omega}} \widetilde{\chi}^2 e^{2s\alpha} (s\phi)^{2(r+2)} |f|^2 + \int_0^T \int_\Omega e^{2s\alpha} (s\phi)^{2(r+2-k)} |\Phi|^2 \right) \tag{298}$$
$$+ C_\delta \left(\int_0^T \int_{\widetilde{\omega}} e^{2s\alpha} (s\phi)^{2(k+2)} |g|^2 + \int_0^T \int_{\widetilde{\omega}} e^{2s\alpha} (s\phi)^{2k} |\nabla g|^2 \right). \tag{299}$$

Proof. We integrate by parts with respect to the spatial variable and we use (284), (283),

$$\int_0^T \int_{\widetilde{\omega}} \chi e^{s\alpha} \psi \Delta g$$
$$= - \int_0^T \int_{\widetilde{\omega}} \frac{\psi}{(s\phi)^k} \nabla (\chi e^{s\alpha} (s\phi)^k) . \nabla g - \int_0^T \int_{\widetilde{\omega}} \chi e^{s\alpha} (s\phi)^k \nabla \left(\frac{\psi}{(s\phi)^k} \right) . \nabla g$$
$$\leq \delta \left\| \frac{\psi}{(s\phi)^k} \right\|^2_{L^2(0,T;H_0^1(\Omega))} + C_\delta \int_0^T \int_{\widetilde{\omega}} e^{2s\alpha} (s\phi)^{2(k+1)} |\nabla g|^2$$
$$\leq \delta \left(\|\eta\|^2_{L^2(Q)} + \int_0^T \int_\Omega e^{2s\alpha} (s\phi)^{2(r+2-k)} |\Phi|^2 \right) + C_\delta \int_0^T \int_{\widetilde{\omega}} e^{2s\alpha} (s\phi)^{2(k+1)} |\nabla g|^2$$
$$\leq \delta \left(\int_0^T \int_{\widetilde{\omega}} \widetilde{\chi}^2 e^{2s\alpha} (s\phi)^{2(r+2)} |f|^2 + \int_0^T \int_\Omega e^{2s\alpha} (s\phi)^{2(r+2-k)} |\Phi|^2 \right)$$
$$+ C_\delta \int_0^T \int_{\widetilde{\omega}} e^{2s\alpha} (s\phi)^{2(k+1)} |\nabla g|^2.$$

We integrate by parts with respect to the time variable and we use (284), (283),

$$\int_0^T \int_{\widetilde{\omega}} \chi e^{s\alpha} \psi \partial_t g$$
$$= - \left\langle \left(\frac{\psi}{(s\phi)^k} \right)_t , \chi e^{s\alpha} (s\phi)^k g \right\rangle_{L^2(0,T;H^{-1}(\Omega)), L^2(0,T;H_0^1(\Omega))}$$
$$- \int_0^T \int_{\widetilde{\omega}} \frac{\psi}{(s\phi)^k} \chi \partial_t (e^{s\alpha} (s\phi)^k) g$$
$$\leq \delta \left\| \left(\frac{\psi}{(s\phi)^k} \right)_t \right\|^2_{L^2(0,T;H^{-1}(\Omega))} + C_\delta \left\| \chi e^{s\alpha} (s\phi)^k g \right\|^2_{L^2(0,T;H_0^1(\Omega))}$$
$$+ \delta \left\| \left(\frac{\psi}{(s\phi)^k} \right) \right\|^2_{L^2(0,T;L^2(\Omega))} + C_\delta \int_0^T \int_{\widetilde{\omega}} |\partial_t (e^{s\alpha} (s\phi)^k)|^2 |g|^2$$
$$\leq \delta \left(\|\eta\|^2_{L^2(Q)} + \int_0^T \int_\Omega e^{2s\alpha} (s\phi)^{2(r+2-k)} |\Phi|^2 \right)$$
$$+ C_\delta \left(\int_0^T \int_{\widetilde{\omega}} e^{2s\alpha} (s\phi)^{2(k+2)} |g|^2 + \int_0^T \int_{\widetilde{\omega}} e^{2s\alpha} (s\phi)^{2k} |\nabla g|^2 \right)$$
$$\leq \delta \left(\int_0^T \int_{\widetilde{\omega}} \widetilde{\chi}^2 e^{2s\alpha} (s\phi)^{2(r+2)} |f|^2 + \int_0^T \int_\Omega e^{2s\alpha} (s\phi)^{2(r+2-k)} |\Phi|^2 \right)$$
$$+ C_\delta \left(\int_0^T \int_{\widetilde{\omega}} e^{2s\alpha} (s\phi)^{2(k+2)} |g|^2 + \int_0^T \int_{\widetilde{\omega}} e^{2s\alpha} (s\phi)^{2k} |\nabla g|^2 \right).$$

