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A sufficient condition for observability of waves by measurable subsets

Emmanuel Humbert* Yannick Privat† Emmanuel Trélat‡

Abstract

We consider the wave equation on a closed Riemannian manifold (M, g) . Given a measurable subset ω of M and given $T > 0$, we establish that, if the metric g is of class C^2 and if ω is regular enough, if the average time over $[0, T]$ of geodesic rays crossing $\bar{\omega}$ is greater than $1/2$ then the Geometric Control Condition is satisfied and thus the wave equation is observable on (ω, T) . Our proof relies on measuring the discrepancy of this average time in $\bar{\omega}$ with respect to $\hat{\omega}$. We show that our assumptions are essentially sharp.

1 Introduction and main result

Let (M, g) be a closed connected Riemannian manifold. Let $T > 0$ be arbitrary. We consider the wave equation

$$\partial_{tt}y - \Delta_g y = 0 \quad \text{in } (0, T) \times M, \quad (1)$$

where Δ_g is the Laplace-Beltrami operator on M for the metric g . Let ω be an arbitrary measurable subset of M . We denote by v_g the canonical Riemannian volume. We define the observability constant $C_T(\omega) \geq 0$ as the largest possible nonnegative constant such that

$$\int_0^T \int_{\omega} |y(t, x)|^2 dv_g(x) dt \geq C \int_{\Omega} (|y(0, x)|^2 + |\partial_t y(0, x)|^2) dv_g(x) \quad (2)$$

for any solution y of (1), that is,

$$C_T(\omega) = \inf \left\{ \int_0^T \int_{\omega} |y(t, x)|^2 dv_g(x) dt \mid \|(y(0, \cdot), \partial_t y(0, \cdot))\|_{L^2(M) \times H^{-1}(M)} = 1 \right\}.$$

When $C_T(\omega) > 0$, the wave equation (1) is said to be *observable* on ω in time T , and when $C_T(\omega) = 0$ we say that observability does not hold for (ω, T) .

It is well known that, for ω open, observability holds if the pair (ω, T) satisfies the *Geometric Control Condition* in M (in short, GCC; see [1, 8]), stipulating that every geodesic ray that propagates in Ω should intersect ω within time T .

*Laboratoire de Mathématiques et de Physique Théorique, UFR Sciences et Technologie, Faculté François Rabelais, Parc de Grandmont, 37200 Tours, France (emmanuel.humbert@lmpt.univ-tours.fr).

†Sorbonne Universités, CNRS UMR 7598, UPMC Univ Paris 06, Laboratoire Jacques-Louis Lions, F-75005, Paris, France (yannick.privat@upmc.fr).

‡Sorbonne Universités, UPMC Univ Paris 06, CNRS UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France (emmanuel.trelat@upmc.fr).

Denoting by Γ the set of geodesic rays, that is, the set of projections onto M of Riemannian geodesic curves in the co-sphere bundle S^*M , given any $T > 0$ and any Lebesgue measurable subset ω of M , we define

$$g_2^T(\omega) = \inf_{\gamma \in \Gamma} \frac{1}{T} \int_0^T \chi_\omega(\gamma(t)) dt. \quad (3)$$

Here, χ_ω is the characteristic function of ω , defined by $\chi_\omega(x) = 1$ if $x \in \omega$ and $\chi_\omega(x) = 0$ if $x \in M \setminus \omega$. When ω is open, GCC reads:

$$g_2^T(\omega) > 0,$$

which is, as recalled above, a *sufficient* condition for observability. It is also well known that this condition is not necessary: indeed, taking $M = \mathbb{S}^2$, the unit sphere in \mathbb{R}^3 endowed with the restriction of the Euclidean structure, and taking ω the open Northern hemisphere, we have $g_2^T(\omega) = 0$ for every $T > 0$ but $C_T(\omega) > 0$ for every $T > \pi$. The latter fact has been established in [7] by an explicit computation exploiting symmetries of solutions. This failure of the functional g_2^T to capture the observability property is due to the existence, here, of a very particular geodesic ray which is *grazing* the open set ω : the equator.

