LOGARITHMIC DEGENERATIONS OF LANDAU-GINZBURG MODELS FOR TORIC ORBIFOLDS AND GLOBAL tt^* GEOMETRY

ETIENNE MANN AND THOMAS REICHELT

Abstract. We discuss the behavior of Landau-Ginzburg models for toric orbifolds near the large volume limit. This enables us to express mirror symmetry as an isomorphism of Frobenius manifolds which aquire logarithmic poles along a boundary divisor. If the toric orbifold admits a crepant resolution we construct a global moduli space on the B-side and show that the associated tt^*-geometry exists globally.

Contents

1. Introduction

The present paper deals with classical Hodge-theoretic mirror symmetry for smooth toric Deligne-Mumford stacks. One of the first mathematical incarnations of this type of mirror symmetry was a theorem of Givental identifying a solution (the so-called J-function) of the Quantum

2010 *Mathematics Subject Classification.* 14J33, 14M25, 32S40, 32G34, 34Mxx, 53D45
Keywords:Hypergeometric $\mathcal{D}$-module, quantum cohomology, Frobenius manifold, mirror symmetry, tt^*-geometry
E.M is supported by the grant of the Agence Nationale de la Recherche "New symmetries on Gromov-Witten theories" ANR- 09-JCJC-0104-01. Th.R. is supported by the Emmy-Noether DFG grant RE 3567/1-1.
Both authors acknowledge partial support by the ANR/DFG grant ANR-13-IS01-0001-01/02, HE 2287/4-1 & SE 1114/5-1 (SISYPH).

$\mathcal{D}$-module of a (complete intersection inside a) smooth toric variety with a generalized hypergeometric function (the I-function). This has laid the foundation to express mirror symmetry as an equivalence of differential systems matching the Quantum $\mathcal{D}$-module on the A-side with certain (Fourier-Laplace transformed) Gauß-Manin systems coming from an algebraic family of maps (the Landau-Ginzburg model) on the B-side. An analytic proof of this fact using oscillating integrals was given by Iritani in [Iri09]. A purely algebraic proof was given in [RS15] where it was also shown that the Frobenius manifold on the A-side, which encodes the big quantum cohomology, is isomorphic to a Frobenius manifold on the B-side which comes from the Landau-Ginzburg model. The construction of Frobenius manifolds is a classical subject in singulartity theory. The first examples arose from the work of K.+ M. Saito on the base space of a semi-universal unfolding and later it was shown by Sabbah partly with Douai [DS03],[DS04],[Sab06] that these results carry over to an algebraic map which satisfies certain tameness assumptions. However, their construction is not unique in the sense that it depends on the choice of a good basis, which provides a solution to a Birkhoff problem, and on the choice of a primitive section. To circumvent this problem a careful analysis of the Fourier-Laplace transformed Gauß-Manin system and its degeneration along a boundary divisor, which contains the large volume limit, was carried out in [RS15] in the case of Landau-Ginzburg models which serve as mirror partners for smooth nef toric varieties. Beyond the smooth case partial results for weighted projective spaces were obtained in [DM13] where mirror symmetry is proven as an isomorphism of logarithmic Frobenius manifolds without pairing.

In this paper we prove mirror symmetry for smooth toric Deligne-Mumford stacks, satisfying a positivity condition, as an isomorphism of logarithmic Frobenius manifolds, which generalizes the theorem obtained in [RS15] for smooth nef toric varieties. In order to ensure a good behavior of the connection and the pairing along the boundary divisor a careful choice of the coordinates on the complexified Kähler moduli space is needed. Since the Fourier-Laplace transformed Gauß-Manin system is a cyclic $\mathcal{D}$-module, the generator is a canonical candidate for the primitve section. The Birkhoff problem is solved at the large volume limit where we identify the fiber of the holomorphic bundle with the orbifold cohomology of the toric Deligne-Mumford stack.

The notion of tt^* geometry was introduced by Cecotti and Vafa in their study of moduli spaces of $N = 2$ supersymmetric quantum field theories. Hertling [Her03] formalized this structure under the name of pure and polarized TERP-structures and showed that the base space of a semi-universal unfolding of an isolated hypersurface singularity carries such a structure. In the case of a tame algebraic map, a theorem of Sabbah [Sab08] shows that the corresponding Fourier-Laplace transformed Gauß-Manin system underlies a pure and polarized TERP-structure. In [RS15] this was used to show that a Zariski open subset of the base space of the Landau-Ginzburg model carries a pure and polarized TERP-structure. Using mirror symmetry this induces tt^* geometry on the quantum $\mathcal{D}$-module. Iritani [Iri09] gave an intrinsic description of the corresponding real structure on the A-side using K-theory. In this paper the result of [RS15] is generalized to toric orbifolds. If the toric orbifold X admits a crepant resolution Z we construct a global base space which contains two limit points corresponding to the large volume limit points of X and Z respectively. We prove that there exists tt^*-geometry on the whole moduli space which, when restricted to some analytic neighborhood of the large volume limits, is isomorphic to the tt^*-geometry coming from the quantum cohomology of X resp. Z. The result here is in the sprit of Y.Ruan's Crepant Transformation conjecture which has stimulated a lot of research: [BMP11, BMP09, Per07, BG09b, BG09a, BG09c, BG08, BGP08, CLLZ14, CLZZ09, CIT09, Coa09, CR13, Rua06, Iwa08, LLW16a, LLW16b, LLQW16].

Shortly after finishing this paper we learned that Coates-Corti-Iritani-Tseng [CCIT20] also studied Hodge-theoretic mirror symmetry for toric orbifolds. While their result is broader in the sense that firstly they allow orbifolds with generic stabilizer, secondly they do not need the nef assumption and thirdly they construct a mirror Landau-Ginzburg model even for the big orbifold quantum cohomology, their construction is inherently local on the complexified Kähler moduli space since they use a formal version of the Gauß-Manin system in the non-nef case and an analytic version of the Gauß-Manin in an analytic neighborhood of the large volume limit. The Gauß-Manin system which we construct in this paper is defined on the whole complexified Kähler moduli space and is algebraic. These two facts are essential in the proof of the existence of tt^*-geometry on the whole Kähler moduli space.

We give a short overview of the contents of this paper: In section 2 we recall some standard facts on toric Deligne-Mumford stacks. An import ingredient in the construction of the mirror Landau-Ginzburg model is the extended stacky fan of Jiang [Jia08]. This enables us to introduce the extended Picard group and the extended Kähler cone which are needed to construct coordinates on the base space of the mirror Landau-Ginzburg model. In section 3 we review the notion of the Fourier-Laplace transformed Gauß-Manin system and cite some results of [RS15],[RS17] which identifies the FL-Gauß-Manin system of a family of Laurent polynomials with a FL-transformed GKZ-system for which an explicit description as a cyclic $\mathcal{D}$-module is available. However, the results for the duality of FL-transformed GKZ-systems and the construction of a pairing do not follow from loc. cit. We therefore prove some statements concerning the duality of algebraically micro-localized GKZ-systems and conclude from these the existence of a pairing. In the fourth section we use the results of the previous section to calculate the FL-transformed Gauß-Manin system corresponding to the Landau-Ginzburg model (cf. Proposition 4.4). As a next step we show that the FL-transformed Brieskorn lattice is coherent over the tame locus of the Landau-Ginzburg model (Theorem 4.10). We then analyze the degeneration behavior along a boundary divisor which contains the large volume limit. Finally we prove that there exists a canonical germ of a logarithmic Frobenius manifold associated to the Landau-Ginzburg model. Section 5 reviews orbifold quantum cohomology and the Givental connection. We show that the big quantum cohomology gives rise to a logarithmic Frobenius manifold (Proposition 5.7). In section 6, using a Givental-style mirror theorem of Coates, Corti, Iritani and Tseng [CCIT15b], we combine the last two sections to express mirror symmetry for toric Deligne-Mumford stacks as an isomorphism of logarithmic Frobenius manifolds (Theorem 6.6). In section 7 we consider a toric orbifold X admitting a crepant resolution Z and construct a global Landau-Ginzburg model. We prove that there exists a pure and polarized variation of TERP structures on the base space $\mathcal{M}$ of this model which gives the tt^*-geometry of the corresponding quantum $\mathcal{D}$-modules in different neighborhoods of $\mathcal{M}$.

Acknowledgements: The second author would like to thank Takuro Mochizuki and Claude Sabbah for continuing support and interest in his work, Christian Sevenheck and Uli Walther for useful discussions. We thank an anonymous referee for noticing a gap in the proof of Proposition 3.18, for pointing out a number of inaccuracies and for suggesting several improvements of the paper.

2. Some toric facts

We start with some notations on groups, homomorphisms of groups and associated tori. Let G be a free abelian group. We associate to it the group ring $\mathbb{C}[G]$ which is generated by the elements χ^g for $g \in G$. Its maximal spectrum $\mathrm{Specm}(\mathbb{C}[G]) = Hom(G, \mathbb{C}^*)$ is naturally a commutative algebraic group (i.e. a torus). Let $\mathfrak{a} : G \to H$ be a group homomorphism between the free

abelian groups G and H. This induces a ring homomorphism

$$\phi_{\mathfrak{a}} : \mathbb{C}[G] \longrightarrow \mathbb{C}[H]\,,$$
$$\chi^g \mapsto \chi^{\mathfrak{a}(g)}$$

and a morphism of algebraic groups

$$\psi_{\mathfrak{a}} : \mathrm{Specm}(\mathbb{C}[H]) \longrightarrow \mathrm{Specm}(\mathbb{C}[G])\,.$$

Choose a basis $g_1, \ldots, g_n$ resp. $h_1, \ldots, h_m$ of G resp. H. The homomorphism $\mathfrak{a}$ is then given by a matrix $A = (a_{ij})$ with $\mathfrak{a}(g_j) = \sum_{j=1}^m a_{ij} h_i$. The bases also determine coordinates $x_j = \chi^{g_j}$ resp. $y_i = \chi^{h_i}$ which identifies $\mathrm{Specm}(\mathbb{C}[G])$ with $(\mathbb{C}^*)^n$ and $\mathrm{Specm}(\mathbb{C}[H])$ with $(\mathbb{C}^*)^m$. In these coordinates the map $\psi_{\mathfrak{a}}$ is given by

$$(\mathbb{C}^*)^m \longrightarrow (\mathbb{C}^*)^n\,,$$
$$(y_1, \ldots, y_m) \mapsto (\underline{y}^{\underline{a}_1}, \ldots, \underline{y}^{\underline{a}_n})$$

where $\underline{y}^{\underline{a}_j} := \prod_{i=1}^n y_i^{a_{ij}}$.

2.1. **Toric Deligne-Mumford stacks.** A toric Deligne-Mumford stack is constructed by a so-called stacky fan which was introduced by [BCS05]. A stacky fan

$$\mathbf{\Sigma} = (N, \Sigma, \mathfrak{a})$$

consists of

- a finitely generated abelian group N of rank d,
- a complete simplicial fan Σ in $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$, where we denote by $\Sigma(k)$ the set of k-dimensional cones of Σ and by $\{\rho_1, \ldots, \rho_m\}$ the rays of Σ,
- a homomorphism $\mathfrak{a} : \mathbb{Z}^m \to N$ given by elements $a_1, \ldots a_m$ of N with $a_i \in \rho_i$ and $\mathfrak{a}(e_i) = a_i$, where $e_1, \ldots, e_m$ is the standard basis of $\mathbb{Z}^m$.

Assumption: In the remainder of the paper we will assume that N is torsion-free. The geometric meaning of this assumption is that the associated toric stack have a trivial generic stabilizer group.

If we choose a basis $v_1, \ldots, v_d$ of N the map $\mathfrak{a}$ is given by a matrix $A = (a_{ki})$.

The morphism $\mathfrak{a}$ gives rise to a triangle in the derived category of $\mathbb{Z}$-modules

$$\mathbb{Z}^m \xrightarrow{\mathfrak{a}} N \longrightarrow Cone(\mathfrak{a}) \xrightarrow{+1}\,.$$

We apply the derived functor $RHom(-, \mathbb{Z})$ and consider the associated long exact sequence

$$0 \longrightarrow N^\star \xrightarrow{\mathfrak{a}^\star} (\mathbb{Z}^m)^\star \longrightarrow \mathrm{Ext}^1(Cone(\mathfrak{a}), \mathbb{Z}) \longrightarrow 0\,, \tag{2.1}$$

where the injectivity of $\mathfrak{a}^\star$ follows from the fact that the image of $\mathfrak{a}$ has finite index in N and the surjectivity of $(\mathbb{Z}^m)^\star \longrightarrow \mathrm{Ext}^1(Cone(\mathfrak{a}), \mathbb{Z})$ follows from $\mathrm{Ext}^1(N, \mathbb{Z}) = 0$, i.e. from our assumption that N is free.

Applying $Hom(-, \mathbb{C}^*)$ to the exact sequence (2.1) gives the short exact sequence

$$0 \longrightarrow G \xrightarrow{\psi_{\mathfrak{a}}} (\mathbb{C}^*)^m \longrightarrow Hom(N^\star, \mathbb{C}^*) \longrightarrow 0\,,$$

where $G := Hom(\mathrm{Ext}^1(Cone(\mathfrak{a}), \mathbb{Z}), \mathbb{C}^*)$ (The notion G was used before for a free abelian group, we will stick to the definition given here for the remainder of the paper). Here we have used the

fact that $\mathbb{C}^*$ is a divisible group, hence $Hom(-,\mathbb{C}^*)$ is exact.

The set of anti-cones is defined to be

$$\mathcal{A} := \left\{ I \subset \{1,\dots,m\} \mid \sum_{i\notin I} \mathbb{Q}_{\geq 0}\rho_i \text{ is a cone in } \Sigma \right\}.$$

Each $I \in \mathcal{A}$ gives rise to a subvariety $\mathbb{C}^I \subset \mathbb{C}^m$ given by $\{(x_1,\dots,x_m) \in \mathbb{C}^m \mid x_i = 0 \text{ for } i \notin I\}$. We set

$$\mathcal{U}_{\mathcal{A}} := \mathbb{C}^m \setminus \bigcup_{I\notin\mathcal{A}} \mathbb{C}^I .$$

The toric Deligne-Mumford stack associated to this data is the following quotient stack:

$$\mathcal{X} := \mathcal{X}(\mathbf{\Sigma}) := [\mathcal{U}_A/G],$$

where G acts on $\mathcal{U}_{\mathcal{A}}$ via $\psi_{\mathfrak{a}}$.

For $\sigma \in \Sigma$ we set

$$I_\sigma := \{i \in \{1,\dots,m\} \mid a_i \in \sigma\}. \tag{2.2}$$

and define

$$\mathrm{Box}(\sigma) = \{a \in N \mid a = \sum_{a_i\in\sigma} r_i a_i, 0 \leq r_i < 1\}.$$

We set

$$Box(\Sigma) = \bigcup_{\sigma\in\Sigma} Box(\sigma).$$

The inertia stack $\mathcal{IX}(\mathbf{\Sigma})$ is the fiber product taken over the diagonal maps $\mathcal{X} \to \mathcal{X}\times\mathcal{X}$. Its components are indexed by the set $Box(\Sigma)$:

$$\mathcal{IX}(\mathbf{\Sigma}) = \bigsqcup_{v\in \mathrm{Box}(\Sigma)} \mathcal{X}_{(v)},$$

where $\mathcal{X}_{(v)}$ is the toric orbifold $\mathcal{X}(\mathbf{\Sigma}/\sigma(v))$ with $\sigma(v)$ being the smallest cone in Σ which contains v (cf. [BCS05] Section 4).

We have the following description of the orbifold cohomology ring of $\mathcal{X}$. As a $\mathbb{Q}$-vector space it is isomorphic to the direct sum of the cohomology groups of the components of its inertia stack:

$$H^*_{orb}(\mathcal{X},\mathbb{Q}) \simeq \bigoplus_{v\in Box(\Sigma)} H^{*-2i_v}(\mathcal{X}_{(v)},\mathbb{Q}),$$

where $i_v := \sum r_i$ for $v \in Box(\Sigma)$. The orbifold cohomology of $\mathcal{X}$ carries a product which makes $H^*_{orb}(\mathcal{X},\mathbb{Q})$ into a graded algebra. A combinatorial description in terms of the fan has been given by Borisov, Chen and Smith [BCS05] and, in the semi-projective case, by Jiang and Tseng [JT08]. We equip

$$\mathbb{Q}[N] := \bigoplus_{c\in N} \mathbb{Q}\chi^c$$

with the product

$$\chi^{c_1}\cdot\chi^{c_2} := \begin{cases} \chi^{c_1+c_2} & \text{if there exists } \sigma\in\Sigma \text{ such that } c_1,c_2\in\sigma, \\ 0 & \text{otherwise}. \end{cases}$$

Let $c \in N$ and $\sigma(c)$ be the minimal cone containing c. Then c can be uniquely expressed as

$$c = \sum_{a_i \in \sigma(c)} r_i a_i \,.$$

We define

$$deg(\chi^c) := deg(c) := \sum r_i \,.$$

Using this graduation $\mathbb{Q}[N]$ becomes a graded ring. By [BCS05] we have the following isomorphism of $\mathbb{Q}$-graded rings:

$$H^*_{orb}(\mathcal{X}, \mathbb{Q}) \simeq \frac{\mathbb{Q}[N]}{\{\sum_{i=1}^m \kappa(a_i)\chi^{a_i} \mid \kappa \in N^\star\}} \,. \tag{2.3}$$

Denote by $PL(\Sigma)$ the free $\mathbb{Z}$-module of continuous piece-wise linear functions on Σ having integer values on N. We have the natural embedding of $N^\star = \mathrm{Hom}(N, \mathbb{Z})$ into $PL(\Sigma)$, where the cokernel of this map is isomorphic to the Picard group of the underlying coarse moduli space $X := X(\Sigma)$:

$$0 \to N^\star \to PL(\Sigma) \to \mathrm{Pic}(X) \longrightarrow 0 \,. \tag{2.4}$$

We have isomorphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N^\star \otimes \mathbb{Q} & \longrightarrow & PL(\Sigma) \otimes \mathbb{Q} & \longrightarrow & \mathrm{Pic}(X) \otimes \mathbb{Q} & \longrightarrow & 0 \\ & & \simeq \uparrow & & \simeq \uparrow & & \simeq \uparrow & & \\ 0 & \longrightarrow & N^\star \otimes \mathbb{Q} & \longrightarrow & \mathbb{Q}^m & \longrightarrow & H^2(X, \mathbb{Q}) & \longrightarrow & 0 \end{array}$$

where the image of the standard generator $\overline{D}_i \in \mathbb{Q}^m$ in $PL(\Sigma) \otimes \mathbb{Q}$ is the piece-wise linear function having value 1 on a_i and 0 on a_j for $j \in \{1, \ldots, m\} \setminus \{i\}$. We denote the image of $\overline{D}_i$ in $\mathrm{Pic}(X) \otimes \mathbb{Q}$ by $[\overline{D}_i]$.

2.2. **Extended stacky fans.** Toric Deligne-Mumford stacks can also be described by a so-called extended stacky fan (cf. [Jia08]). To the datum of a stacky fan $\mathbf{\Sigma} = (N, \Sigma, \mathfrak{a})$ one adds a map $\mathbb{Z}^e \to N$ and writes $S = \{a_{m+1}, \ldots, a_{m+e}\}$ for the image of the standard basis. By abuse of notation we will call the following map still $\mathfrak{a}$:

$$\begin{aligned} \mathfrak{a} : \mathbb{Z}^{m+e} &\longrightarrow N \,, \\ e_i &\mapsto \mathfrak{a}(e_i) = a_i \quad \text{for } i = 1, \ldots m+e \,. \end{aligned}$$

The S-extended stacky fan $\mathbf{\Sigma}^e = (N, \Sigma, \mathfrak{a})$ is given by the free group N, the fan Σ and the map $\mathfrak{a} : \mathbb{Z}^{m+e} \to N$.

Assumption 2.5. In the following we will choose $a_{m+1}, \ldots, a_{m+e}$ in such a way that $\mathfrak{a}$ is surjective *i.e.*, the lattice generated by $a_1, \ldots, a_{m+e}$ in N is saturated.

We denote by $\mathbb{L}$ the kernel of $\mathfrak{a}$. This gives us as above the two exact sequences

$$0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{Z}^{m+e} \xrightarrow{\mathfrak{a}} N \longrightarrow 0 \,, \tag{2.6}$$

$$0 \longrightarrow N^\star \longrightarrow (\mathbb{Z}^{m+e})^\star \longrightarrow \mathbb{L}^\star \longrightarrow 0 \,. \tag{2.7}$$

We denote by $D_1, \ldots, D_{m+e}$ the standard basis of $(\mathbb{Z}^{m+e})^\star$, i.e. $D_i(e_j) = \delta_{ij}$ and by $[D_1], \ldots, [D_{m+e}]$ the images of $D_1, \ldots, D_{m+e}$ in $\mathbb{L}^\star$.

Applying the exact functor $Hom(-, \mathbb{C}^*)$ to the sequence 2.7 gives a map

$$\psi_{\mathfrak{a}} : G^e \to (\mathbb{C}^*)^{m+e},$$

where $G^e := Hom_{\mathbb{Z}}(\mathbb{L}^\star, \mathbb{C}^*)$. We set

$$\mathcal{U}_{\mathcal{A}}^e := \mathcal{U}_{\mathcal{A}} \times (\mathbb{C}^*)^e\,,$$

then G^e acts on $\mathcal{U}_A^e$ via $\psi_{\mathfrak{a}}$ and the quotient stack

$$[\mathcal{U}_{\mathcal{A}}^e/G^e]$$

is isomorphic to the stack $\mathcal{X}(\mathbf{\Sigma})$ by [Jia08].

Remark 2.8. Given a stacky fan $\mathbf{\Sigma} = (N, \Sigma, \mathfrak{a})$ there exists a "canonical" choice of an extended stacky fan $\mathbf{\Sigma}^e$. Let $Gen(\sigma)$ be the subset of $Box(\sigma)$ of elements which are primitive in $\sigma \cap N$, i.e. which can not be generated by other elements in the semigroup $\sigma \cap N$ and set $Gen(\Sigma) := \bigcup_{\sigma\in\Sigma} Gen(\sigma)$.

Assumption 2.9. In the following we will always choose

$$S = Gen(\Sigma)$$

and we will set

$$n := m + e$$

and $\mathcal{G} := \{a_1, \ldots, a_m\} \cup Gen(\Sigma)$. Notice that $\{a_1, \ldots, a_m\} \cap Gen(\Sigma) = \emptyset$, hence the cardinality of $\mathcal{G}$ is n.

Lemma 2.10. *The map $\mathfrak{a}_{\mathbb{N}_0} : \mathbb{N}_0^n \to N$ given by $e_i \mapsto \mathfrak{a}_{\mathbb{N}_0}(e_i) = a_i$ is surjective.*

Proof. We assumed that the fan Σ is complete. Hence every $a \in N$ lies in some cone σ. Denote by $\sigma(a)$ the minimal cone containing a. The element a can be written as a non-negative rational linear combination $\sum_{a_i\in\sigma(a)} q_i a_i$. Write $q_i = n_i + \xi_i$ with $n_i \in \mathbb{N}_0$ and $\xi_i \in [0, 1)$. Hence $a = \sum_{a_i\in\sigma(a)}^m n_i a_i + \sum_{a_i\in\sigma(a)}^m \xi_i a_i$ where the last term lies in $\mathrm{Box}(\sigma)$. By the definition of S this term can be written as a non-negative integer linear combination of $Gen(\sigma)$. □

This choice of an extended stacky fan will allow us to give a different description of the orbifold cohomology ring which will be very useful for our purposes.
We introduce for every $a_i \in \mathcal{G}$ a formal variable $\mathfrak{D}_i$. For a top-dimensional cone $\sigma \in \Sigma(d)$ define in $\mathbb{C}[\mathfrak{D}_1, \ldots, \mathfrak{D}_n]$ the ideal

$$\mathcal{J}(\sigma) := \left\langle \prod_{l_i>0, a_i\in\sigma} \mathfrak{D}_i^{l_i} - \prod_{l_i<0, a_i\in\sigma} \mathfrak{D}_i^{-l_i} \mid \sum_{a_i\in\sigma} l_i a_i = 0, l_i \in \mathbb{N} \right\rangle .$$

We call relations $\underline{l} = (l_1, \ldots, l_n)$ of such type **cone relations**. The ideal

$$\mathcal{J}(\Sigma) := \sum_{\sigma\in\Sigma(d)} \mathcal{J}(\sigma)$$

is called the cone ideal of Σ.
Let $\mathcal{K}(\Sigma)$ be the ideal which is generated by

$$E_k := \sum_{i=1}^m a_{ki}\mathfrak{D}_i \qquad \text{for} \quad k = 1, \ldots, d\,,$$

where a_{ki} is the k-th coordinate of $a_i \in \mathbb{Z}^d$.

We call $I \subset \{1, \ldots, n\}$ a **generalized primitive collection** if the set $\{a_i \mid i \in I\}$ is not contained in a cone of Σ and if any proper subset of $\{a_i \mid i \in I\}$ is contained in some cone of Σ. We

denote by $\mathcal{GP}(\Sigma)$ the set of generalized primitive collections.

The orbifold cohomology of $\mathcal{X}$ can then be expressed in the following way.

Lemma 2.11 ([TW12] Lemma 2.4)**.** *Let $deg(\mathfrak{D}_i) = deg(a_i)$ for $i = 1, \dots, n$, then we have an isomorphism of graded $\mathbb{C}$-algebras*

$$H^*_{orb}(\mathcal{X}) \simeq \frac{\mathbb{C}[\mathfrak{D}_1, \dots, \mathfrak{D}_n]}{\mathcal{J}(\Sigma) + \mathcal{K}(\Sigma) + \langle \prod_{i \in I} \mathfrak{D}_i \mid I \in \mathcal{GP}(\Sigma) \rangle},$$

which sends $\mathfrak{D}_i$ to χ^{a_i} for $i = 1, \dots n$.

Remark 2.12. We would like to remind the reader that the ordinary cohomology ring of the underlying coarse moduli space X is given by

$$H^*(X, \mathbb{C}) \simeq \frac{\mathbb{C}[\mathfrak{D}_1, \dots, \mathfrak{D}_m]}{\mathcal{K}(\Sigma) + \langle \prod_{i \in I} \mathfrak{D}_i \mid I \in \mathcal{P}(\Sigma) \rangle}$$

(see [Ful93, p. 106]) where $\mathcal{P}(\Sigma)$ is the set of primitive collections. Here a collection $I \subset \{1, \dots, m\}$ is primitive if the set $\{a_i \mid i \in I\}$ does not lie a cone of Σ but any proper subset does (see [Ful93]).

2.3. **The extended Picard group.** As $PL(\Sigma)$ are piece-wise linear function on Σ, we can pre-compose them with $\mathfrak{a} : \mathbb{Z}^n \to N$ and we get the morphism

$$\begin{aligned} \Theta : PL(\Sigma) &\hookrightarrow (\mathbb{Z}^n)^\star, \\ \varphi &\mapsto \Theta(\varphi)(e_i) = \varphi(a_i). \end{aligned} \tag{2.13}$$

We have the following commutative diagram (cf. (2.4))

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N^\star & \longrightarrow & PL(\Sigma) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & 0 \\ & & \| & & \downarrow \Theta & & \downarrow & & \\ 0 & \longrightarrow & N^\star & \longrightarrow & (\mathbb{Z}^n)^\star & \longrightarrow & \mathbb{L}^\star & \longrightarrow & 0 \end{array}$$

We want to determine the image of this map. For this we consider the distinguished relations

$$a_{m+k} - \sum_{i \in I_\sigma} r_{ki} a_i = 0 \qquad \text{for } k \in \{1, \dots, e\}, \tag{2.14}$$

which give elements $l_1, \dots, l_e \in \mathbb{L} \otimes \mathbb{Q}$.

Lemma 2.15. *The image of Θ is a saturated subgroup of $(\mathbb{Z}^n)^\star$, i.e.*

$$(\Theta(PL(\Sigma)) \otimes_{\mathbb{Z}} \mathbb{Q}) \cap (\mathbb{Z}^n)^\star = \Theta(PL(\Sigma)).$$

Remark 2.16. This implies that the cokernel of Θ is free.

Proof of Lemma 2.15. Denote by $\mathbf{K}$ the kernel of the map

$$\begin{aligned} (\mathbb{Z}^n)^\star &\longrightarrow \mathbb{Q}^e \\ x &\mapsto (x(l_1), \dots, x(l_e)) \end{aligned}$$

In order to show the claim it is enough to show the following equality

$$\mathbf{K} = \Theta(PL(\Sigma)).$$

since a kernel is always saturated. It is clear that $\Theta(PL(\Sigma)) \subset \mathbf{K}$ since a function in $PL(\Sigma)$ is linear on each cone $\sigma \in \Sigma$. Now let $u \in \mathbf{K}$ and let $\sigma \in \Sigma(d)$ be a maximal cone. Choose a

$\mathbb{Z}$-basis $a_{j_1}, \ldots, a_{j_d}$ in the set $Gen(\sigma) \cup \{a_j \mid a_j \in \sigma\}$. The values $u(a_{j_1}), \ldots, u(a_{j_d})$ determine an element $m_\sigma \in N^\star$. Since $u \in \mathbf{K}$ we get $u(a_i) = m_\sigma(a_i)$ for alle $a_i \in \sigma$ with $j = 1, \ldots, n$. Repeating this for any cone we get an element in $PL(\Sigma)$ whose image under Θ is u. □

Notation 2.17. (1) We denote by $PL(\Sigma^e)$ the full sublattice of $(\mathbb{Z}^n)^\star$ generated by $\Theta(PL(\Sigma))$ and the elements $D_{m+1}, \ldots, D_{m+e} = D_n$
(2) We denote its image in $\mathbb{L}^\star$ by the extended Picard group $\mathrm{Pic}^e(X)$.

We get an exact sequence of $\mathbb{Z}$-free modules

$$0 \longrightarrow N^\star \longrightarrow PL(\Sigma^e) \longrightarrow \mathrm{Pic}^e(X) \longrightarrow 0\,. \tag{2.18}$$

The map Θ induces a map

$$\theta : \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}^e(X). \tag{2.19}$$

Notice that the images of $[\overline{D}_i] \in \mathrm{Pic}(X) \otimes \mathbb{Q}$ for $i = 1, \ldots, m$ in $\mathrm{Pic}^e(X) \otimes \mathbb{Q}$ are given by

$$\theta([\overline{D}_i]) = [D_i] + \sum_{k=1}^{e} r_{ki}[D_{m+k}]\,,$$

which follows from the formulas 2.13 and 2.14.
Using Lemma 2.15, one can summarise the previous objects in the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N^\star & \longrightarrow & PL(\Sigma) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \Theta} & & \downarrow{\scriptstyle \theta} & & \\
0 & \longrightarrow & N^\star & \longrightarrow & PL(\Sigma^e) & \longrightarrow & \mathrm{Pic}^e(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathbb{Z}^e & \longrightarrow & \mathbb{Z}^e & \longrightarrow & 0
\end{array} \tag{2.20}$$

Proposition 2.21. *We have the following isomorphism*

$$\mathrm{Pic}^e(X) \simeq \mathrm{Pic}(X) \oplus \mathbb{Z}^e$$

Proof. From the previous diagram (2.20), we have an exact sequence of free $\mathbb{Z}$-module

$$0 \longrightarrow \mathrm{Pic}(X) \xrightarrow{\theta} \mathrm{Pic}^e(X) \longrightarrow \mathbb{Z}^e \longrightarrow 0$$

As $Ext^1(\mathbb{Z}^e, \mathrm{Pic}(X)) = 0$, then the exact sequence splits. □

2.4. **The extended Kähler cone.** In this section we follow [Iri09]. Inside $PL(\Sigma)$ we consider the cone of convex functions $CPL(\Sigma)$. It has non-empty interior since X is projective. We denote its $\mathbb{Q}_{\geq 0}$-span in $H^2(X, \mathbb{Q})$ by $\overline{\mathcal{K}}$. Consider now the cone $CPL(\Sigma^e)$ generated by $\Theta(CPL(\Sigma))$ and $D_{m+1}, \ldots, D_{m+e}$. We denote the $\mathbb{Q}_{\geq 0}$-span of $\Theta(CPL(\Sigma))$ resp. $CPL(\Sigma^e)$ in $\mathbb{L}^\star \otimes \mathbb{Q}$ by $\mathcal{K}$ resp. $\mathcal{K}^e$ and call it the Kähler resp. extended Kähler cone.
We denote the image of anti-canonical divisor $-K_X$ of X in $\mathrm{Pic}(X) \otimes \mathbb{Q} \simeq H^2(X, \mathbb{Q})$ by $\overline{\rho}$. It is given by $\overline{\rho} = [\overline{D}_1] + \ldots [\overline{D}_m]$. The toric variety X is weak Fano if $\overline{\rho} \in \mathcal{K}$. Consider the following class in $\mathbb{L}^\star$

$$\rho := [D_1] + \ldots + [D_{m+e}]\,.$$

Later we will impose the following condition

$$\rho \in \mathcal{K}^e\,.$$

There is the following characterization of this condition

Lemma 2.22. [Iri09, Lemma 3.3] *We have $\rho \in \mathcal{K}^e$ iff $\overline{\rho} \in \mathcal{K}$ and $deg(a_i) \leq 1$ for $i = m+1, \ldots, m+e$.*

Proof. The element ρ can be expressed in the following way

$$\begin{aligned}\rho = [D_1] + \ldots + [D_{m+e}] &= \theta([\overline{D}_1]) + \ldots + \theta([\overline{D}_m]) + \sum_{k=1}^{e}(1 - \sum_{i=1}^{m} r_{ki})[D_{m+k}] \\ &= \theta([\overline{D}_1]) + \ldots + \theta([\overline{D}_m]) + \sum_{k=1}^{e}(1 - deg(a_{m+k}))[D_{m+k}] \\ \in \qquad & \mathcal{K} \qquad\qquad \oplus \qquad\qquad \bigoplus_{k=1}^{e} \mathbb{Q}[D_{m+k}]\,.\end{aligned}$$

The last term is in $\mathcal{K}^e$ iff $deg(a_{m+k}) \leq 1$ for $k = 1, \ldots, e$. □

Notice that the degree function deg gives rise to a piece-wise linear function φ which is given by

$$\varphi(a_i) = 1 \qquad \text{for} \quad i = 1, \ldots, m\,. \tag{2.23}$$

This piece-wise linear function corresponds to the anti-canonical divisor. If we assume that X is nef (i.e. $\overline{\rho} \in \overline{\mathcal{K}}$) then φ is a convex function.

Remark 2.24. It follows from Lemma 2.11 and Lemma 2.22 that for the choice $S = Gen(\Sigma)$ we have an isomorphism $H^{\leq 2}_{orb}(\mathcal{X}, \mathbb{Q}) \simeq \mathbb{L}^\star \otimes \mathbb{Q}$.

We now introduce the so-called extended Mori cone. Set

$$\mathcal{A}^e := \{I \cup \{m+1, \ldots, m+e\} \mid I \in \mathcal{A}\}$$

and

$$\begin{aligned}\mathbb{K} &:= \{d \in \mathbb{L} \otimes \mathbb{Q} \mid \{i \in \{1, \ldots, m+e\} \mid \langle D_i, d\rangle \in \mathbb{Z}\} \in \mathcal{A}^e\}\,, \\ \mathbb{K}^{\text{eff}} &:= \{d \in \mathbb{L} \otimes \mathbb{Q} \mid \{i \in \{1, \ldots, m+e\} \mid \langle D_i, d\rangle \in \mathbb{Z}_{\geq 0}\} \in \mathcal{A}^e\}\,.\end{aligned}$$

Notice that the lattice $\mathbb{L} \subset \mathbb{K}$ acts on $\mathbb{K}$.
Denote by $\lceil\cdot\rceil$, $\lfloor\cdot\rfloor$ and $\{\cdot\}$ the ceiling, floor and fractional part of a real number.