□

A.3.2 Proof of technical lemmas: Lemma 4.31 and Lemma 4.32

Let $\varepsilon \in (0,1)$. We introduce

$$\theta = \chi_3 e^{s\alpha} (s\phi)^3. \tag{300}$$

The function $\theta\Delta\Delta\varphi_4$ satisfies the following parabolic system (see (156)),

$$\begin{cases} -\partial_t(\theta\Delta\Delta\varphi_4) - d_4\Delta(\theta\Delta\Delta\varphi_4) = \theta\Delta\Delta(m_3(\varphi_1-\varphi_2)) \\ \qquad\qquad -\partial_t\theta\Delta\Delta\varphi_4 - 2d_4\nabla\theta.\nabla(\Delta\Delta\varphi_4) - d_4\Delta\theta\Delta\Delta\varphi_4 & \text{in } (0,T)\times\Omega, \\ \theta\Delta\Delta\varphi_4 = 0 & \text{on } (0,T)\times\partial\Omega, \\ \theta\Delta\Delta\varphi_4(T,.) = 0 & \text{in } \Omega. \end{cases} \tag{301}$$

We decompose

$$\theta\Delta\Delta\varphi_4 = \eta + \psi, \tag{302}$$

where η and ψ solve, respectively,

$$\begin{cases} -\partial_t\eta - d_4\Delta\eta = \theta\Delta\Delta(m_3(\varphi_1-\varphi_2)) & \text{in } (0,T)\times\Omega, \\ \eta = 0 & \text{on } (0,T)\times\partial\Omega, \\ \eta(T,.) = 0 & \text{in } \Omega, \end{cases} \tag{303}$$

$$\begin{cases} -\partial_t\psi - d_4\Delta\psi = -\partial_t\theta\Delta\Delta\varphi_4 - 2d_4\nabla\theta.\nabla(\Delta\Delta\varphi_4) - d_4\Delta\theta\Delta\Delta\varphi_4 & \text{in } (0,T)\times\Omega, \\ \psi = 0 & \text{on } (0,T)\times\partial\Omega, \\ \psi(T,.) = 0 & \text{in } \Omega. \end{cases} \tag{304}$$

A.3.2.1 Proof of Lemma 4.31 We have

$$\int_0^T\int_{\omega_2}(\chi_3(x))^2e^{2s\alpha}(s\phi)^3(\Delta\Delta\varphi_4)(\Delta\Delta\varphi_3)dxdt = \int_0^T\int_{\omega_2}\chi_3(x)e^{s\alpha}(\eta+\psi)(\Delta\Delta\varphi_3)dxdt. \tag{305}$$

The first term in the right-hand side of (305) can be estimated as follows,

$$\begin{aligned}\int_0^T\int_{\omega_2}\chi_3(x)e^{s\alpha}\eta(\Delta\Delta\varphi_3)dxdt &\le \varepsilon\int_0^T\int_\Omega e^{2s\alpha}(s\phi)|\Delta\Delta\varphi_3|^2dxdt + C_\varepsilon\int_0^T\int_{\omega_2}(\chi_3(x))^2(s\phi)^{-1}\eta^2dxdt \\ &\le \varepsilon\int_0^T\int_\Omega e^{2s\alpha}(s\phi)|\Delta\Delta\varphi_3|^2dxdt + C_\varepsilon\int_0^T\int_\Omega\eta^2dxdt. \end{aligned}\tag{306}$$

Lemma A.6. *For every $\delta>0$,*

$$\begin{aligned}\int_0^T\int_\Omega|\eta|^2dxdt &\le \delta\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^4(|\Delta\varphi_1|^2+|\Delta\varphi_2|^2)dxdt \\ &\quad + C_\delta\int_0^T\int_{\omega_2}e^{2s\alpha}\Big\{(s\phi)^{24}(|\varphi_1|^2+|\varphi_2|^2+|\Delta\varphi_3|^2)+(s\phi)^{22}(|\nabla\varphi_1|^2+|\nabla\varphi_2|^2)\Big\}dxdt. \end{aligned}\tag{307}$$

Proof. The idea of the proof is to apply two times Lemma A.4 because the source term of (303) is $\theta\Delta\Delta(\dots)$.