In more general, the existence of such *grazing rays*, which are rays having a contact of infinite order with $\partial\omega = \bar{\omega} \setminus \dot{\omega}$ and may involve an arc entirely contained in $\partial\omega$, adds a serious difficulty to the analysis of observability (see [1]).

It is noticeable that, if one replaces the characteristic function χ_ω of ω , in the integral at the left-hand side of (2) as well as in the definition (3) of the functional g_2 , by a continuous function a , this difficulty disappears and the condition $g_2^T(a) > 0$ becomes a necessary and sufficient condition for observability of (1) on ω in time T (see [4]).

For general measurable subsets $\omega \subset M$, the situation has remained widely open for a long time. Recent advances have been made, which we can summarize as follows. It has been established in [6] that, given any ω measurable, observability holds if and only if $\alpha_T(\omega) > 0$. The quantity $\alpha_T(\omega)$, defined as the limit of highfrequency observability constants, is however not easy to compute and we have, in general, the inequality $g_2^T(\dot{\omega}) \leq \alpha_T(\omega) \leq g_2^T(\bar{\omega})$. In particular, the condition $g_2^T(\omega) > 0$ becomes a necessary and sufficient condition for observability as soon as there are no grazing rays. It has also been shown in [6] that the limit as $T \rightarrow +\infty$ of $\frac{C_T(\omega)}{T}$ is the minimum of two quantities, one of them being $g_2^T(\bar{\omega})$ and the other being of a spectral nature. In [3, 5], the authors consider domains ω that have the shape of a checkerboard, on M that is of dimension 2. They establish a generalized geometric control condition in terms of an ODE on S^*M , meaning roughly that all rays should meet, within time T , the interior of one of the polygons or follow for some time one of the sides of a polygon and there exists $s > 0$ such that all neighbor rays are in the interior of one of the polygons during a time greater than s .

Our main result is the following.

Theorem 1. *Let $T > 0$ be arbitrary and let ω be a measurable subset of M . We make the following assumptions:*

- (i) *the metric g is at least of class C^2 ;*
- (ii) *ω is an embedded C^1 submanifold of M with boundary if $n \geq 3$ and is piecewise C^1 if $n = 2$;*

Then

$$g_2^T(\bar{\omega}) \leq \frac{1}{2} (g_2^T(\dot{\omega}) + 1). \quad (4)$$

Remark 1. Assumption (ii) may be weakened as follows:

- If M is of dimension 2, it suffices to assume that ω is piecewise C^1 . More precisely, we assume that ω is a C^1 stratified submanifold of M (in the sense of Whitney).
- In any dimension, the following much more general assumption is enough: given any grazing ray γ , for almost every $t \in [0, T]$ such that $\gamma(t) \in \partial\omega$, the subdifferential at $\gamma(t)$ of $\partial\omega \cap \gamma(\cdot)^\perp$ is a singleton. This is the case under the (much stronger) assumption that ω be geodesically convex.

Corollary 1. *Under the assumptions of Theorem 1, if $g_2^T(\bar{\omega}) > 1/2$ then $g_2^T(\hat{\omega}) > 0$ and thus the wave equation (1) is observable on ω in time T .*

Corollary 1 is a consequence of the usual fact that, since $\hat{\omega}$ is open, the condition $g_2^T(\hat{\omega}) > 0$ implies observability for $(\hat{\omega}, T)$, and thus $C_T(\omega) \geq C_T(\hat{\omega}) > 0$.

Comments. It is interesting to note that the assumptions made in Theorem 1 are essentially sharp. Remarks are in order.

- The inequality (4) gives a quantitative measure of the discrepancy that can happen for g_2^T when we take the closure of a measurable subset ω of observation or, conversely, when we pass to the interior (this is the sense of Corollary 1). The inequality is sharp, as shown by the following example.

