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EXISTENCE AND ASYMPTOTIC RESULTS FOR AN INTRINSIC MODEL OF SMALL-STRAIN INCOMPATIBLE ELASTICITY

SAMUEL AMSTUTZ AND NICOLAS VAN GOETHEM

ABSTRACT. A general model of incompatible small-strain elasticity is presented and analyzed, based on the linearized strain and its associated incompatibility tensor field. Strain incompatibility accounts for the presence of dislocations, whose motion is ultimately responsible for the plastic behaviour of solids. The specific functional setting is built up, on which existence results are proved. Our solution strategy is essentially based on the projection of the governing equations on appropriate subspaces in the spirit of the Leray decomposition of solenoidal square-integrable velocity fields in hydrodynamics. It is also strongly related with the Beltrami decomposition of symmetric tensor fields in the wake of previous works by the authors. Moreover a novel model parameter is introduced, the incompatibility modulus, that measures the resistance of the elastic material to incompatible deformations. An important result of our study is that classical linearized elasticity is recovered as the limit case when the incompatibility modulus goes to infinity. Several examples are provided to illustrate this property and the physical meaning of the incompatibility modulus in connection with the dissipative nature of the processes under consideration.

1. INTRODUCTION

1.1. **The intrinsic approach to elasticity.** An intrinsic approach to elasticity simply means that the main and primal variable is the strain, together with its derivatives, and that the displacement and rotation fields are possibly recovered in a second step, in case they are needed. This approach is most probably the first historically, being the strain indeed considered to measure deformation, that is, variation in length and in mutual orientation of infinitesimal fibers within a solid body. As a matter of fact, for the geometer the strain is a metric from which all other geometric concepts (such as curvature, torsion and other sophisticated tensors, see [4, 11]) are retrieved. Specifically, given a smooth strain tensor field ε, the classical Kirchhoff-Saint Venant construction (for a historical review, see [4, 28, 36]) in linearized elasticity basically consists in

- introducing the Frank tensor[1] $\mathcal{F} = \operatorname{Curl}^t \varepsilon$, where $\operatorname{Curl}^t \varepsilon$ stands for the transpose of the curl of the symmetric tensor ε (computed row-wise);
- defining the rotation field as $\omega(x) = \omega(x_0) + \int_{x_0}^{x} \mathcal{F}(\xi) dl(\xi)$, on a smooth curve joining the endpoints x_0 and x;
- defining the displacement field as $u(x) = u(x_0) + \int_{x_0}^{x} (\varepsilon - \epsilon(\omega))(\xi) dl(\xi)$, where $\epsilon(\omega)$ stands for the skew-symmetric rotation tensor constructed from ω, namely $\epsilon(\omega)_{il} := \epsilon_{ilk}\omega_k$, with ϵ_{ilk} standing for the Levi-Civita symbol.

Obviously such definitions are a priori path-dependent. In order for u and ω to be well-defined, i.e., to be path-independent, it is immediately seen that a sufficient and necessary condition in a simply connected domain be that

$$\operatorname{inc} \varepsilon := \operatorname{Curl} \operatorname{Curl}^t \varepsilon = \operatorname{Curl} \mathcal{F} = 0,$$

where $\operatorname{inc} \varepsilon$ denotes the strain incompatibility tensor defined index-wise by

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[1]This terminology was introduced in [42] simply because its integral on a closed loop yields the so-called Frank tensor attached to a disclination singularity.

$$(\operatorname{inc}\varepsilon)_{ij} = \epsilon_{ikm}\epsilon_{jln}\partial_k\partial_l\varepsilon_{mn},$$

that is easily seen to be symmetric. When $\operatorname{inc}\varepsilon = 0$ one retrieves the well-known expressions

$$\nabla^S u = \varepsilon \text{ and } \nabla u = \varepsilon - \epsilon(\omega),$$

where $\nabla^S u = (\nabla u + \nabla^t u)/2$ stands for the symmetrized gradient of the displacement field u. On the contrary, if the tensor $\operatorname{inc}\varepsilon$ is not vanishing, this means exactly that the rotation and/or displacement fields exhibit a jump around what is classically called a Burgers circuit (to this respect an important role is played by the choice of the origin x_0 as shown in [42]). So far, it appears clear that the important geometric quantities are

$$\varepsilon,\ \operatorname{Curl}^t \varepsilon,\ \text{and}\ \operatorname{inc}\varepsilon.$$

This is the seminal motivation for our model which is precisely designed from these variables. In particular, this choice makes the model of gradient-type. Note that it appears natural to consider the curl instead of the full gradient, and the inc instead of the full Hessian.

Let us also stress that this approach, despite rarely seen today, has a long history: we date the origin of the intrinsic view to Riemann with ground-breaking applications in general relativity and later in mechanics (in particular see the Hodge and Prager approach in perfect plasticity [34]). In general, as explained in [4], Riemann's view is in contrast with Gauss' standpoint of immersions, that is a displacement or velocity-based formulation. Furthermore, as far as dislocations are involved, this geometric approach was very much developed and enhanced by the physicist E. Kröner in the second half of last century [27]. It should however be mentioned that in finite as well as in linearized elasticity the intrinsic approach was recently considered and developed during the last decades in a systematic way by Ciarlet and co-authors [2, 3, 12–15], and Geymonat and co-authors [20–22] (see also [47] for a geometric approach). In particular, their aim was to write, for the elasticity system, the homogeneous boundary condition on the displacement in terms of the elastic strain only.

Although the incompatibility operator has been used in the engineering literature for a long time, the mathematical study of spaces of square integrable tensor-valued functions with square integrable incompatibility was not yet considered and thus our first step was to dedicate a paper to the subject [5].

1.2. **A model of incompatible elasticity.** The approach we propose was introduced in [6] from a physical standpoint. It aims at accounting for the macroscopic effect of the motion of dislocations, since it is known from the works of E. Kröner that elastic strain incompatibility is related to the dislocation density tensor [27]. The proposed model is expected to ultimately provide an original framework to the modeling of elasto-plastic behaviors. At the current stage of development, it can be termed generalized elasticity or incompatible elasticity, since one important feature is that classical linearized elasticity is recovered as a limit case. The main point of our approach is that the strain is a symmetric tensor, but not necessarily a symmetric gradient. Moreover, our theory relies on the following rationales.

(1) Strain rate is preferred to strain and is given the following, primordial definition. The medium is considered as a collection of infinitesimal cells that individually deform smoothly, so that within each cell one can identify and follow fibers. Denote by a_1, a_2, a_3 three such fibers, which at time t originate from point x and are oriented along the axes of a Cartesian coordinate system and scaled to be of unit lengths. Then the strain rate is defined at x as (see, e.g., [18])

$$d_{ij}(t) = \frac{1}{2}\left(\frac{d}{dt}(a_i \cdot a_j)\right)_t. \tag{1.1}$$

Having fixed an initial time $t_0 = 0$, the time integral of the objective tensor d, called the strain or deformation tensor, reads $\varepsilon(t) = \int_0^t d(s)ds$. Note that this latter expression is only valid in the small strain setting, while (1.1) is general.

(2) The strain rate defined in this way admits a mathematical decomposition: it is the sum of a compatible part and an incompatible part, that is given by a structure theorem called Beltrami decomposition [28]:
$$d = \nabla^S v + E^0. \tag{1.2}$$
This decomposition is unique once boundary conditions for v, which can be seen as a velocity field, are prescribed. Therefore, while d is an objective field by definition, neither $\nabla^S v$ nor E^0 are objective, see the discussion in [6]. For this reason the model will be constructed upon the full strain rate d (or ε if small strain is considered) and its space derivatives.

(3) The governing equations should generalize those of classical linear elasticity in the sense that they must take into account the possible strain incompatibility. The idea behind this is to represent the macroscopic effect of dislocations in the micro-structure, as strain incompatibility is related to the density of dislocations [27, 39, 41–43].

Our model can be briefly described as follows in the simplified case of a homogeneous material (see [6] for details). One considers linearized gradient elasticity in the sense of Mindlin [31]. One assumes that the virtual strain rate $\hat{d}$ and its gradient produce intrinsic work, and by the virtual power principle we write
$$\int_\Omega (\sigma \cdot \hat{d} + \tau \cdot \nabla \hat{d}) dx = \int_\Omega \mathbb{K} \cdot \hat{d}\, dx,$$
where σ, τ are the Cauchy stress and hyperstress tensors, respectively, and $\mathbb{K}$ is a tensor representing external efforts. Assuming first a natural initial configuration (at $t = 0$), constitutive relations are taken as $\sigma = \mathbb{C}\varepsilon$ and $\tau = \mathbb{D}\nabla\varepsilon$, where $\mathbb{C}, \mathbb{D}$ are the isotropic Lamé and Mindlin tensors, respectively (see [31]). Next, we require that the intrinsic power induced by the hyperstress $\int_\Omega \tau \cdot \nabla \hat{d}\, dx$ vanishes as soon as the deformation is compatible, i.e., that it is only due to micro-structural defects in the form of dislocations. Then it was shown in [6] (see also [4] for different arguments) that the components of $\mathbb{D}$ are related through a scalar ℓ, called incompatibility modulus, which eventually yields that $-\operatorname{div}\tau = \ell \operatorname{inc}\varepsilon$. Therefore the virtual power principle leads to the weak form
$$\int_\Omega (\mathbb{C}\varepsilon + \ell \operatorname{inc}\varepsilon) \cdot \hat{d}\, dx = \int_\Omega \mathbb{K} \cdot \hat{d}\, dx, \ \forall \hat{d} \in \mathcal{E}, \tag{1.3}$$
where $\mathcal{E}$ is the set of admissible virtual strain rates.