Step 1: We apply Lemma A.4: (283) with $d=d_4$, $f=m_3\Delta(\varphi_1-\varphi_2)$, $\Phi_T=\Delta\Delta\varphi_{4,T}$, $\widetilde{\omega}=\omega_2$, $\chi=\chi_3$, $r=3$, $\Theta=\theta$, $\Phi=\Delta\Delta\varphi_4$ and the decomposition (302). Then, there exists $\widetilde{\chi_3}\in C^\infty(\overline{\Omega};[0,+\infty[)$ such that $supp(\widetilde{\chi_3})\subset\subset\omega_2$, $\widetilde{\chi_3}=1$ on $supp(\chi_3)$ and

$$\|\eta\|^2_{L^2(Q)} \le C\int_0^T\int_{\omega_2}(\widetilde{\chi_3})^2e^{2s\alpha}(s\phi)^{10}(|\Delta\varphi_1|^2+|\Delta\varphi_2|^2)dxdt. \tag{308}$$

Remark A.7. This estimate is not sufficient because we can not absorb the right hand side term of (308) by the left hand side term of (163).

Step 2: Now, our aim is to prove that for every $i\in\{1,2\}$, $\delta>0$, we have

$$\begin{aligned}&\int_0^T\int_{\omega_2}(\widetilde{\chi_3})^2e^{2s\alpha}(s\phi)^{10}|\Delta\varphi_i|^2dxdt \\ &\le \delta\left(\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^4|\Delta\varphi_i|^2dxdt\right) \\ &\quad + C_\delta\left(\int_0^T\int_{\omega_2}e^{2s\alpha}(s\phi)^{24}(|\varphi_1|^2+|\varphi_2|^2+|\Delta\varphi_3|^2)dxdt + \int_0^T\int_{\omega_2}e^{2s\alpha}(s\phi)^{22}|\nabla\varphi_i|^2dxdt\right). \end{aligned}\tag{309}$$

Remark A.8. This previous estimate is also useful for the proof of the observability inequality with one component (see (168)).

First, we remark that

$$\int_0^T \int_{\omega_2} (\widetilde{\chi_3})^2 e^{2s\alpha}(s\phi)^{10}|\Delta\varphi_i|^2 dxdt = \int_0^T \int_{\omega_2} \widetilde{\chi_3} e^{s\alpha}\widetilde{\theta}\Delta\varphi_i\Delta\varphi_i,$$

with

$$\widetilde{\theta} = \widetilde{\chi_3} e^{s\alpha}(s\phi)^{10}. \tag{310}$$

Moreover, $\widetilde{\theta}\Delta\varphi_i$ satisfies the following parabolic system (see (154) and Lemma 4.23),

$$\begin{cases} -\partial_t(\widetilde{\theta}\Delta\varphi_i) - d_i\Delta(\widetilde{\theta}\Delta\varphi_i) = \widetilde{\theta}\Delta(a_{1i}\varphi_1 + a_{2i}\varphi_2 + \delta_{i2}(d_2-d_3)\Delta\varphi_3) \\ \qquad\qquad -\partial_t\widetilde{\theta}\Delta\varphi_i - 2d_i\nabla\widetilde{\theta}.\nabla(\Delta\varphi_i) - d_i\Delta\widetilde{\theta}\Delta\varphi_i & \text{in } (0,T)\times\Omega, \\ \widetilde{\theta}\Delta\varphi_i = 0 & \text{on } (0,T)\times\partial\Omega, \\ \widetilde{\theta}\Delta\varphi_i(T,.) = 0 & \text{in } \Omega. \end{cases} \tag{311}$$

We decompose

$$\widetilde{\theta}\Delta\varphi_i = \widetilde{\eta_i} + \widetilde{\psi_i}, \tag{312}$$

where $\widetilde{\eta_i}$ and $\widetilde{\psi_i}$ solve, respectively,

$$\begin{cases} -\partial_t\widetilde{\eta_i} - d_i\Delta\widetilde{\eta_i} = \widetilde{\theta}\Delta(a_{1i}\varphi_1 + a_{2i}\varphi_2 + \delta_{i2}(d_2-d_3)\Delta\varphi_3) & \text{in } (0,T)\times\Omega, \\ \widetilde{\eta_i} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \widetilde{\eta_i}(T,.) = 0 & \text{in } \Omega, \end{cases} \tag{313}$$

$$\begin{cases} -\partial_t\widetilde{\psi_i} - d_i\Delta\widetilde{\psi_i} = -\partial_t\widetilde{\theta}\Delta\varphi_i - 2d_i\nabla\widetilde{\theta}.\nabla(\Delta\varphi_i) - d_i\Delta\widetilde{\theta}\Delta\varphi_i & \text{in } (0,T)\times\Omega, \\ \widetilde{\psi_i} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \widetilde{\psi_i}(T,.) = 0 & \text{in } \Omega. \end{cases} \tag{314}$$

