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Camille Laurent-Gengoux, Sylvain Lavau, Thomas Strobl. The universal Lie infinity-algebroid of a singular foliation. *Documenta Mathematica*, 2020, 25, pp.1571-1652. 10.25537/dm.2020v25.1571-1652 . hal-01653761

HAL Id: hal-01653761

<https://hal.science/hal-01653761>

Submitted on 1 Dec 2017

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The universal Lie infinity-algebroid of a singular foliation

Camille Laurent-Gengoux ^{*1}, Sylvain Lavau ^{†2}, and Thomas Strobl ^{‡3}

¹Institut Elie Cartan de Lorraine, Université de Lorraine, Metz, France.

²CMUP, Universidade de Porto, Portugal.

³UMI CNRS-2924, Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, Brasil, and Institut Camille Jordan, Université de Lyon, France.

Abstract

We associate a Lie ∞ -algebroid to every resolution of a singular foliation, where we consider a singular foliation as a locally generated, \mathcal{O} -submodule of vector fields on the underlying manifold closed under Lie bracket. Here \mathcal{O} can be the ring of smooth, holomorphic, or real analytic functions. The choices entering the construction of this Lie ∞ -algebroid, including the chosen underlying resolution, are unique up to homotopy and, moreover, every other Lie ∞ -algebroid inducing the same foliation or any of its subfoliations factorizes through it in an up-to-homotopy unique manner. We thus call it the universal Lie ∞ -algebroid of the singular foliation. It can be chosen, locally, to be a Lie n -algebroid for real analytic or holomorphic singular foliations.

We show that this universal structure encodes several aspects of the geometry of the leaves of a singular foliation. In particular, it contains the holonomy algebroid and groupoid of a leaf in the sense of Androulidakis and Skandalis. But even more, each leaf carries an isotropy Lie ∞ -algebra structure that is unique up to isomorphism and that extends a minimal isotropy Lie algebra that can be associated to each leaf by higher brackets containing additional invariants of the foliation. As a byproduct, we construct an example of a foliation generated by r vector fields for which we show by these techniques that it cannot be generated by the image through the anchor map of a Lie algebroid of the minimal rank r .

Key words: holomorphic or smooth singular foliations, Lie infinity-algebras, Lie algebroids and dg manifolds, holonomy groupoids, higher structures.

AMS 2010 Classification: 37F75, 53C12, 18G10, 22A22

* camille.laurent-gengoux@univ-lorraine.fr

† sylvain.lavau@ens-lyon.fr

‡ thomas[AT]impa.br, strobl[AT]math.univ-lyon1.fr

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Introduction

Regular foliations, i.e. a partition of a manifold into embedded submanifolds of a given dimension, are familiar objects of interest in differential geometry, see e.g. [20]. According to the Frobenius theorem they are equivalent to involutive distributions.

Singular foliations, on the other hand, are much less understood while, at the same time, they appear much more frequently. Typical Lie group actions have orbits of different dimensions. Similarly, the symplectic leaves of a Poisson manifold change dimension whenever the rank of the bivector jumps. Both of these two classes of singular foliations are an example of what one obtains on the base manifold of a Lie algebroid. It is therefore natural to ask if *any* singular foliation arises in this way.

To make this question more precise, we first need to clarify, what we mean by a singular foliation. One way of viewing them would be a partition of the given manifold into embedded submanifolds of possibly different dimensions. While in the case of regular foliations, the description in terms of generating vector fields mentioned above is completely equivalent, here such a characterization contains more information. Consider, for example, vector fields on a line vanishing at the origin up to order k . While the corresponding partition of \mathbb{R} into leaves consists of \mathbb{R}_+ , \mathbb{R}_- , and the origin 0 , the generating module of functions is in addition invariantly characterized by the integer $k \in \mathbb{N}$. We will thus define a singular foliation as an involutive submodule $\mathcal{F} \subset \Gamma(TM)$, where as usual, involutivity means $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$.

Defined like this, it is, however, even not guaranteed that the vector fields \mathcal{F} generate a subdivision of M into leaves such that at each point \mathcal{F} evaluated at this point would agree with the tangent of the leaf containing this point. Also, if we do not restrict \mathcal{F} further, the answer to the question if it is generated by a Lie algebroid A is definitely negative: The image of A with respect to the anchor $\rho: A \rightarrow TM$ gives an involutive module $\mathcal{F} = \rho(\Gamma(A))$, but evidently this is locally finitely generated. Adding this additional condition on \mathcal{F} , namely to be locally finitely generated, Hermann's theorem establishes that M is indeed partitioned by immersed submanifolds, called leaves [21].

Thus in this paper, we define a singular foliation of a manifold M to be a locally finitely generated involutive \mathcal{O} -submodule of vector fields on M . This perspective seems to also become more and more the prevailing one these days [1, 3]. Here \mathcal{O} can be chosen to be the

ring of smooth functions $C^\infty(M)$, or, in the case that M is a real analytic or a complex manifold, the ring of real analytic and holomorphic functions, respectively.

So, now we are in the position to again pose the question about the existence of a finite rank Lie algebroid over M that would induce a given singular foliation on this manifold. This question can be split into a global and a local one. While it is easy to see that the answer to the global problem posed as such is negative—the maximal number of local generators of \mathcal{F} does not need to be finite—the local problem is much more intricate and still open—although implicitly, after shifting our focus, we will be able to give partial answers to this question in the present article as well (we will come back to this below).

Another interesting and probably in fact more important question in the context of singular foliations is to find invariants characterizing them, also locally. In the example of vector fields on \mathbb{R} , we see that even a complete knowledge of the partition of M into leaves is not sufficient for this.

In the present paper we will address a question that is related to both of the above ones and, at least for what concerns the first question, to our mind probably even a better question: is there a Lie ∞ -algebroid generating a given singular foliation? And, if so, can it be used to find invariants of the foliation? Both of these latter two questions will be answered in the positive in our paper: For example, in the case of \mathcal{O} being real analytic, we show that *every* singular foliation \mathcal{F} is locally generated by a Lie ∞ -algebroid. Even more importantly, there is such a Lie ∞ -algebroid whose homotopy class is unique and universal: the one constructed on a resolution of the singular foliation. This is in sharp contrast to the Lie algebroid story: not only is a Lie algebroid A over M for a given singular foliation \mathcal{F} far from unique, if it exists at all, also its homotopy class cannot be unique since homotopies of Lie algebroids do not change the rank of the underlying vector bundle A .

If we take any other Lie ∞ -algebroid whose induced foliation is \mathcal{F} , or even only a submodule of \mathcal{F} , and whose underlying complex is now not necessarily a resolution, there exists a morphism into every Lie ∞ -algebroid with the complex being a resolution and this morphism itself is unique up to homotopy (see Theorem 1.8). So, considering the category of Lie ∞ -algebroids up to homotopy, i.e. the category where objects are Lie ∞ -algebroids over M inducing a sub-singular foliation of \mathcal{F} and where arrows are homotopy classes of Lie ∞ -algebroids morphisms), Lie ∞ -algebroids constructed on resolutions are a terminal object. This justifies to call them *universal Lie ∞ -algebroids over a singular foliation \mathcal{F}* .

The universal Lie ∞ -algebroid over a singular foliation \mathcal{F} turns out to be an efficient tool for the construction of invariants associated to a singular foliation, like different types of cohomology classes associated to a Lie ∞ -algebroid representing its universal class. Let us explain the construction a bit further, starting first with the case that \mathcal{O} is real analytic or holomorphic. Then by Syzygy theorems in the neighbourhood of any point $m \in M$ the \mathcal{O} module \mathcal{F} admits resolutions of finite length by free modules, which we can reinterpret as sections of trivial vector bundles over the neighbourhood:

$$0 \longrightarrow \Gamma(E_{-n-1}) \longrightarrow \dots \longrightarrow \Gamma(E_{-1}) \xrightarrow{\rho} \mathcal{F} \longrightarrow 0$$

where n is the dimension of M . The Lie ∞ -algebroid is then constructed over the corresponding complex of vector bundles $E_{-n-1} \rightarrow \dots \rightarrow E_{-1}$ by showing the existence of s -brackets for $s = 2, \dots, n+1$ so that together with the differential of the complex they satisfy the required higher Jacobi identities. Certainly, the above resolution is not unique; in particular, the ranks of the bundles are far from being fixed. These individual ranks can now be changed by homotopies in the category of Lie ∞ -algebroids, only their total index

$$\text{Ind}(E_\bullet) := \sum_{i=1} (-1)^{i+1} \text{rk}(E_i) \tag{0.1}$$

remains invariant. This index, on the other hand, is nothing but the highest possible dimension of the leaves in the neighbourhood of m . There are, however, much more subtle invariants associated to the foliation and coming from our construction, since we can prove that for any two resolutions as above and for whatever choices of higher brackets the resulting Lie ∞ -algebroid is unique up to homotopies, which will make all its induced cohomologies depend only on the singular foliation. For instance, restricting the universal Lie ∞ -algebroids

over a singular foliation \mathcal{F} to a point m , and taking its cohomology, we get a Lie ∞ -algebra, that we call the isotropy Lie ∞ -algebra of \mathcal{F} at the point $m \in M$. This Lie ∞ -algebra has by construction no 1-ary bracket, i.e. no differential. Therefore, its 3-ary bracket is a class in the Chevalley-Eilenberg cohomology for the isotropy graded Lie algebra bracket given by the 2-ary bracket. We show that there cannot be a generating Lie algebroid of minimal rank in the neighbourhood of a point m where this class does not vanish. An explicit example of such a foliation will be provided in Example 25 below: the vector fields on $\mathbb{C}^4 \ni (x, y, z, t)$ preserving the function $x^3 + y^3 + z^3 + t^3$ form a singular foliation of rank 6, i.e. they need at least six generating vector fields to be defined by means of generators and relations. Since its above 3-class is shown to be non-zero, it follows that this particular singular foliation cannot be defined as the image through the anchor map of a Lie algebroid of rank 6. Notice that the problem of finding, given a singular foliation of rank r , a Lie algebroid of rank r that induces it, has a priori no relation with higher structures. It is interesting to see that it can be answered through the use of those.

The structure of the paper is as follows. Chapter 1 contains the main results about the construction of the universal Lie ∞ -algebroid. In particular, we concentrate all important definitions and results in its first section, Section 1.1. An ordered list of examples of singular foliations and some useful lemmas are presented in Section 1.2. Section 1.3 recalls classical results about Lie ∞ -algebroids, in particular the useful perspective as a differential positively graded manifold, equivalently known under the name of an NQ-manifold. We put particular emphasis on homotopies between Lie ∞ -algebroids morphisms, where precision about boundary conditions is required; we believe that the point of view presented about homotopies will be of interest also in other contexts. In Section 1.4 we address the question of whether and when a singular foliation \mathcal{F} admits a resolution as a module over functions. Here we do not just mean a projective resolution, but a resolution by sections of vector bundles. Around a point, it is equivalent to require the existence of a resolution of \mathcal{F} by free finitely generated \mathcal{O} -modules. In general, the answer is no, and a counter-example is given, but classical results, called syzygy theorems, imply that the answer is yes in the real analytic, algebraic, and holomorphic cases in a neighborhood of a point. Moreover, in the real analytic case, the real-analytic resolution can be proved to be also a smooth resolution by classical results of Malgrange and Tougeron. This part then is followed by examples of such resolutions in Section 1.5. Only then, in Section 1.6, we turn to the proof of the main theorems, Theorem 1.6 about equipping any such a resolution with a Lie ∞ -algebroid structure and Theorem 1.8 about its uniqueness up to homotopy and its universality property. We prove all these theorems by careful step-by-step constructions of brackets, morphisms, and homotopies. We conclude this chapter by providing examples in Section 2.4.

Chapter 2 is devoted to the geometrical meaning of the previously found structures. Since the universal Lie ∞ -algebroid over a singular foliation is unique up to homotopy, most cohomologies constructed out of it do not depend on the choices made in the construction and are thus associated to the initial foliation. In particular, as argued in Section 2.1, the cohomology of the degree 1 vector field Q describing the universal Lie ∞ -algebroid over a singular foliation is canonical, i.e. it depends only on the singular foliation. In Section 2.2 we derive even more interesting cohomological spaces by restricting the universal Lie ∞ -algebroid structure to a point $x \in M$. This is analogous to the familiar situation for Lie algebroids, where the Lie algebroid bracket induces a Lie algebra bracket on the kernel of the anchor map at a given point, called the isotropy Lie algebra of the point or its leaf (since for different points on a given leaf these Lie algebras are isomorphic). Essentially the same construction applies here and allows us to induce a Lie ∞ -algebra bracket on a graded vector space which coincides with the fiber over x of the resolution of the foliation in degree $-2, -3, \dots$ and to the kernel of the anchor map in degree -1 . If the resolution is chosen to be minimal at x —the ranks of the vector bundles that define the resolution are as small as they can be—we obtain a Lie ∞ -algebra that we call the isotropy Lie ∞ -algebra of \mathcal{F} at the point $x \in M$. It has several interesting features: First, its differential or 1-ary bracket vanishes, so that its 2-ary bracket defines an honest graded Lie algebra. But it may still have k -ary brackets for $k \geq 3$. Second, this structure is unique up to isomorphism, its 2-ary bracket being even unique on the nose, cf. Proposition 2.8 below. In Section 2.3 we then

prove that, like for isotropy Lie algebras of Lie algebroids, the isotropy Lie ∞ -algebras of Lie ∞ -algebroids only change by isomorphisms along any leaf of \mathcal{F} . Section 2.4 contains examples of these isotropy algebras.

In Section 2.5 we show that our structure induces the holonomy algebroid and groupoid of Androulidakis and Skandalis [1] by an appropriate truncation (and integration), cf. , in particular, Proposition 2.15 below.

In Section 2.6 we return to the issue of Lie ∞ -algebroid versus Lie algebroid. Evidently, we can always add a non-acting Lie algebra to every Lie algebroid, which increases its isotropy Lie algebras at each point accordingly while not changing the induced foliation. In contrast to the isotropy Lie ∞ -algebras of a singular foliation and its graded Lie algebra introduced in this paper, the isotropy Lie algebras induced from Lie algebroids are far from unique. Moreover, as we will prove, a Lie algebroid inducing a given singular foliation, even if it exists, may in some cases need more generators than the initial singular foliation does. More precisely, we will show that the 3-ary bracket of the isotropy Lie ∞ -algebra of \mathcal{F} at every point $x \in M$ is a Chevalley-Eilenberg cocycle with respect to the 2-ary bracket and that this cocycle is exact if there exists a Lie algebroid of rank r defining the singular foliation (with r being the rank of the foliation). Example 25 then presents a singular foliation for which this Chevalley-Eilenberg class does not vanish.

Section 2.7 concludes the paper with a side remark that every singular foliation admitting a resolution of finite length is the image through the anchor map of a Leibniz algebroid.

Remark: Several results of the present work were also presented in the Ph-D thesis of S.L. [31], defended under the supervision of Henning Semtleben and T.S. in November 2016.

Acknowledgment

We have benefited of several crucial comments by Jim Stasheff and feel honored by the interest he took in this project. In addition to his numerous mathematical comments he corrected many typos and mistakes of English. The present work is also the outcome of several years of discussions and lectures. We mention in the text the very points where we received direct help from Iakovos Androulidakis, François Petit, Jean-Louis Tu, Rupert Yu and Marco Zambon. Finally, we acknowledge important discussions with and comments from Damien Calaque, Claire Debord, Yaël Frégier, Yvette Kosmann-Schwarzbach, Benjamin Hennion, Bruno Vallette, Ted Voronov and Ping Xu.

We were told that results more or less equivalent to Theorem 1.6 were discussed by Ralph Mayer and Chenchang Zhu as well as by Ted Voronov and his collaborators. Tom Lada and Jim Stasheff present in [6] a construction of a Lie ∞ -algebra on the resolution of a Lie algebra: our construction follows the same pattern and is inspired by theirs. Moreover, Johannes Huebschmann suggests such a result also in his introduction to [22], without giving further details though. We claim, however, that we are the first ones to have clearly stated and published a proof for Theorem 1.6 and, more importantly, to have found, formulated and proven unicity, cf. Theorem 1.8. Also, we are not aware of any predecessor for Section 2, where we derive some geometric implications of the construction.

This research was partially supported by CMUP (UID/MAT/00144/2013) funded by FCT (Portugal) with national funds. C.L.-G. is grateful to the Université de Lorraine for according a CRCT of six months. T.S. is grateful to the CNRS for according a delegation and an attachment to beautiful IMPA for one semester and equally to IMPA and its staff for their hospitality during his stay.

1 Existence and unicity of the universal Lie ∞ -algebroid over a singular foliation

Definitions and basic facts about Lie ∞ -algebroids are given in Section 1.3, and singular foliations are defined in Section 1.2. We assume for the moment that the reader is familiar with those and we state the main results of the article. Then we go back to the definitions of these two objects.

1.1 The main results: Theorems 1.6 and 1.8

We intend to state results that are true in the smooth, algebraic, real analytic and holomorphic settings altogether, sometimes with adaptations.

Definition 1.1. *Let $\mathcal{F} \subset \mathfrak{X}(M)$ be a singular foliation on a manifold M . A resolution (E, d, ρ) of the foliation \mathcal{F} is a triple consisting of:*

1. *a collection of vector bundles $E = (\bigoplus E_{-i})_{i \geq 1}$ over M ,*
2. *a collection $d = (d^{(i)})_{i \geq 2}$ of vector bundle morphisms $d^{(i)} : E_{-i} \rightarrow E_{-i+1}$ over the identity of M ,*
3. *a vector bundle morphism $\rho : E_{-1} \rightarrow TM$ over the identity of M called the anchor of the resolution,*

such that the following sequence of sections of \mathcal{O} -modules is an exact sequence of sheaves:

$$\dots \xrightarrow{d^{(3)}} \Gamma(E_{-2}) \xrightarrow{d^{(2)}} \Gamma(E_{-1}) \xrightarrow{\rho} \mathcal{F} \longrightarrow 0$$

A resolution is said to be of length n if $E_{-i} = 0$ for $i \geq n + 1$.

We shall speak of a resolution by trivial bundles when all the vector bundles $(E_{-i})_{i \geq 1}$ are trivial vector bundles.

We shall say that a resolution is minimal at a point $m \in M$ if, for all $i \geq 2$, the vector bundle morphisms $d_m^{(i)} : E_{-i}|_m \rightarrow E_{-i+1}|_m$ is equal to zero¹ at the point m .

Since sections of vector bundles over M are projective \mathcal{O} -modules by the Serre-Swan theorem, resolutions of smooth singular foliations are projective resolutions of \mathcal{F} in the category of \mathcal{O} -modules. It is a classical result that such resolutions always exist. But the projective modules of a projective resolution may not correspond to vector bundles - they may not be locally finitely generated. By the Serre-Swan theorem [37], however:

Lemma 1.2. *For smooth compact manifolds, resolutions of a singular foliation \mathcal{F} are in one-to-one correspondence with resolutions of \mathcal{F} by locally finitely generated projective \mathcal{O} -modules.*

In the smooth, holomorphic, algebraic or real-analytic cases, resolutions by trivial vector bundles are in one-to-one correspondence with resolutions by free finitely generated \mathcal{O} -modules.

There are several contexts in which such resolutions always exist, at least locally, and are of finite length. For instance, for singular foliations generated by polynomial vector fields on \mathbb{C}^n , the existence is due to the fact that the ring of polynomial functions is Noetherian, and finiteness is due to the Hilbert's syzygy theorem. Moreover, a real analytic resolution is also a smooth resolution. This is not an trivial result: the proof uses theorems due to Malgrange and Tougeron [38]. In short:

Proposition 1.3. *The following items hold:*

1. *Any holomorphic (resp. real analytic) singular foliation on a complex (resp. real analytic) manifold M of dimension n admits, in a neighborhood of a point, a resolution by trivial vector bundles whose length is less or equal to $n + 1$ (i.e. $E_{-i} = 0$ for all $i \geq n + 2$).*

¹This of course does not mean that it is zero in a neighborhood of m : it means simply that the ranks of the vector bundles $(E_{-i})_{i \geq 2}$ is as small as can be.

2. Moreover, a real analytic resolution of a real analytic singular foliation \mathcal{F} is also a smooth resolution of \mathcal{F} (seen as a smooth singular foliation).
3. Any algebraic singular foliation on a Zarisky open subset of \mathbb{C}^n admits a resolution by trivial vector bundles whose length is less or equal to $n + 1$.
4. There exists a smooth singular foliation on \mathbb{R} that does not admit resolutions.
5. If a resolution of finite length exists, then for any point $x \in M$, a resolution of finite length and minimal at x exists in a neighborhood of x .

Recall that we say that a leaf is *regular* when all leaves in a neighborhood are of the same dimension, and *singular* otherwise. The dimension of singular leaves is always strictly less than the dimension of the regular ones, and the union of all regular leaves is a dense open subset. On a connected complex or real analytic manifold, all regular leaves are of the same dimension. On a smooth manifold, it is in general not the case (see Example 15). We are thankful to Marco Zambon for leading us to the following result:

Proposition 1.4. *If a singular foliation \mathcal{F} on a connected manifold M admits resolutions of finite length in a neighborhood of all points in M , then all its regular leaves have the same dimension r . Moreover, for every resolution of finite length (E, d, ρ) of \mathcal{F} over an open subset of M :*

$$r = \sum_{i \geq 1} (-1)^{i-1} \text{rk}(E_{-i}).$$

Above rk stands for the rank of a vector bundle.

We refer to Section 1.4 for a proof of this proposition (the first part of which is obvious in the real analytic or holomorphic cases, as already mentioned). We now introduce the main object and the two main theorems of the present article. We assume that the reader is familiar with L_∞ -structures, and understands that a Lie ∞ -algebroid (E, Q) (with grading going from -1 to $-\infty$) admits a linear part, which is a complex (E, d) of vector bundles over M , and an anchor map $\rho : E_{-1} \rightarrow TM$. All these properties are explained in Section 1.3.

Definition 1.5. *Let \mathcal{F} be a singular foliation on a manifold M . We say that a Lie ∞ -algebroid (E, Q) over M is a universal Lie ∞ -algebroid over \mathcal{F} if the linear part of (E, Q) is a resolution² of \mathcal{F} .*

When $E_{-k} = 0$ for all $k \geq n + 1$, we speak of a universal Lie n -algebroid over \mathcal{F} .

Here is the first main result:

Theorem 1.6. *Let \mathcal{F} be a singular foliation of a smooth or real analytic or complex manifold M . A universal Lie ∞ -algebroid over \mathcal{F} exists:*

1. in the smooth case, when a resolution (E, d, ρ) of \mathcal{F} exists, (and, moreover, its linear part can be chosen to be the resolution (E, d, ρ)),
2. in a neighborhood of every point in M in the real analytic and complex cases.

Together with item 1 and 3 in Proposition 1.3, the previous theorem implies:

Corollary 1.7. *Let \mathcal{F} be a singular foliation of a real analytic or complex manifold M of dimension n . Every point in M admits a neighborhood on which a universal Lie n -algebroid over \mathcal{F} exists.*

The use of the word ‘universal’ is justified by the first item in the next theorem:

Theorem 1.8. *Let (E, Q) be a universal Lie ∞ -algebroid over a singular foliation \mathcal{F} on a smooth manifold. Then,*

1. for any Lie ∞ -algebroid (E', Q') defining a sub-singular foliation of \mathcal{F} (i.e. such that $\rho'(\Gamma(E'_{-1})) \subset \mathcal{F}$), there is a Lie ∞ -algebroid morphism from (E', Q') to (E, Q) over the identity of M and any two such Lie ∞ -algebroid morphisms are homotopic.
2. in particular, two universal Lie ∞ -algebroids over the singular foliation \mathcal{F} are isomorphic up to homotopy and two such isomorphisms are homotopic.

² In particular, \mathcal{F} is the foliation associated to (E, Q) , i.e. $\rho(\Gamma(E_{-1})) = \mathcal{F}$, with ρ the anchor map of (E, Q) .

The same results hold in the complex or real-analytic settings in a neighborhood of any point.