To see that this equation generalizes linearized elasticity, take $\hat{d} = \nabla^S \hat{v}$ with $\hat{v} = 0$ on $\Gamma_D \subset \partial\Omega$ and take $\mathbb{K}$ such that $-\operatorname{div}\mathbb{K} = f$ in Ω and $\mathbb{K}N = g$ on $\Gamma_N := \partial\Omega \setminus \Gamma_D$. Then, plugging this into (1.3) immediately yields
$$\left\{ \begin{array}{rcll} -\operatorname{div}(\mathbb{C}\varepsilon + \ell \operatorname{inc}\varepsilon) & = & f & \text{in } \Omega, \\ (\mathbb{C}\varepsilon + \ell \operatorname{inc}\varepsilon)N & = & g & \text{on } \Gamma_N, \end{array} \right. \tag{1.4}$$
which is exactly the system of linearized elasticity in case of compatible strain, i.e. with $\varepsilon = \nabla^S u$ with $u = u_0$ on Γ_D, since for such strains $\operatorname{inc}\varepsilon = 0$.

More generally, we believe that our model of incompatible linearized elasticity is able to represent inhomogeneous material properties and finite deformations through an incremental formulation. Indeed, nonlinear problems in continuum mechanics are classically solved through the finite element method used in conjunction with an incremental solution procedure. In this way, nonlinear problems are reduced to a sequence of iterations consisting of linearized problems. Therefore, in our model we will rather consider the strain increment (later denoted by E) in place of ε, together with the generalized tangent parameters $(\mathbb{C}, \ell)$. We emphasize that the procedure we suggest is Eulerian by essence, with all coordinates related to the deformed configuration, and is not to be confounded with Lagrangian incremental methods (as described e.g. in [10]). In case coordinates in a fixed reference configuration are needed for practical purposes, standard transformation rules might be applied. Of course, the evolution of the tangent moduli between increments should be driven by constitutive laws in order to account, for instance, for hardening phenomena. A possible thermodynamic approach is to relate changes in these coefficients with dissipation. In order to reach this aim, as a first step, a sensitivity analysis of the dissipation functional with respect to a

variation of ℓ within a small inclusion was conducted in [6] for a simplified model. The extension to the full model and the numerical implementation, for which the existence results of the present work constitute a mandatory preliminary step, is an ongoing work. Our approach and model have been put in a historical perspective of intrinsic views in geometry in the survey paper [4]. For philosophical thoughts about modeling in physics, we refer to [29].

1.3. **Summary of our results.** Set $\mathcal{E} = L^2(\Omega, \mathbb{S}^3)$. Subsequently the model variable will be denoted by E, and can represent either a strain, a strain increment or a strain rate by time derivation. The main purpose of this work is to prove that (1.3), or equivalently the associated strong form

$$\mathbb{C}E + \ell \operatorname{inc} E = \mathbb{K} \quad \text{in } \Omega,$$

has a unique solution in the space of square integrable functions with square integrable incompatibility, with the additional condition on the dislocation flux at the boundary $\operatorname{inc} E\, N = h$ on $\partial\Omega$. The main ingredients to achieve the proof are (i) an orthogonal decomposition of $L^2(\Omega, \mathbb{S}^3)$ related to the Beltrami decomposition, and (ii) Fredholm's alternative. It is also to be stressed that our model has no internal variational structure in the sense that the solution is not seen as the minimizer of some energy. Moreover we analyze the limit case $|\ell| \to \infty$, showing that our model reduces to classical linearized elasticity. We conclude by three explicit computations to illustrate our approach.

2. The incompatibility operator: generalities and preliminary results

Let Ω be a sufficiently regular bounded domain of $\mathbb{R}^3$. We denote by $\partial\Omega$ its boundary and by N its outward unit normal. For simplicity we will assume that $\partial\Omega$ is $\mathcal{C}^\infty$, but weaker assumptions could be considered for each specific result, depending on the traces and liftings involved. We recall the definition of the incompatibility of a symmetric second order tensor E:

$$\operatorname{inc} E := \operatorname{Curl} \operatorname{Curl}^t E, \quad (\operatorname{inc} E)_{ij} = \epsilon_{ikm}\epsilon_{jln}\partial_k\partial_l E_{mn},$$

with the operator Curl intended row-wise, and with Curl^t denoting its transpose. In a Cartesian frame, the incompatibility of a symmetric tensor is obtained by taking its curl column-wise then row-wise, or vice-versa by symmetry. Note that some authors define the operator "curl" as our Curl^t , then what they call curl-curl coincides with our inc operator, since by symmetry one has $\operatorname{inc} E = (\operatorname{inc} E)^t = \operatorname{Curl}^t \operatorname{Curl}^t E$.

2.1. **The curvilinear frame.** For all $x \in \partial\Omega$, the system $(\tau^A(x), \tau^B(x))$ is an orthonormal basis of the tangent plane to $\partial\Omega$, that can be naturally extended along $N(x)$ in a tubular neighborhood W of $\partial\Omega$ (see [5]). The curvatures along τ^A and τ^B are denoted by κ^A and κ^B, respectively. Define the normal derivative as $\partial_N := N \cdot \nabla$ and the tangential derivatives as $\partial_R := \tau^R \cdot \nabla$, for $R \in \{A, B\}$. We will also use the notation $R^* = B$ if $R = A$, $R^* = A$ if $R = B$. The following results are proved in [5].

Theorem 2.1. *There exist smooth scalar fields ξ, γ^A, γ^B in W such that*

$$\partial_N N = \partial_N \tau^R = 0, \tag{2.1}$$
$$\partial_R N = \kappa^R \tau^R + \xi \tau^{R^*}, \tag{2.2}$$
$$\partial_R \tau^R = -\kappa^R N - \gamma^{R^*} \tau^{R^*}, \tag{2.3}$$
$$\partial_{R^*} \tau^R = \gamma^R \tau^{R^*} - \xi N. \tag{2.4}$$

If $(\tau^A(x), \tau^B(x))$ are oriented along the principal directions of curvature then $\xi(x) = 0$.

Lemma 2.2. *If f is twice differentiable in W then it holds*

$$\partial_R \partial_N f = \partial_N \partial_R f + \kappa^R \partial_R f + \xi \partial_{R^*} f. \tag{2.5}$$

2.2. **Basic function spaces.** Let $\mathbb{M}^3$ be the set of 3×3 real matrices and $\mathbb{S}^3$ be the subset of symmetric matrices. We define

$$\begin{aligned} H^{\rm curl}(\Omega,\mathbb{M}^3) &:= \{E\in L^2(\Omega,\mathbb{M}^3):\ {\rm Curl}\ E\in L^2(\Omega,\mathbb{M}^3)\},\\ H^{\rm div}(\Omega,\mathbb{S}^3) &:= \{E\in L^2(\Omega,\mathbb{S}^3):\ {\rm div}\, E\in L^2(\Omega,\mathbb{R}^3)\},\\ H^{\rm inc}(\Omega,\mathbb{S}^3) &:= \left\{E\in L^2(\Omega,\mathbb{S}^3):\ {\rm inc}\, E\in L^2(\Omega,\mathbb{S}^3)\right\}. \end{aligned}$$

These spaces, endowed with the norms defined by $\|E\|^2_{H^{\rm curl}}=\|E\|^2_{L^2}+\|{\rm Curl}\ E\|^2_{L^2}$, $\|E\|^2_{H^{\rm div}}=\|E\|^2_{L^2}+\|{\rm div}\, E\|^2_{L^2}$, $\|E\|^2_{H^{\rm inc}}=\|E\|^2_{L^2}+\|{\rm inc}\, E\|^2_{L^2}$, respectively, and the corresponding inner products are obviously Hilbert spaces. Also, by classical regularization arguments (see e.g. [9, 16, 37]), $\mathcal{C}^\infty(\bar\Omega,\mathbb{M}^3)$ [resp. $\mathcal{C}^\infty(\bar\Omega,\mathbb{S}^3)$] is dense in each of these spaces. We also define

$$H^{\rm inc}_0(\Omega,\mathbb{S}^3)=\text{ the closure of } \mathcal{D}(\Omega,\mathbb{S}^3) \text{ in } H^{\rm inc}(\Omega,\mathbb{S}^3),$$

where the notation $\mathcal{D}$ stands for compactly supported $\mathcal{C}^\infty$ functions, as well as the trace space

$$\tilde H^{3/2}(\partial\Omega,\mathbb{S}^3)=\left\{E\in H^{3/2}(\partial\Omega,\mathbb{S}^3):\ \int_{\partial\Omega} E N dS(x)=0\right\}.$$

Theorem 2.3 (Lifting [5]). *Let $\mathbb{E}\in\tilde H^{3/2}(\partial\Omega,\mathbb{S}^3)$, and $\mathbb{G}\in H^{1/2}(\partial\Omega,\mathbb{S}^3)$. There exists $E\in H^2(\Omega,\mathbb{S}^3)$ such that*

$$\begin{cases} E=\mathbb{E} & \text{on } \partial\Omega,\\ (\partial_N E)_T=\mathbb{G}_T & \text{on } \partial\Omega,\\ {\rm div}\, E=0 & \text{in } \Omega, \end{cases}$$

where the subscript T stands for the tangential part given by the components $(\mathbb{G}_T)_{RR'}=\mathbb{G}\tau^R\cdot\tau^{R'}$, $R,R'\in\{A,B\}$. In addition, such a lifting can be obtained through a linear continuous operator

$$\mathcal{L}_{\partial\Omega}:(\mathbb{E},\mathbb{G})\in\tilde H^{3/2}(\partial\Omega,\mathbb{S}^3)\times H^{1/2}(\partial\Omega,\mathbb{S}^3)\mapsto E\in H^2(\Omega,\mathbb{S}^3).$$

Define the subset of $\mathcal{C}^\infty(\partial\Omega,\mathbb{S}^3)$

$$\mathcal{G}=\{V\odot N, V\in\mathbb{R}^3\},$$

with the notation $U\odot V:=(U\otimes V+V\otimes U)/2$.

