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KAM FOR THE NONLINEAR WAVE EQUATION ON THE CIRCLE: SMALL AMPLITUDE SOLUTION

MOUDHAFFAR BOUTHELJA

ABSTRACT. In this paper we consider the nonlinear wave equation on the circle:

$$u_{tt} - u_{xx} + mu = g(x, u), \quad t \in \mathbb{R}, x \in \mathbb{S}^1,$$

where $m \in [1, 2]$ is a mass and $g(x, u) = 4u^3 + O(u^4)$. This equation will be treated as a perturbation of the integrable Hamiltonian:

$$(*) \quad u_t = v, \quad v_t = -u_{xx} + mu.$$

Near the origin and for generic m , we prove the existence of small amplitude quasi-periodic solutions close to the solution of the linear equation (*). For the proof we use an abstract KAM theorem in infinite dimension and a Birkhoff normal form result.

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1. INTRODUCTION AND RESULTS

1.1. Introduction. We consider the cubic wave equation on the circle:

$$(1.1) \quad u_{tt} - u_{xx} + mu = g(x, u), \quad t \in \mathbb{R}, x \in \mathbb{S}^1,$$

where $m \in [1, 2]$ is a mass and g is a real holomorphic function on $\mathbb{S}^1 \times J$, for J some neighborhood of the origin of \mathbb{R} . We suppose that the nonlinearity g satisfies

$$(1.2) \quad g(x, u) = 4u^3 + O(u^4).$$

We prove the existence of small amplitude quasi-periodic solutions close to the solution of the linear equation.

Since the space variable belongs to the circle, we can diagonalize the linear part of the equation in Fourier basis. So we can study the PDE as a perturbation of an integrable Hamiltonian of the following form

$$(\square) \quad H = \sum_{s \in \mathbb{Z}} \lambda_s \xi_s \eta_s + \text{Perturbation} ,$$

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where $\lambda_s = \sqrt{s^2 + m}$. In order to prove the existence of quasi-periodic solutions, we will use an abstract KAM theorem in infinite dimension adapted to our situation and proven in [6]. The KAM theory (Kolmogorov-Arnold-Moser) tells us that, under the effect of a small perturbation and under several conditions of non resonance, an integrable Hamiltonian system continues to exhibit finite-dimensional invariant tori in an infinite dimensional space. The existence of these invariant tori gives us the existence of quasi-periodic solutions. The main issue here is that frequencies λ_s do not satisfy the standard non resonance hypotheses ¹. In the Dirichlet case, the sum in (□) is restricted to positive indices (see [15]). In this case, the standard non resonance conditions can be verified with the mass m . In the periodic case, both positive and negative indices are allowed. Note that $\lambda_s = \lambda_{-s}$. So we obtain a resonant Hamiltonian system. For this purpose, the KAM theorem that we will use must deal with the case of multiple eigenvalues.

The existence of quasi-periodic solutions for nonlinear Hamiltonian PDEs have interested many authors. The first result related to preserving such solutions, after the perturbation of an integrable Hamiltonian of infinite dimension, was given by Kuksin in 1987 in [11, 10] for the Schrödinger equation in dimension 1 with Dirichlet conditions.

Concerning the wave equation, the first result is due to Wayne in [15]. He considered the cubic-wave equation in dimension 1 with external potential in $L^2([0, 1])$, and with Dirichlet conditions (which leads to simplicity of the spectrum).

We can also cite the work of Pöschel in [13]. In this paper, the author considers the wave equation in dimension 1 with mass, homogeneous Dirichlet condition, and analytical cubic nonlinearity that does not depends on the space variable.

In 1998; Chierchia and You consider in [7] the wave equation in dimension 1 with analytic periodic potential and an analytic quadratic perturbation that does not depends on the space variable. In this case, the potential acts as an external parameter. This makes verifying the non resonance conditions possible. In particular, the authors do not authorize the case of a vanishing potential.

The most recent work is due to Berti, Biasco and Procesi in 2013 in [5]. In this paper, they consider the derivative wave equation given by:

$$u_{tt} - u_{xx} + mu + f(Du) = 0, \quad m > 0, \quad D := \sqrt{-\partial_{xx}^2 + m}, \quad (t, x) \in \mathbb{R} \times \mathbb{T},$$

where $f(s)$ is a real analytic nonlinearity of the form

$$f(s) = as^3 + \sum_{k \geq 5} f_k s^k, \quad a \neq 0.$$

We remark that the nonlinearity is independent of the space variable x . This implies that the moment $-i \int_{\mathbb{T}} \bar{u} \partial_x u dx$ is preserved. This symmetry simplifies the proof of the KAM theorem.

In our case, there are no external parameters. The space variable belongs to the circle, so we are in the periodic case. The non-linearity g depends on the space variable.

The plan of the paper is the following:

- In the first section, we give the main result of the paper (see Theorem 1.3).
- In the second section, we show that, for an admissible set (see definition 1.2), the small divisors of the wave equation (1.1) admit a positive lower bound. This is proven for $m \in [1, 2] \setminus \mathcal{U}$ where \mathcal{U} is zero Lebesgue measure set.
- In the third section, using a Birkhoff normal form, we transform the resonant Hamiltonian associated to the equation (1.1) into a Hamiltonian that satisfies the hypotheses of the KAM theorem (see Theorem 3.7).
- In the fourth section, we state the KAM theorem (see Theorem 4.1), and we verify the non resonance hypotheses (see Lemma 4.2-4.8).
- In the last part we prove the existence of quasi-periodic solutions of small amplitudes for the equation (1.1).

1.2. Results. We consider the nonlinear wave equation on the circle (1.1) with g in the form (1.2). Introducing the change of variable $v = \dot{u}$, the equation (1.1) becomes:

$$\begin{cases} \dot{u} = v, \\ \dot{v} = -\Lambda^2 u + g(x, u), \end{cases}$$

where $\Lambda := (\sqrt{-\partial_{xx}^2 + m})$. Defining $\psi := \frac{1}{\sqrt{2}}(\Lambda^{\frac{1}{2}}u - i\Lambda^{-\frac{1}{2}}v)$, we get the following equation for ψ :

$$\dot{\psi} = \frac{1}{\sqrt{2}} \left(\Lambda^{1/2} \dot{u} - i\Lambda^{-1/2} \dot{v} \right).$$

¹Hypotheses likes: $|k_1 \lambda_1 + \dots + k_n \lambda_n| \geq \frac{\gamma}{|k|^\tau}$, $k \in \mathbb{Z}^n \setminus \{0\}$.

We note that $u = \Lambda^{-1/2} \left(\frac{\psi + \bar{\psi}}{\sqrt{2}} \right)$; replacing \dot{u} and \dot{v} by their expressions yields:

$$\frac{1}{i} \dot{\psi} = \Lambda \psi - \frac{1}{\sqrt{2}} \Lambda^{-1/2} g \left(x, \Lambda^{-1/2} \left(\frac{\psi + \bar{\psi}}{\sqrt{2}} \right) \right).$$

Let us endow $L^2(\mathbb{S}^1, \mathbb{C})$ with the classical real symplectic form $-i d\psi \wedge d\bar{\psi} = -du \wedge dv$ and consider the following Hamiltonian:

$$H(\psi, \bar{\psi}) = \int_{\mathbb{S}^1} (\Lambda \psi) \bar{\psi} dx + \int_{\mathbb{S}^1} G \left(x, \Lambda^{-1/2} \left(\frac{\psi + \bar{\psi}}{\sqrt{2}} \right) \right) dx,$$

where G is a primitive of g with respect to u :

$$g = \partial_u G, \quad G(x, u) = u^4 + \mathcal{O}(u^5).$$

Then, (1.1) becomes a Hamiltonian system:

$$\dot{\psi} = i \frac{\partial H}{\partial \bar{\psi}}.$$

Consider now the complex Fourier orthonormal basis given by $\{\varphi_s(x) = \frac{e^{isx}}{\sqrt{2\pi}}, s \in \mathbb{Z}\}$. In this base, the operator Λ is diagonal, and we have:

$$\Lambda \varphi_s = \lambda_s \varphi_s,$$

with $\lambda_s = \sqrt{s^2 + m}$. Decomposing ψ and $\bar{\psi}$ in this basis yields:

$$\psi = \sum_{s \in \mathbb{Z}} \xi_s \varphi_s \quad \text{and} \quad \bar{\psi} = \sum_{s \in \mathbb{Z}} \eta_s \varphi_{-s}.$$

By injecting this decomposition into the expression of H , we obtain:

$$(1.3) \quad H = \sum_{s \in \mathbb{Z}} \lambda_s \xi_s \eta_s + \int_{\mathbb{S}^1} G \left(x, \sum_{s \in \mathbb{Z}} \frac{\xi_s \varphi_s + \eta_s \varphi_{-s}}{\sqrt{2\lambda_s}} \right) dx.$$

Let $\mathcal{P}_{\mathbb{C}} := \ell^2(\mathbb{Z}, \mathbb{C}) \times \ell^2(\mathbb{Z}, \mathbb{C})$ that we endow with the complex symplectic form $-i \sum_{s \in \mathbb{Z}} d\xi_s \wedge d\eta_s$. We define

$$\mathcal{P}_{\mathbb{R}} := \{(\xi, \eta) \in \mathcal{P}_{\mathbb{C}} \mid \eta_s = \bar{\xi}_s\}.$$

Then, equation (1.1) is equivalent to the following Hamiltonian system on $\mathcal{P}_{\mathbb{R}}$:

$$(1.4) \quad \begin{cases} \dot{\xi}_s = i \frac{\partial H}{\partial \eta_s}, \\ \dot{\eta}_s = -i \frac{\partial H}{\partial \xi_s}, \end{cases}$$

for $s \in \mathbb{Z}$.

From now, we write $H = H_2 + P$, where

$$(1.5) \quad H_2 = \sum_{s \in \mathbb{Z}} \lambda_s \xi_s \eta_s, \quad \text{and} \quad P = \int_{\mathbb{S}^1} G \left(x, \sum_{s \in \mathbb{Z}} \frac{\xi_s \varphi_s + \eta_s \varphi_{-s}}{\sqrt{2\lambda_s}} \right) dx.$$

Remark 1.1. Recall that $g(x, u) = 4u^3 + \mathcal{O}(u^4)$ and $g = \partial_u G$, so we can decompose P into $P = P_4 + R_5$ where

$$P_4(\xi, \eta) = \int_{\mathbb{S}^1} u^4 dx = \int_{\mathbb{S}^1} \left(\sum_{s \in \mathbb{Z}} \frac{\xi_s \varphi_s + \eta_s \varphi_{-s}}{\sqrt{2\lambda_s}} \right)^4 dx,$$

$$R_5(\xi, \eta, x) = P(\xi, \eta, x) - P_4(\xi, \eta) = \mathcal{O}(\|(\xi, \eta)\|^5).$$

In addition, P_4 reads

$$P_4 = \sum_{i, j, k, l \in \mathbb{Z}} C(i, j, k, l) (\xi_i + \eta_{-i}) (\xi_j + \eta_{-j}) (\xi_k + \eta_{-k}) (\xi_l + \eta_{-l}),$$

where

$$C(i, j, k, l) := \int_{\mathbb{S}^1} \varphi_i(x) \varphi_j(x) \varphi_k(x) \varphi_l(x) dx = \begin{cases} \frac{1}{2\pi} & \text{if } i + j + k + l = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let \mathcal{A} be a finite set of \mathbb{Z} of cardinality n , and a vector $I = (I_a)_{a \in \mathcal{A}}$ with positive components (i.e. $I_a > 0$ for all $a \in \mathcal{A}$). Let T_I^n be the real torus of dimension n defined by

$$T_I^n = \begin{cases} \xi_a = \bar{\eta}_a, |\xi_a|^2 = I_a & \text{if } a \in \mathcal{A}, \\ \xi_s = \eta_s = 0 & \text{if } s \in \mathcal{L} = \mathbb{Z} \setminus \mathcal{A}. \end{cases}$$

This torus is stable by the Hamiltonian flow when the perturbation P is zero. We can even give the analytic expression of the solution of the linear equation.

Our purpose in all the following is to prove the persistence of the torus T_I^n when the perturbation P is no longer zero, while making the crucial assumption that this torus is admissible. A torus is said to be admissible if it is constructed from an admissible set \mathcal{A} .

Definition 1.2. Let \mathcal{A} be a finite set of \mathbb{Z} . \mathcal{A} is admissible if, for all $j \in \mathcal{A} \setminus \{0\}$, we have $-j \notin \mathcal{A} \setminus \{0\}$.

Let us introduce the sets $\mathcal{L} = \mathbb{Z} \setminus \mathcal{A}$ and $\mathcal{A}^- = \{j \in \mathcal{L} \mid -j \in \mathcal{A}\}$. In a neighborhood of the invariant torus T_I^n in \mathbb{C}^{2n} , we define the action-angle variables $(r_a, \theta_a)_{\mathcal{A}}$ by:

$$\begin{cases} \xi_a = \sqrt{(I_a + r_a)} e^{i\theta_a}, \\ \eta_a = \bar{\xi}_a. \end{cases}$$

For $s \in \mathcal{A}$, we denote by ω_s (instead of λ_s) the internal frequencies. In these new variables and notations, the quadratic part H_2 of H becomes, up to a constant,

$$H_2 = \sum_{a \in \mathcal{A}} \omega_a r_a + \sum_{s \in \mathcal{L}} \lambda_s \xi_s \eta_s.$$

In addition, the perturbation becomes:

$$P(r, \theta, \xi, \eta) = \int_{\mathbb{S}^1} G(x, \hat{u}_{I,m}(r, \theta, \xi, \eta)) dx,$$

with

$$\begin{aligned} \hat{u}_{I,m}(r, \theta, \xi, \eta) &= \sum_{a \in \mathcal{A}} \sqrt{(I_a + r_a)} \frac{e^{-i\theta_a} \varphi_a(x) + e^{i\theta_a} \varphi_{-a}(x)}{\sqrt{2}(a^2 + m)^{1/4}} \\ &+ \sum_{s \in \mathcal{L}} \frac{\xi_s \varphi_s(x) + \eta_{-s} \varphi_s(x)}{\sqrt{2}(s^2 + m)^{1/4}}. \end{aligned}$$

We set $u_{I,m}(\theta, x) = \hat{u}_{I,m}(0, \theta, 0, 0)$. Then, for any $I \in \mathbb{R}_+^{\mathcal{A}}$, $m \in [1, 2]$ and $\theta_0 \in \mathbb{S}^1$, the function $(t, x) \mapsto u_{I,m}(\theta_0 + t\omega, x)$ is solution of the linear wave equation. In this case, the torus T_I^n is stable by the Hamiltonian flow. Our goal is to state a similar result when the perturbation is not zero (in the nonlinear case).

Theorem 1.3. Let $\alpha > 1/2$. Assume that \mathcal{A} is an admissible set of cardinality n . Assume also that the perturbation g is real holomorphic on a neighborhood of $\mathbb{S}^1 \times J$ with J some neighborhood of the origin of \mathbb{R} and reads $g(x, u) = 4u^3 + O(u^4)$. There exists a Borel subset $\mathcal{U} \subset [1, 2]$ with zero Lebesgue measure, such that for $m \in ([1, 2] \setminus \mathcal{U})$, there exists ν_0 that depends on \mathcal{A} , m , and the nonlinearity g , such that:

For $0 < \nu \leq \nu_0$ there exists a Borel set $\mathcal{D}' \subset [\nu, 2\nu]^n$ asymptotically of full Lebesgue measure, i.e.

$$\text{mes}([\nu, 2\nu]^n \setminus \mathcal{D}') \leq \nu^{n+\gamma},$$

with $\gamma > 0$ and depending on n . For $m \in ([1, 2] \setminus \mathcal{U})$ and $I \in \mathcal{D}'$, there exists:

(1) a function $u(\theta, x)$ analytic in θ and of class H^α in $x \in \mathbb{S}^1$ such that:

$$\sup_{\theta \in \mathbb{R}} \|u(\theta, \cdot) - u_{I,m}(\theta, \cdot)\|_{H^\alpha} \leq C\nu^{4/5},$$

with C an absolute constant.

(2) a mapping $\omega' : ([1, 2] \setminus \mathcal{U}) \times \mathcal{D}' \rightarrow \mathbb{R}^n$ verifying:

$$\omega' = \omega + MI + O(\nu^{3/2}),$$

such that for any $m \in ([1, 2] \setminus \mathcal{U})$ and $I \in \mathcal{D}'$ the function

$$t \mapsto u(\theta + t\omega', x)$$

is solution of the wave equation (1.1). This solution is linear stable.

The rest of the paper will be devoted to the proof of this theorem.

2. SMALL DIVISORS

2.1. Non resonance of frequencies. In this section, we assume that \mathcal{A} is an admissible set as in Definition 1.2.

We consider the frequency vector

$$\omega \equiv \omega(m) = (\omega_a(m))_{a \in \mathcal{A}},$$

with $\omega_a(m) = \sqrt{a^2 + m}$. The main and only result of this section is the following:

Proposition 2.1. *Consider an admissible set \mathcal{A} of cardinality n that verifies $\mathcal{A} \subset \{a \in \mathbb{Z} \mid |a| \leq N, N \geq 1\}$. Then, for any $k \in \mathbb{Z}^{\mathcal{A}} \setminus \{0\}$, any $\chi > 0$ and any $c \in \mathbb{R}$, we have*

$$\text{mes} \left\{ m \in [1, 2] \mid \left| \sum_{a \in \mathcal{A}} k_a \omega_a(m) + c \right| \leq \chi \right\} \leq C \frac{N^{2n^2} \chi^{1/n}}{|k|},$$

with $|k| := \sum_{a \in \mathcal{A}} |k_a|$ and $C > 0$ is a constant that depends only on n .

The proof uses the same arguments as in Theorem 6.5 of [2] (see also [1] and [3]). For clarity, we recall the main steps of the proof.

Lemma 2.2. *Assume that $\mathcal{A} \subset \{a \in \mathbb{Z} \mid |a| \leq N\}$. For any $p \leq n := \text{Card}(\mathcal{A})$, consider $a_1, \dots, a_p \in \mathcal{A}$. The the following determinant*

$$D := \begin{vmatrix} \frac{d\omega_{a_1}}{dm} & \frac{d\omega_{a_2}}{dm} & \dots & \dots & \frac{d\omega_{a_p}}{dm} \\ \frac{d^2\omega_{a_1}}{dm^2} & \frac{d^2\omega_{a_2}}{dm^2} & \dots & \dots & \frac{d^2\omega_{a_p}}{dm^2} \\ \cdot & \cdot & \dots & \dots & \cdot \\ \cdot & \cdot & \dots & \dots & \cdot \\ \frac{d^p\omega_{a_1}}{dm^p} & \frac{d^p\omega_{a_2}}{dm^p} & \dots & \dots & \frac{d^p\omega_{a_p}}{dm^p} \end{vmatrix}$$

verifies

$$(2.1) \quad |D| \geq CN^{-2p^2},$$

where $C = C(p) > 0$ is a constant that depends only on p .

Proof. An explicit computation gives

$$(2.2) \quad \frac{d^j \omega_i}{dm^j} = \frac{(2j-2)!}{2^{2j-1}(j-1)!} \frac{(-1)^{j+1}}{(a_i^2 + m)^{j-\frac{1}{2}}}.$$

Inserting this formula in D , by factoring from each l -th column $(a_\ell^2 + m)^{-1/2}$, and from j -th row $\frac{(2j-2)!}{2^{2j-1}(j-1)!}$, the determinant is equal up to a sign to

$$\left[\prod_{l=1}^p \omega_{a_\ell}^{-1} \right] \left[\prod_{j=1}^p \frac{(2j-2)!}{2^{2j-1}(j-1)!} \right] \times \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ x_{a_1} & x_{a_2} & x_{a_3} & \dots & x_{a_p} \\ x_{a_1}^2 & x_{a_2}^2 & x_{a_3}^2 & \dots & x_{a_p}^2 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ x_{a_1}^p & x_{a_2}^p & x_{a_3}^p & \dots & x_{a_p}^p \end{vmatrix},$$

where $x_a := (a^2 + m)^{-1} \equiv \omega_a^{-2}$. The above determinant is a Vandermonde determinant. It is equal to

$$\prod_{1 \leq l < k \leq p} (x_{a_\ell} - x_{a_k}) = \prod_{1 \leq l < k \leq p} \frac{a_k^2 - a_\ell^2}{\omega_{a_\ell}^2 \omega_{a_k}^2}.$$

The set \mathcal{A} is admissible, so $\left| \prod_{1 \leq l < k \leq p} (a_k^2 - a_\ell^2) \right| \geq 1$. Since $|a| \leq N$, then for any $a \in \mathcal{A}$ we have $|\omega_a| \leq 2N$. Therefore:

$$\prod_{l=1}^p \omega_{a_\ell}^{-1} \prod_{1 \leq l < k \leq p} \frac{1}{\omega_{a_\ell}^2 \omega_{a_k}^2} \geq \frac{1}{2^{p(2p-1)}} \frac{1}{N^{p(2p-1)}} \geq \frac{1}{2^{p(2p-1)}} \frac{1}{N^{2p^2}},$$

which leads to (2.1). \square

We need the following proposition, presented in appendix B of [4].

Lemma 2.3. Let $(u^{(1)}, \dots, u^{(p)})$ be p independent vector in \mathbb{R}^n such that $\|u^{(j)}\|_{\ell^1} \leq K$ for $j \in [1, \dots, p]$. Let w be a linear combination of $u^{(1)}, \dots, u^{(p)}$. There exists $j \in [1, \dots, p]$ such that:

$$(2.3) \quad |u^{(j)} \cdot w| \geq \frac{\|w\|_{\ell^2} \mathcal{V}_p(u^{(1)}, \dots, u^{(p)})}{pK^{p-1}},$$

where $\mathcal{V}_p(u^{(1)}, \dots, u^{(p)})$ denotes the Euclidean volume of the parallelepiped generated by the p vectors $u^{(1)}, \dots, u^{(p)}$.

Recall that, for $m \in [1, 2]$, the internal frequency vector is given by

$$\omega(m) \equiv (\omega_a(m))_{a \in \mathcal{A}} = (\sqrt{a^2 + m})_{a \in \mathcal{A}}.$$

Corollary 2.4. Let $n = \text{Card}(\mathcal{A})$ and w a nonzero vector in \mathbb{R}^n . Then, for any $m \in [1, 2]$, there exists $j \in [1, \dots, n]$ such that

$$\left| w \cdot \frac{d^j \omega}{dm^j}(m) \right| \geq CN^{-2n^2} \|w\|_{\ell^1},$$

where $C > 0$ is a constant that depends only on n .