Item 1. means that in the category where objects are Lie ∞ -algebroids whose induced singular foliation is included in \mathcal{F} and arrows are homotopy classes of morphisms, any universal Lie ∞ -algebroid over \mathcal{F} is a terminal object. This justifies the name "universal", since terminal objects satisfy what is called a "universal property". It is automatic that item 1. implies item 2.: it is a general fact that two terminal objects are related by a unique invertible arrow. Of course, in most well-known cases (the universal enveloping algebra for instance), this invertible unique arrow is bijective, while here it is only a homotopy class of invertible-up-to-homotopy morphisms.

By item 1. in Theorem 1.8, in particular, for every Lie algebroid A defining a singular foliation \mathcal{F} , a Lie ∞ -algebroid morphism from A to any universal Lie ∞ -algebroid (E, Q) over \mathcal{F} exists and any two such morphisms are homotopic. To our point of view, this explains why the universal Lie ∞ -algebroid (E, Q) over \mathcal{F} is more important than a Lie algebroid (maybe not unique) defining the foliation.

Let us say a few words about the proofs of the previous results. A crucial result is Lemma 1.29, that states that vertical vector fields on a resolution E , seen as a graded manifold, have little cohomology. The proofs are mainly based on step-by-step constructions using that Lemma. For clarity, we have dedicated different sections to the proofs of these various results. Theorem 1.6 is proven in Section 1.6.3, Theorem 1.8 is proven in Section 1.6.4.

As mentioned in the introduction, several invariants and geometric properties of the singular foliation can be derived out of these two theorems: they are to be studied in Section 2.

1.2 Singular foliations: definitions and examples

Let M be a manifold that may be smooth, real analytic or complex. It may also be Zarisky open subset $U \subset \mathbb{C}^n$. Generalizing to affine or projective varieties would be an interesting topic by itself.

Denote by $\mathcal{O}(U)$, with $U \subset M$ an open subset, the algebra of polynomials, smooth, real analytic or holomorphic functions over U and by $\mathfrak{X}(U)$ the $\mathcal{O}(U)$ -module of vector fields over U . The assignment $\mathfrak{X} : U \mapsto \mathfrak{X}(U)$ is a sheaf of Lie-Rinehart algebras (that is, a sheaf of Lie algebras compatible in an obvious sense with the sheaf of algebras $\mathcal{O} : U \mapsto \mathcal{O}(U)$), see [22]. We say that a sheaf $\Gamma : U \mapsto \Gamma(U)$ is *locally finitely generated* if for every $x \in M$ there exists an open neighborhood U_x of x and a finite number of sections $X_1, \dots, X_p \in \Gamma(U)$ such that for every open neighborhood V of x such that $\bar{V} \subset U$, $X_1|_V, \dots, X_p|_V$ span $\Gamma(V)$. We call *rank at a point x* of a finitely generated sheaf the minimal number of local generators.

We define singular foliations in the smooth, complex, real analytic or algebraic contexts as follows:

Definition 1.9. A singular foliation is a subsheaf $\mathcal{F} : U \mapsto \mathcal{F}(U)$ of the sheaf of vector fields \mathfrak{X} , which is locally finitely generated as an \mathcal{O} -submodule and closed under Lie bracket of vector fields.

For smooth compact manifolds, this definition matches exactly the definition in [1, 4, 13]. There is however a difference, since the previously cited works consider compactly supported vector fields. For non-compact smooth manifolds, A. Garmendia and R. Wang [42] simultaneously showed that singular foliations in the sense of Definition 1.9 is an equivalent to singular foliations in the sense of [1, 4, 13].

The restriction of \mathcal{F} at a point x is given by the evaluation of all the sections of \mathcal{F} at the point x and is denoted by \mathcal{F}_x . A *singular sub-foliation* of a singular foliation \mathcal{F} is a singular foliation \mathcal{F}' such that $\mathcal{F}'(U) \subset \mathcal{F}(U)$ for all open subsets $U \subset M$. A singular foliation on a manifold M will be said to be *finitely generated* when there exist k vector fields $X_1, \dots, X_k \in \mathfrak{X}(M)$, globally defined on the whole of M , that generate \mathcal{F} , i.e., such that, for every open subset $U \subset M$, $\mathcal{F}(U)$ is generated over $\mathcal{O}(U)$ by the restrictions to U of X_1, \dots, X_k .

When there exists a (smooth, real analytic or holomorphic) vector bundle A over M and a vector bundle morphism $\rho : A \rightarrow TM$ over the identity of M such that³ $\mathcal{F} = \rho(\Gamma(A))$,

³i.e. the sheaf \mathcal{F} is obtained by sheaffying the presheaf $\rho(\Gamma(A))$, which means in this context that every point

where $\Gamma(A) : U \mapsto \Gamma_U(A)$ is the sheaf of sections of A , we say that \mathcal{F} is covered by (A, ρ) . Notice that for a singular foliation, the Lie bracket of vector fields cannot necessarily be lifted to a Lie bracket on the sections of A , because there is no guarantee that the Jacobi identity is satisfied. This is another argument for the necessity of studying the universal Lie ∞ -algebroid associated to \mathcal{F} .

Singular foliations are generally defined in the smooth category. It is obvious, however, that real analytic or holomorphic singular foliations induce a smooth singular foliation, so that the result that we now describe still hold true in their respective categories.

We call *leaves* of a singular foliation \mathcal{F} the connected submanifolds N of M whose tangent space is, at every point $x \in N$, obtained by evaluating at x all the local sections of the sheaf \mathcal{F} , and which is maximal, with respect to inclusion, among such submanifolds. The following result, now classical, is due to R. Hermann in 1962:

Proposition 1.10. [21] *A singular foliation \mathcal{F} on a manifold M induces a partition of M into leaves.*

Remark 1. Unlike the case of regular foliations, singular foliations are not characterized by their leaves, and two different singular foliations may have the same leaves but differ as sheaves of vector fields. For instance, as noticed in [4], for M a real or complex vector space (supposed, in the real case, to be of dimension greater than or equal to 2) and for each integer $k \geq 1$, consider \mathcal{F}_k to be the module of all smooth, real analytic, or holomorphic vector fields vanishing to order k at the origin. This is clearly a singular foliation for all k , and all such singular foliations have exactly the same two leaves: the origin and the complement of the origin. They are not, however, identical as sub-modules of the module of vector fields.

We would like to convince the reader of the interest of the notion of singular foliations by giving an ordered but wide list of examples of those. The first example of a singular foliation comes as the image of a vector bundle morphism:

Example 1. For A a (smooth or holomorphic [30]) Lie algebroid over M with anchor $\rho : A \rightarrow TM$, the \mathcal{O} -module $\rho(\Gamma(A))$ is a singular foliation. It is a finitely generated foliation when $\Gamma(A)$ is a finitely generated \mathcal{O} -module, which is always true when the vector bundle A is trivial, or when, at least, there exists a vector bundle B such that the direct sum $A \oplus B$ is trivial.

In particular, regular foliations, orbits of a connected Lie group action, orbits of a Lie algebra or a Lie algebroid action, symplectic leaves of a Poisson manifold and foliations induced by Dirac structures, are singular foliations covered by a vector bundle. Example 1 can be enlarged, by generalizing the notion of Lie algebroids.

Definition 1.11. [22] *An almost-Lie algebroid over M is a vector bundle $A \rightarrow M$, equipped with a vector bundle morphism $\rho : A \rightarrow TM$ called the anchor map, and a skew-symmetric bracket $[\cdot, \cdot]_A$ on $\Gamma(A)$, satisfying the Leibniz identity:*

$$\forall x, y \in \Gamma(A), f \in \mathcal{C}^\infty(M) \quad [x, fy]_A = f[x, y]_A + \rho(x)[f]y, \quad (1.1)$$

together with the Lie algebra homomorphism condition:

$$\forall x, y \in \Gamma(A) \quad \rho([x, y]_A) = [\rho(x), \rho(y)]. \quad (1.2)$$

We do not require that the bracket $[\cdot, \cdot]_A$ be a Lie bracket: it may not satisfy the Jacobi identity. However, the Jacobi identity being satisfied for vector fields, Condition (1.2) imposes that the Jacobiator takes values in the kernel of the anchor map at all points. These Marco Zambon mentioned the following result, which seems well-known, although we can not give a precise reference:

Proposition 1.12. *Let M be a smooth, or real analytic or complex manifold, and let (A, ρ) be an anchored vector bundle, where $A \rightarrow M$ is a vector bundle and $\rho : A \rightarrow TM$ is a vector bundle morphism called the anchor map.*

$x \in M$ admits a neighborhood U such that $\mathcal{F}(U) = \rho(\Gamma_U(A))$.

1. For every almost-Lie algebroid structure on $A \rightarrow M$, the image of the anchor map $\rho : \Gamma(A) \rightarrow \mathfrak{X}(M)$ is a singular foliation.
2. Every finitely generated foliation on M is the image under the anchor map of an almost-Lie algebroid, defined on a trivial bundle.
3. In the smooth case, every anchored vector bundle (A, ρ) over M that covers a singular foliation \mathcal{F} can be equipped with an almost-Lie algebroid structure with anchor ρ .

Proof. The first item follows from Conditions (1.1) and (1.2). Let us prove the second item. Let X_1, \dots, X_r be generators of a singular foliation \mathcal{F} . Since \mathcal{F} is closed under the Lie bracket of vector fields, there exist functions $c_{ij}^k \in \mathcal{O}(M)$ satisfying:

$$[X_i, X_j] = \sum_{k=1}^r c_{ij}^k X_k, \quad (1.3)$$

for all indices $i, j \in \{1, \dots, r\}$. Upon replacing c_{ij}^k by $\frac{1}{2}(c_{ij}^k - c_{ji}^k)$ if necessary, we can assume that the functions $c_{ij}^k \in \mathcal{O}(M)$ satisfy the skew-symmetry relations $c_{ij}^k = -c_{ji}^k$ for all possible indices. Define A to be the trivial bundle $A = \mathbb{R}^r \times M \rightarrow M$. Denote its canonical global sections by e_1, \dots, e_r and define:

1. an anchor map by $\rho(e_i) = X_i$ for all $i = 1, \dots, r$,
2. a skew-symmetric bracket by $[e_i, e_j]_A = \sum_{k=1}^r c_{ij}^k e_k$ for all $i, j = 1, \dots, r$,

then extend these structures by, respectively, \mathcal{O} -linearity and Leibniz property. This bracket and anchor define by construction an almost-Lie algebroid structure on A that covers \mathcal{F} .

Let us now prove the third item. Unlike Lie algebroid brackets, almost-Lie algebroid brackets can be glued using partitions of unity. More precisely, let $(\varphi_i)_{i \in I}$ be a partition of unity subordinate to an open covering $(U_i)_{i \in I}$ by open sets trivializing the vector bundle A . By the proof of item 2., we can define an almost-Lie algebroid structure with anchor ρ on the restriction of A to U_i , that is, a bracket $[\cdot, \cdot]_{U_i}$ that satisfies (1.1) and (1.2). For all sections in $\Gamma_{U_i}(A)$. The bracket:

$$[\cdot, \cdot]_A = \sum_{i \in I} \varphi_i [\cdot, \cdot]_{U_i} \quad (1.4)$$

still satisfies (1.1) and (1.2), hence defines an almost-Lie algebroid structure on A with anchor ρ . \square

It has been conjectured (see [3]) that, even locally, not every smooth singular foliation is of the type described in Example 1, i.e. is the image under the anchor map of a Lie algebroid. As far as we know, the question remains open to this day. There are quite a few singular foliations for which the underlying Lie algebroid, if any, is not obvious to find in a natural manner.

Example 2. Consider, for $\mathbb{K} = \mathbb{R}$ or \mathbb{C} :

1. Let $P := (P_1, \dots, P_k)$ be a k -tuple of polynomial functions in d variables over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . The symmetries of P , i.e. all polynomial vector fields $X \in \mathfrak{X}(\mathbb{K}^d)$ that satisfy $X[P_i] = 0$ for all $i \in \{1, \dots, k\}$, form a singular foliation. An interesting case, that appears in the Batalin-Vilkovisky context [17], occurs when considering the symmetries of a polynomial function S , which represents the classical action.
2. *Symmetries of W* i.e. all polynomial vector fields X such that $X[I_W] \in I_W$, I_W being now assumed to be the ideal of polynomial functions vanishing on some affine variety $W \subset \mathbb{K}^d$.
3. Vector fields vanishing at all points of an affine variety W .

All the previous spaces of polynomial vector fields are closed under the Lie bracket, and form a sub-module (over the ring of polynomial functions on \mathbb{K}^d) of the module of algebraic vector fields. Since the latter is finitely generated over the ring of polynomial functions on \mathbb{K}^d , and since the ring of polynomial functions is Noetherian, each of these spaces is a

finitely generated module over the polynomial functions. The $\mathcal{O}(\mathbb{K}^d)$ -module generated by these polynomial vector fields (with \mathcal{O} standing here for smooth, real analytic or holomorphic functions) is therefore also a singular foliation.

Below, we list various singular foliations, which seem not to be of any of the previous types.

Example 3. Vector fields on a manifold M which are tangent to a submanifold L form an example of a singular foliation. Of course, L is the only singular leaf, while connected components of M/L are the regular ones.

Example 4. Let k and d be integers greater than or equal to 1. Vector fields vanishing to order k at the origin of \mathbb{R}^d form a singular foliation. For $k = 1$, it is the singular foliation associated to the action of the group $\mathrm{GL}(d)$ of invertible $d \times d$ -matrices on \mathbb{R}^d . It is therefore the image through the anchor map of a transformation Lie algebroid.

Example 5. In [39], a bivector field $\pi \in \Gamma(\wedge^2 TM)$ on a manifold M is said to be *foliated* when the space of vector fields of the form $\pi^\#(\alpha)$ with $\alpha = \Gamma(T^*M)$ a 1-form, is closed under the Lie bracket of vector fields. Such vectors define therefore a singular foliation. When π is a Poisson bivector, or at least a twisted Poisson bivector (sometimes also referred to as ‘Poisson with background’ [24, 33, 35]), it is known that T^*M comes equipped with a Lie algebroid structure [11] with anchor $\pi^\# : T^*M \rightarrow TM$, but for ‘generic’ foliated bivector fields, no such formula seems to exist, as discussed in [39].

Example 6. For a Leibniz algebroid (see Section 2.7 and [36] for a definition), the image of the anchor map is obviously also a singular foliation. Recall that Courant algebroids [34] are examples of Leibniz algebroids.

For instance, for S a function on M , a Leibniz algebroid structure on the bundle $\wedge^2 TM$ is given by the anchor $P \mapsto P_S := P^\#(dS)$ together with the Leibniz bracket:

$$(P, Q) \mapsto \mathcal{L}_{P_S} Q,$$

for P and Q two sections of $\wedge^2 TM$, i.e., bivector fields. Note that for M a vector space and S a polynomial function, the associated singular foliation is a sub-foliation of the foliation of symmetries of S described in Example 14.

Now we give an example of a sub-sheaf of the sheaf of vector fields, which is closed under the Lie bracket, but which is *not* a singular foliation.

Example 7. On $M = \mathbb{R}$, smooth vector fields vanishing on \mathbb{R}_- are closed under the Lie bracket but are not locally finitely generated (see [15]), hence are not a singular foliation. On $M = \mathbb{R}^2$ with variables (x, y) , the $\mathcal{C}^\infty(M)$ -module generated by the vector field $\frac{\partial}{\partial x}$ and vector fields of the form $\varphi \frac{\partial}{\partial y}$ where φ is a smooth function vanishing on the half-plane $x \leq 0$ is closed under the Lie bracket but it is not locally finitely generated. This counter-example is interesting, since Proposition 1.10 does not hold in that case.

1.3 Lie ∞ -algebroids, their morphisms and homotopies of those

In this article, we think and prove results with the NQ -manifold point of view, which we think to be the most natural here, but we state theorems with Lie ∞ -algebroids, since it seems nowadays to be the notion which is easier to grasp, and therefore the most popular. We define these objects in the smooth, real analytic or holomorphic settings altogether. We first recall the definition of Lie ∞ -algebroids [41]:

Definition 1.13. *Let M be a smooth/real analytic/complex manifold whose sheaf of functions we denote by \mathcal{O} . Let E be a sequence $E = (E_{-i})_{i \leq 1 \leq \infty}$ of vector bundles over M , then a Lie ∞ -algebroid structure on E , is defined by:*

1. a degree 1 vector bundle morphism $\rho : E_{-1} \rightarrow TM$ called the anchor of the Lie ∞ -algebroid,
2. and a family, for all $k \geq 1$, of graded symmetric k -multilinear maps $(\{\dots\}_n)_{n \geq 1}$ of degree +1 on the sheaf of graded vector spaces $\Gamma(E)$,

satisfying a set of constraints. The first ones are the Leibniz conditions:

1. the unary bracket $d := \{.\}_1 : \Gamma(E) \rightarrow \Gamma(E)$ is \mathcal{O} -linear, i.e. forms a family $d^{(i)} : E_{-i} \rightarrow E_{-i+1}$ of vector bundle morphisms, with $i \geq 2$;
2. for all $n \geq 3$, the bracket $\{.\}_n$ is \mathcal{O} -linear in each of its n arguments;
3. the binary bracket obeys different rules, depending on its arguments: for all $x \in \Gamma(E_{-1})$ and $y \in \Gamma(E)$ it satisfies

$$\{x, fy\}_2 = f\{x, y\}_2 + \rho(x)[f]y, \quad (1.5)$$

whereas $\{x, fy\}_2 = f\{x, y\}_2$ for all $x \in \Gamma(E_{-i})$ with $i \geq 2$.

The second ones are the compatibility conditions of the anchor:

1. $\rho \circ d^{(2)} = 0$,
2. $\rho(\{x, y\}) = [\rho(x), \rho(y)]$ for all $x, y \in \Gamma(E_{-1})$,

The third assumptions are the higher Jacobi identities:

1. $d^{(i-1)} \circ d^{(i)} = 0$ for all $i \geq 3$
2. for all $n \geq 2$, and for every homogeneous elements $x_1, \dots, x_n \in \Gamma(E)$:

$$\sum_{i=1}^n \sum_{\sigma \in \text{Un}(i, n-i)} \epsilon(\sigma) \{ \{x_{\sigma(1)}, \dots, x_{\sigma(i)}\}_i, x_{\sigma(i+1)}, \dots, x_{\sigma(n)} \}_{n-i+1} = 0. \quad (1.6)$$

where $\epsilon(\sigma)$ is the sign induced by the permutation of the elements x_1, \dots, x_n :

$$x_1 \odot \dots \odot x_n = \epsilon(\sigma) x_{\sigma(1)} \odot \dots \odot x_{\sigma(n)} \quad (1.7)$$

where \odot is the symmetric product on $\Gamma(S^n(E))$.

A Lie ∞ -algebroid structure over M is said to be a Lie n -algebroid when $E_{-i} = 0$ for all $i \geq n+1$.

Remark 2. This definition relies on the symmetric convention of the L_∞ algebras found e.g. in [23]. The original definition of L_∞ algebras involved graded skew-symmetric brackets [28]. However, they are in one-to-one correspondence, see [32]. In particular, under such conventions, a Lie algebroid $A \rightarrow M$ can be equivalently seen as a vector bundle $A[1]$ whose sections have degree -1 , and are equipped with a graded symmetric bracket $\{.,.\}$ that satisfies the usual identities.

We observe that for degree reasons, if $n < \infty$ there is no bracket of arity higher than or equal to $n+2$. Moreover, we notice that the fact that the differential d is \mathcal{O} -linear implies that the graded vector bundles $(E_{-i})_{i \leq 1 \leq n}$ form a chain complex of vector bundles. For every Lie ∞ -algebroid E over M , it follows from items 1. and 2. in the higher Jacobi identities that the following sequence:

$$\dots \xrightarrow{d^{(4)}} E_{-3} \xrightarrow{d^{(3)}} E_{-2} \xrightarrow{d^{(2)}} E_{-1} \xrightarrow{\rho} TM,$$

is a complex of vector bundles that we call its *linear part*.

Also, for every Lie ∞ -algebroid E over M , the binary bracket restricts to a skew-symmetric bilinear bracket on $\Gamma(E_{-1})$. Together with the anchor map, it defines an almost-Lie algebroid structure on E_{-1} . Therefore, by using the first item of Proposition 1.12, we obtain:

Proposition 1.14. *For every Lie ∞ -algebroid E over M with anchor ρ , the \mathcal{O} -module $\mathcal{F} := \rho(\Gamma(E_{-1}))$ is a singular foliation.*

We call this singular foliation the *singular foliation of the Lie ∞ -algebroid structure on E* .

The definition of Lie ∞ -algebroids above, although elementary, is quite cumbersome and often hard to use - especially when dealing with morphisms. For the sake of clarity, Q -manifolds with purely positive degrees, the so-called NQ -manifolds, are much more efficient

objects and are in one-to-one correspondence with the Lie ∞ -algebroids [41]. Let us define them.

By a N -manifold $E \rightarrow M$, we mean a sequence $E := (E_{-i})_{i \geq 1}$ of vector bundles (all of finite rank) over M , indexed by negative numbers. An element $x \in E_{-i}$ is said to be of *degree* $-i$ and the degree of x is written $|x| = -i$. We call *functions on the N -manifold $E \rightarrow M$* the graded commutative \mathcal{O} -algebra sheaf \mathcal{E} of (smooth, real analytic or holomorphic) sections of the symmetric algebra $S(E^*)$ (with the understanding that $E^* = \bigoplus_{i \geq 1} E_{-i}^*$ and E_{-i}^* is considered to be of degree $+i$).

By construction, \mathcal{E} is a sheaf of graded commutative \mathcal{O} -algebras. For every $k, n \geq 1$, sections of

$$\sum_{i_1 + \dots + i_k = n} E_{-i_1}^* \odot \dots \odot E_{-i_k}^* \subset S(E^*)$$

(with \odot the symmetric product) will be said to be of *degree* n and of *arity* k and are denoted by $\mathcal{E}_n^{(k)}$. The sheaf \mathcal{O} of functions on M can be identified with the sub-sheaf of functions of \mathcal{E} which are of degree 0.

Vector fields on the graded manifold $E \rightarrow M$ are graded derivations of \mathcal{E} . A vector field Q is said to be of *arity* k when, for all functions $F \in \mathcal{E}$ of arity l , the arity of $Q[F]$ is $l + k$. Any vector field Q can be decomposed as an infinite sum:

$$Q = \sum_{k \geq -1} Q^{(k)}$$

with $Q^{(k)}$ being a vector field of arity k . For a vector field of degree $i \geq 0$, the sum above goes from 0 to $+\infty$ for degree reasons. A vector field Q of odd degree commuting with itself, i.e. satisfying $Q^2 := \frac{1}{2}[Q, Q] = 0$, is said to be *homological*.

Definition 1.15. A NQ -manifold is a pair (E, Q) with $E \rightarrow M$ an N -manifold with base M and Q a homological degree $+1$ vector field.

By construction, for every NQ -manifold (E, Q) with sheaf of functions \mathcal{E} , we have an isomorphism of sheaves $\mathcal{E}_0 \simeq \mathcal{O}$, while $\mathcal{E}_1 \simeq \Gamma(E_{-1}^*)$, so that the derivation Q maps \mathcal{O} to $\Gamma(E_{-1}^*)$. By the derivation property, there exists a unique morphism of graded vector bundles $\rho : E_{-1} \rightarrow TM$ such that

$$\langle Q[f], x \rangle = \rho(x)[f] \quad \text{for all } f \in \mathcal{O}, x \in \Gamma(E_{-1}). \quad (1.8)$$

Above, $\langle \cdot, \cdot \rangle$ stands for the duality pairing between sections of a vector bundle and sections of its dual. We call the vector bundle morphism ρ the *anchor map of the NQ -manifold (E, Q)* . The next result is classical [41], and describes the duality between Lie ∞ -algebroids and NQ -manifolds.