Take $M = \mathbb{S}^2$ and ω the open Northern hemisphere. Then $g_2^T(\omega) = 0$ for every $T > 0$ and $g_2^T(\bar{\omega}) = 1/2$ for every $T > \pi$. Hence, here, (4) is an equality.

As a variant, take ω which is the union of the open Northern hemisphere and of a Southern spherical cap, i.e., a portion of the open Southern hemisphere limited by a given latitude $-\varepsilon < 0$. Then we have as well $g_2^T(\omega) = 0$ for every T and $g_2^T(\bar{\omega}) = 1/2$ for every $T > \pi$.

Note that, taking $\varepsilon = 0$ (i.e., ω is the unit sphere $M = \mathbb{S}^2$ minus the equator), we have $g_2^T(\omega) = 0$ and $g_2^T(\bar{\omega}) = 1$ for every $T > 0$ and thus (4) fails. But here, ω is not an embedded C^1 submanifold of M with boundary: Assumption (ii) (which implies local separation between $\hat{\omega}$ and $M \setminus \bar{\omega}$) is not satisfied. More generally, the result does not apply to any subset ω that is M minus a countable number of rays.

- The result fails in general if $\partial\omega$ is piecewise C^1 only, on a manifold M is of dimension $n \geq 3$. Here is a counterexample.

Let γ be a geodesic ray. If $T > 0$ is small enough, it has no conjugate point. In a local chart of coordinates, we have $\gamma(t) = (t, 0, \dots, 0)$ (see the proof of Theorem 1). Now, using this local chart we define a subset ω of M as follows: the section of $\partial\omega$ with the vertical hyperplane $\gamma(\cdot)^\perp$ is locally equal to this entire hyperplane minus a cone of vertex $\gamma(t)$ with small angle $2\pi\varepsilon > 0$, less than $\pi/4$ for instance (see Figure 1). Now, we assume that, as $t > 0$ increases, these sections rotate with such a speed that, along $[0, T]$, the entire vertical hyperplane is scanned by the section with ω . If the speed of rotation is exactly $T/2\pi$ then it can be proved that $g_2^T(\hat{\omega}) = 0$ and $g_2^T(\bar{\omega}) = 1 - \varepsilon$.

This example shows that Assumption (ii), or its generalization given in Remark 1, cannot be weakened too much. The idea here is to consider a subset ω such that the section of $\partial\omega$ with the vertical hyperplane $\gamma(\cdot)^\perp$ has locally the shape of the hypograph of an absolute value, which is rotating along $\gamma(\cdot)$.

Similar examples can as well be designed with checkerboard-shaped domains ω , thus underlining that in [3, 5] it was important to consider checkerboards in dimension 2.

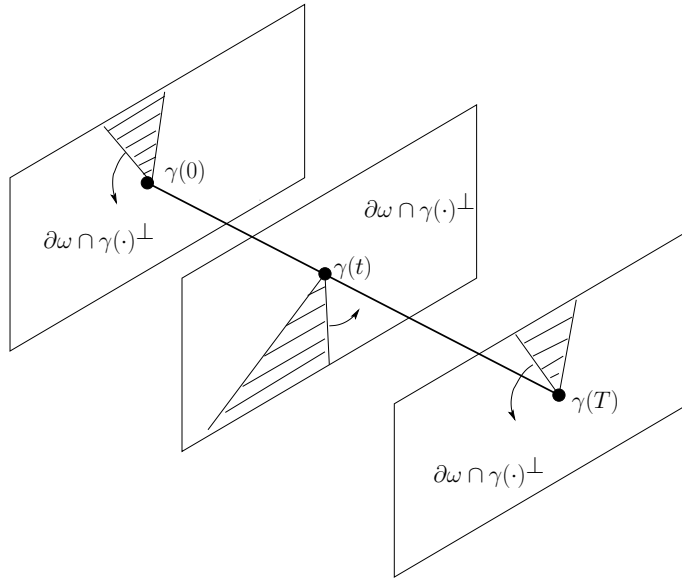


Figure 1: Locally around $\gamma(t)$, $\partial\omega \cap \gamma(t)^\perp$ is the complement of the hatched area.