Lemma 2.25 ([Iri09, Section 3])**.** *The map*

$$\begin{aligned}\mathbb{K}/\mathbb{L} &\longrightarrow Box(\Sigma)\,, \\ d \mapsto v(d) &:= \sum_{i=1}^{m+e} \lceil\langle D_i, d\rangle\rceil a_i\end{aligned}$$

is bijective.

Proof. We first notice that $\sum_{i=1}^{m+e}\lceil\langle D_i, d\rangle\rceil a_i \in N$. From the definition of $\mathbb{K}$ and the exact sequence 2.6 we get

$$v(d) = \sum_{i=1}^{m+e}(\{-\langle D_i, d\rangle\} + \langle D_i, d\rangle)a_i = \sum_{i=1}^{m+e}\{-\langle D_i, d\rangle\}a_i = \sum_{a_i \in \sigma}\{-\langle D_i, d\rangle\}a_i \tag{2.26}$$

for some σ. This shows $v(d) \in Box(\sigma)$. From the formula 2.26 we easily see that the map $d \mapsto v(d)$ factors through $\mathbb{K} \to \mathbb{K}/\mathbb{L}$. Choose $v \in Box(\Sigma)$. We can express v either as $v = \sum_{a_i \in \sigma} n_i a_i$ with $n_i \in \mathbb{N}$ since $S = Gen(\Sigma)$ or as $v = \sum_{i \in I_\sigma} r_i a_i \in Box(\Sigma)$ with $r_i \in [0, 1)$ (cf. (2.2) for the definition of I_σ). The equation $\sum_{a_i \in \sigma} n_i a_i - \sum_{i \in I_\sigma} r_i a_i = 0$ gives rise to an element in $\mathbb{K} \subset \mathbb{L} \otimes \mathbb{Q}$,

this shows the surjectivity. In order to show injectivity let $d, d' \in \mathbb{K}$ with $v(d) = v(d')$. This means there exists a $\sigma, \sigma' \in \Sigma$ such that

$$\sum_{a_i \in \sigma} \{-\langle D_i, d\rangle\} a_i = v(d) = v(d') = \sum_{a_i \in \sigma'} \{-\langle D_i, d\rangle\} a_i$$

since both cones are simplicial we find a cone $\sigma'' \subset \sigma \cap \sigma'$ such that

$$v(d) = \sum_{a_i \in \sigma''} \{-\langle D_i, d\rangle\} a_i = \sum_{a_i \in \sigma''} \{-\langle D_i, d'\rangle\} a_i = v(d')$$

and therefore $\{-\langle D_i, d\rangle\} = \{-\langle D_i, d'\rangle\}$ for all $i = 1, \dots m + e$. Hence $\{\langle D_i, d - d'\rangle\} = 0$ and there fore $d - d' \in \mathbb{L}$. This shows the injectivity. □

Denote by $\mathrm{Pic}^e(X)^\star \subset \mathbb{L} \otimes \mathbb{Q}$ the lattice dual to the extended Picard group $\mathrm{Pic}^e(X) \subset \mathbb{L}^\star$ which gives the following short exact sequence of free $\mathbb{Z}$-modules

$$0 \longrightarrow \mathrm{Pic}^e(X)^\star \longrightarrow PL(\Sigma^e)^* \longrightarrow N \longrightarrow 0\,. \tag{2.27}$$

Lemma 2.28. *There are the following inclusions*

$$\mathbb{L} \subset \mathbb{K} \subset \mathrm{Pic}^e(X)^\star\,.$$

Proof. The first inclusion follows from the fact that $\{1, \dots, m + e\} \in \mathcal{A}^e$. In order to prove the second equation we notice that for each $v \in Box(\Sigma)$ we can write $v = \sum_{a_i \in \sigma} n_i a_i = \sum_{i \in I_\sigma} r_i a_i$. Denote by $d_v \in \mathbb{L} \otimes \mathbb{Q}$ the element in $\mathbb{K} \subset \mathbb{L} \otimes \mathbb{Q}$ which corresponds to the relation $\sum_{a_i \in \sigma} n_i a_i - \sum_{i \in I_\sigma} r_i a_i =$. The proof of Lemma 2.25 shows that $\mathbb{K}$ is the union of the sets $d_v + \mathbb{L}$ for $v \in Box(\Sigma)$. So in order to prove the second inclusion it is enough to show that $d_v \in \mathrm{Pic}^e(X)^\star$. For this we need to check that $L(d_v) \in \mathbb{Z}$ for every $L \in \mathrm{Pic}^e(X)$. Recall that $\mathrm{Pic}^e(X)$ is the image of $PL(\Sigma^e) \subset (\mathbb{Z}^n)^\star$ which in turn is generated by $\Theta(PL(\Sigma))$ and $D_{m+1}, \dots, D_{m+e}$. Take a lift $\varphi^e = \Theta(\varphi) + \sum_{i=m+1}^{m+e} t_i D_i = \sum_{i=1}^{m} \varphi(a_i) D_i + \sum_{i=m+1}^{m+e} t_i D_i$ of L in $PL(\Sigma^e)$. We have

$$L(d_v) = \varphi^e\Big(\sum_{a_i \in \sigma} n_i e_i - \sum_{i \in I_\sigma} r_i e_i\Big) = \sum_{i=1}^{m} n_i \varphi(a_i) + \sum_{i=m+1}^{m+e} t_i n_i - \varphi\Big(\sum_{i \in I_\sigma} r_i a_i\Big) \in \mathbb{Z}$$

since φ is integer-valued on N. □

3. Laurent polynomials and GKZ-systems

In this section we review some results from [RS15] and [RS17] concerning the relationship between (Fourier-Laplace-transformed) Gauß-Manin systems of families of Laurent polynomials and (Fourier-Laplace-transformed) GKZ-systems.

Notation 3.1. We will first review some notations from the theory of algebraic $\mathcal{D}$-modules. Let X be a smooth algebraic variety over $\mathbb{C}$ of dimension $d \geq 1$. We denote by $\mathcal{D}_X$ the sheaf of algebraic differential operators and by $D_X = \Gamma(X, \mathcal{D}_X)$ its sheaf of global sections. Recall when X is affine there is an equivalence of categories between $\mathcal{D}$-modules on X which are quasi-coherent as $\mathcal{O}_X$-modules and the corresponding D_X-module of global sections. If $\mathcal{M}$ is a $\mathcal{D}$-module on X we will write M for its module of global sections. We denote by $M(\mathcal{D}_X)$ the abelian category of algebraic $\mathcal{D}_X$-modules and the abelian subcategory of (regular) holonomic $\mathcal{D}_X$-modules by $M_h(\mathcal{D}_X)$ (resp. $M_{rh}(\mathcal{D}_X)$). The full triangulated subcategory of $D^b(\mathcal{D}_X)$ which consists of objects with (regular) holonomic cohomology is denoted by $D^b_h(\mathcal{D}_X)$ (resp. $D^b_{rh}(\mathcal{D}_X)$). Let $f : X \to Y$ be a map between smooth algebraic varieties and let $\mathcal{M} \in D^b(\mathcal{D}_X)$ and $\mathcal{N} \in D^b(\mathcal{D}_Y)$. The direct (resp. inverse) image functors are defined by $f_+\mathcal{M} := Rf_*(\mathcal{D}_{Y \leftarrow X} \overset{L}{\otimes} \mathcal{M})$ (resp.

$f^+\mathcal{N} := \mathcal{D}_{X\to Y} \overset{L}{\otimes} f^{-1}\mathcal{N})$.

Let $\mathcal{V}' := \mathbb{C}_t \times X$ be a trivial vector bundle on X of rank one and denote by $\mathcal{V} = \mathbb{C}_\tau \times X$ its dual. Denote by $can : \mathcal{V}' \times_X \mathcal{V} \to \mathbb{C}$ the canonical pairing between its fibers.

Definition 3.2. *Let $\mathcal{L} := \mathcal{O}_{\mathcal{V}'\times_X\mathcal{V}} e^{-can}$ be the free rank one module with differential given by the product rule. Denote by $p_1 : \mathcal{V}' \times_X \mathcal{V} \to \mathcal{V}'$, $p_2 : \mathcal{V}' \times_X \mathcal{V} \to \mathcal{V}$ the canonical projections. The Fourier-Laplace transformation is defined by*

$$FL_X(\mathcal{M}) := p_{2+}(p_1^+\mathcal{M} \overset{L}{\otimes} \mathcal{L}) \quad \textit{for} \quad M \in D_h^b(\mathcal{D}_{\mathcal{V}'}) .$$

Set $z = 1/\tau$ and denote by $j_\tau : \mathbb{C}_\tau^* \times X \hookrightarrow \mathbb{C}_\tau \times X$ and $j_z : \mathbb{C}_\tau^* \times X \hookrightarrow \hat{\mathcal{V}} := \mathbb{C}_z \times X = \mathbb{P}^1_\tau \setminus \{\tau = 0\} \times X$ the canonical embeddings. The partial, localized Fourier-Laplace transformation is defined by

$$FL_X^{loc}(\mathcal{M}) := j_{z+}j_\tau^+ FL_X(\mathcal{M}) \qquad \text{for} \quad \mathcal{M} \in D_h^b(\mathcal{D}_{\mathcal{V}'}) .$$

Set $\hat{V} := \mathbb{C}_z \times \Lambda$, where $\Lambda = \mathbb{C}^n$ with coordinates $\lambda_1, \dots, \lambda_n$. Let A be a $d \times n$ integer matrix with columns $(\underline{a}_1, \dots, \underline{a}_n)$ and entries a_{ki} for $k = 1, \dots, d$, $i = 1, \dots, n$. Let β be a vector $(\beta_1, \dots, \beta_d)$ in $\mathbb{C}^d$. We denote by $\mathbb{L} \subset \mathbb{Z}^n$ the $\mathbb{Z}$-submodule of relations among the columns A, i.e. $(l_1, \dots, l_n) \in \mathbb{L}$ iff $\sum_i l_i \underline{a}_i = 0$.

Assumption 3.3. Later, we will construct the matrix A from an extended stacky fan. Hence we can and will assume that each element of $\mathbb{Z}^d$ can be written as a non-negative integer linear combination of the vectors $(\underline{a}_i)_{i\in\{1,\dots,n\}}$ since this will be guaranted by the Assumption 2.9. We write $\mathbb{N}A = \mathbb{Z}^d$ for this condition.

Definition 3.4. *The Fourier-Laplace-transformed GKZ-system $\widehat{M}_A^{(\beta_0,\beta)}$ is the left $D_{\hat{V}}$-module $D_{\hat{V}}[z^{-1}]/I$, where I is the left ideal generated by the operators $\hat{\Box}_{\underline{l}}$, $\hat{E}_k - \beta_k z$ and $\hat{E} - \beta_0 z$, which are defined by*

$$\hat{\Box}_{\underline{l}} := \prod_{i:l_i<0} (z\cdot\partial_{\lambda_i})^{-l_i} - \prod_{i:l_i>0} (z\cdot\partial_{\lambda_i})^{l_i} \quad \textit{for } \underline{l} \in \mathbb{L} ,$$

$$\hat{E} := z^2\partial_z + \sum_{i=1}^m z\lambda_i\partial_{\lambda_i} ,$$

$$\hat{E}_k := \sum_{i=1}^m a_{ki} z\lambda_i\partial_{\lambda_i} .$$

We denote the corresponding $\mathcal{D}_{\hat{V}}$-module by $\widehat{\mathcal{M}}_A^{(\beta_0,\beta)}$.

Remark 3.5. 1.) Set $V = \mathbb{C}_{\lambda_0} \times \Lambda$. Denote by $\widetilde{A}$ the $(d+1)\times(n+1)$ matrix whose columns are given by $\widetilde{a}_0 := (1, \underline{0}), \widetilde{a}_1 := (1, \underline{a}_1), \dots, \widetilde{a}_n := (1, \underline{a}_n)$. Denote by $\mathbb{L}_{\widetilde{A}} \subset \mathbb{Z}^{n+1}$ the $\mathbb{Z}$-submodule of relations among the columns $\widetilde{A}$, i.e. $(l_0, \dots, l_n) \in \mathbb{L}_{\widetilde{A}}$ iff $\sum_i l_i \widetilde{\underline{a}}_i = 0$. The GKZ-system $M_{\widetilde{A}}^{(\beta_0,\beta)}$ is the left D_V-module $D_V/I_{\widetilde{A}}$, where $I_{\widetilde{A}}$ is the left ideal generated by the operators $\Box_{\underline{l}}$, $E_k - \beta_k$ and $E - \beta_0$, which are defined by

$$\Box_{\underline{l}} := \prod_{i:l_i<0} (\partial_{\lambda_i})^{-l_i} - \prod_{i:l_i>0} (\partial_{\lambda_i})^{l_i} \quad \text{for } \underline{l} \in \mathbb{L}_{\widetilde{A}} ,$$

$$E_0 := \sum_{i=0}^n \lambda_i\partial_{\lambda_i} ,$$

$$E_k := \sum_{i=1}^m a_{ki}\lambda_i\partial_{\lambda_i} .$$

An easy computation shows that the $D_{\widehat{V}}$-module $\widehat{M}_A^{(\beta_0,\beta)}$ is the localized Fourier-Laplace transform of the GKZ-system $M_{\widetilde{A}}^{(\beta_0-1,\beta)}$.

2.) Right Multiplication with z^ℓ gives an isomorphism

$$\widehat{M}_A^{(\beta_0,\beta)} \longrightarrow \widehat{M}_A^{(\beta_0-\ell,\beta)}\,.$$

Let $Y = (\mathbb{C}^*)^d$, we define a related family of Laurent polynomials:

$$\varphi_A = (\phi_A, pr_2) : Y \times \Lambda \longrightarrow V := \mathbb{C}_{\lambda_0} \times \Lambda\,,$$

$$(\underline{y}, \lambda_1, \ldots, \lambda_n) \mapsto (-\sum_{i=1}^{n} \lambda_i \underline{y}^{\underline{a}_i}, \lambda_1, \ldots, \lambda_n)\,.$$

The Gauß-Manin system is the zeroth cohomology of the direct image of the structure sheaf $\mathcal{O}_{Y\times\Lambda}$ in the category of $\mathcal{D}$-modules:

$$\mathcal{H}^0(\varphi_{A,+}\mathcal{O}_{Y\times\Lambda})\,.$$

We now consider the localized partial Fourier-Laplace transform of the Gauß-Manin system of φ_A:

$$\mathcal{G}^+ := FL_\Lambda^{loc}\mathcal{H}^0(\varphi_{A,+}\mathcal{O}_{Y\times\Lambda})\,.$$

Write $G^+ := H^0(\widehat{V}, \mathcal{G}^+)$ for its module of global sections. Then there is an isomorphism of $D_{\widehat{V}}$-modules (cf. e.g. [RS17, Lemma 3.4])

$$G^+ \cong H^0\left(\Omega^{\bullet+d}_{Y\times\Lambda/\Lambda}[z^{\pm}], d - z^{-1}\cdot d_y\phi_A\wedge\right),$$

where d is the differential in the relative de Rham complex $\Omega^\bullet_{Y\times\Lambda/\Lambda}$.
The following result relates the localized FL-transform of the Gauß-Manin system of φ_A with a FL-transformed GKZ-system with a special parameter.

Proposition 3.6. *Under Assumption 3.3, we have an isomorphism*

$$\mathcal{G}^+ \simeq \widehat{\mathcal{M}}_A^{(0,0)}\,.$$

Proof. This follows from [RS17, Proposition 3.3] and Assumption 3.3. The latter one guarantees that $\mathbb{R}_{\geq 0}A \cap \mathbb{Z}^d = \mathbb{Z}^d$ hence the former reference gives us for $(\beta_0, \beta) = (0, 0)$ the stated isomorphism. □

In the following we set $\widehat{\mathcal{M}}_A := \widehat{\mathcal{M}}_A^{(0,0)}$ resp. $\widehat{M}_A := \widehat{M}^{(0,0)}$.

For certain parameters $\lambda \in \Lambda$ the fibers of $\varphi_A(\cdot, \lambda)$ acquire singularities at infinity. Outside this set the singularities of the $\mathcal{D}$-module $\mathcal{G}^+$ are particularly simple.

Let Q be the convex hull of the set $\{\underline{a}_0 := 0, \underline{a}_1, \ldots, \underline{a}_n\}$

$$Q := conv(0, \underline{a}_1, \ldots, \underline{a}_n)\,.$$

The volume of Q is denoted by

$$\mu := vol(Q)\,,$$

where the volume of a hypercube $[0, 1]^d$ is normalized to $d!$.
Let Γ be a face of Q and denote by $Y_\Gamma^{crit,(\lambda_0,\underline{\lambda})}$ the set

$$\{(y_1, \ldots, y_d) \in Y \mid \sum_{\underline{a}_i\in\Gamma} \lambda_i \underline{y}^{\underline{a}_i} = 0\ ;\ y_k\partial_{y_k}(\sum_{\underline{a}_i\in\Gamma} \lambda_i \underline{y}^{\underline{a}_i}) = 0 \quad \text{for all} \quad k \in \{1, \ldots, d\}\}\,.$$

We say that the fiber $\varphi_A^{-1}(\lambda_0, \underline{\lambda})$ has a singularity at infinity if there exists a **proper** face Γ of the Newton polyhedron Q such that $Y_\Gamma^{crit,(\lambda_0,\underline{\lambda})} \neq \emptyset$. The set

$$\Delta_A^\infty := \{(\lambda_0, \underline{\lambda}) \in \mathbb{C}_{\lambda_0} \times \Lambda \mid \exists \Gamma \neq Q \text{ such that } Y_\Gamma^{crit,(\lambda_0,\underline{\lambda})} \neq \emptyset\}$$

is called the non-tame locus of φ_A. Notice that Δ_A^∞ is independent of λ_0 since 0 lies in the interior of Q (see Assumption 3.3.), hence no proper face of Q contains 0. Denote the projection of Δ_A^∞ to Λ by Λ^{bad}. Let $\Lambda^* := \Lambda \setminus \{\lambda_1 \cdot \ldots \cdot \lambda_n = 0\}$ and define

$$\Lambda^\circ := \Lambda^* \setminus \Lambda^{bad}.$$

The following was proven in [RS17] Lemma 3.13:

Lemma 3.7. *Consider $\widehat{\mathcal{M}}_A$ as a $\mathcal{D}_{\mathbb{P}^1 \times \overline{\Lambda}}$-module, where $\overline{\Lambda}$ is a smooth projective compactification of Λ. Then $\widehat{\mathcal{M}}_A$ is regular outside $(\{z = 0\} \times \Lambda) \cup (\mathbb{P}^1_z \times (\overline{\Lambda} \setminus \Lambda^\circ))$ and smooth on $\mathbb{C}^*_z \times \Lambda^\circ$.*

Next we want to consider natural lattices in $\widehat{\mathcal{M}}_A$. For this we need the notion of $\mathcal{R}$-modules.

Definition 3.8. *Let X be a smooth variety . Then the sheaf of (non-commutative) rings $\mathcal{R}_{\mathbb{C}_z \times X}$ is by definition the $\mathcal{O}_{\mathbb{C}_z \times X}$-subalgebra of $\mathcal{D}_{\mathbb{C}_z \times X}$ locally generated by $z^2\partial_z, z\partial_{x_1}, \ldots, z\partial_{x_n}$, where $(x_1, \ldots, x_n)$ are local coordinates on X.*

Definition 3.9.

(1) *Let $\mathcal{I}$ be the left ideal in $\mathcal{R}_{\mathbb{C}_z \times \Lambda^*}$ generated by $(\hat{\Box}_{\underline{l}})$ and $(\hat{E}_k)_{k=0,\ldots,d}$ (cf . Definition 3.4). Write ${}_0\widehat{\mathcal{M}}_A$ resp. ${}_0^*\widehat{\mathcal{M}}_A$ for the cyclic $\mathcal{R}$-module $\mathcal{R}_{\mathbb{C}_z \times \Lambda}/\mathcal{I}$ resp. $\mathcal{R}_{\mathbb{C}_z \times \Lambda^*}/\mathcal{I}$.*

(2) *Consider the open inclusions $\Lambda^\circ \subset \Lambda^* \subset \Lambda$ and define the $\mathcal{D}_{\mathbb{C}_z \times \Lambda^\circ}$-module*

$$^\circ\widehat{\mathcal{M}}_A := \left(\widehat{\mathcal{M}}_A\right)_{|\mathbb{C}_z \times \Lambda^\circ}$$

and the $\mathcal{R}_{\mathbb{C}_z \times \Lambda^\circ}$-module

$${}_0^\circ\widehat{\mathcal{M}}_A := \left({}_0^*\widehat{\mathcal{M}}_A\right)_{|\mathbb{C}_z \times \Lambda^\circ}.$$

Remark 3.10. It follows from Lemma 3.7 that $^\circ\widehat{\mathcal{M}}_A$ is a meromorphic connection with poles exactly along $\{0\} \times \Lambda^\circ$.

We now list some properties of the $\mathcal{R}$-module ${}_0^\circ\widehat{\mathcal{M}}_A$:

Proposition 3.11. [RS17, Proposition 3.18, Corollary 3.19]

(1) *The sheaf ${}_0^\circ\widehat{\mathcal{M}}_A$ is a locally free $\mathcal{O}_{\mathbb{C}_z \times \Lambda^\circ}$-module of rank μ.*

(2) *The natural map ${}_0^\circ\widehat{\mathcal{M}}_A \to {}^\circ\widehat{\mathcal{M}}_A$, which is induced from the inclusion $\mathcal{R}_{\mathbb{C}_z \times \Lambda^\circ} \to \mathcal{D}_{\mathbb{C}_z \times \Lambda^\circ}$, is injective.*

The so-called Fourier-Laplace transformed Brieskorn lattice of the FL-transformed Gauß-Manin system $\mathcal{G}^+$ is given by the following $R_{\mathbb{C}_z \times \Lambda^\circ}$-module:

$$H^0\left(\Omega^{\bullet+d}_{Y \times \Lambda^\circ/\Lambda^\circ}[z], zd - d_y\phi_A\wedge\right).$$

By Assumption 3.3, the semigroup $\mathbb{N}A$ is saturated then we have the following identification:

Proposition 3.12. [RS17, Proposition 3.20] *There exists the following $R_{\mathbb{C}_z \times \Lambda^\circ}$-linear isomorphism*

$$H^0\left(\Omega^{\bullet+d}_{Y \times \Lambda^\circ/\Lambda^\circ}[z], zd - d_y\phi_A\wedge\right) \simeq {}_0^\circ\widehat{M}_A.$$

Before constructing the pairing on ${}_0^{\circ}\widehat{\mathcal{M}}_A$, we need to introduce some notations. The basic ingredient for the proof is the so-called Euler-Koszul complex. (cf. [MMW05]). We put $R = \mathbb{C}[\partial_{\lambda_0}, \dots, \partial_{\lambda_n}]$ and $S = R/J_{\widetilde{A}}$ where $J_{\widetilde{A}}$ is the toric ideal generated by $(\Box_{\underline{l}})_{\underline{l} \in \mathbb{L}_{\widetilde{A}}}$. Both rings are $\mathbb{Z}^{d+1}$-graded, where $\deg(\partial_{\lambda_i}) := -\underline{\widetilde{a}}_i \in \mathbb{Z}^{d+1}$.
We also have a natural $\mathbb{Z}^{d+1}$-grading on D defined by $\deg(\lambda_i) = \underline{\widetilde{a}}_i$ and $\deg(\partial_{\lambda_i}) = -\underline{\widetilde{a}}_i$.
Let P be any $\mathbb{Z}^{d+1}$-graded D-module, and $\widetilde{\alpha} \in \mathbb{C}^{d+1}$ arbitrary, then by putting

$$(E_k - \widetilde{\alpha}_k) \circ y := (E_k - \widetilde{\alpha}_k - \deg_k(y))(y)$$

for $k = 0, \dots, d$ and for any homogeneous element $y \in P$ and by extending $\mathbb{C}$-linearly, we obtain a D-linear endomorphism of P. We also have that the commutator $[(E_i - \widetilde{\alpha}_i)\circ, (E_j - \widetilde{\alpha}_j)\circ]$ vanishes for any $i, j \in \{0, \dots, d\}$. Therefore one can define the Euler-Koszul complex $\mathcal{K}_\bullet(E - \widetilde{\alpha}, P)$, a complex of $\mathbb{Z}^{d+1}$-graded left D-modules, to be the Koszul complex of the endomorphisms $(E_0 - \widetilde{\alpha}_0)\circ, \dots, (E_d - \widetilde{\alpha}_d)\circ$ on P. Notice that here E is an abbreviation for the vector $(E_0, E_1, \dots, E_n)$ and should not be confused with the single vector field used in the definition of the modules $\widehat{\mathcal{M}}_A^{(\beta_0,\beta)}$. The definition of the Euler-Koszul complex applies in particular to the case $N := D \otimes_R T$, where T is a so-called toric R-module (see [MMW05, definition 4.5]), in which case we also write $\mathcal{K}_\bullet(E - \widetilde{\alpha}, T)$ for the Euler-Koszul complex instead of $\mathcal{K}_\bullet(E - \widetilde{\alpha}, N)$. Similarly one defines the Euler-Koszul cocomplex, denoted by $\mathcal{K}^\bullet(E - \widetilde{\alpha}, T)$, where $\mathcal{K}^i(E - \widetilde{\alpha}, T) = \mathcal{K}_{n+1-i}(E - \widetilde{\alpha}, T)$ and the signs of the differentials are changed accordingly. In particular, we have $H^i(\mathcal{K}^\bullet(E - \widetilde{\alpha}, T)) = H_{n+1-i}(\mathcal{K}_\bullet(E - \widetilde{\alpha}, T))$.
We will use the construction of the Euler-Koszul complex resp. cocomplex for example in the case of the toric R-module S. In this case $H_0(\mathcal{K}_\bullet(E - \widetilde{\alpha}, S)) \simeq M_{\widetilde{A}}^{\widetilde{\alpha}}$. Notice however that $\mathcal{K}_\bullet(E - \widetilde{\alpha}, S)$ does not provide a resolution of $M_{\widetilde{A}}^{\widetilde{\alpha}}$ in general because the complex has higher homologies if the R-module S is not Cohen-Macaulay.

The components of the Euler-Koszul complex $\mathcal{K}_\bullet(E - \widetilde{\alpha}, T)$ are direct sums of $D \otimes_R T$. Denote by $U_\bullet$ a resolution of T in the category of $\mathbb{Z}^{d+1}$-graded R-modules. We get a double complex $\mathcal{K}_\bullet(E - \widetilde{\alpha}, U_\bullet)$ of free D-modules, so that the 0-th homology of its total complex is isomorphic to $H_0(\mathcal{K}_\bullet(E - \widetilde{\alpha}, T))$. Applying the functor $Hom_D(-, D)$ to the double complex and taking first cohomology in the $U_\bullet$ direction gives row-wise Euler-Koszul co-complexes $\mathcal{K}^\bullet(E + \widetilde{\alpha} + \varepsilon_{\widetilde{A}}, Ext_R^\bullet(T, R)) \simeq \mathcal{K}^\bullet(E + \widetilde{\alpha}, Ext_R^\bullet(T, \omega_R))(-\varepsilon_{\widetilde{A}})$, where $\varepsilon_{\widetilde{A}} := \sum_{k=0}^n \underline{\widetilde{a}}_i \in \mathbb{Z}^{d+1}$. Taking cohomology in the Euler-Koszul direction afterwards one gets the second page of a spectral sequence. More precisely we get the following Lemma.

Lemma 3.13 ([MMW05, theorem 6.3])**.** *There is a spectral sequence*
(3.14)

$$E_2^{p,q} = H^q(\mathcal{K}^\bullet(E + \widetilde{\alpha}, Ext_R^p(T, \omega_R))(-\underline{\varepsilon_{\widetilde{a}}}) \Longrightarrow H^{p+q-(n+1)}\mathbb{D}\left(H_{p+q-(n+1)}(\mathcal{K}_\bullet(E - \widetilde{\alpha}, T))\right)^- .$$

Here $(-)^-$ is the auto-equivalence of D-modules induced by the involution $\lambda_i \mapsto -\lambda_i$ and $\partial_{\lambda_i} \mapsto -\partial_{\lambda_i}$. Notice that it is shown in [MMW05, lemma 6.1] *that $Ext_R^p(T, \omega_R)$ is toric. Notice also that the dualizing module ω_R is nothing but the ring R, placed in $\mathbb{Z}^{d+1}$-degree $\underline{\varepsilon_{\widetilde{a}}}$ (see, e.g.,* [MS05, definition 12.9 and corollary 13.43] *or* [BH93, corollary 6.3.6] *for this).*

We now introduce a localized version of this Euler-Koszul complex. For this notice that $D[\partial_{\lambda_0}^{-1}]$ is also $\mathbb{Z}^{d+1}$-graded with $\deg(\partial_{\lambda_0}^{-1}) = \underline{\widetilde{a}}_0 \in \mathbb{Z}^{d+1}$. Set $R^{loc} := R[\partial_{\lambda_0}^{-1}]$ and $S^{loc} := R^{loc}/J_{\widetilde{A}}$ and more general for all toric R-modules T we set $T^{loc} := R^{loc} \otimes_R T$. We have $D[\partial_{\lambda_0}^{-1}] \otimes_D D \otimes_R T \simeq D[\partial_{\lambda_0}^{-1}] \otimes_{R^{loc}} T^{loc}$. Because $D[\partial_{\lambda_0}^{-1}]$ is D-flat the localization of the Euler-Koszul (co-)complex gives

$$D[\partial_{\lambda_0}^{-1}] \otimes_D \mathcal{K}_\bullet(E - \widetilde{\alpha}, T) \simeq \mathcal{K}_\bullet^{loc}(E - \widetilde{\alpha}, T^{loc})$$

where $\mathcal{K}^{loc}_\bullet(E-\widetilde{\alpha},\bullet)$ is the corresponding localized Euler-Koszul complex $D[\partial_{\lambda_0}^{-1}]\otimes_D\mathcal{K}_\bullet(E-\widetilde{\alpha},\bullet)$. We get the following localized version of the spectral sequence above

Lemma 3.15. *There is a spectral sequence*
(3.16)
$$E_2^{p,q}=H^q(\mathcal{K}^\bullet_{loc}(E+\widetilde{\alpha},Ext^p_{R^{loc}}(T^{loc},\omega_{R^{loc}}))(-\underline{\varepsilon}_{\underline{\widetilde{a}}})\Longrightarrow H^{p+q-(n+1)}\mathbb{D}^{loc}\left(H_{p+q-(n+1)}(\mathcal{K}^{loc}_\bullet(E-\widetilde{\alpha},T^{loc}))\right)^-.$$

where $\mathbb{D}^{loc}(-)$ is the derived functor $RHom_{D[\partial_{\lambda_0}^{-1}]}(-,D[\partial_{\lambda_0}^{-1}])[n+1]$ together with a right-left transformation.

Proof. Applying $D[\partial_{\lambda_0}^{-1}]\otimes_D$ to the double complex $\mathcal{K}_\bullet(E-\widetilde{\alpha},U_\bullet)$ gives $\mathcal{K}^{loc}_\bullet(E-\widetilde{\alpha},U^{loc}_\bullet)$ where $U^{loc}_\bullet$ is a resolution of T^{loc} in the category of $\mathbb{Z}^{d+1}$-graded R^{loc}-modules. Applying $Hom_{D^{loc}}(-,D^{loc})$ to this double complex and taking cohomology in the $U^{loc}_\bullet$ direction gives row-wise localized Euler-Koszul co-complexes $\mathcal{K}^\bullet_{loc}(E+\widetilde{\alpha}+\varepsilon_{\widetilde{A}},Ext^\bullet_{R^{loc}}(T^{loc},R^{loc}))\simeq\mathcal{K}^\bullet(E+\widetilde{\alpha},Ext^\bullet_{R^{loc}}(T^{loc},\omega_{R^{loc}}))(-\varepsilon_{\widetilde{A}})$. Taking again cohomology in the Euler-Koszul direction afterwards one gets the second page of a spectral sequence stated above. Its abutement is just the localization of the abutment stated in Lemma 3.13. □

Let (M,F) be a filtered D_V-module. Denote by $\widehat{\mathrm{loc}}:M\to M[\partial_{\lambda_0}^{-1}]$ the localization map. We define the following filtration on $M[\partial_{\lambda_0^{-1}}]$:

$$F^{loc}_kM[\partial_{\lambda_0^{-1}}]:=\sum_{j\geq0}\partial_{\lambda_0}^{-j}\widehat{\mathrm{loc}}(F_{k+j}M).$$

Proposition 3.17. (1) *Let $\widetilde{\beta}=(\beta_0,\beta),\widetilde{\gamma}=(\gamma_0,\gamma)\in\mathbb{Z}^{d+1}$. Right multiplication by $z^{-\gamma_0}(z\partial)^\gamma$ induces an isomorphism*

$$\cdot z^{-\gamma_0}(z\partial)^\gamma:\widehat{M}^{\widetilde{\beta}}_A\xrightarrow{\simeq}\widehat{M}^{\widetilde{\beta}+\widetilde{\gamma}}_A$$

where $(z\partial)^\gamma:=(z\partial_{\lambda_1})^{\kappa_1}\dots(z\partial_{\lambda_n})^{\kappa_n}$ for $\kappa\in\mathbb{N}^n$ satisfying $\gamma=A\kappa$ (notice that the morphism is independent of the choice of κ).

(2) *There exists $\tilde{c}=(c_0,c)\in\mathbb{Z}^{d+1}$ such that for any $\widetilde{\beta}\in\mathbb{Z}^{d+1}$ we have an isomorphism of holonomic left $\mathcal{D}_{\widehat{V}}$-modules*

$$\widehat{\mathcal{M}}_A^{-\widetilde{\beta}-\tilde{c}+(2,\underline{0})}\simeq\iota^*\mathbb{D}\widehat{\mathcal{M}}^{\widetilde{\beta}}_A$$

where ι is the automorphism sending (z,λ) to $(-z,\lambda)$.