Theorem 1.16. Let $E = (E_{-i})_{i \geq 1}$ be a sequence of vector bundles over a manifold M . There is a one-to-one correspondence between NQ -manifolds (E, Q) with anchor ρ and Lie ∞ -algebroid structures on E with anchor ρ . Under this correspondence:

1. the differential d of the linear part of the Lie ∞ -algebroid structure is obtained by dualizing the component $Q^{(0)}$ of arity 0 of Q , i.e. for all $\alpha \in \Gamma(E^*)$ and $x \in \Gamma(E)$:

$$\langle Q^{(0)}[\alpha], x \rangle = (-1)^{|\alpha|} \langle \alpha, d(x) \rangle, \quad (1.9)$$

2. the binary bracket $\{ \cdot, \cdot \}_2$ and the component $Q^{(1)}$ of arity 1 are related by:

$$\langle Q^{(1)}[\alpha], x \odot y \rangle = \rho(x)[\langle \alpha, y \rangle] - \rho(y)[\langle \alpha, x \rangle] - \langle \alpha, \{x, y\}_2 \rangle, \quad (1.10)$$

for all homogeneous elements $x, y \in \Gamma(E)$ and $\alpha \in \Gamma(E^*)$, with the understanding that ρ vanishes on E_{-i} for $i \neq 1$,

3. for every $n \geq 3$, the n -ary brackets $\{ \dots \}_n : \Gamma(S^n(E)) \mapsto \Gamma(E)$ and the component $Q^{(n-1)} : \Gamma(E^*) \mapsto \Gamma(S^n E^*)$ of arity $n - 1$ of Q are dual one to the other.

This theorem justifies the following convention:

Convention. We will denote Lie ∞ -algebroids as a pair (E, Q) , with Q the homological vector field of the corresponding NQ -manifold with sheaf of functions $\mathcal{E} = \Gamma(S(E^*))$.

When dealing with morphisms, NQ -manifolds are much more practical than Lie ∞ -algebroids. By a *morphism* from a N -manifold $E \rightarrow M$ to a N -manifold $E' \rightarrow M'$, we mean a degree 0 morphism Φ of sheaves of graded commutative algebras from the graded commutative algebras \mathcal{E}' of functions on $E' \rightarrow M'$ to the graded commutative algebra \mathcal{E} of functions on $E \rightarrow M$. The notion is then adapted to NQ -manifolds:

Definition 1.17. A Lie ∞ -algebroid morphism from a Lie ∞ -algebroid (E', Q') to a Lie ∞ -algebroid (E, Q) , with sheaves of functions \mathcal{E}' and \mathcal{E} respectively, is an algebra morphism Φ of degree 0 from \mathcal{E} to \mathcal{E}' which intertwines Q and Q' :

$$\Phi \circ Q = Q' \circ \Phi. \quad (1.11)$$

When Φ is an algebra isomorphism, we may speak of a strict isomorphism.

Every Lie ∞ -algebroid morphism Φ induces a smooth map ϕ from M' to M that we call the *base morphism*, and, for each $i \geq 1$, vector bundle morphisms $\phi_i : E'_{-i} \rightarrow E_{-i}$ over ϕ . The condition that $\Phi : \mathcal{E} \rightarrow \mathcal{E}'$ is an algebra morphism implies:

$$\Phi(fG) = \phi^*(f) \Phi(G), \quad (1.12)$$

for every $f \in \mathcal{O}$ and $G \in \mathcal{E}$. When $M = M'$, we say that a Lie ∞ -algebroid morphism Φ is *over* M if Φ is \mathcal{O} -linear, i.e. if the base morphism ϕ is the identity map.

Remark 3. Equation (1.11), restricted to terms of arity 0, implies that the induced graded vector bundle morphism $(\phi_i)_{i \geq 1}$ is a chain map between their respective linear part structures:

$$\begin{array}{ccccccc} \dots & \xrightarrow{d'} & E'_{-3} & \xrightarrow{d'} & E'_{-2} & \xrightarrow{d'} & E'_{-1} & \xrightarrow{\rho'} & TM' \\ & & \downarrow \phi_3 & & \downarrow \phi_2 & & \downarrow \phi_1 & & \downarrow \phi_* \\ \dots & \xrightarrow{d} & E_{-3} & \xrightarrow{d} & E_{-2} & \xrightarrow{d} & E_{-1} & \xrightarrow{\rho} & TM \end{array}$$

We call this chain map the *linear part of* Φ .

A \mathcal{O} -linear map (not necessarily now a Lie ∞ -algebroid morphism) Φ from $\mathcal{E} := \Gamma(S(E^*))$ to $\mathcal{E}' := \Gamma(S(E'^*))$ is said to be of *arity* k (resp. *degree* k) if it sends functions of arity l in \mathcal{E} (resp. degree l) to functions of arity $l+k$ (resp. degree $l+k$) in \mathcal{E}' . The arity of such a map is necessarily positive, for it has to send smooth functions on M (of arity zero) to elements of $\Gamma(S(E'^*))$ (of positive arity). In particular, any Lie ∞ -algebroid morphism Φ over M from (E', Q') to (E, Q) can be decomposed into components according to their arity, which allows us to consider Φ as a formal sum:

$$\Phi = \sum_{k \geq 0} \Phi^{(k)} \quad (1.13)$$

Since Φ is \mathcal{O} -linear, and since it is determined by its restriction to functions of arity 1, i.e. sections of E^* , the component of arity k , namely $\Phi^{(k)}$, can be identified with an element in $\Gamma(S^{k+1}(E'^*) \otimes E)$ that we denote by $\widehat{\Phi}^{(k)}$. When Φ is a morphism between N -manifolds, taking the arity into account, the morphism condition $\Phi(FG) = \Phi(F)\Phi(G)$ (valid for any $F, G \in \mathcal{E}$) becomes:

$$\Phi^{(n)}(F_1 \odot \dots \odot F_k) = \sum_{i_1 + \dots + i_k = n} \widehat{\Phi}^{(i_1)}(F_1) \odot \dots \odot \widehat{\Phi}^{(i_k)}(F_k), \quad (1.14)$$

for all $F_1, \dots, F_k \in \Gamma(E^*)$.

Beyond strict isomorphisms (see Definition 1.17) between Lie ∞ -algebroids, there is a broader notion of morphisms ‘invertible up to homotopy’. To begin with, it is not that

easy to define what homotopy of Lie ∞ -algebroid morphisms are. There has been several attempts to define them [5]. We make this definition very precise in the coming lines.

We proceed step-by-step. Let (E, Q) and (E', Q') be two Lie ∞ -algebroids over M with sheaves of functions \mathcal{E} and \mathcal{E}' respectively. We define an operator $[Q, \cdot]$ on the space of maps $\text{Map}(\mathcal{E}, \mathcal{E}')$ from \mathcal{E} to \mathcal{E}' by:

$$\begin{aligned} [Q, \cdot] : \text{Map}(\mathcal{E}, \mathcal{E}') &\rightarrow \text{Map}(\mathcal{E}, \mathcal{E}') \\ \alpha &\mapsto Q' \circ \alpha - (-1)^{|\alpha|} \alpha \circ Q \end{aligned}$$

for every map of graded manifolds $\alpha : \mathcal{E} \rightarrow \mathcal{E}'$ of homogeneous degree $|\alpha| \in \mathbb{Z}$, and we extend it by derivation. It squares to zero because both vector fields are homological. Then it defines a differential on the space of maps between the graded manifolds E' and E .

Definition 1.18. *Let Φ be a Lie ∞ -algebroid morphism from (E', Q') to (E, Q) . A Φ -derivation of degree k is a \mathcal{O} -linear homogeneous map δ of degree k from \mathcal{E} to \mathcal{E}' satisfying:*

$$\delta(FG) = \delta(F)\Phi(G) + (-1)^{k|F|}\Phi(F)\delta(G), \quad (1.15)$$

for all functions $F, G \in \mathcal{E}$.

Notice that a Φ -derivation can be decomposed as a sum $\delta = \sum_{k=0}^{\infty} \delta^{(k)}$ according to the arity. By \mathcal{O} -linearity, we can again identify $\delta^{(k)}$ with an element $\widehat{\delta}^{(k)} \in \Gamma(S^k(E'^*) \otimes E)$. It is easy to check that for every Φ -derivation δ of degree k , the linear map $[Q, \delta]$ is a Φ -derivation again, but is of degree $k + 1$. Of course, the relation $[Q, [Q, \delta]] = 0$ holds true, so that:

Lemma 1.19. *For every Lie ∞ -algebroid morphism Φ over M from (E', Q') to (E, Q) , the space of Φ -derivations forms a complex when equipped with the differential $[Q, \cdot]$.*

Now let us define what we mean by piecewise- C^1 paths valued in Lie ∞ -algebroid morphisms from (E', Q') to (E, Q) . Recall that a *piecewise- C^1* (resp. *piecewise continuous*) path valued in $\Gamma(B)$, with B a vector bundle over M , is a continuous map $\psi : M \times I \rightarrow B$ to the manifold B (with $I = [0, 1]$) such that for all fixed $t \in I$, the map $m \mapsto \psi(m, t)$ is a section of B and there exists a subdivision $a = x_0 < \dots < x_k = b$ of $I = [a, b]$ such that for every $m \in M$, the map $\psi : M \times]x_i, x_{i+1}[\rightarrow B$ is of class C^1 (resp. continuous).

Definition 1.20. *Let (E, Q) and (E', Q') be Lie ∞ -algebroids over M . A path $t \mapsto \Phi_t$ valued in Lie ∞ -algebroid morphisms from E' to E is said to be *piecewise- C^1* when for all $k \in \mathbb{N}$, its component $t \mapsto \widehat{\Phi}_t^{(k)}$ of arity k is a *piecewise- C^1* path valued in $\Gamma(S^{k+1}((E')^*) \otimes E)$.*

*Given a piecewise- C^1 path $t \mapsto \Phi_t$ valued in Lie ∞ -algebroid morphisms from (E', Q') to (E, Q) , we say that a path $t \mapsto \delta_t$, with δ_t a Φ_t -derivation, is *piecewise continuous* if its component $t \mapsto \widehat{\delta}_t^{(k)}$ of arity k is a *piecewise-continuous* path valued in $\Gamma(S^{k+1}((E')^*) \otimes E)$.*

Remark 4. A subtle point in this definition is that the subdivision of I for which $\widehat{\Phi}_t^{(k)}$ is C^1 may depend on k . Notice that the derivative $\frac{d}{dt}\Phi_t$ is well-defined for all $t \in I$ except on the countable set of the points delimiting these subdivisions. For Lie n -algebroids, since $(S(E'^*) \otimes E)_0$ is a vector bundle of finite rank, this finite subdivision of $[0, 1]$ can be chosen to be the same for all values of $k \geq 0$. Also, notice that, for us, *piecewise- C^1* implies *continuous* - even at the junction points.

It is routine to check that $\frac{d}{dt}\Phi_t$ is a Φ_t -derivation of degree 0 for each value of t for which it is defined, that satisfies $[Q, \frac{d}{dt}\Phi_t] = 0$, i.e. it is a cocycle for the complex of Lemma 1.19. This justifies the following definition, whose rough idea is that homotopies are families of Lie ∞ -algebroid morphisms whose derivatives are coboundaries for the complex of Φ -derivations:

Definition 1.21. *Let Φ and Ψ be two Lie ∞ -algebroid morphisms from (E', Q') to (E, Q) covering the identity morphism. A homotopy between Φ and Ψ is a pair (Φ_t, H_t) consisting in:*

1. a *piecewise- C^1* path $t \mapsto \Phi_t$ valued in Lie ∞ -algebroid morphisms between E' and E such that:

$$\Phi_0 = \Phi \quad \text{and} \quad \Phi_1 = \Psi,$$

2. a piecewise continuous path $t \mapsto H_t$ valued in Φ_t -derivations of degree -1 , such that the following equation:

$$\frac{d\Phi_t^{(k)}}{dt} = [Q, H_t]^{(k)} \quad (1.16)$$

holds for every $k \in \mathbb{N}$ and every $t \in [0, 1]$ where it is defined.

Remark 5. Although presented here for singular foliations only, Theorems 1.6 and 1.8 can be adapted to any locally finitely generated sheaf of Lie-Rinehart algebras over the ring of smooth functions on a manifold M .

Remark 6. Although it may seem different at first look, Definition 1.21 is in fact almost similar to a more classical and natural definition of homotopy given by [8, 40]. It consists in defining homotopies between two morphisms as being Lie ∞ -algebroid morphisms of differential graded algebras from \mathcal{E} to the tensor product $\mathcal{E}' \otimes \Omega^\bullet([0, 1])$ (where $\Omega^\bullet([0, 1])$ stands for forms on $[0, 1]$ equipped with de Rham differential) whose restrictions to 0 and 1 are the two given Lie ∞ -algebroid morphisms.

Both definitions match when the data (Φ_t, H_t) in Definition 1.21 depend smoothly on the parameter t , as we will show below. There is, however, a technical issue in the proof of Theorem 1.8 that imposes to make use of continuous piecewise C^1 -paths.

Let us explain the correspondence between both definitions. Let (E, Q) , (E', Q') , and (Φ_t, H_t) be as in Definition 1.21. Let us equip the tensor product $\mathcal{E}' \otimes \Omega^\bullet([0, 1])$ with the differential given for all $F \in \mathcal{E}'_i$ and $\omega \in \Omega^\bullet([0, 1])$ by

$$D : F \otimes \omega \mapsto Q'(F) \otimes \omega + (-1)^{|i|} F \otimes d_{\text{dR}}\omega.$$

Elements in $\mathcal{E}' \otimes \Omega^\bullet([0, 1])$ can be seen as sums $F_t \otimes 1 + G_t \otimes dt$ with F_t, H_t families of functions in \mathcal{E} depending smoothly on the parameter $t \in [0, 1]$. The previous operator D then reads:

$$D(F_t \otimes 1 + G_t \otimes dt) = (Q'(F_t)) \otimes 1 + \frac{dF_t}{dt} \otimes dt + (Q'(G_t)) \otimes dt. \quad (1.17)$$

Consider the map of degree 0 given by:

$$\begin{aligned} \tilde{\Phi} &:= \mathcal{E} \mapsto \mathcal{E}' \otimes \Omega^\bullet([0, 1]) \\ F &\rightarrow t \mapsto \Phi_t(F) \otimes 1 + (-1)^i H_t(F) \otimes dt \end{aligned}$$

This map is a graded algebra morphism if and only if Φ_t is an algebra morphism and H_t is a Φ_t -derivation for all t . This point follows from the following direct computation, valid for all $F \in \mathcal{E}_i, G \in \mathcal{E}_j$:

$$\begin{aligned} \tilde{\Phi}(FG) &= \Phi_t(FG) \otimes 1 + (-1)^{i+j} H_t(FG) \otimes dt \\ &= \Phi_t(F)\Phi_t(G) \otimes 1 + ((-1)^i H_t(F)\Phi_t(G) + (-1)^j \Phi_t(F)H_t(G)) \otimes dt \\ &= (\Phi_t(F) \otimes 1 + (-1)^i H_t(F) \otimes dt) \cdot (\Phi_t(G) \otimes 1 + (-1)^j H_t(G) \otimes dt) \\ &= \tilde{\Phi}(F) \cdot \tilde{\Phi}(G). \end{aligned}$$

Now let us show that $\tilde{\Phi}$ is a chain map if and only if Φ_t is a chain map and Equation (1.16) holds. On the one hand, we have

$$\tilde{\Phi} \circ Q(F) = \Phi_t \circ Q(F) \otimes 1 + (-1)^i H_t \circ Q(F) \otimes dt = Q \circ \Phi_t(F) \otimes 1 + (-1)^i Q \circ H_t(F) \otimes dt$$

and on the other hand, a direct computation gives:

$$D \circ \tilde{\Phi}(F) = Q \circ \Phi_t(F) \otimes 1 + \frac{d\Phi_t(F)}{dt} \otimes dt + (-1)^i Q \circ H_t(F) \otimes dt.$$

This gives the result and proves the equivalence of both definitions. For a more enhanced discussion about homotopy of Lie ∞ -algebroid morphisms, we refer to Norbert Poncin [9] or Bruno Vallette [40].

The following fact is obvious:

Proposition 1.22. *Homotopy of Lie ∞ -algebroid morphisms is an equivalence relation, denoted by \sim , which is compatible with composition.*

Proof. Let us show that homotopy defines an equivalence relation \sim between Lie ∞ -algebroid morphisms:

- *reflexivity:* $\Phi \sim \Phi$, as can be seen by choosing $\Phi_t = \Phi$ and $H_t = 0$ for every $t \in [0, 1]$.
- *symmetry:* $\Phi \sim \Psi$ implies that $\Psi \sim \Phi$ by reversing the flow of time, i.e. by considering the homotopy (Φ_{1-t}, H_{1-t}) .
- *transitivity:* if $\Phi \sim \Psi$ and $\Psi \sim \Xi$ then there exists a homotopy (Θ_{1t}, H_{1t}) between Φ and Ψ and a homotopy (Θ_{2t}, H_{2t}) joining Ψ and Ξ . It is then sufficient to glue Θ_1 and Θ_2 and rescale the time variables, so that the new time variable takes values in the closed interval $[0, 1]$. (Notice that the resulting data will be continuous at the junction, but not differentiable in general at that point.)

Now assume that $\Phi, \Psi : \mathcal{E} \rightarrow \mathcal{E}'$ are homotopic Lie ∞ -algebroid morphisms between (E', Q') and (E, Q) , and that $\alpha, \beta : \mathcal{E}' \rightarrow \mathcal{E}''$ are homotopic Lie ∞ -algebroid morphisms between (E'', Q'') and (E', Q') . Let us denote by (Φ_t, H_t) the homotopy between Φ and Ψ , and (α_t, Θ_t) the homotopy between α and β . Then $\alpha \circ \Phi$ and $\beta \circ \Psi$ are homotopic via $(\alpha_t \circ \Phi_t, \Theta_t \circ \Phi_t + \alpha_t \circ H_t)$. This completes the proof. \square

We now give an important example, that shall be used in the sequel:

Example 8. Let (E, Q) and (E', Q') be Lie ∞ -algebroids over M and let $i \geq -1$. Let δ be a section of $\Gamma(S^{i+1}(E'^*) \otimes E)$. Let us consider δ as a map from $\Gamma(E^*)$ to $\Gamma(S^{i+1}(E'^*))$. For every Lie ∞ -algebroid morphism $\Xi : \mathcal{E} \rightarrow \mathcal{E}'$ from (E', Q') to (E, Q) , we denote by $\delta(\Xi)$ the \mathcal{O} -linear Ξ -derivation whose restriction to $\Gamma(E^*)$ is δ . For any Lie ∞ -algebroid morphism Φ from (E', Q') to (E, Q) , the differential equation:

$$\frac{d\Phi_t}{dt} = [Q, \delta(\Phi_t)] \quad \text{and} \quad \Phi_0 = \Phi,$$

has solutions defined for all $t \in \mathbb{R}$. This follows from the simple observation that $\frac{d\Phi_t^{(0)}}{dt}$ is equal to 0 if $i \geq 0$ and to $Q^{(0)} \circ \delta - \delta \circ Q^{(0)}$ for $i = -1$, so that $\Phi_t^{(0)}$ is therefore constant in the first case and affine in the second case. Now, $\delta(\Phi_t)^{(k)}$, in all cases above, depends only on δ and $\Phi_t^{(k')}$ for $k' = 0, \dots, k-1$. In view of the relation:

$$\frac{d\Phi_t^{(k)}}{dt} = \sum_{j=0}^k Q'^{(k-j)} \circ \delta(\Phi_t)^{(j)} - \delta(\Phi_t)^{(j)} \circ Q^{(k-j)},$$

an immediate recursion proves that $\delta(\Phi_t)^{(k)}$ is a polynomial in t . For all $k \in \mathbb{N}$, $\Phi_t^{(k)}$ is therefore polynomial in t , and therefore defines an homotopy between the Lie ∞ -algebroid morphisms Φ and Φ_1 .

The importance of Definition 1.21 relies on the following result, which says that two homotopic Lie ∞ -algebroid morphisms are related by a $[Q, \cdot]$ -exact term:

Proposition 1.23. *Let (E, Q) and (E', Q') be Lie ∞ -algebroids over M . For any two homotopic Lie ∞ -morphisms Φ, Ψ from (E', Q') to (E, Q) , there exists a \mathcal{O} -linear map $H : \mathcal{E} \rightarrow \mathcal{E}'$ of degree -1 such that:*

$$\Psi - \Phi = [Q, H] \tag{1.18}$$

Proof. The variation of piecewise- C^1 paths are equal to the integral of their derivatives (recall that, for us, piecewise- C^1 paths are also continuous by definition). From the relation

$\frac{d}{dt}\Phi_t = [Q, H_t]$ and from the fact that the path $t \mapsto \widehat{\Phi}_t^{(k)}$ is a continuous piecewise- C^1 for all $k \in \mathbb{N}$, we therefore obtain:

$$\begin{aligned}\Psi - \Phi &= \int_0^1 [Q, H_t] dt \\ &= \int_0^1 (Q \circ H_t + H_t \circ Q') dt \\ &= Q \circ \left(\int_0^1 H_t dt \right) + \left(\int_0^1 H_t dt \right) \circ Q'\end{aligned}$$

Hence $H = \int_0^1 H_t dt$ satisfies Condition (1.18). Also, H is \mathcal{O} -linear because so is H_t for all $t \in [0, 1]$. \square

It deserves to be noticed that the operator H introduced in the previous proposition is neither an algebra morphism, nor a derivation of any sort.

Remark 7. By isolating the components of arity 0 of Φ and Ψ in the above equations, we find that their respective linear parts $(\phi_i)_{i \geq 1}$ and $(\psi_j)_{j \geq 1}$ are homotopic in the usual sense:

$$\begin{array}{ccccccc} \dots & \xrightarrow{d} & E'_{-3} & \xrightarrow{d} & E'_{-2} & \xrightarrow{d} & E'_{-1} \\ & & \Downarrow \psi_3 & \searrow h & \Downarrow \psi_2 & \searrow h & \Downarrow \psi_1 \\ & & \phi_3 & & \phi_2 & & \phi_1 \\ \dots & \xrightarrow{d'} & E_{-3} & \xrightarrow{d'} & E_{-2} & \xrightarrow{d'} & E_{-1} \end{array}$$

where h is the dual map of the component of arity 0 of H .

We can then define what we precisely mean when we say that two Lie ∞ -algebroids are isomorphic up to homotopy:

Definition 1.24. Let (E, Q) and (E', Q') be two Lie ∞ -algebroids over M , and let $\Phi : \mathcal{E}' \rightarrow \mathcal{E}$ and $\Psi : \mathcal{E} \rightarrow \mathcal{E}'$ be Lie ∞ -algebroid morphisms between (E, Q) and (E', Q') . We say that Φ and Ψ are isomorphisms up to homotopy if:

$$\Phi \circ \Psi \sim \text{id}_{\mathcal{E}} \quad \text{and} \quad \Psi \circ \Phi \sim \text{id}_{\mathcal{E}'}$$

When such Φ, Ψ exist, we say that the Lie ∞ -algebroids (E, Q) and (E', Q') are isomorphic up to homotopy.

1.4 Existence of resolutions of a singular foliation

This section is devoted to the proof of several results of Section 1.1 related to the existence and the properties of resolutions of singular foliations. We start with Proposition 1.3. We address our special thanks to François Petit, whose knowledge of the matter was a crucial help.

Proof. The first and third item simply come from Hilbert's syzygy theorem, which is valid for finitely generated \mathcal{O} -modules, with \mathcal{O} being the ring of holomorphic functions in a neighborhood of a point in \mathbb{C}^n , or the ring of polynomial functions on \mathbb{C}^n . See for references Theorem 4 page 137 in [19] for the holomorphic case or [16] for the algebraic case.

Also, any real analytic manifold admits a complexification, such that the original manifold is the fixed point of some anti-holomorphic involution. A real-analytic singular foliation on a real analytic manifold induces a holomorphic singular foliation on the complexification. Then Hilbert's syzygy theorem applies to that complexification. Restricting on the initial manifold, the henceforth obtained resolution is still a resolution with the same length. This proves the claim.