Lemma 2.4 (Dual trace space [5]). *Every $\mathbb{E}\in H^{-3/2}(\partial\Omega,\mathbb{S}^3)/\mathcal{G}$ admits a unique representative $\tilde{\mathbb{E}}$ such that*

$$\int_{\partial\Omega}\tilde{\mathbb{E}}N dS(x)=0. \tag{2.6}$$

Moreover, the dual space of $\tilde H^{3/2}(\partial\Omega,\mathbb{S}^3)$ is canonically identified with $H^{-3/2}(\partial\Omega,\mathbb{S}^3)/\mathcal{G}$.

Here and in the sequel, for the sake of readability, duality pairings are denoted by integrals.

2.3. **Green formula and applications.** Recall that the Green formula for the divergence allows us to define, for any $E\in H^{\rm div}(\Omega,\mathbb{S}^3)$, its normal trace $EN\in H^{-1/2}(\partial\Omega,\mathbb{R}^3)$ by

$$\int_{\partial\Omega}(EN)\cdot\varphi dS(x):=\int_\Omega({\rm div}\, E\cdot\tilde\varphi+E\cdot\nabla^S\tilde\varphi)dx\qquad\forall\varphi\in H^{1/2}(\partial\Omega,\mathbb{R}^3),$$

with $\tilde\varphi\in H^1(\Omega,\mathbb{R}^3)$ an arbitrary lifting of φ, see e.g. [23, 37]. For the incompatibility operator one has the following counterpart.

Lemma 2.5 (Green formula for the incompatibility [5]). *Suppose that $E\in\mathcal{C}^2(\overline\Omega,\mathbb{S}^3)$ and $\eta\in H^2(\Omega,\mathbb{S}^3)$. Then*

$$\int_\Omega E\cdot{\rm inc}\,\eta dx=\int_\Omega{\rm inc}\, E\cdot\eta dx+\int_{\partial\Omega}\mathcal{T}_1(E)\cdot\eta\ dS(x)+\int_{\partial\Omega}\mathcal{T}_0(E)\cdot\partial_N\eta\ dS(x) \tag{2.7}$$

with the trace operators defined as

$$\mathcal{T}_0(E) := (E\times N)^t\times N, \tag{2.8}$$

$$\mathcal{T}_1(E) := \left({\rm Curl}\ (E\times N)^t\right)^S+((\partial_N+k)E\times N)^t\times N+\left({\rm Curl}^t\ E\times N\right)^S, \tag{2.9}$$

where $k := \kappa^A + \kappa^B$ is twice the mean curvature of $\partial\Omega$, $E^S := (E + E^t)/2$ is the symmetric part of E, and cross products are computed row-wise. In addition, it holds

$$\int_{\partial\Omega} \mathcal{T}_1(E) N dS(x) = 0. \tag{2.10}$$

Alternative expressions for $\mathcal{T}_1(E)$ are given in [5], like

$$\mathcal{T}_1(E) = -\sum_R \kappa^R (E \times \tau^R)^t \times \tau^R - \sum_R \xi (E \times \tau^R)^t \times \tau^{R^*} + ((-\partial_N + k) E \times N)^t \times N \\ - 2 \left(\sum_R (\partial_R E \times N)^t \times \tau^R \right)^S . \tag{2.11}$$

For a general symmetric tensor E, with components $E_{RR'} := E\tau^{R'} \cdot \tau^R$ in the curvilinear frame, one has:

$$E = \begin{pmatrix} E_{AA} & E_{AB} & E_{AN} \\ E_{BA} & E_{BB} & E_{BN} \\ E_{NA} & E_{NB} & E_{NN} \end{pmatrix}, \qquad (E \times N)^t \times N = \begin{pmatrix} E_{BB} & -E_{AB} & 0 \\ -E_{AB} & E_{AA} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.12}$$

$$(E \times \tau^A)^t \times \tau^A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & E_{NN} & -E_{BN} \\ 0 & -E_{BN} & E_{BB} \end{pmatrix}, (E \times \tau^B)^t \times \tau^B = \begin{pmatrix} E_{NN} & 0 & -E_{AN} \\ 0 & 0 & 0 \\ -E_{AN} & 0 & E_{AA} \end{pmatrix}, \tag{2.13}$$

$$(E \times N)^t \times \tau^A = \begin{pmatrix} 0 & E_{BN} & -E_{BB} \\ 0 & -E_{AN} & E_{AB} \\ 0 & 0 & 0 \end{pmatrix}, (E \times N)^t \times \tau^B = \begin{pmatrix} -E_{BN} & 0 & E_{AB} \\ E_{AN} & 0 & -E_{AA} \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.14}$$

As shown in [5], we can define the traces $\mathcal{T}_0(E) \in H^{-1/2}(\partial\Omega, \mathbb{S}^3)$ and $\mathcal{T}_1(E) \in H^{-3/2}(\partial\Omega, \mathbb{S}^3)/\mathcal{G}$ for every $E \in H^{\mathrm{inc}}(\Omega, \mathbb{S}^3)$ by

$$\int_{\partial\Omega} \mathcal{T}_0(E) \cdot \varphi_0 \, dS(x) = \int_\Omega E \cdot \operatorname{inc} \eta_0 dx - \int_\Omega \operatorname{inc} E \cdot \eta_0 dx, \qquad \forall \varphi_0 \in H^{1/2}(\partial\Omega, \mathbb{S}^3),$$

$$\int_{\partial\Omega} \mathcal{T}_1(E) \cdot \varphi_1 \, dS(x) = \int_\Omega E \cdot \operatorname{inc} \eta_1 dx - \int_\Omega \operatorname{inc} E \cdot \eta_1 dx, \qquad \forall \varphi_1 \in \tilde{H}^{3/2}(\partial\Omega, \mathbb{S}^3),$$

with $\eta_0 = \mathcal{L}_{\partial\Omega}(0, \varphi_0)$ and $\eta_1 = \mathcal{L}_{\partial\Omega}(\varphi_1, 0)$ (recall that $\mathcal{L}_{\partial\Omega}$ is the lifting operator defined in Theorem 2.3, and observe that, by Lemma 2.5 and density of $\mathcal{C}^\infty(\bar{\Omega}, \mathbb{S}^3)$ in $H^{\mathrm{inc}}(\Omega, \mathbb{S}^3)$, these definitions are independent of the choices of liftings). In addition, by Lemma 2.4, $\mathcal{T}_1(E)$ admits a unique representative satisfying (2.10). By linearity of $\mathcal{L}_{\partial\Omega}$, this extends formula (2.7) to any functions $E \in H^{\mathrm{inc}}(\Omega, \mathbb{S}^3)$ and $\eta \in H^2(\Omega, \mathbb{S}^3)$.

Remark 2.1. *We have defined $\mathcal{T}_1(E)$ against test functions which admit divergence-free liftings, because spaces of divergence-free tensors arise naturally in problems involving the incompatibility, see the Beltrami decomposition and its consequences in the next sections. But we could also have defined $\mathcal{T}_1(E) \in H^{-3/2}(\partial\Omega, \mathbb{S}^3)$ by using a classical lifting in $H^2(\Omega, \mathbb{S}^3)$. Upon adopting the convention that representatives in $H^{-3/2}(\partial\Omega, \mathbb{S}^3)/\mathcal{G}$ satisfying the gauge condition (2.10) are chosen, the two definitions are equivalent.*

3. Properties of trace operators in $H^{\mathrm{inc}}(\Omega, \mathbb{S}^3)$

In this section, homogeneous displacement-like boundary conditions are analyzed in terms of traces of the symmetric strain. These results should be put in perspective with previous results about this problem obtained by Ciarlet and co-authors by means of change-of-metric and change-of-curvature tensors (see [14,15]). Though our characterization is different, the objective of expressing Dirichlet boundary conditions in terms of intrinsic quantities is the same.

To begin with, as particular cases of the two Green formulae recalled in the previous section, one readily obtains the following.

Lemma 3.1. *1. For all $v \in H^1(\Omega,\mathbb{R}^3)$, one has $\operatorname{inc}\nabla^S v = 0$ in the sense of distributions.*
2. For all $E \in H^{\operatorname{inc}}(\Omega,\mathbb{S}^3)$, one has $\operatorname{div}\operatorname{inc} E = 0$ in the sense of distributions.

Consequently, if $E \in H^{\operatorname{inc}}(\Omega,\mathbb{S}^3)$, then $\operatorname{inc} E\, N$ is defined in $H^{-1/2}(\partial\Omega,\mathbb{R}^3)$ by

$$\int_{\partial\Omega} (\operatorname{inc} E\, N)\cdot\varphi dS(x) = \int_\Omega \operatorname{inc} E\cdot\nabla^S\varphi dx \qquad \forall\varphi\in H^1(\Omega,\mathbb{R}^3).$$

Hereafter, we consider an open and sufficiently regular ($\mathcal{C}^\infty$ for simplicity) subset $\omega \subset\subset \Omega$. If u is a vector or tensor field defined over Ω with well-defined traces on each side of $\partial\omega$, we denote by $[\![u]\!]$ the jump of u across $\partial\omega$ with inner term counted positively.