(3) *Write $F^{\mathbb{D}^{loc}}_\bullet\mathbb{D}^{loc}M^{\widetilde{\beta}}_{\widetilde{A}}[\partial_{\lambda_0}^{-1}]$ for the dual filtration of $F^{loc}_\bullet M^{\widetilde{\beta}}_{\widetilde{A}}[\partial_{\lambda_0}^{-1}]$, i.e. $\mathbb{D}^{loc}(M^{\widetilde{\beta}}_{\widetilde{A}}[\partial_{\lambda_0}^{-1}],F^{loc}_\bullet)=(\mathbb{D}^{loc}M^{\widetilde{\beta}}_{\widetilde{A}}[\partial_{\lambda_0}^{-1}],F^{\mathbb{D}^{loc}}_\bullet)$ (see e.g.* [Sai94, page 55]

$$F^{loc}_kM^{-\widetilde{\beta}-\tilde{c}+(2,\underline{0})}_{\widetilde{A}}=F^{\mathbb{D}^{loc}}_{k-d+n+c_0}\mathbb{D}^{loc}M^{\widetilde{\beta}}_{\widetilde{A}}\qquad\forall\widetilde{\beta}\in\mathbb{Z}^{d+1}.$$

(4) *For any $\widetilde{\beta}\in\mathbb{Z}^{d+1}$, $F_\bullet M^{\widetilde{\beta}}_{\widetilde{A}}$ induces a filtration $G^\beta_\bullet$ by $\mathcal{O}_{\widehat{V}}$-modules on the $\mathcal{D}_{\widehat{V}}$-module $\widehat{\mathcal{M}}^{\widetilde{\beta}}_A$ and we have an isomorphism of $\mathcal{O}_{\mathbb{C}_z\times\Lambda}$-modules*

$$G_0\widehat{\mathcal{M}}_A\simeq{}_0\widehat{\mathcal{M}}_A.$$

Moreover, for any $k\in\mathbb{Z}$ $\left(G_k\widehat{\mathcal{M}}_A\right)_{|\mathbb{C}\times\Lambda^\circ}$ is $\mathcal{O}_{\mathbb{C}_z\times\Lambda^\circ}$-locally free.

We obtain from the dual filtration $F^{\mathbb{D}^{loc}}_\bullet$ on $\mathbb{D}^{loc}M^{\widetilde{\beta}}_{\widetilde{A}}$ a filtration $G^{\mathbb{D}}_\bullet$ by $\mathcal{O}_{\mathbb{C}_z\times\Lambda}$-modules on $\mathbb{D}\widehat{\mathcal{M}}^{\widetilde{\beta}}_A$. Moreover, for any $k\in\mathbb{Z}$ $\left(G^{\mathbb{D}}_k\,\mathbb{D}\widehat{\mathcal{M}}_A\right)_{|\mathbb{C}\times\Lambda^\circ}$ is $\mathcal{O}_{\mathbb{C}_z\times\Lambda^\circ}$-locally free.

(5) *Consider the isomorphism*

$$\phi : \iota^*\mathbb{D}\widehat{\mathcal{M}}_A \simeq \widehat{\mathcal{M}}_A^{-\widetilde{c}+(2,\underline{0})} \longrightarrow \widehat{\mathcal{M}}_A$$

then $\phi(\iota^* G_\bullet^{\mathbb{D}}\, \mathbb{D}\widehat{\mathcal{M}}_A) = G_{\bullet+d-n-2}\widehat{\mathcal{M}}$, *where* ϕ *is the concatenation of the isomorphism in* (2) *for* $\widetilde{\beta} = 0$ *and the isomorphism in* (1) *for* $\widetilde{\beta} = -\widetilde{c} + (2,0)$ *and* $\widetilde{\gamma} = -\widetilde{\beta}$.

Proof. (1) One can conclude from [RS17, Theorem 2.11] that there exists a $\delta_{\widetilde{A}} \in \mathbb{N}\widetilde{A}$ such that $\mathcal{M}_{\widetilde{A}}^{\widetilde{\alpha}} \simeq \mathcal{M}_{\widetilde{A}}^{\widetilde{\alpha}'}$ for $\widetilde{\alpha}, \widetilde{\alpha}' \in \delta_{\widetilde{A}} + (\mathbb{R}_{\geq 0}\widetilde{A} \cap \mathbb{Z}^{d+1})$. Furthermore, it follows from [Rei14, Proposition 1.15] that for $\widetilde{\rho} \in \mathbb{R}_{\geq 0}\widetilde{A} \cap \mathbb{Z}^{d+1}$ the isomorphism $\mathcal{M}_{\widetilde{A}}^{\widetilde{\alpha}} \to \mathcal{M}_{\widetilde{A}}^{\widetilde{\alpha}+\widetilde{\rho}}$ is given by right multiplication with $\partial^{\widetilde{\rho}}$.

Given $\widetilde{\beta} = (\beta_0, \beta), \widetilde{\gamma} = (\gamma_0, \gamma) \in \mathbb{Z}^{d+1}$, the shape of the matrix $\widetilde{A}$ and Assumption 3.3 guarantees that we can find $c, d \in \mathbb{Z}$ such that $(\beta_0 + c, \beta), (\gamma_0 + d, \gamma) \in \delta_{\widetilde{A}} + (\mathbb{R}_{\geq 0}\widetilde{A} \cap \mathbb{Z}^{d+1})$. Hence we get an isomorphism

$$\cdot\partial^{(\gamma_0+d,\gamma)} : \mathcal{M}_{\widetilde{A}}^{(\beta_0+c,\beta)} \to \mathcal{M}_{\widetilde{A}}^{(\beta_0+c+\gamma_0+d,\beta+\gamma)},$$

where $\partial^{(\gamma_0+d,\gamma)} := \partial_0^{\kappa_0} \dots \partial_n^{\kappa_n}$ for $(\gamma_0 + d, \gamma)^{tr} = A\kappa$ (again Assumption 2.9 and the shape of A guarantees that $(\kappa_0, \dots, \kappa_n) \in \mathbb{Z} \times \mathbb{N}_0^n$. By enlarging d we can even achieve $\kappa_0 \in \mathbb{N}_0$).

Applying a localized Fourier-transformation multpilication with $\partial^{(\gamma_0+d,\gamma)}$ becomes

$$\partial^{(\gamma_0+d,\gamma)} \to z^{-\kappa_0}\partial_1^{\kappa_{\lambda_1}} \dots \partial_n^{\kappa_{\lambda_n}} = z^{-\kappa_0-|\kappa|}(z\partial_{\lambda_1})^{\kappa_1} \dots (z\partial_{\lambda_n})^{\kappa_n} = z^{-\gamma_0-d}(z\partial)^{\gamma},$$

where $|\kappa| := \sum_{i=1}^n \kappa_i$. Since left multiplication with z^k gives an isomorphism $\widehat{M}_A^{\beta_0,\beta} \to \widehat{M}_A^{\beta_0-k,\beta}$ we get a commutative diagram where all maps are isomorphisms:

$$\begin{array}{ccc} \widehat{M}_A^{(\beta_0+c,\beta)} & \xrightarrow{z^{-\gamma_0-d}(z\partial)^{\gamma}} & \widehat{M}_A^{\beta_0+c+\gamma_0+d,\beta+\gamma} \\ \downarrow{\scriptstyle z^c} & & \downarrow{\scriptstyle z^{c+d}} \\ \widehat{M}_A^{\beta_0,\beta} & \xrightarrow{z^{-\gamma_0}(z\partial)^{\gamma}} & \widehat{M}_A^{\beta_0+\gamma_0,\beta+\gamma} \end{array}.$$

(2) We will use Lemma 3.15 for $T = S$. Since the columns of A generate $\mathbb{Z}^d$ (cf. Assumption 3.3) the ring S^{loc} is isomorphic to $\mathbb{C}[\mathbb{Z}^{d+1}]$, in particular it is Gorenstein, hence Cohen-Macaulay. Therefore $Ext^p_{R^{loc}}(S^{loc}, \omega_{R^{loc}})$ vanishes for $p \neq n-d$ and $Ext^{n-d}_{R^{loc}}(S^{loc}, \omega_{R^{loc}})$ is isomorphic to S^{loc} in the category of noetherian R^{loc}-modules. Since we can compute $Ext^{n-d}_{R^{loc}}(S^{loc}, \omega_{R^{loc}})$ with a $\mathbb{Z}^{d+1}$-graded filtration the Ext-group is graded as well and is therefore isomorphic to $S^{loc}(\tilde{c})$ for some $\tilde{c} \in \mathbb{Z}^{d+1}$. Hence the spectral sequence degenerates at the second page, so that

$$E_2^{n-d,q} = H^{n-d+q}\mathbb{D}^{loc}\left(H_{(n-d)+q-(n+1)}(\mathcal{K}_\bullet^{loc}(E - \widetilde{\alpha}, S^{loc}))\right)^{-}.$$

On the other hand, since $Ext^{n-d}_{R^{loc}}(S^{loc}, \omega_{R^{loc}})) \simeq S^{loc}[\tilde{c}]$ we get

$$E_2^{n-d,q} = H^q(\mathcal{K}_{loc}^\bullet(E + \widetilde{\alpha}, S^{loc}(\tilde{c}))) \simeq H_{d+1-q}(\mathcal{K}_\bullet^{loc}(E + \widetilde{\alpha} + \tilde{c}, S^{loc}))(\tilde{c}).$$

Since S^{loc} is Cohen-Macaulay the Euler-Koszul complex $\mathcal{K}_\bullet^{loc}(E - \widetilde{\alpha}, S^{loc})$ can only have homology in degree zero (cf. [MMW05, remark 6.4], hence $E_2^{n-d,q} = 0$ unless $q = d+1$. Summarizing we get an isomorphism of $\mathbb{Z}^{d+1}$-graded D^{loc}-modules

$$H^0(\mathbb{D}^{loc}(H_0(\mathcal{K}_\bullet^{loc}(E - \widetilde{\alpha}, S^{loc}))))^{-} \simeq H_0(\mathcal{K}_\bullet^{loc}(E + \widetilde{\alpha} + \tilde{c}, S^{loc}))(\tilde{c}).$$

Since $\mathrm{FL}^{loc} M \simeq \mathrm{FL}^{loc} D_V[\partial_{\lambda_0^{-1}}] \otimes_{D_V} M$ for all holonomic D_V-modules M we get

$$\iota^* \mathbb{D} \widehat{\mathcal{M}}_A^{\widetilde{\alpha}+(1,0)} \simeq \widehat{\mathcal{M}}_A^{-\widetilde{\alpha}-\tilde{c}+(1,0)}.$$

Notice that the extra term $(1,0)$ appears beacuse of the localized Fourier-Laplace transformation where $z\partial_z - 1$ acts as $\lambda_0 \partial_{\lambda_0}$. Setting $\widetilde{\beta} = \widetilde{\alpha} + (1,0)$, we obtain $-\widetilde{\alpha} - \tilde{c} + (1,0) = -\widetilde{\beta} - \tilde{c} + (2,0)$. This gives the statement we claimed.

(3) Let $L_\bullet$ be the total complex of $L_{\bullet,\bullet} := \mathcal{K}_\bullet(E - \widetilde{\beta}, U_\bullet)$ where $U_\bullet$ is a $\mathbb{Z}^{d+1}$-graded resolution of S. In particular this resolution is $\mathbb{Z}$-graded for the grading of $R = \mathbb{C}[\partial_{\lambda_0}, \dots, \partial_{\lambda_n}]$ with $\deg(\partial_{\lambda_i}) = 1$. On the other hand, the differentials of the Euler-Koszul complex are constructed from linear differential operators. We put on each term of the double complex (which is D-free) a filtration $F_\bullet$, which is on each factor of such a module the order filtration on D, shifted appropriately. The corresponding total complex is also filtered.

We now consider the localized Euler-Koszul complex $L^{loc}_{\bullet,\bullet} := \mathcal{K}^{loc}_\bullet(E - \widetilde{\beta}, U_\bullet)$. On each term of this double complex we construct a filtration

$$F_k^{loc} L^{loc}_{\bullet,\bullet} := \sum_{j \geq 0} \partial_{\lambda_0^{-1}}^{-j} \widehat{\mathrm{loc}}(F_{k+j} L_{\bullet,\bullet})$$

where $\widehat{\mathrm{loc}} : L_{\bullet,\bullet} \to L^{loc}_{\bullet,\bullet}$ is the canonical localization morphism. Of course, this also gives a filtration on each term of the total complex $L^{loc}_\bullet$ making this complex a filtered one.

We already know that the localized total complex $L^{loc}_\bullet$ is a resolution of $M_{\widetilde{A}}^{\widetilde{\beta}}[\partial_{\lambda_0}^{-1}]$. Now we want to prove that this resolution is also strictly filtered where $M_{\widetilde{A}}^{\widetilde{\beta}}[\partial_{\lambda_0}^{-1}]$ carries the localized order filtration F_k^{loc}. It suffices to show that $\mathrm{gr}_\cdot^{F^{loc}}(L^{loc}_\bullet) \twoheadrightarrow \mathrm{gr}_\cdot^{F^{loc}} M_{\widetilde{A}}^{\widetilde{\beta}}[\partial_{\lambda_0}^{-1}]$. For this we consider the graded double complex $\mathrm{gr}_\bullet^{F^{loc}}(\mathcal{K}_\bullet(E - \widetilde{\beta}, U^{loc}_\bullet)$. Clearly, we have $\mathrm{Tot}\, \mathrm{gr}_\cdot^{F^{loc}}(\mathcal{K}^{loc}_\bullet(E - \widetilde{\beta}, U^{loc}_\bullet)) = \mathrm{gr}_\cdot^{F^{loc}} L^{loc}_\bullet$.

The cohomology of $\mathrm{Tot}\, \mathrm{gr}_\cdot^{F^{loc}}(\mathcal{K}^{loc}_\bullet(E - \widetilde{\beta}, U^{loc}_\bullet))$ can be obtained from the spectral sequence of the double complex $\mathrm{gr}_\cdot^{F^{loc}}(\mathcal{K}^{loc}_\bullet(E - \widetilde{\beta}, U^{loc}_\bullet))$, where we first take cohomology in the $U^{loc}_\bullet$-direction. However, the differentials of the R^{loc}-free resolution $U^{loc}_\bullet \twoheadrightarrow S^{loc}$ are homogeneous, hence taking the graded object has simply no effect and so we conclude that this spectral sequence degenerates at the E_1-term. Hence we are left to show that $\mathrm{gr}_\cdot^{F^{loc}} \mathcal{K}^{loc}_\bullet(E - \widetilde{\beta}, S^{loc}) \twoheadrightarrow \mathrm{gr}_\cdot^{F^{loc}}(M_{\widetilde{A}}^{\widetilde{\beta}}[\partial_{\lambda_0^{-1}}])$ is a quasi-isomorphism. It is shown in [BZGM15, Theorem 1.2] that the elements $\sum_{i=0}^n \widetilde{a}_{ki} \lambda_i \xi_i$ which give the Koszul complex $\mathrm{gr}_\cdot^F \mathcal{K}_\bullet(E - \widetilde{\beta}, S)$ are a system of parameters for $\mathrm{gr}_\cdot^F D \otimes_R S \simeq k[\lambda_0, \dots, \lambda_n][\mathbb{N}\widetilde{A}]$, hence they are also a system of parameters for $\mathrm{gr}_\cdot^{F^{loc}} D[\partial_{\lambda_0}^{-1}] \otimes_{R^{loc}} S^{loc} \simeq k[\lambda_0, \dots, \lambda_n][\mathbb{Z}^{d+1}]$ which is just a localization of the former ring. However, since $k[\lambda_0, \dots, \lambda_n][\mathbb{Z}^{d+1}]$ is Cohen-Macaulay this system of parameters is a regular sequence. Hence

$$H_i(\mathrm{gr}_\cdot^{F^{loc}} \mathcal{K}^{loc}_\bullet(E - \widetilde{\beta}, S^{loc})) = 0 \quad \text{for } i > 0$$

and

$$H_0(\mathrm{gr}_\cdot^{F^{loc}} \mathcal{K}^{loc}_\bullet(E - \widetilde{\beta}, S^{loc})) \simeq \mathrm{gr}_\cdot^{F^{loc}}(M_{\widetilde{A}}^{\widetilde{\beta}}[\partial_{\lambda_0^{-1}}]).$$

From the construction of the strictly filtered resolution $L^{loc}_\bullet \twoheadrightarrow M_{\widetilde{A}}^{\widetilde{\beta}}[\partial_{\lambda_0}^{-1}]$ we see that $L^{loc}_\bullet = 0$ for all $k > n+1$. We have seen that the filtration on L^{loc}_{n+1} is the order filtration on D^{loc}, shifted appropriately. This shift is the sum of the length of the Euler-Koszul complex (i.e. $d+1$) and the degree (with respect to the grading of R^{loc} for which $\deg(\partial_{\lambda_i}) = 1$ and $\deg(\partial_{\lambda_0}^{-1}) = -1$ of $Ext_{R^{loc}}^{n-d}(S^{loc}, \omega_{R^{loc}})$. The latter is equal to $n+1-c_0$

which is the first component of the difference between the canonical degree of R^{loc} (i.e $\varepsilon_{\widetilde{A}}$ and the canonical degree of S^{loc} (i.e. $\tilde{c}$). Hence the filtration on L_{n+1} is $F^{loc}_{\bullet-(d+n-c_0+2)}$. By definition (see, e.g.[Sai94, page 55] one has

$$\mathbb{D}^{loc}(M_{\widetilde{A}}, F^{loc}_\bullet) = H^{n+1} Hom_{D^{loc}}\left((L^{loc}_\bullet, F^{loc}_\bullet), ((D^{loc}\otimes\Omega_V^{n+1})^\vee, F^{loc}_{\bullet-2(n+1)}D^{loc}\otimes(\Omega_V^{n+1})^\vee)\right)$$

where $F^{Loc}_\bullet$ is the strict filtration on $L^{loc}_\bullet$ constructed above. This implies the formula for $F^{\mathbb{D}^{loc}}_\bullet \mathbb{D}^{loc} M^{\widetilde{\beta}}_{\widetilde{A}}[\partial_{\lambda_0}^{-1}]$.

(4) One easily sees that

$$F^{loc}_k M^{\widetilde{\beta}}_{\widetilde{A}} = Im(\partial^k_{\lambda_0}\mathbb{C}[\lambda_0,\lambda_1,\dots,\lambda_n]\langle\partial_{\lambda_0}^{-1}, \partial_{\lambda_0}^{-1}\partial_{\lambda_1},\dots,\partial_{\lambda_0}^{-1}\partial_{\lambda_n}\rangle \ \subset\ M^{\widetilde{\beta}}_{\widetilde{A}}[\partial_{\lambda_0}^{-1}].$$

The filtration $F^{loc}_\bullet M^{\widetilde{\beta}}_{\widetilde{A}}$ induces a filtration $G_\bullet$ on $\widehat{M}^{\widetilde{\beta}}_A = \Gamma(\hat{V}, \widehat{\mathcal{M}}^{\widetilde{\beta}}_A)$ with

$$G_k\widehat{M}^{\widetilde{\beta}}_A = Im\left(z^{-k}\mathbb{C}[z,\lambda_1,\dots,\lambda_n]\langle z\partial_{\lambda_1},\dots,z\partial_{\lambda_n}, z^2\partial_z\rangle\right) \ \subset\ \widehat{M}^{\widetilde{\beta}}_A.$$

Hence we obtain a filtration $G_\bullet$ on the sheaf $\widehat{\mathcal{M}}^{\widetilde{\beta}}_A$ and we have $G_0\widehat{M}^{\widetilde{\beta}}_A = {}_0\widehat{\mathcal{M}}^{\widetilde{\beta}}_A$ as required.

Moreover the left multiplication with z^k gives an isomorphism $z^k\cdot : G_p\widehat{\mathcal{M}}^{\widetilde{\beta}}_A \xrightarrow{\simeq} G_{p-k}\widehat{\mathcal{M}}^{\widetilde{\beta}}_A$. It follows from Propositon 3.11 that $\left(G_0\widehat{\mathcal{M}}_A\right)_{|\mathbb{C}_z\times\Lambda^\circ} = {}_0^\circ\widehat{\mathcal{M}}_A$ is $\mathcal{O}_{\mathbb{C}_z\times\Lambda^\circ}$-locally free and hence all $\left(G_p\widehat{\mathcal{M}}_A\right)_{|\mathbb{C}_z\times\Lambda^\circ}$ are also $\mathcal{O}_{\mathbb{C}_z\times\Lambda^\circ}$-locally free.

Concerning the filtration $G^{\mathbb{D}}_\bullet$ we consider the filtered module $(\mathbb{D}^{loc}M_{\widetilde{A}}[\partial_{\lambda_0}^{-1}], F^{\mathbb{D}^{loc}}_\bullet)$. From the second statement follows that it is equal to $(M^{-\tilde{c}}_{\widetilde{A}}[\partial_{\lambda_0}^{-1}], F^{loc}_{\bullet+d-n-c_0})$. This induces $G^{\mathbb{D}}_\bullet$ on $\widehat{\mathcal{M}}_A$ which under the isomorphism from statement (2) is given by

$$G_{k+d-n-c_0}\widehat{\mathcal{M}}^{-\tilde{c}+(2,\underline{0})} \simeq \iota^* G^{\mathbb{D}}_k\, \mathbb{D}\widehat{\mathcal{M}}_A.$$

We conclude that the restrictions $\left(G^{\mathbb{D}}_k\,\mathbb{D}\widehat{\mathcal{M}}_A\right)_{|\mathbb{C}_z\times\Lambda^\circ}$ are $\mathcal{O}_{\mathbb{C}_z\times\Lambda^\circ}$-locally free.

(5) This follows from the isomorphisms

$$\iota^* G^{\mathbb{D}}_k\,\mathbb{D}\widehat{\mathcal{M}}_A \simeq G_{k+d-n-c_0}\widehat{\mathcal{M}}_A^{-\tilde{c}+(2,\underline{0})} \xrightarrow{z^{-c_0+2}(z\partial)^c} G_{k+d-n-2}\widehat{\mathcal{M}}_A\ .$$

□

This proposition is a direct generalization of [RS15, Corollary 2.19]. For the convenience of the reader we copy the proof with adapted notations.

Proposition 3.18. (1) *There is a non-degenerate flat $(-1)^d$-symmetric pairing*

$$P : {}^\circ\widehat{\mathcal{M}}_A \otimes \iota^*\left({}^\circ\widehat{\mathcal{M}}_A\right) \to \mathcal{O}_{\mathbb{C}_z\times\Lambda^\circ}(*\{0\}\times\Lambda^\circ).$$

where ι is the automorphism sending (z,λ) to $(-z,\lambda)$.

(2) *We have that $P({}_0^\circ\widehat{\mathcal{M}}_A, {}_0^\circ\widehat{\mathcal{M}}_A) \subset z^d\mathcal{O}_{\mathbb{C}_z\times\Lambda^\circ}$, and P is non-degenerate on ${}_0^\circ\widehat{\mathcal{M}}_A$. i.e., it induces a non-degenerate symmetric pairing*

$$[z^{-d}P] : \left[\frac{{}_0^\circ\widehat{\mathcal{M}}_A}{z\cdot{}_0^\circ\widehat{\mathcal{M}}_A}\right]\otimes\left[\frac{{}_0^\circ\widehat{\mathcal{M}}_A}{z\cdot{}_0^\circ\widehat{\mathcal{M}}_A}\right] \to \mathcal{O}_{\Lambda^\circ}.$$

Proof. (1) The statement can be reformulated as the existence of an isomorphism

$$\psi : {}^\circ\widehat{\mathcal{M}}_A \longrightarrow \iota^*\left({}^\circ\widehat{\mathcal{M}}_A\right)^*,$$

where $(-)^*$ denotes the dual meromorphic bundle with its dual connection. We deduce from [DS03, lemma A.11] that $\mathbb{D}({}^\circ\widehat{\mathcal{M}}_A)((*\{0\}\times\Lambda^\circ) = ({}^\circ\widehat{\mathcal{M}}_A)^*$. On the other hand Proposition 3.17 (1) and (2) give an isomorphism ${}^\circ\widehat{\mathcal{M}}_A \simeq \iota^*\mathbb{D}({}^\circ\widehat{\mathcal{M}}_A)$ so that the latter module is already localized, i.e. equal to $({}^\circ\widehat{\mathcal{M}}_A)^*$, which gives the existence of the isomorphism ψ from above.

(2) We have seen in point 1. that the duality isomorphism

$$\phi : \iota^*\mathbb{D}\widehat{\mathcal{M}}_A \simeq \widehat{\mathcal{M}}_A^{-\widetilde{c}+(2,\underline{0})} \longrightarrow \widehat{\mathcal{M}}_A$$

yields an isomorphism

$$\psi : {}^\circ\widehat{\mathcal{M}}_A \longrightarrow \iota^*\left({}^\circ\widehat{\mathcal{M}}_A\right)^*$$

of meromorphic bundles with connection. Now it follows from [Sai89, formula 2.7.5] that we have

$$\mathcal{H}om_{\mathcal{O}_{\mathbb{C}_z\times\Lambda^\circ}}\left(G_k{}^\circ\widehat{\mathcal{M}}_A, \mathcal{O}_{\mathbb{C}_z\times\Lambda^\circ}\right) = G^{\mathbb{D}}_{k+n+2}\mathbb{D}^\circ\widehat{\mathcal{M}}_A.$$

Hence by Proposition 3.17 (5) from above we conclude that ψ sends the module

$$G_0{}^\circ\widehat{\mathcal{M}}_A = {}^\circ_0\widehat{\mathcal{M}}_A$$

isomorphically into

$$\begin{aligned}\mathcal{H}om_{\mathcal{O}_{\mathbb{C}_z\times\Lambda^\circ}}\left(G_{-d}{}^\circ\widehat{\mathcal{M}}_A, \mathcal{O}_{\mathbb{C}_z\times\Lambda^\circ}\right) &= z^d\mathcal{H}om_{\mathcal{O}_{\mathbb{C}_z\times\Lambda^\circ}}\left(G_0{}^\circ\widehat{\mathcal{M}}_A, \mathcal{O}_{\mathbb{C}_z\times\Lambda^\circ}\right)\\ &= z^d\mathcal{H}om_{\mathcal{O}_{\mathbb{C}_z\times\Lambda^\circ}}\left({}^\circ_0\widehat{\mathcal{M}}_A, \mathcal{O}_{\mathbb{C}_z\times\Lambda^\circ}\right),\end{aligned}$$

which is equivalent to the statement to be shown.

□

4. Construction of the Landau-Ginzburg model

4.1. **Local Landau-Ginzburg models.** Let $\mathcal{X}$ be a projective toric Deligne-Mumford stack with extended fan $\mathbf{\Sigma}^e = (N, \Sigma, \mathfrak{a})$ (cf. Remark 2.8). In this section we explain the construction of a (Zariski local) Landau-Ginzburg model which will serve as a mirror partner for $\mathcal{X}$.

Recall the sequence 2.6

$$0 \longrightarrow \mathbb{L} \xrightarrow{\mathfrak{t}} \mathbb{Z}^n \xrightarrow{\mathfrak{a}} N \longrightarrow 0\,.$$

We apply the functor $\mathrm{Hom}(-,\mathbb{C}^*)$ to the sequence above which gives the following sequence of algebraic tori:

$$1 \longrightarrow \mathrm{Hom}(N,\mathbb{C}^*) \longrightarrow \mathcal{Y}' := \mathrm{Hom}(\mathbb{Z}^n,\mathbb{C}^*) \xrightarrow{\psi_\mathfrak{t}} \mathcal{T} := \mathrm{Hom}(\mathbb{L},\mathbb{C}^*) \longrightarrow 1\,.$$

We denote the standard basis of $\mathbb{Z}^n$ by $e_1,\ldots,e_n$. This equips $\mathcal{Y}'$ with coordinates $w_1,\ldots,w_n$. Consider the map

$$\begin{aligned}W' : \mathcal{Y}' &\longrightarrow \mathbb{C}_t\times\mathcal{T}\,,\\ (w_1,\ldots,w_n) &\mapsto (-\sum_{i=1}^n w_i, \psi_\mathfrak{t}(\underline{w}))\,.\end{aligned}$$

Usually this map W' is called the Landau-Ginzburg model of the toric orbifold $\mathcal{X}$. However to ensure a correct limit behavior we need to consider a covering of this model. Consider the inclusion

$$\mathfrak{c} : \mathbb{L} \hookrightarrow \mathrm{Pic}^e(X)^\star$$

(the inclusion Lemma 2.28), which gives rise to a covering $\psi_{\mathfrak{c}} : \mathcal{M}_{\mathcal{X}} := \mathrm{Hom}(\mathrm{Pic}^e(X)^\star, \mathbb{C}^*) \to \mathcal{T}$. We get the following cartesian diagram

(4.1)
$$\begin{array}{ccc} \mathcal{Y}' & \longleftarrow & \mathcal{Y} \\ \downarrow{\scriptstyle W'} & & \downarrow{\scriptstyle W} \\ \mathbb{C}_t \times \mathcal{T} & \xleftarrow[id\times\psi_{\mathfrak{c}}]{} & \mathbb{C}_t \times \mathcal{M}_{\mathcal{X}} \end{array}$$

Definition 4.2. *Let $\mathcal{X}$ be a projective toric orbifold. The mirror Landau-Ginzburg model of $\mathcal{X}$ is given by*

$$W = (F, pr_2) : \mathcal{Y} \longrightarrow \mathbb{C}_t \times \mathcal{M}_{\mathcal{X}} .$$

We have to compute the Fourier-Laplace transformed Gauß-Manin system and the Brieskorn lattice for the map W. For this we will construct a map $\mathbb{C}_t \times \Lambda^* \leftarrow \mathbb{C}_t \times \mathcal{T}$ such that W becomes the pull-back of the map φ_A from Section 3 (recall that A is the matrix corresponding to $\mathfrak{a}$ after the choice of a basis for N) under the cocatenated morphisms $\Lambda^* \leftarrow \mathbb{C}_t \times \mathcal{T} \leftarrow \mathbb{C}_t \times \mathcal{M}_{\mathcal{X}}$.
Here we will identify $\Lambda^* := \mathrm{Hom}(\mathbb{Z}^n, \mathbb{C}^*)$ with $\Lambda \setminus \{\lambda_1 \cdot \ldots \cdot \lambda_n = 0\}$.

Since we assumed that N is free, we have $\mathrm{Ext}^1(N, \mathbb{Z}) = 0$ which gives us the following commutative diagram whose vertical maps are isomorphismsm

(4.3)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{L} & \xrightarrow{\mathfrak{t}} & \mathbb{Z}^n & \xrightarrow{\mathfrak{a}} & N & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \mathfrak{k}} & & \| & & \\ 0 & \longrightarrow & \mathbb{L} & \longrightarrow & \mathbb{L} \oplus N & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

where $\mathfrak{k} = \mathfrak{s} + \mathfrak{a}$ with $\mathfrak{s} : \mathbb{Z}^n \to \mathbb{L}$ and $\mathfrak{k}^{-1} = \mathfrak{t} + \mathfrak{g}$ with $\mathfrak{g} : N \to \mathbb{Z}^n$. The maps satisfy the following relations

(1) $\mathfrak{a} \circ \mathfrak{t} = 0$ and $\mathfrak{s} \circ \mathfrak{g} = 0$
(2) $\mathfrak{a} \circ \mathfrak{g} = id_N$ and $\mathfrak{s} \circ \mathfrak{t} = id_{\mathbb{L}}$

We have a push-out diagram (see below for the arguments)

(4.4)
$$\begin{array}{ccccc} -\sum_{i=1}^n (\chi^{a_i} \otimes \chi^{e_i} \chi^{e_k}) & & \mathbb{C}[N] \otimes \mathbb{C}[\mathbb{Z}^n] \longrightarrow \mathbb{C}[\mathbb{Z}^n] \simeq \mathbb{C}[\mathbb{L} \oplus N] \text{(by isomorphism } \mathfrak{k}) & & (-\sum_{i=1}^n \chi^{e_i})\, \chi^{\mathfrak{t}(l)} \\ \uparrow & & \uparrow \qquad\qquad\qquad\qquad\qquad\qquad \uparrow & & \uparrow \\ t \otimes \chi^{e_k} & & \mathbb{C}[t] \otimes \mathbb{C}[\mathbb{Z}^n] \longrightarrow \mathbb{C}[t] \otimes \mathbb{C}[\mathbb{L}] & & t \otimes \chi^{l} \end{array}$$

$$t \otimes \chi^{e_k} \longrightarrow t \otimes \chi^{\mathfrak{s}(e_k)}$$

Here we made the following identifications

$$\begin{aligned} \mathbb{C}[N] \otimes \mathbb{C}[\mathbb{Z}^n] \otimes_{\mathbb{C}[t]\otimes\mathbb{C}[\mathbb{Z}^n]} \mathbb{C}[t] \otimes \mathbb{C}[\mathbb{L}] &\xrightarrow{\simeq} \mathbb{C}[N] \otimes \mathbb{C}[\mathbb{L}] \\ 1 \otimes \chi^{e_i} \otimes 1 \otimes 1 &\mapsto 1 \otimes \chi^{\mathfrak{s}(e_i)} \\ 1 \otimes 1 \otimes t \otimes 1 &\mapsto -\sum_{i=1}^n (\chi^{a_i} \otimes \chi^{\mathfrak{s}(e_i)}) \end{aligned}$$

and

$$\begin{aligned}\mathbb{C}[N]\otimes\mathbb{C}[\mathbb{L}] &\longrightarrow \mathbb{C}[\mathbb{Z}^n]\\ \chi^n\otimes\chi^l &\mapsto \chi^{\mathfrak{g}(n)}\chi^{\mathfrak{t}(l)}\,.\end{aligned}$$

From the pushout diagram (4.4) and Diagram (4.1), we get the following cartesian diagrams, where $Y := \operatorname{Spec}\mathbb{C}[N]$.

$$\begin{array}{ccccc} Y\times\Lambda^* & \longleftarrow & \mathcal{Y}'\simeq\mathcal{T}\times Y & \longleftarrow & \mathcal{Y}\simeq\mathcal{M}_{\mathcal{X}}\times Y\\ \downarrow{\scriptstyle\varphi_A} & & \downarrow{\scriptstyle W'} & & \downarrow{\scriptstyle W}\\ \mathbb{C}_t\times\Lambda^* & \xleftarrow[id\times\psi_{\mathfrak{s}}]{} & \mathbb{C}_t\times\mathcal{T} & \xleftarrow[\mathrm{id}\times\psi_{\mathfrak{c}}]{} & \mathbb{C}_t\times\mathcal{M}_{\mathcal{X}}\end{array} \tag{4.5}$$

We denote by $\psi := \psi_{\mathfrak{s}}\circ\psi_{\mathfrak{c}}$.

Proposition 4.6. *Let $\mathcal{X}$ be weak Fano. We can choose a $\mathbb{Z}$-basis $p_1,\dots,p_{r+e}$ in $\operatorname{Pic}^e(\mathcal{X})$ compatible with the splitting that is it satisifes the following conditions:*

(1) $p_1,\dots,p_r\in\theta(\mathcal{K})\subset\mathcal{K}^e$,
(2) $p_{r+i}=[D_{m+i}]$ *for* $i\in\{1,\dots,e\}$,
(3) $\rho\in Cone(p_1,\dots,p_{r+e})$.

Proof. From Proposition 2.21, we have that $\operatorname{Pic}^e(X)\simeq\operatorname{Pic}(X)\times\mathbb{Z}^e$. It is a general fact that one can resolve singularity in toric geometry (see [Ful93, §2.6]). This means that one can find a subdivision of a cone in smooth cones. We fix such a smooth subdivision of the cone $\mathcal{K}$. As $\mathcal{X}$ is weak Fano, that is $\overline{\rho}\in\mathcal{K}$, there is a maximal smooth cone $\sigma_{\overline{\rho}}$ that contains $\overline{\rho}$. Hence, we choose the primitive vectors of the rays of $\sigma_{\overline{\sigma}}$ and they form a $\mathbb{Z}$-basis of $\operatorname{Pic}(\mathbb{X})$. Then we apply θ to this $\mathbb{Z}$-basis and we denote them by $p_1,\dots,p_r$. Then we set $p_{r+i}:=[D_{m+i}]$ for $i\in\{1,\dots,e\}$ and by definition of ρ, we have $\rho\in Cone(p_1,\dots,p_{r+e})$. □

The dual $\mathbb{Z}$-basis of $(p_a)_{a=1,\dots,r+e}$ equips $\mathcal{M}_{\mathcal{X}}=Hom(\operatorname{Pic}^e(X)^\star,\mathbb{C}^*)$ with coordinates $\chi_1,\dots,\chi_{r+e}$.