Now, we have to prove the second item. According to Theorem 4 in [38], germs of smooth functions at a given point are a flat module over germs of real analytic functions at the same point. By definition of flatness, it means that given a complex $E_{-k-1} \rightarrow E_{-k} \rightarrow E_{-k+1}$

of vector bundles on the base manifold such that germs of real analytic functions have no cohomology at degree $-k$, the sheaf of germs of smooth sections also has no cohomology at degree $-k$. Let us choose $e \in \Gamma_U(E_{-k})$ be a local smooth section of E_{-k} defined on an open subset U which is in the kernel of $d^{(k)} : E_{-k} \rightarrow E_{-k+1}$ at every point of U . According to the previous discussion, for every point $x \in U$, and for any neighborhood $U_x \subset U$ of x , there exists a smooth section $f_x \in \Gamma_{U_x}(E_{-k-1})$ such that $d^{(k+1)}(f_x) = e$. A locally finite open cover $(U_{x_i})_{i \in I}$ indexed by I admitting a partition of unity $(\chi_i)_{i \in I}$ can be extracted from the family $(U_x)_{x \in U}$. Since $d^{(k+1)}$ is \mathcal{O} -linear $f := \sum_{i \in I} \chi_i f_i$ is a section over U of E_{-k-1} that satisfies $d^{(k+1)}(f) = e$ by construction. This proves the item.

The fourth item is proved in Example 15 below.

For the last item, one can proceed as follows. Let (E, d, ρ) a resolution of \mathcal{F} , and let $x \in M$ be a point. Let $e_1, \dots, e_k \in E_{-1}|_x$ be a basis of $d^{(2)}(E_{-2}|_x)$. Denote by $\tilde{e}_1, \dots, \tilde{e}_k$ local sections of E_{-2} whose image through $d^{(2)}$ goes through e_1, \dots, e_k . In a neighborhood U_1 of x , the sections $\tilde{e}_1, \dots, \tilde{e}_k$, as well as their image through $d^{(2)}$ in the neighborhood of x in the last step does not have empty interior images $d^{(2)}\tilde{e}_1, \dots, d^{(2)}\tilde{e}_k$ are independent at every point, and therefore define sub-vector bundles $F_{-2} \subset E_{-2}$ and $F_{-1} \subset E_{-1}$ respectively. It is easy to check that (E', d', ρ') is again a resolution of \mathcal{F} , where $E'_{-i} := E_{-i}$ for $i \neq 1, 2$ and $E'_{-i} := E_{-i}/F_{-i}$ for $i = 1, 2$ and where d' and ρ' are the unique induced map on these quotient spaces. For this new resolution, $d^{(2)}$ is zero at the point x . The operation can then be repeated for the indices $i = 2$ to find a new resolution such that $d^{(3)}$ is zero at the point x and can be continued by recursion. Each step may require to shrink the neighborhood of x on which the resolution is defined, but since the resolution is of finite length, only finitely many such operations are required, and the procedure gives a resolution defined in a neighborhood of x and which is minimal at x by construction. \square

Let us now prove Proposition 1.4:

Proof. Let $x \in M$ be a point and let (E, d, ρ) be a resolution of finite length \mathcal{F} , defined on a neighborhood U of x . Let y be a regular point of \mathcal{F} contained in U and $V \subset U$ a neighborhood of y on which the foliation is regular.

Let r be its rank on V . By definition, if V is small enough, the restriction of \mathcal{F} to V is generated by r vector fields X_1, \dots, X_r .

The restriction of the foliation \mathcal{F} to V admits therefore two different resolutions: one is given by the restriction of (E, d, ρ) to V , and the other one is given by $E'_{-1} := \mathbb{R}^r$, $E'_{-i} = 0$ for $i \neq 1$ and $\rho' : (\lambda_1, \dots, \lambda_r) \mapsto \sum_{i=1}^r \lambda_i X_i$ for all sections of E'_{-1} , sections that we consider as an r -tuple of functions on V . Since two resolutions of the same \mathcal{O} -module are isomorphic up to homotopies, the alternate sum of the ranks of the vector bundles of both resolutions are equal. In particular, the relation $r = \sum_{i \geq 1} (-1)^{i-1} \text{rk}(E_{-i})$ holds.

Since y is an arbitrary regular point of U , this proves that the ranks of all regular leaves of \mathcal{F} contained in U are equal. The manifold M being connected, the dimensions of all the regular leaves of \mathcal{F} have to be equal to that integer r . This completes the proof. \square

1.5 Examples of resolutions of singular foliations

We give several examples of resolutions of singular foliations:

Example 9. Regular foliations are singular foliations. For a regular foliation \mathcal{F} , the tangent space of the foliation $E_{-1} = T\mathcal{F}$ is a resolution, when equipped with the inclusion map as anchor. It is a Lie algebroid called the *foliation Lie algebroid associated to \mathcal{F}* .

Example 10. Quasi-graphoids (as defined by C. Debord [13]), that is: singular foliations which are projective $\mathcal{O}(M)$ -modules, are precisely singular foliations that admit, around each point, resolutions of length 1, i.e. such that $E_{-i} = 0$ for all $i \geq 2$. According to Proposition 2 in [13], they can be seen as Lie algebroids whose anchor map is injective on an open subset.

Example 11. The Lie algebra \mathfrak{sl}_2 (with its canonical generators h, e, f) acts on \mathbb{R}^2 (equipped with coordinates x, y) through the vector fields:

$$\underline{h} = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}, \quad \underline{e} = x \frac{\partial}{\partial y}, \quad \underline{f} = y \frac{\partial}{\partial x}. \quad (1.19)$$

Since $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$, the $\mathcal{O}(\mathbb{R}^2)$ -module generated by $\underline{h}, \underline{e}, \underline{f}$ is a singular foliation (that can be seen as smooth or real-analytic). The vector fields in (1.19) are not independent over $\mathcal{C}^\infty(\mathbb{R}^2)$, since:

$$xy\underline{h} + y^2\underline{e} - x^2\underline{f} = 0. \quad (1.20)$$

We use this equation to define a resolution by the following elements:

1. E_{-1} is the trivial vector bundle of rank 3 generated by 3 sections that we denote by $\tilde{e}, \tilde{f}, \tilde{h}$,
2. then we define an anchor by

$$\rho(\tilde{e}) = \underline{e}, \quad \rho(\tilde{f}) = \underline{f}, \quad \rho(\tilde{h}) = \underline{h}, \quad (1.21)$$

3. E_{-2} is the trivial vector bundle of rank 1, generated by a section that we call $\mathbf{1}$,
4. and we define a vector bundle morphism from E_{-2} to E_{-1} by:

$$d(\mathbf{1}) = xy\tilde{h} + y^2\tilde{e} - x^2\tilde{f}, \quad (1.22)$$

5. and we set $E_{-i} = 0$ and $d = 0$ for $i \geq 3$.

The triple (E, d, ρ) is a resolution of the singular foliation given by the action of \mathfrak{sl}_2 on \mathbb{R}^2 .

Example 12. We owe this conjectural example to a discussion with Rupert Yu. The adjoint action of a Lie algebra \mathfrak{g} on itself defines a singular foliation \mathcal{F}_{ad} on the manifold $M := \mathfrak{g}$ that can be seen as smooth, real analytic, holomorphic or algebraic.

Let P_1, \dots, P_l be generators of $S(\mathfrak{g})^{\mathfrak{g}}$, i.e. the algebra of polynomial functions on \mathfrak{g} invariant under adjoint action. According to Chevalley's theorem, these generators can be chosen to be independent (and l coincides with the rank of \mathfrak{g}). Let E_{-1} be the trivial bundle over $M = \mathfrak{g}$ with typical fiber \mathfrak{g} , and E_{-2} to be the trivial bundle over M with typical fiber \mathbb{R}^l . The map $\rho : E_{-1} \rightarrow TM$ is, at a point $m \in M = \mathfrak{g}$, obtained by mapping $a \in (E_{-1})_m \simeq \mathfrak{g}$ to $[a, m] \in T_m M \simeq \mathfrak{g}$, while $d^{(2)}$ is the vector bundle morphism mapping, for all $m \in M$, an l -tuple $(\lambda_1, \dots, \lambda_l) \in (E_{-2})_m$ to $\sum_{i=1}^l \lambda_i \text{grad}_m(P_i) \in (E_{-1})_m \simeq \mathfrak{g}$, where grad stands for the gradient computed with the help of the Killing form.

It is clear that $\rho(\Gamma(E_{-1})) = \mathcal{F}_{ad}$. It is clear that $\rho \circ d^{(2)} = 0$ and that the image of $d^{(2)}$ coincides with the kernel of ρ at all points on regular orbits. We conjecture that the previous complex is a resolution of \mathcal{F}_{ad} , but we have not been able to prove it.

Example 13. Let \mathcal{F} be the singular foliation of all vector fields vanishing at the origin 0 of a vector space V of dimension n . A resolution of \mathcal{F} is given by choosing the following data:

1. for all $i \in \mathbb{Z}$, E_{-i} is the trivial bundle over V with fiber $\wedge^i V^* \otimes V$,
2. at a given point $e \in V$, the anchor map $V^* \otimes V \rightarrow T_e V \simeq V$ is given by $\rho(\alpha \otimes v) = \langle \alpha, e \rangle v$,
3. at a given point $e \in V$, the differential $d^{(i+1)} : E_{-i-1} \rightarrow E_{-i}$ is given by:

$$d^{(i+1)}(\alpha \otimes u) := (i_e \alpha) \otimes u, \quad (1.23)$$

for all $\alpha \in \wedge^{i+1} V^*, u \in V$.

This sequence is isomorphic to $\dim(V)$ copies of a Koszul resolution, hence it is a resolution.

Example 14. Let φ be weight homogeneous polynomial function on $M := \mathbb{C}^n$ with an isolated singularity at the origin 0 . Let $\mathfrak{X}^i := \Gamma(\wedge^i TM)$ stand for the sheaf of i -multivector fields on M . It is classical that multivector fields on M , equipped with the differential given by the contraction by $d\varphi$:

$$\dots \xrightarrow{\iota_{d\varphi}} \mathfrak{X}^3 \xrightarrow{\iota_{d\varphi}} \mathfrak{X}^2 \xrightarrow{\iota_{d\varphi}} \mathfrak{X}^1 \xrightarrow{\iota_{d\varphi}} \mathcal{O},$$

form an exact complex except in degree 0, where the image of the map $\mathfrak{X}^1(M) \mapsto \mathcal{O}(M)$ is the ideal generated by $\left(\frac{\partial\varphi}{\partial x_1}, \dots, \frac{\partial\varphi}{\partial x_n}\right)$. We call *Koszul complex* the previous complex.

The Koszul complex can be seen as a resolution of two different singular foliations. Tensoring the previous resolution with the sheaf \mathfrak{X}^1 of vector fields on M , one can see it as being a resolution of the singular foliation of vector fields vanishing on the singular locus of φ , that is to say the singular foliation generated by

$$\left\{ \frac{\partial\varphi}{\partial x_i} \frac{\partial}{\partial x_j}, \text{ with } i, j = 1, \dots, n \right\}.$$

But the Koszul complex can also be truncated in degree 1 to get a resolution of the singular foliation \mathcal{F}_φ of all vector fields X on M with $X[\varphi] = 0$, that is the singular foliation generated by the vector fields:

$$\mathcal{F}_\varphi = \left\{ \frac{\partial\varphi}{\partial x_i} \frac{\partial}{\partial x_j} - \frac{\partial\varphi}{\partial x_j} \frac{\partial}{\partial x_i}, \text{ with } 1 \leq i < j \leq n \right\}$$

In the first case, $\Gamma(E_{-i}) \simeq \mathfrak{X}^i \otimes_{\mathcal{O}} \mathfrak{X}^1$ and $d := \iota_{d\varphi} \circ \text{id}$, while in the second case $\Gamma(E_{-i}) \simeq \mathfrak{X}^{i+1}$ and $d := \iota_{d\varphi}$.

Example 15. The following example, that we owe to Jean-Louis Tu, provides a smooth singular foliation that does not admit smooth resolutions. Let χ be a smooth real-valued function on $M := \mathbb{R}$ vanishing identically on \mathbb{R}_- and strictly positive on \mathbb{R}_+ . Consider the singular foliation \mathcal{F} generated by the vector field V on \mathbb{R} defined by:

$$V := \chi(t) \frac{d}{dt}.$$

Assume that a smooth resolution of \mathcal{F} exists. By (an obvious adaptation of) the last item in Proposition 1.3, we can replace it on a neighborhood of 0 by a resolution (E, d, ρ) such that $d_0^{(2)} = 0$. Since $\rho(\Gamma(E_{-1})) = \mathcal{F}$, in a neighborhood U of $0 \in \mathbb{R}$, E_{-1} admits a nowhere vanishing section e_t such that $\rho(e_t(s)) = V_s$ for every s . Since an open interval of \mathbb{R} is a contractible manifold, each of the vector spaces $(E_{-i})_{i \in \mathbb{N}}$ must be trivial. Denote by n_i the rank of E_{-i} . In particular, $\Gamma(E_{-1})$ is generated by n_1 canonical generators e_1, \dots, e_{n_1} . Without any loss of generality, we can assume that $e_1 = e$. Moreover, since the image of the anchor map is V , we have for every $1 \leq k \leq n_1$: $\rho(e_k) = g_k V = g_k \rho(e_1)$ for some function $g_k \in \mathcal{C}^\infty(U)$. It implies that $\rho(e_k - g_k e_1) = 0$. Since $\text{Im}(d^{(2)}) = \text{Ker}(\rho)$, there exist sections $f_2, \dots, f_{n_1} \in E_{-2}$ such that $d^{(2)}(f_k) = e_k - g_k e_1$ for all $k = 2, \dots, n_1$. But this contradicts the fact that $d^{(2)}$ vanishes at 0, unless $n_1 = 1$. But if $n_1 = 1$, then the kernel of $\rho : \Gamma_U(E_{-1}) \rightarrow \mathfrak{X}(U)$ is made of all real-valued functions vanishing on $\mathbb{R}_+ \cap U$, which is not finitely generated. As a conclusion \mathcal{F} does not admit smooth resolutions.

1.6 Proof of Theorems 1.6 and 1.8

1.6.1 Arity and linear part

The notion of arity will be at the core of most of the proofs of Theorems 1.6 and 1.8, and as such it deserves to be studied separately. Let E be a positively graded manifold, that is a family of vector bundles $(E_{-i})_{i \geq 1}$ over a base manifold M (positively graded means that coordinate functions on E have positive degree). Recall that, by a *vector field*, we mean a derivation of the sheaf of functions \mathcal{E} over E and by a *vertical vector field* we mean a \mathcal{O} -linear derivation of \mathcal{E} (which geometrically means that the vector field is tangent to the fibers of $E \rightarrow M$).

Recall from Section 1.3 that a function $F \in \mathcal{E}$ is of *arity* n and *degree* k if $F \in \Gamma(S^n(E^*)_k)$, i.e. if it is a section of $\sum_{i_1 + \dots + i_n = k} E_{-i_1}^* \odot \dots \odot E_{-i_n}^*$, where \odot denotes the graded symmetric product. Recall that a vector field is said to be of *arity* n and of *degree* $k \in \mathbb{Z}$ when, seen as a derivation of \mathcal{E} , it increases the arity by n and the degree by k . The following proposition states the main properties of the arity of a function and of a vector field:

Proposition 1.25. *Let $E \rightarrow M$ be a positively graded manifold.*

1. The allowed values of arity of a function range from 0 to $+\infty$, and that of a vector field range from -1 to $+\infty$.
2. The arity of a function is less than or equal to its degree.
3. Vector fields of arity -1 are vertical and of negative degree.
4. Vector fields of arity 0 and of non zero degree are vertical.
5. Vector fields of arity $n \neq 1$ and of degree $+1$ are vertical.
6. The Lie bracket of vector fields of arity n and n' is of arity $n + n'$.

Now, assume that a homological vector field Q of degree $+1$ is given on E , and let us decompose it as:

$$Q = \sum_{i=0}^{\infty} Q^{(i)}$$

with $Q^{(i)}$ its component of arity i . The component of arity 0 is \mathcal{O} -linear. It corresponds therefore to a degree $+1$ linear endomorphism of E , that is to say, to a collection of maps:

$$d^{(i)} : E_{-i} \longrightarrow E_{-i+1},$$

for all $i \geq 2$. For any $\xi \in E_{-i+1}^*$ and $u \in E_{-i}$, the action of $d^{(i)}$ is defined as:

$$\langle \xi, d^{(i)}(u) \rangle = (-1)^{i-1} \langle Q^{(0)}(\xi), u \rangle. \quad (1.24)$$

The component of arity zero in the relation $[Q, Q] = 0$ gives $[Q^{(0)}, Q^{(0)}] = 0$, which, in turn, proves that $d^{(i-1)} \circ d^{(i)} = 0$. We deduce the following Lemma:

Lemma 1.26. *Let $E = (E_{-i})_{i \geq 1}$ be a positively graded manifold over a manifold M . There is a one-to-one correspondence between homological vertical vector fields of arity 0 and collections of maps $d = (d^{(i)} : E_{-i} \rightarrow E_{-i+1})_{i \geq 2}$ making E a complex.*

Now let us say a few words about vertical vector fields. Vertical vector fields form a graded Lie subalgebra of the graded Lie algebra of vector fields (the grading being given by the degree). The following proposition is an easy generalization of Batchelor [7] or Kotov-Strobl [26]. This result that will be used for the proof of the main theorem:

Proposition 1.27. *Let A be a vector bundle over M . There is a one-to-one correspondence between almost-Lie algebroid structures on $A \rightarrow M$ and vector fields Q of degree $+1$ on the graded manifold $A^*[1]$ whose self-commutator $[Q, Q]$ is vertical.*

1.6.2 A fundamental lemma on vertical vector fields

Let E be a positively graded manifold, with base manifold M . Recall that a *vertical vector field* on E is a vector field which is \mathcal{O} -linear, which geometrically means that it is parallel to the fibers of the projection from E onto its base M . Identifying the tangent space of E at each point with the fiber E , we obtain that for every $n \geq 1$, there is an isomorphism between the vector space of vertical vector fields of arity $n - 1$ and elements of the direct sum:

$$\mathfrak{V}^{(n-1)} = \bigoplus_{k=-\infty}^{+\infty} \bigoplus_{\substack{i-j=k \\ i, j \geq 1}} \Gamma(S^n(E^*)_i \otimes E_{-j})$$

Sections of $S^n(E^*)_i \otimes E_{-j}$ are said to be of *height* i and *depth* j . Since homogeneous elements of E^* have at least degree one, the height is valued in $\{n, n + 1, \dots\}$, whereas the depth is valued in $\{1, 2, \dots\}$, so that vertical vector fields of arity $n - 1$ and degree k can be represented as infinite sums of elements in the anti-diagonals $i - j = k$ ('height - depth = k ') in the sections of the bicomplex:

$$\begin{array}{ccccccc}
& & \cdots & & \cdots & & \cdots \\
\hline
\cdots & \longrightarrow & S^n(E^*)_{n+2} \otimes E_{-3} & \longrightarrow & S^n(E^*)_{n+2} \otimes E_{-2} & \longrightarrow & S^n(E^*)_{n+2} \otimes E_{-1} \longrightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longrightarrow & S^n(E^*)_{n+1} \otimes E_{-3} & \longrightarrow & S^n(E^*)_{n+1} \otimes E_{-2} & \longrightarrow & S^n(E^*)_{n+1} \otimes E_{-1} \longrightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longrightarrow & S^n(E^*)_n \otimes E_{-3} & \longrightarrow & S^n(E^*)_n \otimes E_{-2} & \longrightarrow & S^n(E^*)_n \otimes E_{-1} \longrightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
& & 0 & & 0 & & 0
\end{array}$$

The horizontal lines correspond to the action of $\text{id} \otimes d$, whereas the vertical lines correspond to the action of $Q^{(0)} \otimes \text{id}$. Summing the vertical and the horizontal differentials gives a total differential:

$$\partial = Q^{(0)} \otimes \text{id} + \text{id} \otimes d, \quad (1.25)$$

on the total bicomplex $\mathfrak{U}^{(n-1)}$.

The *depth* (resp. *height*) of a vertical vector field of a fixed degree is the minimum of the depths (resp. heights) of all its non zero components. In the diagram above it would coincide with the depth and height of the lowest non-zero element of the anti diagonal which symbolizes the given vector field of degree k . The *root* of a vertical vector field of degree k is its component of depth 1 – the *root* of X is denoted by $rt(X)$. It can be zero, and in that case the depth of X is strictly higher than 1. A *root-free* element is a vertical vector field whose root is zero (in particular it means that its depth is strictly higher than 1), otherwise it is said to be *rooted*. For degree reason, any vertical vector field of arity n and degree less than or equal to $n - 1$ is root-free.

Now, it is clear that, when (E, Q) is a Lie ∞ -algebroid, and $Q^{(0)}$ is the component of arity 0 of Q whose dual differential we denote by d (as in Lemma 1.26), then $X \mapsto [Q^{(0)}, X]$ squares to zero and therefore makes vertical vector fields a complex. This complex restricts to vertical vector fields of a given arity. Moreover, upon decomposing vertical vector fields of a arity $n - 1$ with respect to their height and depth, we obtain this operator is the total differential ∂ defined in Equation (1.25). the following lemma:

Lemma 1.28. *For every $n \geq 0$, the space of vertical vector fields of arity n , equipped with the adjoint action $X \mapsto [Q^{(0)}, X]$, is, as a complex, isomorphic to the bicomplex $(\mathfrak{U}^{(n)}, \partial)$.*

The vertical lines in this bicomplex may not be exact, whereas the exactness of the sequence:

$$\dots \xrightarrow{d} \Gamma(E_{-2}) \xrightarrow{d} \Gamma(E_{-1}) \xrightarrow{\rho} \mathcal{F} \longrightarrow 0$$

implies that the horizontal lines are exact, except maybe at depth 1. More precisely, by exactness of the short sequence $\Gamma(E_{-2}) \xrightarrow{d} \Gamma(E_{-1}) \xrightarrow{\rho} \mathcal{F}$, a rooted element is a coboundary if and only if its image under $\text{id} \otimes \rho$ is zero. By diagram chasing, this leads to the following fundamental lemma.

Lemma 1.29. *Let $n \geq 1$ be an integer, and consider the bicomplex $(\mathfrak{U}^{(n)}, [Q^{(0)}, \cdot])$ of vertical vector fields of arity n .*

1. *A root-free cocycle is a coboundary.*
2. *A cocycle whose root is in the kernel of $\text{id} \otimes \rho$ is a coboundary.*

Remark 8. The first item implies that the cohomology of this bicomplex is zero in degree less than or equal to $n - 1$.

1.6.3 Construction of Lie ∞ -algebroid structures on a resolution

We now intend to prove Theorem 1.6. We present a proof for the case of smooth resolutions over smooth manifolds, but the arguments below also work when working in the real analytic or holomorphic case in a neighborhood of a point.

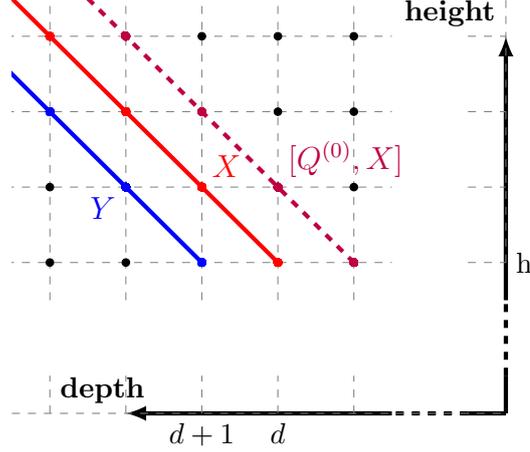


Figure 1: A root-free cocycle of depth d is actually a coboundary: $X = [Q^{(0)}, Y]$.

We have to prove that, given a resolution (E, d, ρ) of \mathcal{F} , there is a homological degree +1 vector field Q on the graded manifold E whose linear part is the given resolution of \mathcal{F} . The proof of the theorem consists in finding a homological degree +1 vector field Q on $E \rightarrow M$ with the following arity decomposition:

$$Q = Q^{(0)} + Q^{(1)} + Q^{(2)} + \dots \quad (1.26)$$

where $Q^{(i)}$ is a vector field on $E \rightarrow M$ of degree 1 and arity i . The homological condition $[Q, Q] = 0$ gives the following set of equations:

$$[Q^{(0)}, Q^{(0)}] = 0 \quad (1.27)$$

$$[Q^{(0)}, Q^{(1)}] = 0 \quad (1.28)$$

$$\forall n \geq 2 \quad [Q^{(0)}, Q^{(n)}] = -\frac{1}{2} \sum_{\substack{1 \leq i, j \leq n-1 \\ i+j=n}} [Q^{(i)}, Q^{(j)}] \quad (1.29)$$

We define $Q^{(0)}$ by dualizing the differential d defined on the resolution (E, d, ρ) associated to the singular foliation \mathcal{F} as in Equation (1.24). The assumption that d squares to zero implies dually that $Q^{(0)}$ is a cohomological vector field on the graded manifold E , i.e. $[Q^{(0)}, Q^{(0)}] = 0$.