We have

$$\int_0^T \int_{\omega_2} (\widetilde{\chi_3})^2 e^{2s\alpha}(s\phi)^{10}|\Delta\varphi_i|^2 dxdt = \int_0^T \int_{\omega_2} \widetilde{\chi_3} e^{s\alpha}(\widetilde{\eta_i} + \widetilde{\psi_i})(\Delta\varphi_i) dxdt. \tag{315}$$

The first term in the right-hand side of (315) can be estimated as follows,

$$\int_0^T \int_{\omega_2} \widetilde{\chi_3} e^{s\alpha}\widetilde{\eta_i}(\Delta\varphi_i) dxdt \le \delta \int_0^T \int_\Omega e^{2s\alpha}(s\phi)^4|\Delta\varphi_i|^2 dxdt + C_\delta \int_0^T \int_\Omega \widetilde{\eta_i}^2 dxdt. \tag{316}$$

Then, we apply Lemma A.4: (283) with $d = d_i$, $f = a_{1i}\varphi_1 + a_{2i}\varphi_2 + \delta_{i2}(d_2-d_3)\Delta\varphi_3 \in Y_2$ (because $A \in \mathcal{M}_4(C_0^\infty(Q))$), $\Phi_T = \Delta\varphi_{i,T}$, $\widetilde{\omega} = \omega_2$, $\chi = \widetilde{\chi_3}$, $r = 10$, $\Theta = \widetilde{\theta}$, $\Phi = \Delta\varphi_i$ and the decomposition (312). There exists ${\chi_3}^\sharp \in C^\infty(\overline{\Omega};[0,+\infty[)$ such that $supp({\chi_3}^\sharp) \subset\subset \omega_2$ and C which depends on $\|A\|_{L^\infty(Q)}$

$$\int_0^T \int_\Omega |\widetilde{\eta_i}|^2 dxdt \le C \int_0^T \int_{\omega_2} ({\chi_3}^\sharp)^2 e^{2s\alpha}(s\phi)^{24}(|\varphi_1|^2 + |\varphi_2|^2 + |\Delta\varphi_3|^2) dxdt. \tag{317}$$

Then, (316) and (317) give

$$\begin{aligned} &\int_0^T \int_{\omega_2} \widetilde{\chi_3} e^{s\alpha}\widetilde{\eta_i}(\Delta\varphi_i) dxdt \\ &\le \delta \int_0^T \int_\Omega e^{2s\alpha}(s\phi)^4|\Delta\varphi_i|^2 dxdt + C_\delta \int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)^{24}(|\varphi_1|^2 + |\varphi_2|^2 + |\Delta\varphi_3|^2) dxdt. \end{aligned} \tag{318}$$

For the second term in the right-hand side of (315), we use Corollary A.5: (297) with $d = d_i$, $f = a_{1i}\varphi_1 + a_{2i}\varphi_2 + \delta_{i2}(d_2-d_3)\Delta\varphi_3 \in Y_2$, $\Phi_T = \Delta\varphi_{i,T}$, $\widetilde{\omega} = \omega_2$, $\chi = \widetilde{\chi_3}$, $(r,k) = (10,10)$, $\Theta = \widetilde{\theta}$, $\Phi = \Delta\varphi_i$ and the decomposition (312)). Then, we have

$$\begin{aligned} &\int_0^T \int_{\omega_2} \widetilde{\chi_3} e^{s\alpha}\widetilde{\psi_i}\Delta\varphi_i \\ &\le \delta \left(\int_0^T \int_{\omega_2} ({\chi_3}^\sharp)^2 e^{2s\alpha}(s\phi)^{24}(|\varphi_1|^2 + |\varphi_2|^2 + |\Delta\varphi_3|^2) + \int_0^T \int_\Omega e^{2s\alpha}(s\phi)^4|\Delta\varphi_i|^2 \right) \\ &\quad + C_\delta \int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)^{22}|\nabla\varphi_i|^2. \end{aligned} \tag{319}$$

Gathering (315), (318) and (319), we have (309).