Using the coordinates χ_a the map ψ is given by

$$\begin{aligned}\psi=\psi_{\mathfrak{n}}:\mathcal{M}_{\mathcal{X}} &\longrightarrow \Lambda^*\,,\\ (\chi_1,\dots,\chi_r) &\mapsto (\underline{\chi}^{\underline{n}_1},\dots,\underline{\chi}^{\underline{n}_{m+e}})\end{aligned}$$

where $\mathfrak{n}:=\mathfrak{c}\circ\mathfrak{s}$ with

$$\mathfrak{n}(e_i)=\sum_{a=1}^{r}n_{ai}p_a\,.$$

After the choice of the splitting $\mathfrak{k}$ in Diagram (4.3) (see also Diagram (4.5)) above the Landau-Ginzburg model for $\mathcal{X}$ is given by

$$\begin{aligned}W:\mathcal{Y}\simeq Y\times\mathcal{M}_{\mathcal{X}} &\longrightarrow \mathbb{C}_t\times\mathcal{M}_{\mathcal{X}}\,,\\ (\underline{y},\underline{\chi}) &\mapsto (-\sum_{i=1}^{n}\underline{\chi}^{n_i}\underline{y}^{a_i},\underline{\chi})\,.\end{aligned} \tag{4.7}$$

Since $\mathbb{L}$ is a full sublattice in $\operatorname{Pic}^e(X)^\star$ the map $\mathfrak{c}$ becomes an isomorphism after tensoring with $\mathbb{Q}$. Let

$$\mathfrak{r}:\operatorname{Pic}^e(X)^\star\otimes\mathbb{Q}\longrightarrow\mathbb{L}\otimes\mathbb{Q}$$

be its inverse. We denote by

$$\mathfrak{m}:\operatorname{Pic}^e(X)^\star\otimes_{\mathbb{Z}}\mathbb{Q}\longrightarrow\mathbb{Z}^n\otimes_{\mathbb{Z}}\mathbb{Q}$$

the concatenation $\mathfrak{t} \circ \mathfrak{r}$. With respect to the basis $e_1, \dots, e_n$ and the dual basis of $p_1, \dots, p_{r+e}$ the map $\mathfrak{m}$ is given by the matrix $M = (m_{ia})$. It follows that

$$[D_i] = \sum_{a=1}^{r+e} m_{ia} p_a \,. \tag{4.8}$$

Now we want to compute the Fourier-Laplace transformed Gauß-Manin system of the map W. We do this by computing an inverse image of the FL-transformed Gauß-Manin system of φ_A.

Proposition 4.9. *Let $\widetilde{\psi} := (id_z, \psi) : \mathbb{C}_z \times \mathcal{M}_\mathcal{X} \to \mathbb{C}_z \times \Lambda^*$.*

(1) *The inverse image $\mathcal{QM}_A := \widetilde{\psi}^+({}^*\widehat{\mathcal{M}}_A)$ is isomorphic to the quotient of $\mathcal{D}_{\mathbb{C}_z \times \mathcal{M}_\mathcal{X}}/\mathcal{I}$, where $\mathcal{I}$ is the left ideal generated by*

$$\widetilde{\Box}_{\underline{l}} := \prod_{a: p_a(\underline{l})>0} {\chi_a}^{p_a(\underline{l})} \prod_{i: l_i<0} \prod_{\nu=0}^{-l_i-1} (\sum_{a=1}^{r+e} m_{ia} z \chi_a \partial_{\chi_a} - \nu z) - \prod_{a: p_a(\underline{l})<0} {\chi_a}^{-p_a(\underline{l})} \prod_{i: l_i>0} \prod_{\nu=0}^{l_i-1} (\sum_{a=1}^{r+e} m_{ia} z \chi_a \partial_{\chi_a} - \nu z)$$

for any $\underline{l} \in \mathbb{L}$ and by the single operator

$$\check{E} := z^2 \partial_z + \sum_{a=1}^{r+e} \sum_{i=1}^{m+e} m_{ia} z \chi_a \partial_{\chi_a} \,.$$

(2) *There is an isomorphism of $\mathcal{D}_{\mathbb{C}_z \times \mathcal{M}_\mathcal{X}}$-modules*

$$\mathcal{QM}_A \simeq \mathrm{FL}^{loc}_{\mathcal{M}_\mathcal{X}}(\mathcal{H}^0 W_+ \mathcal{O}_{Y \times \mathcal{M}_\mathcal{X}}) \,.$$

Proof. We first choose bases $w_1, \dots, w_{r+e}$ of $\mathbb{L}$ and $v_1, \dots, v_d$ of N and denote the dual bases by $w_1^*, \dots, w_{r+e}^*$ and $v_1^*, \dots, v_d^*$, respectively. This gives rise to coordinates $\tau_1, \dots, \tau_{r+e}$ on $\mathcal{T} = \mathrm{Hom}(\mathbb{L}, \mathbb{C}^*)$ and $h_1, \dots, h_d$ on $H = \mathrm{Hom}(N, \mathbb{C}^*)$. We set $a_i = \mathfrak{a}(e_i)$, $s_i = \mathfrak{s}(e_i)$ for $i = 1, \dots, n$ and $t_a = \mathfrak{t}(w_a)$ resp. $g_j = \mathfrak{g}(v_j)$ for $a = 1, \dots, r+e$ resp. $j = 1, \dots, d$.

We will first compute the inverse image under the map $\psi_\mathfrak{s}$, which was induced by the linear morphism $\mathfrak{s} : \mathbb{Z}^n \to \mathbb{L}$. We factor $\mathfrak{s}$ in the following way:

$$\mathfrak{s} : \mathbb{Z}^n \xrightarrow{(\mathfrak{s},\mathfrak{a})} \mathbb{L} \oplus N \xrightarrow{\mathfrak{p}_1} \mathbb{L}$$

where $\mathfrak{p}_1$ is the projection to the first factor. Hence we get a factorization of $\psi_\mathfrak{s}$ given by

$$\begin{aligned} \psi_{\mathfrak{p}_1} : \mathcal{T} &\longrightarrow \mathcal{T} \times H \,, \\ (\tau_1, \dots, \tau_r) &\mapsto (\tau_1, \dots, \tau_r, 1, \dots, 1) \,, \end{aligned}$$

$$\begin{aligned} \psi_{(\mathfrak{s},\mathfrak{a})} : \mathcal{T} \times H &\longrightarrow \Lambda^* \,, \\ (\tau_1, \dots, \tau_r, h_1, \dots, h_d) &\mapsto (\underline{\tau}^{s_1} \cdot \underline{h}^{a_1}, \dots, \underline{\tau}^{s_{m+e}} \cdot \underline{h}^{a_{m+e}}) \end{aligned}$$

where $\underline{\tau}^{s_1} := \prod_{b=1}^{r+e} \tau_b^{s_{b1}}$ etc. and the inverse of $\psi_{(\mathfrak{s},\mathfrak{a})}$ is given by

$$\begin{aligned} \psi^{-1}_{(\mathfrak{s},\mathfrak{a})} : \Lambda^* &\longrightarrow \mathcal{T} \times H \,, \\ (\lambda_1, \dots, \lambda_{m+e}) &\mapsto (\underline{\lambda}^{\underline{t}_1}, \dots, \underline{\lambda}^{\underline{t}_r}, \underline{\lambda}^{\underline{g}_1}, \dots, \underline{\lambda}^{\underline{g}_d}) \,. \end{aligned}$$

Notice that the ideal $\mathcal{I}$ in Definition 3.4 is generated by

$$\left(\prod_{i:l_i>0}\lambda_i^{l_i}\right)\cdot\hat{\Box}_{\underline{l}}=\prod_i\lambda_i^{l_i}\cdot\prod_{i:l_i<0}(\lambda_i)^{-l_i}(z\partial_{\lambda_i})^{-l_i}-\prod_{i:l_i>0}(\lambda_i)^{l_i}(z\partial_{\lambda_i})^{l_i}\quad\text{for }\underline{l}\in\mathbb{L}\,,$$

$$=\prod_i\lambda_i^{l_i}\cdot\prod_{i:l_i<0}\prod_{\nu=0}^{-l_i-1}(z\lambda_i\partial_{\lambda_i}-\nu z)^{-l_i}-\prod_{i:l_i>0}\prod_{\nu=0}^{l_i-1}(z\lambda_i\partial_{\lambda_i}-\nu z)^{l_i}\quad\text{for }\underline{l}\in\mathbb{L}\,,$$

$$\hat{E}:=z^2\partial_z+\sum_{i=1}^{m+e}z\lambda_i\partial_{\lambda_i}\,,$$

$$\hat{E}_k:=\sum_{i=1}^{m+e}a_{ki}z\lambda_i\partial_{\lambda_i}\,.$$

We have the following transformation rules for the coordinate change $\psi_{(\mathfrak{s},\mathfrak{a})}$:

$$\lambda_i\partial_{\lambda_i}\mapsto\sum_{b=1}^{r+e}t_{ib}\tau_b\partial_{\tau_b}+\sum_{k=1}^{d}g_{ik}h_k\partial_{h_k}$$

where $t_{ib}=D_i(\mathfrak{t}(w_b))$.

Since $\psi_{(\mathfrak{s},\mathfrak{a})}$ is an isomorphic, we have that $\psi^+_{(\mathfrak{s},\mathfrak{a})}({}^*\widehat{\mathcal{M}}_A)$ is isomorphic to $\mathcal{D}_{\mathbb{C}_z\times\Lambda^*}/\mathcal{I}'$ where the left ideal $\mathcal{I}'$ is generated by

$$\Box'_{\underline{l}}:=\prod_{b=1}^{r+e}\tau_b^{w_b^*(\underline{l})}\prod_{i:l_i<0}\prod_{\nu=0}^{-l_i-1}(\sum_{b=1}^{r+e}D_i(\mathfrak{t}(w_b))z\tau_b\partial_{\tau_b}-\nu z)-\prod_{i:l_i>0}\prod_{\nu=0}^{l_i-1}(\sum_{b=1}^{r+e}D_i(\mathfrak{t}(w_b))z\tau_b\partial_{\tau_b}-\nu z)$$

for any $\underline{l}\in\mathbb{L}$ and by the Euler operators

$$E':=z\partial_z+\sum_{b=1}^{r+e}\left(\sum_{i=1}^{m+e}D_i(\mathfrak{t}(w_b))\right)\tau_b\partial_{\tau_b}\,,$$

$$E'_k:=h_k\partial_{h_k}\qquad\text{for }k=1,\dots,d\,.$$

Notice that we used the relations $h_k\partial_{h_k}=$ in the presentation of $\Box'_{\underline{l}}$. We also used the formulas

$$\prod_{i=1}^{m+e}{\lambda_i}^{l_i}=\prod_{i=1}^{m+e}\underline{\tau}^{l_i\cdot\underline{s}_i}\cdot\underline{h}^{l_i\cdot\underline{a}_i}=\prod_{b=1}^{r+e}\tau_b^{\sum_i l_i s_{bi}}\prod_{k=1}^{d}h_k^{\sum_i l_i a_{ki}}=\prod_{b=1}^{r+e}\tau_b^{\sum_i l_i s_{bi}}$$

and

$$\sum_i l_i s_{ib}=\sum_i l_i w_b^*(\mathfrak{s}(e_i))=w_b^*(\mathfrak{s}(\mathfrak{t}(\underline{l}))=w_b^*(\underline{l}).$$

It is now easy to see that the inverse image $\psi^+_{\mathfrak{s}}({}^*\widehat{\mathcal{M}}_A)\simeq\psi^+_{\mathfrak{p}_1}\psi^+_{(\mathfrak{s},\mathfrak{a})}({}^*\widehat{\mathcal{M}}_A)$ is isomorphic to $\mathcal{D}_{\mathbb{C}_z\times\mathcal{T}}/\mathcal{I}''$ where the left ideal $\mathcal{I}''$ is generated by $\Box'_{\underline{l}}$ and E'.

We will now compute the inverse image under $\psi_{\mathfrak{c}}$. Denote by $q_1,\dots,q_{r+e}$ the basis in $\mathrm{Pic}^e(X)^\star$ dual to $p_1,\dots,p_{r+e}\subset\mathrm{Pic}^e(X)$. With respect to the bases $w_1,\dots,w_{r+e}$ resp. $q_1,\dots,q_{r+e}$ the $\mathbb{Z}$-linear map $\mathfrak{c}$ is given by a matrix $C=(c_{ab})$, i.e. $\mathfrak{c}(w_b)=\sum_{a=1}^{r+e}c_{ab}q_a$. We factorize this matrix to obtain

$$C=C_1\cdot D\cdot C_2$$

with $C_1=(c^1_{ab}),C_2=(c^2_{ba})\in GL(r,\mathbb{Z})$ and $D=diag(d_1,\dots,d_{r+e})$ a diagonal matrix with strictly positive integer entries.

The factorization of C gives also a factorization of $R = C^{-1}$, i.e.

$$R = R_2 \cdot D^{-1} \cdot R_1$$

with $R_i = (r^i_{ab}) = C_i^{-1} \in Gl(r, \mathbb{Z})$. We define new bases

$$w'_h = \sum_{b=1}^{r+e} r^2_{bh} w_b \qquad \text{and} \qquad q'_h = \sum_{a=1}^{r+e} c^1_{ah} q_a \,.$$

With respect to these bases $\mathfrak{c}$ is diagonal, i.e. $\mathfrak{c}(w'_h) = d_h \cdot q'_h$.

The choice of these bases gives rise to coordinate changes on $\mathcal{T}$ and $\mathcal{M}_\mathcal{X}$:

$$\tau'_h = \prod_{b=1}^{r+e} \tau_b^{r^2_{bh}} \qquad \text{and} \qquad \chi'_h = \prod_{a=1}^{r+e} \chi_a^{c^1_{ah}}$$

with inverses

$$\tau_b = \prod_{h=1}^{r+e} (\tau'_h)^{c^2_{hb}} \qquad \text{and} \qquad \chi_a = \prod_{h=1}^{r+e} (\chi'_h)^{r^1_{ha}} \,.$$

Hence we get a factorization of $\psi_\mathfrak{c} = \psi_2 \circ \kappa \circ \psi_1$, where the maps are given by

$$\begin{aligned}
\psi_1 : \mathcal{T} &\longrightarrow \mathcal{T} \,, \\
(\tau'_1, \ldots, \tau'_{r+e}) &\mapsto (\tau_1 = \prod_{h=1}^{r+e} (\tau'_h)^{c^1_{h1}}, \ldots, \tau_{r+e} = \prod_{h=1}^{r+e} (\tau'_h)^{c^1_{h,r+e}}) \,, \\
\kappa : \mathcal{M}_\mathcal{X} &\longrightarrow \mathcal{T} \,, \\
(\chi'_1, \ldots, \chi'_{r+e}) &\mapsto (\tau'_1 = (\chi'_1)^{d_1}, \ldots, \tau'_{r+e} = (\chi'_{r+e})^{d_{r+e}}) \,, \\
\psi_2 : \mathcal{M}_\mathcal{X} &\longrightarrow \mathcal{M}_\mathcal{X} \,, \\
(\chi_1, \ldots, \chi_{r+e}) &\mapsto (\chi'_1 = \prod_{a=1}^{r+e} \chi_a^{c^2_{a1}}, \ldots, \chi'_{r+e} = \prod_{a=1}^{r+e} (\chi_a)^{c^2_{a,r+e}}) \,.
\end{aligned}$$

Notice that we have the following transformation rule:

$$\tau_b \partial_{\tau_b} \mapsto \sum_{h=1}^{r+e} r^2_{bh} \tau'_h \partial_{\tau'_h} \,.$$

Since ψ_1 is an isomorphism $\psi_1^+ \mathcal{D}_{\mathbb{C}_z \times \mathcal{T}} / \mathcal{I}''$ is isomorphic to $\mathcal{D}_{\mathbb{C}_z \times \mathcal{T}} / \mathcal{J}''$ where $\mathcal{J}''$ is generated by

$$\Box'_{\underline{l}} := \prod_{h=1}^{r+e} (\tau'_h)^{(w'_h)^*(\underline{l})} \prod_{i: l_i < 0} \prod_{\nu=0}^{-l_i-1} (\sum_{h=1}^{r+e} D_i(\mathfrak{t}(w'_h)) z \tau'_h \partial_{\tau'_h} - \nu z) - \prod_{i: l_i > 0} \prod_{\nu=0}^{l_i-1} (\sum_{h=1}^{r+e} D_i(\mathfrak{t}(w'_h)) z \tau'_h \partial_{\tau'_h} - \nu z)$$

where $\underline{l} \in \mathbb{L}$ and

$$z^2 \partial_z + \sum_{h=1}^{r+e} \left(\sum_{i=1}^{m+e} D_i(\mathfrak{t}(w'_h)) \right) z \tau'_h \partial_{\tau'_h} \,.$$

Here we have used that

$$\prod_{b=1}^{r+e} \tau_b^{w_b^*(\underline{l})} = \prod_{b=1}^{r+e} \left(\prod_{h=1}^{r+e} (\tau'_h)^{c^2_{hb}} \right)^{w_b^*(\underline{l})} = \prod_{h=1}^{r+e} (\tau'_h)^{\sum_{b=1}^{r+e} c^2_{hb} w_b^*(\underline{l})} = \prod_{h=1}^{r+e} (\tau'_h)^{(w'_h)^*(\underline{l})}$$

and

$$\begin{aligned}\sum_{b=1}^{r+e} D_i(\mathfrak{t}(w_b))z\tau_b\partial_{\tau_b} &= \sum_{b=1}^{r+e} D_i(\mathfrak{t}(w_b))z\left(\sum_{h=1}^{r+e} r_{bh}^2\tau'_h\partial_{\tau'_h}\right)\\ &= \sum_{h=1}^{r+e} D_i(\mathfrak{t}(\sum_{b=1}^{r+e} r_{bh}^2 w_b))z\tau'_h\partial_{\tau'_h}\\ &= \sum_{h=1}^{r+e} D_i(\mathfrak{t}(w'_h))z\tau'_h\partial_{\tau'_h}\,.\end{aligned}$$

In order to compute $\kappa^+\psi_1^+\mathcal{D}_{\mathbb{C}_z\times\mathcal{T}}/\mathcal{I}'' \simeq \kappa^+\mathcal{D}_{\mathbb{C}_z\times\mathcal{T}}/\mathcal{J}$, we first notice that

$$\kappa^+\mathcal{D}_{\mathbb{C}_z\times\mathcal{T}}/\mathcal{J} \simeq \mathcal{O}_{\mathcal{M}_\mathcal{X}}\otimes\kappa^{-1}(\mathcal{D}_{\mathbb{C}_z\times\mathcal{T}}/\mathcal{J})$$

where the operators χ'_h resp. $\chi'_h\partial_{\chi'_h}$ act by

$$(\chi'_h)^{d_n}(f\otimes P) = f\otimes\tau'_h P$$

resp.

$$\chi'_h\partial_{\chi'_h}(f\otimes P) = \chi'_h\partial_{\chi'_h}(f)\otimes P + f\otimes(d_h\tau'_h\partial_{\tau'_h})P\,.$$

An easy computation shows that $\kappa^+\mathcal{D}_{\mathbb{C}_z\times\tau}/\mathcal{J}''$ is isomorphic to the quotient $\mathcal{D}_{\mathbb{C}_z\times\mathcal{T}}/\mathcal{J}'$ where the left ideal $\mathcal{J}'$ is generated by

$$\prod_{h=1}^{r+e}(\chi'_h)^{\mathfrak{c}^*(q'_h)^*(\underline{l})}\prod_{i:l_i<0}\prod_{\nu=0}^{-l_i-1}(\sum_{h=1}^{r} D_i(\mathfrak{t}\circ\mathfrak{r}(q'_h))z\chi'_h\partial_{\chi'_h}-\nu z) - \prod_{i:l_i>0}\prod_{\nu=0}^{l_i-1}(\sum_{h=1}^{r} D_i(\mathfrak{t}\circ\mathfrak{r}(q'_h))z\chi'_h\partial_{\chi'_h}-\nu z)$$

for any $\underline{l}\in\mathbb{L}$ and by the single operator

$$z^2\partial_z + \sum_{h=1}^{r+e}\left(\sum_{i=1}^{m+e} D_i(\mathfrak{t}\circ\mathfrak{r}(q'_h))\right)z\chi'_h\partial_{\chi'_h}\,.$$

Here we used $\mathfrak{c}(w'_h) = d_h\cdot p'_h$, i.e. $\mathfrak{c}^*((q'_h)^*) = d_h\cdot(w'_h)^*$.
The final step consists in computing $\psi_2^+\kappa^+\psi_1^+\mathcal{D}_{\mathbb{C}_z\times\mathcal{T}}/\mathcal{I}'$ which is completely parallel to the computation of the inverse image under ψ_1. Therefore the first claim follows.

For the second claim consider the cartesian diagram

$$\begin{array}{ccc} Y\times\mathcal{M}_\mathcal{X} & \longrightarrow & Y\times\Lambda^* \\ {\scriptstyle W}\downarrow & & \downarrow{\scriptstyle\varphi_A} \\ \mathbb{C}_t\times\mathcal{M}_\mathcal{X} & \xrightarrow{(id\times\psi)} & \mathbb{C}_t\times\Lambda^* \end{array}$$

We have the following isomorphisms

$$\begin{aligned}\mathcal{QM}_A &\simeq \widetilde{\psi}^+({}^*\widehat{\mathcal{M}}_A)\\ &\simeq \widetilde{\psi}^+\,\mathrm{FL}_\Lambda^{loc}\,\mathcal{H}^0(\varphi_{A,+}\mathcal{O}_{Y\times\Lambda^*}) && \text{cf. Proposition 3.6}\\ &\simeq \mathrm{FL}_{\mathcal{M}_\mathcal{X}}^{loc}(id_{\mathbb{C}_t}\times\psi)^+\mathcal{H}^0(\varphi_{A,+}\mathcal{O}_{Y\times\Lambda})\\ &\simeq \mathrm{FL}_{\mathcal{M}_\mathcal{X}}^{loc}(\mathcal{H}^0 W_+\mathcal{O}_{Y\times\mathcal{M}_\mathcal{X}})\end{aligned}$$

where the third isomorphism follows from the compatibility of the localized Fourier-Laplace transform with base change and the fourth isomorphism is base change with respect to the diagram above.

□

Write
(4.10)

$$\mathcal{M}_{\mathcal{X}}^{\circ} := \psi^{-1}(\Lambda^{\circ}) = \{(\chi_1, \dots, \chi_{r+e}) \in \mathcal{M}_{\mathcal{X}} \mid W = -\sum_{i=1}^{m+e} \underline{\chi}^{n_i} \underline{y}^{a_i} \text{ is Newton non-degenerate}\}.$$

We have the following statement for the Brieskorn-lattice:

Proposition 4.11.

(1) *The inverse image (in the category of $\mathcal{O}$-modules)*

$$ {}_0^{\circ}\mathcal{QM}_A := \widetilde{\psi}^*({}_0^{\circ}\widehat{\mathcal{M}}_A) = \mathcal{O}_{\mathbb{C}_z \times \mathcal{M}_{\mathcal{X}}^{\circ}} \otimes \widetilde{\psi}^{-1}({}_0^{\circ}\widehat{\mathcal{M}}_A) \tag{4.12}$$

carries a natural structure of an $\mathcal{R}_{\mathbb{C}_z \times \mathcal{M}_{\mathcal{X}}^{\circ}}$-module. It is isomorphic to the quotient $\mathcal{R}_{\mathbb{C}_z \times \mathcal{M}_{\mathcal{X}}^{\circ}}/\mathcal{I}_0$, where $\mathcal{I}_0$ is the left ideal generated by $(\widetilde{\Box}_{\underline{l}})_{\underline{l} \in \mathbb{L}}$ and $\check{E}$.

(2) *There exists the following $R_{\mathbb{C}_z \times \mathcal{M}_{\mathcal{X}}^{\circ}}$-linear isomorphism*

$$H^0\left(\Omega^{\bullet+r+e}_{Y \times \mathcal{M}_{\mathcal{X}}^{\circ}/\mathcal{M}_{\mathcal{X}}^{\circ}}[z], zd - d_y W \wedge\right) \simeq {}_0^{\circ}QM_A$$

where ${}_0^{\circ}QM_A = \Gamma(\mathcal{M}_{\mathcal{X}}^{\circ}, {}_0^{\circ}\mathcal{QM}_A)$.

(3) *There is a non-degenerate flat $(-1)^d$-symmetric pairing*

$$P : (\mathcal{QM}_A)_{|\mathbb{C}_z^* \times \mathcal{M}_{\mathcal{X}}^{\circ}} \otimes \iota^* (\mathcal{QM}_A)_{|\mathbb{C}_z^* \times \mathcal{M}_{\mathcal{X}}^{\circ}} \to \mathcal{O}_{\mathbb{C}_z^* \times \mathcal{M}_{\mathcal{X}}^{\circ}}.$$

(4) *$P({}_0^{\circ}\mathcal{QM}_A, {}_0^{\circ}\mathcal{QM}_A) \subset z^d \mathcal{O}_{\mathbb{C}_z \times \mathcal{M}_{\mathcal{X}}^{\circ}}$, and P is non-degenerate on ${}_0^{\circ}\mathcal{QM}_A$.*

Proof. First notice that the map $\widetilde{\psi}$ factorizes as $(id \times \psi_{\mathfrak{s}}) \circ (id \times \psi_{\mathfrak{c}})$. The map $(id \times \psi_{\mathfrak{s}})$ is non-characteristic with respect to ${}^{\circ}\widehat{\mathcal{M}}_A$ since the singular locus of ${}^{\circ}\widehat{\mathcal{M}}_A$ is contained in $(\{0, \infty\} \times \Lambda^{\circ})$ and the map $(id \times \psi_{\mathfrak{c}})$ is non-characteristic with respect to any coherent $\mathcal{D}_{\mathbb{C}_z \times \mathcal{T}}$-module since $(id \times \psi_{\mathfrak{c}})$ is smooth. Hence the inverse image is nothing but the inverse image in the category of meromorphic connections. The inverse image of the lattice ${}_0^{\circ}\widehat{\mathcal{M}}_A$ is then simply given by the formula (4.12).
The second point follows by base change and the fact that $\widetilde{\psi}^* = (id \times \psi_{\mathfrak{c}})^* \circ (id \times \psi_{\mathfrak{s}})^*$ is exact. The third and fourth point follow from Proposition 3.18. □

The following Lemma will be used later (See the proof of Lemma 6.2) to prove that the I-function is killed by the operators $\Box_{\underline{l}}^{\mathcal{X}}$.

Lemma 4.13.

(1) *The $\mathcal{D}_{\mathbb{C}_z \times \mathcal{M}_{\mathcal{X}}}$-module $\mathcal{QM}_A$ is isomorphic to the quotient $\mathcal{D}_{\mathbb{C}_z \times \mathcal{T}}/\mathcal{I}$ where $\mathcal{J}$ is the left ideal generated by $\check{E}$ and*

$$\Box_{\underline{l}}^{\mathcal{X}} := \prod_{\substack{a=1 \\ p_a(\underline{l})>0}}^{r} \chi_a^{p_a(\underline{l})} \prod_{\substack{i=m+1 \\ l_i<0}}^{m+e} \mathscr{D}_i^{-l_i} \prod_{\substack{i=1 \\ l_i<0}}^{m} \prod_{\nu=0}^{-l_i-1} (\mathscr{D}_i - \nu z) - \prod_{\substack{a=1 \\ p_a(\underline{l})<0}}^{r} \chi_a^{-p_a(\underline{l})} \prod_{\substack{i=m+1 \\ l_i>0}}^{m+e} \mathscr{D}_i^{l_i} \prod_{\substack{i=1 \\ l_i>0}}^{m} \prod_{v=0}^{l_i-1} (\mathscr{D}_i - \nu z)$$

where

$$\mathscr{D}_i = \begin{cases} \sum_{a=1}^{r+e} m_{ia} z \chi_a \partial_{\chi_a} & \text{for } i = 1, \dots, m \\ z \partial_{\chi_{i-m+r}} & \text{for } i = m+1, \dots, m+e\,. \end{cases}$$

(2) *The $\mathcal{R}_{\mathbb{C}_z \times \mathcal{M}_{\mathcal{X}}^{\circ}}$-module ${}_0^{\circ}\mathcal{QM}_A$ is isomorphic to the quotient $\mathcal{R}_{\mathbb{C}_z \times \mathcal{M}_{\mathcal{X}}^{\circ}}/\mathcal{J}_0$ where $\mathcal{J}_0$ is the left ideal generated by $\Box_{\underline{l}}^{\mathcal{X}}$ and $\check{E}$.*

Proof. Notice that we have the following identifications

(1) $p_{r+i}(\underline{l}) = [D_{m+i}](\underline{l}) = l_{m+i}$ for $i = 1, \dots, e$,

(2) $m_{m+i,a} = \delta_{r+i,a}$ for $i = 1, \dots, e$

where the second point follows from formula 4.8 and our choic of the basis $(p_a)_{a=1,\dots,r+e}$. We can therefore write

$$\begin{aligned}
\widetilde{\Box}_{\underline{l}} &= \prod_{\substack{a=1\\ p_a(\underline{l})>0}}^{r+e} \chi_a{}^{p_a(\underline{l})} \prod_{i:l_i<0} \prod_{\nu=0}^{-l_i-1} (\sum_{a=1}^{r+e} m_{ia} z\chi_a\partial_{\chi_a} - \nu z) - \prod_{\substack{a=1\\ p_a(\underline{l})<0}}^{r+e} \chi_a{}^{-p_a(\underline{l})} \prod_{i:l_i>0} \prod_{\nu=0}^{l_i-1} (\sum_{a=1}^{r+e} m_{ia} z\chi_a\partial_{\chi_a} - \nu z) \\
&= \prod_{\substack{a=1\\ p_a(\underline{l})>0}}^{r} \chi_a^{p_a(\underline{l})} \prod_{\substack{i=1\\ l_{m+i}>0}}^{e} \chi_{r+i}^{l_{m+i}} \prod_{\substack{i=1\\ l_{m+i}<0}}^{e} \chi_{r+i}^{-l_{m+i}} \prod_{\substack{i=m+1\\ l_i<0}}^{m+e} \mathscr{D}_i^{-l_i} \prod_{\substack{i=1\\ l_i<0}}^{m} \prod_{\nu=0}^{-l_i-1} (\mathscr{D}_i - \nu z) \\
&- \prod_{\substack{a=1\\ p_a(\underline{l})<0}}^{r} \chi_a^{-p_a(\underline{l})} \prod_{\substack{i=1\\ l_{m+i}<0}}^{e} \chi_{r+i}^{-l_{m+i}} \prod_{\substack{i=1\\ l_{m+i}>0}}^{e} \chi_{r+i}^{l_{m+i}} \prod_{\substack{i=m+1\\ l_i>0}}^{m+e} \mathscr{D}_i^{l_i} \prod_{\substack{i=1\\ l_i>0}}^{m} \prod_{\nu=0}^{l_i-1} (\mathscr{D}_i - \nu z) \\
&= \prod_{i=1}^{e} \chi_{r+i}^{|l_{m+i}|} \cdot \Box_{\underline{l}}^{\mathcal{X}} .
\end{aligned}$$

Since the χ_a are invertible on $\mathbb{C}_z \times \mathcal{M}_{\mathcal{X}}$ this shows the first and second point. □

4.2. **Logarithmic extension.** Let Y be a smooth variety and D be a reduced normal-crossing divisor in Y. Denote by $\mathcal{R}_{\mathbb{C}_z\times Y}(logD)$ the subsheaf of $\mathcal{R}_{\mathbb{C}_z\times Y}$ generated by $\mathcal{O}_{\mathbb{C}_z\times Y}$, $z^2\partial_z$ and $z \cdot p^{-1}Der(logD)$, where $p : \mathbb{C}_z \times Y \to Y$ is the canonical projection and $Der(logD)$ the sheaf of logarithmic vector fields along D.

Recall the definition of the base space $\mathcal{M}_{\mathcal{X}} = \mathrm{Hom}(\mathrm{Pic}^e(X)^\star, \mathbb{C}^*) \simeq (\mathbb{C}^*)^{r+e}$ of the Landau-Ginzburg model from section 4.1. The choice of a basis $p_1, \dots, p_{r+e}$ determines a partial compactification $\overline{\mathcal{M}}_{\mathcal{X}} \simeq \mathbb{C}^{r+e}$. Let $D_{\mathcal{X}} \subset \overline{\mathcal{M}}_{\mathcal{X}}$ be the normal crossing divisor given by $\chi_1 \cdots \chi_r = 0$. Recall from last page that

$$\mathcal{M}_{\mathcal{X}}^\circ := \psi^{-1}(\Lambda^\circ) = \{(\chi_1, \dots, \chi_{r+e}) \in \mathcal{M}_{\mathcal{X}} \ \mid W = -\sum_{i=1}^{m+e} \underline{x}^{\underline{n}_i} \underline{y}^{\underline{a}_i} \text{ is Newton non-degenerate}\}.$$

Denote by $\Delta := \mathcal{M}_{\mathcal{X}} \setminus \mathcal{M}_{\mathcal{X}}^\circ$ (*i.e.*, W is degenerated on Δ) and let $\overline{\Delta}$ be the closure of Δ in $\overline{\mathcal{M}}_{\mathcal{X}}$. Define $\overline{\mathcal{M}}_{\mathcal{X}}^\circ := \overline{\mathcal{M}}_{\mathcal{X}} \setminus \overline{\Delta}$. We denote by $p_{\mathcal{X}}$ the point with coordinates $\chi_1 = \ldots = \chi_{r+e} = 0$.

Remark 4.14. Notice that until now any object with index A (like ${}_0\mathcal{QM}_A$) only depends on the matrix A which is given by the generators of the rays of the extended stacky fan. Moreover there is no difference between the rays coming from the original stacky fan and the rays which are added for the extended stacky fan.

Lemma 4.15. *The point $p_{\mathcal{X}}$ is contained in $\overline{\mathcal{M}}_{\mathcal{X}}^\circ$.*

Proof. Iritani proves in [Iri09, Lemma 3.8] that in $\mathcal{M}_{\mathcal{X}} := (\mathbb{C}^*)^{r+e} \subset \overline{\mathcal{M}}_{\mathcal{X}} := \mathbb{C}^{r+e}$, there exists an open neighbourhood $U_{p_{\mathcal{X}}}$ of $p_{\mathcal{X}} := \underline{0} \in \overline{\mathcal{M}}_{\mathcal{X}}$ such that when one restricts W to $U_{p_{\mathcal{X}}} \cap \mathcal{M}_{\mathcal{X}}$ is non degenerated. This means that Δ does not meet $U_{p_{\mathcal{X}}}$. This implies that $p_{\mathcal{X}}$ is not in $\overline{\Delta}$. □

Notice that the operators $(\Box_{\underline{l}}^{\mathcal{X}})_{\underline{l}\in\mathbb{L}}$ and $\check{E}$ are in $\mathcal{R}_{\mathbb{C}_z\times\overline{\mathcal{M}}_{\mathcal{X}}^\circ}(logD_{\mathcal{X}})$.