Proof. From Lemma 2.2, $(\frac{d\omega}{dm}(m), \dots, \frac{d^n \omega}{dm^n}(m))$ is a basis of \mathbb{R}^n . Therefore w is as a linear combination of these vectors. According to Lemma 2.3, there exists $j \in [1, \dots, n]$ such that

$$\begin{aligned} \left| w \cdot \frac{d^j \omega}{dm^j}(m) \right| &\geq \frac{\|w\|_{\ell^2} \mathcal{V}_n\left(\frac{d\omega}{dm}(m), \dots, \frac{d^n \omega}{dm^n}(m)\right)}{nK^{n-1}} \\ &\geq \frac{\|w\|_{\ell^1} \mathcal{V}_n\left(\frac{d\omega}{dm}(m), \dots, \frac{d^n \omega}{dm^n}(m)\right)}{n^{3/2} K^{n-1}}. \end{aligned}$$

Note that $(\frac{d\omega}{dm}(m), \dots, \frac{d^n \omega}{dm^n}(m))$ is a n -family vector in \mathbb{R}^n . So

$$\mathcal{V}_n\left(\frac{d\omega}{dm}(m), \dots, \frac{d^n \omega}{dm^n}(m)\right) = D,$$

where D is the determinant defined in the Lemma 2.2. Let us now give the expression of K . For $j \in [1, \dots, n]$ we have

$$\begin{aligned} \left\| \frac{d^j \omega}{dm^j}(m) \right\|_{\ell^1} &= \sum_{1 \leq k \leq n} \left| \frac{(2j-2)!}{2^{2j-1}(j-1)!} \frac{(-1)^{j+1}}{(k^2+m)^{j-\frac{1}{2}}} \right| \\ &\leq \sum_{1 \leq k \leq n} \frac{(2n-2)!}{(n-1)!} = \frac{n(2n-2)!}{(n-1)!} =: K. \end{aligned}$$

Then

$$\left| w \cdot \frac{d^j \omega}{dm^j}(m) \right| \geq \frac{(n-1)!}{n^{5/2}(2n-2)!} D \|w\|_{\ell^1} \geq CN^{-2n^2} \|w\|_{\ell^1},$$

which ends the proof of the corollary. \square

We need the following lemma 2.1 from [16]:

Lemma 2.5. Assume that $g(\tau)$ is a p -th differentiable on $J \subset \mathbb{R}$. For $h > 0$, we define the open set J_h

$$J_h := \{\tau \in J \mid |g(\tau)| < h\}.$$

If $|g^{(p)}(\tau)| \geq d > 0$, for $\tau \in J$, then

$$(2.4) \quad \text{mes}(J_h) \leq Mh^{1/p},$$

where $M := 2(2 + 3 + \dots + p + d^{-1})$.

Now we have all the tools to give a proof of the Proposition 2.1.

Proof of Proposition 2.1. Let $k \in \mathbb{R}^n$, where $n = \text{Card}(\mathcal{A})$, and consider the function $g \in \mathcal{C}^\infty([1, 2], \mathbb{R})$ defined by:

$$g(m) = k \cdot \omega(m) + c.$$

From Corollary 2.4, there exists $j \in [1, \dots, n]$ such that

$$\left| k \cdot \frac{d^j \omega}{dm^j}(m) \right| \geq CN^{-2n^2} |k|.$$

Using Lemma 2.5 with $h = CN^{-2n^2} |k|$, we obtain

$$\text{mes} \{m \in [1, 2] \mid |k \cdot \omega(m) + c| \leq \chi\} \leq M\chi^{1/n},$$

where $M \leq \tilde{C}N^{2n^2} |k|^{-1}$. The constant \tilde{C} is strictly positive and depends only on n . This ends the proof of the proposition. \square

2.2. Small Divisors Estimates. For $m \in [1, 2]$, recall that the internal frequencies are denoted by $\omega \equiv \omega(m) = (\sqrt{a^2 + m})_{a \in \mathcal{A}}$, while the external frequencies are denoted by $\lambda_s \equiv \lambda_s(m) = \sqrt{s^2 + m}$ for $s \in \mathcal{L} = \mathbb{Z} \setminus \mathcal{A}$. We Note that for $s \in \mathcal{L} \setminus \{0\}$ we have

$$(2.5) \quad |\lambda_s(m) - |s|| \leq \frac{m}{2|s|}.$$

We Recall that $\mathcal{A}^- := \{s \in \mathcal{L} \mid -s \in \mathcal{A}\}$. We denote by \mathcal{L}^∞ the complementary of \mathcal{A}^- in \mathcal{L} and n the cardinality of \mathcal{A} .

In this part, we will give a lower bound of the modulus of the following small divisors:

$$\begin{aligned} D_0 &= \omega \cdot k, \quad k \in \mathbb{Z}^n \setminus \{0\}, \\ D_1 &= \omega \cdot k + \lambda_a, \quad k \in \mathbb{Z}^n, a \in \mathcal{L}, \\ D_2 &= \omega \cdot k + \lambda_a + \lambda_b, \quad k \in \mathbb{Z}^n, a, b \in \mathcal{L}, \\ D_3 &= \omega \cdot k + \lambda_a - \lambda_b, \quad k \in \mathbb{Z}^n, a, b \in \mathcal{L}. \end{aligned}$$

Definition 2.6. Let $k \in \mathbb{Z}^n$, $a, b \in \mathcal{L}$.

- (i) The vector k is D_0 resonant if $k = 0$.
- (ii) The couple $(k; a)$ is D_1 resonant if $|a| = |s|$ where $s \in \mathcal{A}$ and $\omega \cdot k = -\omega_s$
- (iii) The triplet $(k; a, b)$ is D_2 resonant if $|a| = |s|$, $|b| = |s'|$ where $s, s' \in \mathcal{A}$ and $\omega \cdot k = -\omega_s - \omega_{s'}$
- (iv) The triplet $(k; a, b)$ is D_3 resonant if $|a| = |s|$, $|b| = |s'|$ where $s, s' \in \mathcal{A}$ and $\omega \cdot k = -\omega_s + \omega_{s'}$.

Remark 2.7. Note that $(k; a, b)$ can be D_2 or D_3 resonant only if $(a, b) \in \mathcal{A}^- \times \mathcal{A}^-$. Similarly, $(k; a)$ can be D_1 resonant only if $a \in \mathcal{A}^-$.

Our goal is to give a lower bound to the modulus of small divisors D_0 , D_1 , D_2 and D_3 when they are not resonant, for $m \in [1, 2] \setminus \mathcal{C}$, with \mathcal{C} an open set of zero Lebesgue measure. In this section, C will denotes a constant that depends only on the admissible set \mathcal{A} . Let us start with the small divisors D_0 , D_1 and D_2 .

Proposition 2.8. Let $\kappa > 0$ and an integer $N > 1$. Then there is an open set $\mathcal{C} \subset [1, 2]$ that verifies

$$\text{mes}(\mathcal{C}) \leq C\kappa^\tau N^\iota,$$

where $\tau, \iota > 0$ and depend only on $n = \text{Card}(\mathcal{A})$, such that for all $m \in ([1, 2] \setminus \mathcal{C})$, all $0 < |k| \leq N$ and all $a, b \in \mathcal{L}$ we have:

$$(2.6) \quad |\omega \cdot k| \geq \kappa,$$

except when k is D_0 resonant;

$$(2.7) \quad |\omega \cdot k + \lambda_a| \geq \kappa \langle a \rangle,$$

except when $(k; a)$ is D_1 resonant;

$$(2.8) \quad |\omega \cdot k + \lambda_a + \lambda_b| \geq \kappa (\langle a \rangle + \langle b \rangle),$$

except when $(k; a, b)$ is D_2 resonant. The constant C depends only on the admissible set \mathcal{A} .

Proof. We start by proving (2.6). Let $\kappa > 0$ and an integer $N > 1$. Consider

$$\mathcal{U} = \{m \in [1, 2] \mid |\omega \cdot k| < \kappa, k \in \mathbb{Z}^n \text{ for } 0 < |k| \leq N\}.$$

For $k \in \mathbb{Z}^n$ we consider the sets:

$$\mathcal{U}_k = \{m \in [1, 2] \mid |\omega \cdot k| < \kappa\}$$

and

$$\mathcal{B}_0 = \{k \in \mathbb{Z}^n \mid |k| \leq N\}.$$

Then

$$\mathcal{U} = \bigcup_{k \in \mathcal{B}_0} \mathcal{U}_k.$$

Thanks to Proposition 2.1, we have $\text{mes}(\mathcal{U}_k) \leq C \frac{\kappa^{1/n}}{|k|}$. We Note that there are at most N^n points in \mathcal{B}_0 . So we obtain:

$$\text{mes}(\mathcal{U}) \leq C\kappa^{1/n} N^n.$$

Let us now look at the second small divisor (2.7). Consider

$$C_{\mathcal{A}} = (\max\{|a| \mid a \in \mathcal{A}\})^{1/2}.$$

There are two cases: if $|a| \geq 2C_{\mathcal{A}}N$, then

$$|\omega \cdot k + \lambda_a| \geq \lambda_a - |\omega \cdot k| \geq |a| - C_{\mathcal{A}}|k| \geq |a| - \frac{1}{2}|a| \geq \kappa \langle a \rangle,$$

for $0 < \kappa \leq 1/2$. If $|a| < 2C_{\mathcal{A}}N$, let

$$\mathcal{V} = \{m \in [1, 2] \mid |\omega \cdot k + \lambda_a| < \kappa \langle a \rangle, (k; a) \in \mathcal{L} \times \mathbb{Z}^n \\ \text{with } 0 \leq |k| \leq N \text{ and } (k; a) \text{ non } D_1 \text{ resonant} \}.$$

We want to give an upper bound of the Lebesgue measure of \mathcal{V} . Consider for $k \in \mathbb{Z}^n$ and $a \in \mathcal{L}$ the sets

$$\mathcal{V}_{k,a} = \{m \in [1, 2] \mid |\omega \cdot k + \lambda_a| < \kappa \langle a \rangle, (k; a) \text{ non } D_1 \text{ resonant} \}$$

and

$$\mathcal{B}_1 = \{(k, a) \in \mathbb{Z}^n \times \mathcal{L} \mid |k| \leq N \text{ et } |a| < 2C_{\mathcal{A}}N\}.$$

Then we have

$$(2.9) \quad \mathcal{V} \subset \bigcup_{(k,a) \in \mathcal{B}_1} \mathcal{V}_{k,a}.$$

We note that there are at most $4C_{\mathcal{A}}N^{n+1}$ points in \mathcal{B}_1 . It remains to give an upper bound of the Lebesgue measure of $\mathcal{V}_{k,a}$. There are two cases:

- If $\{a, -a\} \not\subset \mathcal{A}$, then $\mathcal{A}' = \mathcal{A} \cup \{a\}$ is still an admissible set of cardinality $n+1$. In addition, we have $\mathcal{A}' \subset \{a \in \mathbb{Z} \mid |a| \leq CN\}$. Applying Proposition 2.1 to the new admissible set, we have

$$\text{mes}(\mathcal{V}_{k,a}) \leq C \frac{\kappa^{1/(n+1)} N^{2(n+1)^2+1/(n+1)}}{|k+1|}.$$

- If $|a| \in \mathcal{A}$ but $(k; a)$ is not D_1 resonant, then by applying Proposition 2.1 without changing \mathcal{A} we have

$$\text{mes}(\mathcal{V}_{k,a}) \leq C \frac{\kappa^{1/n} N^{2n^2+1/n}}{|k|}.$$

So

$$\text{mes}(\mathcal{V}) \leq C \kappa^{1/(n+1)} N^{(n+1)(2n+3)+1/(n+1)}.$$

With the same argument we show (2.8). We end the proof of Proposition 2.8 by taking $\mathcal{C} = \mathcal{U} \cup \mathcal{V} \cup \mathcal{W}$ where \mathcal{W} is the open set where (2.8) is not verified. \square

It remains to control $D_3 = \omega \cdot k + \lambda_a - \lambda_b$.

Lemma 2.9. *Let $\tilde{\kappa} \in]0, 1]$ and an integer $N \geq 1$. We have*

$$\text{mes}\{m \in [1, 2] \mid |\omega \cdot k - e| < 2\tilde{\kappa}, (k, e) \in \mathbb{Z}^{n+1} \text{ for } 0 < |k| \leq N\} \leq C \tilde{\kappa}^{\frac{1}{n}} N^{n+1},$$

where $C > 0$ and depends only on the admissible set \mathcal{A} .

Proof. Let $(k, e) \in \mathbb{Z}^n \times \mathbb{Z}$ such that $0 < |k| \leq N$. Using Proposition 2.1, we have

$$\text{mes}\{m \in [1, 2] \mid |\omega \cdot k - e| < 2\tilde{\kappa}, \} \leq C \frac{\tilde{\kappa}^{\frac{1}{n}}}{|k|}.$$

Since $\kappa \leq 1$, we can restrict ourselves to

$$|e| \leq |\omega \cdot k - e| + |\omega \cdot k| \leq CN.$$

Then

$$\text{mes} \bigcup_{\substack{|k| \leq N \\ (k,e) \in \mathbb{Z}^{n+1}}} \{m \in [1, 2] \mid |\omega \cdot k - e| < 2\tilde{\kappa}\} \leq CN^{n+1} \tilde{\kappa}^{\frac{1}{n}}.$$

The proof is thus concluded. \square

Proposition 2.10. *Let $\kappa > 0$ and an integer $N > 0$. Then there is an open set $\mathcal{C} \subset [1, 2]$ satisfying*

$$\text{mes}(\mathcal{C}) \leq C \kappa^\tau N^\iota,$$

where τ and ι are two strictly positive exponents which depend only on $n = \text{Card}(\mathcal{A})$, such that for all $m \in ([1, 2] \setminus \mathcal{C})$, all $0 < |k| \leq N$ and all $a, b \in \mathcal{L}$ we have

$$(2.10) \quad |\omega \cdot k + \lambda_a - \lambda_b| \geq \kappa(1 + ||a| - |b||),$$

except when $(k; a, b)$ is D_3 resonant. The constant C depends only on the admissible set \mathcal{A} .

Proof. Using (2.5) for $|b| \geq |a| > 0$, we remark that

$$|\lambda_a - \lambda_b - (|a| - |b|)| \leq \frac{m}{|a|} \leq 2|a|^{-1}.$$

So we have

$$|\omega \cdot k + \lambda_a - \lambda_b| \geq |\omega \cdot k + |a| - |b|| - 2|a|^{-1}.$$

In Lemma 2.9, we denote $\tilde{\kappa} = \bar{\kappa}^\varrho$ where ϱ is an exponent in $]0, 1[$ which will be determined later. According to this Lemma, there is an open set $\mathcal{C}_1 = \mathcal{C}_1(N, \bar{\kappa}^\varrho)$ whose Lebesgue measure is smaller than $C\bar{\kappa}^{\frac{\varrho}{n}}N^{n+1}$, where C is a constant that depends on \mathcal{A} . For all $m \in ([1, 2] \setminus \mathcal{C}_1)$, all $0 < |k| \leq N$ and all $a, b \in \mathcal{L}$ where $|b| \geq |a| \geq 2\bar{\kappa}^{-\varrho}$, we have:

$$(2.11) \quad |\omega \cdot k + \lambda_a - \lambda_b| \geq \bar{\kappa}^\varrho \geq \bar{\kappa}.$$

Let us look at the remaining cases where the previous estimate does not hold. These cases are included in the set:

$$\mathcal{C}_2 = \{m \in [1, 2] \mid |\omega \cdot k + \lambda_a - \lambda_b| < \bar{\kappa}, (a, b) \in \mathcal{L}^2, |a| \leq 2\bar{\kappa}^{-\varrho}, 0 < |k| \leq N\}.$$

We note that, if $|\omega \cdot k + \lambda_a - \lambda_b| < \bar{\kappa}$, $|a| \leq 2\bar{\kappa}^{-\varrho}$ and $|k| \leq N$, then we have:

$$\begin{aligned} |b| \leq \lambda_b &\leq |\omega \cdot k + \lambda_a - \lambda_b| + |\omega \cdot k| + \lambda_a \\ &\leq 2\bar{\kappa}^{-\varrho} + (C_{\mathcal{A}} + 3)N, \end{aligned}$$

where $C_{\mathcal{A}} = (\max\{|a| \mid a \in \mathcal{A}\})^{1/2}$. Consider the set

$$\mathcal{B} = \{(a, b) \in \mathbb{Z}^2 \mid |a| \leq |b| \leq 2\bar{\kappa}^{-\varrho} + (C_{\mathcal{A}} + 3)N\}.$$

There are at most $4(2\bar{\kappa}^{-\varrho} + (C_{\mathcal{A}} + 3)N)^2$ points in \mathcal{B} . So we have

$$\mathcal{C}_2 \subset \{m \in [1, 2] \mid |\omega \cdot k + \lambda_a - \lambda_b| < \bar{\kappa}, (a, b) \in \mathcal{B}, 0 < |k| \leq N\} := \mathcal{C}_3$$

Recall that $\mathcal{L}^\infty = \mathcal{L} \setminus \mathcal{A}^-$ where $\mathcal{A}^- = -\mathcal{A}$. For $a \in \mathcal{L}$, we define the set $]a[$ by : $]a[= \{a\}$ if $a \in \mathcal{L}^\infty$ and $]a[= \emptyset$ if $a \in \mathcal{A}^-$. We define

$$\mathcal{A}' = \mathcal{A} \cup]a[\cup]b[.$$

The set \mathcal{A}' is admissible. In addition, we have

$$\mathcal{A}' \subset \{a \in \mathbb{Z} \mid |a| \leq C(2\bar{\kappa}^{-\varrho} + (C_{\mathcal{A}} + 3)N)\}.$$

The triplet $(k; a, b)$ is D_3 non resonant. By applying the Proposition 2.1 with the admissible set \mathcal{A}' , we have:

$$\begin{aligned} \text{mes}(\mathcal{C}_2) &\leq \text{mes}(\mathcal{C}_3) \leq C\bar{\kappa}^{1/(n+2)}N^n (2\bar{\kappa}^{-\varrho} + (C_{\mathcal{A}} + 3)N)^{2(n+2)^2} \text{Card } \mathcal{B} \\ &\leq C\bar{\kappa}^{1/(n+2)}\bar{\kappa}^{-2\varrho((n+2)^2+1)}N^{(n+2)(2n+5)}. \end{aligned}$$

Let

$$\varrho = \frac{1}{4((n+2)^2+1)(n+2)}, \quad \tau = \frac{1}{2(n+2)}, \quad \iota' = (n+2)(2n+5),$$

and consider $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, then we have:

$$\text{mes}(\mathcal{C}) \leq C\bar{\kappa}^\tau N^{\iota'}.$$

Moreover, for all $m \in [1, 2] \setminus \mathcal{C}_4$, all $0 < |k| \leq N$ and all $a, b \in \mathcal{L}$ non D_3 resonant, the estimation (2.11) is satisfied.

To conclude the proof of the proposition, we need to estimate the difference $||a| - |b||$. Without loss of generality, assume that $|a| > |b|$.

- If $|a| - |b| \geq 8C_{\mathcal{A}}N \geq 8|\omega \cdot k|$, then for $m \in [1, 2]$ and $0 < |k| \leq N$, we have:

$$\begin{aligned} |\omega \cdot k + \lambda_a - \lambda_b| &\geq \lambda_a - \lambda_b - |\omega \cdot k| \geq \frac{1}{4}(|a| - |b|) - |\omega \cdot k| \\ &\geq \frac{1}{8}(|a| - |b|) \geq \frac{1}{16}(1 + |a| - |b|) \geq \kappa(1 + |a| - |b|), \end{aligned}$$

if we assume that $\kappa \leq \frac{1}{16}$.

- If $|a| - |b| < 8C_{\mathcal{A}}N$, then for $m \in [1, 2] \setminus \mathcal{C}$, we have:

$$\begin{aligned} |\omega \cdot k + \lambda_a - \lambda_b| &\geq \frac{\bar{\kappa}}{1 + 8C_{\mathcal{A}}N}(1 + |a| - |b|) \\ &\geq \kappa(1 + |a| - |b|). \end{aligned}$$

Thus, for all $m \in [1, 2] \setminus \mathcal{C}$, all $0 < |k| \leq N$ and all $a, b \in \mathcal{L}$ non D_3 resonant, the estimation (2.10) holds. Moreover,

$$\text{mes}(\mathcal{C}) \leq C\kappa^\tau N^{\ell'+\tau} = C\kappa^\tau N^\ell.$$

□

It remains to treat the case where $k = 0$ in D_3 .

Lemma 2.11. *Let $m \in [1, 2]$ and $a, b \in \mathcal{L}$ such that $|a| \neq |b|$. Therefore*

$$|\lambda_a - \lambda_b| \geq \frac{1}{8}(1 + ||a| - |b||).$$

Proof. Without loss of generality, assume that $|a| > |b|$. Then for all $m \in [1, 2]$, we have:

$$\lambda_a - \lambda_b = \frac{(|a| - |b|)(|a| + |b|)}{\sqrt{a^2 + m} + \sqrt{b^2 + m}} \geq \frac{1}{4}(|a| - |b|) \geq \frac{1}{8}(1 + ||a| - |b||),$$

which concludes the proof. □

3. NORMAL FORM

In this section, we construct a symplectic change of variable that puts the Hamiltonian (1.5) in normal form to which we can apply our KAM theorem.

3.1. Class of Hamiltonian function. In this part, we begin by recalling some notations introduced in [6]. For \mathcal{L} a set of \mathbb{Z} and $\alpha \geq 0$, we define the ℓ_2 weighted space:

$$Y_\alpha := \left\{ \zeta = \left(\zeta_s = \begin{pmatrix} \xi_s \\ \eta_s \end{pmatrix} \in \mathbb{C}^2, s \in \mathcal{L} \right) \mid \|\zeta\|_\alpha < \infty \right\},$$

where

$$\|\zeta\|_\alpha^2 = \sum_{s \in \mathcal{L}} |\zeta_s|^2 \langle s \rangle^{2\alpha}, \text{ où } \langle s \rangle = \max(|s|, 1).$$

We endow \mathbb{C}^2 with the euclidean norm, i.e. if $\zeta_s = {}^t(\xi_s, \eta_s)$ then $|\zeta_s| = \sqrt{|\xi_s|^2 + |\eta_s|^2}$. For $\beta \geq 0$, we define the ℓ_∞ weighted space

$$L_\beta = \left\{ \left(\zeta_s = \begin{pmatrix} \xi_s \\ \eta_s \end{pmatrix} \in \mathbb{C}^2, s \in \mathcal{L} \right) \mid |\zeta|_\beta < \infty \right\},$$

where

$$|\zeta|_\beta = \sup_{s \in \mathcal{L}} |\zeta_s| \langle s \rangle^\beta.$$

We remark that for, $\beta \leq \alpha$, we have $Y_\alpha \subset L_\beta$.

Infinite matrices. Consider the orthogonal projector Π defined on the set of square matrices by

$$\Pi : \mathcal{M}_{2 \times 2}(\mathbb{C}) \rightarrow \mathbb{S},$$

where

$$\mathbb{S} = \mathbb{C}I + \mathbb{C}\sigma_2, \quad \text{with } \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We introduce \mathcal{M} the set of infinite symmetric matrices $A : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{M}_2(\mathbb{R})$, that verify, for any $s, s' \in \mathcal{L}$,

$$A_s^{s'} \in \mathcal{M}_2(\mathbb{R}), A_s^{s'} = {}^t A_s^s, \text{ and } \Pi A_s^{s'} = A_s^{s'}.$$

We also define \mathcal{M}_α , a subset of \mathcal{M} , by:

$$A \in \mathcal{M}_\alpha \Leftrightarrow |A|_\alpha := \sup_{s, s' \in \mathcal{L}} \langle s \rangle^\alpha \langle s' \rangle^\alpha \|A_s^{s'}\|_\infty < \infty.$$

Let $n \in \mathbb{N}$, $\rho > 0$ and B be a Banach space. We define:

$$\mathbb{T}_\rho^n = \{\theta \in \mathbb{C}^n / 2\pi\mathbb{Z}^n \mid |Im\theta| < \rho\}$$

and

$$\mathcal{O}_\rho(B) = \{x \in B \mid \|x\|_B < \rho\}.$$

For $\sigma, \mu \in]0, 1[$, we define

$$\mathcal{O}^\alpha(\sigma, \mu) = \mathbb{T}_\sigma^n \times \mathcal{O}_{\mu^2}(\mathbb{C}^n) \times \mathcal{O}_\mu(Y_\alpha) = \{(\theta, r, \zeta)\},$$

$$\mathcal{O}^{\alpha, \mathbb{R}}(\sigma, \mu) = \mathcal{O}^\alpha(\sigma, \mu) \cap \{\mathbb{T}^n \times \mathbb{R}^n \times Y_\alpha^{\mathbb{R}}\},$$

where $Y_\alpha^{\mathbb{R}} = \left\{ \zeta \in Y_\alpha \mid \zeta = \left(\zeta_s = \begin{pmatrix} \xi_s \\ \eta_s \end{pmatrix}, \xi_s = \bar{\eta}_s, s \in \mathcal{L} \right) \right\}$.