The estimates (308) and (309) give (307). □

Applying Lemma A.6 with $\delta = \varepsilon/C_\varepsilon$, we find

$$\begin{aligned}
&\int_0^T \int_\Omega |\eta|^2 dxdt \\
&\leq \frac{\varepsilon}{C_\varepsilon} \int_0^T \int_\Omega e^{2s\alpha}(s\phi)^4(|\Delta\varphi_1|^2 + |\Delta\varphi_2|^2)dxdt \\
&\quad + C'_\varepsilon \int_0^T \int_{\omega_2} e^{2s\alpha}\Big\{(s\phi)^{24}(|\varphi_1|^2 + |\varphi_2|^2 + |\Delta\varphi_3|^2) + (s\phi)^{22}(|\nabla\varphi_1|^2 + |\nabla\varphi_2|^2)\Big\}dxdt. \qquad (320)
\end{aligned}$$

Then, we put (320) in (306) to get

$$\begin{aligned}
&\int_0^T \int_{\omega_2} \chi_3(x) e^{s\alpha}\eta(\Delta\Delta\varphi_3)dxdt \\
&\leq \varepsilon\left(\int_0^T \int_\Omega e^{2s\alpha}(s\phi)|\Delta\Delta\varphi_3|^2 dxdt + \int_0^T \int_\Omega e^{2s\alpha}(s\phi)^4(|\Delta\varphi_1|^2 + |\Delta\varphi_2|^2)dxdt\right) \\
&+ C_\varepsilon\left(\int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)^{24}(|\varphi_1|^2 + |\varphi_2|^2 + |\Delta\varphi_3|^2)dxdt + \int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)^{22}(|\nabla\varphi_1|^2 + |\nabla\varphi_2|^2)dxdt\right). \qquad (321)
\end{aligned}$$

Lemma A.9. *For every $\delta > 0$,*

$$\begin{aligned}
&\int_0^T \int_{\omega_2} \chi_3 e^{s\alpha}\psi(\Delta\Delta\varphi_3)dxdt \\
&\leq \delta\left(\int_0^T \int_{\omega_2} (\widetilde{\chi_3})^2 e^{2s\alpha}(s\phi)^{10}(|\Delta\varphi_1|^2 + |\Delta\varphi_2|^2) + \int_0^T \int_\Omega e^{2s\alpha}(s\phi)^3|\Delta\Delta\varphi_4|^2\right) \\
&\quad + C_\delta \int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)^9|\nabla\Delta\varphi_3|^2. \qquad (322)
\end{aligned}$$

Proof. We apply Corollary A.5: (297) with $d = d_4$, $f = m_3\Delta(\varphi_1 - \varphi_2)$, $\Phi_T = \Delta\Delta\varphi_{4,T}$, $\widetilde{\omega} = \omega_2$, $\chi = \chi_3$, $(r,k) = (3,7/2)$, $\Theta = \theta$, $\Phi = \Delta\Delta\varphi_4$, the decomposition (302) and $g = \Delta\varphi_3$. □

Applying Lemma A.9 with $\delta = \varepsilon$, we find

$$\begin{aligned}
&\int_0^T \int_{\omega_2} \chi_3 e^{s\alpha}\psi(\Delta\Delta\varphi_3)dxdt \\
&\leq \varepsilon\left(\int_0^T \int_{\omega_2} (\widetilde{\chi_3})^2 e^{2s\alpha}(s\phi)^{10}(|\Delta\varphi_1|^2 + |\Delta\varphi_2|^2) + \int_0^T \int_\Omega e^{2s\alpha}(s\phi)^3|\Delta\Delta\varphi_4|^2\right) \\
&\quad + C_\varepsilon \int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)^9|\nabla\Delta\varphi_3|^2. \qquad (323)
\end{aligned}$$

Then, we put (309) with $\delta = \varepsilon$ in (323) to get

$$\begin{aligned}
&\int_0^T \int_{\omega_2} \chi_3 e^{s\alpha}\psi(\Delta\Delta\varphi_3)dxdt \\
&\leq \varepsilon\left(\int_0^T \int_\Omega e^{2s\alpha}\Big\{(s\phi)^4(|\Delta\varphi_1|^2 + |\Delta\varphi_2|^2) + (s\phi)^3|\Delta\Delta\varphi_4|^2\Big\}dxdt\right) \\
&+ C_\varepsilon\left(\int_0^T \int_{\omega_2} e^{2s\alpha}\Big\{(s\phi)^{24}(|\varphi_1|^2 + |\varphi_2|^2 + |\Delta\varphi_3|^2) + (s\phi)^{22}(|\nabla\varphi_1|^2 + |\nabla\varphi_2|^2 + |\nabla\Delta\varphi_3|^2)\Big\}dxdt\right). \qquad (324)
\end{aligned}$$

Therefore, recalling (305), (321), (324), we get (166) and consequently Lemma 4.31.