Lemma 3.2. *If $E \in H^{\operatorname{inc}}(\Omega,\mathbb{S}^3)$, then $[\![\operatorname{inc} E\, N]\!] = 0$ across $\partial\omega$.*

Proof. Let $\varphi\in\mathcal{D}(\Omega,\mathbb{R}^3)$. By definition and Lemma 3.1, one has

$$\int_{\partial\omega}[\![\operatorname{inc} E\, N]\!]\cdot\varphi dx = \int_\Omega \operatorname{inc} E\cdot\nabla^S\varphi dx = 0.$$

By density this is also true for any $\varphi\in H^1_0(\Omega,\mathbb{R}^3)$, and subsequently for any trace of φ in $H^{1/2}(\partial\omega,\mathbb{R}^3)$. □

Lemma 3.3. *Let $E\in L^2(\Omega,\mathbb{S}^3)$ be such that $E_{|\omega}\in H^{\operatorname{inc}}(\omega,\mathbb{S}^3)$ and $E_{|\Omega\setminus\bar\omega}\in H^{\operatorname{inc}}(\Omega\setminus\bar\omega,\mathbb{S}^3)$. Then $[\![\mathcal{T}_0(E)]\!] = [\![\mathcal{T}_1(E)]\!] = 0$ across $\partial\omega$ if and only if $E\in H^{\operatorname{inc}}(\Omega,\mathbb{S}^3)$.*

Proof. Let $\Phi\in\mathcal{D}(\Omega,\mathbb{S}^3)$. We have in the sense of distributions $\langle\operatorname{inc} E,\Phi\rangle = \int_\Omega E\cdot\operatorname{inc}\Phi dx$, and the Green formula yields

$$\langle\operatorname{inc} E,\Phi\rangle = \int_\omega \operatorname{inc} E\cdot\Phi dx + \int_{\Omega\setminus\bar\omega}\operatorname{inc} E\cdot\Phi dx + \int_{\partial\omega}[\![\mathcal{T}_0(E)]\!]\cdot\partial_N\Phi dS(x) + \int_{\partial\omega}[\![\mathcal{T}_1(E)]\!]\cdot\Phi dS(x).$$

In the forward implication the last two integrals vanish by assumption, hence the distribution $\operatorname{inc} E\in\mathcal{D}'(\Omega,\mathbb{S}^3)$ is actually an L^2 function. In the converse implication the distribution $\operatorname{inc} E$ identifies with an L^2 function, whereby the last two integrals must vanish. □

If $E\in H^{\operatorname{inc}}(\Omega,\mathbb{S}^3)$ and $\mathcal{T}_0(E)=\mathcal{T}_1(E)=0$ on $\partial\Omega$, then extending E by 0 and applying Lemmas 3.3 and 3.2 yields $\operatorname{inc} E\, N = 0$ on $\partial\Omega$. If $v\in H^1_0(\Omega,\mathbb{R}^3)$, then by density of $\mathcal{D}(\Omega,\mathbb{R}^3)$ and continuity of the trace operators in $H^{\operatorname{inc}}(\Omega,\mathbb{S}^3)$, it follows $\mathcal{T}_0(\nabla^S v)=\mathcal{T}_1(\nabla^S v)=0$ on $\partial\Omega$. These two remarks admit the following local versions.

Considering a relatively open subset Γ of $\partial\Omega$ and given $E\in H^{\operatorname{inc}}(\Omega,\mathbb{S}^3)$, we say that $\mathcal{T}_0(E)=\mathcal{T}_1(E)=0$ on Γ if the corresponding distributions vanish on Γ, namely

$$\int_{\partial\Omega}\mathcal{T}_0(E)\cdot\varphi_0\, dS(x) = \int_{\partial\Omega}\mathcal{T}_1(E)\cdot\varphi_1\, dS(x) = 0 \qquad \forall\varphi_0,\varphi_1\in\mathcal{C}^\infty(\partial\Omega,\mathbb{S}^3),\ \operatorname{spt}\varphi_0\subset\Gamma,\ \operatorname{spt}\varphi_1\subset\Gamma, \tag{3.1}$$

and similarly that $\operatorname{inc} E\, N = 0$ on Γ if

$$\int_{\partial\Omega}(\operatorname{inc} E\, N)\cdot\varphi dS(x) = 0 \qquad \forall\varphi\in\mathcal{C}^\infty(\partial\Omega,\mathbb{R}^3),\ \operatorname{spt}\varphi\subset\Gamma. \tag{3.2}$$

Lemma 3.4. *If $E\in H^{\operatorname{inc}}(\Omega,\mathbb{S}^3)$ satisfies $\mathcal{T}_0(E)=\mathcal{T}_1(E)=0$ on Γ then $\operatorname{inc} E\, N = 0$ on Γ.*

Proof. Let $z\in\Gamma$ and B be an open ball of center z such that $\partial\Omega\cap B=\Gamma\cap B$ and $\Omega\cap B$ is on one side of $\Gamma\cap B$. Let $v\in\mathcal{C}^\infty(\bar\Omega,\mathbb{R}^3)$ with $\operatorname{spt} v\subset B$. We have by the Green formulae

$$\int_{\partial\Omega}(\operatorname{inc} E\, N)\cdot v dS(x) = \int_\Omega\operatorname{inc} E\cdot\nabla^S v dx = -\int_{\partial\Omega}\left(\mathcal{T}_0(E)\cdot\partial_N\nabla^S v + \mathcal{T}_1(E)\cdot\nabla^S v\right)dS(x) = 0.$$

By lifting, this holds true for any $v\in\mathcal{C}^\infty(\partial\Omega,\mathbb{R}^3)$ with support in B. By linearity, covering and partition of unity this extends to any $v\in\mathcal{C}^\infty(\partial\Omega,\mathbb{R}^3)$ with support in Γ. □

Lemma 3.5. *If $v\in H^1(\Omega,\mathbb{R}^3)$ satisfies $v=0$ on Γ in the sense of traces, then $\mathcal{T}_0(\nabla^S v) = \mathcal{T}_1(\nabla^S v) = 0$ on Γ.*

Proof. Let $z \in \Gamma$ and B be an open ball of center z such that $\partial\Omega \cap B = \Gamma \cap B$ and $\Omega \cap B$ is on one side of $\Gamma \cap B$. Let $\varphi \in \mathcal{C}^\infty(\bar\Omega, \mathbb{S}^3)$ with spt $\varphi \subset B$. We have

$$\langle \mathcal{T}_0(\nabla^S v), \partial_N \varphi\rangle + \langle \mathcal{T}_1(\nabla^S v), \varphi\rangle = \int_\Omega \nabla^S v \cdot \operatorname{inc} \varphi dx = \int_{\partial\Omega} (\operatorname{inc} \varphi\, N) \cdot v dS(x) = 0.$$

We conclude as in Lemma 3.4. □

Corollary 3.6. *Let $v \in H^1(\Omega, \mathbb{R}^3)$ be such that $v = r$ on Γ in the sense of traces, with r a rigid displacement field. Then $\mathcal{T}_0(\nabla^S v) = \mathcal{T}_1(\nabla^S v) = 0$ on Γ.*

Proof. On Γ it holds

$$\mathcal{T}_i(\nabla^S v) = \mathcal{T}_i(\nabla^S (v - r)) = 0, \qquad i = 0, 1,$$

by Lemma 3.5. □

Lemma 3.7. *Let $E \in H^2(\Omega, \mathbb{S}^3)$ such that $E = 0$ on Γ. Then $\mathcal{T}_1(E) = 0$ on Γ if and only if $\mathcal{T}_0(\partial_N E) = 0$ on Γ.*

Proof. Using (2.11) and (2.12)-(2.14) one obtains the expression of $\mathcal{T}_1(E)$ in the basis of principal curvatures for simplicity ($\xi = 0$), as follows:

$$\mathcal{T}_1(E) = \begin{pmatrix} -\partial_N E_{BB} + 2(\partial_B E)_{BN} + kE_{BB} - \kappa^B E_{NN} & \partial_N E_{AB} - kE_{AB} - (\partial_B E)_{AN} - (\partial_A E)_{BN} & \kappa^B E_{AN} + (\partial_A E)_{BB} - (\partial_B E)_{AB} \\ \partial_N E_{AB} - kE_{AB} - (\partial_B E)_{AN} - (\partial_A E)_{BN} & -\partial_N E_{AA} + 2(\partial_A E)_{AN} + kE_{AA} - \kappa^A E_{NN} & \kappa^A E_{BN} - (\partial_A E)_{AB} + (\partial_B E)_{AA} \\ \kappa^B E_{AN} + (\partial_A E)_{BB} - (\partial_B E)_{AB} & \kappa^A E_{BN} - (\partial_A E)_{AB} + (\partial_B E)_{AA} & -\kappa^A E_{BB} - \kappa^B E_{AA} \end{pmatrix}. \quad (3.3)$$

From $E = 0$ on Γ one infers that $\partial_R E = 0$ on Γ, with $R = A, B$. Thus, by (3.3), one obtains

$$\mathcal{T}_1(E) = \begin{pmatrix} -\partial_N E_{BB} & \partial_N E_{AB} & 0 \\ \partial_N E_{AB} & -\partial_N E_{AA} & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\mathcal{T}_0(\partial_N E)$$

on Γ, achieving the proof. □

We remark that the condition $\mathcal{T}_0(\partial_N E) = 0$ is equivalent to $(\partial_N E \times N)^t \times N = \operatorname{Curl}^t E \times N = 0$ on Γ. In particular one sees the role of the boundary condition expressed in terms of the Frank tensor $\operatorname{Curl}^t E$, namely $E = 0$ and $\operatorname{Curl}^t E \times N = 0$ is equivalent to $E = 0$ and $\mathcal{T}_1(E) = 0$.