Let us denote a point in $\mathcal{O}^\alpha(\sigma, \mu)$ as $x = (\theta, r, \zeta)$. A function on $\mathcal{O}^\alpha(\sigma, \mu)$ is real if it has a real value for any real x . We define:

$$\|(r, \theta, \zeta)\|_\alpha = \max(|r|, |\theta|, \|\zeta\|_\alpha).$$

Class of Hamiltonian functions. Let \mathcal{D} be a compact set of \mathbb{R}^p , called the parameters set from now on. Let $f : \mathcal{O}^\alpha(\delta, \mu) \times \mathcal{D} \rightarrow \mathbb{C}$ be a \mathcal{C}^1 function, real and holomorphic in the first variable, such that for all $\rho \in \mathcal{D}$, the maps

$$\mathcal{O}^\alpha(\delta, \mu) \ni x \mapsto \nabla_\zeta f(x, \rho) \in Y_\alpha \cap L_\beta$$

and

$$\mathcal{O}^\alpha(\delta, \mu) \ni x \mapsto \nabla_\zeta^2 f(x, \rho) \in \mathcal{M}_\beta,$$

are holomorphic. We define:

$$\begin{aligned} |f(x, \cdot)|_{\mathcal{D}} &= \sup_{\rho \in \mathcal{D}} |f(x, \rho)|, & \left\| \frac{\partial f}{\partial \zeta}(x, \cdot) \right\|_{\mathcal{D}} &= \sup_{\rho \in \mathcal{D}} \|\nabla_\zeta f(x, \rho)\|_\alpha, \\ \left| \frac{\partial f}{\partial \zeta}(x, \cdot) \right|_{\mathcal{D}} &= \sup_{\rho \in \mathcal{D}} |\nabla_\zeta f(x, \rho)|_\beta, & \left| \frac{\partial^2 f}{\partial \zeta^2}(x, \cdot) \right|_{\mathcal{D}} &= \sup_{\rho \in \mathcal{D}} |\nabla_\zeta^2 f(x, \rho)|_\beta. \end{aligned}$$

We denote by $\mathcal{T}^{\alpha, \beta}(\mathcal{D}, \sigma, \mu)$ the space of functions f that verify, for all $x \in \mathcal{O}^\alpha(\sigma, \mu)$, the following estimates:

$$|f(x, \cdot)|_{\mathcal{D}} \leq C, \quad \left\| \frac{\partial f}{\partial \zeta}(x, \cdot) \right\|_{\mathcal{D}} \leq \frac{C}{\mu}, \quad \left| \frac{\partial f}{\partial \zeta}(x, \cdot) \right|_{\mathcal{D}} \leq \frac{C}{\mu}, \quad \left| \frac{\partial^2 f}{\partial \zeta^2}(x, \cdot) \right|_{\mathcal{D}} \leq \frac{C}{\mu^2}.$$

For $f \in \mathcal{T}^{\alpha, \beta}(\mathcal{D}, \sigma, \mu)$, we denote by $\llbracket f \rrbracket_{\sigma, \mu, \mathcal{D}}^{\alpha, \beta}$ the smallest constant C that satisfies the above estimates. If $\partial_\rho^j f \in \mathcal{T}^{\alpha, \beta}(\mathcal{D}, \sigma, \mu)$ for $j \in \{0, 1\}$, then for $\gamma > 0$ we define:

$$\llbracket f \rrbracket_{\sigma, \mu, \mathcal{D}}^{\alpha, \beta, \gamma} = \llbracket f \rrbracket_{\sigma, \mu, \mathcal{D}}^{\alpha, \beta} + \gamma \llbracket \partial_\rho f \rrbracket_{\sigma, \mu, \mathcal{D}}^{\alpha, \beta}.$$

We also denote by

$$\mathcal{T}^{\alpha, \beta}(\mu) = \{f(\zeta) \mid f \in \mathcal{T}^{\alpha, \beta}(\mathcal{D}, \sigma, \mu)\},$$

the set of functions of $\mathcal{T}^{\alpha, \beta}(\mathcal{D}, \sigma, \mu)$ that do not depend on r, θ and ρ . The norm of such functions will be denoted by $\llbracket f \rrbracket_\mu^{\alpha, \beta}$.

We finish this part by defining the space $\mathcal{T}^{\alpha, \beta+}(\mathcal{D}, \sigma, \mu)$. Consider the following spaces

$$L_{\beta+} = \{\zeta = (\zeta_s = (p_s, q_s), s \in \mathcal{L}) \mid |\zeta|_{\beta+} < \infty\},$$

where $|\zeta|_{\beta+} = \sup_{s \in \mathcal{L}} |\zeta_s| \langle s \rangle^{\beta+1}$, and

$$\mathcal{M}_{\beta+} = \{A \in \mathcal{M} \mid |A|_{\beta+} < \infty\},$$

where $|A|_{\beta+} = \sup_{s, s' \in \mathcal{L}} (1 + ||s| - |s'||) \langle s \rangle^\beta \langle s' \rangle^\beta \|A_s^{s'}\|_\infty$.

We remark that $L_{\beta+} \subset L_\beta$ and $\mathcal{M}_{\beta+} \subset \mathcal{M}_\beta$. We define $\mathcal{T}^{\alpha, \beta+}(\mathcal{D}, \sigma, \mu)$ the same way that we defined $\mathcal{T}^{\alpha, \beta}(\mathcal{D}, \sigma, \mu)$, but replacing L_β by $L_{\beta+}$ and \mathcal{M}_β by $\mathcal{M}_{\beta+}$. So, we have $\mathcal{T}^{\alpha, \beta+}(\mathcal{D}, \sigma, \mu) \subset \mathcal{T}^{\alpha, \beta}(\mathcal{D}, \sigma, \mu)$. For $f, g \in \mathcal{T}^{\alpha, \beta}(\mu)$, we define the Poisson bracket by:

$$\{f, g\} = i \langle \nabla_\zeta f, J \nabla_\zeta g \rangle.$$

Lemma 3.1. Consider $f \in \mathcal{T}^{\alpha, \beta}(\mu)$ and $g \in \mathcal{T}^{\alpha, \beta+}(\mu)$. Then, for any $0 < \mu' < \mu$, $\{f, g\} \in \mathcal{T}^{\alpha, \beta}(\mu')$, we have:

$$\llbracket \{f, g\} \rrbracket_{\mu'}^{\alpha, \beta} \leq \frac{C}{\mu(\mu - \mu')} \llbracket f \rrbracket_\mu^{\alpha, \beta} \llbracket g \rrbracket_\mu^{\alpha, \beta+},$$

where the constant C depends on α on β .

For the proof, we recall the following lemma from [14] (appendix A).

Lemma 3.2. Let E and F be two complex Banach spaces, $f : E \rightarrow F$ and $v \in E$. Assume that there exists $r > 0$ such that f is holomorphic on the open ball of center v and radius r and satisfies $\|f\|_F \leq M$ on this ball. Then $d_v f \in \mathcal{L}(E, F)$, and we have:

$$\|d_v f\|_{\mathcal{L}(E, F)} \leq \frac{M}{r}.$$

Proof of Lemma 3.1. Let $x \in \mathcal{O}_{\mu'}(Y_\alpha)$. Our goal is to prove that

- (i) $|\{f, g\}(x)| \leq \frac{C}{\mu(\mu - \mu')} \llbracket f \rrbracket_\mu^{\alpha, \beta} \llbracket g \rrbracket_\mu^{\alpha, \beta+};$
- (ii) $\|\nabla_\zeta \{f, g\}(x)\|_\alpha \leq \frac{C}{\mu\mu'(\mu - \mu')} \llbracket f \rrbracket_\mu^{\alpha, \beta} \llbracket g \rrbracket_\mu^{\alpha, \beta+};$
- (iii) $|\nabla_\zeta \{f, g\}(x)|_\beta \leq \frac{C}{\mu\mu'(\mu - \mu')} \llbracket f \rrbracket_\mu^{\alpha, \beta} \llbracket g \rrbracket_\mu^{\alpha, \beta+};$
- (iv) $|\nabla_\zeta^2 \{f, g\}(x)|_\beta \leq \frac{C}{\mu\mu'^2(\mu - \mu')} \llbracket f \rrbracket_\mu^{\alpha, \beta} \llbracket g \rrbracket_\mu^{\alpha, \beta+}.$

Let us begin with the first estimation (i). We have

$$\begin{aligned} |\{f, g\}(x)| &= |\langle \nabla_\zeta f(x), J\nabla_\zeta g(x) \rangle| \leq \|\nabla_\zeta f(x)\|_\alpha \|J\nabla_\zeta g(x)\|_\alpha \\ &\leq \frac{1}{\mu^2} \llbracket f \rrbracket_\mu^{\alpha, \beta} \llbracket g \rrbracket_\mu^{\alpha, \beta+} \\ &\leq \frac{1}{\mu(\mu - \mu')} \llbracket f \rrbracket_\mu^{\alpha, \beta} \llbracket g \rrbracket_\mu^{\alpha, \beta+}. \end{aligned}$$

Let us now turn to the ζ -gradient of the Poisson bracket:

$$\nabla_\zeta \{f, g\}(x) = \langle \nabla_\zeta^2 f(x), J\nabla_\zeta g(x) \rangle + \langle \nabla_\zeta f(x), J\nabla_\zeta^2 g(x) \rangle =: \Sigma_1 + \Sigma_2.$$

For Σ_1 , we have $\nabla_\zeta f : \mathcal{O}_{\mu'}(Y_\alpha) \rightarrow Y_\alpha$. Moreover, $x \mapsto \nabla_\zeta f(x)$ is holomorphic, so $\nabla_\zeta^2 f(x) \in \mathcal{L}(Y_\alpha, Y_\alpha)$ for $x \in \mathcal{O}_{\mu'}(Y_\alpha)$. Using Lemma 3.2, we have:

$$\begin{aligned} \|\Sigma_1\|_\alpha &\leq \|\nabla_\zeta^2 f(x)\|_{\mathcal{L}(Y_\alpha, Y_\alpha)} \|J\nabla_\zeta g(x)\|_\alpha \\ &\leq \frac{1}{\mu - \mu'} \sup_{y \in \mathcal{O}_\mu(Y_\alpha)} (\|\nabla_\zeta f(y)\|_\alpha) \|\nabla_\zeta g(x)\|_\alpha \\ &\leq \frac{1}{\mu^2(\mu - \mu')} \llbracket f \rrbracket_\mu^{\alpha, \beta} \llbracket g \rrbracket_\mu^{\alpha, \beta+} \\ &\leq \frac{1}{\mu\mu'(\mu - \mu')} \llbracket f \rrbracket_\mu^{\alpha, \beta} \llbracket g \rrbracket_\mu^{\alpha, \beta+}. \end{aligned}$$

We use the same arguments for Σ_2 , which ends the proof of (ii). To prove (iii), we use estimations 2. and 3. from Lemma 2.1 in [6]. So we have:

$$\begin{aligned} |\nabla_\zeta \{f, g\}(x)|_\beta &\leq |\langle \nabla_\zeta^2 f(x), J\nabla_\zeta g(x) \rangle|_\beta + |\langle \nabla_\zeta f(x), J\nabla_\zeta^2 g(x) \rangle|_\beta \\ &\leq C |\nabla_\zeta^2 f(x)|_\beta |J\nabla_\zeta g(x)|_{\beta+} + C |\nabla_\zeta f(x)|_\beta |J\nabla_\zeta^2 g(x)|_{\beta+} \\ &\leq \frac{C}{\mu^3} \llbracket f \rrbracket_\mu^{\alpha, \beta} \llbracket g \rrbracket_\mu^{\alpha, \beta+} \\ &\leq \frac{C}{\mu\mu'(\mu - \mu')} \llbracket f \rrbracket_\mu^{\alpha, \beta} \llbracket g \rrbracket_\mu^{\alpha, \beta+}. \end{aligned}$$

It remains to prove estimation (iv). We start by computing the second derivative of the Poisson bracket:

$$\begin{aligned} \nabla_\zeta^2 \{f, g\}(x) &= \langle \nabla_\zeta^3 f(x), J\nabla_\zeta g(x) \rangle + \langle \nabla_\zeta^2 f(x), J\nabla_\zeta^2 g(x) \rangle \\ &\quad + \langle \nabla_\zeta^2 f(x), J\nabla_\zeta^2 g(x) \rangle + \langle \nabla_\zeta f(x), J\nabla_\zeta^3 g(x) \rangle \\ &=: \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4. \end{aligned}$$

For Γ_1 , we have $\nabla_\zeta^2 f : \mathcal{O}_{\mu'}(Y_\alpha) \rightarrow \mathcal{M}_\beta$. Moreover, $x \mapsto \nabla_\zeta^2 f(x)$ is holomorphic, so $\nabla_\zeta^3 f(x) \in \mathcal{L}(Y_\alpha, \mathcal{M}_\beta)$ for $x \in \mathcal{O}_{\mu'}(Y_\alpha)$. By Lemma 3.2, we have

$$\begin{aligned} |\Gamma_1| &\leq \|\nabla_\zeta^3 f(x)\|_{\mathcal{L}(Y_\alpha, \mathcal{M}_\beta)} |J\nabla_\zeta g(x)|_\alpha \\ &\leq \frac{1}{\mu - \mu'} \sup_{y \in \mathcal{O}_\mu(\mathcal{M}_\beta)} |\nabla_\zeta^2 f(y)|_\beta \|\nabla_\zeta g(x)\|_\alpha \\ &\leq \frac{1}{\mu^3(\mu - \mu')} \llbracket f \rrbracket_\mu^{\alpha, \beta} \llbracket g \rrbracket_\mu^{\alpha, \beta+} \\ &\leq \frac{1}{\mu\mu'^2(\mu - \mu')} \llbracket f \rrbracket_\mu^{\alpha, \beta} \llbracket g \rrbracket_\mu^{\alpha, \beta+}. \end{aligned}$$

We use the same arguments for Γ_4 . It remains to estimate Γ_2 and Γ_3 . The two cases are treated in the same way. Let us look for example at Γ_3 . Using the first estimation from Lemma 2.1 in [6], we have

$$\begin{aligned} |\Gamma_3| &\leq C |\nabla_\zeta^2 f(x)|_\beta |J\nabla_\zeta^2 g(x)|_{\beta+} \\ &\leq \frac{C}{\mu^4} \llbracket f \rrbracket_\mu^{\alpha, \beta} \llbracket g \rrbracket_\mu^{\alpha, \beta+} \\ &\leq \frac{C}{\mu\mu'^2(\mu - \mu')} \llbracket f \rrbracket_\mu^{\alpha, \beta} \llbracket g \rrbracket_\mu^{\alpha, \beta+}. \end{aligned}$$

The proof is thus concluded. \square

Consider g a \mathcal{C}^1 -function on $\mathcal{O}_\mu(Y_\alpha)$. We denote by Φ_g the Hamiltonian flow of g at $t = 1$, i.e.

$$\zeta = \zeta(1) = \Phi_g(\zeta_0) = \Phi_g^{t=1}(\zeta_0),$$

where

$$\dot{\zeta} = iJ\nabla_{\zeta}g(\zeta) \quad \text{and} \quad \zeta(0) = \zeta_0.$$

Corollary 3.3. *Let $f \in \mathcal{T}^{\alpha,\beta}(\mu)$, $g \in \mathcal{T}^{\alpha,\beta+}(\mu)$ and $0 < \mu' < \mu$ such that:*

$$\llbracket g \rrbracket_{\mu}^{\alpha,\beta+} \leq \frac{1}{C}\mu(\mu - \mu').$$

Then $\Phi_g : \mathcal{O}_{\mu'}(Y_{\alpha}) \rightarrow \mathcal{O}_{\mu}(Y_{\alpha})$ and $f \circ \Phi_g \in \mathcal{T}^{\alpha,\beta}(\mu')$. In addition, we have:

$$(3.1) \quad \llbracket f \circ \Phi_g \rrbracket_{\mu'}^{\alpha,\beta} \leq C' \llbracket f \rrbracket_{\mu}^{\alpha,\beta},$$

where C is a constant that depends on α and β while C' is an absolute constant.

Proof. Let us first prove that Φ_g is well-defined and that $\Phi_g : \mathcal{O}_{\mu'}(Y_{\alpha}) \rightarrow \mathcal{O}_{\mu}(Y_{\alpha})$. Consider

$$\bar{t} = \sup\{t > 0 \mid \zeta(a) \text{ is well-defined for } 0 \leq a \leq t \text{ and } \|\zeta(a)\|_{\alpha} < \mu\}.$$

Let us prove that $\bar{t} \geq 1$. We have:

$$\zeta(a) = \zeta_0 + i \int_0^a J\nabla_{\zeta}g(\zeta(s))ds.$$

Recall that $g \in \mathcal{T}^{\alpha,\beta+}(\mu)$, then for $\zeta \in \mathcal{O}_{\mu}(Y_{\alpha})$ we have

$$\|\nabla_{\zeta}g(\zeta)\|_{\alpha} \leq \frac{\llbracket g \rrbracket_{\mu}^{\alpha,\beta+}}{\mu} \leq \frac{1}{C}(\mu - \mu').$$

It follows that, for $a = 1$,

$$\begin{aligned} \|\zeta(a)\|_{\alpha} &\leq \|\zeta_0\|_{\alpha} + \frac{a}{C}(\mu - \mu') \\ &\leq \mu' + \frac{a}{C}(\mu - \mu') < \mu, \end{aligned}$$

We deduce that $\bar{t} \geq 1$, Φ_g is well-defined and $\Phi_g : \mathcal{O}_{\mu'}(Y_{\alpha}) \rightarrow \mathcal{O}_{\mu}(Y_{\alpha})$. Let us prove now that $f \circ \Phi_g \in \mathcal{T}^{\alpha,\beta}(\mu')$, as well as the estimation (3.1). We recall that, for two holomorphic functions f and g , we have:

$$\begin{aligned} f \circ \Phi_g &= f + \{f, g\} + \frac{1}{2!} \{\{f, g\}, g\} + \frac{1}{3!} \{\{\{f, g\}, g\}, g\} + \dots \\ &= \sum_{n \geq 0} \frac{1}{n!} P_g^n f, \end{aligned}$$

where $P_g^0 f = f$, $P_g^1 f = \{f, g\}$ et $P_g^2 f = \{\{f, g\}, g\} \dots$

Using Lemma 3.1, we have:

$$\llbracket P_g^n f \rrbracket_{\mu'}^{\alpha,\beta} \leq C^n \llbracket f \rrbracket_{\mu}^{\alpha,\beta} (\mu(\mu - \mu') \llbracket g \rrbracket_{\mu}^{\alpha,\beta+})^n.$$

Thanks to the assumption made on g , we obtain that $f \circ \Phi_g$ is a convergent series. So $f \circ \Phi_g \in \mathcal{T}^{\alpha,\beta}(\mu')$ and satisfies estimation (3.1). \square

We define the real finite-dimensional torus:

$$(3.2) \quad T_{\rho} = \left\{ \zeta = ((\xi_s, \eta_s), s \in \mathbb{Z}) \mid \xi_s = \bar{\eta}_s, |\zeta_a|^2 = \nu\rho_a \text{ si } a \in \mathcal{A}, \xi_s = 0 \text{ si } s \in \mathcal{L} \right\},$$

where $\nu > 0$ is small and $\rho = (\rho_a, a \in \mathcal{A})$ is a parameter vector that belongs to $\mathcal{D} = [1, 2]^{\mathcal{A}}$.

Let $n = \text{Card}(\mathcal{A})$. The n -dimensional torus T_{ρ} is invariant for the linear wave equation. We wish to put the Hamiltonian $H_2 + P$ in a suitable normal form to which we will apply the KAM theorem 4.1. This normal form will be defined on tori constructed on the space Y_{α} and in the vicinity of the real torus T_{ρ} .

In the vicinity of the real torus T_{ρ} , we change from variables $(\zeta_a, a \in \mathcal{A})$ to action-angle variables defined by:

$$\xi_a = \sqrt{I_a} e^{i\theta_a}, \quad \eta_a = \sqrt{I_a} e^{-i\theta_a}, \quad a \in \mathcal{A}.$$

So we change from variables (ξ, η) to the new variables (I, θ, ξ, η) where $I = (I_a, a \in \mathcal{A})$, $\theta = (\theta_a, a \in \mathcal{A})$, $\xi = (\xi_s, s \in \mathcal{L})$ and $\eta = (\eta_a, a \in \mathcal{L})$. The new variable vector is real if $I = \bar{I}$, $\theta = \bar{\theta}$, and $\xi = \bar{\eta}$.

We now define a complex toroidal vicinity of the real torus T_{ρ} by

$$(3.3) \quad \mathfrak{T}_{\rho} = \mathfrak{T}_{\rho}(\nu, \sigma, \mu, \alpha) = \left\{ (I, \theta, \xi, \eta) \mid |I - \nu\rho| < \nu\mu^2, |Im\theta| < \sigma, \|\zeta^{\mathcal{L}}\|_{\alpha} < \nu^{1/2}\mu \right\},$$

where $I = (I_a, a \in \mathcal{A})$, $\theta = (\theta_a, a \in \mathcal{A})$, $\zeta^{\mathcal{L}} = (\zeta_s, s \in \mathcal{L})$.

Lemma 3.4. *Assume that g is a real holomorphic function on $\mathbb{S}^1 \times J$, for J some neighborhood of the origin on \mathbb{R} . Let $\alpha > 0$ and $\nu > 0$ small. There exist $\sigma^* > 0$ and $\mu^* > 0$ such that the perturbation P is well-defined and analytic on $\mathfrak{T}_{\rho}(\nu, \sigma, \mu, \alpha)$ for $0 < \sigma \leq \sigma^*$ and $0 < \mu \leq \mu^*$. The parameters σ^* and μ^* depend on the nonlinearity g , the admissible set \mathcal{A} , ν and α .*

Remark 3.5.