The proof will then develop in three steps: 1) find $Q^{(1)}$ (= the binary bracket + the anchor map), 2) find $Q^{(2)}$ (= the 3-ary bracket), 3) find $Q^{(n)}$ (= the n -ary bracket) for every $n \geq 3$. We have to separate the cases $n = 1$ and 2 because the methods are different. We shall define at each step a degree 1 vector field Q_n as:

$$Q_n = \sum_{0 \leq i \leq n} Q^{(i)} = Q_{n-1} + Q^{(n)} \quad (1.30)$$

which has the following property: the commutator $[Q_n, Q_n]$ is a sum of vector fields of arity higher than or equal to $n + 1$. The direct sum $Q = \sum_{n \geq 0} Q^{(n)}$ is then a vector field of degree +1 such that $[Q, Q] = 0$, hence defining a Lie ∞ -algebroid structure on E .

According to Proposition 1.12, there exists an almost-Lie algebroid structure on E_{-1} whose anchor is ρ . According to Proposition 1.27, this almost-Lie almost algebroid structure corresponds to a vector field X on the graded manifold E_{-1} , which can be extended to a vector field of arity 1 and degree 1 on E that we still denote by X .

Remark 9. Considering a vector field X of arity 1 and degree 1 on the graded manifold E_{-1} as a vector field of arity 1 and degree 1 on E is an operation which is possible over a smooth

manifold or in the real-analytic and holomorphic case in a neighborhood of a point. Indeed, X can be completed by choosing a local trivialization of E_{-i} for all $i \geq 2$, and defining X to be zero on the dual sections. In the smooth case, such local extensions can be glued together using partitions of unity. But this operation can not be completed globally in the holomorphic or real analytic cases.

The vector field $[Q^{(0)}, X]$ is vertical and is a coboundary in the space of vertical vector fields. By Proposition 1.27, the vector field $[X, X]$ is vertical. Hence, $[Q^{(0)}, [X, X]]$ is vertical as well. Applying this vector field to a function $f \in \mathcal{O}$ we have:

$$0 = \frac{1}{2} [Q^{(0)}, [X, X]](f) = [[Q^{(0)}, X], X](f) = [Q^{(0)}, X] \circ \rho^*(d_{\text{dR}}f). \quad (1.31)$$

Equation (1.31) is equivalent to the following condition (recall that rt stand for the root of a vertical vector field, see Section 1.6.2):

$$(\text{id} \otimes \rho) \circ (rt[Q^{(0)}, X]) = 0. \quad (1.32)$$

As a consequence, the vertical vector field $[Q^{(0)}, X]$ satisfies the assumptions of item 2 of Proposition 1.29. Hence there exists a vertical vector field Y of arity 1 and depth greater than or equal to 2 such that:

$$[Q^{(0)}, X] = -[Q^{(0)}, Y] \quad (1.33)$$

The vector field $Q^{(1)} = X + Y$, satisfies Equation (1.28).

Remark 10. Condition (1.32) is equivalent to the fact that the kernel of ρ is stable under the adjoint action:

$$\forall x \in \Gamma(E_{-2}), y \in \Gamma(E_{-1}) \quad \rho(\{d(x), y\}_{E_{-1}}) = 0. \quad (1.34)$$

From now on we write $Q_1 = Q^{(0)} + Q^{(1)}$ and we are interested in the commutator $[Q_1, Q_1]$. By Equation (1.28), it is naturally equal to $[Q^{(1)}, Q^{(1)}]$. In the decomposition $Q^{(1)} = X + Y$, the vector field Y is vertical. Hence $[Y, Y]$ is vertical. Also, $Y[\alpha] = 0$ for every $\alpha \in \Gamma(E_{-1}^*)$ (because the depth of Y is greater than or equal to 2), so $[X, Y](f) = 0$ for every function f on the base manifold M , and $[X, Y]$ is vertical. Since $[X, X]$ is vertical, this implies that $[Q^{(1)}, Q^{(1)}]$ is vertical.

The Jacobi identity for graded vector fields implies that the vector field $[Q^{(1)}, Q^{(1)}]$ is a cocycle:

$$[Q^{(0)}, [Q^{(1)}, Q^{(1)}]] = 2[[Q^{(0)}, Q^{(1)}], Q^{(1)}] = 0. \quad (1.35)$$

We show that it is in fact a coboundary in the bicomplex $\mathfrak{U}^{(2)}$. Recall that for every odd vector fields, the relation $[[U, U], U] = 0$ holds. For $U = Q^{(1)}$ and for every function $f \in \mathcal{O}$, this relation gives:

$$0 = [[Q^{(1)}, Q^{(1)}], Q^{(1)}](f) = [Q^{(1)}, Q^{(1)}] \circ \rho^*(d_{\text{dR}}f). \quad (1.36)$$

The above equation means (cf Remark 11):

$$(\text{id} \otimes \rho) \circ (rt[Q^{(1)}, Q^{(1)}]) = 0. \quad (1.37)$$

Being a cocycle whose root is the kernel of the anchor map, $[Q^{(1)}, Q^{(1)}]$ is a coboundary by item 2. in Lemma 1.29. Hence there exists a degree 1 element $Q^{(2)} \in \mathfrak{U}^{(2)}$ such that:

$$\frac{1}{2}[Q^{(1)}, Q^{(1)}] = -[Q^{(0)}, Q^{(2)}]. \quad (1.38)$$

Equation (1.29) is therefore satisfied for $n = 2$.

Remark 11. The commutator $[Q^{(1)}, Q^{(1)}]$ corresponds by duality to the Jacobiator of the 2-bracket. In particular it may not identically vanish (this is why $Q^{(2)}$ is needed). However the Jacobi identity of the bracket $[\cdot, \cdot]_{E_{-1}}$ on the sections of E_{-1} should lie in the kernel of the anchor map:

$$\forall x, y, z \in \Gamma(E_{-1}) \quad \rho(\text{Jac}(x, y, z)) = 0, \quad (1.39)$$

where $\text{Jac}(x, y, z)$ is the Jacobiator of the bracket on E_{-1} . The above relation is equivalent to Equation (1.37) (by duality).

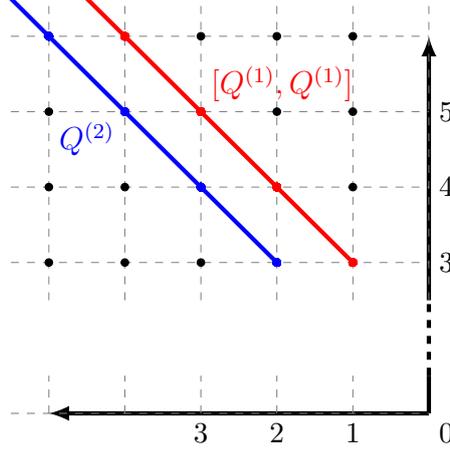


Figure 2: The commutator $[Q^{(1)}, Q^{(1)}]$ is a cocycle in the space of vertical vector fields, whose root lies in the kernel of the anchor map. As such it is a coboundary, i.e. there exists $Q^{(2)} \in \mathfrak{U}^{(2)}$ such that: $\frac{1}{2}[Q^{(1)}, Q^{(1)}] = -[Q^{(0)}, Q^{(2)}]$.

Now assume that we have built all $Q^{(i)}$ satisfying Equations (1.29) up to some order $n \geq 2$. Let $D_{n+1} := \sum_{\substack{1 \leq i, j \leq n \\ i+j=n+1}} [Q^{(i)}, Q^{(j)}]$. A routine computation gives that D_{n+1} commutes

with $Q^{(0)}$. Also, it is a vertical vector field: $[Q^{(i)}, Q^{(j)}]$ is vertical for $i \neq 1$ and $j \neq 1$ because both $Q^{(i)}$ and $Q^{(j)}$ are, and $[Q^{(1)}, Q^{(n)}]$ is vertical because the depth of $Q^{(n)}$ is greater than or equal to 2. Moreover the bracket $[Q^{(i)}, Q^{(n+1-i)}]$ has height $n+2$ and degree 2 then D_{n+1} is a root-free element of the bicomplex $\mathfrak{U}^{(n+1)}$ of vertical vector fields depth at least n .

Thus this cocycle is automatically a coboundary by the first item in Lemma 1.29 (cf. Figure 1). Then there exists an element $Q^{(n+1)}$ of arity $n+1$ and depth at least $n+1$ such that:

$$\frac{1}{2} \sum_{\substack{1 \leq i, j \leq n \\ i+j=n+1}} [Q^{(i)}, Q^{(j)}] = -[Q^{(0)}, Q^{(n+1)}] \quad (1.40)$$

that is precisely Equation (1.29) for $n+1$. The result then follows by induction. This completes the proof of Theorem 1.6.

1.6.4 Universality of the Lie ∞ -algebroid over a foliation

In this section, we prove the first item of Theorem 1.8. The second item is a simple corollary of the first one. We prove the theorem in the smooth case. The real analytic and holomorphic cases are similar, upon restricting to a neighborhood of a point, as can be checked step-by-step.

Assume that we are given a singular foliation \mathcal{F} that admits a resolution (E, d, ρ) . By Theorem 1.6, the resolution (E, d, ρ) can be endowed with a Lie ∞ -algebroid structure with linear part (E, d, ρ) . Let (E', Q') be a Lie ∞ -algebroid whose induced singular foliation \mathcal{F}' is a sub-foliation of \mathcal{F} , that is:

$$\rho'(\Gamma(E'_{-1})) \subset \rho(\Gamma(E_{-1})) = \mathcal{F},$$

and let (E', d', ρ') be its linear part.

By definition, the complex

$$\dots \xrightarrow{d^{(3)}} \Gamma(E_{-2}) \xrightarrow{d^{(2)}} \Gamma(E_{-1}) \xrightarrow{\rho} \mathcal{F} \longrightarrow 0$$

is a resolution of \mathcal{F} in the category of \mathcal{O} -modules. By a classical theorem of algebraic topology, given a complex of \mathcal{O} -modules, for instance:

$$\dots \xrightarrow{d'^{(3)}} \Gamma(E'_{-2}) \xrightarrow{d'^{(2)}} \Gamma(E'_{-1}) \xrightarrow{\rho'} \mathcal{F}' \longrightarrow 0$$

there exists a family of vector bundle morphisms $\phi_i : E'_{-i} \rightarrow E_{-i}$ forming a chain map

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{d^{(4)}} & E'_{-3} & \xrightarrow{d^{(3)}} & E'_{-2} & \xrightarrow{d^{(2)}} & E'_{-1} & \xrightarrow{\rho'} & TM \\
& & \downarrow \phi_3 & & \downarrow \phi_2 & & \downarrow \phi_1 & & \downarrow \text{id} \\
\cdots & \xrightarrow{d^{(4)}} & E_{-3} & \xrightarrow{d^{(3)}} & E_{-2} & \xrightarrow{d^{(2)}} & E_{-1} & \xrightarrow{\rho} & TM
\end{array}$$

This classical result of algebraic topology is summarized in the following Lemma:

Lemma 1.30. *Let (E, d, ρ) be a resolution of a singular foliation \mathcal{F} . Let (E', Q') be a Lie ∞ -algebroid whose associated singular foliation is a subfoliation of \mathcal{F} . Then there exists a chain map $\phi = (\phi_i)_{i \geq 1}$ from (E', d', ρ') to (E, d, ρ) such that $\rho \circ \phi_1 = \rho'$.*

The proof of the existence part of the first item of Theorem 1.8 relies on a variation of Lemma 1.29. For all $n \geq 2$, there is a natural bicomplex structure on $\Gamma(S^n(E'^*) \otimes E)$:

$$\begin{array}{ccccccc}
& & \cdots & & \cdots & & \cdots \\
& & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longrightarrow & S^n(E'^*)_{n+2} \otimes E_{-3} & \longrightarrow & S^n(E'^*)_{n+2} \otimes E_{-2} & \longrightarrow & S^n(E'^*)_{n+2} \otimes E_{-1} & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & S^n(E'^*)_{n+1} \otimes E_{-3} & \longrightarrow & S^n(E'^*)_{n+1} \otimes E_{-2} & \longrightarrow & S^n(E'^*)_{n+1} \otimes E_{-1} & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & S^n(E'^*)_n \otimes E_{-3} & \longrightarrow & S^n(E'^*)_n \otimes E_{-2} & \longrightarrow & S^n(E'^*)_n \otimes E_{-1} & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}$$

where the horizontal lines correspond to the action of $\text{id} \otimes d$ and the vertical lines to the action of $Q'^{(0)} \otimes \text{id}$. Their sum $\tilde{\partial} = Q'^{(0)} \otimes \text{id} + \text{id} \otimes d$ is a differential, and we denote by $(\mathfrak{V}^{(n-1)}, \tilde{\partial})$ this complex. We say again that an element in $S^n(E'^*)_l \otimes E_{-k}$ is of depth k and height l . Lemma 1.29 can be adapted:

Lemma 1.31. *For any $n \geq -1$, let $\mathfrak{V}^{(n)} = \Gamma(S^{n+1}(E'^*) \otimes E)$. Then:*

1. *for all $i \geq 2$, any cocycle in $\mathfrak{V}^{(i)}$ of degree 1 is a coboundary,*
2. *a cocycle in $\mathfrak{V}^{(1)} := \Gamma(S^2(E'^*) \otimes E)$ of degree 1 whose component in $\Gamma(S^2(E'^*)_{-1} \otimes E_{-1})$ is in the kernel of $\text{id} \otimes \rho$ is a coboundary.*

and also:

3. *for all $i \geq 1$, any cocycle in $\mathfrak{V}^{(i)}$ of degree 0 is a coboundary,*
4. *a cocycle in $\mathfrak{V}^{(0)} = \Gamma(E'^* \otimes E)$ of degree 0 whose component in $\Gamma(E'^*_{-1} \otimes E_{-1})$ is in the kernel of $\text{id} \otimes \rho$ is a coboundary.*

Proof. Since (E, d, ρ) is a resolution, since $S^i(E'^*)$ is a projective \mathcal{O} -module for all $i \geq 0$, and since tensoring over \mathcal{O} preserves exactness, all lines in the bicomplex above are exact except maybe in depth 1, where exact elements are given by the kernel of $\text{id} \otimes \rho$.

Now, for degree reasons, cocycles of degree 1 (resp. 0) have no components in sections of $S^{i+1}(E'^*) \otimes E_{-1}$ when $i \geq 2$ (resp. $i \geq 1$). Hence, such a cocycle takes only values in a sub-bicomplex of $\mathfrak{V}^{(i)}$ where all lines are exact, i.e. the bicomplex of elements of depth greater than or equal to 2, hence it is a coboundary by simple diagram chasing. This proves the first and third item. The second item comes from the simple observation that a cocycle in $\Gamma(S^2(E'^*) \otimes E)$ of degree 1 may only have one term of depth 1, and this term is a section of $S^2(E'^*)_{-1} \otimes E_{-1}$. If this term lies in the kernel of $\text{id} \otimes \rho$, it is in the image of $\text{id} \otimes d^{(2)}$ and the result then follows by diagram chasing. This proves the second item. The proof of the fourth item is similar. \square

Let us now give the meaning of this bicomplex. Consider a linear map $\Theta : \mathcal{E} \rightarrow \mathcal{E}'$. We say that Θ is of arity $i \in \mathbb{Z}$ when it increases the arity by i , i.e. if $F \in \mathcal{E}$ is of arity $j \in \mathbb{N}$, then $\Theta(F)$ is of arity $i + j$. More generally, a linear map $\Theta : \mathcal{E} \rightarrow \mathcal{E}'$ decomposes as a direct sum:

$$\Theta = \sum_{i \in \mathbb{Z}} \Theta^{(i)}$$

where $\Theta^{(i)}$ is a \mathcal{O} -linear map of arity i for all $i \in \mathbb{Z}$. Of course, the following relation holds for any two linear maps $\Theta_1 : \mathcal{E} \rightarrow \mathcal{E}'$ and $\Theta_2 : \mathcal{E}' \rightarrow \mathcal{E}''$ and all $n \in \mathbb{Z}$:

$$(\Theta_1 \circ \Theta_2)^{(n)} = \sum_{i \in \mathbb{Z}} \Theta_1^{(i)} \circ \Theta_2^{(n-i)} \quad (1.41)$$

The previous infinite sum is of course finite when applied to a particular element of \mathcal{E} , since there is no element of negative arity.

Remark 12. Two \mathcal{O} -linear graded commutative algebra morphisms: $\Phi, \Psi : \mathcal{E} \rightarrow \mathcal{E}'$ satisfy $\Phi^{(i)} = \Psi^{(i)}$ for all $i = 0, \dots, n$ if and only if the linear operators $\Phi^{(i)}$ and $\Psi^{(i)}$ coincide on functions of arity 1, i.e. on $\Gamma(E^*) \subset \mathcal{E}$, for all $i \in \{1, \dots, n\}$.

Consider a chain map ϕ from (E', d', ρ') to (E, d, ρ) and let $\Phi^{(0)} : \mathcal{E} \rightarrow \mathcal{E}'$ be the corresponding dual graded algebra morphism, i.e. the unique algebra morphism whose restriction to a map $\Gamma(E^*) \rightarrow \Gamma(E'^*)$ is the dual of $\phi : E' \rightarrow E$. By construction, $\Phi^{(0)}$ is of arity 0. Since $\Phi^{(0)}$ -derivations are determined by their restrictions to sections of E^* and are \mathcal{O} -linear, they can be identified with sections of $S(E'^*) \otimes E$. Under this identification, $\Phi^{(0)}$ -derivations of arity i are in one-to-one correspondence with elements of $S^{i+1}(E'^*) \otimes E$. In other words, they are in one-to-one correspondence with sections of $\mathfrak{A}^{(i)}$.

Moreover, since $\Phi^{(0)} : \mathcal{E} \rightarrow \mathcal{E}'$ arises from a chain map from the complex (E', d', ρ') to the complex (E, d, ρ) , $\Phi^{(0)}$ is a chain map with respect to $Q^{(0)}$ and $(Q')^{(0)}$, i.e. with respect to the components of arity 0 of Q and Q' . Now, a simple computation gives that if δ is a $\Phi^{(0)}$ -derivation of degree d and arity i , then

$$\Delta(\delta) := Q'^{(0)} \circ \delta - (-1)^d \delta \circ Q^{(0)} \quad (1.42)$$

is a $\Phi^{(0)}$ -derivation of degree $d + 1$ and arity i again. A straightforward calculation shows that $\Delta^2 = 0$. Hence, the differential Δ turns the space of $\Phi^{(0)}$ -derivations of arity i into a complex. The following lemma is a simple computation.

Lemma 1.32. *Let $\Phi^{(0)} : \mathcal{E} \rightarrow \mathcal{E}'$ be the graded algebra morphism dualizing a chain map ϕ from (E', d', ρ') to (E, d, ρ) . Then the space of $\Phi^{(0)}$ -derivations of arity i , equipped with the differential Δ coincides, as a complex, with the bicomplex $(\mathfrak{A}^{(i)}, \tilde{\partial})$.*

To prove the first item of Theorem 1.8, we have to show that there exists a Lie ∞ -algebroid morphism from (E', Q') to (E, Q) over M , that is: a graded commutative algebra morphism $\Phi : \mathcal{E} \rightarrow \mathcal{E}'$ whose term of arity 0 coincides with $\Phi^{(0)}$ and which intertwines the homological vector fields Q and Q' :

$$Q' \circ \Phi - \Phi \circ Q = 0. \quad (1.43)$$

It is sufficient for that purpose to construct a sequence $(\Phi_n)_{n \in \mathbb{N}}$ of \mathcal{O} -linear algebra morphisms $\Phi_n : \mathcal{E} \rightarrow \mathcal{E}'$ whose first term is $\Phi_0 = \Phi^{(0)}$ and that satisfy the following properties:

1. The linear map $Q' \circ \Phi_n - \Phi_n \circ Q$ has no components of arity i for $i = 0, \dots, n$,
2. The components $(\Phi_n)^{(i)}$ of arity $i = 0, \dots, n - 1$ of Φ_n coincide with those of Φ_{n-1} ,
3. The restriction to $\Gamma(E^*)$ of the component $(\Phi_n)^{(n+1)}$ of arity $n + 1$ is vanishing.

The morphism $\Phi : \mathcal{E} \rightarrow \mathcal{E}'$ is then defined to be the ‘limit’ of the Φ_n . By ‘limit’, we mean precisely the following: for all $n \in \mathbb{N}$ and for all F of arity $k \in \mathbb{N}$, $\Phi(F)$ is defined to be the element of \mathcal{E}' whose component $\Phi(F)^{(n+k)}$ of arity $n + k$ is defined to be $\Phi_m(F)^{(n+k)}$, with m being any integer greater than or equal to n . The second item of the definition of the sequence $(\Phi_n)_{n \in \mathbb{N}}$ implies that $m \mapsto \Phi_m(F)^{(n+k)}$ is constant for $m \geq n$ and justifies this

definition. Such a morphism Φ is a graded algebra morphism that satisfies Equation (1.43) by construction.

We construct this sequence by recursion. First, we choose $\Phi_0 := \Phi^{(0)}$, the dual of the chain map ϕ from (E', d', ρ') to (E, d, ρ) as in Lemma 1.30. The latter being the dual of a chain map, $Q' \circ \Phi^{(0)} - \Phi^{(0)} \circ Q$ has no component of arity 0, so the required property holds at $n = 0$.

Assume now Φ_i constructed for $i = 0$ to n . Let us construct Φ_{n+1} . Consider the linear operator from \mathcal{E} to \mathcal{E}' given by:

$$\Delta_{\Phi_n} := Q' \circ \Phi_n - \Phi_n \circ Q. \quad (1.44)$$

An easy computation gives the following relation:

$$\Delta_{\Phi_n}(FG) = \Delta_{\Phi_n}(F)\Phi_n(G) - (-1)^{|F|}\Phi_n(F)\Delta_{\Phi_n}(G), \quad (1.45)$$

for all degree-homogeneous $F, G \in \mathcal{E}$. Let us prove that the operator Δ_{Φ_n} is \mathcal{O} -linear. In view of (1.45), it is sufficient to prove that it vanishes when applied to any $f \in \mathcal{O}$. Now, the relation that $\Phi^{(0)} \circ Q(f) = Q'(f)$ holds for all $f \in \mathcal{O}$, since, by Lemma 1.30, it dualizes the relation $\rho \circ \phi_1 = \rho'$. Recall that $Q(f) = \rho^* \circ \text{d}_{\text{DR}}(f)$ takes values in $\Gamma(E_{-1}^*)$. Now for every operator $\Psi : \mathcal{E} \rightarrow \mathcal{E}'$ of degree 0 and every $F \in \Gamma(E_{-1}^*)$, $\Psi(F)$ has to be in $\Gamma(E_{-1}^*)$, hence $\Psi(F) = \Psi^{(0)}(F)$. As a consequence:

$$\Phi_n \circ Q(f) = \Phi_n^{(0)} \circ Q(f) = \Phi^{(0)} \circ Q(f) = Q'(f). \quad (1.46)$$

Since $f = \Phi_0(f)$ by definition, and that $\Phi_0(f) = \Phi_n(f)$ by induction, we have that $Q'(f) = Q' \circ \Phi_n(f)$ and thus the relation $\Phi_n \circ Q(f) = Q' \circ \Phi_n(f)$ holds true for all $f \in \mathcal{O}$, and proves that Δ_{Φ_n} is \mathcal{O} -linear.