Definition 4.16. *Let ${}_0\mathcal{QM}_A^{log,\mathcal{X}}$ be the quotient $\mathcal{R}_{\mathbb{C}_z\times\overline{\mathcal{M}}_{\mathcal{X}}^\circ}(logD_{\mathcal{X}})/\mathcal{I}_{\mathcal{X}}$ where $\mathcal{I}_{\mathcal{X}}$ is the ideal generated by $(\Box_{\underline{l}}^{\mathcal{X}})_{\underline{l}\in\mathbb{L}}$ and $\check{E}$ (notice that these operators are global sections of $\mathcal{R}_{\mathbb{C}_z\times\overline{\mathcal{M}}_{\mathcal{X}}^\circ}(logD_{\mathcal{X}})$).*

Theorem 4.17. *There is a Zariski open subset $\mathcal{U}_{\mathcal{X}} \subset \overline{\mathcal{M}}_{\mathcal{X}}^\circ$ containing the point $p_{\mathcal{X}}$ such that ${}_0\mathcal{QM}_A^{log,\mathcal{X}} := {}_0\mathcal{QM}_A^{log,\mathcal{X}}{}_{|\mathbb{C}_z\times\mathcal{U}_{\mathcal{X}}}$ is $\mathcal{O}_{\mathbb{C}_z\times\mathcal{U}_{\mathcal{X}}}$-coherent.*

Proof. Let $\mathcal{R}'(logD_{\mathcal{X}})$ be the sheaf associated to the ring

$$\mathbb{C}[z,\chi_1,\dots,\chi_r]\langle z\chi_1\partial_{\chi_1},\dots,z\chi_r\partial_{\chi_r},z\partial_{\chi_{r+1}},\dots,z\partial_{\chi_{r+e}}\rangle.$$

Notice that ${}_0\mathcal{QM}_A^{log,\mathcal{X}}$ carries a natural $\mathcal{R}'(logD_{\mathcal{X}})$-module structure. We denote the corresponding $\mathcal{R}'(logD_{\mathcal{X}})$-module by $For_{z^2\partial_z}({}_0\mathcal{QM}_A^{log,\mathcal{X}})$. Notice that it is enough to prove the coherence for $For_{z^2\partial_z}({}_0\mathcal{QM}_A^{log,\mathcal{X}})$ in the neighbourhood of the point $p_{\mathcal{X}}$ since it is isomorphic to ${}_0\mathcal{QM}_A^{log,\mathcal{X}}$ as a $\mathcal{O}_{\mathbb{C}_z\times\overline{\mathcal{M}}^\circ_{\mathcal{X}}}$-module.

Because of the operator $\check{E}=z^2\partial_z+\sum_{a=1}^{r+e}\sum_{i=1}^{m+e}m_{ia}z\chi_a\partial_{\chi_a}$ the module $For_{z^2\partial_z}({}_0\mathcal{QM}_A^{log,\mathcal{X}})$ is isomorphic to $\mathcal{R}'(logD_{\mathcal{X}})/(\Box_{\underline{l}}^{\mathcal{X}})_{\underline{l}\in\mathbb{L}}$. Now consider on $\mathcal{R}'(logD_{\mathcal{X}})$ the natural filtration $F_\bullet$ given by the orders of operators, i.e. the filtration $F_\bullet\mathcal{R}'(logD_{\mathcal{X}})$ is given on the level of global sections by

$$F_k\mathbb{C}[z,\chi_1,\dots,\chi_r]\langle z\chi_1\partial_{\chi_1},\dots,z\chi_r\partial_{\chi_r}\rangle:=\left\{P\mid P=\sum_{|\underline{s}|\le k}g_{\underline{s}}(z,\underline{\chi})(z\chi_1\partial_{\chi_1})^{s_1}\cdot\ldots\cdot(z\chi_r\partial_{\chi_r})^{s_r}\right\}.$$

This filtration induces a filtration $F_\bullet$ on $For_{z^2\partial_z}({}_0\mathcal{QM}_A^{log,\mathcal{X}})$ which is good, i.e.

$$F_k\mathcal{R}'(logD_{\mathcal{X}})\cdot F_lFor_{z^2\partial_z}({}_0\mathcal{QM}_A^{log,\mathcal{X}})=F_{k+l}For_{z^2\partial_z}({}_0\mathcal{QM}_A^{log,\mathcal{X}}).$$

We have a natural identification

$$gr_\bullet^F(\mathcal{R}'(logD_{\mathcal{X}}))=\pi_*\mathcal{O}_{\mathbb{C}_z\times T^*\overline{\mathcal{M}}^\circ_{\mathcal{X}}(logD_{\mathcal{X}})}$$

where $T^*\overline{\mathcal{M}}^\circ_{\mathcal{X}}(logD_{\mathcal{X}})$ is the total space of the vector bundle associated to the locally free sheaf $\Omega^1_{\overline{\mathcal{M}}^\circ_{\mathcal{X}}}(logD_{\mathcal{X}})$ and $\pi:\mathbb{C}_z\times T^*\overline{\mathcal{M}}^\circ_{\mathcal{X}}(logD_{\mathcal{X}})\to\mathbb{C}_z\times\overline{\mathcal{M}}^\circ_{\mathcal{X}}$ is the projection. The symbols of all operators $\Box_{\underline{l}}^{\mathcal{X}}$ for $\underline{l}\in\mathbb{L}$ cut out a subvariety $\mathbb{C}_z\times S$ of $\mathbb{C}_z\times T^*\overline{\mathcal{M}}^\circ_{\mathcal{X}}(logD_{\mathcal{X}})$.
It will be sufficient to show that the fiber over $\underline{\chi}=0$ of $S\to\overline{\mathcal{M}}^\circ_{\mathcal{X}}$ is quasi-finite since this implies that $S\to\overline{\mathcal{M}}^\circ_{\mathcal{X}}$ is quasi-finite in a Zariski open neighborhood of $\underline{\chi}=0$, denoted by $\mathcal{U}_{\mathcal{X}}$. Since S is homogeneous this shows that S is equal to the zero section of $T^*\mathcal{U}_{\mathcal{X}}(logD_{\mathcal{X}})$ over this neighborhood. Adapting a well-known argument from the theory of $\mathcal{D}$-modules (see, e.g. [Pha79]) we see that the filtration $F_\bullet$ will become eventually stationary and we conclude by the fact that all $F_kFor_{z^2\partial_z}({}_0^\circ\mathcal{QM}_A^{log,\mathcal{X}})$ are stationary in this neighborhood.

Therefore, it remains now to prove that that the fiber over $z=\underline{\chi}=0$ of $S\to\overline{\mathcal{M}}^\circ_{\mathcal{X}}$ is quasi-finite.

First notice that in the limit $z=\underline{\chi}=0$ the operators

$$\mathscr{D}_i=\begin{cases}\sum_{a=1}^{r+e}m_{ia}z\chi_a\partial_{\chi_a} & \text{for } i=1,\dots,m\\ z\partial_{\chi_{i-m+r}} & \text{for } i=m+1,\dots,m+e\end{cases}$$

in $\mathcal{R}'(logD_{\mathcal{X}})$ degenerate to

$$\mathbf{D}_i=\begin{cases}\sum_{a=1}^{r}m_{ia}z\chi_a\partial_{\chi_a} & \text{for } i=1,\dots,m\\ z\partial_{\chi_{i-m+r}} & \text{for } i=m+1,\dots,m+e.\end{cases}\tag{4.18}$$

Since the fan Σ is simplicial, we have for each a_j, $j=m+1,\dots,m+e$ a cone relation $\underline{l}_{C_j}$:

$$\sum_{i=1}^{m}l_ia_i-l_ja_j=0\tag{4.19}$$

with $l_i, l_j \in \mathbb{Z}_{\geq 0}$. Because of $p_1, \ldots, p_r \in \theta(\mathcal{K}) \subset \theta(\mathrm{Pic}(X))$ and the definition of the map Θ one easily sees that

$$p_a(\underline{l}) = 0 \tag{4.20}$$

for all (cone) relations $\underline{l}$. Hence the corresponding Box operator is

$$\Box^{\mathcal{X}}_{\underline{l}_{C_j}} = (\mathscr{D}_j)^{l_j} - \prod_{i=1}^{m} \prod_{\nu=0}^{l_i-1} (\mathscr{D}_i - \nu z)\,.$$

Because $\deg(a_i) = 1$ for $i = 1, \ldots, m$ and $\deg(a_j) \leq 1$ by Lemma 2.22 we get from (4.19) the inequality $l_j \geq \sum_{i=1}^{m} l_i$. Hence the symbol of $\Box^{\mathcal{X}}_{\underline{l}_{C_j}}$ is

$$\sigma(\Box^{\mathcal{X}}_{\underline{l}_{C_j}}) = \begin{cases} \sigma(\mathscr{D}_j)^{l_j} & \text{if } \deg(a_j) < 1 \\ \sigma(\mathscr{D}_j)^{l_j} - \prod_{i=1}^{m} \sigma(\mathscr{D}_i)^{l_i} & \text{if} \deg(a_j) = 1 \end{cases} \tag{4.21}$$

which degenerates to

$$\sigma(\Box^{\mathcal{X}}_{\underline{l}_{C_j}})_{z=\underline{\chi}=0} = \begin{cases} \sigma(\mathbf{D}_j)^{l_j} & \text{if } \deg(a_j) < 1 \\ \sigma(\mathbf{D}_j)^{l_j} - \prod_{i=1}^{m} \sigma(\mathbf{D}_i)^{l_i} & \text{if} \deg(a_j) = 1\,. \end{cases} \tag{4.22}$$

Now suppose that $\{a_i \mid i \in I\}$ for $I \subset \{1, \ldots, m\}$ is a primitive collection. Denote by $\underline{l}_I \in \mathbb{L} \otimes \mathbb{Q}$ a primitive relation

$$\sum_{i \in I} a_i - \sum_{j \in I_\sigma}^{m+e} l_j a_j = 0$$

where I_σ was defined in (2.2).
We claim that

$$p_a(\underline{l}_I) \geq 0 \qquad \text{for} \quad a = 1, \ldots, r\,. \tag{4.23}$$

Recall that $p_a \in \mathcal{K}$ and that $\mathcal{K}$ is the image of $\Theta(CPL(\Sigma))$. Let $\varphi_a \in CPL(\Sigma)$ be a convex, piece-wise linear function such that $\Theta(\varphi_a)$ is a preimage of p_a. By the definition of Θ (cf. (2.13)) we have

$$\begin{aligned} p_a(\underline{l}_I) &= \sum_{i \in I} \varphi_a(a_i) - \sum_{j \in I_\sigma} l_j \varphi_a(a_j) = \sum_{i \in I} \varphi_a(a_i) - \varphi_a(\sum_{j \in I_\sigma} l_j a_j) \\ &\geq \varphi_a(\sum_{i \in I} a_i) - \varphi_a(\sum_{j \in I_\sigma} l_j a_j) = 0\,. \end{aligned}$$

Additionally, the following inequality

$$\#I = \sum_{i \in I} 1 \geq \sum_{j \in I_\sigma} l_j$$

is true for the relation $\underline{l}_I$, because $\deg(a_i) = 1$ for $i = 1, \ldots, m$. Clearing denominators we get a relation $\underline{l}'_I := c \cdot \underline{l}_I \in \mathbb{L}$ for some $c \in \mathbb{Z}_{>1}$. The symbol of a box operator with respect this relation is

$$\sigma(\Box^{\mathcal{X}}_{\underline{l}'_I}) = \prod_{a=1}^{r} \chi_a^{p_a(\underline{l}'_I)} \prod_{j \in I_\sigma} \sigma(\mathscr{D}_j)^{l'_j} - \prod_{i \in I} \sigma(\mathscr{D}_i)^c \qquad \text{for} \quad \#I = \sum_{j \in I_\sigma} l_j$$

and

$$\sigma(\Box^{\mathcal{X}}_{\underline{l}'_I}) = -\prod_{i \in I} \sigma(\mathscr{D}_i)^c \qquad \text{for} \quad \#I > \sum_{j \in I_\sigma} l_j.$$

For $\underline{\chi} = (\chi_1, \dots, \chi_{r+e}) = 0$ this gives

$$\sigma(\Box^{\mathcal{X}}_{\underline{l}_I})_{|z=\underline{\chi}=0} = -\prod_{i\in I} \sigma(\mathbf{D}_i)^c \tag{4.24}$$

in both cases.
Notice that for $i = 1, \dots, r$ the $\mathbf{D}_i$, and therefore also the $\sigma(\mathbf{D}_i)$ satisfy

$$\sum_{i=i}^{m} a_{ki}\mathbf{D}_i = \sum_{i=1}^{m} a_{ki} \sum_{a=1}^{r} m_{ia} z\chi_a \partial_{\chi_a} = \sum_{a=1}^{r} \left(\sum_{i=1}^{m} a_{ki} m_{ia} \right) z\chi_a \partial_{\chi_a} = 0\,. \tag{4.25}$$

If we keep Remark 2.12 in mind, identify $\mathfrak{D}_i$ with $\sigma(\mathbf{D_i})$ for $i = 1, \dots, r$ and use the relations (4.21), (4.25), (4.24) we see that the dimension of the reduced ring $(S_{|z=\underline{\chi}=0})^{red}$ satisfies the bound:

$$\dim_{\mathbb{C}} \left((S_{|z=\underline{\chi}=0})^{red} \right) \leq H^*(X(\Sigma), \mathbb{C}) \cdot \prod_{j=m+1}^{m+e} l_j$$

where here we denote by l_j the j-th component of $\underline{l}_{C_j}$.
In conclusion, this shows that the variety S over $z = \underline{\chi} = 0$ is zero dimensional. This finishes the proof.

□

Proposition 4.26. *Let ${}_0^{\circ}\mathcal{QM}_A^{log,\mathcal{X}}$ be the $\mathcal{R}(logD)$-module defined above. We have the following isomorphism of finite-dimensional commutative algebras:*

$${}_0^{\circ}\mathcal{QM}_A^{log,\mathcal{X}}{}_{|z=\underline{\chi}=0} \simeq H^*_{orb}(\mathcal{X}, \mathbb{C})\,.$$

Proof. Let $l \in \mathbb{L}$ be a cone relation. The corresponding box operator in the limit $z = \underline{\chi} = 0$ is equal to

$$(\Box^{\mathcal{X}}_{\underline{l}})_{|z=\underline{\chi}=0} := \prod_{\substack{i=1 \\ l_i<0}}^{m+e} \mathbf{D}_i^{-l_i} - \prod_{\substack{i=1 \\ l_i>0}}^{m+e} \mathbf{D}_i^{l_i} \tag{4.27}$$

where we have used the fact that for a cone relation $\underline{l}$ we have $p_a(\underline{l}) = 0$ for $a = 1, \dots, r$.

Now suppose that $I \subset \{1, \dots, m+e\}$ is a generalized primitive collection and consider a primitive relation $\underline{l}_I$:

$$\sum_{i\in I} a_i - \sum_{\substack{j=1 \\ a_j\in\sigma_I}}^{m+e} l_j a_j = 0\,.$$

where σ_I is the unique minimal cone containing $\sum_{i\in I} a_i$. Notice that we have $p_a(\underline{l}_I) \geq 0$ for $a = 1, \dots, r$ which can be shown similarly to (4.23).
We now claim that there exists an $a \in 1, \dots, r$ such that $p_a(\underline{l}_I) > 0$. Notice that the kernel of the map

$$\begin{aligned} \mathbb{L} &\longrightarrow \mathbb{Z}^r\,, \\ \underline{l} &\mapsto (p_1(\underline{l}), \dots, p_r(\underline{l}))\,. \end{aligned}$$

is e-dimensional, since $p_1, \dots, p_r$ is part of a basis of $\mathrm{Pic}^e(X)$. Because the p_a vanish on all cone relations for $a = 1, \dots, r$ (cf. (4.20) and the space of cone relations has rank e, the claim follows by dimensional reasons. We therefore get

$$(\Box^{\mathcal{X}}_{\underline{l}_I})_{|z=\underline{\chi}=0} = -\prod_{i\in I} \mathbf{D}_i\,. \tag{4.28}$$

Using Lemma 2.11 and the formulas (4.25), (4.27) and (4.28) we get the following surjective map

$$H^*_{orb}(\mathcal{X},\mathbb{C}) \twoheadrightarrow {}_0\mathring{\mathcal{Q}}\mathcal{M}_A^{log,\mathcal{X}}{}_{|z=\underline{\chi}=0}\,, \tag{4.29}$$

by sending $\mathfrak{D}_i$ to $\mathbf{D}_i$.

Notice that ${}_0\mathring{\mathcal{Q}}\mathcal{M}_A^{log,\mathcal{X}}$ is coherent by the theorem above and its generic rank is equal to μ by Proposition 3.11. Since $H^*_{orb}(\mathcal{X},\mathbb{C})$ is also μ-dimensional and the dimension of the fibers of a coherent sheaf is upper-semi-continuous, we conclude that the map above is an isomorphism. □

Corollary 4.30. *The $\mathcal{O}_{\mathbb{C}_z\times\mathcal{U}_\mathcal{X}}$-module ${}_0\mathring{\mathcal{Q}}\mathcal{M}_A^{log,\mathcal{X}}$ is locally free of rank μ.*

Proof. Since $D_\mathcal{X}\subset\mathcal{M}_\mathcal{X}$ is a normal crossing divisor it carries a natural stratification $\{S_i\}_{i\in I}$ by smooth subvarieties. The restriction of ${}_0\mathring{\mathcal{Q}}\mathcal{M}_A^{log,\mathcal{X}}$ to $\mathbb{C}_z^*\times S_i$ is equipped with a $\mathcal{D}_{\mathbb{C}_z^*\times S_i}$-module structure, so that it must be locally free. Since each stratum contains $p_\mathcal{X}$ in its closure the claim follows again by semi-continuity of the dimension of the fibers of a coherent sheaf and from Proposition 4.26 above. □

The $\mathcal{R}(logD)$-module structure on ${}_0\mathring{\mathcal{Q}}\mathcal{M}_A^{log,\mathcal{X}}$ induces a flat meromorphic connection denoted by ∇. Let $\mathcal{E}_\mathcal{X}$ be the restriction $({}_0\mathring{\mathcal{Q}}\mathcal{M}_A^{log,\mathcal{X}})_{|\mathbb{C}_z\times\{\underline{\chi}=0\}}$ and let $E_\mathcal{X}=\Gamma(\mathbb{C}_z,\mathcal{E}_\mathcal{X})$ be its module of global sections.

Lemma 4.31. *There is a canonical isomorphism*

$$\alpha_\mathcal{X}:\mathcal{O}_{\mathbb{C}_z}\otimes_{\mathbb{C}} H^*_{orb}(\mathcal{X},\mathbb{C})\xrightarrow{\simeq}\mathcal{E}_\mathcal{X}\,.$$

It comes equipped with a connection

$$\nabla^{res,\underline{\chi}}:\mathcal{E}_\mathcal{X}\longrightarrow\mathcal{E}_\mathcal{X}\otimes z^{-2}\Omega^1_{\mathbb{C}_z}$$

induced by the residue connection of ∇. Let $\pi_\mathcal{X}:{}_0^\circ QM_A^{log,\mathcal{X}}\to E_\mathcal{X}$ be the canonical projection. Set $F_\mathcal{X}:=\pi_\mathcal{X}(\mathbb{C}[z\chi_1\partial_{\chi_1},\dots,z\chi_r\partial_{\chi_r},z\partial_{\chi_{r+1}},\dots,z\partial_{\chi_{r+e}}]$ and denote by $\mathcal{F}_\mathcal{X}\subset\mathcal{E}_\mathcal{X}$ the corresponding sheaf a $\mathbb{C}$-vector spaces. Then $\alpha_\mathcal{X}(1\otimes H^*_{orb}(\mathcal{X},\mathbb{C}))=\mathcal{F}_\mathcal{X}$. The connection operator $\nabla^{res,\underline{\chi}}_{\partial_z}$ sends $\mathcal{F}_\mathcal{X}$ into $z^{-2}\mathcal{F}_\mathcal{X}\oplus z^{-1}\mathcal{F}_\mathcal{X}$.

Proof. Recall that $E_\mathcal{X}$ is a quotient of $\mathbb{C}[z,z\chi_1\partial_{\chi_1},\dots z\chi_r\partial_{\chi_r},z\partial_{\chi_{r+1}},\dots,z\partial_{\chi_{r+e}}]$ and $E_\mathcal{X}/zE_\mathcal{X}$ is canonically isomorphic to $H^*_{orb}(\mathcal{X},\mathbb{C})$. Denote by $w_1,\dots,w_\mu$ a basis of $H^*_{orb}(\mathcal{X},\mathbb{C})$ which can be represented as monomials $\underline{s}:=(s_1,\dots,s_\mu)$ in $\mathbb{C}[z\chi_1\partial_{\chi_1},\dots z\chi_r\partial_{\chi_r},z\partial_{\chi_{r+1}},\dots,z\partial_{\chi_{r+e}}]$ of degree $d_1,\dots,d_\mu$. Denote by $(E_\mathcal{X})_{(0)}$ the localization of $E_\mathcal{X}$ at 0. By Nakayama's lemma the basis $s_1,\dots,s_\mu$ lifts to a basis in $(E_\mathcal{X})_{(0)}$ and hence provides a basis in a Zariski open neighborhood of $0\in\mathbb{C}_z$. Since the (s_i) are global sections we have to show that they are nowhere vanishing. From the presentation of ${}_0\mathring{\mathcal{Q}}\mathcal{M}_A^{log,\mathcal{X}}$ we see that

$$\begin{aligned}
&(z^2\nabla^{res,\underline{\chi}}_{\partial_z})\prod_{a=1}^{r}(z\chi_a\partial_{\chi_a})^{k_a}\cdot\prod_{b=r+1}^{r+e}(z\partial_{\chi_b})^{k_b}\\
=&(z^2\partial_z)\prod_{a=1}^{r}(z\chi_a\partial_{\chi_a})^{k_a}\cdot\prod_{b=r+1}^{r+e}(z\partial_{\chi_b})^{k_b}\\
=&\prod_{a=1}^{r}(z\chi_a\partial_{\chi_a})^{k_a}\cdot\prod_{b=r+1}^{r+e}(z\partial_{\chi_b})^{k_b}\cdot\left((z^2\partial_z)+\sum_{c=1}^{r+e}k_c\cdot z\right)\\
=&\left(-\sum_{a=1}^{r+e}\sum_{i=1}^{m}m_{ia}z\chi_a\partial_{\chi_a}+z\cdot\left(\sum_{a=1}^{r}k_a+\sum_{b=r+1}^{r+e}k_b\cdot\left(1-\sum_{i=1}^{m+e}m_{ib}\right)\right)\right)\prod_{a=1}^{r}(z\chi_a\partial_{\chi_a})^{k_a}\cdot\prod_{b=r+1}^{r+e}(z\partial_{\chi_b})^{k_b}\,.
\end{aligned}$$

Hence, we have

$$(z^2\nabla^{res,\underline{\chi}}_{\partial_z})(\underline{s}) = \underline{s}\cdot(A_0 + zA_\infty) \tag{4.32}$$

where $A_0, A_\infty \in M(\mu\times\mu,\mathbb{C})$ and A_∞ is a diagonal matrix with entries $d_1,\dots,d_\mu$. Since the connection has no singularities in $\mathbb{C}^*_z$ we conclude that $\underline{s}$ is nowhere vanishing, hence is a $\mathbb{C}[z]$-basis of $E_{\mathcal{X}}$. This contruction gives the isomorphism $\alpha_{\mathcal{X}}$ which is of course independent of the choice of the basis $\underline{s}$. □

Remark 4.33. Notice the the entries $d_1,\dots,d_\mu$ of the matrix A_∞ are the (algebraic) degrees of the cohomology classes corresponding to the operators $\prod_{a=1}^r(z\chi_a\partial_{\chi_a})^{k_a}\cdot\prod_{b=r+1}^{r+e}(z\partial_{\chi_b})^{k_b}$. For the first factor this is clear because elements in the untwisted sector have (algebraic) degree one. For the second factor we recall that the m_ia are matrix entries of the map $\mathfrak{m}:\mathrm{Pic}^e(X)^\star\otimes\mathbb{Q}\longrightarrow\mathbb{Z}^n\otimes\mathbb{Q}$ with respect to the basis $e_1\dots,e_n$ and the dual basis of $p_1,\dots,p_r$. It follows from Lemma 2.15 that the elements $l_1,\dots,l_e$ defined in (2.14) are part of the dual basis of $p_1,\dots,p_{r+e}$, namely $l_i = p^*_{r+i}$. Hence

$$\sum_{i=1}^{m+e} m_{ib} = 1 - \sum_{i\in I_\sigma} r_{ki} \qquad \text{where} \qquad b = r+k$$

Since $\sum_{i\in I_\sigma} r_{ki}$ is the (algebraic) degree of $\mathfrak{D}_{m+k}$ the claim follows.

Lemma 4.34. *Write $\mathcal{QM}^{log,\mathcal{X}}_A$ for the restriction $({}_0\mathcal{QM}^{log,\mathcal{X}}_A)_{|\mathbb{C}^*_z\times\mathcal{U}_{\mathcal{X}}}$. Then for any $a\in\{1,\dots,r\}$ the residue endomorphisms*

$$z\chi_a\partial_{\chi_a} \in \mathcal{E}nd_{\mathcal{O}_{\mathbb{C}^*_z}}((\mathcal{QM}^{log,\mathcal{X}}_A)_{|\mathbb{C}^*_z\times\{\underline{\chi}=0\}})$$

are nilpotent.

Proof. Under the identification of $({}_0\mathring{\mathcal{Q}}\mathcal{M}^{log,\mathcal{X}}_A)_{z=\underline{\chi}=0}$ with $H^*_{orb}(X,\mathbb{C})$ the action of the operator $z\chi_a\partial_{\chi_a}$ corresponds to the cup product with $\overline{\mathfrak{D}}_a$. Hence the class of $z\chi_a\partial_{\chi_a}$ is nilpotent in $End_{\mathbb{C}}(({}_0\mathring{\mathcal{Q}}\mathcal{M}^{log,\mathcal{X}}_A)_{|z=\underline{\chi}=0})$. On the other hand, the class of $z\chi_a\partial_{\chi_a}$ gives rise to a well-defined element of $\mathcal{E}nd_{\mathcal{O}_z}\left(({}_0\mathring{\mathcal{Q}}\mathcal{M}^{log,\mathcal{X}}_A)_{\underline{\chi}=0}\right)$, which is flat on $\mathbb{C}^*_z$ with respect to the residue connection. Its eigenvalues are algebraic functions on $\mathbb{C}_z$ which are constant on $\mathbb{C}^*_z$ and take the value zero at the origin. This implies that the eigenvalues are zero over all of $\mathbb{C}_z$, hence the residue endomorphisms are nilpotent as required. □

Denote by $D_{\mathcal{X}}$ the reduced normal-crossing divisor in $\mathcal{U}_{\mathcal{X}}$ given by $\{\chi_1\cdot\ldots\cdot\chi_r=0\}$ and denote its components by D_a for $a=1,\dots,r$.

Proposition 4.35. *There is a non-degenerate flat $(-1)^d$-symmetric pairing*

$$P:\ {}_0\mathring{\mathcal{Q}}\mathcal{M}^{log,\mathcal{X}}_A\otimes\iota^*\,{}_0\mathring{\mathcal{Q}}\mathcal{M}^{log,\mathcal{X}}_A\longrightarrow z^d\mathcal{O}_{\mathbb{C}_z\times\mathcal{U}_{\mathcal{X}}}$$

i.e. P is flat on $\mathbb{C}^*_z\times(\mathcal{U}_{\mathcal{X}}\setminus D_{\mathcal{X}})$ and the induced pairings

$$P:({}_0\mathring{\mathcal{Q}}\mathcal{M}^{log,\mathcal{X}}_A)_{|z=0}\otimes\iota^*({}_0\mathring{\mathcal{Q}}\mathcal{M}^{log,\mathcal{X}}_A)_{|z=0}\longrightarrow z^d\mathcal{O}_{\mathcal{U}}$$

and $P:({}_0\mathring{\mathcal{Q}}\mathcal{M}^{log,\mathcal{X}}_A)_{|D_a}\otimes\iota^*({}_0\mathring{\mathcal{Q}}\mathcal{M}^{log,\mathcal{X}}_A)_{D_a}\longrightarrow z^d\mathcal{O}_{\mathbb{C}_z\times D_a}$ are non-degenerate.

Proof. Denote by M_a, $a=1,\dots,r$, the unipotent monodromy automorphism corresponding to a counter-clockwise loop around the divisor $\mathbb{C}^*\times D_a$ and by M_z the monodromy automorphisms corresponding to a counter-clockwise loop around $z=0$. Denote by $M_{z,u}$ resp. $M_{z,s}$ their unipotent and semi-simple part. We set $N_a=\log M_a$ and $N_z=\log M_{z,u}$. Denote by H^∞ the space of multi-valued flat sections on which the monodromy operators M_a and M_z act. Let $f_1,\dots,f_\mu$ be a basis of flat multi-valued sections of $\mathcal{QM}_{A|\mathbb{C}^*_z\times\mathcal{U}_{\mathcal{X}}}$ which is adapted to the

generalized eigenspace decomposition of the space H^∞ with respect to the automorphisms M_z and M_a. We define the single-valued sections

$$\widetilde{s}_i = e^{-\log z(\rho_i + \frac{N_z}{2\pi i})} \prod_{a=1}^{r} e^{-\log \chi_a \frac{N_a}{2\pi i}} f_i$$

for some ρ_i such that $e^{2\pi i \rho_i}$ is the generalized eigenvalue of f_i with respect to M_z. These sections provide a basis for ${}_0^{\circ}\mathcal{QM}_A^{\log,\mathcal{X}}{}_{|\mathbb{C}_z^* \times \mathcal{U}_\mathcal{X}}$. Notice that $P(\widetilde{s}_i, \widetilde{s}_j)$ is holomorphic on $\mathbb{C}_z^* \times (\mathcal{U}_\mathcal{X} \setminus D_\mathcal{X})$. By the flatness of P we get that $P(\widetilde{s}_i, \widetilde{s}_j) = z^{-\rho_i - \rho_j} P(f_i, f_j)$ which shows that $P(\widetilde{s}_i, \widetilde{s}_j)$ extends over $\mathbb{C}_z^* \times \mathcal{U}_\mathcal{X}$ and is non-degenerate. Together with Proposition 4.11 we get a non-degenerate pairing on the restriction ${}_0^{\circ}\mathcal{QM}_A^{log,\mathcal{X}}{}_{|(\mathbb{C}_z \times \mathcal{U}_\mathcal{X}) \setminus \{0\} \times D_\mathcal{X}}$. Since, $\{0\} \times D_\mathcal{X}$ has codimension two in $\mathbb{C}_z \times \mathcal{U}_\mathcal{X}$, P extends to a non-degenerate pairing on ${}_0^{\circ}\mathcal{QM}_A^{log,\mathcal{X}}$. □

Lemma 4.36. *The induced pairing $P : E_\mathcal{X} \otimes \iota^* E_\mathcal{X} \to z^d\mathbb{C}[z]$, restricts to a pairing $P : F_\mathcal{X} \times F_\mathcal{X} \to z^d\mathbb{C}$. The pairing $z^{-d}P$ on $F_\mathcal{X}$ coincides under the identification made in Lemma 4.31 with the orbifold Poincaré pairing on $H^*_{orb}(X, \mathbb{C})$ up to a non-zero constant.*

Proof. Let $\{d_0, \dots, d_t\} = \{q \in \mathbb{Q} \mid H^{2q}_{orb}(\mathcal{X}, \mathbb{C}) \neq 0\}$ where $d_i < d_j$ for $i < j$. Set $r_k = dim H^{2d_k}_{orb}(X, \mathbb{C})$ and notice that $d_0 = 0$, $d_t = d$ and $r_0 = r_t = 1$. Choose a homogeneous basis

$$w_{1,d_0}, w_{1,d_1}, \dots, w_{r_1,d_1}, \dots, w_{1,d_{t-1}}, \dots, w_{r_{t-1},d_{t-1}}, w_{1,d_t}$$

where $w_{i,d_k} \in H^{2d_k}_{orb}(X, \mathbb{C})$. Denote by $s_{1,d_0}, s_{1,d_1}, \dots, s_{r_1,d_1}, \dots, s_{1,d_{t-1}}, \dots, s_{r_{t-1},d_{t-1}}, s_{1,d_t}$ the corresponding sections of $\mathcal{E}_\mathcal{X}$ under the isomorphism $\alpha_\mathcal{X}$ of Lemma 4.31. By Lemma 4.34 and a construction similar to the one in the proof of Proposition 4.35 we can find sections $\tilde{s}_{1,d_0}, \tilde{s}_{1,d_1}, \dots, \tilde{s}_{r_1,d_1}, \dots, \tilde{s}_{1,d_{t-1}}, \dots, \tilde{s}_{r_{t-1},d_{t-1}}, \tilde{s}_{1,d_t}$ which satisfy $(\tilde{s}_{i,d_k})_{|\mathbb{C}_z \times \{\underline{\chi}=0\}} = s_{i,d_k}$ and

$$\nabla_{z\chi_a\partial_{\chi_a}} \left(\prod_{c=1}^{r} e^{\log \chi_c \frac{N_a}{2\pi i}} \tilde{s}_{i,d_k} \right) = 0 \qquad \text{for} \quad a = 1, \dots, r\,,$$
$$\nabla_{z\partial_{\chi_b}} \left(\prod_{c=1}^{r} e^{\log \chi_c \frac{N_a}{2\pi i}} \tilde{s}_{i,d_k} \right) = 0 \qquad \text{for} \quad b = r+1, \dots, r+e\,.$$

From the definition of the sections $\tilde{s}_{i,d_k}$ and the flatness of P then follows

$$P(\tilde{s}_{i,d_k}, \tilde{s}_{j,d_l})(z, \underline{\chi}) = P(s_{i,d_k}, s_{j,d_l})(z)$$

and therefore

$$0 = z\chi_a\partial_{\chi_a} P(\tilde{s}_{i,d_k}, \tilde{s}_{j,d_l}) = P(\nabla_{z\chi_a\partial_{\chi_a}} \tilde{s}_{i,d_k}, \tilde{s}_{j,d_l}) - P(\tilde{s}_{i,d_k}, \nabla_{z\chi_a\partial_{\chi_a}} \tilde{s}_{j,d_l})\,,$$
$$0 = z\partial_{\chi_b} P(\tilde{s}_{i,d_k}, \tilde{s}_{j,d_l}) = P(\nabla_{z\partial_{\chi_b}} \tilde{s}_{i,d_k}, \tilde{s}_{j,d_l}) - P(\tilde{s}_{i,d_k}, \nabla_{z\partial_{\chi_b}} \tilde{s}_{j,d_l})\,.$$

By continuity this holds on $\mathbb{C}_z \times \{\underline{\chi} = 0\}$. This shows the multiplication invariance of the corresponding pairing on $E_X \simeq \mathbb{C}[z] \otimes_\mathbb{C} H^*_{orb}(\mathcal{X}, \mathbb{C})$. It follows from equation 4.32 that

$$z\nabla^{res,\underline{q}}_{\partial_z}(s_{i,d_k}) = d_k \cdot s_{i,d_k} + \frac{1}{z} \sum_{m=1}^{r_{k+1}} \Theta_{m,i,k} s_{m,d_k+1} \quad \text{for} \quad k < t\,,$$
$$z\nabla^{res,\underline{q}}_{\partial_z}(s_{1,d_t}) = d \cdot s_{1,d_t}\,,$$

where $\Theta_{m,i,k} := (\check{A}_0)_{u,v}$ with $u = m + \sum_{l=1}^{k} r_l$ and $v = i + \sum_{l=1}^{k-1} r_l$ and $\check{A}_0$ is the matrix with respect to the basis $s_{1,0}, \dots, s_{1,d_t}$ of the endomorphism $-c_1(\mathcal{X})\cup$. Since the pairing is

multiplication invariant it is clear that $P(s_{i,d_k}, s_{j,d_l}) = 0$ for $d_k + d_l > d$ since $H^{2(d_k+d_l)}_{orb}(\mathcal{X}, \mathbb{C})$ is zero in this case. For $d_k + d_l \leq d$ we compute

$$
\begin{aligned}
(4.37)\quad & z\partial_z P(s_{i,d_k}, s_{j,d_l}) \\
=& P(d_k \cdot s_{i,d_k} + \frac{1}{z}\sum_{m=1}^{r_{k+1}} \Theta_{m,i,k} s_{i,d_k+1}, s_{j,d_l}) + P(s_{i,d_k}, d_l \cdot s_{j,d_l} + \frac{1}{z}\sum_{m=1}^{r_{l+1}} \Theta_{m,j,l} s_{j,d_l+1}) \\
=& (d_k + d_l)P(s_{i,d_k}, s_{j,d_l}) + \frac{1}{z}\left(P(\sum_{m=1}^{r_{k+1}} \Theta_{m,i,k} s_{i,d_k+1}, s_{j,d_l}) + P(s_{i,d_k}, \sum_{m=1}^{r_{l+1}} \Theta_{m,j,l} s_{j,d_l+1}) \right).
\end{aligned}
$$

In the case $d_k + d_l = d$ this amounts to

$$
(z\partial_z - d)P(s_{i,d_k}, s_{j,d_l}) = -\frac{1}{z}\left(P(\sum_{m=1}^{r_{k+1}} \Theta_{m,i,k} s_{i,d_k+1}, s_{j,d_l}) + P(s_{i,d_k}, \sum_{m=1}^{r_{l+1}} \Theta_{m,j,l} s_{j,d_l+1}) \right).
$$

Hence $P(s_{i,d_k}, s_{j,d_l}) \in z^d\mathbb{C}$. For $d_j + d_k < d$ we get by induction

$$
(z\partial_z - (d_k+d_l))P(s_{i,d_k}, s_{j,d_l}) = -\frac{1}{z}\left(P(\sum_{m=1}^{r_{k+1}} \Theta_{m,i,k} s_{i,d_k+1}, s_{j,d_l}) + P(s_{i,d_k}, \sum_{m=1}^{r_{l+1}} \Theta_{m,j,l} s_{j,d_l+1}) \right) \in z^{d_k+d_l}\mathbb{C}
$$

Since $P(s_{i,d_k}, s_{j,d_l}) \in z^d\mathbb{C}[z]$ we conclude that $P(s_{i,d_k}, s_{j,d_l}) = 0$ for $d_k + d_l < d$.
It remains to show that the pairing $z^{-n}P$ coincides, under the isomorphism $\alpha : 1 \otimes H^*_{orb}(X, \mathbb{C}) \to F$ and possibly up to a non-zero constant, with the Poincaré pairing on the cohomology algebra. First notice that by construction, $z^{-n}P$, seen as defined on $H^*_{orb}(X, \mathbb{C})$ is again multiplication invariant. It suffices now to show that $P(1, a)$ equals the value of the Poincaré pairing on 1 and a. But as we have seen above, $P(1, a)$ can only be non-zero if $a \in H^{2n}_{orb}(X, \mathbb{C})$. Since $dim H^{2n}_{orb}(X, \mathbb{C}) = 1$, the P on $H^*_{orb}(X, \mathbb{C})$ is entirely determined by the non-zero complex number $P(s_{1,d_0}, s_{1,d_t})$.