- We can fix σ^* ($\sigma^* = 1$ for example) and explicitly determine μ^* .
- For small ν , we have:

$$\mathfrak{T}_\rho(\nu, \sigma, \mu, \alpha) \subset \mathcal{O}^\alpha(\sigma, \mu^*).$$

Proof. The nonlinearity g is real holomorphic on $\mathbb{S}^1 \times J$, for J some neighborhood of the origin on \mathbb{R} . Assume that $I = [-M, M]$ for $M > 0$. We can extend holomorphically g on $\mathbb{S}^1 \times I_{\mathbb{C}}$, for some $I_{\mathbb{C}}$ of the form

$$I_{\mathbb{C}} = \{u \in \mathbb{C} \mid |Re(u)| \leq M, |Im(u)| < K\}.$$

Recall that

$$\begin{aligned} u(\zeta)(x) &= \sum_{s \in \mathbb{Z}} \frac{\xi_s \varphi_s + \eta_s \varphi_{-s}}{\sqrt{2\lambda_s}} \\ &= \sum_{a \in \mathcal{A}} \frac{\xi_a \varphi_a + \eta_a \varphi_{-a}}{\sqrt{2\lambda_a}} + \sum_{s \in \mathbb{Z}} \frac{\xi_s \varphi_s + \eta_s \varphi_{-s}}{\sqrt{2\lambda_s}}. \end{aligned}$$

We want to control these two sums. For the first sum we have:

$$\begin{aligned} \left| \sum_{a \in \mathcal{A}} \frac{\xi_a \varphi_a + \eta_a \varphi_{-a}}{\sqrt{2\lambda_a}} \right| &\leq \sum_{a \in \mathcal{A}} (|\xi_a| + |\eta_a|) \\ &\leq 2 \sum_{a \in \mathcal{A}} \sqrt{I_a} e^{|Im(\theta_a)|} \leq C_{\mathcal{A}} |I|^{1/2} e^{|Im(\theta)|}, \end{aligned}$$

where $C_{\mathcal{A}}$ is a constant that depends on \mathcal{A} . For the second sum, using the Cauchy-Schwarz inequality, we have:

$$\left| \sum_{s \in \mathcal{L}} \frac{\xi_s \varphi_s + \eta_s \varphi_{-s}}{\sqrt{2\lambda_s}} \right| \leq \sum_{s \in \mathcal{L}} \frac{|\xi_s| + |\eta_s|}{\sqrt{2\lambda_s}} \leq C(\alpha) \|\zeta\|_{\alpha}.$$

So we have:

$$\begin{aligned} |u(\zeta)| &\leq C(\alpha) \|\zeta\|_{\alpha} + C_{\mathcal{A}} |I|^{1/2} e^{|Im(\theta)|}, \\ &\leq C(\alpha) \|\zeta\|_{\alpha} + C_{\mathcal{A}} |I - \nu\rho|^{1/2} e^{|Im(\theta)|} + C_{\mathcal{A}} (\nu\rho)^{1/2} e^{|Im(\theta)|}. \end{aligned}$$

We want to prove that, if $(I, \theta, \zeta) \in \mathfrak{T}_\rho(\nu, \sigma, \mu, \alpha)$, then $u(\zeta)(x) \in I_{\mathbb{C}}$ for all $x \in \mathbb{S}^1$. This is true if we assume

$$C(\alpha) \nu^{1/2} \mu + C_{\mathcal{A}} \nu^{1/2} \mu e^{\sigma} + C_{\mathcal{A}} \sqrt{2} \nu^{1/2} e^{\sigma} \leq \min(M, K).$$

For example, if we assume that $\sigma^* = 1$, we have

$$\mu^* \leq \frac{\min(M, K) - C_{\mathcal{A}} \sqrt{2} \nu^{1/2} e}{\nu^{1/2} (C(\alpha) + C_{\mathcal{A}})}.$$

The proof is thus achieved. \square

Now, we are interested in the perturbation P . We will prove that P belongs to the right class of Hamiltonian functions. Recall that

$$P(\zeta) = \int_{\mathbb{S}^1} G(x, u(\zeta)(x)) dx,$$

where $g = \partial_u G$ and $g(x, u) = 4u^3 + O(u^4)$.

Lemma 3.6. *Assume that $(x, u) \mapsto g(x, u)$ is real holomorphic on a neighborhood of $\mathbb{S}^1 \times J$, for J some neighborhood of the origin of \mathbb{R} . Then for $\alpha > 0$, there exists $\mu^* > 0$ such that, for $0 < \mu \leq \mu^*$, the perturbation*

$$\begin{aligned} P : \mathcal{O}_{\mu}(Y_{\alpha}) &\rightarrow \mathbb{C} \\ \zeta &\mapsto P(\zeta) \end{aligned}$$

belongs to $\mathcal{T}^{\alpha, 1/2}(\mu)$.

Proof. Recall that for $x \in \mathbb{S}^1$, we have:

$$u(x) = \sum_{s \in \mathbb{Z}} \frac{\xi_s \varphi_s(x) + \eta_s \varphi_{-s}(x)}{\sqrt{2\lambda_s}} = u(\zeta)(x).$$

Using the Cauchy-Schwarz inequality and the fact that $\alpha > 0$, there exists a constant C_{α} that depends on α , such that for $\zeta \in \mathcal{O}_{\mu}(Y_{\alpha})$ we have

$$|u(\zeta)(x)| \leq C_{\alpha} \|\zeta\|_{\alpha} \leq C_{\alpha} \mu.$$

To prove that $P \in \mathcal{T}^{\alpha, 1/2}(\mu)$, it is enough to show that

$$\nabla P \in Y^\alpha \cap L_{1/2} \quad \text{and} \quad \nabla^2 P \in \mathcal{M}_{1/2}.$$

For $\alpha \geq 0$, we define the following space:

$$Z_\alpha = \{v = (v_s \in \mathbb{C}, s \in \mathbb{Z}) \mid (|v_s| \langle s \rangle^\alpha)_s \in \ell^2(\mathbb{Z})\}.$$

For $v \in Z_\alpha$, we define the Fourier transform $\mathcal{F}(v)$ of v by $u(x) = \mathcal{F}(v) := \sum v_s e^{isx}$. We also define the discrete Sobolev space by

$$H^\alpha(\mathbb{S}^1) = \left\{ u \mid u(x) = \sum_{s \in \mathbb{Z}} \hat{u}(s) e^{isx} \mid (|\hat{u}(s)| \langle s \rangle^\alpha)_s \in \ell^2(\mathbb{Z}) \right\}.$$

If $\alpha \in \mathbb{N}$, then

$$H^\alpha(\mathbb{S}^1) = \left\{ u \mid u(x) = \sum_{s \in \mathbb{Z}} \hat{u}(s) e^{isx} \mid \left(\widehat{\partial^\alpha u}(s) \right)_s \in \ell^2(\mathbb{Z}) \right\}.$$

So we have the following equivalence:

$$(3.4) \quad u \in H^\alpha(\mathbb{S}^1) \iff (\hat{u}(s))_s \in Z_\alpha.$$

- To prove that $\nabla_\zeta P \in Y_\alpha$, it is sufficient to prove, for example, that $\frac{\partial P}{\partial \xi} \in Z_\alpha$. We have

$$\frac{\partial P}{\partial \xi_s}(\zeta) = \frac{1}{\sqrt{2\lambda_s}} \int_{\mathbb{S}^1} \partial_u G(x, u(\zeta)(x)) \varphi_s(x) dx.$$

The map $(x, u) \mapsto g(x, u)$ is real holomorphic on a neighborhood of $\mathbb{S}^1 \times J$, so $x \mapsto \partial_u f(x, u(\zeta)(x)) \in H^\alpha(\mathbb{S}^1)$. We deduce from equivalence (3.4) that $\frac{\partial P}{\partial \xi} \in Z_\alpha$.

- Let us prove now that $\nabla^2 P \in \mathcal{M}_{1/2}$. Recall that:

$$|\nabla^2 P|_{1/2} = \sup_{s, s' \in \mathbb{Z}} \langle s \rangle^{1/2} \langle s' \rangle^{1/2} \left\| \frac{\partial^2 P}{\partial \zeta_s \partial \zeta_{s'}} \right\|_\infty.$$

We have

$$\frac{\partial^2 P}{\partial \zeta_s \partial \zeta_{s'}} = \frac{1}{2\lambda_s^{1/2} \lambda_{s'}^{1/2}} \int_{\mathbb{S}^1} \partial_u^2 G(x, u(\zeta)(x)) \varphi_s(x) \varphi_{s'}(x) dx.$$

Then

$$\frac{\partial^2 P}{\partial \zeta_s \partial \zeta_{s'}} = \frac{1}{2\lambda_s^{1/2} \lambda_{s'}^{1/2}} \begin{pmatrix} \widehat{\partial_u^2 G}(s+s') & \widehat{\partial_u^2 G}(s-s') \\ \widehat{\partial_u^2 G}(-s+s') & \widehat{\partial_u^2 G}(-s-s') \end{pmatrix},$$

which leads to

$$|\nabla^2 P|_{1/2} = \sup_{s \in \mathbb{Z}} \left| \widehat{\partial_u^2 G}(s) \right| < \infty.$$

- To conclude the proof, we have to show that $\nabla P \in L_{1/2}$. Recall that for $\beta \leq \alpha$, we have $Y_\alpha \subset L_\beta$. So $\nabla P \in Y_1 \subset L_{1/2}$, and the proof is achieved. \square

Now, we are able to give the symplectic change of variable which puts the Hamiltonian (1.5) into a normal form that satisfies the hypotheses of the KAM theorem.

Let $0 < \mu < \mu^*$ and $0 < \sigma < \sigma^*$ where μ^* and σ^* are defined in Lemma 3.2.

Theorem 3.7. *Let \mathcal{A} be an admissible set. There exists a Borel set of zero Lebesgue measure $\mathcal{U} \subset [1, 2]$, such that for any $m \in ([1, 2] \setminus \mathcal{U})$ there exists ν_0 that depends on \mathcal{A} , m and g such that:*

- (i) For $0 < \nu \leq \nu_0$, $\alpha > 1/2$ and $\rho \in \mathcal{D}$ there exists a real symplectic holomorphic change of variable

$$\Psi_\rho : \mathcal{O}^\alpha \left(\frac{\sigma}{2}, \frac{\mu}{2} \right) \rightarrow \mathfrak{T}_\rho(\nu, \sigma, \mu, \alpha),$$

that transforms the symplectic form $-id\xi \wedge d\eta$ on $\mathfrak{T}_\rho(\nu, \sigma, \mu, \alpha)$ into

$$-\nu \sum_{a \in \mathcal{A}} dr_a \wedge d\theta_a - i\nu \sum_{s \in \mathcal{L}} d\xi_s \wedge d\eta_s.$$

- (ii) For $c > \frac{1}{2}$, the change of variable Φ_ρ can be extended holomorphically on the following complex domain

$$\mathcal{D}_c = \left\{ \rho \in \mathbb{C}^{\mathcal{A}} \mid \left| \rho_j - \frac{3}{2} \right| \leq c, 1 \leq j \leq \text{Card}(\mathcal{A}) \right\}.$$

It transforms the perturbed Hamiltonian $H = H_2 + P$ into the following normal form

$$(3.5) \quad \nu^{-1} H \circ \Psi_\rho = \Omega(\rho) \cdot r + \sum_{a \in \mathcal{L}} \Lambda_a(\rho) \xi_a \eta_a + f(r, \theta, \zeta, \rho),$$

for all $\rho \in \mathcal{D}_c$. The internal frequency vector Ω and the external frequencies Λ_a , $a \in \mathcal{L}$, are given explicitly in (3.22) and (3.23). Moreover, they are linear in ρ and verify:

$$(3.6) \quad |\Omega(\rho) - \omega(\rho)| \leq C\nu, \quad |\Lambda_a(\rho) - \lambda_a(\rho)| \leq C\nu|a|^{-1},$$

for all $\rho \in \mathcal{D}_c$. The constant C depends on the admissible set \mathcal{A} .

(iii) The perturbation f is real holomorphic on \mathcal{D}_c , belongs to $\mathcal{T}^{\alpha, 1/2}(\mathcal{D}, \frac{\sigma}{2}, \frac{\mu}{2})$ and satisfies

$$\begin{aligned} \llbracket f \rrbracket_{\frac{\sigma}{2}, \frac{\mu}{2}, \mathcal{D}}^{\alpha, 1/2, \gamma} &\leq C_1(1 + \gamma)\nu\mu^4, \\ \llbracket f^T \rrbracket_{\frac{\sigma}{2}, \frac{\mu}{2}, \mathcal{D}}^{\alpha, 1/2, \gamma} &\leq C_1(1 + \gamma)\nu^{3/2}\mu^5. \end{aligned}$$

The constant C_1 depends on the admissible set \mathcal{A} , the mass m and the nonlinearity g .

Remark 3.8. In (iii), we need to estimate the derivative of the perturbation f with respect to the parameter ρ . This is possible thanks to the Cauchy estimate. So we need to extend Ψ_ρ holomorphically onto \mathcal{D}_c (see 3.6).

All the rest of this section will be dedicated to the proof of the previous theorem.

3.2. Resonance. We consider H_4 , the quartic part of the Hamiltonian H defined by:

$$H_4 = H_2 + P_4,$$

where

$$H_2 = \sum_{s \in \mathbb{Z}} \lambda_s \xi_s \eta_s,$$

and

$$P_4 = \sum_{(i, j, k, l) \in \mathbb{Z}} C(i, j, k, l) \frac{(\xi_i + \eta_{-i})(\xi_j + \eta_{-j})(\xi_k + \eta_{-k})(\xi_l + \eta_{-l})}{4\sqrt{\lambda_i \lambda_j \lambda_k \lambda_l}}.$$

The constant $C(i, j, k, l)$ is defined by:

$$C(i, j, k, l) := \int_{\mathbb{S}^1} \varphi_i(x) \varphi_j(x) \varphi_k(x) \varphi_l(x) dx = \begin{cases} \frac{1}{2\pi} & \text{if } i + j + k + l = 0, \\ 0 & \text{if } i + j + k + l \neq 0. \end{cases}$$

We define the following subset of \mathbb{Z}^4 :

$$\mathcal{J} := \{(i, j, k, l) \in \mathbb{Z}^4 \mid i + j = k + l\}.$$

So

$$P_4 = \frac{1}{2\pi} \sum_{(i, j, k, l) \in \mathcal{J}} \frac{(\xi_i + \eta_{-i})(\xi_j + \eta_{-j})(\xi_k + \eta_{-k})(\xi_l + \eta_{-l})}{4\sqrt{\lambda_i \lambda_j \lambda_k \lambda_l}}.$$

We can decompose P_4 in three parts $P_4 = P_4^0 + P_4^1 + P_4^2$ where:

$$\begin{aligned} P_4^0 &= \frac{1}{8\pi} \sum_{(i, j, k, l) \in \mathcal{J}} \frac{\xi_i \xi_j \xi_k \xi_l + \eta_i \eta_j \eta_k \eta_l}{\sqrt{\lambda_i \lambda_j \lambda_k \lambda_l}}, \\ P_4^1 &= \frac{1}{2\pi} \sum_{(i, j, k, -l) \in \mathcal{J}} \frac{\xi_i \xi_j \xi_k \eta_l + \eta_i \eta_j \eta_k \xi_l}{\sqrt{\lambda_i \lambda_j \lambda_k \lambda_l}}, \\ P_4^2 &= \frac{3}{4\pi} \sum_{(i, j, -k, -l) \in \mathcal{J}} \frac{\xi_i \xi_j \eta_k \eta_l}{\sqrt{\lambda_i \lambda_j \lambda_k \lambda_l}}. \end{aligned}$$

For $(i, j, k, l) \in \mathbb{Z}^4$, we define the small divisors:

$$\Omega_0(i, j, k, l) = \lambda_i + \lambda_j + \lambda_k + \lambda_l; \quad \Omega_1(i, j, k, l) = \lambda_i + \lambda_j + \lambda_k - \lambda_l; \quad \Omega_2(i, j, k, l) = \lambda_i + \lambda_j - \lambda_k - \lambda_l.$$

Definition 3.9. A monomial $\xi_i \xi_j \xi_k \eta_l$ or $\eta_i \eta_j \eta_k \xi_l$ is resonant if $\Omega_1(i, j, k, l) = 0$. In this case, we denote $\mathcal{R}_1 := \{(i, j, k, l) \in \mathbb{Z}^4 \mid \Omega_1(i, j, k, l) = 0\}$. A monomial $\xi_i \xi_j \eta_k \eta_l$ is resonant if $\Omega_2(i, j, k, l) = 0$. In this case, we denote $\mathcal{R}_2 := \{(i, j, k, l) \in \mathbb{Z}^4 \mid \Omega_2(i, j, k, l) = 0\}$. Let \mathcal{R} be the union of \mathcal{R}_1 and \mathcal{R}_2 .

We define also

$$\mathcal{J}_2 = \{(i, j, k, l) \in \mathcal{J} \mid \#\{i, j, k, l\} \cap \mathcal{A} \geq 2\},$$

and

$$\mathcal{J}_2^c = \mathcal{J} \setminus \mathcal{J}_2 = \{(i, j, k, l) \in \mathcal{J} \mid \#\{i, j, k, l\} \cap \mathcal{L} \geq 3\}$$

Lemma 3.10. There exists a Borel set $\mathcal{U} \subset [1, 2]$ of full Lebesgue measure, such that for $m \in \mathcal{U}$ we have:

- (i) $\mathcal{R}_1 = \emptyset$,
- (ii) $\mathcal{R}_2 \subset \{(i, j, k, l) \in \mathbb{Z}^4 \mid \{|i|, |j|\} = \{|k|, |l|\}\}$,
- (iii) There exists $\gamma(m) > 0$ such that for any $(i, j, k, l) \in \mathcal{J}_2 \setminus \mathcal{R}$, we have:

$$|\Omega_1(i, j, k, l)| \geq \gamma(m); \quad |\Omega_2(i, j, k, l)| \geq \gamma(m).$$

Proof. We begin by proving assertions (i) and (ii). Let us fix $(i, j, k, l) \in \mathbb{Z}^4$. Consider, for $\delta = \pm 1$, the function:

$$f(m) = \sqrt{i^2 + m} + \sqrt{j^2 + m} + \delta\sqrt{k^2 + m} - \sqrt{l^2 + m}.$$

The function f is analytic on $[1, 2]$, and it can be extended to an algebraical multi-valued function of $m \in \mathbb{C}$. If $j = k = l = 0$, then $-i^2$ is a branching point of f . Branching points for f are $-i^2, -j^2, -k^2$ and $-l^2$.

If $\delta = 1$, then f is not identically zero on $[1, 2]$, and admits finitely many zeros. It follows that, there is a Borel set $\mathcal{U}_1 \subset [1, 2]$ of full Lebesgue measure, such that for $m \in \mathcal{U}_1$, we have $\mathcal{R}_1 = \emptyset$.

Now, for $\delta = -1$, if $f|_{[1, 2]} \equiv 0$, then $f \equiv 0$, and we have $\{i^2, j^2\} = \{k^2, l^2\}$. If f is not identically zero on $[1, 2]$, then f admits finitely many zeros. So there is a Borel set $\mathcal{U}_2 \subset [1, 2]$ of full Lebesgue measure, such that for any $m \in \mathcal{U}_2$, we have $\mathcal{R}_2 \subset \{(i, j, k, l) \in \mathbb{Z}^4 \mid \{|i|, |j|\} = \{|k|, |l|\}\}$.

It remains to prove the last assertion. Let us begin with Ω_2 . Thanks to Proposition 2.10, for $\kappa > 0$, there is an open set $\mathcal{C}_\kappa \subset [1, 2]$ such that

$$\text{mes}(\mathcal{C}_\kappa) < C\kappa^\tau,$$

where $\tau > 0$ and depends on $\text{Card}(\mathcal{A})$. The constant C depends on the admissible set \mathcal{A} . For $m \in [1, 2] \setminus \mathcal{C}_\kappa$ and any $(i, j, k, l) \in \mathcal{J}_2 \setminus \mathcal{R}$, we have:

$$|\Omega_2(i, j, k, l)| > \kappa.$$

If $\kappa' \leq \kappa$, then $\mathcal{C}_{\kappa'} \subset \mathcal{C}_\kappa$. So $\mathcal{C} := \bigcap_{0 < \kappa < 1} \mathcal{C}_\kappa$ is a Borel set, and we have

$$\text{mes}(\mathcal{C}) = 0.$$

Moreover, for $m \in \mathcal{U}_3 \equiv ([1, 2] \setminus \mathcal{C}) = \bigcup_{0 < \kappa < 1} ([1, 2] \setminus \mathcal{C}_\kappa)$, there exists a constant $\gamma(m)$ such that for each $(i, j, k, l) \in \mathcal{J}_2 \setminus \mathcal{R}$, we have:

$$|\Omega_2(i, j, k, l)| > \gamma(m).$$

To control $\Omega_1(i, j, k, l)$ we follow the same procedure, but we use Proposition 2.8 instead of Proposition 2.10. Finally we denote $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3 \cap \mathcal{U}_4$, where \mathcal{U}_4 is the Borel set of full Lebesgue measure that we obtain after controlling $\Omega_1(i, j, k, l)$. \square

3.3. Birkhoff's PrecEDURE. For $\alpha > 0$, we recall the definition of the following space:

$$Z_\alpha = \{v = (v_s \in \mathbb{C}, s \in \mathbb{Z}) \mid (|v_s| \langle s \rangle^\alpha)_s \in \ell^2(\mathbb{Z})\}.$$

We endow Z_α with the norm:

$$\|v\|_\alpha^2 = \sum_{s \in \mathbb{Z}} |v_s|^2 \langle s \rangle^{2\alpha}, \quad \langle s \rangle = \max(|s|, 1).$$

We denote by $v * y$ the convolution in $\ell^2(\mathbb{Z})$ defined by $(v * w)_l = \sum_{i+j=l} v_i w_j$. We recall Lemma 2 from [12].

Lemma 3.11. *Consider $v, w \in Z_\alpha$ for $\alpha > \frac{1}{2}$. Then, $v * w \in Z_\alpha$, and*

$$(3.7) \quad \|v * w\|_\alpha < C(\alpha) \|v\|_\alpha \|w\|_\alpha,$$

where C is a constant that depends only on α .

Proof. Consider $v, w \in Z_\alpha$. We have:

$$\begin{aligned} \|v * w\|_\alpha^2 &= \sum_{s \in \mathbb{Z}} \langle s \rangle^{2\alpha} \left| \sum_{i+j=s} v_i w_j \right|^2 \\ &= \sum_{s \in \mathbb{Z}} \langle s \rangle^{2\alpha} \left| \sum_{i+j=s} \frac{\langle s \rangle^\alpha}{\langle i \rangle^\alpha \langle j \rangle^\alpha} \frac{\langle i \rangle^\alpha \langle j \rangle^\alpha}{\langle s \rangle^\alpha} v_i w_j \right|^2 \\ &\leq \sum_{s \in \mathbb{Z}} \langle s \rangle^{2\alpha} \left(\sum_{i+j=s} \left(\frac{\langle s \rangle}{\langle i \rangle \langle j \rangle} \right)^{2\alpha} \right) \left(\sum_{i+j=s} \frac{\langle i \rangle^{2\alpha} \langle j \rangle^{2\alpha}}{\langle s \rangle^{2\alpha}} |v_i|^2 |w_j|^2 \right), \end{aligned}$$

and

$$\sum_{i+j=s} \left(\frac{\langle s \rangle}{\langle i \rangle \langle j \rangle} \right)^{2\alpha} \leq \sum_{i, j \in \mathbb{Z}} \left(\frac{\langle i \rangle + \langle j \rangle}{\langle i \rangle \langle j \rangle} \right)^{2\alpha} \leq 4^\alpha \sum_{i, j \in \mathbb{Z}} \frac{1}{\langle i \rangle^{2\alpha}} + \frac{1}{\langle j \rangle^{2\alpha}} \leq C^2(\alpha).$$

Then

$$\begin{aligned} \|v * w\|_\alpha^2 &\leq C(\alpha)^2 \sum_{s \in \mathbb{Z}} \sum_{i+j=s} \langle i \rangle^{2\alpha} |v_i|^2 \langle j \rangle^{2\alpha} |w_j|^2 \\ &\leq \sum_{i,j \in \mathbb{Z}} \langle i \rangle^{2\alpha} |v_i|^2 \langle j \rangle^{2\alpha} |w_j|^2 \\ &= C(\alpha)^2 \|v\|_\alpha^2 \|w\|_\alpha^2. \end{aligned}$$

The proof is thus achieved. \square

We endow the phase space with symplectic structure $-i \sum d\xi_k \wedge d\eta_k$. For $\alpha > 1/2$, Y_α is an algebra for the convolution.