Equation (1.45), applied to elements $F, G \in \mathcal{E}$ of arities i and j , implies, when considering the component of arity $n + 1 + i + j$:

$$(\Delta_{\Phi_n}(FG))^{(n+1+i+j)} = (\Delta_{\Phi_n}(F))^{(n+1+i)}(\Phi_n(G))^{(j)} - (-1)^{|F|}(\Phi_n(F))^{(i)}(\Delta_{\Phi_n}(G))^{(n+1+j)}, \quad (1.47)$$

while the remaining terms disappear in view of the recursion assumption. By definition of the components of arity $n + 1$ and 0 of an operator, this implies:

$$\Delta_{\Phi_n}^{(n+1)}(FG) = \Delta_{\Phi_n}^{(n+1)}(F)\Phi^{(0)}(G) - (-1)^{|F|}\Phi^{(0)}(F)\Delta_{\Phi_n}^{(n+1)}(G). \quad (1.48)$$

In other words, $\Delta_{\Phi_n}^{(n+1)}$ is a \mathcal{O} -linear $\Phi^{(0)}$ -derivation. Applied to a function $F \in \Gamma(E^*)$, the map $\Delta_{\Phi_n}^{(n+1)}$ decomposes as:

$$\Delta_{\Phi_n}^{(n+1)}(F) = \sum_{i=0}^n Q'^{(n+1-i)} \circ \Phi_n^{(i)}(F) - \Phi_n^{(i)} \circ Q^{(n+1-i)}(F), \quad (1.49)$$

because the restriction to $\Gamma(E^*)$ of $\Phi_n^{(n+1)}$ is vanishing.

Moreover, by definition of Δ_{Φ_n} , and since Q and Q' square to zero, the following relation holds:

$$Q' \circ \Delta_{\Phi_n} + \Delta_{\Phi_n} \circ Q = 0. \quad (1.50)$$

Taking the component of arity $n + 1$ in the previous relation, and using the recursion assumption that Δ_{Φ_n} has no components of arity i for $i = 0$ to n , we obtain by Equation (1.41):

$$Q'^{(0)} \circ \Delta_{\Phi_n}^{(n+1)} + \Delta_{\Phi_n}^{(n+1)} \circ Q^{(0)} = 0. \quad (1.51)$$

In other words, the $\Phi^{(0)}$ -derivation $\Delta_{\Phi_n}^{(n+1)}$ is a Δ -cocycle. In view of Lemma 1.32, this cocycle can be seen as a cocycle of the bi-complex $(\mathfrak{B}^{(n+1)}, \tilde{\partial})$.

For $n \geq 1$, this cocycle is a coboundary in view of the first item in Lemma 1.31. For $n = 0$, this cocycle is also a coboundary in view of the second item of Lemma 1.31. This deserves some justification: it is routine to check that the component in $S^2(E_{-1}^* \otimes E_{-1})$ of

$$\Delta_{\Phi_0}^{(1)} = Q'^{(1)} \circ \Phi_0 - \Phi_0 \circ Q^{(1)} \quad (1.52)$$

is given by $\{\phi_1(x), \phi_1(y)\} - \phi_1(\{x, y\}')$ for all $x, y \in \Gamma(E'_{-1})$, where $\phi_1 : E'_{-1} \rightarrow E_{-1}$ is the first component of the chain map ϕ dual to $\Phi^{(0)}$ and $\{., .\}$ and $\{., .\}'$ are the almost Lie algebroid brackets on E_{-1} and E'_{-1} dualizing $Q^{(1)}$ and $Q'^{(1)}$. This element is in the kernel of ρ because $\rho \circ \phi_1 = \rho'$ by construction of ϕ_1 . The condition in the second item of Lemma 1.31 is satisfied, and gives that $\Delta_{\Phi_0}^{(1)}$ is a coboundary.

For every value of n therefore, there exists a $\Phi^{(0)}$ -derivation δ_{n+1} of degree 0 and of arity $n + 1$ such that:

$$\Delta_{\Phi_n}^{(n+1)} = -\Delta(\delta_{n+1}). \quad (1.53)$$

Now, we construct a \mathcal{O} -linear graded algebra morphism Φ_{n+1} by requiring that its restrictions to $\Gamma(E^*) \subset \mathcal{E}$ has components of arity $0, \dots, n$ that coincide with those of Φ_n and a term of arity $n + 1$ that coincides with δ_{n+1} . We extend these restrictions to all of \mathcal{E} by using Equation (1.14). The henceforth defined graded algebra morphism Φ_{n+1} has all components of arity $0, \dots, n$ that coincide with those of Φ_n by Remark 12. Hence the second assumption of the recursion is satisfied at rank $n + 1$. By Equation (1.41), it implies in turn that the operator:

$$\Delta_{\Phi_{n+1}} := Q' \circ \Phi_{n+1} - \Phi_{n+1} \circ Q \quad (1.54)$$

has no components of arity $0, \dots, n$, and that the restriction to $\Gamma(E^*)$ of the component of arity $n + 1$ is:

$$\Delta_{\Phi_{n+1}}^{(n+1)} = \Delta_{\Phi_n}^{(n+1)} + Q'^{(0)} \circ \Phi_{n+1}^{(n+1)} - \Phi_{n+1}^{(n+1)} \circ Q^{(0)}, \quad (1.55)$$

because the restriction to $\Gamma(E^*)$ of the component of arity $n + 1$ of Φ_n is vanishing, see Equation (1.49). Since $\Phi_{n+1}^{(n+1)}(F) = \delta_{n+1}(F)$ for all $F \in \Gamma(E^*)$, $\Delta_{\Phi_{n+1}}^{(n+1)}$ vanishes on this space. It also vanishes on \mathcal{O} . Now, $\Delta_{\Phi_{n+1}}$ satisfies a derivation relation of the type of Equation (1.45) - upon replacing n by $n + 1$. If the components of $\Delta_{\Phi_{n+1}}$ of arity $0, \dots, n + 1$ are zero when applied to elements in $\Gamma(E^*) \subset \mathcal{E}$, they have to vanish on the whole graded algebra \mathcal{E} . Finally, the restriction to $\Gamma(E^*)$ of the component $\Phi_{n+1}^{(n+2)}$ is vanishing, hence all three items in the recursion relation are therefore satisfied at rank $n + 1$, which completes the proof of the first point: there exist a Lie ∞ -algebroid morphism from (E', Q') to (E, Q) whose linear part is an arbitrary chain map from (E', d', ρ') to (E, d, ρ) .

We now have to show that any two such morphisms are homotopic. Let Φ and Ψ be \mathcal{O} -linear Lie ∞ -algebroid morphisms from (E', Q') to a universal Lie ∞ -algebroid (E, Q) of a singular foliation \mathcal{F} . Let us first show that there exists a Lie ∞ -algebroid morphism which is homotopic to Φ and has the same linear part a Ψ . Since (E, d, ρ) is a resolution of \mathcal{F} in the sense of Definition (1.1), the complex

$$\dots \xrightarrow{d^{(3)}} \Gamma(E_{-2}) \xrightarrow{d^{(2)}} \Gamma(E_{-1}) \xrightarrow{\rho} \mathcal{F} \longrightarrow 0$$

is a resolution of \mathcal{F} in the category of \mathcal{O} -modules (as already stated in Lemma 1.2). It is classical that two chain maps from the chain complex (E', d', ρ') valued in E in the category of \mathcal{O} -modules are homotopic. In particular, the linear parts of Φ and Ψ are homotopic through an homotopy $h : \Gamma(E') \rightarrow \Gamma(E)$:

$$\begin{array}{ccccccc} \dots & \xrightarrow{d} & E'_{-3} & \xrightarrow{d} & E'_{-2} & \xrightarrow{d} & E'_{-1} \\ & & \downarrow \psi & \searrow h & \downarrow \psi & \searrow h & \downarrow \psi \\ & & \phi & & \phi & & \phi \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \xrightarrow{d'} & E_{-3} & \xrightarrow{d'} & E_{-2} & \xrightarrow{d'} & E_{-1} \end{array}$$

Since h is \mathcal{O} -linear of degree -1 , it comes from a vector bundle morphism, from E' to E which is also of degree -1 . The dual map $h^* : \Gamma(E^*) \rightarrow \Gamma(E'^*)$ satisfies by construction:

$$\Psi^{(0)}(F) - \Phi^{(0)}(F) = Q'^{(0)} \circ h^*(F) + h^* \circ Q^{(0)}(F) \quad (1.56)$$

for all $F \in \Gamma(E^*)$.

Consider the differential equation⁴:

$$\frac{d\Phi_t}{dt} = [Q, h^*(\Phi_t)] \quad \text{and} \quad \Phi_0 = \Phi,$$

where $h^*(\Phi_t)$ is the unique Φ_t -derivation whose restriction to $\Gamma(E^*)$ is the dual h^* of h . It admits a solution according to Example 8. By construction, $(\Phi_t, h^*(\Phi_t))$ is a homotopy between Φ and Φ_1 . Moreover, since the restriction of $h^*(\Phi_t)$ to the linear part of Φ_t is h^* for all t , the following differential equation is satisfied:

$$\begin{aligned} \frac{d\Phi_t^{(0)}}{dt}(F) &= Q'^{(0)} \circ h^*(F) + h^* \circ Q^{(0)}(F) & \text{and} & \quad \Phi_0 = \Phi^{(0)}, \\ &= \Psi^{(0)}(F) - \Phi^{(0)}(F) \end{aligned} \quad (1.57)$$

for all $F \in \Gamma(E^*)$. In particular, the map $\frac{d\Phi_t^{(0)}}{dt}$ does not depend on t and coincides with the component of arity 0 of $\Psi - \Phi$. In view of the initial condition, $\Phi_t^{(0)}$ is therefore equal to $\Phi^{(0)} + t(\Psi^{(0)} - \Phi^{(0)})$ and the component of arity 0 of Φ_1 is equal to $\Psi^{(0)}$.

In view of this first point, we are left with the task of finding a homotopy between Φ_1 and Ψ , which are Lie ∞ -algebroid morphisms that have the same linear part ψ whose dual we shall simply denote by $\Psi^{(0)}$. We are now going to build a sequence $(\Phi_n)_{n \geq 1}$ of Lie ∞ -algebroid morphisms from (E', Q') to (E, Q) such that:

1. Φ_n and Ψ have components of arity k that coincides for all $k \leq n - 1$.
2. Φ_n and Φ_{n+1} are homotopic through homotopies $(\Phi_t)_{t \in [n, n+1]}$ obtained out of Φ_t -derivations $(\delta_t)_{t \in [n, n+1]}$ whose components of arity less or equal to n vanish.

The construction of such a sequence of homotopies completes the proof, since then the pair $t \mapsto \Phi_{f(t)}$ and $t \mapsto \delta_{f(t)}$ with $f : [0, 1[\rightarrow [1, +\infty[$ a strictly increasing surjective C^1 -function is a homotopy between Φ_1 and Ψ . And since homotopy is an equivalence relation by Proposition 1.22, then Φ and Ψ are homotopic.

Let us explain this point carefully. Due to the first assumption, it is easy to see that the component $t \mapsto \Phi_{f(t)}^{(n)}$ of arity n is constant (that is, does not depend on t) in a neighborhood of 1, and coincides with $\Psi^{(n)}$ in that neighborhood. Also, it is continuous and piecewise- C^1 . For the same reason, components of a given fixed arity of $t \mapsto \delta_{f(t)}$ are equal to 0 in a neighborhood of 1, hence are piecewise continuous. A homotopy between Φ_1 and Ψ is therefore constructed. Since Φ_1 and Φ are homotopic as well, a homotopy between Φ and Ψ exists. Now, the following lemma allows to construct the family of homotopies as above linking Φ_1 and Ψ as above and completes the proof of Theorem 1.8.

Lemma 1.33. *Let $\Phi, \Psi : \mathcal{E} \rightarrow \mathcal{E}'$ be two Lie ∞ -algebroid morphisms from (E', Q') to (E, Q) such that $\Phi^{(i)} = \Psi^{(i)}$ for every $0 \leq i \leq n$ for some n . Then there exists a Lie ∞ -algebroid morphism Ξ which is homotopic to Φ and which satisfies $\Xi = \Psi$ up to arity $n + 1$.*

Proof. For all $F, G \in \mathcal{E}$, one has:

$$(\Psi - \Phi)(FG) = (\Psi - \Phi)(F)\Phi(G) + \Psi(F)(\Psi - \Phi)(G). \quad (1.58)$$

In view of Equation (1.14), this relation implies, when F, G are of arity i, j respectively, and when only the component of arity $n + 1 + i + j$ of the previous relation is considered, that:

$$(\Psi - \Phi)^{(n+1)}(FG) = (\Psi - \Phi)^{(n+1)}(F)\Phi^{(0)}(G) + \Phi^{(0)}(F)(\Psi - \Phi)^{(n+1)}(G). \quad (1.59)$$

Above, we have used the assumption that Φ and Ψ coincide up to arity n . The previous relation means that $(\Psi - \Phi)^{(n+1)} : \mathcal{E} \rightarrow \mathcal{E}'$ is a $\Phi^{(0)}$ -derivation. Moreover, since $\Phi^{(i)} = \Psi^{(i)}$ for every $i = 0, \dots, n$, the relation:

$$(\Psi - \Phi) \circ Q = Q' \circ (\Psi - \Phi), \quad (1.60)$$

⁴We recall that $[Q, L]$ is a shorthand for $Q' \circ L - (-1)^l L \circ Q$ for all linear maps $L : \mathcal{E} \rightarrow \mathcal{E}'$ of degree $l \in \mathbb{Z}$

evaluated at arity $n + 1$ implies $(\Psi - \Phi)^{(n+1)} \circ Q^{(0)} = Q'^{(0)} \circ (\Psi - \Phi)^{(n+1)}$, i.e. that the $\Phi^{(0)}$ -derivation $(\Psi - \Phi)^{(n+1)}$ is a Δ -cocycle. This implies by the third item in Lemma 1.31 that it is a coboundary, i.e. that there exists a $\Phi^{(0)}$ -derivation δ_{n+1} of arity $n + 1$ and degree -1 such that:

$$(\Psi - \Phi)^{(n+1)} = \Delta(\delta_{n+1}). \quad (1.61)$$

Now, for every Lie ∞ -algebroid morphism $\Xi : \mathcal{E} \rightarrow \mathcal{E}'$, let us denote by $\delta_{n+1}(\Xi)$ the unique Ξ -derivation whose restriction to $\Gamma(E^*) \subset \mathcal{E}$ coincides with δ_{n+1} . Then consider the solution of the following initial value problem (it admits a solution defined on \mathbb{R} in view of Example 8):

$$\frac{d\Phi_t}{dt} = [Q, \delta_{n+1}(\Phi_t)] \quad \text{and} \quad \Phi_0 = \Phi.$$

The pair $(\Phi_t, \delta_{n+1}(\Phi_t))$ is by construction a homotopy between Φ and $\Xi = \Phi_{t=1}$, and it is obtained through a family of Φ_t -derivations $t \mapsto \delta_{n+1}(\Phi_t)$ whose components of arity k vanish for all $k = 0, \dots, n$. In particular, for every $k = 0, \dots, n$:

$$\frac{d\Phi_t^{(k)}}{dt} = [Q, \delta_{n+1}(\Phi_t)]^{(k)} = 0 \quad \text{and} \quad \Phi_0^{(k)} = \Phi^{(k)}.$$

Hence $\Xi^{(k)} = \Phi^{(k)} = \Psi^{(k)}$ for all $k = 0, \dots, n$. In arity $n + 1$, the previous relation gives:

$$\frac{d\Phi_t^{(n+1)}}{dt} = [Q, \delta_{n+1}(\Phi_t)]^{(n+1)} = \Delta(\delta_{n+1}(\Phi_t))^{(n+1)} \quad \text{and} \quad \Phi_0^{(n+1)} = \Phi^{(n+1)}.$$

Since $\delta_{n+1}(\Phi_t)$ is a Φ_t -derivation, since the component of arity 0 of Φ_t is constant and equal to $\Phi^{(0)}$ for all $t \in [0, 1]$, and since the restriction to $\Gamma(E^*)$ of $\delta_{n+1}(\Phi_t)$ coincides with the $\Phi^{(0)}$ -derivation $\delta_{n+1} = \delta_{n+1}(\Phi^{(0)})$, we have:

$$\frac{d\Phi_t^{(n+1)}}{dt} = Q'^{(0)} \circ \delta_{n+1} + \delta_{n+1} \circ Q^{(0)} \quad \text{and} \quad \Phi_0^{(n+1)} = \Phi^{(n+1)}.$$

By definition of δ_{n+1} therefore:

$$\frac{d\Phi_t^{(n+1)}}{dt} = \Delta(\delta_{n+1}) = (\Psi - \Phi)^{(n+1)} \quad \text{and} \quad \Phi_0^{(n+1)} = \Phi^{(n+1)},$$

so that $\Xi^{(n+1)} = (\Psi - \Phi)^{(n+1)} + \Phi^{(n+1)} = \Psi^{(n+1)}$. The proof then continues by recursion. \square

1.7 Examples of universal Lie ∞ -algebroid structures over a singular foliation

In this section, we give examples of universal Lie ∞ -algebroid structures over a given singular foliation.

Example 16. For a regular foliation \mathcal{F} on a manifold M , it is clear that the tangent space $T\mathcal{F}$ is a resolution of \mathcal{F} , when equipped with inclusion map as an anchor map. The Lie algebroid $T\mathcal{F}$ is a universal Lie ∞ -algebroid over \mathcal{F} .

Example 17. More generally, when a foliation is Debord (see Proposition 2.14), then a resolution of length 1 exists and comes equipped with a Lie algebroid structure. This Lie algebroid is a universal Lie ∞ -algebroid over \mathcal{F} .

Example 18. In Example 11, we gave a resolution of length 2 of the singular foliation coming from the action of \mathfrak{sl}_2 on \mathbb{R}^2 . Let us compute now the Lie ∞ -algebroid structure on that resolution. We define the bracket between two constant sections of $E_{-1} \simeq \mathfrak{sl}_2$ as being their bracket in \mathfrak{sl}_2 . Then we extend it to every section of E_{-1} by the Leibniz identity (1.5). To define the bracket between sections of E_{-1} and E_{-2} , we notice that:

$$\{\tilde{e}, dr\} = xy\{\tilde{e}, \tilde{h}\} + \rho(\tilde{e})(xy)\tilde{h} + \rho(\tilde{e})(y^2)\tilde{e} - x^2\{\tilde{e}, \tilde{f}\} = 0. \quad (1.62)$$

Since d is injective on a dense open subset, this imposes $\{\tilde{e}, r\} = 0$. The same argument also gives $[\tilde{f}, r] = [\tilde{h}, r] = 0$. We then extend these brackets to a bracket between sections of E_{-1} and E_{-2} by the Leibniz property (1.5). There is no k -ary bracket for $k \geq 3$.

Example 19. Let \mathcal{F} be the singular foliation of all vector fields vanishing at the origin 0. Let us consider the resolution given in Example 13.

The Lie ∞ -algebroid structure on that resolution can be described explicitly. We let all k -ary brackets to vanish for $k \geq 3$. Let us then define the binary bracket on constant sections. For all $\alpha \in \wedge^i V^*$, $\beta \in \wedge^j V^*$, and $u, v \in V$, we define a (graded symmetric) Lie algebra bracket by:

$$\{\alpha \otimes u, \beta \otimes v\}_2 = \alpha \wedge \mathbf{i}_u \beta \otimes v + (-1)^{ij} \beta \wedge \mathbf{i}_v \alpha \otimes u \quad (1.63)$$

and then extend it to all sections with the help of the anchor map. This bracket is graded symmetric by construction, and the Jacobi identity is a direct computation (since the Lie bracket previously defined preserves constant sections, it suffices to check it on constant sections). The compatibility with the differential is also a matter of computation:

$$\begin{aligned} d^{(i+j-1)}\{\alpha \otimes u, \beta \otimes v\}_2 &= \mathbf{i}_e (\alpha \wedge \mathbf{i}_u \beta \otimes v + (-1)^{ij} \beta \wedge \mathbf{i}_v \alpha \otimes u) \\ &= (\mathbf{i}_e \alpha) \wedge \mathbf{i}_u \beta \otimes v + (-1)^{ij+j+1} \beta \wedge \mathbf{i}_v (\mathbf{i}_e \alpha) \otimes u \\ &\quad + (-1)^{i+1} \alpha \wedge \mathbf{i}_u \mathbf{i}_e \beta \otimes v + (-1)^{ij} (\mathbf{i}_e) \beta \wedge \mathbf{i}_v \alpha \otimes u \\ &= \{d^{(i)}(\alpha \otimes u), \beta \otimes v\}_2 - (-1)^i \{\alpha \otimes u, d^{(j)}(\beta \otimes v)\}_2 \end{aligned}$$

The computation above implies that the compatibility holds for two sections of E_{-i} and E_{-j} with $i, j \geq 2$. For $i = 1$ or $j = 1$, the computation above has to be done differently, because the differential is zero on E_{-1} , but the anchor map enters then into the computation and makes the formula valid again: we leave it to the reader.

Example 20. This is a continuation of Example 14, where we explain that the Koszul complex is a resolution of the singular foliation \mathcal{F}_φ of all algebraic vector fields X on \mathbb{C}^n satisfying $X[\varphi] = 0$ for some weight homogeneous function φ with isolated singularities.

By example 14, there is a resolution (E, d, ρ) such that $\Gamma(E_{-i})$ is the sheaf of $i + 1$ -multivector fields on \mathbb{C}^n . We describe the brackets giving the Lie ∞ -algebroid structure as follows:

$$\{\partial_{I_1}, \dots, \partial_{I_k}\}_k := \sum_{i_1 \in I_1, \dots, i_k \in I_k} \epsilon(i_1, \dots, i_k) \varphi_{i_1, \dots, i_k} \partial_{\frac{I_1}{\{i_1\}} \cup \dots \cup \frac{I_k}{\{i_k\}}}, \quad (1.64)$$

where:

1. φ is a function on \mathbb{C}^n ,
2. for all $I = \{i_1, \dots, i_k\}$ a sub-list of elements in $\{1, \dots, n\}$, the set $\frac{I}{\{i_p\}}$, where $1 \leq p \leq k$, is the sub-list $\{i_1, \dots, i_{p-1}, i_{p+1}, \dots, i_k\}$, and ∂_I is a shorthand for $\frac{\partial}{\partial x_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x_{i_k}}$,
3. $\epsilon(i_1, \dots, i_k) \in \{-1, 1\}$ is the signature of the permutation of the list I_1, \dots, I_k which sends i_1, \dots, i_k , in that order, in front of the list,
4. $\varphi_{i_1, \dots, i_k}$ is a shorthand for $\frac{\partial^k \varphi}{\partial x_{i_1} \dots \partial x_{i_k}}$.

The brackets (1.64) are then defined for any multivector fields by \mathcal{O} -linearity for $k \neq 2$. For $k = 2$, it extends with the help of the anchor map ρ which is given by:

$$\rho(\delta_{ij}) := \varphi_i \frac{\partial}{\partial x_j} - \varphi_j \frac{\partial}{\partial x_i}. \quad (1.65)$$

A brutal computation that we leave to the reader gives that (1.64) is a Lie ∞ -algebroid. For φ weight homogeneous with isolated singularities, we saw in Example 14 that the previous Lie ∞ -algebroid structure is built on a resolution of the foliation \mathcal{F}_φ of all vector fields X on $M = \mathbb{C}^n$ such that $X[\varphi] = 0$.

Notice that in this example, the k -ary brackets for $k = 3, 4, \dots$ are in general not zero.

2 The geometry of a singular foliation through its universal Lie ∞ -algebroid

The purpose of this section is to exploit the universal Lie ∞ -algebroid over a singular foliation to understand its geometry. For this purpose we must associate objects to the latter structure which do not depend on the many choices made into the construction. In the next sections, we shall study first the global invariants, and then turn to local ones, attached to the leaves.

2.1 Universal foliated cohomology

We refer to [27] for an insight about the importance of Q -manifold cohomology in relation with characteristic classes. (e.g. an interpretation of Chern-Weyl maps with the help of Q -manifolds).

Let \mathcal{F} be a singular foliation, and (E, Q) a universal Lie ∞ -algebroid over it, with sheaf of functions $\mathcal{E} = \bigoplus_{k \geq 0} \mathcal{E}_k$. The homological vector field Q makes $\mathcal{E} = \bigoplus_{k \geq 0} \mathcal{E}_k$ a complex, whose cohomology is the Q -manifold cohomology.