□

Proposition 4.38. *Consider the $\mathcal{O}_{\mathbb{C}_z}$-module $\mathcal{E}_\mathcal{X}$ with the connection $\nabla^{res,\underline{\chi}}$ and the subsheaf $\mathcal{F}_\mathcal{X} \subset \mathcal{E}_\mathcal{X}$ from lemma 4.31.*

1. *Let $\widehat{\mathcal{E}}_\mathcal{X} := \mathcal{O}_{\mathbb{P}^1_z \times \{\underline{0}\}} \otimes_{\mathbb{C}} F_\mathcal{X}$ be an extension of $\mathcal{E}_\mathcal{X}$ to a trivial $\mathbb{P}^1$-bundle. Then the connection $\nabla^{res,\underline{\chi}}$ has a logarithmic pole at $z = \infty$ with spectrum (i.e., set of residue eigenvalues) equal to the (algebraic) degrees of the cohomology classes of $H^*_{orb}(X, \mathbb{C})$.*
2. *The pairing P on $\mathcal{E}_\mathcal{X}$ extends to a non-degenerate pairing $P : \widehat{\mathcal{E}}_\mathcal{X} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \iota^*\widehat{\mathcal{E}}_\mathcal{X} \to \mathcal{O}_{\mathbb{P}^1}(-d, d)$, where $\mathcal{O}_{\mathbb{P}^1}(a, b)$ is the subsheaf of $\mathcal{O}_{\mathbb{P}^1}(*\{0, \infty\})$ consisting of meromorphic functions with a pole of order a at 0 and a pole of order b at ∞.*

Proof. The formula 4.32 shows that the connection $\nabla^{res,\underline{\chi}}$ has a logarithmic pole at $z = \infty$ and Remark 4.33 shows that the residue eigenvalues are equal to the (algebraic) degrees of the cohomology classes of $H^*_{orb}(\mathcal{X})$. This shows the first point. The second point follows from Proposition 4.35 and the definition of $\hat{\mathcal{E}}_\mathcal{X}$.

□

Set $j_{\frac{1}{z}} : \mathbb{C}^*_z \to \mathbb{P}^1_z \setminus \{0\}$ and $E^\infty := \psi_{\frac{1}{z}} j_{\frac{1}{z},!}(\mathcal{E}^{an}_\mathcal{X})^{\nabla^{res,\underline{\chi}}}_{|\mathbb{C}^*_z}$ (where $\psi_{\frac{1}{z}}$ is the nearby cycle functor at $z = \infty$). It is known (cf. e.g. [Her02, Lemma 7.6, Lemma 8.14] that there is a correspondence between logarithmic extensions of flat bundles and filtrations on the corresponding local system of flat sections. With respect to the connection $(z^2\nabla^{res,\underline{\chi}}_{\partial_z})(\underline{s}) = \underline{s} \cdot (A_0 + zA_\infty)$, the isomorphism

$$
(4.39)\qquad F_\mathcal{X} \xrightarrow{\simeq} E^\infty
$$

is given by multiplication with $z^{-A_\infty}z^{-A_0}$. Hence, this correspondence gives a filtration, denoted by $F_\bullet$ on $F_{\mathcal{X}}$.

Lemma 4.40. *The filtration $F_\bullet$ on $F_{\mathcal{X}}$ is given by*

$$F_p = \sum_{|k|\geq -p} \mathbb{C}\left((z\chi_1\partial_{\chi_1})^{k_1}\cdot\ldots\cdot(z\chi_r\partial_{\chi_r})^{k_r}\cdot(z\partial_{\chi_{r+1}})^{l_1}\cdot\ldots(z\partial_{\chi_{r+e}})^{l_e}\right).$$

The residue endomorphism N_a of $\mathcal{QM}_A^{log,\mathcal{X}}$ along $\mathbb{C}_z^*\times D_a$ acts on E^∞ and satisfies $N_aF_\bullet\subset F_{\bullet-1}$.

Proof. The first claim follows from the identification of $H^*_{orb}(\mathcal{X})$ with $F_{\mathcal{X}}$ and the computation of the residue connection $(z^2\nabla^{res,\underline{\chi}}_{\partial_z})(\underline{s})$ in Lemma 4.31. The second claim is immediate since the residue endomorphism is induce by left multiplication with $z\chi_a\partial_{\chi_a}$. □

The next result gives an extension of ${}_0\mathcal{QM}_A^{log}$ to a family of trivial $\mathbb{P}^1$-bundles, possibly after restricting to a smaller analytic open subset inside U^{an}. Set $r := \inf\{|\underline{\chi}| : \underline{\chi}\notin\overline{\mathcal{M}}^\circ_{\mathcal{X}}\}$ and let $B := B_r(0)\subset\mathcal{U}_{\mathcal{X}}^{an}$ the open ball with radius r. Notice that $r>0$ by Lemma 4.15.

Proposition 4.41. *There is an analytic open subset $V_{\mathcal{X}}\subset\mathcal{U}_{\mathcal{X}}^{an}$ still containing the point $p_{\mathcal{X}}$ and a trivial holomorphic bundle ${}_0\widehat{\mathcal{QM}}_A^{log,\mathcal{X}}\to\mathbb{P}^1_z\times V_{\mathcal{X}}$*
such that

1. $({}_0\widehat{\mathcal{QM}}_A^{log,\mathcal{X}})_{|\mathbb{C}_z\times V_{\mathcal{X}}}\cong(({}_0\mathring{\mathcal{QM}}_A^{log,\mathcal{X}})^{an})_{|\mathbb{C}_z\times V_{\mathcal{X}}}$,
2. $({}_0\widehat{\mathcal{QM}}_A^{log,\mathcal{X}})_{|\mathbb{P}^1_z\times\{0\}}\cong(\widehat{\mathcal{E}}_{\mathcal{X}})^{an}$,
3. *The connection ∇ has a logarithmic pole along $\widehat{D}_{\mathcal{X}}$ on ${}_0\widehat{\mathcal{QM}}_A^{log,\mathcal{X}}$, where $\widehat{D}_{\mathcal{X}}$ is the normal crossing divisor $(\{z=\infty\}\cup\bigcup_{a=1}^r\{\chi_a=0\})\cap\mathbb{P}^1_z\times V_{\mathcal{X}}$,*
4. *The given pairings $P:{}_0\mathring{\mathcal{QM}}_A^{log,\mathcal{X}}\otimes\iota^*{}_0\mathring{\mathcal{QM}}_A^{log,\mathcal{X}}\to z^d\mathcal{O}_{\mathbb{C}_z\times\mathcal{U}^\circ_{\mathcal{X}}}$ and $P:\widehat{\mathcal{E}}_{\mathcal{X}}\otimes_{\mathcal{O}_{\mathbb{P}^1_z}}\iota^*\widehat{\mathcal{E}}_{\mathcal{X}}\to\mathcal{O}_{\mathbb{P}^1_z}(-d,d)$ extend to a non-degenerate pairing*

$$P:{}_0\widehat{\mathcal{QM}}_A^{log,\mathcal{X}}\otimes_{\mathcal{O}_{\mathbb{P}^1_z\times V_{\mathcal{X}}}}\iota^*{}_0\widehat{\mathcal{QM}}_A^{log,\mathcal{X}}\to\mathcal{O}^{an}_{\mathbb{P}^1_z\times V_{\mathcal{X}}}(-d,d),$$

   where the latter sheaf is defined as in point 3. of proposition 4.38,
5. *The residue connection along $\frac{1}{z}=\tau=0$*

$$\nabla^{res,z=\infty}:{}_0\widehat{\mathcal{QM}}_A^{log,\mathcal{X}}/\tau\cdot{}_0\widehat{\mathcal{QM}}_A^{log,\mathcal{X}}\longrightarrow{}_0\widehat{\mathcal{QM}}_A^{log,\mathcal{X}}/\tau\cdot{}_0\widehat{\mathcal{QM}}_A^{log,\mathcal{X}}\otimes\Omega^{1,an}_{\{\infty\}\times V_{\mathcal{X}}}(\log(\{\infty\}\times D)).$$

   has trivial monodromy around $\{\infty\}\times D$ and the element of $1\in F\subset H^0(\mathbb{P}^1_z\times U^0,{}_0\widehat{\mathcal{QM}}_A^{log,\mathcal{X}})$ is horizontal for $\nabla^{res,z=\infty}$.

Proof. Set $\tilde{D}:=\bigcup_{a=1}^r\{\chi_a=0\}\cap B$. A logarithmic extension of $(\mathcal{QM}_A^{log,\mathcal{X}})^{an}_{|\mathbb{C}_z^*\times(B\setminus\tilde{D})}$ over $(\{z=\infty\}\times B)\cup(\mathbb{P}^1_z\setminus\{0\}\times\tilde{D})$ is given by a $\mathbb{Z}^{r+1}$-filtration on the local system $\mathcal{L}=(\mathcal{QM}_A^{log,\mathcal{X}})^{an,\nabla}_{|\mathbb{C}_z^*\times(B\setminus\tilde{D})}$, which is split iff the extension is locally free (cf. [Her02, Lemma 8.14].

We are looking for an holomorphic vector bundle $\widehat{\mathcal{QM}_A}\to(\mathbb{P}^1_z\setminus\{0\})\times B$ which should satisfy two constraints. First, $\widehat{\mathcal{QM}_A}$ should restrict to $({}_0\mathring{\mathcal{QM}}_A^{log,\mathcal{X}})^{an}_{|\mathbb{C}_z^*\times B}$ on $\mathbb{C}_z^*\times B$ and second it should restrict to $(\hat{\mathcal{E}}_{\mathcal{X}})^{an}_{|\mathbb{P}^1_z\setminus\{0\}}$ over $\mathbb{P}^1_z\setminus\{0\}\times\{\underline{\chi}=0\}$.
The $\mathbb{Z}^r$-filtration $P_\bullet$ corresponding to the extension over $\mathbb{C}_z^*\times\tilde{D}$ is trivial since its the Deligne extension due to Lemma 4.34. Let L^∞ be the space of multi-valued flat sections of $(\mathcal{QM}_A^{log,\mathcal{X}})^{an}_{|\mathbb{C}_z^*\times(B\setminus\tilde{D})}$ and let E^∞ be the space of multi-valued flat sections of $\mathcal{E}_{\mathcal{X}}^{an}$ from above. We have an isomorphism $L^\infty\to E^\infty$ which is given by multiplication with $\prod_{a=1}^r\chi_a^{\frac{N_a}{2\pi i}}$, where N_a is the logarithm of the

(uni-potent part of the) monodromy, and restriction to $\{\underline{\chi}=0\}$. This allows us to shift the filtration $F_\bullet$ on E^∞ (resp. $F_{\mathcal{X}}$) to a filtration, denoted by $F'_\bullet$, on L^∞. This gives a $\mathbb{Z}^{r+1}$-filtration $(F_\bullet, P_\bullet)$ which is split, since $P_\bullet$ is trivial. The corresponding extension $\widehat{\mathcal{QM}_A}$ has logarithmic poles along $(\{z=\infty\}\times B)\cup(\mathbb{P}^1\setminus\{0\}\times\tilde{D})$ and restricts to $({}_0\mathring{\mathcal{QM}}_A^{log,\mathcal{X}})^{an}_{|\mathbb{C}^*_z\times B}$ on $\mathbb{C}^*_z\times B$ resp. $(\hat{\mathcal{E}}_{\mathcal{X}})^{an}_{|\mathbb{P}^1_z\setminus\{0\}}$ on $\mathbb{P}^1_z\setminus\{0\}\times\{\underline{\chi}=0\}$. We therefore can glue $\widehat{\mathcal{QM}_A}$ and $({}_0\mathring{\mathcal{QM}}_A^{log,\mathcal{X}})^{an}_{|\mathbb{C}_z\times B}$ to a holomorphic bundle on $\mathbb{P}^1_z\times B$, which is trivial on on $\mathbb{P}^1_z\times\{\underline{\chi}=0\}$ since its restriction is isomorphic to $\mathcal{E}_{\mathcal{X}}$. Because triviality is an open condition there exists a subset $V_{\mathcal{X}}\subset B$ such that the restriction of the glued bundle to $\mathbb{P}^1_z\times V_{\mathcal{X}}$ is trivial. This shows the points 1. to 3. . For the fourth point notice that the flat pairing P gives rise to a pairing on L^∞ which in turn gives rise to a pairing on E^∞. The pole order property of this pairing on $\hat{\mathcal{E}}_{\mathcal{X}}$ at $z=\infty$ can be encoded by an orthogonality property of the filtration $F_\bullet$ with respect to that pairing (see e.g. [Her03, Theorem 7.17, Definition 7.18]). Hence the same property must hold for P and $F_\bullet$ seen as defined on L^∞, so we conclude $P: {}_0\widehat{\mathcal{QM}}_A^{log,\mathcal{X}}\otimes_{\mathcal{O}_{\mathbb{P}^1_z\times V_{\mathcal{X}}}}\iota^*{}_0\widehat{\mathcal{QM}}_A^{log,\mathcal{X}}\to\mathcal{O}^{an}_{\mathbb{P}^1_z\times V_{\mathcal{X}}}(-d,d)$ as required.
The last statement follows from the fact that the residue connection $\nabla^{res,z=\infty}$ defined on ${}_0\widehat{\mathcal{QM}}_A^{log,\mathcal{X}}/z^{-1}{}_0\widehat{\mathcal{QM}}_A^{log,\mathcal{X}}$ has trivial monodromy around $D_{\mathcal{X}}\cap\{z=\infty\}\times B$ if for any $a=1,\dots,r$ the nilpotent part N_a of the monodromy of $\mathcal{L}$ kills $gr_\bullet^{F'}$, i.e. $N_aF_\bullet\subset F_{\bullet-1}$. Using the identification $(L^\infty,F'_\bullet)$ with $(E^\infty,F^\bullet)$ this has been shown in Lemma 4.40. From this follows that the element 1 is a global sections over $\mathbb{P}^1_z\times V_{\mathcal{X}}$ and flat with respect to the residue connection.

□

4.3. **Frobenius structures.** We begin with a definition from [Rei09] which formalizes the structure which we obtained in Proposition 4.41.

Definition 4.42. *Let M be a complex manifold of dimension bigger or equal than one and $D\subset M$ be a simple normal crossing divisor.*

(1) *A* log-TEP(d)*-structure on M is a holomorphic vector bundle $\mathcal{H}\to\mathbb{P}^1_z\times M$ which is equipped with an integrable connection ∇ with a pole of order two along $\{0\}\times M$ and a logarithmic pole along $(\mathbb{C}_z\times D)$ and a flat, $(-1)^d$-symmetric, non-degenerate pairing $P:\mathcal{H}\otimes\iota^*\mathcal{H}\to z^d\mathcal{O}_{\mathbb{C}_z\times M}$. If D is empty we will simply denote it as a* TEP(d)*-structure.*

(2) *A* log-trTLEP(d)*-structure on M is a holomorphic vector bundle $\widehat{\mathcal{H}}\to\mathbb{P}^1_z\times M$ such that $p^*p_*\widehat{\mathcal{H}}=\widehat{\mathcal{H}}$ (where $p:\mathbb{P}^1_z\times M\twoheadrightarrow M$ is the projection) which is equipped with an integrable connection ∇ with a pole of order two along $\{0\}\times M$ and a logarithmic pole along $(\mathbb{P}^1_z\times D)\cup(\{\infty\}\times M)$ and a flat, $(-1)^d$-symmetric, non-degenerate pairing $P:\widehat{\mathcal{H}}\otimes\iota^*\widehat{\mathcal{H}}\to\mathcal{O}_{\mathbb{P}^1_z\times M}(-d,d)$. If D is empty we will simply denote it as a* trTLEP(d)*-structure.*

Here, ι is the automorphism sending (z,m) to $(-z,m)$.

Proposition 4.43. *Let $\mathcal{X}(\mathbf{\Sigma})$ be a projective toric Deligne-Mumford stack with an S-extended stacky fan $\mathbf{\Sigma}^e$ with $S=Gen(\Sigma)$ and let $W: Y\times\mathcal{M}_{\mathcal{X}}\longrightarrow\mathbb{C}\times\mathcal{M}_{\mathcal{X}}$ the corresponing Landau-Ginzburg model. Then the tuple $({}_0\widehat{\mathcal{QM}}_A^{log,\mathcal{X}},\nabla,P)$ from Proposition 4.41 is a* log-trTLEP(d)*-structure on $V_{\mathcal{X}}\subset\mathcal{M}^{an}_{\mathcal{X}}$.*

Proof. This follows from Proposition 4.41. □

The following theorem which is a combination of Proposition 1.20 and Theorem 1.22 in [Rei09] gives sufficient conditions when a given log-trTLEP(d)-structure can be unfolded to a logarithmic Frobenius manifold.

Theorem 4.44. *Let $(M,0)$ be a germ of a complex manifold and $(D,0)\subset(M,0)$ be a normal crossing divisor. Let $(\mathcal{H},0),\nabla,P)$ be a germ of a* log-trTLEP(d)*-structure on $\mathbb{P}^1\times(M,0)$.*

Suppose that there is a section $\xi \in H^0(\mathbb{P}^1 \times (M,0), \mathcal{H})$ *whose restriction to* $\{\infty\} \times (M,0)$ *is horizontal for the residue connection* $\nabla^{res} : \mathcal{H}/z^{-1}\mathcal{H} \to \mathcal{H}/z^{-1}\mathcal{H} \otimes \Omega^1_{\{\infty\}\times M}(log(\infty\} \times D))$ *and which satisfies the conditions*

- (IC) *The map* $\Theta(logD)_{|0} \to p_*\mathcal{H}_{|0}$ *induced by* $[z\nabla_\bullet](\xi) : \Theta(logD) \to p_*\mathcal{H}$ *is injective.*
- (GC) *The vector space* $p_*\mathcal{H}_{|0}$ *is generated by* ξ *and its images under iteration of the maps* $\mathcal{U}$ *and* $[z\nabla_X]$ *for any* $X \in \Theta(log)D$.
- (EC) ξ *is an eigenvector for the residue endomorphism* $\mathcal{V} \in \mathcal{E}nd_{\mathcal{O}_{\{\infty\}\times M}}(\mathcal{H}/z^{-1}\mathcal{H})$.

Then there exists a unique (up to canonical isomorphism) gern of a logarithmic Frobenius manfold on $(\tilde{M}, \tilde{D})$ *with a unique embedding* $i : M \hookrightarrow \tilde{M}$ *with* $i(M) \cap \tilde{D} = i(D)$ *and a unique isomorphism* $\mathcal{H} \to p^*\Theta_{\tilde{M}}(log\tilde{D})_{|i(M)}$ *of* log-trTLEP(d)*-structures.*

Using the theorem above we are now able to construct a logarithmic Frobenius manifold from the Landau-Ginzburg model corresponding to a projective toric Deligne-Mumford stack.

Theorem 4.45. *Let* $W : Y \times \mathcal{M}_{\mathcal{X}} \to \mathbb{C} \times \mathcal{M}_{\mathcal{X}}$ *be the Landau-Ginzburg model corresponding to a projective toric Deligne-Mumford stack. Then there exists a canonical logarithmic Frobenius manifold on* $(V_{\mathcal{X}} \times \mathbb{C}^{\mu-r}, 0)$ *with logarithmic pole along* $(D \times \mathbb{C}^{\mu-r}, 0)$.

Proof. In order to apply Theorem 4.44 to the log-trTLEP(d)-structure obtained in Proposition 4.41 we define the section ξ to be the class of 1. Because of Proposition 4.41 5. this section is flat with respect to the residue connection along $\frac{1}{z} = \tau = 0$. The conditions (IC) and (GC) follow from the identification of $({}_0^{\circ}\mathcal{QM}_A^{log,\mathcal{X}})_{|0}$ with the cohomology ring $(H^*_{orb}(X_\Sigma, \mathbb{C}), \cup)$ (cf. Proposition 4.26 and Formula 4.29), the definition of the $\mathbf{D}_i$ for $i = 1, \ldots, n$ (cf. Formula 4.18) and the representation of $H^*_{orb}(\mathcal{X}_\Sigma, \mathbb{C})$ in Lemma 2.11. The condition (EC) follows from Proposition 4.38 1. □

5. Orbifold Quantum cohomology

In this section we review some constructions from orbifold quantum cohomology in a very general context (In particular, we do not assume any positivity assumptions in this section.
Let $\mathcal{X}$ be a smooth proper Deligne-Mumford stack over $\mathbb{C}$. The inertia stack of $\mathcal{X}$ is defined by

$$I\mathcal{X} := \mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X}$$

with respect to the diagonal morphism $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$. A geometric point on $I\mathcal{X}$ is given by a geometric point $x \in \mathcal{X}$ and an element $g \in Aut(\mathcal{X})$ of the isotropy group. We call g the stabilizer of $(x, g) \in I\mathcal{X}$. The inertia stack is a smooth Deligne-Mumford stack but different components will in general have different dimensions. Let T be the the index set of the components of $I\mathcal{X}$. Let $0 \in T$ be the distinguished element corresponding to the trivial stabilizer. We thus have

$$\mathcal{I} = \bigsqcup_{v\in T} \mathcal{X}_v \,.$$

The orbifold cohomology of $\mathcal{X}$ is defined, as a vector space, by $H^*_{orb}(\mathcal{X}, \mathbb{C}) := H^*(I\mathcal{X}, \mathbb{C})$, hence we have

$$H^*_{orb}(\mathcal{X}, \mathbb{C}) = H^*(\mathcal{X}, \mathbb{C}) \oplus \bigoplus_{v\in T'} H^*(\mathcal{X}_v, \mathbb{C})$$

where $T' := T \setminus \{0\}$ is the index set of the twisted sectors.
In order to define a grading on the orbifold cohomology , we associate to any $v \in T$ a rational number called the age of $\mathcal{X}_v$.
The genus zero Gromov-Witten invariants with descendants are defined by

$$\langle \alpha_1\psi_1^{k_1}, \ldots, \alpha_l\psi_l^{k_l}\rangle_{0,l,d} := \int_{[\overline{\mathcal{M}}_{0,l}(\mathcal{X},d)]^{vir}} \prod_{i=1}^{l} ev_i^*(\alpha_i)\psi_i^{k_i}$$

where $\alpha_i \in H^*_{orb}(\mathcal{X})$, $d \in H_2(X,\mathbb{Z})$, k_i is a non-negative integer, $\overline{\mathcal{M}}_{0,l}(\mathcal{X},d)$ is the moduli stack of genus zero, l-pointed stable maps to $\mathcal{X}$ of degree d, $[\overline{\mathcal{M}}_{0,l}(\mathcal{X},d)]^{vir}$ is the virtual fundamental class, ev_i is the evaluation map at the i-th marked point

$$ev_i : \overline{\mathcal{M}}_{0,l}(\mathcal{X},d) \to I\mathcal{X}$$

and $\psi_i = c_1(L_i)$ where L_i is the line bundle over $\overline{\mathcal{M}}_{0,l}(\mathcal{X},d)$ whose fiber at a stable map is the cotagent space of the coarse curve at the i-th marked point. The correlator $\langle \alpha_1\psi_1^{k_1},\dots,\alpha_l\psi_l^{k_l}\rangle_{0,l,d}$ is non-zero only if $d \in \mathrm{Eff}_{\mathcal{X}} \subset H_2(X;\mathbb{Z})$, where $\mathrm{Eff}_{\mathcal{X}}$ is the semigroup generated by effective stable maps.
We choose a homogeneous basis $T_0,\dots,T_\mu$ of $H^*_{orb}(\mathcal{X})$, where $T_0 = 1 \in H^0(\mathcal{X},\mathbb{C})$, $T_1,\dots,T_r \in H^2(\mathcal{X})$ and $T_i \in \bigoplus_{k\neq 0,2} H^k(\mathcal{X}) \oplus \bigoplus_{v\in T'} H^*(\mathcal{X}_v)$. We denote by $T^0,\dots,T^\mu$ the basis of $H^*(\mathcal{X})$ which is dual with respect to the orbifold Poincaré pairing.
Let $\alpha,\beta,\tau \in H^*_{orb}(\mathcal{X},\mathbb{C})$ and write $\tau = \tau' + \delta$ where $\delta \in H^2(\mathcal{X},\mathbb{C})$ and $\tau' \in \bigoplus_{k\neq 2} H^k(\mathcal{X}) \oplus \bigoplus_{v\in T'} H^*(\mathcal{X}_v)$. We define the the big orbifold quantum product $\circ_\tau$ as the formal family of commutative and associative products on $H^*_{orb}(\mathcal{X}) \otimes \mathbb{C}[\![\mathrm{Eff}_{\mathcal{X}}]\!]$:

$$\begin{aligned}\alpha \circ \gamma &:= \sum_{d\in \mathrm{Eff}_{\mathcal{X}}} \sum_{l,k\geq 0} \frac{1}{l!} \langle \alpha,\gamma,\underbrace{\tau,\dots,\tau}_{l-\text{times}},T_k\rangle_{0,l+3,d}\, T^k Q^d \\ &= \sum_{d\in \mathrm{Eff}_{\mathcal{X}}} \sum_{l,k\geq 0} \frac{e^{\delta(d)}}{l!} \langle \alpha,\gamma,\underbrace{\tau',\dots,\tau'}_{l-\text{times}},T_k\rangle_{0,l+3,d}\, T^k Q^d\end{aligned}$$

where the last equality follows from the divisor axiom. The Novikov ring $\mathrm{Eff}_{\mathcal{X}}$ was introduced to split the contribution of the different $d \in \mathrm{Eff}_{\mathcal{X}}$. However, we will make the following assumption:

Assumption 5.1. The orbifold quantum product $\circ_\tau$ is convergent over an open subset $U \subset H^*_{orb}(\mathcal{X})$:

$$U = \{\tau \in H^*_{orb}(\mathcal{X}) \mid \Re(\delta(d)) < -M, \forall d \in \mathrm{Eff}_{\mathcal{X}} \setminus \{0\}, ||\tau'|| < e^{-M}\}$$

for some $M \gg 0$ (here $||\cdot||$ is the standard hermitan norm on $H^*_{orb}(\mathcal{X})$).

Using this assumption, we can set $Q = 1$. We will denote this product on $H^*_{orb}(\mathcal{X},\mathbb{C})$ parametrized by $\tau \in U$ by $(H^*(\mathcal{X},\mathbb{C}),\circ_\tau)$.
Let $t_0,\dots,t_\mu$ be the coordinates on $H^*_{orb}(\mathcal{X})$ determined by the homogeneous basis.

Definition 5.2. *The Givental connection is the tuple $(\mathcal{F}^{big},\nabla^{Giv},P)$ which consists of the trivial holomorphic vector bundle $\mathcal{F}^{big} := H^*(\mathcal{X},\mathbb{C}) \times (U \times \mathbb{P}^1_z)$, the connection ∇^{Giv}*

$$\begin{aligned}\nabla_{\partial_{t_k}} &:= \frac{\partial}{\partial_{t_k}} - \frac{1}{z} T_k \circ_\tau , \\ \nabla_{z\partial_z} &:= z\frac{\partial}{\partial_z} + \frac{1}{z} E \circ_\tau + \mu\end{aligned}$$

where $\mu : H^*_{orb}(\mathcal{X},\mathbb{C}) \to H^*_{orb}(\mathcal{X},\mathbb{C})$ is the grading operator given by $\mu(T_k) = deg(T_k)/2$ and the holomorphic Euler vector field E is given by

$$c_1(T\mathcal{X}) + \sum_{k=1}^{\mu} \left(1 - \frac{deg(T_i)}{2}\right) t_k T_k$$

and the pairing

$$\begin{aligned}P : \mathcal{F}^{big} \otimes \iota^*\mathcal{F}^{big} &\longrightarrow \mathcal{O}_{\mathbb{P}^1_z \times U}(-d,d) , \\ (a,b) &\mapsto z^d (a(z), b(-z))_{orb}\end{aligned}$$

where $\iota(z,\underline{t}) = (-z,\underline{t})$ and $(-,-)_{orb}$ is the Orbifold Poincaré pairing.

Notice that the connection ∇^{Giv} is flat (*cf.* . [Iri09, § 2.2]) and the pairing P is non-degenerate, $(-1)^d$-symmetric and ∇^{Giv}-flat.

Let $H^{gen}_{orb}(\mathcal{X}) \supset H^2(\mathcal{X})$ be a minimal homogeneous subspace which generates $H^*_{orb}(\mathcal{X})$ with respect to the orbifold cup-product. We write $H^{gen}_{orb}(\mathcal{X}) = H^2(\mathcal{X}) \oplus H'_{orb}(\mathcal{X})$.

Definition 5.3. (1) *Let $\alpha, \gamma \in H^*_{orb}(\mathcal{X})$ and $\tau \in H^{gen}_{orb}(\mathcal{X}) \cap U$. Define the semi-small quantum product as the restricton of the quantum product to parameter space $H^{gen}_{orb}(\mathcal{X})$:*

$$\alpha \circ \gamma = \sum_{d \in Eff_{\mathcal{X}}} \sum_{l,k \geq 0} \frac{e^{\delta(d)}}{l!} \langle \alpha, \gamma, \underbrace{\tau', \dots, \tau'}_{l-times}, T_k \rangle_{0,l+3,d} T^k Q^d$$

for $\tau = \delta + \tau' \in H^2(\mathcal{X}) \oplus H'_{orb}(\mathcal{X})$.

(2) *The semi-small Givental connection $(\mathcal{F}^{ss}, \nabla^{Giv}, P)$ is the restriction of the Givental connection to $(H^{gen}_{\mathrm{orb}}(\mathcal{X}) \cap U) \times \mathbb{P}^1_z$.*

Let $L_\xi \to \mathcal{X}$ be a orbifold line bundle corresponding to $\xi \in \mathrm{Pic}(\mathcal{X})$. For any point $(x, g) \in \mathcal{X}_v \subset \mathcal{I}\mathcal{X}$ the stabilizer g acts on the fiber L_x by a rational number. This number depends only on the sector v, hence we denote the number by $f_v(\xi)$ and call it the age of L_ξ along $\mathcal{X}_v$.
Iritani defined an action of $\mathrm{Pic}(\mathcal{X})$ on $(\mathcal{F}^{big}, \nabla^{Giv}, P)$ and showed that it is equivariant with respect to this action:

Proposition 5.4. *For each $\xi \in \mathrm{Pic}(\mathcal{X})$ there is an isomorphism of $\mathcal{G}^{big}$*

$$H^*_{orb}(\mathcal{X}, \mathbb{C}) \times (U \times \mathbb{C}) \longrightarrow H^*_{orb}(\mathcal{X}, \mathbb{C}) \times (U \times \mathbb{C}),$$
$$(\alpha, \tau, z) \mapsto (dG(\xi)\alpha, G(\xi)\tau, z)$$

which preserves the connection ∇^{Giv} and the pairing P, where $G(\xi), dG(\xi) : H^*_{orb}(\mathcal{X}) \longrightarrow H^*_{orb}(\mathcal{X})$ are defined by

$$G(\xi)(\tau_0 + \sum_{v \in T'} \tau_v) = (\tau_0 - 2\pi i \xi_0) + \sum_{v \in T'} e^{2\pi i f_v(\xi)} \tau_v ,$$
$$dG(\xi)(\tau_0 + \sum_{v \in T'} \tau_v) = \tau_0 + \sum_{v \in T'} e^{2\pi i f_v(\xi)} \tau_v$$

where $\tau_v \in H^*(\mathcal{X}_v)$ and ξ_0 is the image of ξ in $H^2(\mathcal{X}, \mathbb{Q})$.