Lemma 3.12. *Let $\alpha > 1/2$ and P^4 a real homogeneous polynomial on Y_α , of degree 4, indexed by \mathcal{J} . We assume that P^4 is of the form:*

$$P^4(\zeta) = \sum_{(j_1, j_2, j_3, j_4) \in \mathcal{J}} \sum_{1 \leq r \leq 4} a_{j_1, j_2, j_3, j_4}^r \xi_{j_1} \cdots \xi_{j_r} \eta_{j_{r+1}} \cdots \eta_{j_4},$$

where $|a_{j_1, j_2, j_3, j_4}^r| < M$ for any $(j_1, j_2, j_3, j_4) \in \mathcal{J}$. Then we have:

$$\|\nabla P^4\|_\alpha \leq C(\alpha, M) \|\zeta\|_\alpha^3.$$

So, for $t \leq 1$, the flow $\Phi_{P^4}^t$ of the Hamiltonian vector $X_{P^4} = iJ\nabla P^4$ is well-defined, real and analytic on the ball:

$$\mathcal{O}_\delta(Y_\alpha) = \{\zeta \in Y_\alpha \mid \|\zeta\|_\alpha < \delta = \delta(M)\}.$$

Moreover, for any $\zeta \in \mathcal{O}_\delta(Y_\alpha)$,

$$\|\Phi_{P^4}^t(\zeta) - \zeta\|_\alpha \leq C(M) \|\zeta\|_\alpha^3.$$

Proof. We recall that $X_{P^4} = i \begin{pmatrix} \nabla_\xi P^4 \\ \nabla_\eta P^4 \end{pmatrix}$. Since $|a_{j_1, j_2, j_3, j_4}^r| < M$, we have:

$$\left| \frac{\partial P}{\partial \eta_l} \right| \leq M \sum_{(i,j,k,l) \in \mathcal{J}} |\xi_i \xi_j \xi_k| + |\xi_i \xi_j \eta_k| + |\xi_i \eta_j \eta_k| + |\eta_i \eta_j \eta_k|.$$

We remark that

$$\sum_{(i,j,k,l) \in \mathcal{J}} |\xi_i \xi_j \xi_k| = (\xi \star \xi \star \xi)_l + (\xi \star \xi \star \tilde{\xi})_l + (\xi \star \tilde{\xi} \star \tilde{\xi})_l + (\tilde{\xi} \star \tilde{\xi} \star \tilde{\xi})_l + (\xi \star \tilde{\xi} \star \xi)_l + (\tilde{\xi} \star \tilde{\xi} \star \xi)_l,$$

where $\tilde{\xi} = (\tilde{\xi}_j)_{j \in \mathbb{Z}}$ and $\tilde{\xi}_j = \xi_{-j}$. Using Lemma 3.11, we have

$$\begin{aligned} \left\| \frac{\partial P^4}{\partial \eta} \right\|_\alpha^2 &= \sum_{l \in \mathbb{Z}} \langle l \rangle^{2\alpha} \left| \frac{\partial P^4}{\partial \eta_l} \right|^2 \\ &\leq C(M) (\|\xi \star \xi \star \xi\|_\alpha^2 + \|\xi \star \xi \star \eta\|_\alpha^2 + \|\xi \star \eta \star \eta\|_\alpha^2 + \|\eta \star \eta \star \eta\|_\alpha^2) \\ &\leq C(\alpha, M) (\|\xi\|_\alpha^6 + \|\xi\|_\alpha^4 \|\eta\|_\alpha^2 + \|\xi\|_\alpha^2 \|\eta\|_\alpha^4 + \|\eta\|_\alpha^6) \\ &\leq C(\alpha, M) (\|\xi\|_\alpha^2 + \|\eta\|_\alpha^2)^3 = C(\alpha, M) \|\zeta\|_\alpha^6. \end{aligned}$$

We prove the same way that:

$$\left\| \frac{\partial P^4}{\partial \xi} \right\|_\alpha^2 \leq C(\alpha, M) \|\zeta\|_\alpha^6.$$

So, we have:

$$\|\nabla_\zeta P^4\|_\alpha = \left(\left\| \frac{\partial P^4}{\partial \xi} \right\|_\alpha^2 + \left\| \frac{\partial P^4}{\partial \eta} \right\|_\alpha^2 \right)^{1/2} \leq C(\alpha, M) \|\zeta\|_\alpha^3.$$

This concludes the proof of the lemma. \square

Lemma 3.13. *Consider D^- the following bounded operator on Y_α*

$$D^- = \text{diag} \left\{ \lambda_s^{-1/2} I_2, s \in \mathbb{Z} \right\}.$$

Define $Q^4(\zeta) := P^4(D^-(\zeta))$, where P^4 is the polynomial defined in the Lemma 3.12. Then $\nabla_\zeta^2 Q^4 \in \mathcal{M}_{1/2}$, and we have:

$$|\nabla_\zeta^2 Q^4|_{1/2} \leq C(\alpha, M) \|\zeta\|_\alpha^2.$$

Proof. We recall that

$$P^4(\zeta) = \sum_{(j_1, j_2, j_3, j_4) \in \mathcal{J}} \sum_{1 \leq r \leq 4} a_{j_1, j_2, j_3, j_4}^r \xi_{j_1} \cdots \xi_{j_r} \eta_{j_{r+1}} \cdots \eta_{j_4},$$

where $|a_{j_1, j_2, j_3, j_4}^r| < M$ for $(j_1, j_2, j_3, j_4) \in \mathcal{J}$. So we have

$$Q^4(\zeta) = \sum_{(j_1, j_2, j_3, j_4) \in \mathcal{J}} \sum_{1 \leq r \leq 4} \frac{a_{j_1, j_2, j_3, j_4}^r}{\sqrt{\lambda_{j_1} \lambda_{j_2} \lambda_{j_3} \lambda_{j_4}}} \xi_{j_1} \cdots \xi_{j_r} \eta_{j_{r+1}} \cdots \eta_{j_4},$$

and

$$|\nabla_{\zeta}^2 Q^4|_{1/2} = \sup_{s, s' \in \mathbb{Z}} \langle s \rangle^{1/2} \langle s' \rangle^{1/2} \left\| \frac{\partial^2 Q^4}{\partial \zeta_s \partial \zeta_{s'}} \right\|_{\infty}.$$

For any s and s' in \mathbb{Z} , we have:

$$\langle s \rangle^{1/2} \langle s' \rangle^{1/2} \left\| \frac{\partial^2 Q^4}{\partial \zeta_s \partial \zeta_{s'}} \right\|_{\infty} \leq \|A\|_{\infty},$$

where A is a real square matrix of size 2, whose coefficients are homogeneous polynomials of Y_{α} of degree 2, of the form:

$$P^2(\zeta) = \sum_{\bar{j} \in \mathcal{J}} \sum_{k, l=1}^4 \frac{1}{\sqrt{\lambda_{j_{\sigma(k)}} \lambda_{j_{\sigma(l)}}}} (a_1 \xi_{j_{\sigma(k)}} \xi_{j_{\sigma(l)}} + a_2 \xi_{j_{\sigma(k)}} \eta_{j_{\sigma(l)}} + a_3 \eta_{j_{\sigma(k)}} \eta_{j_{\sigma(l)}}),$$

where $\bar{j} = (j_{\sigma(1)}, j_{\sigma(2)}, j_{\sigma(3)}, j_{\sigma(4)})$ and σ is a permutation from the symmetric group S_4 . Using Lemma 3.11, we have:

$$|P^2(\zeta)| \leq C(\alpha, M) \|\zeta\|_{\alpha}^2.$$

So

$$|\nabla_{\zeta}^2 Q|_{1/2} \leq C(\alpha, M) \|\zeta\|_{\alpha}^2.$$

□

Remark 3.14. We recall that $P_4 = P_4^0 + P_4^1 + P_4^2$ where

$$\begin{aligned} P_4^0 &= \frac{1}{8\pi} \sum_{(i, j, k, l) \in \mathcal{J}} \frac{\xi_i \xi_j \xi_k \xi_l + \eta_i \eta_j \eta_k \eta_l}{\sqrt{\lambda_i \lambda_j \lambda_k \lambda_l}}, \\ P_4^1 &= \frac{1}{2\pi} \sum_{(i, j, k, -l) \in \mathcal{J}} \frac{\xi_i \xi_j \xi_k \eta_l + \eta_i \eta_j \eta_k \xi_l}{\sqrt{\lambda_i \lambda_j \lambda_k \lambda_l}}, \\ P_4^2 &= \frac{3}{4\pi} \sum_{(i, j, -k, -l) \in \mathcal{J}} \frac{\xi_i \xi_j \eta_k \eta_l}{\sqrt{\lambda_i \lambda_j \lambda_k \lambda_l}}. \end{aligned}$$

The coefficients of each monomial are bounded by $3/4\pi$. Using 3.12 and 3.13, we have $P_4 \in \mathcal{T}^{\alpha, 1/2}(\mu)$, for $\alpha > 1/2$ and $\zeta \in \mathcal{O}_{\mu}(Y_{\alpha})$.

Let $\mathcal{U} \subset [1, 2]$ be the Borel full Lebesgue measure set from Lemma 3.10. For $m \in \mathcal{U}$, we want to construct a holomorphic real symplectic change of variable in the neighborhood of the origin of Y_{α} which transforms the quartic part of the Hamiltonian H into a Birkhoff normal form up to order 5. This transformation extracts the integrable terms from the quartic part of the perturbation P and cubic terms in the direction of $\mathcal{L} = \mathbb{Z} \setminus \mathcal{A}$.

Proposition 3.15. For $m \in \mathcal{U}$, there is a holomorphic real symplectic change of variable τ on $\mathcal{O}_{\delta(m)}(Y_{\alpha})$, for some $\delta(m) > 0$ and $\alpha > 1/2$. The change of variable τ satisfies:

$$(3.8) \quad \|\tau(\zeta) - \zeta\|_{\alpha} \leq C(m) \|\zeta\|_{\alpha}^3, \quad \forall \zeta \in \mathcal{O}_{\delta(m)}(Y_{\alpha}).$$

The mapping τ transforms the Hamiltonian $H = H_2 + P = H_2 + P_4 + R_5$ into :

$$(3.9) \quad H \circ \tau = (H_2 + P) \circ \tau = H_2 + Z_4 + Q_4 + R_6 + R_5 \circ \tau,$$

where

$$Z_4 = \frac{3}{\pi} \sum_{(i, j, k, l) \in \mathcal{J}_2 \cap \mathcal{R}_2} \frac{\xi_i \xi_j \eta_k \eta_l}{\lambda_i \lambda_j},$$

and $Q_4 = Q_{4,1} + Q_{4,2}$ for

$$\begin{aligned} Q_{4,1} &= \frac{1}{2\pi} \sum_{(i, j, -k, l) \in \mathcal{J}_2^c} \frac{\xi_i \xi_j \xi_k \eta_l + \eta_i \eta_j \eta_k \xi_l}{\sqrt{\lambda_i \lambda_j \lambda_k \lambda_l}}, \\ Q_{4,2} &= \frac{3}{4\pi} \sum_{(i, j, k, l) \in \mathcal{J}_2^c} \frac{\xi_i \xi_j \eta_k \eta_l}{\sqrt{\lambda_i \lambda_j \lambda_k \lambda_l}}. \end{aligned}$$

The polynomial Z_4 contains integrable terms while Q_4 is cubic or quartic in the direction of \mathcal{L} . Moreover, Z_4 , Q_4 , R_6 and $R_5 \circ \tau$ are real holomorphic on $\mathcal{O}_{\delta(m)}(Y_\alpha)$. The remainder terms $R_5 \circ \tau$ and R_6 are respectively of order 5 and 6 at the origin of Y_α . Moreover, for any $0 < \mu \leq \delta(m)$, Z_4 , Q_4 , $R_5 \circ \tau$ and R_6 belong to $\mathcal{T}^{\alpha,1/2}(\mu)$ and satisfy:

$$(3.10) \quad \llbracket Z_4 \rrbracket_\mu^{\alpha,1/2} + \llbracket Q_4 \rrbracket_\mu^{\alpha,1/2} \leq C\mu^4,$$

$$(3.11) \quad \llbracket R_6 \rrbracket_\mu^{\alpha,1/2} \leq C\mu^6,$$

$$(3.12) \quad \llbracket R_5 \circ \tau \rrbracket_\mu^{\alpha,1/2} \leq C\mu^5,$$

where the constant C depends on m , the nonlinearity g and the admissible set \mathcal{A} .

We recall that the Poisson bracket associated to the symplectic form $-i \sum_{s \in \mathbb{Z}} d\xi_s \wedge d\eta_s$ is

$$\{f, g\}(\xi, \eta) = i \sum_{j \in \mathbb{Z}} \frac{\partial f}{\partial \eta_j} \frac{\partial g}{\partial \xi_j} - \frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial \eta_j},$$

for $f, g \in \mathcal{C}^1(Z_\alpha \times Z_\alpha)$.

Lemma 3.16. *Let P be a homogeneous polynomial of degree 4 defined by:*

$$P(\xi, \eta) = \sum_{|\alpha|+|\beta|=4} a_{\alpha,\beta} \xi^\alpha \eta^\beta,$$

where $\xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \xi_3^{\alpha_3} \xi_4^{\alpha_4}$. Then:

$$\{H_2, P\}(\xi, \eta) = i \sum_{|\alpha|+|\beta|=4} a_{\alpha,\beta} \Omega_{\min(|\alpha|, |\beta|)}(\alpha, \beta) \xi^\alpha \eta^\beta.$$

Proof. We prove the previous lemma by using the expression of the Hamiltonian H_2 , the Poisson bracket and the frequencies $\Omega_p(\alpha, \beta)$ for $0 \leq p \leq 2$. \square

Proof of the Proposition 3.15. We want to construct a holomorphic real symplectic change of variable τ in the neighborhood of the origin of Y_α for $\alpha > 1/2$. The mapping τ puts the Hamiltonian H into a Birkhoff normal form up to order 5. To do this, we use a classical method: τ will be the time one flow of a Hamiltonian χ_4 (ie $\tau = \Phi_{\chi_4}^1$ where $\Phi_{\chi_4}^t$ is the flow of χ_4 at time t). The Hamiltonian χ_4 will be a solution of a certain homological equation. Using the Taylor formula, we obtain:

$$\begin{aligned} (H_2 + P_4 + R_5) \circ \tau &= (H_2 + P_4) \circ \tau + R_5 \circ \tau \\ &= H_2 + P_4 + \{H_2, \chi_4\} + \{P_4, \chi_4\} \\ &\quad + \int_0^1 (1-t) \{\{H_2 + P_4, \chi_4\}, \chi_4\} \circ \Phi_{\chi_4}^t dt + R_5 \circ \tau. \end{aligned}$$

We want that

$$(H_2 + P_4 + R_5) \circ \tau = H_2 + Z_4 + Q_4 + R_6 + R_5 \circ \tau.$$

So, by taking

$$R_6 = \{P_4, \chi_4\} + \int_0^1 (1-t) \{\{H_2 + P_4, \chi_4\}, \chi_4\} \circ \Phi_{\chi_4}^t dt,$$

the Hamiltonian χ_4 satisfies the following homological equation:

$$(3.13) \quad \{H_2, \chi_4\} = Z_4 + Q_4 - P_4.$$

Using Lemma 3.16, the Hamiltonian χ_4 is given by:

$$\begin{aligned} \chi_4 &= \frac{i}{8\pi} \sum_{(i,j,-k,-l) \in \mathcal{J}} \frac{\xi_i \xi_j \xi_k \xi_l - \eta_i \eta_j \eta_k \eta_l}{\Omega_0(i, j, k, l) \sqrt{\lambda_i \lambda_j \lambda_k \lambda_l}} \\ &\quad + \frac{i}{2\pi} \sum_{(i,j,-k,l) \in \mathcal{J}_2} \frac{\xi_i \xi_j \xi_k \eta_l - \eta_i \eta_j \eta_k \xi_l}{\Omega_1(i, j, k, l) \sqrt{\lambda_i \lambda_j \lambda_k \lambda_l}} \\ &\quad + \frac{3i}{4\pi} \sum_{(i,j,k,l) \in \mathcal{J}_2 \setminus \mathcal{R}_2} \frac{\xi_i \xi_j \eta_k \eta_l}{\Omega_2(i, j, k, l) \sqrt{\lambda_i \lambda_j \lambda_k \lambda_l}}. \end{aligned}$$

By Lemma 3.10, there is a Borel set $\mathcal{U} \subset [1, 2]$ of full Lebesgue measure set, such that for $m \in \mathcal{U}$, there is a constant $\gamma(m) > 0$ smaller than $|\Omega_1(i, j, k, l)|$ and $|\Omega_2(i, j, k, l)|$. Remark also that $\Omega_0(i, j, k, l) > 4$. Then χ_4 is a homogeneous polynomial of degree 4 indexed by \mathcal{J} , with bounded coefficients. So, the

Hamiltonian vector field X_{χ_4} is real and holomorphic on Y_α . Using Lemma 3.12, for $m \in \mathcal{U}$ and $\alpha > 1/2$, there exists $C(\alpha, m) > 0$ such that:

$$\|X_{\chi_4}\|_\alpha \leq C(\alpha, m) \|\zeta\|_\alpha^3.$$

So, there is $\delta(m) > 0$ such that τ is real holomorphic symplectic change of variable on $\mathcal{O}_{\delta(m)}(Y_\alpha)$.

By Lemma 3.16, we have

$$\begin{aligned} \{H_2, \chi_4\} &= -\frac{1}{8\pi} \sum_{(i,j,-k,-l) \in \mathcal{J}} \frac{\xi_i \xi_j \xi_k \xi_l + \eta_i \eta_j \eta_k \eta_l}{\sqrt{\lambda_i \lambda_j \lambda_k \lambda_l}} \\ &\quad - \frac{1}{2\pi} \sum_{(i,j,-k,l) \in \mathcal{J}_2} \frac{\xi_i \xi_j \xi_k \eta_l + \eta_i \eta_j \eta_k \xi_l}{\sqrt{\lambda_i \lambda_j \lambda_k \lambda_l}} \\ &\quad - \frac{3}{4\pi} \sum_{(i,j,k,l) \in \mathcal{J}_2 \setminus \mathcal{R}_2} \frac{\xi_i \xi_j \eta_k \eta_l}{\sqrt{\lambda_i \lambda_j \lambda_k \lambda_l}}. \end{aligned}$$

So

$$(H_2 + P_4) \circ \tau = H_2 + Z_4 + Q_4 + R_6,$$

where Z_4, Q_4 is defined as in the proposition. They are two homogeneous polynomials of degree 4 with bounded coefficients. From Lemma 3.11 and 3.12, for $\alpha > 1/2$ and $0 < \mu \leq \delta(m)$, these two polynomials belong to $\mathcal{T}^{\alpha, 1/2}(\mu)$ and satisfy 3.10.

Let us study the remainder terms R_6 and $R_5 \circ \tau$. Concerning R_6 , by construction, R_6 is a holomorphic Hamiltonian of order 6 in the neighborhood of the origin of Y_α . We recall that

$$R_6 = H \circ \tau - H_2 - Z_4 - Q_4 - R_5 \circ \tau.$$

The right-hand side of the equation is real, so R_6 is also real. Let us prove that R_6 belongs to $\mathcal{T}^{\alpha, 1/2}(\mu)$, with $0 < \mu \leq \delta(m)$.

We begin by proving that $\chi_4 \in \mathcal{T}^{\alpha, 1/2+}(\mu)$. We remark that, for $i \in \mathbb{Z}$ such that $(i, j, k, l) \in \mathcal{J}$, we have

$$\frac{\langle i \rangle^{3/2}}{|\Omega_\iota(i, j, k, l)| \sqrt{\lambda_i \lambda_j \lambda_k \lambda_l}} \leq C(\mathcal{A}, m), \quad \iota = 0, 1, 2.$$

Using this estimate and the same method as in the proof of the Lemma 3.12, we get that $\nabla_\zeta \chi_4 \in L_{\frac{1}{2}+}$. It remains to prove that $\nabla_\zeta^2 \chi_4 \in \mathcal{M}_{\frac{1}{2}+}$. The first terms of χ_4 are indexed by \mathcal{J} . For $i, j \in \mathbb{Z}$, we have

$$\frac{\langle i \rangle^{1/2} \langle j \rangle^{1/2} (1 + ||i| - |j||)}{\Omega_0(i, j, k, l) \sqrt{\lambda_i \lambda_j \lambda_k \lambda_l}} \leq 1.$$

The next terms of χ_4 are indexed by \mathcal{J}_2 . By Proposition 2.8, we have:

$$\frac{\langle i \rangle^{1/2} \langle j \rangle^{1/2} (1 + ||i| - |j||)}{\Omega_\iota(i, j, k, l) \sqrt{\lambda_i \lambda_j \lambda_k \lambda_l}} \leq C'(m, \mathcal{A}), \quad \iota = 1, 2.$$

Using these two estimates and the same method as in the proof of the Lemma 3.13, we get that $\nabla_\zeta^2 \chi_4 \in \mathcal{M}_{\frac{1}{2}+}$. So we proved that $\chi_4 \in \mathcal{T}^{\alpha, 1/2+}(\mu)$ for $\alpha > 1/2$ and $0 < \mu \leq \delta(m)$.

By Lemma 3.1, we have $\{\mathcal{T}^{\alpha, 1/2}(\delta(m)), \mathcal{T}^{\alpha, 1/2+}(\delta(m))\} \in \mathcal{T}^{\alpha, 1/2}(\frac{1}{2}\delta(m))$. So $\{P_4, \chi_4\} \in \mathcal{T}^{\alpha, 1/2}(\frac{1}{2}\delta(m))$ and for $0 < \mu \leq \frac{1}{2}\delta(m)$, we have:

$$\llbracket \{P_4, \chi_4\} \rrbracket_\mu^{\alpha, 1/2} \leq C\mu^{-2} \llbracket P_4 \rrbracket_\mu^{\alpha, 1/2} \llbracket \chi_4 \rrbracket_\mu^{\alpha, 1/2+} \leq C\mu^6.$$

Due to the homological equation (3.13), we have:

$$\{H_2 + P_4, \chi_4\} = Z_4 + Q_4 - P_4 + \{P_4, \chi_4\} \in \mathcal{T}^{\alpha, 1/2}(\frac{1}{2}\delta(m)).$$

Using Lemma 3.1 again, for $0 < \mu \leq \frac{1}{4}\delta(m)$, we have:

$$\{\{H_2 + P_4, \chi_4\}, \chi_4\} \in \mathcal{T}^{\alpha, 1/2}(\frac{1}{4}\delta(m)) \quad \text{and} \quad \llbracket \{\{H_2 + P_4, \chi_4\}, \chi_4\} \rrbracket_\mu^{\alpha, 1/2} \leq C\mu^6.$$

Since $\chi_4 \in \mathcal{T}^{\alpha, 1/2+}(\delta(m))$ and $\llbracket \chi_4 \rrbracket_\mu^{\alpha, 1/2+} \leq C\mu^4$, we have by Corollary 3.3

$$\mathcal{T}^{\alpha, 1/2}(\frac{1}{4}\delta(m)) \circ \Phi_{\chi_4}^t \in \mathcal{T}^{\alpha, 1/2}(\frac{1}{8}\delta(m)).$$

So, $R_6 \in \mathcal{T}^{\alpha, 1/2}(\mu)$ and satisfies (3.11) for $0 < \mu \leq \frac{1}{8}\delta(m)$.