Lemma 3.8. *We have the characterization*

$$H_0^{\rm inc}(\Omega, \mathbb{S}^3) = \left\{ E \in H^{\rm inc}(\Omega, \mathbb{S}^3) : \mathcal{T}_0(E) = \mathcal{T}_1(E) = 0 \text{ on } \partial\Omega \right\}.$$

Proof. Suppose $E_n \in \mathcal{D}(\Omega, \mathbb{S}^3)$, $E \in H^{\rm inc}(\Omega, \mathbb{S}^3)$, $E_n \to E$ in $H^{\rm inc}(\Omega, \mathbb{S}^3)$. Of course, $\mathcal{T}_0(E_n) = \mathcal{T}_1(E_n) = 0$ on $\partial\Omega$. Then by continuity $\mathcal{T}_0(E) = \mathcal{T}_1(E) = 0$ on $\partial\Omega$.

Suppose now $E \in H^{\rm inc}(\Omega, \mathbb{S}^3)$ with $\mathcal{T}_0(E) = \mathcal{T}_1(E) = 0$ on $\partial\Omega$. Extend E by 0 to get $\tilde E \in H^{\rm inc}(\mathbb{R}^3, \mathbb{S}^3)$. By local charts, shifting and convolution with mollifiers, we can define through a standard construction $E_n \in \mathcal{D}(\mathbb{R}^3, \mathbb{S}^3)$ such that $E_n \to \tilde E$ in $H^{\rm inc}(\mathbb{R}^3, \mathbb{S}^3)$ and spt $E_n \subset \Omega$. Hence $E_n \to E$ in $H^{\rm inc}(\Omega, \mathbb{S}^3)$, which yields $E \in H_0^{\rm inc}(\Omega, \mathbb{S}^3)$. □

4. Compatibility conditions and Beltrami decomposition

In this section we first recall the Beltrami decomposition of symmetric tensor fields, stated here in an L^p version for the sake of generality. A specific proof of the L^2 version, which in fact is our main concern, can be found in, e.g., [21, 22]. This structure theorem is named after Eugenio Beltrami (1835-1900), an Italian physicist and mathematician known in particular for his works

on elasticity, in particular by stating the equilibrium equations of a body in terms of the stress in place of the strain [8][2], but also in non-Euclidean geometries in the wake of Gauss and Riemann[3].

We need to introduce first the so-called Saint-Venant-Beltrami condition, originally considered by Saint-Venant in [7], then extended by Donati [17], Ting [38], Moreau [32], Ciarlet and Ciarlet in [13], Geymonat and Krasucki [20], and eventually by Amrouche at al. [3]. Below we give the version found in [28] (originally from [3]).

Theorem 4.1 (Saint-Venant-Beltrami compatibility conditions)**.** *Assume that Ω is simply-connected. Let $p \in (1, +\infty)$ be a real number and let $E \in L^p(\Omega, \mathbb{S}^3)$. Then,*

$$\operatorname{inc} E = 0 \text{ in } W^{-2,p}(\Omega, \mathbb{S}^3) \iff E = \nabla^S v$$

for some $v \in W^{1,p}(\Omega, \mathbb{R}^3)$. Moreover, v is unique up to rigid displacements.

Let us also refer to [25] and [33] for more details and references on this topic. The following decomposition will show crucial in our model. Pioneer version of this result can be found in [22] for $p = 2$.

Theorem 4.2 (Beltrami decomposition [28])**.** *Assume that Ω is simply-connected. Let $p \in (1, +\infty)$ be a real number and let $E \in L^p(\Omega, \mathbb{S}^3)$. Then, for any $v_0 \in W^{1/p,p}(\partial\Omega)$, there exists a unique $v \in W^{1,p}(\Omega, \mathbb{R}^3)$ with $v = v_0$ on $\partial\Omega$ and a unique $F \in L^p(\Omega, \mathbb{S}^3)$ with $\operatorname{Curl} F \in L^p(\Omega, \mathbb{R}^{3\times 3})$, $\operatorname{inc} F \in L^p(\Omega, \mathbb{S}^3)$, $\operatorname{div} F = 0$ and $FN = 0$ on $\partial\Omega$ such that*

$$E = \nabla^S v + \operatorname{inc} F. \tag{4.1}$$

We call v and F the velocity and incompatibility fields, respectively, associated with E. The following result is the dual counterpart of Saint-Venant's conditions.

Corollary 4.3 (Representation of solenoidal symmetric tensors)**.** *Assume that Ω is simply-connected. If $E \in L^2(\Omega, \mathbb{S}^3)$ satisfies $\operatorname{div} E = 0$ in $H^{-1}(\Omega, \mathbb{R}^3)$, then there exists a unique $F \in L^2(\Omega, \mathbb{S}^3)$ with $\operatorname{Curl} F \in L^2(\Omega, \mathbb{S}^3)$, $\operatorname{div} F = 0$ and $FN = 0$ on $\partial\Omega$ such that $E = \operatorname{inc} F$.*

Proof. Theorem 4.2 yields

$$E = \nabla^S v + \operatorname{inc} F,$$

with the appropriate F and $v \in H^1_0(\Omega, \mathbb{S}^3)$. The condition $0 = \operatorname{div} E = \operatorname{div} \nabla^S v$ entails $v = 0$. □

We now specialize Saint-Venant's decomposition in the case of homogeneous boundary conditions.

Proposition 4.4 (Saint-Venant with boundary conditions)**.** *Assume that Ω is simply-connected. If $E \in L^2(\Omega, \mathbb{S}^3)$ satisfies*

$$\begin{cases} \operatorname{inc} E = 0 \text{ in } \Omega, \\ \mathcal{T}_0(E) = \mathcal{T}_1(E) = 0 \text{ on } \partial\Omega, \end{cases} \tag{4.2}$$

then there exists $v \in H^1_0(\Omega, \mathbb{R}^3)$ such that $\nabla^S v = E$. Moreover, the map $E \in L^2(\Omega, \mathbb{S}^3) \mapsto v \in H^1_0(\Omega, \mathbb{R}^3)$ is linear and continuous.

Proof. Let $A : H^{-1}(\Omega, \mathbb{R}^3) \to L^2(\Omega, \mathbb{S}^3)$ be the linear map defined by $A\varphi = \nabla^S u$ with

$$\begin{cases} -\operatorname{div} \nabla^S u = \varphi \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Let $A^* : L^2(\Omega, \mathbb{S}^3) \to H^1_0(\Omega, \mathbb{R}^3)$ be the adjoint operator of A. Let $v = A^*E \in H^1_0(\Omega, \mathbb{R}^3)$. Let $\Phi \in \mathcal{D}(\Omega, \mathbb{S}^3)$. By definition we have

$$-\int_\Omega A^*E \operatorname{div} \Phi dx = -\int_\Omega E \cdot A(\operatorname{div} \Phi) dx.$$

[2]Here, Beltrami also showed a new proof of the conditions when six given functions are the components of an elastic deformation.

[3]Beltrami was indeed a friend of Riemann whom he met at Pisa university where he had a chair. Moreover, he was later professor in Rome, and his position was transmitted to Volterra in 1900. Vito Volterra (1860-1940) is presumably the first who gave a correct definition of dislocations and disclinations in [44].

Set $\Psi = A(\operatorname{div}\Phi)$. We have $-\operatorname{div}\Psi = \operatorname{div}\Phi$. By Corollary 4.3, $\Psi = -\Phi + \operatorname{inc}\zeta$ for some $\zeta \in H^{\operatorname{inc}}(\Omega, \mathbb{S}^3)$. We obtain

$$-\int_\Omega A^*E \cdot \operatorname{div}\Phi dx = \int_\Omega E \cdot \Phi dx - \int_\Omega E \cdot \operatorname{inc}\zeta dx.$$

By the Green formula and the assumptions it holds

$$\int_\Omega E \cdot \operatorname{inc}\zeta dx = 0.$$

We arrive at

$$-\int_\Omega A^*E \operatorname{div}\Phi dx = \int_\Omega E \cdot \Phi dx,$$

thus

$$\nabla^S(A^*E) = E$$

in the sense of distributions. □

We can now state a converse to Lemma 3.5.

Proposition 4.5. *Assume that Ω is simply connected. If $v \in H^1(\Omega, \mathbb{R}^3)$ is such that $\mathcal{T}_0(\nabla^S v) = \mathcal{T}_1(\nabla^S v) = 0$ on $\partial\Omega$ then there exists a rigid displacement field r such that $v = r$ on $\partial\Omega$.*

Proof. By Proposition 4.4, there exists $w \in H^1_0(\Omega, \mathbb{R}^3)$ such that $\nabla^S v = \nabla^S w$. Hence there exists a rigid displacement field r such that $v = w + r$. On $\partial\Omega$ this reduces to $v = r$. □

5. Orthogonal decompositions of symmetric tensors in L^2

We assume in this section that Ω is simply-connected.