- The result is wrong if the metric g is not C^2 . A counterexample is the following.

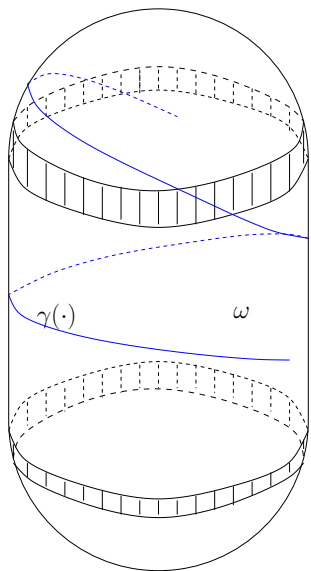


Figure 2: M is pill-shaped and ω is the complement of the hatched area.

Let M be a pill-shaped two-dimensional manifold given by the union of a cylinder of finite length, at the extremities of which we glue two hemispheres (domain also obtained by rotating a 2D stadium in \mathbb{R}^3 around its longest symmetry axis; or, take the unit sphere in \mathbb{R}^3 , cut it

at the equator, separate the two hemispheres and glue them with, inbetween, a cylinder of arbitrary length), and endow it with the induced Euclidean metric (see Figure 2). Then the metric is not C^2 at the gluing circles. Now, take ω defined as the union of the open cylinder with two open spherical caps (i.e., the union of the two hemispheres of which we remove latitudes between 0 and some $\varepsilon > 0$). Then $g_2^T(\omega) = 0$ for every $T > 0$, because ω does not contain the rays consisting of the circles at the extremities of the cylinder. In contrast, $g_2^T(\bar{\omega})$ may be arbitrarily close to 1 as T is large enough and ε is small enough, and thus (4) fails. This is because any ray of M spending a time π in $M \setminus \omega$ spends then much time over the cylinder.

This shows that Assumption (i) is sharp. In the above example, the metric is only $C^{1,1}$.

To conclude, note that Corollary 1 does not apply to the (limit) case where $M = \mathbb{S}^2$ and ω is the open Northern hemisphere. It does not apply, too, to the case where M is the two-dimensional torus and ω is an appropriate (half-covering) open checkerboard on it, as in [3, 5]. Indeed, in these two cases, we have $g_2^T(\omega) = 0$ for every $T > 0$ but $C_T(\omega) > 0$ (i.e., we have observability) for T large enough. This is due to the fact that trapped rays are the weak limit of Gaussian beams that oscillate *on both side* of the limit ray, spreading on one side and on the other a sufficient amount of energy so that indeed observability holds true. In full generality, having information on the way that semi-classical measures, supported on a grazing ray, can be approached by highfrequency wave packets such as Gaussian beams, is a difficult question. In the case of the sphere, symmetry arguments give the answer (see [7]). In the case of the torus, a much more involved analysis is required, based on second microlocalization arguments (see [3, 5]).

Anyway, our result can as well be applied for instance to any kind of checkerboard domain ω on the two-dimensional torus, as soon as the measure of ω is large enough so that $g_2^T(\bar{\omega}) > 1/2$.

Our proof, given in Section 2 hereafter, does not use any microlocal analysis but only elementary arguments of Riemannian geometry. It essentially relies on Lemma 2, in which we establish that, given a grazing ray (i.e., a ray propagating in $\partial\omega$), thanks to our assumption on ω we can always construct neighbor rays, one of which being inside ω and the other being outside of ω for all times.

2 Proof of Theorem 1

Without loss of generality, we take $\omega \subset M$ open. We will use several well known facts of Riemannian geometry, for which we refer, e.g., to [2].