Now, consider the remainder term $R_5 \circ \tau$. Recall that $R_5 = P - P_4$, so R_5 is real and holomorphic, of order 5 at the origin and belongs to $\mathcal{T}^{\alpha, 1/2}(\delta(m))$. Using Corollary 3.3 again, we obtain that $R_5 \circ \tau \in \mathcal{T}^{\alpha, 1/2}(\frac{1}{2}\delta(m))$, so $R_5 \circ \tau \in \mathcal{T}^{\alpha, 1/2}(\mu)$ and satisfies for $0 < \mu \leq \frac{1}{2}\delta(m)$:

$$\llbracket R_5 \circ \tau \rrbracket_\mu^{\alpha, 1/2} \leq \llbracket P - P_4 \rrbracket_\mu^{\alpha, 1/2} \leq C\mu^5.$$

We finish the proof by replacing $\frac{1}{8}\delta(m)$ by $\delta(m)$. \square

Lemma 3.17. *For $m \in \mathcal{U}$ and $\alpha > 1/2$, the change of variable τ defined on the Proposition 3.15 satisfies:*

$$\tau \left(\mathfrak{F}_\rho(\nu, \frac{\sigma}{2}, \frac{\mu}{2}, \alpha) \right) \subset \mathfrak{F}_\rho(\nu, \sigma, \mu, \alpha),$$

for $0 < \sigma \leq 1$, $0 < \mu \leq 1$ and $\nu \leq 4\mu^{-2}e^{-\sigma}\delta^2(m)$.

Proof. Consider $m \in \mathcal{U} \subset [1, 2]$ and $\alpha > 1/2$, the change of variable τ satisfies:

$$\|\tau(\zeta) - \zeta\|_\alpha \leq C(m)\|\zeta\|_\alpha^3, \quad \forall \zeta \in \mathcal{O}_{\delta(m)}(Y_\alpha).$$

Recall that

$$\mathfrak{F}_\rho(\nu, \sigma, \mu, \alpha) = \left\{ (I, \theta, \xi, \eta) \mid |I - \nu\rho| < \nu\mu^2, |Im\theta| < \sigma, \|\zeta^\mathcal{L}\|_\alpha < \nu^{1/2}\mu \right\}.$$

Let $\tilde{\zeta} = \tau(\zeta)$. Then, for $\nu \leq 4\mu^{-2}e^{-\sigma}\delta^2(m)$, we have:

$$(3.14) \quad \|\tilde{\zeta} - \zeta\|_\alpha \leq C'(m)\mu^3 e^{\frac{3\delta}{2}} \nu^{\frac{3}{2}},$$

where $C'(m)$ is a multiple constant of $C(m)$. Using the previous estimate, let us prove that:

$$\tau \left(\mathfrak{F}_\rho(\nu, \frac{\sigma}{2}, \frac{\mu}{2}, \alpha) \right) \subset \mathfrak{F}_\rho(\nu, \sigma, \mu, \alpha).$$

- On \mathcal{L} , we have:

$$\|\tilde{\zeta}\|_\alpha \leq \|\zeta\|_\alpha + C'(m)\mu^3 e^{\frac{3\delta}{2}} \nu^{\frac{3}{2}} < \frac{1}{2}\nu^{\frac{1}{2}} + C'(m)\mu^3 e^{\frac{3\delta}{2}} \nu^{\frac{3}{2}} < \nu\mu.$$

- For $a \in \mathcal{A}$, we have:

$$\begin{aligned} |\tilde{I}_a - \nu\rho_a| &\leq |I_a - \nu\rho_a| + |\tilde{I}_a - I_a| \\ &< \frac{1}{4}\nu\mu^2 + |\tilde{\xi}_a\tilde{\eta}_a - \xi_a\eta_a| \\ &< \frac{1}{4}\nu\mu^2 + |\tilde{\xi}_a||\tilde{\eta}_a - \eta_a| + |\eta_a||\tilde{\xi}_a - \xi_a|. \end{aligned}$$

As $\zeta \in \mathfrak{F}_\rho(\nu, \frac{\sigma}{2}, \frac{\mu}{2}, \alpha)$, we have

$$|\eta_a| < \left(\frac{1}{2}\mu + \sqrt{2}\right)e^{\frac{\sigma}{2}}\nu^{\frac{1}{2}}.$$

Using estimate (3.14), we have:

$$\begin{aligned} |\tilde{\xi}_a| &< \left(\frac{1}{2}\mu + \sqrt{2}\right)e^{\frac{\sigma}{2}}\nu^{\frac{1}{2}} + C'(m)\mu^3 e^{\frac{3\delta}{2}} \nu^{\frac{3}{2}}, \\ |\tilde{\eta}_a - \eta_a| + |\tilde{\xi}_a - \xi_a| &< C'(m)\mu^3 e^{\frac{3\delta}{2}} \nu^{\frac{3}{2}}. \end{aligned}$$

So

$$|\tilde{I}_a - \nu\rho_a| < \nu\mu^2.$$

- It remains to verify that $|Im(\tilde{\theta}_a)| < \sigma$ for $a \in \mathcal{A}$. On the one hand we have:

$$|\tilde{\xi}_a| = \left| \sqrt{\tilde{I}_a} e^{i\tilde{\theta}_a} \right| \leq \sqrt{\tilde{I}_a} e^{|Im(\tilde{\theta}_a)|} < \nu^{\frac{1}{2}}(\mu + \sqrt{2})e^{|Im(\tilde{\theta}_a)|}.$$

On the other hand, using estimate (3.14), we have:

$$|\tilde{\xi}_a| < \left(\frac{1}{2}\mu + \sqrt{2}\right)e^{\frac{\sigma}{2}}\nu^{\frac{1}{2}} + C'(m)\mu^3 e^{\frac{3\delta}{2}} \nu^{\frac{3}{2}}.$$

So

$$e^{|Im(\tilde{\theta}_a)|} \leq \frac{\mu + 2\sqrt{2}}{2(\mu + \sqrt{2})} e^{\frac{\sigma}{2}} + \nu \frac{C'(m)\mu^3}{\mu + \sqrt{2}} e^{\frac{3\delta}{2}},$$

and we get that $|Im(\tilde{\theta})| \leq \sigma$. \square

3.4. Normal form on admissible sets. We recall that Z_4 is given by:

$$Z_4 = 3/4\pi \sum_{(i,j,k,l) \in \mathcal{J}_2 \cap \mathcal{R}_2} \frac{\xi_i \xi_j \eta_k \eta_l}{\lambda_i \lambda_j},$$

where

$$\mathcal{J}_2 = \{(i, j, k, l) \in \mathcal{J} \mid \#\{i, j, k, l\} \cap \mathcal{A} \geq 2\}.$$

We note that Z_4 contains integrable terms formed by the monomials of the form $\xi_i \xi_j \eta_k \eta_l = I_i I_j$. Those monomials depend only on actions defined by $I_n = \xi_n \eta_n$ for $n \in \mathbb{Z}$. We denote those terms by Z_4^+ and $Z_4^- = Z_4 - Z_4^+$. After straightforward computations, we obtain that:

$$(3.15) \quad Z_4^+ = \frac{3}{4\pi} \sum_{l \in \mathcal{A}, k \in \mathbb{Z}} \frac{4 - 3\delta_{l,k}}{\lambda_l \lambda_k} I_l I_k.$$

Concerning Z_4^- , we have $Z_4^- = \sum_{0 \leq r \leq 4} Z_4^{-r}$ where $r = \text{Card}(\{i, j, k, l\} \cap \mathcal{A})$. Using the definition of \mathcal{J}_2 yields $Z_4^{-0} = Z_4^{-1} = 0$.

Lemma 3.18. *Assume that \mathcal{A} is an admissible set. Then, for $m \in \mathcal{U}$, we have*

$$Z_4^{-4} = 0.$$

Proof. Consider $\mathcal{F} = \{(i, j, k, l) \in \mathcal{J}_2 \cap \mathcal{R}_2 \cap \mathcal{A}\}$. Then, for $(i, j, k, l) \in \mathcal{F}$ and $m \in \mathcal{U}$,

- ★ $i + j = k + l$ using the definition of \mathcal{J} ,
- ★ $\{|i|, |j|\} = \{|k|, |l|\}$ using Lemma 3.10,
- ★ $\{i, j\} \neq \{k, l\}$ since \mathcal{A} is an admissible set.

So $\mathcal{F} = \emptyset$ and $Z_4^{-4} = 0$. □

Lemma 3.19. *Assume that \mathcal{A} is an admissible set. Then for $m \in \mathcal{U}$*

$$Z_4^{-3} = 0.$$

Proof. Suppose that there is $(i, j, k, l) \in \mathcal{J}_2 \cap \mathcal{R}_2$ such that $\#\{i, j, k, l\} \cap \mathcal{A} = 3$. Without loss of generality, we can suppose that $i, j, k \in \mathcal{A}$ and $l \in \mathcal{L}$. Due to Lemma 3.10, we have $|i| = |k|$ or $|j| = |k|$. Moreover, since \mathcal{A} is an admissible set, we have $i = k$ or $j = k$. Suppose that $i = k$. Since $i + j = k + l$, we have $j = l$ and $l \in \mathcal{A}$, which contradicts the fact that \mathcal{A} is an admissible set. So $Z_4^{-3} = 0$. □

Lemma 3.20. *Assume that \mathcal{A} is an admissible set. Then, for any $m \in \mathcal{U}$,*

$$Z_4^{-2} = 0.$$

Proof. Consider $\mathfrak{N} = \{(i, j, k, l) \in \mathcal{J}_2 \cap \mathcal{R}_2 \mid \#\{i, j, k, l\} \cap \mathcal{A} = 2\}$.

- ★ Assume that $i, j \in \mathcal{A}$ and $k, l \in \mathcal{L}$. Then, by Lemma 3.10, we have $i = -k$ and $j = -l$ or $i = -l$ and $j = -k$. Without loss of generality, we can suppose that $i = -k$ and $j = -l$. We have $(i, j, k, l) \in \mathcal{J}$, so $i + j = k + l = -k - l$ and $i = -j$. It contradicts the fact that \mathcal{A} is an admissible set. The case where $k, l \in \mathcal{A}$ and $i, j \in \mathcal{L}$ is treated in the same way.
- ★ Assume now that $i, l \in \mathcal{A}$ and $j, k \in \mathcal{L}$. By Lemma 3.10, we have $|i| = |k|$ and $|j| = |l|$ or $i = l$ and $|j| = |k|$. Let us first consider the case where $i = l$ and $|j| = |k|$. since $i + j = k + l$, we have $j = k$. So the monomial $\xi_i \xi_j \eta_k \eta_l$ will be in Z_4^+ , which leads to a contradiction. Consider now the case where $|i| = |k|$ and $|j| = |l|$. Since \mathcal{A} is an admissible set, we have $i = -k$ and $l = -j$. As $i + j = k + l$, necessarily $i = l$ and $j = k$, which lead to the previous case. The cases where $k, l \in \mathcal{A}$ and $i, j \in \mathcal{L}$ or $i, k \in \mathcal{A}$ and $j, l \in \mathcal{L}$ are treated in the same way.

Thus, we deduce that $\mathfrak{N} = \emptyset$ and $Z_4^{-2} = 0$. □

3.5. Action-angle variables. As in (3.3), we pass from variables (ξ, η) to $(I, \theta, \zeta^{\mathcal{L}})$, where $I = (I_a, a \in \mathcal{A})$, $\theta = (\theta_a, a \in \mathcal{A})$ and $\zeta^{\mathcal{L}} = (\zeta_s, s \in \mathcal{L})$. We recall that, for $a \in \mathcal{A}$, action-angle variables I and θ are given by:

$$\xi_a = \sqrt{I_a} e^{i\theta_a}, \quad \eta_a = \sqrt{I_a} e^{-i\theta_a}.$$

In these new variables, the symplectic form $-id\xi \wedge d\eta$ becomes:

$$(3.16) \quad - \sum_{a \in \mathcal{A}} dI_a \wedge d\theta_a - i \sum_{s \in \mathcal{L}} d\xi_s \wedge d\eta_s.$$

Moreover I is of order 2, θ is of order zero, ξ and η are of order 1.

Using expression (3.15) Z_4^+ and the lemmas 3.18-3.20, the Hamiltonian 3.9 becomes:

$$\begin{aligned} H \circ \tau &= \sum_{a \in \mathcal{A}} \omega_a I_a + \frac{3}{4\pi} \sum_{l, a \in \mathcal{A}} \frac{4 - 3\delta_{l,a}}{\lambda_l \lambda_a} I_l I_a + \sum_{s \in \mathcal{L}} \lambda_s \xi_s \eta_s + \frac{3}{\pi} \sum_{l \in \mathcal{A}, s \in \mathcal{L}} \frac{1}{\lambda_l \lambda_s} I_l \xi_s \eta_s \\ &+ Q_4 + R_5 \circ \tau + R_6. \end{aligned}$$

The first line contains the integrable terms. The second one contains:

- Q_4 , of order 4 and at least of order 3 in the direction of \mathcal{L} ;
- R_6 , which comes from the Birkhoff normal form and of order 6;
- $R_5 \circ \tau$, which comes from term of order 5 of the nonlinearity (1.2).

The Hamiltonian $H \circ \tau$ depends on variables $(I, \theta, \zeta^{\mathcal{L}})$. For the rest of the paper, we will drop the multi-index \mathcal{L} (i.e. $\zeta^{\mathcal{L}}$ will be replaced by ζ).

3.6. Rescaling the variables. We want to study the Hamiltonian H_1 . To do this, we will rescale the variables (I, θ, ξ, η) by considering the following change of variables:

$$(3.17) \quad \chi_\rho : (\tilde{r}, \tilde{\theta}, \tilde{\xi}, \tilde{\eta}) \mapsto (I, \theta, \xi, \eta),$$

where

$$\begin{aligned} I &= \nu\rho + \nu\tilde{r}, & \theta &= \tilde{\theta}, \\ \xi &= \nu^{1/2}\tilde{\xi}, & \eta &= \nu^{1/2}\tilde{\eta}. \end{aligned}$$

We have

$$\chi_\rho : \mathcal{O}^\alpha \left(\frac{\sigma}{2}, \frac{\mu}{2} \right) \rightarrow \mathfrak{T}_\rho \left(\nu, \frac{\sigma}{2}, \frac{\mu}{2}, \alpha \right).$$

In these new variables, the symplectic form (3.16) becomes

$$(3.18) \quad -\nu \sum_{a \in \mathcal{A}} d\tilde{r}_a \wedge d\tilde{\theta}_a - i\nu \sum_{s \in \mathcal{L}} d\tilde{\xi}_s \wedge d\tilde{\eta}_s.$$

Consider

$$\check{\Psi} = \check{\Psi}_\rho = \tau \circ \chi_\rho.$$

The change of variables χ_ρ is linear on ρ . We can extend Φ holomorphically on

$$\mathcal{D}_c = \left\{ \rho \in \mathbb{C}^{\mathcal{A}} \mid \left| \rho_j - \frac{3}{2} \right| \leq c, 1 \leq j \leq \text{Card}(\mathcal{A}) \right\}.$$

To simplify notations, we will drop the tilde. In these rescaled variables, the Hamiltonian H becomes up to a constant:

$$\begin{aligned} H \circ \check{\Psi} &= \nu \sum_{a \in \mathcal{A}} \omega_a r_a + \nu^2 \frac{3}{2\pi} \sum_{a, l \in \mathcal{A}} \frac{4 - 3\delta_{a, l}}{\lambda_a \lambda_l} \rho_l r_a \\ &+ \nu \sum_{s \in \mathcal{L}} \lambda_s \xi_s \eta_s + \nu^2 \frac{3}{\pi} \sum_{\substack{l \in \mathcal{A} \\ s \in \mathcal{L}}} \frac{1}{\lambda_l \lambda_s} \rho_l \xi_s \eta_s \\ &+ \nu^2 \frac{3}{4\pi} \sum_{a, l \in \mathcal{A}} \frac{4 - 3\delta_{a, l}}{\lambda_a \lambda_l} r_l r_a + \nu^2 \frac{3}{\pi} \sum_{\substack{l \in \mathcal{A} \\ s \in \mathcal{L}}} \frac{1}{\lambda_l \lambda_s} r_l \xi_s \eta_s \\ &+ (Q_4 + R_5 \circ \tau + R_6) \circ \chi_\rho. \end{aligned}$$

By dividing by ν , we can rewrite the previous Hamiltonian under the following form:

$$(3.19) \quad \nu^{-1} H \circ \check{\Psi} = h_0 + f,$$

where $h_0 \equiv h_0(r, \xi, \eta; \rho, \nu)$ and contains linear terms in r , quadratic terms in ξ, η and independent from the angle variable θ . The new perturbation f contains all the rest and depends on the angle variable. More precisely

$$(3.20) \quad f \equiv \nu \frac{3}{4\pi} \sum_{a, l \in \mathcal{A}} \frac{4 - 3\delta_{a, l}}{\lambda_a \lambda_l} r_l r_a + \nu \frac{3}{\pi} \sum_{\substack{l \in \mathcal{A} \\ s \in \mathcal{L}}} \frac{1}{\lambda_l \lambda_s} r_l \xi_s \eta_s + \nu^{-1} (Q_4 + R_5 \circ \tau + R_6) \circ \chi_\rho.$$

We can rewrite the new Hamiltonian h under the following form

$$(3.21) \quad h_0 = \Omega \cdot r + \sum_{a \in \mathcal{L}} \Lambda_a \xi_a \eta_a,$$

where $\Omega = (\Omega_k)_{k \in \mathcal{A}}$, and

$$(3.22) \quad \Omega_k = \Omega_k(\rho, \nu) = \omega_k + \nu \tilde{\omega}_k = \omega_k + \nu \frac{3}{2\pi} \frac{1}{\lambda_k} \sum_{l \in \mathcal{A}} \frac{4 - 3\delta_{l, k}}{\lambda_l} \rho_l,$$

$$(3.23) \quad \Lambda_a = \Lambda_a(\rho, \nu) = \lambda_a + \nu \tilde{\lambda}_a = \lambda_a + \nu \frac{3}{\pi} \frac{1}{\lambda_a} \sum_{l \in \mathcal{A}} \frac{\rho_l}{\lambda_l},$$

for $\rho \in \mathcal{D}_c$. We remark that for the internal frequencies, we have:

$$(3.24) \quad \tilde{\omega}_k = \sum_{l \in \mathcal{A}} M_k^l \rho_l, \quad M_k^l = \frac{3}{2\pi} \frac{4 - 3\delta_{l,k}}{\lambda_k \lambda_l}.$$

M is an invertible matrix, since

$$\det M = \left(\frac{3}{2\pi}\right)^n \left(\prod_{l \in \mathcal{A}} \lambda_l^{-2}\right) (4n - 3)(-3)^{n-1}, \quad n = \text{Card}(\mathcal{A}).$$

Recall that $\rho \in [1, 2]^{\mathcal{A}}$ and $\mathcal{A} \subset \{a \in \mathbb{Z}^d \mid |a| \leq N\}$. For $\rho \in \mathcal{D}_c$, we have

$$|\Omega(\rho) - \omega(\rho)| \leq \nu \left| \frac{3}{2\pi} \sum_{l \in \mathcal{A}} \frac{4 - 3\delta_{l,k}}{\lambda_l} \rho_l \right|, \quad \text{and} \quad |\Lambda_a(\rho) - \lambda_a(\rho)| \leq \nu |a|^{-1} \left| \frac{3}{\pi} \sum_{l \in \mathcal{A}} \frac{\rho_l}{\lambda_l} \right|.$$

This proves estimations (3.6) and concludes the proof of the first and second points of Theorem 3.7.

We recall that $\zeta = (\zeta_a)_{a \in \mathcal{L}}$. The quadratic part of the Hamiltonian h is given by the following infinite matrix:

$$A(\rho, \nu) = \text{diag} \left(\begin{pmatrix} 0 & \Lambda_a(\rho, \nu) \\ \Lambda_a(\rho, \nu) & 0 \end{pmatrix}, a \in \mathcal{L} \right).$$

We can put the Hamiltonian h under the following form

$$h = \Omega(\rho, \nu) \cdot r + \frac{1}{2} \langle A(\rho, \nu) \zeta, \zeta \rangle.$$

The Hamiltonian operator is given by:

$$iJA(\rho, \mu) = \text{diag} \left(\begin{pmatrix} -i\Lambda_a(\rho, \nu) & 0 \\ 0 & i\Lambda_a(\rho, \nu) \end{pmatrix}, a \in \mathcal{L} \right).$$

The spectrum of the Hamiltonian operator is:

$$\sigma(iJA) = \{\pm i\Lambda_a(\rho, \nu), a \in \mathcal{L}\}.$$

Let us study now the perturbation (3.20). Due to Proposition 3.15, the perturbation f is real holomorphic and belongs to $\mathcal{T}^{\alpha, 1/2}(\mathcal{D}, \frac{\sigma}{2}, \frac{\mu}{2})$. Using estimations (3.10)-(3.12) and for $x = (r, \theta, \zeta) \in \mathcal{O}^\alpha(\frac{\sigma}{2}, \frac{\mu}{2})$, we have:

$$\begin{aligned} |f| &\leq C\nu\mu^4, \\ \|\nabla_\zeta f\|_\alpha &\leq C\nu\mu^3, \\ |\nabla_\zeta f|_{1/2} &\leq C\nu\mu^3, \\ |\nabla_\zeta^2 f|_{1/2} &\leq C\nu\mu^2, \end{aligned}$$

where C is a constant that depends on the admissible set \mathcal{A} , the mass m and the nonlinearity g . So we have:

$$\|f\|_{\frac{\sigma}{2}, \frac{\mu}{2}, \mathcal{D}}^{\alpha, 1/2} \leq C\nu\mu^4.$$

Let us study now the jet of the perturbation f . Recall that the jet function f^T is defined by:

$$f^T = f(\theta, 0, 0, \rho) + \nabla_r f(\theta, 0, 0, \rho)r + \langle \nabla_\zeta f(\theta, 0, 0, \rho), \zeta \rangle + \frac{1}{2} \langle \nabla_{\zeta\zeta}^2 f(\theta, 0, 0, \rho)\zeta, \zeta \rangle.$$

We look for the terms of (3.20) which can contribute to f^T . Clearly the first two terms do not contribute to f^T . The third term is indexed by \mathcal{J}_2^c and does not contribute to f^T . Let us now look at $R_5 \circ \tau$. According to 3.15, $R_5 \circ \tau$ is of order 5. Moreover $R_5 \circ \tau$ depends on the action variable (of order 2), on the angle θ , on ξ and η (of order 1). So $R_5 \circ \tau$ can contains terms like:

- $I^{5/2}$ will contribute to $f(\theta, 0, 0, \rho)$ and $\nabla_r f(\theta, 0, 0, \rho)$;
- $I^2 \xi, I^2 \eta$ will contribute to $\nabla_\xi f(\theta, 0, 0, \rho)$ or $\nabla_\eta f(\theta, 0, 0, \rho)$;
- $I^{3/2} \xi \eta, I^{3/2} \xi \xi, I^{3/2} \eta \eta$ will contribute to $\nabla_{\xi\eta}^2 f(\theta, 0, 0, \rho), \nabla_{\xi\xi}^2 f(\theta, 0, 0, \rho)$ or $\nabla_{\eta\eta}^2 f(\theta, 0, 0, \rho)$.