5.1. **Orthogonal decomposition of $L^2(\Omega, \mathbb{S}^3)$.** In this section we obtain a decomposition of $L^2(\Omega, \mathbb{S}^3)$ into orthogonal subspaces, in the same spirit as in [22], but to account for more general boundary conditions. We define the spaces

$$\begin{aligned} \mathcal{V} &= \left\{E \in L^2(\Omega, \mathbb{S}^3) \ : \ \operatorname{inc} E = 0\right\}, \\ \mathcal{W} &= \left\{E \in L^2(\Omega, \mathbb{S}^3) \ : \ \operatorname{div} E = 0\right\}, \end{aligned}$$

and, given a subset Γ of $\partial\Omega$,

$$\begin{aligned} \mathcal{V}^0_\Gamma &= \{E \in \mathcal{V} \ : \ \mathcal{T}_0(E) = \mathcal{T}_1(E) = 0 \text{ on } \Gamma\}, \\ \mathcal{V}^{00}_\Gamma &= \left\{\nabla^S v \ : \ v \in H^1(\Omega, \mathbb{R}^3), v = 0 \text{ on } \Gamma\right\}, \\ \mathcal{W}^0_\Gamma &= \{E \in \mathcal{W} \ : \ EN{=}0 \text{ on } \Gamma\}. \end{aligned}$$

Recall that $\mathcal{V}^0_\Gamma$ is well-defined by (3.1) if Γ is a relatively open subset of $\partial\Omega$. In the definition of $\mathcal{W}^0_\Gamma$, $EN \equiv 0$ on Γ means

$$\int_{\partial\Omega} EN \cdot \varphi dS(x) = 0 \qquad \forall \varphi \in H^{1/2}(\partial\Omega, \mathbb{R}^3), \varphi_{|\partial\Omega\setminus\Gamma} = 0.$$

This is usually stronger than vanishing in the sense of distributions, see e.g. [24] for density and extension properties in fractional Sobolev spaces.

Remark 5.1. *By Theorem 4.1 and Corollary 4.3 we have*

$$\mathcal{V} = \left\{\nabla^S v, v \in H^1(\Omega, \mathbb{R}^3)\right\}, \tag{5.1}$$

$$\mathcal{W} = \left\{\operatorname{inc} F, F \in L^2(\Omega, \mathbb{S}^3), \operatorname{Curl} F \in L^2(\Omega, \mathbb{S}^3), \operatorname{div} F = 0 \text{ in } \Omega, FN = 0 \text{ on } \partial\Omega\right\}. \tag{5.2}$$

Moreover, the velocity field v in (5.1) is unique up to a rigid displacement field. The incompatibility field F in (5.2) is unique.

Remark 5.2. *If $|\Gamma| > 0$ then the velocity field v in the definition of $\mathcal{V}^{00}_\Gamma$ is unique.*

Theorem 5.1 (Orthogonal decomposition of L^2)**.** *Assume that $\partial\Omega$ admits the partition $\partial\Omega = \Gamma_1 \cup \Gamma_2$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$. We have the orthogonal decomposition*

$$L^2(\Omega, \mathbb{S}^3) = \mathcal{V}^{00}_{\Gamma_1} \oplus \mathcal{W}^0_{\Gamma_2}.$$

Proof. i) Let $\hat{E} \in \mathcal{V}^{00}_{\Gamma_1}$, $E \in \mathcal{W}^0_{\Gamma_2}$. We have $\hat{E} = \nabla^S \hat{v}$ for some $\hat{v} \in H^1(\Omega)$, $\hat{v} = 0$ on Γ_1. The Green formula entails

$$\int_\Omega \hat{E} \cdot E dx = \int_\Omega \nabla^S \hat{v} \cdot E dx = -\int_\Omega \hat{v} \cdot \operatorname{div} E dx + \int_{\partial\Omega} \hat{v} \cdot EN dS(x) = 0.$$

ii) Let $E \in L^2(\Omega, \mathbb{S}^3)$. Write the Beltrami decomposition of Theorem 4.2 as $E = \nabla^S v + \operatorname{inc} F$ with $v = 0$ on $\partial\Omega$. Let $w \in H^1(\Omega, \mathbb{R}^3)$ be the solution of

$$\begin{cases} -\operatorname{div} \nabla^S w = 0 \text{ in } \Omega, \\ w = 0 \text{ on } \Gamma_1, \\ \nabla^S w N = \operatorname{inc} F N \text{ on } \Gamma_2, \end{cases}$$

that is, $w \in H^1_{\Gamma_1}(\Omega, \mathbb{R}^3) := \{\varphi \in H^1(\Omega, \mathbb{R}^3) : \varphi_{|\Gamma_1} = 0\}$,

$$\int_\Omega \nabla^S w \cdot \nabla^S \varphi dx = \int_\Omega \operatorname{inc} F \cdot \nabla^S \varphi dx \qquad \forall \varphi \in H^1_{\Gamma_1}(\Omega, \mathbb{R}^3).$$

We infer

$$E = \nabla^S(v + w) + (\operatorname{inc} F - \nabla^S w) \in \mathcal{V}^{00}_{\Gamma_1} + \mathcal{W}^0_{\Gamma_2},$$

since by definition

$$\int_{\partial\Omega} (\operatorname{inc} F - \nabla^S w) N \cdot \varphi dS(x) = \int_\Omega (\operatorname{inc} F - \nabla^S w) \cdot \nabla^S \varphi dx = 0 \qquad \forall \varphi \in H^1_{\Gamma_1}(\Omega, \mathbb{R}^3).$$

This completes the proof. □

Remark 5.3. *By Lemma 3.5 we have*

$$\mathcal{V}^{00}_\Gamma \subset \mathcal{V}^0_\Gamma,$$

whenever Γ is a relatively open subset of $\partial\Omega$, and we infer from Proposition 4.4 that

$$\mathcal{V}^{00}_{\partial\Omega} = \mathcal{V}^0_{\partial\Omega}.$$

In this case the decomposition of Theorem 5.1 is the same as in [22, Theorem 2.1.].

We have the following additional property.

Lemma 5.2. *If $\mathbb{K} \in \mathcal{V}^{00}_{\Gamma_1}$ and $\operatorname{inc} \hat{F} \in \mathcal{W}^0_{\Gamma_2}$ then it holds*

$$\int_{\partial\Omega} \left(\mathcal{T}_1(\mathbb{K}) \cdot \hat{F} + \mathcal{T}_0(\mathbb{K}) \cdot \partial_N \hat{F} \right) dS(x) = 0.$$

Proof. By the Green formula, we obtain

$$\int_{\partial\Omega} \left(\mathcal{T}_1(\mathbb{K}) \cdot \hat{F} + \mathcal{T}_0(\mathbb{K}) \cdot \partial_N \hat{F} \right) dS(x) = \int_\Omega \left(\mathbb{K} \cdot \operatorname{inc} \hat{F} - \operatorname{inc} \mathbb{K} \cdot \hat{F} \right) dx = \int_\Omega \mathbb{K} \cdot \operatorname{inc} \hat{F} dx.$$

Writing $\mathbb{K} = \nabla^S w$ and applying the Green formula for the divergence yields

$$\int_{\partial\Omega} \left(\mathcal{T}_1(\mathbb{K}) \cdot \hat{F} + \mathcal{T}_0(\mathbb{K}) \cdot \partial_N \hat{F} \right) dS(x) = \int_{\partial\Omega} \operatorname{inc} \hat{F} N \cdot w dS(x).$$

However, we have $w = 0$ on Γ_1 while $\operatorname{inc} \hat{F} N = 0$ on Γ_2, achieving the proof. □

5.2. **Orthogonal decomposition of $H^{\mathrm{inc}}(\Omega,\mathbb{S}^3)$ and related results.** Define

$$\begin{aligned}\mathcal{Z} &= \{E\in H^{\mathrm{inc}}(\Omega,\mathbb{S}^3)\ :\ \operatorname{div} E=0 \text{ in } \Omega, EN=0 \text{ on } \partial\Omega\},\\ \mathcal{Z}_0 &= \{E\in \mathcal{Z}\ :\ \operatorname{inc} EN=0 \text{ on } \partial\Omega\},\\ \mathcal{F} &= \{E\in H^{\mathrm{inc}}(\Omega,\mathbb{S}^3)\ :\ \operatorname{inc} EN=0 \text{ on } \partial\Omega\}.\end{aligned} \tag{5.3}$$

These spaces are endowed with the norm of $H^{\mathrm{inc}}(\Omega,\mathbb{S}^3)$. By virtue of Theorem 5.1 we infer the following decompositions.

Proposition 5.3 (Orthogonal decomposition of H^{inc})**.** *We have the orthogonal decompositions*

$$\begin{aligned}H^{\mathrm{inc}}(\Omega,\mathbb{S}^3) &= \mathcal{V}\oplus\mathcal{Z},\\ \mathcal{F} &= \mathcal{V}\oplus\mathcal{Z}_0.\end{aligned} \tag{5.4}$$

We now gather some properties of the spaces $\mathcal{Z}$ and $\mathcal{Z}_0$.