Lemma 1. *There exists $\gamma \in \Gamma$ such that $g_2^T(\omega) = \frac{1}{T} \int_0^T \chi_\omega(\gamma(t)) dt$, i.e., the infimum in the definition (3) of $g_2^T(\omega)$ is reached.*

Proof. Let $(\gamma_k)_{k \in \mathbb{N}}$ be a sequence of rays such that $\frac{1}{T} \int_0^T \chi_\omega(\gamma_k(t)) dt \rightarrow g_2^T(\omega)$. By compactness of geodesics, $\gamma_k(\cdot)$ converges uniformly to some ray $\gamma(\cdot)$ on $[0, T]$.

Let $t \in [0, T]$ be arbitrary. If $\gamma(t) \in \omega$ then for k large enough we have $\gamma_k(t) \in \omega$, and thus $1 = \chi_\omega(\gamma(t)) \leq \chi_\omega(\gamma_k(t)) = 1$. If $\gamma(t) \in M \setminus \omega$ then $0 = \chi_\omega(\gamma(t)) \leq \chi_\omega(\gamma_k(t))$ for any k . In all cases, we have obtained the inequality

$$\chi_\omega(\gamma(t)) \leq \liminf_{k \rightarrow +\infty} \chi_\omega(\gamma_k(t))$$

for every $t \in [0, T]$.

By the Fatou lemma, we infer that

$$g_2^T(\omega) \leq \frac{1}{T} \int_0^T \chi_\omega(\gamma(t)) dt \leq \frac{1}{T} \int_0^T \liminf_{k \rightarrow +\infty} \chi_\omega(\gamma_k(t)) dt \leq \liminf_{k \rightarrow +\infty} \frac{1}{T} \int_0^T \chi_\omega(\gamma_k(t)) dt = g_2^T(\omega).$$

The lemma follows. \square

If the ray γ given by Lemma 1 is not grazing, i.e., if $\int_0^T \chi_{\partial\omega}(\gamma(t)) dt = 0$, then $\int_0^T \chi_\omega(\gamma(t)) dt = \int_0^T \chi_{\bar{\omega}}(\gamma(t)) dt$ and thus $g_2^T(\bar{\omega}) \leq \frac{1}{T} \int_0^T \chi_{\bar{\omega}}(\gamma(t)) dt \leq g_2^T(\omega)$ and hence $g_2^T(\bar{\omega}) = g_2^T(\omega)$. So in this case there is nothing to prove.

In what follows we assume that the ray γ given by Lemma 1 is grazing, i.e., $\int_0^T \chi_{\partial\omega}(\gamma(t)) dt > 0$. Assume that $\gamma(t) = \pi \circ \varphi_t(x_0, \xi_0)$ with $x_0 \in M$ and $\xi_0 \in S_{x_0}^*M$. Here, $S_{x_0}^*M$ denotes the unit cotangent bundle at x_0 (i.e., $\|\xi_0\|_{g^*} = 1$), φ_t is the geodesic flow on S^*M and $\pi : S^*M \rightarrow M$ is the canonical projection.

Lemma 2. *There exists a continuous path of points $s \mapsto x_s \in M$, passing through x_0 at $s = 0$, such that, setting $\gamma_s(t) = \pi \circ \varphi_t(x_s, \xi_0)$, we have*

$$\lim_{s \rightarrow 0} (\chi_{\bar{\omega}}(\gamma_s(t)) + \chi_{\bar{\omega}}(\gamma_{-s}(t))) = 1 \quad (5)$$

for almost every $t \in [0, T]$ such that $\gamma(t) \in \partial\omega$.