So $R_5 \circ \tau \circ \chi_\rho$ will contribute to f^T . A similar result holds for $R_6 \circ \chi_\rho$. We deduce that:

$$\|f^T\|_{\frac{\sigma}{2}, \frac{\mu}{2}, \mathcal{D}}^{\alpha, 1/2} \leq C\nu^{3/2}\mu^5.$$

To finish the proof of the third point of Theorem 3.7, we have to look at the derivative in ρ of the perturbation f and its jet f^T . Note that, according to (3.20), the dependence of the perturbation f on ρ comes from the change of variable χ_ρ via the relation $I = \nu\rho + \nu\tilde{r}$. We can extend f holomorphically on \mathcal{D}_c with the same estimates. Using a Cauchy estimate on \mathcal{D}_c , we obtain

$$\|\partial_\rho f\|_{\frac{\sigma}{2}, \frac{\mu}{2}, \mathcal{D}}^{\alpha, 1/2} \leq c^{-1} \|f\|_{\frac{\sigma}{2}, \frac{\mu}{2}, \mathcal{D}}^{\alpha, 1/2}, \quad \|\partial_\rho f^T\|_{\frac{\sigma}{2}, \frac{\mu}{2}, \mathcal{D}}^{\alpha, 1/2} \leq c^{-1} \|f^T\|_{\frac{\sigma}{2}, \frac{\mu}{2}, \mathcal{D}}^{\alpha, 1/2}.$$

Then $\partial_\rho f$ and $\partial_\rho f^T$ satisfy the same estimates as f and f^T .

3.7. Real variables. In the normal form (3.5), the quadratic part is expressed in complex variables. However, the KAM theorem is expressed in real variables. In order to remedy to this, we consider the following symplectic change of variable:

$$\Upsilon(r, \theta, \zeta) = (r, \theta, \check{\zeta}),$$

where $\check{\zeta} = (p, q) = (\check{\zeta}_s = (p_s, q_s), s \in \mathcal{L})$. Variables p and q are given by:

$$\xi_s = \frac{1}{\sqrt{2}}(p_s + iq_s), \quad \eta_s = \frac{1}{\sqrt{2}}(p_s - iq_s).$$

Under the hypotheses of Theorem 3.7, consider $\Psi_\rho = \check{\Psi}_\rho \circ \Upsilon$. This change of variables is real holomorphic and satisfies:

$$\Psi_\rho : \mathcal{O}^\alpha \left(\frac{\sigma}{2}, \frac{\mu}{2} \right) \rightarrow \mathfrak{X}_\rho(\nu, \sigma, \mu, \alpha).$$

It transforms the symplectic form $-id\xi \wedge d\eta$ into $-dr \wedge d\theta - dp \wedge dq$. In the new variables, the normal form (3.5) becomes:

$$\begin{aligned} h(r, \theta, \check{\zeta}; \rho) &= H \circ \Psi_\rho \\ (3.25) \quad &= \Omega(\rho) \cdot r + \frac{1}{2} \sum_{a \in \mathcal{L}} \Lambda_a(\rho)(p_a^2 + q_a^2) + f(r, \theta, \check{\zeta}; \rho) \\ &= \Omega(\rho) \cdot r + \frac{1}{2} \sum_{a \in \mathcal{L}} \langle \check{A}(\rho, \nu) \check{\zeta}, \check{\zeta} \rangle + f(r, \theta, \check{\zeta}; \rho), \end{aligned}$$

where

$$\check{A}(\rho, \nu) = \text{diag} \left(\left(\begin{array}{cc} \Lambda_a(\rho, \nu) & 0 \\ 0 & \Lambda_a(\rho, \nu) \end{array} \right), a \in \mathcal{L} \right).$$

In these new variables, the perturbation f satisfies the estimates of the third point of Theorem 3.7.

4. KAM FOR THE WAVE EQUATION

4.1. Abstract KAM theorem. In this section we state a KAM theorem and adapt the notations for the cubic nonlinear wave equation on the circle.

Consider a real Hamiltonian h_ρ on normal form and depends on a parameter ρ , given by:

$$(4.1) \quad h(r, p, q; \rho) = \Omega(\rho) \cdot r + \frac{1}{2} \sum_{a \in \mathcal{L}} \Lambda_a(\rho) (p_a^2 + q_a^2),$$

with

- $\rho \in \mathcal{D}$, a compact set of \mathbb{R}^p ;
- $\Omega : \mathcal{D} \rightarrow \mathbb{R}^n$ a C^1 internal frequency vector;
- \mathcal{L} a set of \mathbb{Z} ;
- for $a \in \mathcal{L}$, Λ_a an external frequency of class C^1 on \mathcal{D} .

The internal frequencies Ω and the external frequencies Λ satisfy certain hypotheses which will be stated in the following paragraph. Let us fix two parameters $0 < \delta_0 \leq \delta \leq 1$ and consider \mathcal{A}^- a finite set of \mathcal{L} . The set $\mathcal{L} \setminus \mathcal{A}^-$ shall be denoted by \mathcal{L}^∞ .

Hypothesis A1: Separation condition. Assume that, for all $\rho \in \mathcal{D}$, we have:

- ★ for all $a \in \mathcal{L}$,

$$(4.2) \quad \Lambda_a(\rho) \geq c_0 \langle a \rangle;$$

- ★ for all $a, b \in \mathcal{L}$ and $|a| \neq |b|$, we have

$$(4.3) \quad |\Lambda_a(\rho) - \Lambda_b(\rho)| \geq c_1 ||a| - |b||.$$

Hypothesis A2: Transversality condition. Assume that for all $\Omega' \in \mathcal{C}^1(\mathcal{D}, \mathbb{R}^n)$ that satisfies

$$|\Omega - \Omega'|_{\mathcal{C}^1(\mathcal{D})} < \delta_0,$$

for all $k \in \mathbb{Z}^n$, there exists a unit vector $z_k \in \mathbb{R}^p$, and all $a, b \in \mathcal{L}$ with $|a| > |b|$ the following holds:

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$$|k \cdot \Omega'(\rho)| \geq \delta, \quad \forall \rho \in \mathcal{D},$$

or

$$\langle \partial_\rho(k \cdot \Omega'(\rho)), z_k \rangle \geq \delta \quad \forall \rho \in \mathcal{D};$$

where $k \neq 0$

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$$|k \cdot \Omega'(\rho) \pm \Lambda_a(\rho)| \geq \delta \langle a \rangle, \quad \forall \rho \in \mathcal{D},$$

or

$$\langle \partial_\rho(k \cdot \Omega'(\rho) \pm \Lambda_a(\rho)), z_k \rangle \geq \delta \quad \forall \rho \in \mathcal{D};$$

★

$$|k \cdot \Omega'(\rho) + \Lambda_a(\rho) + \Lambda_b(\rho)| \geq \delta(\langle a \rangle + \langle b \rangle), \quad \forall \rho \in \mathcal{D},$$

or

$$\langle \partial_\rho(k \cdot \Omega'(\rho) + \Lambda_a(\rho) + \Lambda_b(\rho)), z_k \rangle \geq \delta \quad \forall \rho \in \mathcal{D};$$

★

$$|k \cdot \Omega'(\rho) + \Lambda_a(\rho) - \Lambda_b(\rho)| \geq \delta(1 + ||a| - |b||), \quad \forall \rho \in \mathcal{D},$$

or

$$\langle \partial_\rho(k \cdot \Omega'(\rho) + \Lambda_a(\rho) - \Lambda_b(\rho)), z_k \rangle \geq \delta \quad \forall \rho \in \mathcal{D};$$

Hypothesis A3: Second Melnikov condition. Assume that for all $\Omega' \in \mathcal{C}^1(\mathcal{D}, \mathbb{R}^n)$ that satisfies

$$|\Omega - \Omega'|_{\mathcal{C}^1(\mathcal{D})} < \delta_0,$$

the following holds:

for each $0 < \kappa < \delta$ and $N > 1$ there exists a closed set $\mathcal{D}' \subset \mathcal{D}$ that satisfies

$$(4.4) \quad \text{mes}(\mathcal{D} \setminus \mathcal{D}') \leq C(\delta^{-1}\kappa)^\tau N^\iota;$$

for some $\tau, \iota > 0$, such that for all $\rho \in \mathcal{D}'$, all $0 < |k| < N$ and all $a, b \in \mathcal{L}$ with $|a| \neq |b|$ we have:

$$(4.5) \quad |\Omega'(\rho) \cdot k + \Lambda_a(\rho) - \Lambda_b(\rho)| \geq \kappa(1 + ||a| - |b||).$$

We denote $A_0 = \text{diag}\{\Lambda_a I_2, a \in \mathcal{L}\}$. Now we are able to state the abstract KAM theorem proved in [6]:

Theorem 4.1. *Assume that h is a Hamiltonian given by (4.1) and satisfies hypotheses A1, A2 and A3 for fixed δ and δ_0 and all $\rho \in \mathcal{D}$. Fix $\alpha, \beta > 0$ and $0 < \sigma, \mu \leq 1$. Then there is ε_0 depending on $\alpha, \beta, \sigma, n, \mu, |\omega_0|_{\mathcal{C}^1(\mathcal{D})}$ and $|A_0|_{\beta, \mathcal{C}^1(\mathcal{D})}$ such that, if $\partial_\rho^j f \in \mathcal{T}^{\alpha, \beta}(\mathcal{D}, \sigma, \mu)$ for $j = 0, 1$, if*

$$\|f^T\|_{\sigma, \mu, \mathcal{D}}^{\alpha, \beta, \kappa} = \varepsilon < \min(\varepsilon_0, \frac{1}{8}\delta_0) \text{ and } \|f\|_{\sigma, \mu, \mathcal{D}}^{\alpha, \beta, \kappa} = O(\varepsilon^\tau),$$

for $0 < \tau < 1$, then there is a Borel set $\mathcal{D}' \subset \mathcal{D}$ with $\text{mes}(\mathcal{D} \setminus \mathcal{D}') \leq c\varepsilon^\gamma$ such that for all $\rho \in \mathcal{D}'$:

- there is a symplectic analytical change of variable

$$\Phi = \Phi_\rho : \mathcal{O}^\alpha\left(\frac{\sigma}{2}, \frac{\mu}{2}\right) \rightarrow \mathcal{O}^\alpha(\sigma, \mu)$$

- there is a new internal frequency vector $\tilde{\Omega}(\rho) \in \mathbb{R}^n$, a matrix $A \in \mathcal{M}_\beta$ and a perturbation $\tilde{f} \in \mathcal{T}^{\alpha, \beta}(\mathcal{D}', \sigma/2, \mu/2)$ such that

$$(h_\rho + f) \circ \Phi = \tilde{\Omega}(\rho) \cdot r + \frac{1}{2}\langle \zeta, A(\rho)\zeta \rangle + \tilde{f}(\theta, r, \zeta; \rho),$$

where $A : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{M}_{2 \times 2}(\mathbb{R})$ is a block diagonal symmetric infinite matrix in \mathcal{M}_β (ie $A_{[a]}^{[b]} = 0$ if $[a] \neq [b]$). Moreover $\partial_r \tilde{f} = \partial_\zeta \tilde{f} = \partial_{\zeta \zeta}^2 \tilde{f} = 0$ for $r = \zeta = 0$. The change of variables $\Phi = (\Phi_\theta, \Phi_r, \Phi_\zeta)$ is close to identity, and for all $x \in \mathcal{O}^\alpha(\frac{\sigma}{2}, \frac{\mu}{2})$ and all $\rho \in \mathcal{D}'$, we have:

$$(4.6) \quad \|\Phi - Id\|_\alpha \leq C\varepsilon^{4/5}.$$

For all $\rho \in \mathcal{D}'$, the new frequencies $\tilde{\omega}$ and matrix A satisfy

$$(4.7) \quad |A(\rho) - A_0(\rho)|_\alpha \leq C\varepsilon, \quad |\tilde{\Omega}(\rho) - \Omega(\rho)|_{\mathcal{C}^1(\mathcal{D}')} \leq C\varepsilon,$$

where C is a constant that depends on ε_0 .

4.2. Verification of the hypotheses of the KAM theorem.

4.2.1. *Non resonance.* In this section, we verify that the real normal form (3.25) satisfies the hypotheses of Theorem 4.1. We start by verifying the separation hypothesis A1, then the transversality condition A2 and finally the second Melnikov condition A3.

Lemma 4.2. *For all $\rho \in \mathcal{D}$, and all $a, b \in \mathcal{L}$, we have*

- (i) $\Lambda_a(\rho) \geq \langle a \rangle$;
- (ii) $|\Lambda_a(\rho) - \Lambda_b(\rho)| \geq \frac{1}{8}||a| - |b||$, with $|a| \neq |b|$.

Proof. Recall that, for $a \in \mathcal{L}$, the external frequencies are given by:

$$\Lambda_a(\rho) = \lambda_a + \nu \frac{3}{\pi} \frac{1}{\lambda_a} \sum_{l \in \mathcal{A}} \frac{\rho_l}{\lambda_l} = \lambda_a + \nu \frac{C}{\lambda_a}.$$

Estimation (i) is obvious. For (ii), we remark that, for ν small enough, we have:

$$C\nu \left| \frac{1}{\lambda_a} - \frac{1}{\lambda_b} \right| = \frac{C\nu}{\lambda_a \lambda_b} |\lambda_a - \lambda_b| \leq \frac{1}{2} |\lambda_a - \lambda_b|.$$

So

$$|\Lambda_a - \Lambda_b| \geq \frac{1}{2} |\lambda_a - \lambda_b| \geq \frac{1}{8} ||a| - |b||,$$

which concludes the proof of the lemma. \square

The non resonance hypothesis A2 will be verified in three steps. We begin by recalling the results obtained in the Propositions 2.8 and 2.10. For $\kappa = \nu^{1/2}$, we have the following lemma:

Lemma 4.3. *For $\gamma > 0$ small enough, $|k| \leq \nu^{-\gamma}$ and $(a, b) \in \mathcal{L}^2$ we have:*

$$|\omega \cdot k| \geq 2\nu^{1/2},$$

except when k is D_0 resonant.

$$|\omega \cdot k + \lambda_a| \geq 2\nu^{1/2} \langle a \rangle,$$

except when (k, a) is D_1 resonant.

$$|\omega \cdot k + \lambda_a + \lambda_b| \geq 2\nu^{1/2} (\langle a \rangle + \langle b \rangle),$$

except when (k, a, b) when D_2 resonant.

$$|\omega \cdot k + \lambda_a - \lambda_b| \geq 2\nu^{1/2} (1 + ||a| - |b||),$$

where $|a| \neq |b|$ and (k, a, b) is not D_3 resonant.

Remark 4.4. *In the previous lemma, we have applied the Propositions 2.8 and 2.10 with $\kappa = \nu^{1/2}$, $N = \nu^{-\gamma}$ and $m \in [1, 2] \setminus \mathcal{C}$. The Lebesgue measure of \mathcal{C} satisfies:*

$$\text{mes}(\mathcal{C}) \leq C\kappa^\tau N^\iota,$$

where $\tau = O(\frac{1}{n})$ and $\iota = O(n^2)$. With this choice of parameter, the Lebesgue measure of \mathcal{C} still small, if we assume that $\gamma < O(\frac{1}{n^3})$.

Now we will verify the transversality hypothesis A2 for k small. Recall that the internal frequencies are given by:

$$\Omega = \omega + \nu M\rho,$$

where M is the symmetric invertible matrix defined in (3.24). We denotes $C_{\mathcal{A}} = \|M^{-1}\|_2$.

Lemma 4.5. *For $\gamma > 0$ small enough, $k \in \mathbb{Z}^n$ with $|k| \leq \nu^{-\gamma}$, $(a, b) \in \mathcal{L}^2$, consider $\delta_0 = \frac{1}{2} C_{\mathcal{A}}^{-1} \nu$. Then for all $\Omega' \in \mathcal{C}^1(\mathcal{D}, \mathbb{R}^n)$ that satisfies*

$$|\Omega - \Omega'|_{\mathcal{C}^1(\mathcal{D})} < \delta_0,$$

and all $\rho \in \mathcal{D}$, we have

$$|\Omega' \cdot k| \geq \nu^{1/2},$$

except when k is D_0 resonant.

$$|\Omega' \cdot k + \Lambda_a| \geq \nu^{2/3} \langle a \rangle,$$

except when (k, a) is D_1 resonant.

$$|\Omega' \cdot k + \Lambda_a + \Lambda_b| \geq \nu^{2/3} (\langle a \rangle + \langle b \rangle),$$

except when (k, a, b) is D_2 resonant.

$$|\Omega' \cdot k + \Lambda_a - \Lambda_b| \geq \nu^{2/3} (1 + ||a| - |b||),$$

whith $|a| \neq |b|$ and (k, a, b) is not D_3 resonant.

Proof. Let $k \in \mathbb{Z}^n$ such that $|k| \leq \nu^{-\gamma}$ for $\gamma > 0$ small enough. We begin with the first estimate. We have

$$|\Omega' - \omega| \leq |\Omega' - \Omega| + |\Omega - \omega| \leq \frac{1}{2}C_{\mathcal{A}}^{-1}\nu + c\nu \leq C\nu.$$

So, for $a \in \mathcal{A}$

$$|\Omega' - \omega| \leq C\nu\langle a \rangle.$$

By the Cauchy-Schwarz inequality, for all $(a, b) \in \mathcal{L}$, we have:

$$\begin{aligned} |\Omega' \cdot k - \omega \cdot k| &\leq C\nu^{1-\gamma} \leq \nu^{1/2}. \\ |\Omega' \cdot k - \omega \cdot k| &\leq \nu^{1/2}\langle a \rangle. \\ |\Omega' \cdot k - \omega \cdot k| &\leq \nu^{1/2}(\langle a \rangle + \langle b \rangle). \\ |\Omega' \cdot k - \omega \cdot k| &\leq \nu^{1/2}(1 + ||a| - |b||), \quad |a| \neq |b|. \end{aligned}$$

To conclude the proof of the first case, we use the fact that:

$$|\Omega' \cdot k| \geq |\omega \cdot k| - |\Omega' \cdot k - \omega \cdot k| \geq 2\nu^{1/2} - \nu^{1/2} = \nu^{1/2}.$$

Let us now look at the second estimate. We note that for $a \in \mathcal{L}$, we have:

$$|\lambda_a - \Lambda_a| \leq \tilde{c}\nu\langle a \rangle.$$

So

$$\begin{aligned} |\Omega' \cdot k + \Lambda_a| &\geq |\Omega' \cdot k + \lambda_a| - |\lambda_a - \Lambda_a| \\ &\geq |\omega \cdot k + \lambda_a| - |\Omega' \cdot k - \omega \cdot k| - |\lambda_a - \Lambda_a| \\ &\geq 2\nu^{1/2}\langle a \rangle - \nu^{1/2}\langle a \rangle - \tilde{c}\nu\langle a \rangle \geq \nu^{2/3}\langle a \rangle. \end{aligned}$$

Consider now the third estimate. For $(a, b) \in \mathcal{L}^2$, we have:

$$\begin{aligned} |\Omega' \cdot k + \Lambda_a + \Lambda_b| &\geq |\Omega' \cdot k + \lambda_a + \lambda_b| - |\lambda_a - \Lambda_b| - |\lambda_b - \Lambda_a| \\ &\geq |\omega \cdot k + \lambda_a + \lambda_b| - |\Omega' \cdot k - \omega \cdot k| - |\lambda_a - \Lambda_a| - |\lambda_b - \Lambda_b| \\ &\geq (2\nu^{1/2} - \nu^{1/2} - 2\tilde{c}\nu)(\langle a \rangle + \langle b \rangle) \geq \nu^{2/3}(\langle a \rangle + \langle b \rangle). \end{aligned}$$

Let us now look at the last small divisor. Using (3.23), we remark that, for $(a, b) \in \mathcal{L}^2$ with $|a| \neq |b|$, we have:

$$\begin{aligned} |\Lambda_a - \Lambda_b - (\lambda_a - \lambda_b)| &\leq \tilde{c}\nu |\lambda_a^{-1} - \lambda_b^{-1}| \\ &= \tilde{c}\nu \frac{||a| + |b||}{(\lambda_a + \lambda_b)\lambda_a\lambda_b} ||a| - |b|| \\ &\leq \tilde{c}\nu(1 + ||a| - |b||). \end{aligned}$$

Which leads to

$$\begin{aligned} |\Omega' \cdot k + \Lambda_a - \Lambda_b| &\geq |\Omega' \cdot k + \lambda_a - \lambda_b| - |\Lambda_a - \Lambda_b - (\lambda_a - \lambda_b)| \\ &\geq |\omega \cdot k + \lambda_a - \lambda_b| - |\Omega' \cdot k - \omega \cdot k| - |\Lambda_a - \Lambda_b - (\lambda_a - \lambda_b)| \\ &\geq (2\nu^{1/2} - \nu^{1/2} - \tilde{c}\nu)(1 + ||a| - |b||) \geq \nu^{2/3}(1 + ||a| - |b||). \end{aligned}$$

The proof is thus achieved. \square

We have verified the non resonance hypotheses for $|k| \leq \nu^{-\gamma}$, $\delta_0 = \frac{1}{2}C_{\mathcal{A}}^{-1}\nu$ and $\delta = \nu^{2/3}$. For large k , i.e. $|k| > \nu^{-\gamma}$, we verify the separation conditions A2 on the derivatives in ρ of the small divisors. More precisely, we have:

Lemma 4.6. *For $\gamma > 0$ small enough, $k \in \mathbb{Z}^n$ with $|k| > \nu^{-\gamma}$, $(a, b) \in \mathcal{L}^2$, we consider $\delta_0 = \frac{1}{2}C_{\mathcal{A}}^{-1}\nu$. Then for all $\Omega' \in \mathcal{C}^1(\mathcal{D}, \mathbb{R}^n)$ that satisfies*

$$|\Omega - \Omega'|_{\mathcal{C}^1(\mathcal{D})} < \delta_0,$$

there exists a unit vector z_k , such that for all $\rho \in \mathcal{D}$ we have

- (i) $|\langle \partial_\rho(k \cdot \Omega'(\rho)), z_k \rangle| \geq C\nu^{1-\gamma} \geq \nu$,
- (ii) $|\langle \partial_\rho(k \cdot \Omega'(\rho) \pm \Lambda_a(\rho)), z_k \rangle| \geq \nu$ for all $a \in \mathcal{L}$,
- (iii) $|\langle \partial_\rho(k \cdot \Omega'(\rho) + \Lambda_a(\rho) + \Lambda_b(\rho)), z_k \rangle| \geq \nu$ for all $(a, b) \in \mathcal{L}^2$,
- (iv) $|\langle \partial_\rho(k \cdot \Omega'(\rho) + \Lambda_a(\rho) - \Lambda_b(\rho)), z_k \rangle| \geq \nu$ for all $(a, b) \in \mathcal{L}^2$.