Proposition 5.4. *If $E\in\mathcal{Z}$ then* $\operatorname{Curl} E\in L^2(\Omega,\mathbb{S}^3)$. *Moreover there exists $c>0$ such that*

$$\|E\|_{L^2}+\|\operatorname{Curl} E\|_{L^2}\le c\|\operatorname{inc} E\|_{L^2}\qquad \forall E\in\mathcal{Z}.$$

Proof. Let

$$X=\left\{F\in L^2(\Omega,\mathbb{S}^3),\ \operatorname{Curl} F\in L^2,\ \operatorname{inc} F\in L^2,\ \operatorname{div} F=0, FN=0 \text{ on } \partial\Omega\right\},$$

$$Y=\left\{F\in L^2(\Omega,\mathbb{S}^3),\ \operatorname{div} F=0\right\}$$

and define the linear map $\Phi: X\to Y$ by $\Phi(E)=\operatorname{inc} E$. Equip X and Y with the norms

$$\|F\|_X=\|F\|_{L^2}+\|\operatorname{Curl} F\|_{L^2}+\|\operatorname{inc} F\|_{L^2},$$

$$\|F\|_Y=\|F\|_{L^2}.$$

Clearly, X and Y are Banach spaces and Φ is continuous. If $E\in X$ and $\Phi(E)=0$ then $\operatorname{inc} E=0$ and E is a symmetric gradient by Theorem 4.1. From $\operatorname{div} E=0$ and $EN=0$ on $\partial\Omega$, one obtains $E=0$. Hence Φ is injective. By Corollary 4.3, Φ is also surjective. The open mapping theorem entails that Φ^{-1} is continuous. Hence there exists $c>0$ such that

$$\|\Phi^{-1}(F)\|_X\le c\|F\|_{L^2}\qquad \forall F\in Y.$$

Let $E\in\mathcal{Z}$. Then $\operatorname{inc} E\in Y$. Set $F=\Phi^{-1}(\operatorname{inc} E)$. From $\operatorname{inc} F=\operatorname{inc} E$, $\operatorname{div} F=\operatorname{div} E=0$ and $FN=EN=0$ on $\partial\Omega$ we infer $F=E$. From $E=\Phi^{-1}(\operatorname{inc} E)\in X$ we obtain

$$\|E\|_{L^2}+\|\operatorname{Curl} E\|_{L^2}+\|\operatorname{inc} E\|_{L^2}=\|E\|_X=\|\Phi^{-1}(\operatorname{inc} E)\|_X\le c\|\operatorname{inc} E\|_{L^2} \tag{5.5}$$

and the result follows. □

The following result is proved in [23, Theorem 3.8.], see [26,35,45] for L^p versions, generalizations and extensions to non simply connected domains.

Theorem 5.5. *There exists a constant $c>0$ such that*

$$\|u\|_{H^1}\le c(\|\operatorname{div} u\|_{L^2}+\|\operatorname{Curl} u\|_{L^2})$$

for all $u\in L^2(\Omega,\mathbb{R}^3)$ such that $\operatorname{div} u\in L^2$, $\operatorname{Curl} u\in L^2$ *and $u\cdot N=0$ on $\partial\Omega$.*

Proposition 5.6 (Poincaré's inequality in $\mathcal{Z}$)**.** *There exists $c_P>0$ such that for all $E\in\mathcal{Z}$*

$$\|E\|_{H^1}\le c_P\|\operatorname{inc} E\|_{L^2}.$$

Proof. Let $E\in\mathcal{Z}$. By Proposition 5.4 we already have

$$\|E\|_{L^2}+\|\operatorname{Curl} E\|_{L^2}\le c\|\operatorname{inc} E\|_{L^2}.$$

Then Theorem 5.5 yields

$$\|\nabla E\|_{L^2}\le c\|\operatorname{Curl} E\|_{L^2}$$

for some other constant c. This completes the proof. □

We infer in particular that $\mathcal{Z}$ is imbedded in $H^1(\Omega,\mathbb{S}^3)$ and compactly imbedded in $L^2(\Omega,\mathbb{S}^3)$.

Proposition 5.7. *We have the representation*

$$\mathcal{W}^0_{\partial\Omega} = \operatorname{inc} \mathcal{Z}_0.$$

Proof. Of course, if $F \in \mathcal{Z}_0$, then $\operatorname{inc} F \in \mathcal{W}^0_{\partial\Omega}$. Take $E \in \mathcal{W}^0_{\partial\Omega}$. By Corollary 4.3 there exists $F \in H^{\operatorname{inc}}(\Omega, \mathbb{S}^3)$ with $\operatorname{div} F = 0$ and $FN = 0$ on $\partial\Omega$ such that $E = \operatorname{inc} F$. The condition $EN = 0$ on $\partial\Omega$ yields $F \in \mathcal{Z}_0$. □

Lemma 5.8. *Given a symmetric uniformly positive definite fourth order tensor field $\mathbb{B}$ (i.e. $\mathbb{B}(x)T \cdot T > \alpha|T|^2$ $\forall T \in \mathbb{S}^3$ for some $\alpha > 0$ independent of x) such that $|\mathbb{B}| \in L^\infty(\Omega)$, define the linear map $L_{\mathbb{B}} : \mathcal{Z} \to \mathcal{Z}'$ by*

$$\langle L_{\mathbb{B}} E, \Phi \rangle = \int_\Omega \mathbb{B} \operatorname{inc} E \cdot \operatorname{inc} \Phi dx \qquad \forall E, \Phi \in \mathcal{Z}.$$

Then $L_{\mathbb{B}}$ is an isomorphism from $\mathcal{Z}$ into $\mathcal{Z}'$.

Proof. By Proposition 5.4, $\langle L_{\mathbb{B}} E, E\rangle$ defines a norm in $\mathcal{Z}$ equivalent to the H^{inc}-norm. Let $\mathcal{T} \in \mathcal{Z}'$. By the Riesz representation theorem, there exists $T \in \mathcal{Z}$ such that $\langle \mathcal{T}, \Phi\rangle = \langle L_{\mathbb{B}} T, \Phi\rangle$ for all $\Phi \in \mathcal{Z}$. Therefore $L_{\mathbb{B}}$ is an isomorphism. □

We define the inverse map $L^{-1}_{\mathbb{B}} : \mathcal{Z}' \to \mathcal{Z}$, that is continuous by Banach's continuous inverse theorem. Since $\mathcal{Z} \subset L^2(\Omega, \mathbb{S}^3) \subset \mathcal{Z}'$, the restriction $L^{-1}_{\mathbb{B}} : L^2(\Omega, \mathbb{S}^3) \to L^2(\Omega, \mathbb{S}^3)$ is also well-defined.

Lemma 5.9. *The operator $L^{-1}_{\mathbb{B}} : L^2(\Omega, \mathbb{S}^3) \to L^2(\Omega, \mathbb{S}^3)$ is self-adjoint positive definite and compact.*

Proof. The compactness stems from the compact embedding $\mathcal{Z} \hookrightarrow L^2(\Omega, \mathbb{S}^3)$, consequence of Proposition 5.6. One has for all $E, F \in L^2(\Omega, \mathbb{S}^3)$

$$\int_\Omega L^{-1}_{\mathbb{B}} E \cdot F dx = \langle F, L^{-1}_{\mathbb{B}} E\rangle = \langle L_{\mathbb{B}} L^{-1}_{\mathbb{B}} F, L^{-1}_{\mathbb{B}} E\rangle = \int_\Omega \mathbb{B} \operatorname{inc}(L^{-1}_{\mathbb{B}} E) \cdot \operatorname{inc}(L^{-1}_{\mathbb{B}} F) dx.$$

It follows that $L^{-1}_{\mathbb{B}}$ is self-adjoint and positive definite, achieving the proof. □

6. Two elliptic boundary value problems for the incompatibility

Lemmas 5.8 and 5.9 yield the following proposition.

Proposition 6.1 (Weak form in $\mathcal{Z}$)**.** *Let $\mathbb{K} \in L^2(\Omega, \mathbb{S}^3)$ and $\mathbb{B}$ a symmetric uniformly positive definite fourth order tensor field. There exists a unique $E \in \mathcal{Z}$ such that*

$$\int_\Omega \mathbb{B} \operatorname{inc} E \cdot \operatorname{inc} \hat{E} dx = \int_\Omega \mathbb{K} \cdot \hat{E} dx \qquad \forall \hat{E} \in \mathcal{Z}. \tag{6.1}$$

Moreover, the solution map $\Phi : \mathbb{K} \in L^2(\Omega, \mathbb{S}^3) \to E \in L^2(\Omega, \mathbb{S}^3)$ is linear and compact.

Similarly we have the following.

Proposition 6.2 (Weak form in $\mathcal{Z}_0$)**.** *Let $\mathbb{K} \in L^2(\Omega, \mathbb{S}^3)$ and $\mathbb{B}$ a symmetric uniformly positive definite fourth order tensor field. There exists a unique $E \in \mathcal{Z}_0$ such that*

$$\int_\Omega \mathbb{B} \operatorname{inc} E \cdot \operatorname{inc} \hat{E} dx = \int_\Omega \mathbb{K} \cdot \hat{E} dx \qquad \forall \hat{E} \in \mathcal{Z}_0. \tag{6.2}$$

Moreover, the solution map $\Phi_0 : \mathbb{K} \in L^2(\Omega, \mathbb{S}^3) \to E \in L^2(\Omega, \mathbb{S}^3)$ is linear and compact.

Proposition 6.3 (Strong form in $\mathcal{Z}$)**.** *Let $\mathbb{K}$ be such that $\operatorname{div} \mathbb{K} = 0$ in Ω and $\mathbb{K}N = 0$ on $\partial\Omega$. Then, the strong form of (6.1) reads*

$$\begin{cases} \operatorname{inc}(\mathbb{B} \operatorname{inc} E) &= \mathbb{K} \quad \text{in } \Omega, \\ \operatorname{div} E &= 0 \quad \text{in } \Omega, \\ EN &= 0 \quad \text{on } \partial\Omega, \\ \mathcal{T}_0(\mathbb{B} \operatorname{inc} E) = \mathcal{T}_1(\mathbb{B} \operatorname{inc} E) &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{6.3}$$

whose solution coincides with the solution of the weak form.