This cohomology makes sense, i.e. does not depend on the choice of a universal Lie ∞ -algebroid over \mathcal{F} , in view of the following corollary of Theorem 1.8:

Corollary 2.1. *Let \mathcal{F} be a singular foliation on M . Let (E, Q) and (E', Q') be universal Lie ∞ -algebroids over \mathcal{F} with sheaves of functions \mathcal{E} and \mathcal{E}' . The cohomologies of (\mathcal{E}, Q) and (\mathcal{E}', Q') are canonically isomorphic as graded commutative algebras.*

Proof. By Theorem 1.8, there exist Lie ∞ -algebroid morphisms $\Phi : \mathcal{E}' \rightarrow \mathcal{E}$ and $\Psi : \mathcal{E} \rightarrow \mathcal{E}'$ whose compositions are homotopic to the identity maps of \mathcal{E} and \mathcal{E}' respectively. Any two choices of morphisms are moreover homotopic.

Proposition 1.23 implies that $\tilde{\Phi}$ and $\tilde{\Psi}$ are inverse to one another at the level of cohomology, and that any two choices for Φ and Ψ give the same morphisms at the level of cohomology. \square

By Corollary 2.1, the following definition therefore makes sense (i.e. does not depend on the choice of a universal Lie ∞ -algebroid (E, Q) over \mathcal{F}):

Definition 2.2. *Let \mathcal{F} be a singular foliation on M that admits at least a universal Lie ∞ -algebroid over it. We call universal foliated cohomology of \mathcal{F} and denote by $H_{\mathcal{U}}(\mathcal{F})$ the cohomology of the complex (\mathcal{E}, Q) , where \mathcal{E} is the sheaf of functions on any universal Lie ∞ -algebroid (E, Q) over \mathcal{F} .*

Remark 13. It is clear that the first quotient space $H_{\mathcal{U}}^0(\mathcal{F})$ consists in smooth functions on M constant along the leaves of \mathcal{F} .

Let us give some interpretations of the universal foliated cohomology of \mathcal{F} . Let us call *forms on \mathcal{F}* and denote by $\Omega(\mathcal{F})$ the space of \mathcal{O} -multi-linear skew-symmetric assignments from \mathcal{F} to \mathcal{O} :

$$\Omega(\mathcal{F}) := \text{Hom}_{\mathcal{O}}(\wedge_{\mathcal{O}} \mathcal{F}, \mathcal{O}) = \bigoplus_{k \geq 0} \text{Hom}(\wedge_{\mathcal{O}}^k \mathcal{F}, \mathcal{O})$$

Note that 0-forms on \mathcal{F} are just functions on M . Also, a k -form α on \mathcal{F} induces a k -form α_L on each regular leaf L , but maybe not on singular ones. A *foliated de Rham operator* d_{dR} on $\Omega(\mathcal{F})$ is defined by the usual formula:

$$\begin{aligned} d_{\text{dR}}(\alpha)(X_0, \dots, X_k) &= \sum_{i=0}^k (-1)^i X_i [\alpha(X_0, \dots, \widehat{X}_i, \dots, X_k)] \\ &\quad + \sum_{0 \leq i < j \leq k} (-1)^{i+j} \alpha([X_i, X_j], X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_k), \end{aligned} \tag{2.1}$$

with the understanding that \widehat{X}_i means that the term X_i is omitted. We call the cohomology of this operator the *foliated de Rham cohomology of \mathcal{F}* and denote it by $H_{\text{dR}}(\mathcal{F})$.

Let \mathcal{F} be a singular foliation on M and let (E, Q) , with sheaf of functions \mathcal{E} , be a universal Lie ∞ -algebroid over \mathcal{F} . There is a natural map ρ^* from $\Omega(\mathcal{F})$ to \mathcal{E} given by associating to each $\alpha \in \Omega^k(\mathcal{F})$ the element $\rho^*\alpha \in \Gamma(S^k(E_{-1}^*)) \subset \mathcal{E}_k$ such that for all $x_1, \dots, x_k \in \Gamma(E_1)$:

$$\rho^*\alpha(x_1, \dots, x_k) = \alpha(\rho(x_1), \dots, \rho(x_k)). \quad (2.2)$$

It is routine to check that $\alpha \mapsto \rho^*(\alpha)$ is a chain map and a graded commutative algebra morphism, inducing therefore an algebra morphism, still denoted by ρ^* , from $H_{\text{dR}}(\mathcal{F})$ (the foliated de Rham cohomology of \mathcal{F}) to $H_{\mathcal{U}}(\mathcal{F})$ (the universal foliated cohomology of \mathcal{F}).

Proposition 2.3. *Let \mathcal{F} be a singular foliation on M that admits a universal Lie ∞ -algebroid over it. The algebra morphism ρ^* from the foliated de Rham cohomology of \mathcal{F} to the universal foliated cohomology of \mathcal{F} given by Equation (2.2) is canonical. In other words, for every two universal Lie ∞ -algebroids (E, Q) and (E', Q') over the singular foliation \mathcal{F} , the following diagram is commutative:*

$$\begin{array}{ccc} & H_{\text{dR}}^{\bullet}(\mathcal{F}) & \\ \rho^* \swarrow & & \searrow \rho'^* \\ H^{\bullet}(\mathcal{E}, Q) & \xleftarrow{\Phi_{\mathcal{E}, \mathcal{E}'}} & H^{\bullet}(\mathcal{E}', Q') \end{array}$$

where $\Phi_{\mathcal{E}, \mathcal{E}'}$ is the canonical isomorphism given in Corollary 2.1.

Proof. Let (E, Q) and (E', Q') be two universal Lie ∞ -algebroids over \mathcal{F} , and denote by \mathcal{E} and \mathcal{E}' their sheaves of graded functions.

Let $\Phi_{\mathcal{E}, \mathcal{E}'} : \mathcal{E}' \rightarrow \mathcal{E}$ be a Lie ∞ -algebroid morphism from (E, Q) and (E', Q') as in Theorem 1.8. Let $\phi_1 : E_{-1} \rightarrow E'_{-1}$ be the linear part of $\Phi_{\mathcal{E}, \mathcal{E}'}$. The relation $\rho' \circ \phi_1 = \rho$ holds, with ρ and ρ' the respective anchors of the Lie ∞ -algebroids (E, Q) and (E', Q') , which in turn implies that the relation $\Phi_{\mathcal{E}, \mathcal{E}'}^* \circ (\rho')^* = \rho^*$ holds at the chain level. Together with Corollary 2.1, this proves the claim. \square

2.2 The isotropy Lie ∞ -algebra at a point

In [1, 3], the isotropy Lie algebra of a singular foliation at a point x is defined. In this section, we show that this Lie algebra is the first component (in degree -1) of a Lie ∞ -algebra with no differential, which is canonically associated to the singular foliation.

Let \mathcal{F} be a singular foliation on a manifold M and (E, Q) be a universal Lie ∞ -algebroid over it. Choose an arbitrary point $x \in M$. Denote by $i_x^* E_{-i}$ the fiber of E_{-i} at x , and consider the complex:

$$\dots \xrightarrow{d^{(4)}} i_x^* E_{-3} \xrightarrow{d^{(3)}} i_x^* E_{-2} \xrightarrow{d^{(2)}} \text{Ker}(\rho_x)$$

where ρ_x stands for the anchor map $\rho_x : i_x^* E_{-1} \rightarrow T_x M$ at the point x . The previous sequence may have cohomology: the exactness of the complex in Definition 1.1 at the level of sections does not imply that it is exact at all points. The resolutions constructed in Examples 11-14-13 for instance have cohomology at the origin of the vector space on which they are constructed.

The linearity properties of the brackets defining a Lie ∞ -algebroid (E, Q) in Definition 1.13, and the fact that the kernel of the anchor map is an ideal with respect to the 2-bracket, imply that the Lie ∞ -algebroid structure (E, Q) restricts to yield a Lie ∞ -algebra structure on the graded vector space:

$$V_x = \text{Ker}(\rho_x) \oplus \bigoplus_{i \geq 2} i_x^* E_{-i}$$

We call this Lie ∞ -algebra the *isotropy Lie ∞ -algebra at $x \in M$ of the universal Lie ∞ -algebroid (E, Q) . The following lemma is an obvious consequence of the second item of Theorem 1.8: it suffices to restrict the morphisms described in that item to the point x .*

Lemma 2.4. *Let (E, Q) and (E', Q') be two universal Lie ∞ -algebroids over a singular foliation \mathcal{F} . For every point $x \in M$, the isotropy Lie ∞ -algebra at $x \in M$ of the universal Lie ∞ -algebroid (E, Q) and (E', Q') are isomorphic up to homotopy. Moreover, there is a distinguished homotopy class of isomorphisms up to homotopy relating them.*

Taking the cohomology of the isotropy Lie ∞ -algebra of (E, Q) at a point $x \in M$, we obtain a graded Lie algebra. By the previous lemma, different choices of universal Lie ∞ -algebroids over \mathcal{F} lead to canonically isomorphic graded Lie algebra.

Proposition 2.5. *Let \mathcal{F} be a singular foliation on M that admits two universal Lie ∞ -algebroids (E, Q) and (E', Q') over it. The isotropy graded Lie algebras of \mathcal{F} at $x \in M$, computed with respect to these structures, are canonically isomorphic.*

The following definition therefore makes sense:

Definition 2.6. *We call isotropy graded Lie algebra of \mathcal{F} at the point x the graded Lie algebra described in Proposition 2.5 and denote it by $H^{\mathcal{F}}(x) = \bigoplus_{i \geq 1} H_{-i}^{\mathcal{F}}(x)$.*

Remark 14. Notice that in this article, graded Lie algebras are considered in their symmetric version, due to the conventions established in Definition 1.13. See also Remark 2.

Now, we claim that we can have much more structures than just a graded Lie algebra. Recall that for a singular foliation \mathcal{F} that admits a resolution of finite length, any point $x \in M$ admits a neighborhood of x in M on which a resolution minimal at x exists by item 5. in Proposition 1.3. The following lemma is then an obvious result of Abelian categories:

Lemma 2.7. *For any resolution (E, d, ρ) of a singular foliation \mathcal{F} which is minimal at x and all $i \geq 2$, the vector space $\mathfrak{i}_x^* E_{-i}$ is canonically isomorphic to $H_{-i}^{\mathcal{F}}(x)$ and $\text{Ker}(\rho_x) \subset \mathfrak{i}_x^* E_{-1}$ is canonically isomorphic to $H_{-1}^{\mathcal{F}}(x)$.*

It follows from the previous lemma that the isotropy Lie ∞ -algebra structure at $x \in M$ of a universal Lie ∞ -algebroid (E, Q) over \mathcal{F} is, when the resolution (E, d, ρ) is minimal at x , constructed on a graded vector space which is canonically isomorphic to $H^{\mathcal{F}}(x) = \bigoplus_{i \geq 1} H_{-i}^{\mathcal{F}}(x)$. It can therefore be transported on that space.

By definition of a resolution minimal at x , this Lie ∞ -algebra has no unary bracket, i.e. its differential is the zero map. Since the differential is zero, its binary bracket induces a graded Lie algebra structure (i.e. satisfies the graded Jacobi identity). By construction, it coincides with the bracket of the isotropy graded Lie algebra at x . But it may still have k -ary brackets for $k \geq 3$ that may depend on the construction. Let us study this dependence. A morphism of Lie ∞ -algebras from (V, Q) to (V', Q') is by definition a graded algebra morphism $\Phi : S((V')^*) \rightarrow S(V^*)$. Its linear part is the dual of the linear part of Φ : it is a graded morphism from V to V' . When $\Phi : S((V')^*) \rightarrow S(V^*)$ is a strict isomorphism, we shall speak of a strict isomorphism of Lie ∞ -algebras. It is routine to check that a morphism is a strict isomorphism if and only if the linear part is a graded vector space isomorphism. Now:

Proposition 2.8. *Let \mathcal{F} be a singular foliation on M that admits a resolution of finite length. Then the graded cohomological spaces $H^{\mathcal{F}}(x)$ obtained by restricting an arbitrary resolution of \mathcal{F} to a point $x \in M$:*

1. *comes equipped with a Lie ∞ -algebra structure for every choice of universal Lie ∞ -algebroid (E, Q) over \mathcal{F} ,*
2. *this Lie ∞ -algebra admits a trivial 1-ary bracket (that is, the differential is zero) and its binary bracket is the Lie bracket of the isotropy graded Lie algebra (in particular, it satisfies the graded Jacobi identity),*
3. *its k -ary brackets, for $k \geq 3$, depend on the choice of a universal Lie ∞ -algebroid (E, Q) constructed on a resolution (E, d, ρ) minimal at x , but any two such Lie ∞ -algebras are strictly isomorphic, with respect to a Lie ∞ -algebra isomorphism whose linear part is the identity map.*

Proof. Items 1. and 2. follow from Lemma 2.7, Definition 2.9 and the discussion before Proposition 2.8. Item 3. is a consequence of Lemma 2.4 (which is itself a consequence of Theorem 1.8) when used at a point where the differential d of the resolution of \mathcal{F} vanishes. \square

Item 3. in Proposition 2.8 implies that the Lie ∞ -algebra obtained by considering resolutions minimal at x is not unique: it is only unique up to a Lie ∞ -algebra isomorphism whose linear part is the identity. By contrast with the "isotropy Lie ∞ -algebra of (E, Q) at $x \in M$ ", which depends on the choice of (E, Q) , its class up to isomorphisms whose linear part is the identity depends only on the foliation \mathcal{F} .

Definition 2.9. We call isotropy Lie ∞ -algebra of \mathcal{F} at the point x the isomorphism class of Lie ∞ -algebra structures on $H^{\mathcal{F}}(x)$ described in Proposition 2.8.

Since $H^{\mathcal{F}}(x)$ comes equipped with a canonical graded Lie algebra structure, $H_{-1}^{\mathcal{F}}(x)$ is in particular a Lie algebra. We show that it is isomorphic to the isotropy Lie algebra \mathfrak{g}_x constructed by I. Androulidakis and G. Skandalis in [1], defined to be the quotient \mathfrak{g}_x of the Lie algebra $\mathcal{F}(x)$ of local sections in \mathcal{F} vanishing at $x \in M$ by the Lie ideal $I_x\mathcal{F}$ (here I_x stands for the ideal of local functions vanishing at x). It is equipped with a quotient Lie algebra structure.

Proposition 2.10. Let \mathcal{F} be a singular foliation that admits a universal Lie ∞ -algebroid over it. For every $x \in M$, the isotropy Lie algebra \mathfrak{g}_x of the singular foliation at x as defined by Androulidakis and Skandalis is isomorphic to the component $H_{-1}^{\mathcal{F}}(x)$ of degree -1 in the isotropy graded Lie algebra of \mathcal{F} at x .

Proof. The isomorphism τ is defined as follows. For all $e \in \mathfrak{i}_x^*E_{-1}$ in the kernel of ρ_x , let \tilde{e} be a local section through e . Then $\rho(\tilde{e})$ is a local section of \mathcal{F} that vanishes at x . Its class modulo the Lie ideal $I_x\mathcal{F}$ is well-defined, since another choice for \tilde{e} would differ from the first one by a section in $I_x\Gamma(E_{-1})$. If $e = d^{(2)}(h)$ for some $h \in \mathfrak{i}_x^*E_{-2}$, then \tilde{e} can be chosen to be $d^{(2)}(\tilde{h})$ with \tilde{h} any section through h , so that $\rho(\tilde{e}) = \rho \circ d^{(2)}(\tilde{h}) = 0$.

This defines a well-defined Lie algebra morphism τ from $H_{-1}^{\mathcal{F}}(x)$ to \mathfrak{g}_x . It is clear that τ is surjective, since any local section of \mathcal{F} vanishing at $x \in M$ is of the form $\rho(\tilde{e})$ with \tilde{e} a local section of E_{-1} whose value at x is in the kernel of ρ . Now, let us prove injectivity. Let $e \in \mathfrak{i}_x^*E_{-1}$ with $\tau(e) = 0$. Then for any local section \tilde{e} of E_{-1} through e , we have that $\rho(\tilde{e})$ is in the ideal $I_x\mathcal{F}$, i.e. it is a finite sum of the form $\sum_{i=1}^r f_i X_i$, with $X_i \in \mathcal{F}$ and $f_i \in I_x$ for all $i = 1, \dots, r$. This implies $\rho(\tilde{e} - \sum_i f_i \tilde{e}_i) = 0$, where \tilde{e}_i is, for every $i = 1, \dots, n$, a local section of E_{-1} mapped to X_i through ρ . By definition of the resolution (E, d, ρ) , there exists a local section $\tilde{h} \in \Gamma(E_{-2})$ such that:

$$\tilde{e} - \sum_i f_i \tilde{e}_i = d^{(2)}\tilde{h}. \quad (2.3)$$

Evaluating this last relation at $x \in M$ gives that $\tilde{e}|_x = e$ is in the image of $d^{(2)} : \mathfrak{i}_x^*E_{-2} \rightarrow \mathfrak{i}_x^*E_{-1}$. This proves the injectivity of τ and completes the proof. \square

2.3 The isotropy Lie ∞ -algebra along a leaf

We let \mathcal{F} be a singular foliation on a smooth, real analytic or holomorphic manifold M with sheaf of functions \mathcal{O} . Assume that \mathcal{F} comes equipped with a universal Lie ∞ -algebroid (E, Q) over it. Let us consider a leaf L of \mathcal{F} . We start with a proposition:

Proposition 2.11. Let \mathcal{F} be a singular foliation on a manifold M and (E, Q) be a universal Lie ∞ -algebroid over it. The isotropy Lie ∞ -algebras of (E, Q) associated to two points x and y in the same leaf are isomorphic.

Proof. It suffices to prove this relation for all points y that lie in some neighborhood of x , in the same leaf. Let $x \in M$ and let y be a point in a neighborhood of x that can be reached from x as the time-1 flow of a time-dependent section $t \mapsto X_t$ of \mathcal{F} . (We want to present a proof valid in the real-analytic or holomorphic case, hence we can not extend this vector field X_t to the whole manifold as in the smooth case.)

Upon restricting this neighborhood if necessary, one can assume that there exists a time-dependent section e_t of E_{-1} such that $\rho(e_t) = X_t$ for all $t \in [0, 1]$.

For all $t \in [0, 1]$, let ∂_{e_t} stand for the (vertical) vector field of degree -1 on the graded manifold E defined by contraction by e_t . Said otherwise, let ∂_{e_t} be the derivation of the graded algebra \mathcal{E} of functions on the graded manifold E defined on generators by:

$$\partial_{e_t}[\xi] = \langle \xi, e_t \rangle, \quad (2.4)$$

for every $\xi \in \Gamma(E_{-1}^*) \subset \mathcal{E}$ and $\partial_{e_t}[\xi] = 0$ for all $\xi \in \Gamma(E_{-k}^*)$ with $k \geq 2$.

For all $t \in I = [0, 1]$, consider the vector field $V_t = [Q, \partial_{e_t}]$. The family $(V_t)_{t \in [0, 1]}$ is a 1-parameter family of vector fields of degree 0 on the graded manifold E . The 1-parameter family $(\Phi_t)_{t \in I}$ of endomorphisms of \mathcal{E} obtained by solving for all $F \in \mathcal{E}$ the differential equation:

$$\frac{d\Phi_t(F)}{dt} = V_t \circ \Phi_t(F), \quad (2.5)$$

is a family of graded algebra morphisms, defined in a neighborhood of x . By construction, they commute with Q . Hence, Φ_t is a Lie ∞ -algebroid morphism for all $t \in I$, defined in a neighborhood of x .

Moreover, for every function $f \in \mathcal{O}$:

$$V_t[f] = \partial_{e_t} \circ \rho^*(d_{\text{dR}}f) = \rho(e_t)[f] = X_t[f]. \quad (2.6)$$

Said differently, the derivation $V_t : \mathcal{E} \rightarrow \mathcal{E}$, being of degree 0, restricts to a derivation of \mathcal{O} , which is the derivation associated to the vector field $X_t = \rho(e_t)$. As a consequence, the 1-parameter family of strict Lie ∞ -algebroid isomorphisms Φ_t is over the 1-parameter family of diffeomorphisms $(\phi_t)_{t \in I}$ of M which satisfies $\frac{d}{dt}(\phi_t(m)) = X_t|_{\phi_t(m)}$ for all m in a sufficiently small neighborhood of x . By definition of X_t , we have $\phi_1(x) = y$ so that Φ_1 is a strict isomorphism of the Lie ∞ -algebroid (E, Q) to itself, defined in a neighborhood of x , that maps x to y . This completes the proof. \square

As an immediate consequence of this result, Lemma 2.4 and Proposition 2.5, we have the following result:

Corollary 2.12. *Let \mathcal{F} be a singular foliation that admits a universal Lie ∞ -algebroid over it in the neighborhood of every point. For any two points x and y in the same leaf of a singular foliation \mathcal{F} , the isotropy graded Lie algebras $H^{\mathcal{F}}(x)$ and $H^{\mathcal{F}}(y)$ are isomorphic as graded Lie algebras and the isotropy Lie ∞ -algebra structures at these points are isomorphic.*

The codimension of the leaf also gives important restrictions about the possible degrees of the isotropy Lie ∞ -algebra.

Proposition 2.13. *Let L be a leaf of a holomorphic or real analytic singular foliation \mathcal{F} . The isotropy graded Lie algebra $H^{\mathcal{F}}(x)$ at a point $x \in L$ is concentrated in degrees $-1, \dots, -\text{codim}(L) - 1$.*

Proof. According to Proposition 1.12 in [1], every singular foliation is, in a neighborhood of a point x in a leaf L , the trivial product of a singular foliation on a neighborhood of 0 in $\mathbb{R}^{n-\text{dim}(L)}$ (called the *transverse foliation*) with the foliation $T\mathcal{B}$, with \mathcal{B} an open ball of dimension $\text{dim}(L)$. A resolution of the singular foliation in a neighborhood of x is simply given by adding $T\mathcal{B}$ (in degree -1) to a resolution of the transverse foliation. Its length is the length of the resolution of the transverse foliation. In the real analytic or holomorphic cases, the transverse foliation admits resolutions of length less or equal to $\text{codim}(L) + 1$ (see item 1. in Proposition 1.3) in the neighborhood of every point. Hence so does \mathcal{F} in a neighborhood of x and the result follows. \square

2.4 Examples of isotropy Lie ∞ -algebras at a point

For regular foliations, the isotropy graded Lie algebra is trivial at all points. For Debord foliations (i.e. foliations which are, locally, the image of a Lie algebroid whose anchor is injective on a dense open subset) it is equal to the kernel of the anchor map.

Example 21. Consider the singular foliation given by the action of \mathfrak{sl}_2 on \mathbb{R}^2 . According to Example 18, at any point $x \in \mathbb{R}^2 \setminus \{0\}$, the resolution introduced in Example 18 is exact, and there is no cohomology. On the contrary, at the origin $0 \in \mathbb{R}^3$, both d and ρ vanish. The following cohomologies then appear: $H_{\mathcal{F}}^{-1}(0) \simeq \mathbb{R}^3 \simeq \mathfrak{sl}_2$ and $H_{\mathcal{F}}^{-2}(0) \simeq \mathbb{R}$. The isotropy L_∞ -algebra of the foliation is given by the graded vector space $H_{\mathcal{F}}(0) \simeq \mathbb{R}[2] \oplus \mathfrak{sl}_2[1]$. It admits on $H_{\mathcal{F}}^1(0) \simeq \mathfrak{sl}_2[1]$ the usual Lie algebra bracket, at and all k -ary brackets vanish for $k \geq 3$.

Example 22. Consider the singular foliation given by all vector fields on a vector space V vanishing at the origin. According to Example 19, the isotropy L_∞ -algebra of the foliation at any point which is not the origin is zero. The isotropy L_∞ -algebra at the origin is the graded Lie algebra $\bigoplus_{i \geq 1} \wedge^i V^* \otimes V$ equipped with the (graded symmetric) Lie bracket defined as in (1.63). There is no k -ary bracket for $k \geq 3$.

Example 23. In the conjectural Example 12, the result is very similar. Recall that we are not able to prove that the sequence we construct in that example is a resolution. But we are able to equip it with a Lie ∞ -algebroid structure that induces the singular foliation given by the adjoint foliation of \mathfrak{g} on itself. At the origin 0, again, the L_∞ -algebra associated to that Lie ∞ -algebroid has all the brackets which are trivial except in degree -1 where we recover the Lie algebra \mathfrak{g} . There is no k -ary bracket for $k \geq 3$.