The constant C depends on the admissible set \mathcal{A} .

Proof. Let us begin with (i). First of all, we remark that:

$$\langle \partial_\rho(k \cdot \Omega'(\rho)), z_k \rangle = \langle \partial_\rho(k \cdot \Omega(\rho)), z_k \rangle + \langle \partial_\rho(k \cdot \Omega'(\rho)) - k \cdot \Omega(\rho), z_k \rangle.$$

However

$$|\langle \partial_\rho(k \cdot \Omega'(\rho)) - k \cdot \Omega(\rho), z_k \rangle| \leq \frac{1}{2} |k| C_{\mathcal{A}}^{-1} \nu.$$

Recall that the matrix M is symmetric and $|k| > \nu^{-\gamma}$. Assume that $z_k = \frac{Mk}{|Mk|}$. Then, we have:

$$\begin{aligned} |\langle \partial_\rho(k \cdot \Omega'(\rho)), z_k \rangle| &\geq \nu \left| \langle Mk, \frac{Mk}{|Mk|} \rangle \right| - |\langle \partial_\rho(k \cdot \Omega'(\rho)) - k \cdot \Omega(\rho), z_k \rangle| \\ &= \nu |Mk| - \frac{1}{2} |k| C_{\mathcal{A}}^{-1} \nu \\ &\geq \nu \left(C_{\mathcal{A}}^{-1} - \frac{1}{2} C_{\mathcal{A}}^{-1} \right) |k| \\ &\geq \frac{1}{2} C_{\mathcal{A}}^{-1} \nu^{1-\gamma} \geq \nu. \end{aligned}$$

Let us now consider the second estimation (ii). For $a, l \in \mathcal{L}$, we have:

$$|\partial_{\rho_l} \Lambda_a(\rho)| = \left| \nu \frac{3}{\pi} \frac{1}{\lambda_a \lambda_l} \right| \leq c\nu.$$

Using the same unit vector z_k , we get:

$$\begin{aligned} |\langle \partial_\rho(k \cdot \Omega'(\rho) \pm \Lambda_a(\rho)), z_k \rangle| &\geq |\langle \partial_\rho(k \cdot \Omega'(\rho)), z_k \rangle| - |\partial_\rho(\Lambda_a(\rho))| |z_k| \\ &\geq \frac{1}{2} C_{\mathcal{A}}^{-1} \nu^{1-\gamma} - c'\nu \geq \nu. \end{aligned}$$

Applying the same principle for (iii) and (iv), we obtain:

$$\begin{aligned} |\langle \partial_\rho(k \cdot \Omega'(\rho) + \Lambda_a(\rho) \pm \Lambda_b(\rho)), z_k \rangle| &\geq |\langle \partial_\rho(k \cdot \Omega'(\rho)), z_k \rangle| - |\partial_\rho(\Lambda_a(\rho))| |z_k| \\ &\quad - |\partial_\rho(\Lambda_b(\rho))| |z_k| \\ &\geq \frac{1}{2} C_{\mathcal{A}}^{-1} \nu^{1-\gamma} - 2c'\nu \geq \nu, \end{aligned}$$

and the proof is thus concluded. \square

To finish the verification of the transversality condition, it remains to consider the cases where (k, a) is D_1 resonant and (k, a, b) is D_2 or D_3 resonant.

Lemma 4.7. *Let $k \in \mathbb{Z}^n$ and $(a, b) \in \mathcal{L}^2$. Consider $\delta_0 = \frac{1}{4} \nu \check{C}_{\mathcal{A}}$, where $\check{C}_{\mathcal{A}}$ is a constant that depends on the admissible set \mathcal{A} . Then for all $\Omega' \in \mathcal{C}^1(\mathcal{D}, \mathbb{R}^n)$ that satisfies*

$$|\Omega - \Omega'|_{\mathcal{C}^1(\mathcal{D})} < \delta_0,$$

and all $\rho \in \mathcal{D}$, we have:

$$|\Omega' \cdot k + \Lambda_a| \geq \check{C}_{\mathcal{A}} \nu \langle a \rangle,$$

if (k, a) is D_1 resonant;

$$|\Omega' \cdot k + \Lambda_a + \Lambda_b| \geq \check{C}_{\mathcal{A}} \nu (\langle a \rangle + \langle b \rangle),$$

if (k, a, b) is D_2 resonant;

$$|\Omega' \cdot k + \Lambda_a - \Lambda_b| \geq \check{C}_{\mathcal{A}} \nu (1 + ||a| - |b||),$$

where $|a| \neq |b|$ and (k, a, b) is D_3 resonant.

Proof. Consider $a \in \mathcal{A}$. From (3.22) and (3.23), we have

$$\tilde{\omega}_a - \tilde{\lambda}_a = \frac{3}{2\pi} \frac{1}{\lambda_a} \sum_{l \in \mathcal{A}} \frac{2 - 3\delta_{l,a}}{\lambda_l} \rho_l.$$

Assume that (k, a) is D_1 resonant. Then we have

$$\Omega' \cdot k + \Lambda_a = (\Omega'_a - \Omega_a) - \nu(\tilde{\omega}_a - \tilde{\lambda}_a).$$

We remark that there exists a constant C_1 , that depends on the admissible set \mathcal{A} , such that:

$$|\tilde{\omega}_a - \tilde{\lambda}_a| \geq \frac{C_1}{\lambda_a}.$$

Consider $\delta_0 \leq \frac{1}{2}\nu \frac{C_1}{\max(\mathcal{A})^2 + 2} := \frac{1}{2}\nu\tilde{C}_1$, then we have:

$$\begin{aligned} |\Omega' \cdot k + \Lambda_a| &\geq C_1\nu \frac{1}{\lambda_a} - \frac{1}{2}\tilde{C}_1\nu \\ &\geq \tilde{C}_1\nu\langle a \rangle - \frac{1}{2}\tilde{C}_1\nu \\ &\geq \frac{1}{2}\tilde{C}_1\nu\langle a \rangle. \end{aligned}$$

Let us consider now the case where (k, a, b) is D_2 resonant. There is a constant C_2 , that depends on the admissible set \mathcal{A} , such that for all $(a, b) \in \mathcal{A}^2$, we have:

$$\begin{aligned} |(\tilde{\omega}_a - \tilde{\lambda}_a) + (\tilde{\omega}_b - \tilde{\lambda}_b)| &\geq C_2\left(\frac{1}{\lambda_a} + \frac{1}{\lambda_b}\right) \\ &\geq \tilde{C}_2(\langle a \rangle + \langle b \rangle), \end{aligned}$$

where $\tilde{C}_2 := \frac{C_2}{\max(\mathcal{A})^2 + 2}$. So, if (k, a, b) is D_2 resonant and $\delta_0 \leq \frac{1}{4}\nu\tilde{C}_2$, then we have:

$$\begin{aligned} |\Omega' \cdot k + \Lambda_a + \Lambda_b| &= |(\Omega' - \Omega) \cdot k - \nu((\tilde{\omega}_a - \tilde{\lambda}_a) + (\tilde{\omega}_b - \tilde{\lambda}_b))| \\ &\geq \nu|(\tilde{\omega}_a - \tilde{\lambda}_a) + (\tilde{\omega}_b - \tilde{\lambda}_b)| - 2|\Omega' - \Omega| \\ &\geq \frac{1}{2}\tilde{C}_2\nu(\langle a \rangle + \langle b \rangle). \end{aligned}$$

It remains to look at the last small divisor in the case where (k, a, b) is D_3 resonant. We note that there exists a constant C_3 , which depends on the admissible set \mathcal{A} , such that for all $(a, b) \in \mathcal{A}^2$, we have:

$$\begin{aligned} |(\tilde{\omega}_a - \tilde{\lambda}_a) - (\tilde{\omega}_b - \tilde{\lambda}_b)| &\geq C_3 \left| \frac{1}{\lambda_a} - \frac{1}{\lambda_b} \right| \\ &\geq \tilde{C}_3(1 + ||a| - |b||), \end{aligned}$$

where $\tilde{C}_3 := \frac{C_3}{8(\max(\mathcal{A})^2 + 2)}$. So, if (k, a, b) is D_3 resonant and $\delta_0 \leq \frac{1}{4}\nu\tilde{C}_3$, then we have:

$$\begin{aligned} |\Omega' \cdot k + \Lambda_a - \Lambda_b| &= |k \cdot (\Omega' - \Omega) - \nu((\tilde{\omega}_a - \tilde{\lambda}_a) - (\tilde{\omega}_b - \tilde{\lambda}_b))| \\ &\geq \nu|(\tilde{\omega}_a - \tilde{\lambda}_a) - (\tilde{\omega}_b - \tilde{\lambda}_b)| - 2|\Omega' - \Omega| \\ &\geq \frac{1}{2}\tilde{C}_3\nu(1 + ||a| - |b||). \end{aligned}$$

We conclude the proof by choosing $\check{C}_{\mathcal{A}} = \frac{1}{2} \min(\tilde{C}_1, \tilde{C}_2, \tilde{C}_3)$. \square

The last hypothesis to verify to apply the KAM theorem is the second Melnikov condition. Recall that $n = \text{Card}(\mathcal{A})$.

Lemma 4.8. *For $\delta = \nu$, $\delta_0 \leq \delta$, $\tau = \frac{1}{3}$ and $\iota = n + \frac{3}{2} + \frac{2}{3\gamma}$, the second Melnikov condition is satisfied.*

Proof. Consider $\gamma > 0$ small enough and $N \geq 0$. If $N \leq \nu^{-\gamma}$, then, using Lemma 4.5, the second Melnikov condition is satisfied for all $\rho \in \mathcal{D}$. If $N > \nu^{-\gamma}$, then from Lemma 4.5 and for $|k| \leq \nu^{-\gamma}$ the second Melnikov condition is satisfied for all $\rho \in \mathcal{D}$. Assume now that $|k| > \nu^{-\gamma}$, then from Lemma 4.6 there is a unit vector z_k such that:

$$|\langle \partial_\rho(k \cdot \Omega'(\rho) + \Lambda_a(\rho) - \Lambda_b(\rho)), z_k \rangle| \geq \nu.$$

For $0 < \kappa < \nu$, consider the set

$$J(k, a, b) = \{\rho \in \mathcal{D} \mid |\Omega'(\rho) \cdot k + \Lambda_a(\rho) - \Lambda_b(\rho)| < \kappa\}.$$

Then we have

$$\text{mes } J(k, a, b) \leq C\kappa\nu^{-1},$$

where C is a constant that depends on \mathcal{D} . For $p \in \mathbb{Z}$ and $k \in \mathcal{Z}^n$, we consider the following set

$$W(k, p) = \{\rho \in \mathcal{D} \mid |\Omega' \cdot k + p| < 5\kappa^{1/3}\}.$$

By the first estimate (i) from Lemma 4.6, we have

$$\text{mes } W(k, p) \leq C\kappa^{1/3}\nu^{-1},$$

where C is a constant that depends on \mathcal{D} . Consider

$$W = \{\rho \in \mathcal{D} \mid |\Omega' \cdot k + p| < 5\kappa^{1/3}\}.$$

We remark that $|\Omega' \cdot k + p| < 5\kappa^{1/3}$ for $|k| < N$, so we have $|p| \leq C|k| < CN$. This leads to

$$\text{mes}(W) \leq \sum_{\substack{k \in \mathbb{Z}^n \\ |k| \leq N}} \sum_{\substack{p \in \mathbb{Z} \\ |p| < CN}} W(k, p) \leq CN^{n+1} \kappa^{1/3} \nu^{-1}.$$

For $a \in \mathcal{L}$, we recall that $\Lambda_a(\rho) = \lambda_a + \nu C(\rho) \lambda_a^{-1}$. Consider ν small enough, then for $\rho \in \mathcal{D}$, we have:

$$|\Lambda_a(\rho) - |a|| = \left| \lambda_a(\rho) - |a| + \nu \frac{C(\rho)}{\lambda_a} \right| = \left| \frac{m}{\lambda_a + |a|} + \nu \frac{C(\rho)}{\lambda_a} \right| \leq \frac{2}{|a|}.$$

If $|a| > |b| > \kappa^{-1/3}$, we obtain

$$|\Lambda_a(\rho) - \Lambda_b(\rho) - (|a| - |b|)| \leq \frac{4}{|b|} \leq 4\kappa^{1/3}.$$

So, if $\rho \in \mathcal{D} \setminus W$ and $|a| > |b| > \kappa^{-1/3}$, we obtain

$$\begin{aligned} |k \cdot \Omega'(\rho) + \Lambda_a(\rho) - \Lambda_b(\rho)| &\geq |k \cdot \Omega'(\rho) + (|a| - |b|)| - |\Lambda_a(\rho) - \Lambda_b(\rho) - (|a| - |b|)| \\ &\geq 5\kappa^{1/3} - 4\kappa^{1/3} = \kappa^{1/3}. \end{aligned}$$

It remains to look at the cases where $|a| \leq \kappa^{-1/3}$ or $|b| \leq \kappa^{-1/3}$. Then, there is $k \in \mathbb{Z}^n$ such that

$$|k \cdot \Omega'(\rho) + \Lambda_a(\rho) - \Lambda_b(\rho)| < 1,$$

for $\nu^{-\gamma} < |k| < N$. We remark that, in those cases, $||a| - |b|| \leq CN$. Consider the set:

$$\mathcal{Q} = \left\{ (a, b) \in \mathbb{Z}^2 \mid \min(|a|, |b|) \leq \kappa^{-1/3} \text{ et } ||a| - |b|| \leq CN \right\}.$$

We have

$$\text{Card}(\mathcal{Q}) \leq CN\kappa^{-2/3}.$$

Let

$$\mathcal{D}' = \mathcal{D} \setminus \left(W \bigcup_{\substack{|k| \leq N \\ (a, b) \in \mathcal{Q}}} (J(k, a, b)) \right).$$

Then for all $\rho \in \mathcal{D}'$ we have:

$$|k \cdot \Omega'(\rho) + \Lambda_a(\rho) - \Lambda_b(\rho)| \geq \kappa.$$

Moreover,

$$\begin{aligned} \text{mes}(\mathcal{D} \setminus \mathcal{D}') &\leq \text{mes}(W) + \sum_{\substack{k \in \mathbb{Z}^n \\ |k| \leq N}} \sum_{(a, b) \in \mathcal{Q}} \text{mes} J(k, a, b) \\ &\leq CN^{n+1} \kappa^{1/3} \nu^{-1} + CN^n N \kappa^{-2/3} \kappa \nu^{-1} \\ &\leq CN^{n+1} (\kappa \nu^{-1})^{1/3}. \end{aligned}$$

Recall that $N > \nu^{-\gamma}$. This leads to

$$\text{mes}(\mathcal{D} \setminus \mathcal{D}') \leq CN^{n+1+2/3\gamma} \nu^{-2/3} (\kappa \nu^{-1})^{1/3}.$$

Now it remains to show that for all $\rho \in \mathcal{D}'$ we have:

$$|\Omega'(\rho) \cdot k + \Lambda_a(\rho) - \Lambda_b(\rho)| \geq \kappa(1 + |a| - |b|).$$

We will prove this estimation in two steps. Assume at first that $||a| - |b|| \geq 16|\Omega' \cdot k|$. Then, using the second separation condition from Lemma 4.2, we have:

$$\begin{aligned} |\Omega'(\rho) \cdot k + \Lambda_a(\rho) - \Lambda_b(\rho)| &\geq |\Lambda_a - \Lambda_b| - |\Omega' \cdot k| \\ &\geq \frac{1}{16} ||a| - |b|| \geq \frac{1}{32} (1 + ||a| - |b||) \\ &\geq \kappa(1 + ||a| - |b||), \end{aligned}$$

for all $0 \leq \kappa \leq \frac{1}{32}$ and all $\rho \in \mathcal{D}$.

Assume now that $||a| - |b|| < 16|\Omega' \cdot k| < 16C_{\mathcal{A}}N$. Then for all $\rho \in \mathcal{D}'$, $0 < |k| < N$ and $a, b \in \mathcal{L}$ with $|a| \neq |b|$, we have:

$$\begin{aligned} |\Omega' \cdot k + \Lambda_a - \Lambda_b| &\geq \frac{\kappa}{1 + 16C_{\mathcal{A}}N} (1 + |a| - |b|) \\ &\geq \tilde{\kappa} (1 + |a| - |b|), \end{aligned}$$

where

$$\text{mes}(\mathcal{D} \setminus \mathcal{D}') \leq CN^{n+3/2+2/3\gamma} \left(\tilde{\kappa} \nu^{-7/5} \right)^{1/2},$$

which ends the proof and concludes the verification of the assumptions made on the frequencies. \square

4.2.2. *Application of KAM theorem 4.1.* Using lemmas 4.2-4.8, the separation conditions, the Transversality condition and the second Melnikov condition are satisfied for $\delta = \nu$ and $\delta_0 \leq \delta$. From Theorem 3.7, for $0 < \mu \leq 1$, we have:

$$(4.8) \quad \llbracket f^T \rrbracket_{\frac{\sigma}{2}, \frac{\mu}{2}, \mathcal{D}}^{\alpha, 1/2, \kappa} \leq 2C\nu^{3/2}.$$

Moreover we have:

$$\llbracket f \rrbracket_{\sigma, \mu, \mathcal{D}}^{\alpha, \gamma, \kappa} = O\left(\left(\llbracket f^T \rrbracket_{\frac{\sigma}{2}, \frac{\mu}{2}, \mathcal{D}}^{\alpha, 1/2, \kappa}\right)^{3/2}\right).$$

To use Theorem 4.1, we need that:

$$\llbracket f \rrbracket_{\sigma, \mu, \mathcal{D}}^{\alpha, \gamma, \kappa} \ll \delta = \nu^{7/5}$$

and

$$\llbracket f \rrbracket_{\sigma, \mu, \mathcal{D}}^{\alpha, \gamma, \kappa} = O\left(\left(\llbracket f^T \rrbracket_{\frac{\sigma}{2}, \frac{\mu}{2}, \mathcal{D}}^{\alpha, 1/2, \kappa}\right)^\tau\right), \text{ for } \tau \in]0, 1[.$$

These last two conditions are indeed verified. We can therefore apply the Theorem 4.1.

5. PROOF OF THEOREM 1.3

Now we have all the tools to prove Theorem 1.3. For $m \in \mathcal{U}$ and $\rho \in (\mathcal{D} \setminus \mathcal{D}')$, consider:

$$\Pi_\rho = \Psi_\rho \circ \Phi_\rho = \tau \circ \chi_\rho \circ \Upsilon \circ \Phi_\rho,$$

where

- τ is the Birkhoff change of variable constructed in the Proposition 3.15.
- χ_ρ is the rescaling defined in 3.17;
- Υ is the transition to real variables;
- Φ_ρ is the KAM change of variable from Theorem 4.1.

So Π_ρ is a real holomorphic symplectic change variable

$$\Pi_\rho : \mathcal{O}^\alpha\left(\frac{\sigma}{4}, \frac{\mu}{4}\right) \rightarrow \mathfrak{T}_\rho(\nu, \sigma, \mu, \alpha),$$

that transforms the perturbed Hamiltonian (1.5) into

$$H \circ \Pi_\rho = \omega'(\rho) \cdot r + \frac{1}{2} \langle \zeta, A(\rho)\zeta \rangle + \tilde{f}(\theta, r, \zeta; \rho).$$

where $A : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{M}_{2 \times 2}(\mathbb{R})$ is a block diagonal symmetric infinite matrix in \mathcal{M}_β (ie $A_{[a]}^{[b]} = 0$ if $[a] \neq [b]$). Moreover $\partial_r \tilde{f} = \partial_\zeta \tilde{f} = \partial_{\zeta^2}^2 \tilde{f} = 0$ for $r = \zeta = 0$. From (3.24) and (4.7), the internal frequencies are given by

$$\omega' = \omega + \nu M\rho + O(\nu^{3/2}).$$

For the following, let $I = \nu\rho$ and $\mathcal{D} = [\nu, 2\nu]^n$. Then, for $m \in \mathcal{U}$, there is a Borel set $\mathcal{D}' \subset [\nu, 2\nu]^n$ such that:

$$\text{mes}([\nu, 2\nu]^n \setminus \mathcal{D}') \leq \nu^{\gamma+n},$$

for $\gamma > 0$ and depends on n . For $X = (\theta, r, \zeta)$, we denotes $[X]_\alpha = \left(\sum_{a \in \mathcal{A}} |r_a e^{2i\theta_a}|\right)^{1/2} + \|\zeta\|_\alpha$. Let $X \in \mathbb{T}^n \times \{I\} \times \{0\}$. So $\chi_\rho \circ \Upsilon(X) \in \mathfrak{T}_\rho(\nu, [X]_\alpha, \alpha)$, and we have:

$$\text{dist}_\alpha(\chi_\rho \circ \Upsilon(X), X) \leq 2\nu^{1/2}[X]_\alpha \leq 4\nu^{3/2}.$$

Using (3.8), we obtain that

$$\text{dist}_\alpha(\tau \circ \chi_\rho \circ \Upsilon(X), X) \leq \nu^{3/2} + \text{dist}_\alpha(\chi_\rho \circ \Upsilon(X), X) \leq 5\nu^{3/2}.$$

Then, thanks to (4.6) and (4.8), we have

$$\text{dist}_\alpha(\Pi_\rho(X), X) \leq C\nu^{4/5},$$

where C is an absolute constant. For $m \in ([1, 2] \setminus \mathcal{U})$ and $I \in \mathcal{D}'$, consider $(\tilde{\theta}, \tilde{I}, \tilde{\zeta}) = \Pi^{-1}(X)$; and let

$$u(\tilde{\theta}, \tilde{I}, x) = \sum_{a \in \mathcal{A}} \sqrt{\tilde{I}_a} \frac{e^{-i\tilde{\theta}_a} \varphi_a(x) + e^{i\tilde{\theta}_a} \varphi_{-a}(x)}{\sqrt{2}\omega^{1/4}}.$$

The function $t \mapsto u(\tilde{\theta} + t\omega', \tilde{I}, x)$ is a quasi-periodic solution of the wave equation (1.1). Let $\zeta_{I, \theta} = (\xi_{I, \theta}, \eta_{I, \theta})$ where

$$\begin{aligned} (\xi_{I, \theta})_a &= \sqrt{\tilde{I}_a} e^{i\theta_a}, & (\eta_{I, \theta})_a &= \sqrt{\tilde{I}_a} e^{-i\theta_a}, & \text{if } a \in \mathcal{A}, \\ (\xi_{I, \theta})_s &= (\eta_{I, \theta})_s = 0, & & & \text{if } s \in \mathcal{L}. \end{aligned}$$

Then we have:

$$\sup_{\theta_0 \in \mathbb{T}^n} \| u(\theta_0, I, \cdot) - u_{I,m}(\theta_0, \cdot) \|_{H^\alpha} \leq \| \Pi_\rho(X) - \zeta_{I,\theta} \|_\alpha \leq C\nu^{4/5}.$$

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