It follows from the Proposition above that the Givental connection is invariant under the action of $\mathrm{Pic}(\mathcal{X})$, however, as observed in [DM13], the functions $t_0, \chi_1 = e^{t_1}, \dots \chi_r = e^{t_r}, t_{r+1}, \dots, t_s$ are not coordinates on $H^*_{orb}(\mathcal{X}, \mathbb{C})/\mathrm{Pic}(\mathcal{X})$. Therefore we mod out only a subgroup namely the subgroup $\mathrm{Pic}(X)$ of line bundles with zero age , i.e. $f_v(\xi) = 0$. The set U is invariant under the action of $\mathrm{Pic}(X)$.

Let V be the quotient of U by the action of $\mathrm{Pic}(X)$ and denote by $\pi : U \to V$ the natural projection. Set $\chi_i = e^{t_i}$ for $i = 1, \dots, r$, then $t_0, \chi_1, \dots, \chi_r, t_{r+1}, \dots, t_s$ are coordinates for V.

Lemma 5.5.

(1) *There is a* trTLEP(d)*-structure $(\mathcal{G}^{big}, \nabla^{Giv}, P)$ on V such that $\pi^*(\mathcal{G}^{big}, \nabla^{big}, P) = (\mathcal{F}^{big}, \nabla^{big}, P)$.*
(2) *Set $V_{gen} := \pi(H^{gen}_{\mathrm{orb}}(\mathcal{X}) \cap U)$. There is a* trTLEP(d)*-structure $(\mathcal{G}^{ss}, \nabla^{Giv}, P)$ on V_{gen} such that $\pi^*(\mathcal{G}^{ss}, \nabla^{Giv}, P) = (\mathcal{F}^{ss}, \nabla^{Giv}, P)$ and $(\mathcal{G}^{ss}, \nabla^{Giv}, P) = (\mathcal{G}^{big}, \nabla^{Giv}, P)_{|V_{gen}}$.*

Proof. The statements are a direct consequence of Proposition 4.4. The connection of $(\mathcal{G}^{big}, \nabla^{big}, P)$ is given by

$$\begin{aligned} \nabla_{\partial_{t_k}} &:= \frac{\partial}{\partial_{t_k}} - \frac{1}{z} T_k \circ_\kappa , \\ \nabla_{\chi_j \partial_{\chi_j}} &:= \chi_j \frac{\partial}{\partial_{\chi_j}} - \frac{1}{z} T_j \circ_\kappa \qquad \text{for} \quad j = 1, \dots, r , \\ \nabla_{z\partial_z} &:= z\frac{\partial}{\partial_z} + \frac{1}{z} E \circ_\kappa + \mu \end{aligned} \tag{5.6}$$

for $\kappa \in V$. □

Let $T_0, \dots, T_\mu$ be a homogeneous basis of $H^*_{orb}(\mathcal{X})$ as above. We assume that $1 = T_0 \in H^0(\mathcal{X}, \mathbb{Z})$, $T_1, \dots, T_r \in H^2(\mathcal{X})$ and $T_{r+1}, \dots, T_\mu$ is a basis of $\bigoplus_{k \neq 0,2} H^k(\mathcal{X}) \oplus \bigoplus_{v \in T'} H^*(\mathcal{X}_{(v)})$. Additionally we assume that $T_1, \dots, T_r$ is a $\mathbb{Z}$-basis of $\mathrm{Pic}(X) \subset H^2(\mathcal{X}, \mathbb{Z})$ and lies in the Kähler cone $\overline{\mathcal{K}} \subset H^2(\mathcal{X})$.

The choice of the basis $T_0, \dots, T_\mu$ gives rise to an embedding $j : H^2(X, \mathbb{C})/\mathrm{Pic}(X) \hookrightarrow \mathbb{C}^r$. Let V'_{gen} resp. V' be the closure of image of $j \times id$.

Proposition 5.7. *There exist extensions $(\overline{\mathcal{G}}^{big}, \hat{\nabla}^{Giv}, P)$ resp. $(\overline{\mathcal{G}}^{ss}, \hat{\nabla}^{Giv}, P)$ of $(\mathcal{G}^{big}, \hat{\nabla}^{Giv}, P)$ resp. $(\mathcal{G}^{ss}, \hat{\nabla}^{Giv}, P)$ to a log-trTLEP(n)-structure on V' resp. V'_{gen}. Moreover, there is a structure of a logarithmic Frobenius manifold on V'.*

Proof. The first statement follows from the form of the connection 5.6. The second statement follows from [Rei09, Proposition 1.10 and Proposition 1.11], where the vector ξ in loc. cit. corresponds to $T_0 = 1$ here. □

We now recall the fundamental solution of the Givental connection. Define

$$L(\tau, z)\alpha := e^{-\delta/z}\alpha - \sum_{\substack{d \in Eff_{\mathcal{X}} \setminus \{0\} \\ l > 0, 0 \leq k \leq s}} \frac{1}{l!} \langle \frac{e^{-\delta/z}\alpha}{z + \psi}, \tau', \dots, \tau', T_k \rangle_{0, l+2, d} e^{\delta(d)} T^k$$

where $\tau = \delta + \tau'$. The following proposition summarizes the properties of the fundamental solution.

Proposition 5.8 ([Iri09, Proposition 2.4])**.**

(1) *$L(\tau, z)$ satisfies the following differential equations:*

$$\nabla_{\partial_{t_k}} L(\tau, z)\alpha = 0, \quad \nabla_{z\partial_z} L(\tau, z)\alpha = L(\tau, z)(\mu\alpha - \frac{\rho}{z}\alpha)$$

where $\alpha \in H^*_{\mathrm{orb}}(\mathcal{X})$, $\rho := c_1(TX) \in H^2(\mathcal{X})$ and μ is the grading operator from Definition 5.2. If we put $z^{-\mu} z^\rho := \exp(-\mu \log z) \exp(\rho \log z)$, then

$$\nabla_{\partial_{t_k}} L(\tau, z) z^{-\mu} z^\rho \alpha = 0, \quad \nabla_{z\partial_z} L(\tau, z) z^{-\mu} z^\rho \alpha = 0 .$$

(2) *$L(\tau, z)$ is convergent and invertible on $U \times \mathbb{C}^*$.*

(3) *$(L(\tau, -z)\alpha, L(\tau, z)\beta)_{orb} = (\alpha, \beta)_{orb}$*

(4) *$dG(\xi) L(G(\xi)^{-1}\tau, z)\alpha = L(\tau, z) e^{2\pi i \xi_0} e^{2\pi i f_v(\xi)} \alpha$ for $\alpha \in H^*(\mathcal{X}_v)$. In particular*

$$dG(\xi) L(G(\xi)^{-1}\tau, z)\alpha = L(\tau, z)\alpha$$

for $\xi \in \mathrm{Pic}(X)$.

(5) *Define $\tilde{L}(\tau, z) := L(\tau, z) z^{-\mu} z^\rho$, then*

$$\nabla_{\partial_{t_k}} \tilde{L}(\tau, z)\alpha = 0, \quad \nabla_{z\partial_z} \tilde{L}(\tau, z)\alpha = 0 .$$

Since L and therefore also $\tilde{L}$ is invertible, then the sections $s_i := \tilde{L}(T_i)$ are a basis of flat sections. The J-function of $\mathcal{X}$ is given by $L(\tau,z)^{-1}1 = L(\tau,z)^{-1}T_0$. We set $\tilde{J} := \sum_{i=0}^{s} \tilde{J}_i T_i := \sum_{i=0}^{s}(s_i,T_0)_{orb}T_i = \tilde{L}(\tau,z)^{-1}(T_0)$ and get the formula

$$1 = T_0 = \sum_{i=0}^{s} \tilde{J}_i s_i \qquad \text{on} \quad \mathbb{C}_z^* \times U\,. \tag{5.9}$$

6. Mirror correspondence

Let $\mathcal{X}$ be projective, toric orbifold. In order to state the mirror theorem for toric orbifolds we have to introduce the I-function. In this section, we will use the assumption that $\rho \in \mathcal{K}^e$ which is a bit stronger that weak Fano (see discussion in the beginning of §2.4). We recall this assumption in all statements when needed.

Definition 6.1. *The I-function of a toric orbifold $\mathcal{X}$ is defined by*

$$I(\underline{\chi},z) = e^{\sum_{a=1}^{r+e} e^{\overline{p}_a}\log\chi_a/z} \sum_{d\in\mathbb{K}} \chi^d \frac{\prod_{\nu=\lceil\langle D_i,d\rangle\rceil}^{\infty}(\overline{D}_i + (\langle D_i,d\rangle - \nu)z)}{\prod_{\nu=0}^{\infty}(\overline{D}_i + (\langle D_i,d\rangle - \nu)z)}\mathbf{1}_{v(d)}\,.$$

We collect a few facts about the I-function.

Lemma 6.2. *We have:*

(1) $e^{-\sum_{a=1}^{r+e} e^{\overline{p}_a}\log\chi_a/z} I(\underline{\chi},z) \in H^*_{orb}(\mathcal{X})[z,z^{-1}][\![\chi_1,\dots,\chi_{r+e}]\!]$.

(2) *The function $e^{-\sum_{a=1}^{r+e} e^{\overline{p}_a}\log\chi_a/z} I(\underline{\chi},z)$ is a convergent power series in $\chi_1,\dots,\chi_{r+e}$ if and only if $\rho \in \mathcal{K}^e$. In this case, the I-function has the asymptotics*

$$I(\underline{\chi},z) = 1 + \frac{\tau(\underline{\chi})}{z} + o(z^{-1})\,.$$

The function τ, which take values in $H^{\leq 2}_{orb}(\mathcal{X})$, is a local embedding and is called the mirror map.

(3) *Set $\tilde{I} := Iz^{-\rho}z^{\mu}$ then*

$$\check{E}(\tilde{I}) = 0 \quad \textit{and} \quad \Box^X_{\underline{l}}(\tilde{I}) = 0 \quad \textit{for} \quad \underline{l} \in \mathbb{L}\,.$$

Proof. The first point follows directly from the definition of the I-function. The second point is [Iri09, Lemma 4.2]. The third point follows from $\tilde{\Box}_{\underline{l}} = \prod_{i=1}^{e} \chi_{r+i}^{|l_{m+i}|} \cdot \Box^{\mathcal{X}}_{\underline{l}}$ (cf. Lemma 4.13) and the [Iri09, lemma 4.19] (Note that Iritani proves this for the equivariant I-function. In order to to get the statement one simply has to consider the equivariant limit $\lambda \to 0$). □

There is the following theorem which compares the I-function from above and the J-function which has been introduced in Section 5.

Theorem 6.3. [CCIT15a, Theorem 31, Remark 33] *Let $\rho \in \mathcal{K}^e$, then the I-function and the J-function coincide up to a coordinate change given by the mirror map τ,i.e.*

$$I(\underline{\chi},z) = J(\tau(\underline{\chi}),z)\,.$$

We can now identify the two log-trTLEP(d)-structures $({}_0\widehat{\mathcal{QM}}_A^{log,\mathcal{X}},\nabla,P)$ and $(\mathcal{G}^{ss},\nabla^{Giv},P)$.

Proposition 6.4. *Let $\mathcal{X}$ such that $\rho \in \mathcal{K}^e$. There exists an analytic neighborhood $W_{\mathcal{X}}$ of 0 in $V_{\mathcal{X}}$ such that there is an isomorphism*

$$\theta : ({}_0\widehat{\mathcal{QM}}_A^{log,\mathcal{X}})_{|\mathbb{P}^1_z\times W_{\mathcal{X}}} \longrightarrow (id_{\mathbb{P}^1_z}\times\tau)^*\mathcal{G}^{ss}_{|\mathbb{P}^1_z\times W_{\mathcal{X}}}$$

of log-trTLEP(d)*-structures on $W_{\mathcal{X}}$.*

Proof. As a first step we define a morphism of holomorphic vector bundles with meromorphic connections

$$\gamma : \left(({}_0\mathring{\mathcal{Q}}\mathcal{M}_A^{log,\mathcal{X}})^{an}_{|\mathbb{C}_z\times W_{\mathcal{X}}}, \nabla \right) \longrightarrow (id_{\mathbb{C}_z}\times\tau)^* \left(\mathcal{G}^{ss}_{|\mathbb{C}_z\times W_{\mathcal{X}}}, \hat{\nabla}^{Giv} \right) ,$$
$$1 \mapsto 1 = T_0 .$$

We set $\tilde{\Box}^X_{\underline{l}} := (id_{\mathbb{C}_z}\times\tau)_*\Box^{\mathcal{X}}_{\underline{l}}$ and $\tilde{E} := (id_{\mathbb{C}_z}\times\tau)_*\check{E}$. In order to show that the morphism above is well-defined, the following equations have to hold:

$$\tilde{\Box}^X_{\underline{l}}(\chi_1,\dots,\chi_{r+e},z,\hat{\nabla}^{Giv}_{z\chi_1\partial_{\chi_1}},\dots,\hat{\nabla}^{Giv}_{z\chi_r\partial_{\chi_r}},\hat{\nabla}^{Giv}_{z\partial_{\chi_{r+1}}},\dots,\hat{\nabla}^{Giv}_{z\partial_{\chi_{r+e}}})(1) = 0 \quad \text{for all } \underline{l}\in\mathbb{L},$$
$$\tilde{E}(\chi_1,\dots,\chi_{r+e},z,\hat{\nabla}^{Giv}_{z\chi_1\partial_{\chi_1}},\dots,\hat{\nabla}^{Giv}_{z\chi_r\partial_{\chi_r}},\hat{\nabla}^{Giv}_{z\partial_{\chi_{r+1}}},\dots,\hat{\nabla}^{Giv}_{z\partial_{\chi_{r+e}}})(1) = 0 . \tag{6.5}$$

We are using the presentation $1 = \sum_{i=1}^s \tilde{J}_i s_i$ of the section 1 on $\mathbb{C}^*_z\times W_{\mathcal{X}}$. Since the s_i are flat sections the equations above are equivalent to

$$\Box^{\mathcal{X}}_{\underline{l}}(\chi_1,\dots,\chi_{r+e},z,z\chi_1\partial_{\chi_1},\dots,z\chi_r\partial_{\chi_r},z\partial_{\chi_{r+1}},\dots,z\partial_{\chi_{r+e}})((id_{\mathbb{C}_z}\times\tau)^*\tilde{J}_i) = 0,$$
$$\check{E}(\chi_1,\dots,\chi_{r+e},z,z\chi_1\partial_{\chi_1},\dots,z\chi_r\partial_{\chi_r},z\partial_{\chi_{r+1}},\dots,z\partial_{\chi_{r+e}})((id_{\mathbb{C}_z}\times\tau)^*\tilde{J}_i) = 0 .$$

But this follows from Theorem 6.3 and Lemma 6.2. Since the equations (6.5) hold on $\mathbb{C}^*_z\times W_{\mathcal{X}}$ they hold on $\mathbb{C}_z\times W_{\mathcal{X}}$ by continuity. In order to show that they are isomorphic it is enough to prove this on the germs at 0 (since we are allowed to shrink $W_{\mathcal{X}}$ if necessary). By Nakayama's lemma it is even enough to show this on the fiber over 0. But this is clear since both fibers are isomorphic to $H^*_{orb}(\mathcal{X})$ and the action of $\hat{\nabla}^{Giv}_{z\chi\partial_{\chi_i}}$ and $\hat{\nabla}^{Giv}_{z\partial_{\chi_j}}$ resp. $z\chi\partial_{\chi_i}$ resp. $z\partial_{\chi_j}$ for $i=1,\dots,r$ and $j=r+1,\dots,r+e$ generate the fibers at 0. It remains to show that this isomorphism extends to an isomorphism of log-trTLEP(d)-structures.
Denote by $D\subset W_{\mathcal{X}}$ the normal-crossing divisor given by $\chi_1\cdot\ldots\cdot\chi_r = 0$. We will show that the extensions to $\{z=\infty\}\times\mathbb{P}^1_z\setminus\{0\}\times D$ coincide under the isomorphism γ. First notice that γ gives an identification of local systems $(({}_0\mathring{\mathcal{Q}}\mathcal{M}_A^{log,\mathcal{X}})^{an}_{|\mathbb{C}^*_z\times W_{\mathcal{X}}})^{\nabla} \simeq ((id_{\mathbb{C}^*_z}\times\tau)^*\mathcal{G}^{ss}_{|\mathbb{C}^*_z\times W_{\mathcal{X}}})^{\nabla}$. The extension is then encoded by the $\mathbb{Z}^{r+1}$-filtrations $(F'_\bullet, P_\bullet)$ resp. $(\tilde{F}'_\bullet,\tilde{P}_\bullet)$. Since we already know that the extension over $\mathbb{C}^*_z\times D$ coincide we conclude that $P_\bullet = \tilde{P}_\bullet$. Hence it is enough to show $F'_\bullet = \tilde{F}'_\bullet$. Arguing as in Proposition 4.41 it is enough to show that the extensions over $\mathbb{P}^1_z\times\{\underline{\chi}=0\}$ coincide. But this is clearly the case since the subspace $F_{\mathcal{X}}$ which generates the extension of $\mathcal{E}_{\mathcal{X}}$ is identified under γ with the subspace generated by $T_1,\dots,T_{r+e}$ which in turn generates the extension of $\mathcal{G}^{ss}_{|\mathbb{C}_z\times\{\underline{\chi}=0\}}$.

□

Using the proposition above we can now deduce an isomorphism of logarithmic Frobenius manifolds.

Theorem 6.6. *Let $\mathcal{X}$ a toric DM stack such that $\rho\in\mathcal{K}^e$. There is a unique germ $Mir : (W_{\mathcal{X}}\times\mathbb{C}^{\mu-(r+e)},0)\longrightarrow(V,0)$ which identifies the logarithmic Frobenius manifold coming from the big orbifold quantum cohomology (cf. Proposition 5.7 to the one coming from the Landau-Ginzburg model (cf. Theorem 4.45). Its restriction to $W_{\mathcal{X}}$ corresponds to the isomorphism θ of* log-trTLEP(d)*-structures.*

Proof. This follows from Proposition 6.4 and the uniqueness statement in Theorem 4.44. □

7. Crepant resolutions and global tt^*-geometry

In this section we will first recall the notion of a (pure and polarized) variation of TERP-structures. If a TERP-structure is pure and polarized it gives rise to tt^*-geometry on the underlying space. We will show that the quantum $\mathcal{D}$-module of a toric orbifold X underlies such a

variation of pure and polarized TERP-structures. Our main result is that if X admits a crepant resolution Z than the pure and polarized TERP-structures glue which gives global tt^*-geometry.

7.1. **Log-TERP structures.**

Definition 7.1 ([Her03, Definition 2.12], [HS10, Definition 2.1])**.** *Let $\mathcal{M}$ be a complex manifold and $d \in \mathbb{Z}$. A variation of* TERP*-structures on $\mathcal{M}$ of weight d consists of the following set of data*

(1) *A holomorphic vector bundle $\mathcal{H}$ on $\mathbb{C}_z \times \mathcal{M}$*
(2) *A $\mathbb{R}$-local system $\mathcal{L}$ on $\mathbb{C}_z^* \times \mathcal{M}$, together with an isomorphism*

$$\mathcal{L} \otimes_{\mathbb{R}} \mathcal{O}^{an}_{\mathbb{C}_z^* \times \mathcal{M}} \longrightarrow \mathcal{H}^{an}_{|\mathbb{C}_z^* \times \mathcal{M}}$$

such that the induced connection extends to a meromorphic connection ∇ on $\mathcal{H}$ such that ∇ has a pole of Poincaré rank 1 along $\{0\} \times \mathcal{M}$.
(3) *A polarization $P : \mathcal{L} \otimes \iota^* \mathcal{L} \longrightarrow i^d \underline{\mathbb{R}}_{\mathbb{C}_z^* \times \mathcal{M}}$, which is $(-1)^d$ symmetric and which induces a non-degenerate pairing*

$$P : \mathcal{H} \otimes_{\mathbb{C}_z \times \mathcal{M}} \iota^* \mathcal{H} \longrightarrow z^d \mathcal{O}_{\mathbb{C}_z \times \mathcal{M}}$$

where non-degenerate means that the induced symmetric pairing $[z^{-d}P] : \mathcal{H}/z\mathcal{H} \otimes \mathcal{H}/z\mathcal{H} \to \mathcal{O}_{\mathcal{M}}$ is non-degenerate.

Definition 7.2. *Let $\mathcal{M}$ be a complex manifold, D a simple normal crossing divisor and $n \in \mathbb{Z}$. A* log-TERP*-structures is a holomorphic vector bundle $\mathcal{H}$ on $\mathbb{C}_z \times \mathcal{M}$, a $\mathbb{R}$-local system $\mathcal{L}$ on $\mathbb{C}_z^* \times (\mathcal{M} \setminus D)$ which induces an integrable connection ∇ so that $\nabla_{z^2\partial_z}(\mathcal{H}) \subset \mathcal{H}$ and $\nabla_{zX}(\mathcal{H}) \subset \mathcal{H}$ for any holomorphic vector field X on M which is logarithmic along D and a pairing $P : \mathcal{L} \otimes \iota^* \mathcal{L} \to i^d \underline{\mathbb{R}}_{\mathbb{C}_z^* \times \mathcal{M}}$ which induces a pairing $P : \mathcal{H} \otimes \iota^* \mathcal{H} \to z^d \mathcal{O}_{\mathbb{C}_z \times M}$ such that $(\mathcal{H}, \nabla, P, d)$ is a* log-TEP(d)*-structure and $(\mathcal{H}, \mathcal{L}, P, d)_{|\mathbb{C}_z \times (\mathcal{M} \setminus D)}$ is a variation of* TERP*-structures.*

We now state the definition of a pure and polarized variation of TERP-structures.

Definition 7.3. *Let $(\mathcal{H}, \mathcal{L}, P, d)$ be a variation of* TERP*-structures on $\mathcal{M}$. Let $\overline{\mathcal{M}}$ be the complex manifold with the conjugate complex structure and $\gamma : \mathbb{P}^1 \times \mathcal{M} \to \mathbb{P}^1 \times \mathcal{M}$ be the involution $(z, x) \mapsto (\overline{z}^{-1}, x)$. Consider $\overline{\gamma^* \mathcal{H}}$ which is a holomorphic vector bundle on $(\mathbb{P}^1 \setminus \{0\}) \times \overline{\mathcal{M}}$. Let $\mathcal{O}_{\mathbb{P}^1} \mathcal{C}^{an}_{\mathcal{M}}$ be the subsheaf of $\mathcal{C}^{an}_{\mathbb{P}^1 \times \mathcal{M}}$ consisting of functions which are annihilated by $\partial_{\overline{z}}$. Define a locally free $\mathcal{O}_{\mathbb{P}^1} \mathcal{C}^{an}_{\mathcal{M}}$-module $\hat{\mathcal{H}}$ by glueing $\mathcal{H}$ and $\overline{\gamma^* \mathcal{H}}$ via the following identification on $\mathbb{C}_z^* \times \mathcal{M}$: Let $x \in \mathcal{M}$ and $z \in \mathbb{C}_z^*$ and define*

$$\begin{aligned} c : \mathcal{H}_{|(z,x)} &\longrightarrow (\overline{\gamma^* \mathcal{H}})_{|(z,x)}\,, \\ a \quad &\mapsto \quad \nabla\text{-parallel transport of } \overline{z^{-d} \cdot a}\,. \end{aligned}$$

Then c is an anti-linear involution and identifies $\mathcal{H}_{|\mathbb{C}_z^* \times \mathcal{M}}$ with $\overline{\gamma^* \mathcal{H}}_{|\mathbb{C}_z^* \times \overline{\mathcal{M}}}$. The involution c restricts to complex conjugation (with respect to $\mathcal{L}$) in the fibres over $S^1 \times \mathcal{M}$.

(1) *$(\mathcal{H}, \mathcal{L}, P, d)$ is called pure iff $\hat{\mathcal{H}} = p^* p_* \hat{\mathcal{H}}$, where $p : \mathbb{P}^1 \times \mathcal{M} \to \mathcal{M}$.*
(2) *Let $(\mathcal{H}, \mathcal{L}, P, d)$ be pure, then*

$$\begin{aligned} h : p_* \hat{\mathcal{H}} \otimes_{\mathcal{C}^{an}_{\mathcal{M}}} p_* \hat{\mathcal{H}} &\longrightarrow \mathcal{C}^{an}_{\mathcal{M}} \\ (s, t) &\mapsto z^{-d} P(s, c(t)) \end{aligned}$$

is a hermitian form on $p_* \hat{\mathcal{H}}$. We call $(\mathcal{H}, \mathcal{L}, P, d)$ a pure and polarized TERP*-structure if this form is positive definite.*

A log-TERP*-structure is called pure resp. pure and polarized if its underlying variation of* TERP*-structures is pure resp. pure and polarized.*

Theorem 7.4. *The restriction of the quantum $\mathcal{D}$-module $\mathcal{G}$ of a toric orbifold to $\mathbb{C}_z \times W_{\mathcal{X}} \setminus D_{\mathcal{X}}$ underlies a variation of pure and polarized* TERP*-structures of weight d.*

Proof. The proof carries over almost word for word from the manifold case in [RS15, theorem 5.3]. So we just give a sketch of the proof and refer the reader to loc. cit. for details. Using the mirror isomorphism $\theta : ({}_0\widehat{\mathcal{QM}}_A^{log,\mathcal{X}})_{|\mathbb{P}^1_z \times W_{\mathcal{X}}} \to (id_{\mathbb{P}^1_z} \times \tau)^* \mathcal{G}^{ss}_{|\mathbb{P}^1_z \times W_{\mathcal{X}}}$ it is enough to show that ${}_0{}^{\circ}\mathcal{QM}_A$ underlies a variation of pure and polarized TERP-structures. Notice that the underlying $\mathcal{D}$-module ${}^{\circ}\mathcal{QM}_A$ is isomorphic to $\mathrm{FL}^{loc}_{\mathcal{M}^\circ_{\mathcal{X}}} \mathcal{H}^0 W_+ \mathcal{O}_{Y \times \mathcal{M}^\circ_{\mathcal{X}}}$ by the description (4.7) and Proposition 4.9 (2). The Riemann-Hilbert correspondence gives $DR(\mathcal{H}^0 W_+ \mathcal{O}_{Y \times \mathcal{M}^\circ_{\mathcal{X}}}) \simeq {}^p\mathcal{H}^0 RW_* \underline{\mathbb{C}}_{Y \times \mathcal{M}^\circ_{\mathcal{X}}}$. Therefore $DR(\mathcal{H}^0 W_+ \mathcal{O}_{Y \times \mathcal{M}^\circ_{\mathcal{X}}})$ carries a real structure ${}^p\mathcal{H}^0 RW_* \underline{\mathbb{R}}_{Y \times \mathcal{M}^\circ_{\mathcal{X}}}$. It follows from [Sab97, Theorem 2.2] that the local system of flat sections of $({}^{\circ}\mathcal{QM}_A, \nabla)$ is equipped with a real structure. That ${}^{\circ}\mathcal{QM}_A$ is pure and polarized follows from [Sab08, Theorem 4.10]. □

The proof of the theorem above shows that variation of pure and polarized TERP-structures exists on a Zariski open subset of the complexified Kähler moduli space $\mathcal{M}_{\mathcal{X}}$. In the remaining part of the paper we glue the complexified Kähler moduli space of a toric Deligne-Mumford stack $\mathcal{X}$ to the complexified Kähler moduli space of a crepant resolution and show that the corresponding variation of TERP-structures also glue on the common domain of definition. This gives the global tt^* geometry.

7.2. **Construction of the global Kähler moduli spaces.** Let X be a simplical, numerical-effective toric variety with fan Σ_X. Let $a_1, \ldots, a_m \in N$ be the primitive generators of the ray $\rho_1, \ldots, \rho_m$. The canonical stacky fan $\mathbf{\Sigma} = (N, \Sigma_X, \mathfrak{a})$ is given by $\mathfrak{a}(e_i) = a_i$ for $i = 1, \ldots, m$. We denote the corresponding Deligne-Mumford stack by $\mathcal{X}$. Assume that there exists a crepant toric resolution $\pi : Z \to X$ of X. We denote by Σ_Z the corresponding fan. The rays of Σ_Z are denoted by $\rho_1, \ldots, \rho_m, \rho_{m+1}, \ldots, \rho_{m+e}$ and the primitive integral generators by $a_1, \ldots, a_m, a_{m+1}, \ldots, a_{m+e}$ (notice that the first m primitive generators are the same as the primitive generators of X since Z is a resolution of X). The following lemma is well-known, but the authors could not find a suitable reference.

Lemma 7.5. *We have the following equivalence*

$$\pi : Z \longrightarrow X \quad \textit{is crepant} \qquad \Leftrightarrow \qquad a_{m+1}, \ldots, a_{m+e} \in \partial \mathrm{conv}(a_1, \ldots, a_m)$$

where $\mathrm{conv}(a_1, \ldots, a_m)$ *is the convex hull of* $a_1, \ldots, a_m$.

Proof. Notice that the fan of Z is a subdivision of the fan of X. Recall that the canonical divisor of X and Z are

$$K_Z = D_1 + \cdots + D_{m+e}, \quad K_X = D_1 + \cdots + D_m.$$

Denote by ψ_{K_X} (resp. ψ_{K_Z}) the piece-wise linear function $N_{\mathbb{Q}} \to \mathbb{Q}$ corresponding to the $\mathbb{Q}$-Cartier divisor K_X (resp. K_Z). As both fans are simplicial, these piece-linear functions are determined by their values on the primitive generators of the rays. We can interpret the function ψ_{K_X} as a piece-wise linear in the fan of Z which is exactly $\psi_{\pi^* K_X}$. Then we have

$$\psi_{K_X}(a_i) = \psi_{\pi^* K_X}(a_i) = 1 \text{ for } i \in \{1, \ldots, m\},$$
$$\psi_{K_Z}(a_i) = 1 \text{ for } i \in \{1, \ldots, m+e\}.$$

The morphism π is crepant if and only if these piece-wise linear functions are equal *i.e.*, $\psi(K_X)(a_i) = 1$ for $i \in \{m+1, \dots, m+e\}$. Notice that a point $p \in N$ is in $\partial\text{conv}(a_1, \dots, a_m)$ means exactly that $\psi_{K_X}(p) = 1$ which finishes the proof.

□

The crepantness of π puts several restrictions on X.

Lemma 7.6. *Assume that $\pi : Z \to X$ is a crepant resolution, then*

(1) *X is an SL-orbifold.*
(2) *$Gen(\Sigma_X) = \{a_{m+1}, \dots, a_{m+e}\}$.*

Proof.

(1) Let $c \in N$ be arbitrary and let $\sigma(c)$ be the unique minimal cone of Σ_X containing c. As above we can write uniquely

$$c = \sum_{\substack{i=1 \\ a_i \in \sigma(c)}}^{m} \kappa_i a_i .$$

The claim is equivalent to the fact that $deg(c) := \sum_{a_i \in \sigma(c)} \kappa_i \in \mathbb{N}$. Since Σ_Z is a subdivision of Σ_X, hence also complete, we can find a unique minimal cone $\sigma'(c) \in \Sigma_Z$ containing c and with $\sigma'(c) \subset \sigma(c)$. Because Σ_Z is regular we can uniquely write

$$c = \sum_{\substack{j=1 \\ a_j \in \sigma'(c)}}^{m+e} \kappa'_j a_j$$

with $\kappa'_j \in \mathbb{N}$. Hence we have

$$c = \sum_{\substack{j=1 \\ a_j \in \sigma'(c)}}^{m+e} \kappa'_j a_j = \sum_{\substack{j=1 \\ a_j \in \sigma'(c)}}^{m+e} \kappa'_j \left(\sum_{\substack{i=1 \\ a_i \in \sigma(a_j)}}^{m} \kappa_{ij} a_i \right) .$$

Because the $\sigma(c)$ was chosen to be minimal and because of the lemma above, this gives

$$deg(c) = \sum_{\substack{j=1 \\ a_j \in \sigma'(c)}}^{m+e} \kappa'_j \left(\sum_{\substack{i=1 \\ a_i \in \sigma(a_j)}}^{m} \kappa_{ij} \right) = \sum_{\substack{j=1 \\ a_j \in \sigma'(c)}}^{m+e} \kappa'_j \cdot 1 \in \mathbb{N} .$$

(2) Let σ be a cone of Σ_X. First notice that the degree $deg(c)$ of an element c is additive inside a fixed cone, i.e. for $c, c' \in \sigma$ we have $deg(c+c') = deg(c) + deg(c')$. Because of the first point this shows that $\{a_{m+1}, \dots, a_{m+e}\} \subset Gen(\Sigma_X)$, since their degree is minimal. Now assume that $c \in Gen(\Sigma_X)$. Because Σ_Z is a regular fan, there exists a cone $\sigma' \in \Sigma_Z$ such that $c = \sum_{a_j \in \sigma'} \kappa_j a_j$ with $\kappa_j \in \mathbb{N}$. Since $c \in Gen(\Sigma_X)$, we conclude that $c = a_{i_0}$ for some $i_0 \in \{m+1, \dots, m+e\}$.

□

We get the following statement from $\deg(a_i) = 1$ for $i = m+1, \dots, m+e$.

Corollary 7.7. *The orbifold cohomology $H^*_{orb}(X)$ is H^2_{orb}-generated.*

Consider the sequence

$$0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{Z}^{m+e} \longrightarrow N \longrightarrow 0 . \tag{7.8}$$

Since Z is smooth, we get the exact sequence

$$0 \longrightarrow N^\star \longrightarrow (\mathbb{Z}^{m+e})^\star \simeq PL(\Sigma_Z) \longrightarrow \mathbb{L}^\star \simeq \mathrm{Pic}(Z) \longrightarrow 0$$

when we apply $Hom(-,\mathbb{Z})$ to sequence (7.8).

We get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N^* & \longrightarrow & PL(\Sigma_X) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & 0 \\
 & & \| & & \downarrow \Theta & & \downarrow \theta & & \\
0 & \longrightarrow & N^* & \longrightarrow & PL(\Sigma_X^e) & \longrightarrow & \mathrm{Pic}^e(X) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N^* & \longrightarrow & PL(\Sigma_Z) & \longrightarrow & \mathrm{Pic}(Z) & \longrightarrow & 0
\end{array}$$

The image of the Kähler cone $\mathcal{K}_X$ under the embedding $\mathrm{Pic}(X)\otimes\mathbb{Q} \xrightarrow{\theta} \mathrm{Pic}^e(X)\otimes\mathbb{Q} \simeq \mathrm{Pic}(Z)\otimes\mathbb{Q}$ is a face of the Kähler cone $\mathcal{K}_Z$ by [OP91, Theorem 2.5]. We need the following lemma

Lemma 7.9. *The images of $D_j \in (\mathbb{Z}^{m+e})^\star \simeq PL(\Sigma_Z)$ for $j \in \{m+1,\dots,m+e\}$ do not lie in $\mathcal{K}_Z$.*

Proof. The element D_j seen as a piece-wise linear function on the fan Σ_Z satisfies $D_j(a_i) = \delta_{ij}$. We have

$$a_j = \sum_{\substack{i=1 \\ a_i \in \sigma(a_j)}}^{m} \kappa_i a_i$$

where $\sigma(a_j)$ is the minimal cone in Σ_X containing a_j. Therefore $1 = D_j(a_j) = D_j(\sum_{a_i\in\sigma(a_j)} \kappa_i a_i) > \sum_{a_i\in\sigma(a_j)} \kappa_i D_j(a_i) = 0$ which shows that $D_j \notin CPL(\Sigma_Z)$ for $i \in \{m+1,\dots,m+e\}$. Since we have $N^\star \subset CPL(\Sigma_Z)$ we see that $[D_j] \notin \mathcal{K}_Z$. □

The lemma above shows that we get two $r+e$ dimensional cones in $\mathrm{Pic}(Z)\otimes\mathbb{Q}$, namely $\mathcal{K}_Z$ and $\mathcal{K}_X^e$ which intersect along the face $\mathcal{K}_X$. Now consider the lattice $\mathrm{Pic}^e(X)$ inside $\mathrm{Pic}(Z)\otimes\mathbb{Q}$.