Example 24. We saw in Example 20 that the singular foliation \mathcal{F}_φ of all vector fields X on $M = \mathbb{C}^n$ such that $X[\varphi] = 0$, with φ a weight homogeneous function on $M = \mathbb{C}^n$ with isolated singularities, admits a universal Lie ∞ -algebroid over it.

The origin 0 of \mathbb{C}^n is a leaf. Let us study the Lie ∞ -algebra at this point. Since all partial derivatives of φ vanish at the origin 0, the Koszul resolution (see Example 14) is a minimal resolution at 0, so that $H_{-k}^{\mathcal{F}_\varphi}(0) \simeq \wedge^{k+1} \mathbb{C}^n$. For k -ary brackets of the universal Lie ∞ -algebroid over \mathcal{F}_φ given by Equation (1.64), their restrictions at 0 are given by:

$$\{\partial_{I_1}, \dots, \partial_{I_k}\}_k := \sum_{i_1 \in I_1, \dots, i_k \in I_k} \epsilon(i_1, \dots, i_k) \varphi_{i_1, \dots, i_k}(0) \partial_{\frac{I_1}{\{i_1\}} \cup \dots \cup \frac{I_k}{\{i_k\}}} \quad (2.7)$$

where:

1. for all $I = \{i_1, \dots, i_k\}$ a sub-list of elements in $\{1, \dots, n\}$, the set $\frac{I}{\{i_p\}}$, where $1 \leq p \leq k$ is the sub-sub-list $\{i_1, \dots, i_{p-1}, i_{p+1}, \dots, i_k\}$, and ∂_I is a shorthand for $\frac{\partial}{\partial x_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x_{i_k}} \in H_{-k}^{\mathcal{F}_\varphi}(0) \simeq \wedge^{k+1} \mathbb{C}^n$,
2. $\epsilon(i_1, \dots, i_k) \in \{-1, 1\}$ is the signature of the permutation of the list I_1, \dots, I_k which consists in forcing i_1, \dots, i_k to come, in that order, in front of the list,
3. $\varphi_{i_1, \dots, i_k}$ is a shorthand for $\frac{\partial^k \varphi}{\partial x_{i_1} \dots \partial x_{i_k}}$.

The 3-ary bracket is in general not trivial in this case.

We conclude this section with a characterization of the singular foliations described by C. Debord [13]. We call *Debord foliation* a singular foliation \mathcal{F} which is a projective \mathcal{O} -module, i.e. which is covered by an anchored vector bundle (A, ρ) such that $\rho : \Gamma(A) \rightarrow \mathcal{F}$ is an isomorphism of \mathcal{O} -modules, with ρ injective on a dense open subset of M .

Proposition 2.14. *Let \mathcal{F} be a singular foliation. For every $x \in M$ the following are equivalent:*

- (i) *There is a neighborhood of $x \in M$ on which \mathcal{F} is a Debord foliation.*
- (ii) *There is a neighborhood of $x \in M$ on which \mathcal{F} admits resolutions and $H_{-i}^{\mathcal{F}}(y) = 0$ for all $i \geq 2$ and all y in this neighborhood.*
- (iii) *There is a neighborhood of $x \in M$ on which \mathcal{F} admits resolutions and $H_{-2}^{\mathcal{F}}(x) = 0$.*
- (iv) *There is a neighborhood of $x \in M$ on which \mathcal{F} admits resolutions and the isotropy graded Lie algebra at x is concentrated in degree -1 , i.e. it is a Lie algebra.*

Proof. Every Debord foliation is given by a Lie algebroid A whose anchor is injective on an open and dense subset. A resolution is therefore given by $E_{-1} := A$ and $E_{-i} := 0$ for all $i \geq 2$. Since a resolution of length 1 exists in a neighborhood of any point x , the cohomologies $H_{-i}^{\mathcal{F}}(y)$ are all trivial for all $i \geq 2$ and any point y in the neighborhood of x . Hence (i) implies (ii). It is obvious that (ii) implies (iii). Let us assume that (iii) holds. Let E be a resolution of \mathcal{F} with anchor ρ . Since the dimension of the image of $d^{(3)} : E_{-3} \rightarrow E_{-2}$ around x is greater than or equal to its dimension at $x \in M$, while the dimension of the kernel of $d^{(2)} : E_{-2} \rightarrow E_{-1}$ is, around x , lower than or equal to its dimension at the point x , we indeed have $H_{-2}^{\mathcal{F}}(y) = 0$ in a neighborhood of $x \in M$. Moreover, it implies that the dimension of the kernel of $d^{(2)} : E_{-2} \rightarrow E_{-1}$ is constant in a neighborhood of x . This implies that $E'_{-1} := E_{-1} / d^{(2)}(E_{-2})$ is a vector bundle. The anchor goes to the quotient to define a morphism of \mathcal{O} -modules $\rho : \Gamma(E'_{-1}) \rightarrow \mathcal{F}$ that we still denote by ρ , and which is, by construction, an isomorphism of \mathcal{O} -modules. The equivalence between (iii) and (iv) is obvious. This completes the proof. \square

2.5 Holonomy Lie groupoids

Let us consider a Lie ∞ -algebroid (E, Q) over a singular foliation \mathcal{F} on a manifold M . In Section 2.2, the isotropy Lie ∞ -algebra of (E, Q) at a point $x \in M$ was defined on the vector space $V_x = \text{Ker}(\rho_x) \oplus \bigoplus_{i \geq 2} \mathfrak{i}_x^* E_{-i}$. This construction can be enlarged: instead of restricting (E, Q) to a point, we can restrict it to a leaf, and we therefore obtain a Lie ∞ -algebroid over a leaf L of the singular foliation \mathcal{F} .

Proposition 2.11 implies that for all $i \geq 2$, the vector bundle morphism $d^{(i)} : E_{-i} \rightarrow E_{-i+1}$ is of constant rank at all points of the leaf L . This allows to truncate the Lie ∞ -algebroid at a certain order i , to get a Lie ∞ -algebroid structure on the graded vector bundle:

$$\mathfrak{i}_L^* E_{-i} / d^{(i+1)}(\mathfrak{i}_L^* E_{-i-1}) \longrightarrow \mathfrak{i}_L^* E_{-i+1} \longrightarrow \cdots \longrightarrow \mathfrak{i}_L^* E_{-1} \longrightarrow TL$$

Above, \mathfrak{i}_L^* stands for the restriction to the leaf L of a vector bundle over M . This Lie ∞ -algebroid is a Lie i -algebroid, that we call the *i -th truncation of E* . For $i = 1$, we get a Lie algebroid that we call the *holonomy Lie algebroid of the leaf L* .

Proposition 2.15. *Let \mathcal{F} be a singular foliation that admits a universal Lie ∞ -algebroid over it, and let L be a leaf of \mathcal{F} . The 1-truncation of the Lie ∞ -algebroid over L coincides with the Lie algebroid of the fibers of the foliation over L defined by Androulidakis and Skandalis in [1].*

Proof. In [1], the holonomy Lie algebroid is defined by the vector bundle whose fiber over $x \in L$ is the germ at x of $\mathcal{F}/I_x\mathcal{F}$, with I_x the ideal of functions vanishing at x :

$$A_L^{\mathcal{F}} = \bigcup_{x \in L} \mathcal{F}/I_x\mathcal{F}$$

The anchor map is defined by the evaluation at x of an element in \mathcal{F} and the bracket is induced from the Lie bracket of vector fields. Notice that the kernel of the anchor map is the isotropy Lie algebra at x by construction. \square

To any Lie ∞ -algebroid (E, Q) over a manifold M , one associates a topological groupoid as follows. Let $I = [0, 1]$. Morphisms of Lie ∞ -algebroids from the tangent Lie algebroid TI to (E, Q) are in one-to-one correspondence with paths $a : I \rightarrow E_{-1}$ over a path $\gamma : I \rightarrow M$ such that:

$$\frac{d\gamma(t)}{dt} = \rho(a(t)) \quad (2.8)$$

We call such a path a *E -path*. It is said to be *trivial* when $\gamma(t)$ is a constant path equal to some $x \in M$ and $a(t) = 0_x$ for all $t \in I$. A *homotopy between two Lie ∞ -algebroid morphisms a_0, a_1 from the tangent Lie algebroid TI to (E, Q)* is a Lie ∞ -algebroid morphism from the tangent Lie algebroid TI^2 to (E, Q) whose restrictions to $\{0\} \times I$ and $\{1\} \times I$ in I^2 are a_0 and a_1 respectively, while the restrictions to $I \times \{0\}$ and $I \times \{1\}$ are trivial. The groupoid

product is given by concatenation of paths, which makes sense if we assume them to be trivial in neighborhoods of $t = 0$ and $t = 1$. To obtain a topology on this quotient, we restrict ourselves to C^1 -paths and equip it with a Banach manifold topology, as in [10] or [12]. We call this groupoid the *1-truncated groupoid* of (E, Q) .

Proposition 2.16. *Let (E, Q) be a universal Lie ∞ -algebroid over a singular foliation \mathcal{F} . The 1-truncated groupoid of (E, Q) is a universal cover of the connected component of the manifold of units of the holonomy groupoid described by Androulidakis and Skandalis in [1].*

Proof. Given a singular foliation \mathcal{F} , the holonomy groupoid of \mathcal{F} described in [1] is a topological groupoid that induces the singular foliation \mathcal{F} on M . Moreover, according to [1] again, for any leaf L of \mathcal{F} , its restriction to L is a groupoid integrating the holonomy Lie algebroid A_L of the leaf L , which is shown in Proposition 2.15 to coincide with the 1-truncation $i_L^* E_{-1} / d^{(2)} i_L^* E_{-2}$.

Let us check that the 1-truncated groupoid of (E, Q) satisfies the same property. It admits \mathcal{F} for induced foliation on M . Let us show that its restriction to any leaf L coincides with the universal cover of the groupoid integrating the Lie algebroid $A_L := i_L^* E_{-1} / d^{(2)} i_L^* E_{-2}$.

It is clear that any E -path over L (from now on, we shall speak of E_L -paths) induces a A_L -path in the usual sense of Cattaneo-Felder [10] and Crainic-Fernandes [12]. It is also obvious that if two such paths are homotopic as E_L -paths, their induced A_L -paths are homotopic as A_L -paths. Hence, the 1-truncated groupoid of (E, Q) maps to the source-1-connected Lie groupoid integrating A_L .

Let us check that this map is bijective. Surjectivity is obvious: any A_L -path comes from an E_L -path that we call a *lift*. Now, in order to show that the map is injective, let us check that homotopic A_L -paths are induced from homotopic E_L -paths, i.e. that any Lie ∞ -algebroid morphism from TI^2 to A_L lifts to a Lie ∞ -algebroid morphism from TI^2 to (E, Q) whose boundary values are arbitrary lifts of the initial A -paths. Let α be a Lie algebroid morphism from $TI^2 \rightarrow i_L^* E_{-1} / d^{(2)} i_L^* E_{-2}$ whose restriction to the boundaries satisfy the usual requirements of homotopies relating two A_L -paths a_1 and a_2 . The vector bundle morphism $\alpha : TI \rightarrow i_L^* E_{-1} / d^{(2)} i_L^* E_{-2}$ can be lifted to a vector bundle morphism Φ_α valued in $i_L^* E_{-1}$ that still satisfies the requirements of homotopies of E_L -paths when restricted to boundaries, and that relates to arbitrary lifts of a_1 and a_2 . It is not a Lie ∞ -algebroid morphism a priori, i.e. $\Psi := \Phi_\alpha \circ Q - d_{dR} \circ \Phi_\alpha$ may not be zero. By construction, Ψ is a Φ_α derivation whose only term which may be non-vanishing is a vector bundle morphism from $\Gamma(i_L^* E_{-1}^*)$ to $\Omega^2(I^2)$.

Since Φ_α induces a Lie ∞ -algebroid morphism (in fact, a Lie algebroid morphism) when taking the quotient, Ψ is zero on the image of $\Gamma(A_L^*) \rightarrow \Gamma(i_L^* E_{-1}^*)$, i.e. the conormal of the image of $d^{(2)} : E_{-2} \rightarrow E_{-1}$. This allows us to modify Φ_α by adding a map $\Gamma(i_L^* E_{-2})$ to $\Omega^2(I^2)$ so that the relation $\Phi_\alpha \circ Q = d_{dR} \circ \Phi_\alpha$ holds on $\Gamma(E_{-1}^*)$. This modified Φ_α is a homotopy of E_L -paths by construction. This completes the proof. \square

2.6 The 3-ary bracket of the isotropy Lie ∞ -algebra of a singular foliation

Let \mathcal{F} be a singular foliation that admits a resolution of finite length and let $x \in M$ be a point. In the sequel, we shall assume that the foliation vanishes at x (i.e. all vector fields in \mathcal{F} are zero at x), so that $\{x\}$ is a leaf of \mathcal{F} . According to item 5. in Proposition 1.3, a resolution which is minimal at x exists. According to Proposition 2.8, the graded Lie bracket of the isotropy graded Lie algebra $\bigoplus_{i \geq 1} H_{-i}^{\mathcal{F}}(x)$ at x is part of a Lie ∞ -algebra structure, whose differential (= 1-ary bracket) is zero: the isotropy Lie ∞ -algebra of \mathcal{F} at x .

The following Lemma is trivial:

Lemma 2.17. *Let \mathcal{F} be a singular foliation that admits a resolution of finite length. Consider a leaf that reduces to a point x .*

1. *The binary bracket makes $H_{-2}^{\mathcal{F}}(x)$ a module over the isotropy Lie algebra $H_{-1}^{\mathcal{F}}(x)$,*

2. The restriction to $H_{-1}^{\mathcal{F}}(x)$ of the 3-ary bracket is a 3-cocycle for the Chevalley-Eilenberg cohomology of the isotropy Lie algebra $H_{-1}^{\mathcal{F}}(x)$, valued in the module $H_{-2}^{\mathcal{F}}(x)$,
3. The class of cohomology of this cocycle does not depend on the choices made in the construction.

We refer to this class as the NMRLA 3-class.

Proof. Recall from Proposition 2.8 that the binary bracket of the isotropy Lie ∞ -algebra of \mathcal{F} at x coincides with the graded Lie algebra bracket of the isotropy Lie algebra of \mathcal{F} at x . Now, for every graded Lie algebra $\mathfrak{g}_{\bullet} := \sum_{i \geq 1} \mathfrak{g}_{-i}$ (considered with the symmetric conventions discussed in Remark 2), the vector space \mathfrak{g}_{-1} is a Lie algebra, and the graded Lie bracket $\mathfrak{g}_{-1} \times \mathfrak{g}_{-k} \rightarrow \mathfrak{g}_{-k}$ turns \mathfrak{g}_{-k} into a \mathfrak{g}_{-1} -module. The second item is a general fact for a Lie ∞ -algebra (V, Q_V) with trivial differential: in such a case the Jacobi identity implies that:

$$[Q_V^{(1)}, Q_V^{(2)}] = 0 \quad (2.9)$$

which means precisely that the 3-ary bracket is a Chevalley-Eilenberg cocycle when restricted to $H_{-1}^{\mathcal{F}}(x)$. The third item can be obtained as follows: two different choices made in the construction of the universal Lie ∞ -algebroid give isotropy Lie ∞ -algebra structures which are strictly isomorphic by Proposition 2.8, through isomorphisms whose linear parts are the identity. The quadratic part of this isomorphism has a component which is a map $\tilde{\theta}$ from $S^2(H_{-1}^{\mathcal{F}}(x))$ to $H_{-2}^{\mathcal{F}}(x)$. Writing explicitly the definition of Lie ∞ -algebra morphisms, applied to three elements in $H_{-1}^{\mathcal{F}}(x)$, one obtains that $\{\cdot, \cdot, \cdot\}_3$ and $\{\cdot, \cdot, \cdot\}'_3$ differ by (a multiple of) the Chevalley-Eilenberg differential of $\tilde{\theta}$. \square

The name NMRLA stands for ‘no minimal rank Lie algebroid’. Let us explain this name. According to [1], the rank r of the quotient $\mathcal{F}/I_x\mathcal{F}$ is the minimal number of generators of the foliation \mathcal{F} in a neighborhood of that point. Since the leaf of x reduces to $\{x\}$ itself, $\mathcal{F}/I_x\mathcal{F} = H_{-1}^{\mathcal{F}}(x)$, and $r = \dim(H_{-1}^{\mathcal{F}}(x))$.

Proposition 2.18. *Let \mathcal{F} be a singular foliation on a manifold M that admits a resolution of finite length. Consider a leaf that reduces to a point x , and let r be the rank of \mathcal{F} at the point x .*

If the NMRLA 3-class is not equal to 0, then it is not possible to find a Lie algebroid A defined in a neighborhood U_x of x , that satisfies the two following conditions:

1. the rank of A is r , and
2. the Lie algebroid A induces the foliation \mathcal{F} (i.e. $\rho(\Gamma(A)) = \mathcal{F}$ and the anchor map is compatible with both brackets) on U_x .

Proof. Assume that a Lie algebroid A with anchor ρ satisfying that $\rho(\Gamma(A)) = \mathcal{F}$ exists. Let (E, Q) be a universal Lie ∞ -algebroid over \mathcal{F} in a neighborhood of x , built on a resolution minimal at x . By Theorem 1.8, a Lie ∞ -algebroid morphism Φ (over the identity of M) from A to (E, Q) exists. In that case, the linear, quadratic and cubic terms in the Taylor coefficients of Φ satisfy for all $a, b, c \in \Gamma(A)$:

$$\{\Phi_1(a), \Phi_2(b, c)\}_2 - \Phi_2([a, b], c) + c.p. = \{\Phi_1(a), \Phi_1(b), \Phi_1(c)\}_3 - d\Phi_3(a, b, c)$$

When the rank of A is r , we can assume that $A := E_{-1}$ and that the first order Taylor coefficient $\Phi_1 : A \rightarrow E_{-1}$ of the Taylor coefficient of Φ is the identity map. The previous equation, evaluated at the point x gives then that $\{\cdot, \cdot, \cdot\}_3$ is a coboundary for the Chevalley-Eilenberg complex of $i_x^*E_{-1} \simeq H_{-1}^{\mathcal{F}}(x)$ valued in $i_x^*E_{-2} \simeq H_{-2}^{\mathcal{F}}(x)$. \square

Example 25. For the singular foliation \mathcal{F}_φ on \mathbb{C}^n , associated to the function $\varphi(x_1, \dots, x_n) := \sum_{i=1}^n x_i^3$ as in Examples 14-20-24, it follows from Equation (2.7) that at the origin 0:

1. the binary bracket of the isotropy Lie ∞ -algebra at 0 is zero,
2. in particular, the isotropy Lie algebra is Abelian, and its action on $H_{-2}^{\mathcal{F}}(0)$ is trivial,
3. the 3-ary bracket is not zero for $n \geq 4$, since (with the conventions of Example 24):

$$\{\partial_{12}, \partial_{13}, \partial_{14}\}_3 := \partial_{234}.$$

This implies that the NMRLA 3-class of \mathcal{F}_φ at the origin 0 is not zero. This foliation, therefore, is not induced by a Lie algebroid of rank $n(n-1)/2$.

Example 25 proves the following proposition:

Proposition 2.19. *There exist singular foliations of rank r that can not be the image a Lie algebroid of rank r .*

2.7 The Leibniz algebroid of a singular foliation

As already mentioned in the introduction, it is not easy to decide whether a singular foliation is, locally, the image of a Lie algebroid under the anchor map. It is known not to be the case globally, see [3], while in a neighborhood of a point the question is open. We are not able to bring a positive or negative answer to decide this question, but a direct consequence of Theorem 1.6 is that a Leibniz algebroid defining the singular foliation always exists in the real analytic or holomorphic cases, in a neighborhood of a point. Although it can be easily derived from Theorem 1.6, it is far from being obvious if we do not have such a theorem.

Definition 2.20. [26] *Let L be a vector bundle over M . A Leibniz algebroid structure on L is a bilinear assignment $[\cdot, \cdot]_L : \Gamma(L) \otimes \Gamma(L) \rightarrow \Gamma(L)$ and a vector bundle morphism $\rho : L \rightarrow TM$, satisfying the Loday-Jacobi condition:*

$$[x, [y, z]_L]_L = [[x, y]_L, z]_L + [y, [x, z]_L]_L \quad (2.10)$$

for all $x, y, z \in \Gamma(L)$, and the Leibniz identity:

$$[x, fy]_L = f[x, y]_L + \rho(x)[f]y \quad (2.11)$$

for every $x, y \in L$ and $f \in \mathcal{O}$.

In fact, for every singular foliation arising from a Lie ∞ -algebroid, a Leibniz algebroid defining the foliation exists. This follows from Proposition 5.4 item 1 and Lemma 5.5 in [18]: adapting this result to our case, this construction gives the following result.

Proposition 2.21. *Let \mathcal{F} be a singular foliation that admits a universal Lie ∞ -algebroid structure (E, Q) with anchor ρ . Assume that its associated resolution is of finite length. Then $L = (S(E^*) \otimes E)|_{-1}$ is a vector bundle of finite rank and comes with a Leibniz algebroid structure, when equipped with:*

1. the Leibniz bracket defined by:

$$[X, Y]_L := [[Q, X], Y]$$

for all $X, Y \in \Gamma(L)$ (identified with vertical vector fields ∂_X and ∂_Y of degree -1 on the graded manifold E),

2. the anchor given by the composition:

$$E_{-1} \oplus \left(\bigoplus_{k \geq 1} S^k(E^*) \otimes E \right) |_{-1} \longrightarrow E_{-1} \xrightarrow{\rho} TM.$$

Proof. For every graded Lie algebra, $\mathfrak{g} := \sum_{i \in \mathbb{Z}} \mathfrak{g}_i$ and any homological element $Q \in \mathfrak{g}$ of degree $+1$, \mathfrak{g}_{-1} is a graded Leibniz algebra when equipped with the bracket $(X, Y) \mapsto [[Q, X], Y]$, see [25]. Applied to the graded Lie algebra of derivations of functions \mathcal{E} on the Lie ∞ -algebroid (E, Q) (that is, vector fields on the N -manifold defined by the graded vector space E) and to the vector field Q , the bracket given as above induces a Leibniz algebra bracket on vector fields of degree -1 . Now, vertical vector fields of degree -1 are vertical (i.e. are \mathcal{O} -linear derivations of \mathcal{E}), and therefore can be identified with sections of the vector bundle $L = (S(E^*) \otimes E)|_{-1}$ (that is, the vector bundle of elements of degree -1 in $S(E^*) \otimes E$). Also, since sections of $S^k(E^*) \otimes E$ are of non-negative degree for $k \geq n$, L is a vector bundle of finite rank over M . One checks directly that the anchor η is given as in item 2. By construction, $\eta(\Gamma(L)) = \rho(\Gamma(E_{-1})) = \mathcal{F}$. This proves the proposition. \square

The following proposition is an immediate consequence of Proposition 2.21 and Theorem 1.6.

Proposition 2.22. *Let \mathcal{F} be a singular foliation that admits a resolution of finite length. Then there exists a Leibniz algebroid structure whose induced singular foliation is \mathcal{F} .*

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C. LAURENT-GENGOUX, INSTITUT ELIE CARTAN DE LORRAINE (UMR 7502), UNIVERSITÉ DE LORRAINE, 3 rue Augustin Fresnel, 57000 Metz-Technopôle, France
E-mail address, `camille.laurent-gengoux@univ-lorraine.fr`

S. LAVAU, CMUP, UNIVERSIDADE DE PORTO, 687 Rua do Campo Alegre, 4169-007 Porto, Portugal
E-mail address, `sylvain.lavau@ens-lyon.fr`

T. STROBL, CNRS-2924, INSTITUTO DE MATEMÁTICA PURA E APLICADA (IMPA), Estrada Dona Castorina 110, Rio de Janeiro, 22460-320, Brasil and INSTITUT CAMILLE JORDAN, UNIVERSITÉ CLAUDE BERNARD LYON 1, 43 Boulevard du 11 Novembre 1918, 69622 Villeurbanne Cedex, France.
E-mail address, `thomas[AT]impa.br` and `strobl[AT]math.univ-lyon1.fr`