Proof. To prove this fact, we assume that, in a local chart, $\gamma(t) = (t, 0, \dots, 0)$. This is true at least in a neighborhood of $x_0 = \gamma(0) = 0$, and this holds true along $\gamma(\cdot)$ as long as there is no conjugate point. We also assume that, in this chart, any other geodesic ray starting at $(0, x_2^0, \dots, x_n^0)$ in a neighborhood of $\gamma(0) = (0, \dots, 0)$ is given by (t, x_2^0, \dots, x_n^0) (projection onto M of the extremal field). This classical construction of the so-called *extremal field* can actually be done on any subinterval of $[0, T]$ along which there is no conjugate point. Note that the set of conjugate times along $[0, T]$ is of Lebesgue measure zero¹. Let us search an appropriate $(n-1)$ -tuple $(x_2^0, \dots, x_n^0) \in \mathbb{R}^{n-1} \setminus \{0\}$ such that the family of points $x_s = (0, sx_2^0, \dots, sx_n^0)$, $s \in (-1, 1)$, gives (5). Note that the geodesic ray starting at x_s is $\gamma_s(t) = (t, sx_2^0, \dots, sx_n^0)$ in the local chart.

In what follows, we set $N = \partial\omega = \bar{\omega} \setminus \dot{\omega} = \bar{\omega} \setminus \omega$ (ω is open). By assumption, ω is an embedded C^1 submanifold of M with boundary and one has $\dim N = \dim M - 1$. By assumption, in a neighborhood U of any point of N , the set $N \cap U$ is a codimension-one hypersurface of M , written as $F = 0$ with $F : U \rightarrow \mathbb{R}$ of class C^1 , which is separating ω and $M \setminus \omega$ in the sense that $\omega \cap U = \{F < 0\}$, $N = \{F = 0\}$ and $M \setminus \omega = \{F \geq 0\}$.

It suffices to prove that, for almost every time t at which $\gamma_0(t) = \gamma(t) \in N$ and $\dot{\gamma}(t) \in T_{\gamma(t)}N$, the points $\gamma_s(t)$ and $\gamma_{-s}(t)$ are on different sides with respect to the (locally) separating manifold N for s small enough.

This is obvious when γ is transverse to N . We set $\Omega = \{t \in [0, T] \mid \gamma(t) \in N, \dot{\gamma}(t) \in T_{\gamma(t)}N\}$. It is a closed subset of $[0, T]$. Let $t \in \Omega$. In the local chart the tangent space $T_{\gamma(t)}N$ is an hyperplane of \mathbb{R}^n containing the line $\mathbb{R}(1, 0, \dots, 0)$. Its projection onto $\{0\} \times \mathbb{R}^{n-1}$ (the hyperplane orthogonal to the line $\gamma(\cdot)$) is an hyperplane of $\{0\} \times \mathbb{R}^{n-1}$, of normal vector $(0, v(t))$ with $v(t) \in \mathbb{R}^{n-1}$ of Euclidean norm 1. Since only the direction of $v(t)$ is important, we assume that $v(t) \in \mathbb{P}^{n-2}(\mathbb{R})$, the projective space.

We claim that:

There exists $V \in \mathbb{P}^{n-2}(\mathbb{R})$ such that $\langle V, v(t) \rangle \neq 0$ for almost every $t \in \Omega$.

With this result, setting $V = (x_2^0, \dots, x_n^0)$, the points x_s defined above give the lemma.

Let us now prove the claim. We define $A = \{(t, V) \in \Omega \times \mathbb{P}^{n-2}(\mathbb{R}) \mid \langle V, v(t) \rangle = 0\}$. By definition, given $(t, V) \in \Omega \times \mathbb{P}^{n-2}(\mathbb{R})$ we have $\chi_A(t, V) = 1$ when $V \in v(t)^\perp$. Since $v(t)^\perp \cap \mathbb{P}^{n-2}(\mathbb{R})$

¹This is a general fact in Riemannian geometry. Indeed, a conjugate time is a time at which a non-zero Jacobi field vanishes. Since Jacobi fields are solutions of a second-order ordinary differential equation, such times must be isolated, for otherwise the Jacobi field would vanish at the second order and thus would be identically zero.