A.3.2.2 Proof of Lemma 4.32

We have by (300) and (302)

$$\int_0^T \int_{\omega_2} (\chi_3(x))^2 e^{2s\alpha}(s\phi)^3(\Delta\Delta\varphi_4)(\partial_t\Delta\varphi_3)dxdt = \int_0^T \int_{\omega_2} \chi_3(x) e^{s\alpha}(\eta + \psi)\partial_t(\Delta\varphi_3)dxdt. \qquad (325)$$

We easily have by Young's inequality

$$\int_0^T \int_{\omega_2} \chi_3(x) e^{s\alpha} \eta \partial_t(\Delta\varphi_3) dxdt \leq \varepsilon \int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)|\partial_t(\Delta\varphi_3)|^2 dxdt + C_\varepsilon \int_0^T \int_\Omega |\eta|^2 dxdt. \tag{326}$$

By using Lemma A.6 with $\delta = \varepsilon/C_\varepsilon$, we can deduce from (326) that

$$\begin{aligned}
&\int_0^T \int_{\omega_2} \chi_3(x) e^{s\alpha} \eta (\partial_t \Delta\varphi_3) dxdt \\
&\leq \varepsilon \left(\int_0^T \int_\Omega e^{2s\alpha}(s\phi)|\partial_t \Delta\varphi_3|^2 dxdt + \int_0^T \int_\Omega e^{2s\alpha}(s\phi)^4 (|\Delta\varphi_1|^2 + |\Delta\varphi_2|^2) dxdt \right) \\
&+ C_\varepsilon \left(\int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)^{24} (|\varphi_1|^2 + |\varphi_2|^2 + |\Delta\varphi_3|^2) dxdt + \int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)^{22} (|\nabla\varphi_1|^2 + |\nabla\varphi_2|^2) dxdt \right).
\end{aligned} \tag{327}$$

Then, we estimate the other term in the right hand side of (325).

Lemma A.10. *For every $\delta > 0$,*

$$\begin{aligned}
&\int_0^T \int_{\omega_2} \chi_3 e^{s\alpha} \psi \partial_t \Delta\varphi_3 \\
&\leq \delta \left(\int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)^{10} (|\Delta\varphi_1|^2 + |\Delta\varphi_2|^2) + \int_0^T \int_\Omega e^{2s\alpha}(s\phi)^3 |\Delta\Delta\varphi_4|^2 \right) \\
&\quad + C_\delta \left(\int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)^{11} |\Delta\varphi_3|^2 + \int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)^7 |\nabla\Delta\varphi_3|^2 \right).
\end{aligned} \tag{328}$$

Proof. We apply Corollary A.5: (299) with $d = d_4$, $f = \Delta(\varphi_1 - \varphi_2)$, $\Phi_T = \Delta\Delta\varphi_{4,T}$, $\widetilde{\omega} = \omega_2$, $\chi = \chi_3$, $(r, k) = (3, 7/2)$, $\Theta = \theta$, $\Phi = \Delta\Delta\varphi_4$, the decomposition (302) and $g = \Delta\varphi_3$. □

Then, we put (309) with $\delta = \varepsilon$ in (328) to get

$$\begin{aligned}
&\int_0^T \int_{\omega_2} \chi_3 e^{s\alpha} \psi (\partial_t \Delta\varphi_3) dxdt \\
&\leq \varepsilon \left(\int_0^T \int_\Omega e^{2s\alpha} \Big\{ (s\phi)^4 (|\Delta\varphi_1|^2 + |\Delta\varphi_2|^2) + (s\phi)^3 |\Delta\Delta\varphi_4|^2 \Big\} dxdt \right) \\
&+ C_\varepsilon \left(\int_0^T \int_{\omega_2} e^{2s\alpha} \Big\{ (s\phi)^{24} (|\varphi_1|^2 + |\varphi_2|^2 + |\Delta\varphi_3|^2) + (s\phi)^{22} (|\nabla\varphi_1|^2 + |\nabla\varphi_2|^2 + |\nabla\Delta\varphi_3|^2) \Big\} dxdt \right).
\end{aligned} \tag{329}$$

Recalling (325), (327), (329), we get (167) and consequently Lemma 4.32.

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