Proposition 7.10. *We will choose two different $\mathbb{Z}$-bases for* $\mathrm{Pic}^e(X)$.

(1) *The first one is $p_1,\dots,p_{r+e}$ with the property that*
   - *$p_1,\dots,p_r$ is a $\mathbb{Z}$-basis of the image of θ,*
   - *$p_{r+i} = [D_{m+i}]$ for $i \in \{1,\dots,e\}$*

(2) *The second basis $q_1,\dots,q_{r+e}$ is chosen such that*
   - *$p_i = q_i$ for $i \in \{1,\dots,r\}$,*
   - *q_i lies in $\mathcal{K}_Z$ for $i \in \{1,\dots,r+e\}$.*

(3) *$\rho_X = \rho_Z$ lies in the $Cone(p_1,\dots,p_r)$*

Proof of Proposition 7.10. (1) This is a direct consequence of Proposition 4.6.
(2) This follows from the same arguments of Proposition 4.6.
(3) This follows from Proposition 4.6, the fact that $\rho_X = \theta(\overline{\rho_X})$ (see the proof of Lemma 2.22) and $\rho_Z = \rho_X$.

□

Denote the cones generated by $p_1,\dots,p_{r+e}$ resp. $q_1,\dots,q_{r+e}$ by C_X resp. C_Z. They are both smooth.

Definition 7.11. (1) The *global Kähler moduli space* $\overline{\mathcal{M}}$ is the toric variety corresponding to the fan with respect to the lattice $Pic^e(X)$ generated by the cones C_X and C_Z together with its faces. Each cone corresponds to an affine chart which we denote by $\overline{\mathcal{M}}_{\mathcal{X}}$ resp. $\overline{\mathcal{M}}_Z$ and their dense tori by $\mathcal{M}_{\mathcal{X}}$ resp. $\mathcal{M}_{\mathcal{Z}}$ (notice that we identify the images of $\mathcal{M}_Z$ and $\mathcal{M}_{\mathcal{X}}$ under the embeddings $\overline{\mathcal{M}}_{\mathcal{X}} \hookrightarrow \overline{\mathcal{M}} \hookleftarrow \overline{\mathcal{M}}_Z$.).

(2) The non-degenerate locus $\mathcal{M}^\circ$ of $\overline{\mathcal{M}}$ is defined by $\mathcal{M}^\circ_{\mathcal{X}} = \mathcal{M}^\circ_Z$ (cf. the definition given in (4.10)).

Remark 7.12. Since both C_X and C_Z are cones which are spanned by a $\mathbb{Z}$-bases (cf. Proposition 7.10), the variety $\mathcal{M}$ is smooth and is covered by two charts $\overline{\mathcal{M}}_{\mathcal{X}} \simeq \mathbb{C}^{r+e}$ resp. $\overline{\mathcal{M}}_Z \simeq \mathbb{C}^{r+e}$ corresponding to the cones C_X resp. C_Z (See Figure above).

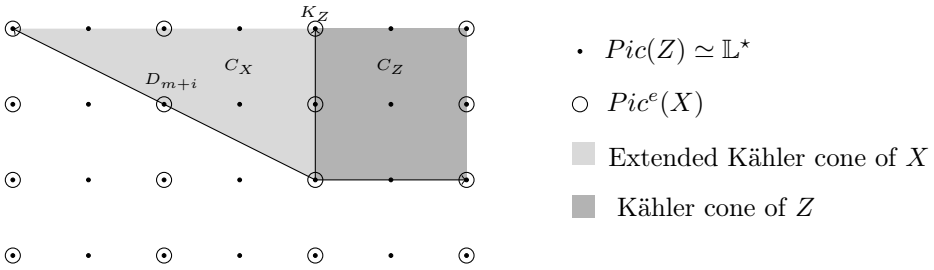


FIGURE 1. The global Kähler moduli space, denoted $\mathcal{M}$.

If we apply the results of Section 4 to the toric orbifold X resp. toric manifold Z we get the following isomorphisms of variations of (pure and polarized) $TERP$-structures from Proposition 6.4 and Theorem 7.4:

$$({}_0^\circ\mathcal{QM}_A^{log,Z})_{|\mathbb{C}_z\times(W_Z\setminus D_Z)} \longrightarrow (id_{\mathbb{C}_z}\times\tau_Z)^*\mathcal{G}^{ss}_{Z|\mathbb{C}_z\times(W_Z\setminus D_Z)},$$
$$({}_0^\circ\mathcal{QM}_A^{log,\mathcal{X}})_{|\mathbb{C}_z\times(W_{\mathcal{X}}\setminus D_{\mathcal{X}})} \longrightarrow (id_{\mathbb{C}_z}\times\tau_{\mathcal{X}})^*\mathcal{G}^{ss}_{\mathcal{X}|\mathbb{C}_z\times(W_{\mathcal{X}}\setminus D_{\mathcal{X}})}$$

where $W_Z \subset U_Z \subset \mathcal{M}_Z^{an}$ resp. $W_{\mathcal{X}} \subset U_{\mathcal{X}} \subset \mathcal{M}_{\mathcal{X}}^{an}$ are analytic neighborhoods of p_Z resp. $p_{\mathcal{X}}$. ${}_0^\circ\mathcal{QM}_A^{log,\mathcal{X}}$

Theorem 7.13. (1) *There exists a pure and polarized* log-TERP*-structure* ${}_0^\circ\mathcal{QM}_A^{log}$ *on a Zariski open neighborhood* $\mathcal{U}$ *of the global Kähler moduli space* $\overline{\mathcal{M}}$ *which contains the large volume limits* $p_Z, p_{\mathcal{X}}$.

(2) *There exist analytic neighborhoods* $W_Z, W_{\mathcal{X}} \subset \overline{\mathcal{M}}^{an}$ *of the large volume limits* $p_Z, p_{\mathcal{X}}$ *such that*

$$({}_0^\circ\mathcal{QM}_A^{log})_{|\mathbb{C}_z\times W_Z} \longrightarrow (id_{\mathbb{C}_z}\times\tau_Z)^*\mathcal{G}^{ss}_{Z|\mathbb{C}_z\times W_Z},$$
$$({}_0^\circ\mathcal{QM}_A^{log})_{|\mathbb{C}_z\times W_{\mathcal{X}}} \longrightarrow (id_{\mathbb{C}_z}\times\tau_{\mathcal{X}})^*\mathcal{G}^{ss}_{\mathcal{X}|\mathbb{C}_z\times W_{\mathcal{X}}}$$

as log-TERP*-structures of weight* d.

Proof. (1) For the first statement we remark that on the chart $\overline{\mathcal{M}}_{\mathcal{X}}$ resp. $\overline{\mathcal{M}}_Z$ of $\overline{\mathcal{M}}$ there exists Zariski open neighborhoods $\mathcal{U}_{\mathcal{X}}$ resp. $\mathcal{U}_Z$ of the large volume limits $p_{\mathcal{X}}$ resp. p_Z and log-TEP(d)-structures ${}_0^\circ\mathcal{QM}_A^{log,\mathcal{X}}$ resp. ${}_0^\circ\mathcal{QM}_A^{log,Z}$ (cf. Theorem 4.17, Corollary 4.30 and Proposition 4.35). Set $\mathcal{W} := \mathcal{U}_{\mathcal{X}} \cap \mathcal{U}_Z$. The restriction of these structures to $\mathcal{W} \cap \mathcal{M}_{\mathcal{X}} = \mathcal{W} \cap \mathcal{M}_Z$ give the same TEP(d)-structure $({}_0^\circ\mathcal{QM}_A)_{|\mathbb{C}_z\times(\mathcal{W}\cap\mathcal{M}_{\mathcal{X}})}$. This

follows from the fact that ${}_0^\circ\mathcal{QM}_A$ only depends on the matrix A whose columns were given by the images $\mathfrak{a}(e_1), \dots, \mathfrak{a}(e_n)$ which by construction are the same for the toric Deligne-Mumford stack $\mathcal{X}$ and the smooth toric variety Z (in contrast to the log-TEP(d)-structures ${}_0^\circ\mathcal{QM}_A^{log,\mathcal{X}}$ resp. ${}_0^\circ\mathcal{QM}_A^{log,Z}$ which depend on the (stacky) fan).

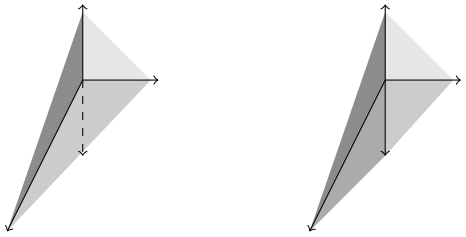
However, the log-TEP(d)-structures ${}_0^\circ\mathcal{QM}_A^{log,\mathcal{X}}$ and ${}_0^\circ\mathcal{QM}_A^{log,Z}$ even agree on $\mathcal{W}$ since the extension along $\mathcal{W}\backslash\mathcal{M}_\mathcal{X} = \mathcal{W}\backslash\mathcal{M}_Z$ are the canonical Deligne extensions as remarked in the proof of Proposition 4.41. Hence we can glue these log-TEP(d)-structures along $\mathcal{W}$ to get a log-TEP(d) structures ${}_0^\circ\mathcal{QM}_A^{log}$ on $\mathcal{U} := \mathcal{U}_\mathcal{X} \cup \mathcal{U}_Z$. From the proof of Theorem 7.4 follows that ${}_0^\circ\mathcal{QM}_A^{log}$ restricted to $\mathcal{U} \cap \mathcal{M}_\mathcal{X} = \mathcal{U} \cap \mathcal{M}_Z$ is a pure and polarized variation of TERP-strucures. This shows the claim.

(2) This follows from Proposition 6.4 and Theorem 7.4 considering the fact that ${}_0^\circ\mathcal{QM}_A^{log}$ restricts to ${}_0^\circ\mathcal{QM}_A^{log,\mathcal{X}}$ resp. ${}_0^\circ\mathcal{QM}_A^{log,Z}$ on $\mathcal{U}_\mathcal{X}$ resp. $\mathcal{U}_Z$.

□

8. The crepant resolution $F_2 \to \mathbb{P}(1,1,2)$

In this section, we explain the construction done in the previous sections for the crepant resolution $F_2 \to \mathbb{P}(1,1,2)$. Consider the following fans

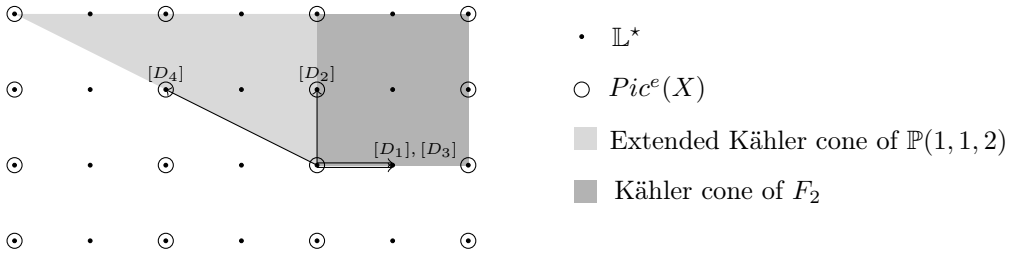


Fan Σ_X of $\mathbb{P}(1,1,2)$ Fan Σ_Z of F_2

The generators of the fan Σ_X are given by $a_1 := (1,0), a_2 := (0,1), a_3 := (-1,-2)$, the generator of the extended ray is $a_4 := (0,-1)$. The short exact sequence (2.7) is given by

$$0 \longrightarrow N^\star \simeq \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -2 \\ 0 & -1 \end{pmatrix}} \bigoplus_{i=1}^4 \mathbb{Z}D_i \xrightarrow{\begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1 \end{pmatrix}} \mathbb{L}^\star \simeq \mathbb{Z}^2 \longrightarrow 0$$

where we have chosen $[D_1], [D_2]$ as a basis for $\mathbb{L}^\star$. The image of $\Theta : PL(\Sigma_X) \to \bigoplus_{i=1}^4 \mathbb{Z}D_i$ is given by $\Theta(PL(\Sigma_X)) = \{aD_1 + bD_2 + cD_3 + dD_4 \mid d = \frac{1}{2}a + \frac{1}{2}c\}$. The lattice $PL(\Sigma^e)$ is given by $\{aD_1 + bD_2 + cD_3 + dD_4 \in \bigoplus_{i=1}^4 \mathbb{Z}D_i \mid \frac{1}{2}a + \frac{1}{2}c \in \mathbb{Z}\}$ and $Pic^e(X)$ is generated by $2[D_1], [D_2]$ or equivalently by $[D_2], [D_4]$.



The global moduli space $\mathcal{M}$ is the gluing of the two grey cones inside $\mathrm{Pic}^e(X)$. It is the total space of $\mathcal{O}_{\mathbb{P}^1}(1)$, the two chart $\mathcal{M}_{\mathbb{P}(1,1,2)}$ and $\mathcal{M}_{F_2}$ correspond to the trivializations of this bundle over $\mathbb{C}$ and $\mathbb{P}^1 \setminus \{0\}$, respectively. The two $\mathbb{Z}$-bases of $\mathrm{Pic}^e(X)$ discussed after Lemma 7.6 are

(1) $p_1 = [D_2]$ and $p_2 = [D_4]$
(2) $q_1 = p_1$ and $q_2 = 2[D_2]$

For a basis of $Pic^e(X)$ we choose $p_1 = [D_2]$ and $p_2 = [D_4]$ inside the extended Kähler cone of $\mathbb{P}(1,1,2)$. Therefore we have

$$[D_1] = \frac{1}{2}p_1 - \frac{1}{2}p_2 \qquad [D_2] = p_1 \qquad \mathscr{D}_1 = \frac{1}{2}z\chi_1\partial_{\chi_1} - \frac{1}{2}z\chi_2\partial_{\chi_2} \qquad \mathscr{D}_2 = z\chi_1\partial_{\chi_1}\,,$$
$$[D_3] = \frac{1}{2}p_1 - \frac{1}{2}p_2 \qquad [D_4] = p_2 \qquad \mathscr{D}_3 = \frac{1}{2}z\chi_1\partial_{\chi_1} - \frac{1}{2}z\chi_2\partial_{\chi_2} \qquad \mathscr{D}_4 = z\partial_{\chi_2}\,.$$

From Proposition 4.9 we get the following operators defining $\mathcal{QM}_A^X$ where χ_1 and χ_2 are coordinates with respect to the basis p_1, p_2:

$$\begin{aligned}
\check{E} &= z^2\partial_z + 2z\chi_1\partial_{\chi_1}\,,\\
\Box^X_{(0,1,0,1)} &= \chi_1 - \mathscr{D}_2\mathscr{D}_4 = \chi_1 - z\chi_1\partial_{\chi_1}z\partial_{\chi_2}\,,\\
\Box^X_{(1,0,1,-2)} &= \mathscr{D}_4^2 - \mathscr{D}_1\mathscr{D}_3 = (z\partial_{\chi_2})^2 - \frac{1}{4}\left(z\chi_1\partial_{\chi_1} - z\chi_2\partial_{\chi_2}\right)^2\,,\\
\Box^X_{(1,1,1,-1)} &= \chi_1\mathscr{D}_4 - \mathscr{D}_1\mathscr{D}_2\mathscr{D}_3 = \chi_1(z\partial_{\chi_2}) - \frac{1}{4}(z\chi_1\partial_{\chi_1} - z\chi_2\partial_{\chi_2})^2 z\chi_1\partial_{\chi_1}\,.
\end{aligned}$$

We now compute the operators for $\mathcal{QM}_A^Z$. For a basis of $Pic^e(X) \subset Pic(Z)$ inside the Kähler cone of F_2 we choose: $q_1 = [D_2]$ and $q_2 = 2[D_1]$ and get

$$[D_1] = \frac{1}{2}q_2 \qquad [D_2] = q_1 \qquad \mathcal{D}_1 = \frac{1}{2}z\eta_2\partial_{\eta_2} \qquad \mathcal{D}_2 = z\eta_1\partial_{\eta_1}\,,$$
$$[D_3] = \frac{1}{2}q_2 \qquad [D_4] = q_1 - q_2 \qquad \mathcal{D}_3 = \frac{1}{2}z\eta_2\partial_{\eta_2} \qquad \mathcal{D}_4 = z\eta_1\partial_{\eta_1} - z\eta_2\partial_{\eta_2}\,.$$

We get the following operators where η_1 and η_2 are coordinates corresponding to the basis q_1, q_2:

$$\begin{aligned}
\check{E} &= z^2\partial_z + 2z\eta_1\partial_{\eta_1}\,,\\
\Box^Z_{(0,1,0,1)} &= \eta_1 - \mathcal{D}_2\mathcal{D}_4 = \eta_1 - (z\eta_1\partial_{\eta_1})(z\eta_1\partial_{\eta_1} - z\eta_2\partial_{\eta_2})\,,\\
\Box^Z_{(1,0,1,-2)} &= \eta_2^2\mathcal{D}_4(\mathcal{D}_4 - z) - \mathcal{D}_1\mathcal{D}_3 = \eta_2^2(z\eta_1\partial_{\eta_1} - z\eta_2\partial_{\eta_2})(z\eta_1\partial_{\eta_1} - z\eta_2\partial_{\eta_2} - z) - \frac{1}{4}(z\eta_2\partial_{\eta_2})^2\,,\\
\Box^Z_{(1,1,1,-1)} &= \eta_1\eta_2^2\mathcal{D}_4 - \mathcal{D}_1\mathcal{D}_2\mathcal{D}_3 = \eta_1\eta_2^2(z\eta_1\partial_{\eta_1} - z\eta_2\partial_{\eta_2}) - \frac{1}{4}(z\eta_1\partial_{\eta_1})(z\eta_2\partial_{\eta_2})^2\,.
\end{aligned}$$

We want to solve the Birkhoff problem in a family (cf. [Sab07, Chapter VI.2]) for $\mathcal{QM}_A^X$ and $\mathcal{QM}_A^Z$ and compare the solutions. Namely, we will explicitly solve the Birkhoff problem in the coordinate χ_1, χ_2 (corresponding to the base (p_1, p_2) *i.e.*, to the extended Kähler cone of $\mathbb{P}(1,1,2)$), then we will do the same with the coordinate η_1, η_2 (corresponding to the base (q_1, q_2) *i.e.*, the Kähler cone of F_2). Finally, as expected by Theorem 7.13, we can glue these two bases using the change of variable given by the Figure above.
First consider the following basis of $\mathcal{QM}_A^X$ that is with the base p_1, p_2 with coordinates χ_1, χ_2:

$$1, (z\chi_1\partial_{\chi_1}), z\partial_{\chi_2}, (z\chi_1\partial_{\chi_1})^2\,.$$

The connection is then given by

$$
\begin{aligned}
d+\frac{1}{z}\begin{pmatrix} 0 & 0 & \chi_1 & 0 \\ 1 & 0 & 0 & 2\chi_1\chi_2 \\ 0 & 0 & 0 & \chi_1\left(4-\chi_2^2\right) \\ 0 & 1 & 0 & 0 \end{pmatrix}\frac{d\chi_1}{\chi_1} + \begin{pmatrix} 0 & 0 & 0 & \chi_1\chi_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\frac{d\chi_1}{\chi_1} \\
+\frac{1}{z}\begin{pmatrix} 0 & \chi_1 & -\frac{2\chi_1\chi_2}{4-\chi_2^2} & 0 \\ 0 & 0 & 0 & \chi_1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4-\chi_2^2} & 0 \end{pmatrix} d\chi_2 + \begin{pmatrix} 0 & 0 & 0 & \chi_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\chi_2}{4-\chi_2^2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} d\chi_2 \\
+\begin{pmatrix} 0 & 0 & -2\chi_1 & 0 \\ -2 & 0 & 0 & -4\chi_1\chi_2 \\ 0 & 0 & 0 & -2\chi_1\left(4-\chi_2^2\right) \\ 0 & -2 & 0 & 0 \end{pmatrix}\frac{dz}{z^2} + \begin{pmatrix} 0 & 0 & 0 & -2\chi_1\chi_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\frac{dz}{z}\,.
\end{aligned}
$$

Notice that this basis does not solve the Birkhoff problem in family since the basis is not flat with respect to the residue connection at $z=\infty$ because

- the matrices in front of $\frac{d\chi_1}{\chi_1}$ and $d\chi_2$ are not vanishing,
- the matrix in front of $\frac{dz}{z}$ is not constant.

Consider now the basis

$$
1,\ (z\chi_1\partial_{\chi_1}),\ \frac{1}{2}\sqrt{4-\chi_2^2}(z\partial_{\chi_2}),\ (z\chi_1\partial_{\chi_1})^2-\chi_1\chi_2\,. \tag{8.1}
$$

With respect to this basis the connection is given by

$$
\begin{aligned}
d+\frac{1}{z}\begin{pmatrix} 0 & \chi_1\chi_2 & \frac{1}{2}\chi_1\sqrt{4-\chi_2^2} & 0 \\ 1 & 0 & 0 & \chi_1\chi_2 \\ 0 & 0 & 0 & 2\chi_1\sqrt{4-\chi_2^2} \\ 0 & 1 & 0 & 0 \end{pmatrix}\frac{d\chi_1}{\chi_1} + \frac{1}{z}\begin{pmatrix} 0 & \chi_1 & \frac{-\chi_1\chi_2}{2\sqrt{4-\chi_2^2}} & 0 \\ 0 & 0 & 0 & \chi_1 \\ \frac{2}{\sqrt{4-\chi_2^2}} & 0 & 0 & -\frac{2\chi_1\chi_2}{\sqrt{\chi_2^2-4}} \\ 0 & 0 & \frac{1}{2\sqrt{4-\chi_2^2}} & 0 \end{pmatrix} d\chi_2 \\
+\begin{pmatrix} 0 & -2\chi_1\chi_2 & -\chi_1\sqrt{4-\chi_2^2} & 0 \\ -2 & 0 & 0 & -2\chi_1\chi_2 \\ 0 & 0 & 0 & -4\chi_1\sqrt{4-\chi_2^2} \\ 0 & -2 & 0 & 0 \end{pmatrix}\frac{dz}{z^2} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\frac{dz}{z}
\end{aligned}
$$

and therefore solves the Birkhoff problem in a family. Notice that one can read of a differential equation for the mirror map from the connection matrices above. It is given by

$$
Mir^{-1}(\chi_1,\chi_2)=(\chi_1,\kappa(\chi_2))
$$

with $\frac{\partial\kappa}{\partial\chi_2}=\frac{\sqrt{4-\chi_2^2}}{2}$. That is

$$
\kappa(\chi_2)=\frac{\chi_2\sqrt{4-\chi_2^2}}{4}+\arcsin(\frac{\chi_2}{2})\,.
$$

A similar computation shows that the basis

$$
1,\ (z\eta_1\partial_{\eta_1}),\ (1-\sqrt{1-4\eta_2^2})\cdot(z\eta_1\partial_{\eta_1})+\sqrt{1-4\eta_2^2}\cdot(z\eta_2\partial_{\eta_2}),\ (z\eta_1\partial_{\eta_1})^2-\eta_1 \tag{8.2}
$$

provides a solution for the Birkhoff problem in a family for $\mathcal{QM}_A^Z$.
The gluing of the Kähler cones given in Figure 8 encodes the following change of coordinates

$$
\chi_1=\eta_1\eta_2,\quad \chi_2=\eta_2^{-1}.
$$

Computing carefully the change of coordinates and the change of basis from (8.1) to (8.2) shows that the two solutions of the Birkhoff problem are the same which means that the extensions at infinity are the same. One should mention that in [CIT09], Corti-Iritani-Tseng prove a similar statement but as we are working on a global D-module, which is defined globally on the B-side, we do not need an analytic continuation argument. In general it is not the case that the extensions at ∞ are the same. However Iritani [Iri10, Theorem 3.22] worked out a sufficient condition, called Hard Lefschetz condition, for the extensions to be the same. Since this condition is met here, our computation is in agreement with this theorem.

References

[BCS05] Lev A. Borisov, Linda Chen, and Gregory G. Smith, *The orbifold Chow ring of toric Deligne-Mumford stacks*, J. Amer. Math. Soc. **18** (2005), no. 1, 193–215 (electronic). MR 2114820 (2006a:14091)

[BG08] Jim Bryan and Amin Gholampour, *Root systems and the quantum cohomology of ADE resolutions*, Algebra Number Theory **2** (2008), no. 4, 369–390. MR 2411404

[BG09a] ———, *Hurwitz-Hodge integrals, the E_6 and D_4 root systems, and the crepant resolution conjecture*, Adv. Math. **221** (2009), no. 4, 1047–1068. MR 2518631

[BG09b] ———, *The quantum McKay correspondence for polyhedral singularities*, Invent. Math. **178** (2009), no. 3, 655–681. MR 2551767

[BG09c] Jim Bryan and Tom Graber, *The crepant resolution conjecture*, Algebraic geometry—Seattle 2005. Part 1, Proc. Sympos. Pure Math., vol. 80, Amer. Math. Soc., Providence, RI, 2009, pp. 23–42. MR 2483931

[BGP08] Jim Bryan, Tom Graber, and Rahul Pandharipande, *The orbifold quantum cohomology of $\mathbb{C}^2/Z_3$ and Hurwitz-Hodge integrals*, J. Algebraic Geom. **17** (2008), no. 1, 1–28. MR 2357679

[BH93] Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993.

[BMP09] S. Boissière, E. Mann, and F. Perroni, *The cohomological crepant resolution conjecture for $\mathbb{P}(1,3,4,4)$*, Internat. J. Math. **20** (2009), no. 6, 791–801. MR 2541935

[BMP11] Samuel Boissière, Étienne Mann, and Fabio Perroni, *Computing certain Gromov-Witten invariants of the crepant resolution of $\mathbb{P}(1,3,4,4)$*, Nagoya Math. J. **201** (2011), 1–22. MR 2772168

[BZGM15] Christine Berkesch Zamaere, Stephen Griffeth, and Ezra Miller, *Systems of parameters and holonomicity of A-hypergeometric systems*, Pacific J. Math. **276** (2015), no. 2, 281–286. MR 3374058

[CCIT15a] Tom Coates, Alessio Corti, Hiroshi Iritani, and Hsian-Hua Tseng, *A mirror theorem for toric stacks*, Compos. Math. **151** (2015), no. 10, 1878–1912. MR 3414388

[CCIT15b] Tom Coates, Alessio Corti, Hiroshi Iritani Iritani, and Hsian-Hua Tseng, *A mirror theorem for toric stacks*, Compos. Math. **151** (2015), no. 10, 1878–1912. MR 3414388

[CCIT20] Tom Coates, Alessio Corti, Hiroshi Iritani, and Hsian-Hua Tseng, *Hodge-theoretic mirror symmetry for toric stacks*, J. Differential Geom. **114** (2020), no. 1, 41–115. MR 4047552

[CIT09] Tom Coates, Hiroshi Iritani, and Hsian-Hua Tseng, *Wall-crossings in toric Gromov-Witten theory. I. Crepant examples*, Geom. Topol. **13** (2009), no. 5, 2675–2744. MR 2529944

[CLLZ14] BoHui Chen, AnMin Li, XiaoBin Li, and GuoSong Zhao, *Ruan's conjecture on singular symplectic flops of mixed type*, Sci. China Math. **57** (2014), no. 6, 1121–1148. MR 3202006

[CLZZ09] Bohui Chen, An-Min Li, Qi Zhang, and Guosong Zhao, *Singular symplectic flops and Ruan cohomology*, Topology **48** (2009), no. 1, 1–22. MR 2553561

[Coa09] Tom Coates, *On the crepant resolution conjecture in the local case*, Comm. Math. Phys. **287** (2009), no. 3, 1071–1108. MR 2486673

[CR13] Tom Coates and Yongbin Ruan, *Quantum cohomology and crepant resolutions: a conjecture*, Ann. Inst. Fourier (Grenoble) **63** (2013), no. 2, 431–478. MR 3112518

[DM13] Antoine Douai and Etienne Mann, *The small quantum cohomology of a weighted projective space, a mirror D-module and their classical limits*, Geom. Dedicata **164** (2013), 187–226. MR 3054624

[DS03] A. Douai and C. Sabbah, *Gauss-Manin systems, Brieskorn lattices and Frobenius structures. I*, Proceedings of the International Conference in Honor of Frédéric Pham (Nice, 2002), vol. 53, 2003, pp. 1055–1116. MR 2033510

[DS04] Antoine Douai and Claude Sabbah, *Gauss-Manin systems, Brieskorn lattices and Frobenius structures. II*, Frobenius manifolds, Aspects Math., E36, Friedr. Vieweg, Wiesbaden, 2004, pp. 1–18. MR 2115764

[Ful93] William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993, The William H. Roever Lectures in Geometry. MR 1234037

[Her02] Claus Hertling, *Frobenius manifolds and moduli spaces for singularities*, Cambridge Tracts in Mathematics, vol. 151, Cambridge University Press, Cambridge, 2002.

[Her03] ———, *tt^* geometry, Frobenius manifolds, their connections, and the construction for singularities*, J. Reine Angew. Math. **555** (2003), 77–161.

[HS10] Claus Hertling and Christian Sevenheck, *Limits of families of Brieskorn lattices and compactified classifying spaces*, Adv. Math. **223** (2010), no. 4, 1155–1224. MR 2581368

[Iri09] Hiroshi Iritani, *An integral structure in quantum cohomology and mirror symmetry for toric orbifolds*, Adv. Math. **222** (2009), no. 3, 1016–1079.

[Iri10] ———, *Ruan's conjecture and integral structures in quantum cohomology*, New developments in algebraic geometry, integrable systems and mirror symmetry. Papers based on the conference "New developments in algebraic geometry, integrable systems and mirror symmetry", Kyoto, Japan, January 7–11, 2008, and the workshop "Quantum cohomology and mirror symmetry", Kobe, Japan, January 4–5, 2008., Tokyo: Mathematical Society of Japan, 2010, pp. 111–166 (English).

[Iwa08] Y. Iwao, *Invariance of Gromov-Witten theory under a simple flop*, Ph.D. thesis, The University of Utah, 2008.

[Jia08] Yunfeng Jiang, *The orbifold cohomology ring of simplicial toric stack bundles*, Illinois J. Math. **52** (2008), no. 2, 493–514. MR 2524648 (2011c:14059)

[JT08] Yunfeng Jiang and Hsian-Hua Tseng, *Note on orbifold Chow ring of semi-projective toric Deligne-Mumford stacks*, Comm. Anal. Geom. **16** (2008), no. 1, 231–250. MR 2411474 (2009h:14092)

[LLQW16] Yuan-Pin Lee, Hui-Wen Lin, Feng Qu, and Chin-Lung Wang, *Invariance of quantum rings under ordinary flops III: A quantum splitting principle*, Camb. J. Math. **4** (2016), no. 3, 333–401. MR 3550285

[LLW16a] Yuan-Pin Lee, Hui-Wen Lin, and Chin-Lung Wang, *Invariance of quantum rings under ordinary flops I: Quantum corrections and reduction to local models*, Algebr. Geom. **3** (2016), no. 5, 578–614. MR 3568338

[LLW16b] ———, *Invariance of quantum rings under ordinary flops II: A quantum Leray-Hirsch theorem*, Algebr. Geom. **3** (2016), no. 5, 615–653. MR 3568339

[MMW05] Laura Felicia Matusevich, Ezra Miller, and Uli Walther, *Homological methods for hypergeometric families*, J. Amer. Math. Soc. **18** (2005), no. 4, 919–941 (electronic).

[MS05] Ezra Miller and Bernd Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2005.

[OP91] Tadao Oda and Hye Sook Park, *Linear gale transforms and Gelfand-Kapranov-Zelevinskij decompositions*, Tohoku Math. J. (2) **43** (1991), no. 3, 375–399.

[Per07] Fabio Perroni, *Chen-Ruan cohomology of ADE singularities*, Internat. J. Math. **18** (2007), no. 9, 1009–1059. MR 2360646

[Pha79] Frédéric Pham, *Singularités des systèmes différentiels de Gauss-Manin*, Progress in Mathematics, vol. 2, Birkhäuser Boston, Mass., 1979, With contributions by Lo Kam Chan, Philippe Maisonobe and Jean-Étienne Rombaldi.

[Rei09] Thomas Reichelt, *A construction of Frobenius manifolds with logarithmic poles and applications*, Comm. Math. Phys. **287** (2009), no. 3, 1145–1187.

[Rei14] ———, *Laurent polynomials, GKZ-hypergeometric systems and mixed Hodge modules*, Compos. Math. **150** (2014), no. 6, 911–941. MR 3223877

[RS15] Thomas Reichelt and Christian Sevenheck, *Logarithmic Frobenius manifolds, hypergeometric systems and quantum $\mathcal{D}$-modules*, J. Algebraic Geom. **24** (2015), no. 2, 201–281.

[RS17] ———, *Non-affine Landau-Ginzburg models and intersection cohomology*, Ann. Sci. Éc. Norm. Supér. (4) **50** (2017), no. 3, 665–753.

[Rua06] Yongbin Ruan, *The cohomology ring of crepant resolutions of orbifolds*, Gromov-Witten theory of spin curves and orbifolds, Contemp. Math., vol. 403, Amer. Math. Soc., Providence, RI, 2006, pp. 117–126. MR 2234886

[Sab97] Claude Sabbah, *Monodromy at infinity and Fourier transform*, Publ. Res. Inst. Math. Sci. **33** (1997), no. 4, 643–685.

[Sab06] ———, *Hypergeometric periods for a tame polynomial*, Port. Math. (N.S.) **63** (2006), no. 2, 173–226. MR 2229875

[Sab07] ———, *Isomonodromic deformations and Frobenius manifolds*, french ed., Universitext, Springer-Verlag London Ltd., London, 2007, An introduction. MR 2368364 (2008i:32015)

[Sab08] ———, *Fourier-Laplace transform of a variation of polarized complex Hodge structure.*, J. Reine Angew. Math. **621** (2008), 123–158.

[Sai89] Morihiko Saito, *On the structure of Brieskorn lattice*, Ann. Inst. Fourier (Grenoble) **39** (1989), no. 1, 27–72.

[Sai94] ———, *On the theory of mixed Hodge modules*, Selected papers on number theory, algebraic geometry, and differential geometry, Amer. Math. Soc. Transl. Ser. 2, vol. 160, Amer. Math. Soc., Providence, RI, 1994, pp. 47–61.

[TW12] H.-H. Tseng and D. Wang, *Seidel Representations and Quantum Cohomology of Toric Orbifolds*, ArXiv e-prints (2012).

Etienne Mann, Univ Angers, CNRS, LAREMA, SFR MATHSTIC, F-49000 Angers, France
Email address: etienne.mann@univ-angers.fr
URL: http://www.math.univ-angers.fr/~mann/

Thomas Reichelt, LS Algebraische Geometrie, Universität Mannheim, B6 26, 68159 Mannheim, Germany
Email address: reichelt@math.uni-mannheim.de
URL: https://www.wim.uni-mannheim.de/hertling/team/pd-dr-thomas-reichelt/