is of codimension one in $\mathbb{P}^{n-2}(\mathbb{R})$, we have $\int_{\mathbb{P}^{n-2}(\mathbb{R})} \chi_A(t, V) d\mathcal{H}^{n-2} = 0$ for every $t \in \Omega$, where we have endowed $\mathbb{P}^{n-2}(\mathbb{R})$ with the Hausdorff measure \mathcal{H}^{n-2} . Therefore, by the Fubini theorem,

$$0 = \int_{\Omega} \int_{\mathbb{P}^{n-2}(\mathbb{R})} \chi_A(t, V) d\mathcal{H}^{n-2} dt = \int_{\mathbb{P}^{n-2}(\mathbb{R})} \int_{\Omega} \chi_A(t, V) dt d\mathcal{H}^{n-2}$$

and thus $\int_{\Omega} \chi_A(t, V) dt = 0$ for almost every $V \in \mathbb{P}^{n-2}(\mathbb{R})$. Fixing such a V , it follows that $\chi_A(t, V) = 0$ for almost every $t \in \Omega$, and the claim is proved. \square

In view of proving Remark 1, note that the argument above still works in dimension 2 with ω piecewise C^1 (but not in dimension greater than or equal to 3: see the counterexample given in Section 1). In more general, in any dimension, the argument above still works if ω is such that, for almost every time t , the subdifferential at $\gamma(t)$ of $\partial\omega \cap \gamma(\cdot)^\perp$ is a singleton.

At this step, we have embedded the ray γ given by Lemma 1 into a family of rays γ_s which enjoy a kind of transversality property with respect to $N = \partial\omega$. Let us consider the partition $[0, T] = A_1 \cup A_2 \cup A_3$ into three disjoint measurable sets, with

$$\begin{aligned} A_1 &= \{t \in [0, T] \mid \gamma(t) \in \omega\}, \\ A_2 &= \{t \in [0, T] \mid \gamma(t) \in M \setminus \bar{\omega}\}, \\ A_3 &= \{t \in [0, T] \mid \gamma(t) \in \partial\omega\}. \end{aligned}$$

Since $\gamma_s(\cdot)$ converges uniformly to $\gamma(\cdot)$ as $s \rightarrow 0$ and since ω and $M \setminus \bar{\omega}$ are open, we have:

- $\lim_{s \rightarrow 0} (\chi_{\bar{\omega}}(\gamma_s(t)) + \chi_{\bar{\omega}}(\gamma_{-s}(t))) = 2$ for every $t \in A_1$;
- $\lim_{s \rightarrow 0} (\chi_{\bar{\omega}}(\gamma_s(t)) + \chi_{\bar{\omega}}(\gamma_{-s}(t))) = 0$ for every $t \in A_2$;
- $\lim_{s \rightarrow 0} (\chi_{\bar{\omega}}(\gamma_s(t)) + \chi_{\bar{\omega}}(\gamma_{-s}(t))) = 1$ for almost every $t \in A_3$ (by Lemma 2).

By the Lebesgue dominated convergence theorem, we infer that

$$\lim_{s \rightarrow 0} \int_0^T (\chi_{\bar{\omega}}(\gamma_s(t)) + \chi_{\bar{\omega}}(\gamma_{-s}(t))) dt = 2|A_1| + |A_3|.$$

Now, on the one part, by the first step we have $\frac{1}{T}|A_1| = \frac{1}{T} \int_0^T \chi_{\omega}(\gamma(t)) dt = g_2^T(\omega)$. On the other part, since A_1 and A_3 are disjoint we have $\frac{1}{T}(|A_1| + |A_3|) \leq 1$. Hence

$$\lim_{s \rightarrow 0} \frac{1}{T} \int_0^T (\chi_{\bar{\omega}}(\gamma_s(t)) + \chi_{\bar{\omega}}(\gamma_{-s}(t))) dt \leq g_2^T(\omega) + 1.$$

Since $g_2^T(\bar{\omega}) \leq \frac{1}{T} \int_0^T \chi_{\bar{\omega}}(\gamma_{\pm s}(t)) dt$ for every s by definition, we infer that $2g_2^T(\bar{\omega}) \leq g_2^T(\omega) + 1$. The theorem is proved.

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