Proof. Eq. (6.1) holds actually true for all $\hat{E} \in \mathcal{Z}+\mathcal{V} = H^{\mathrm{inc}}(\Omega,\mathbb{S}^3)$, in particular for $\hat{E} \in \mathcal{D}(\Omega,\mathbb{S}^3)$ and for $\hat{E}$ with arbitrary traces $\mathcal{T}_0(\partial_N \hat{E})$ and $\hat{E}$ on $\partial\Omega$, by Theorem 2.3. Then the Green formula provides the strong form, which is seen to be equivalent to the weak form. □

Remark 6.1 (Strong form in $\mathcal{Z}_0$)**.** *The solution of* (6.2) *satisfies the strong form*

$$\left\{\begin{array}{rcll} \mathrm{inc}\,(\mathbb{B}\,\mathrm{inc}\,E) & = & \mathbb{K} & \text{in } \Omega, \\ \mathrm{div}\,E & = & 0 & \text{in } \Omega, \\ EN & = & 0 & \text{on } \partial\Omega, \\ (\mathrm{inc}\,E)N & = & 0 & \text{on } \partial\Omega. \end{array}\right. \tag{6.4}$$

In fact, one can take any test function $\hat{E} \in \mathcal{D}(\Omega,\mathbb{S}^3) \subset \mathcal{F} = \mathcal{V} + \mathcal{Z}_0$. One obtains the strong form in Ω. The boundary conditions are given by the essential condition of the space.

7. A model of incompatible small-strain elasticity

7.1. **Internal efforts and incompatibility modulus.** We recall the main steps of the construction of the model introduced in [6].

Assumption 1. The power of the internal efforts within the solid body against the virtual strain rate $\hat{E}$ is is a continuous linear function of $\hat{E} \in L^2(\Omega,\mathbb{S}^3)$. By the Riesz representation theorem we infer the existence of $\Sigma \in L^2(\Omega,\mathbb{S}^3)$ such that

$$W_{\mathrm{int}}(\hat{E}) = \int_\Omega \Sigma \cdot \hat{E} dx \qquad \forall \hat{E} \in L^2(\Omega,\mathbb{S}^3).$$

Assumption 2. There exists a partition of Ω as $\Omega = \bigcup \Omega_p$ such that

$$W_{\mathrm{int}}(\hat{E}) = \int_{\Omega_p} \left(\sigma \cdot \hat{E} + \tau \cdot \nabla \hat{E}\right) dx \qquad \forall \hat{E} \in \mathcal{D}(\Omega_p,\mathbb{S}^3).$$

The second and third order tensor fields σ and τ are called the (Cauchy) stress and hyperstress tensors, respectively. Moreover, the material is supposed to be linear, homogeneous and isotropic within each Ω_p, which is represented by the constitutive laws

$$\sigma = \mathbb{C}_p E, \qquad \tau = \mathbb{D}_p \nabla E \tag{7.1}$$

where E is the strain, $\mathbb{C}_p$ is standard Hooke's tensor and $\mathbb{D}_p$ is Mindlin's tensor [31]. These constitutive laws read componentwise

$$\sigma_{ij} = \lambda \delta_{ij} E_{kk} + 2\mu E_{ij}, \tag{7.2}$$

$$\begin{aligned} \tau_{ijk} &= c_1(\delta_{ki}\partial_l E_{lj} + \delta_{kj}\partial_l E_{li}) + \frac{c_2}{2}(\delta_{ki}\partial_j E_{ll} + \delta_{kj}\partial_i E_{ll} + 2\delta_{ij}\partial_l E_{lk}) + 2c_3\delta_{ij}\partial_k E_{ll} \\ &+ 2c_4\partial_k E_{ij} + c_5(\partial_i E_{jk} + \partial_j E_{ik}), \end{aligned} \tag{7.3}$$

where $\lambda, \mu, c_1, ..., c_5$ are constants assigned to each Ω_p (index p is dropped for readability). Assumption 1 yields $\Sigma = \sigma - \mathrm{div}\,\tau \in L^2(\Omega,\mathbb{S}^3)$.

Assumption 4. The hyperstress τ does not produce any virtual intrinsic power as soon as the strain E is compatible. This means

$$\mathrm{inc}\,E = 0 \Rightarrow \int_\Omega \tau \cdot \nabla \hat{E}\, dx = 0 \ \forall \hat{E} \in \mathcal{D}(\Omega,\mathbb{S}^3),$$

or equivalently $\mathrm{inc}\,E = 0 \Rightarrow -\mathrm{div}\,\tau = 0$ in Ω. From expression (7.3) we derive the existence within each Ω_p of a constant ℓ_p such that $c_1 + c_5 = -\ell_p$, $c_2 = \ell_p$, $c_3 = -\ell_p/2$, $c_4 = \ell_p/2$, leading to $-\mathrm{div}\,\tau = \ell_p\,\mathrm{inc}\,E$ (see details in [6]).

Conclusion. We denote $\ell = \sum \ell_p \chi_{\Omega_p}$ and $\mathbb{C} = \sum \mathbb{C}_p \chi_{\Omega_p}$, whereby $\sigma = \mathbb{C}E$ and $-\mathrm{div}\,\tau = \ell\,\mathrm{inc}\,E$ in Ω. The expression of the internal virtual power is

$$W_{\mathrm{int}}(\hat{E}) = \int_\Omega (\mathbb{C}E + \ell\,\mathrm{inc}\,E) \cdot \hat{E} dx \qquad \forall \hat{E} \in L^2(\Omega,\mathbb{S}^3).$$

The scalar field ℓ is called incompatibility modulus, as it expresses the resistance of the material against incompatible deformations. Subsequently we will extend the model to the case where ℓ is a sufficiently regular function of $x \in \Omega$.

7.2. **Power of external efforts.** The power of external efforts is assumed to be a linear functional on $L^2(\Omega, \mathbb{S}^3)$. By Riesz representation, there exists $\mathbb{K} \in L^2(\Omega, \mathbb{S}^3)$ such that

$$W_{\text{ext}}(\hat{E}) = \int_\Omega \mathbb{K} \cdot \hat{E} dx.$$

We emphasize that the power of external efforts may be at first expressed in terms of the non-objective fields $\hat{v}$ and $\hat{F}$ of the Beltrami decomposition of $\hat{E}$. However, provided attention is paid to the uniqueness of the decomposition, these fields are themeselves linear functions of $\hat{E}$. This will specified in Section 10.1.

7.3. **Virtual power principle.** The virtual power principle in the absence of inertia reads

$$W_{\text{int}}(\hat{E}) = W_{\text{ext}}(\hat{E}),$$

that is

$$\int_\Omega (\mathbb{C}E + \ell \operatorname{inc} E) \cdot \hat{E} dx = \int_\Omega \mathbb{K} \cdot \hat{E} dx, \tag{7.4}$$

for all $\hat{E} \in L^2(\Omega, \mathbb{S}^3)$ satisfying possible kinematical constraints. In the absence of kinematical constraints, (7.4) is obviously equivalent to

$$\mathbb{C}E + \ell \operatorname{inc} E = \mathbb{K}. \tag{7.5}$$

7.4. **Time-evolution of a nonlinear incremental model in generalized elasticity.** Within an incremental formulation, $\mathbb{C}$ and ℓ are generalized elastic tangent moduli. They need to be updated at each increment as soon as nonlinear phenomena occur. The stress-strain relation is therefore piecewise linear. Typically, in a region with plastic deformations, the Lamé coefficients and the incompatibility modulus ℓ are expected to be less than in purely elastic regions. The way these coefficients evolve is driven by nonlinear constitutive laws that substitute to flow rules and hardening models.

8. Solution of elasto-plasticity equations with natural boundary condition

The main problem we address is the following: given $\mathbb{K} \in L^2(\Omega, \mathbb{S}^3)$, find E solution of (7.4).

8.1. **Kinematical setting.** We will limit ourselves to the case where no kinematical constraint is assumed on the virtual strain $\hat{E}$. Therefore, the problem reduces to (7.5). However, we will see that the absence of constraint on E leads to nonunique solutions, and that uniqueness is obtained by prescribing the incompatibility flux $\operatorname{inc} E\, N$ on $\partial\Omega$. The homogeneous case $\operatorname{inc} E\, N = 0$ is studied first. However prescribing a given value, either 0 or not, for the incompatibility flux may seem artificial from a modeling point of view. This issue is related to the characterization of the behavior of dislocations at interfaces, whose difficulty is emphasized in [30] in the case of grain boundaries for polycrystals. An attempt to determine the incompatibility flux through a domain extension technique is proposed in Section 10.3.

Remark 8.1. *A particular kinematical setting is to require $\mathbb{K} \in \mathcal{V}$, and a very special case occurs when $\mathbb{K} = \nabla^S v$ with $\operatorname{div} v = \operatorname{tr} \mathbb{K}$ constant. Then for $\mathbb{C}$ constant a solution to $\mathbb{C}E + \ell \operatorname{inc} E = \mathbb{K}$ is $E = \mathbb{C}^{-1}\mathbb{K}$. Indeed by the structure of $\mathbb{C}^{-1}$ one has E proportional to $\mathbb{K}$ plus a constant tensor hence $\operatorname{inc} E = 0$.*

8.2. **Well-posedness with vanishing incompatibility flux.** For the subsequent mathematical analysis it is not required that $\mathbb{C}$ be an isotropic Hooke tensor. We denote by $\mathbb{S}^{3\times 3}$ the set of symmetric fourth order tensors acting on 3×3 matrices. Our main result is the following.

Theorem 8.1 (First existence result)**.** *Assume Ω is simply connected. Let $\mathbb{K}\in L^2(\Omega,\mathbb{S}^3)$, $\mathbb{C}\in L^\infty(\Omega,\mathbb{S}^{3\times 3})$, $\ell\in L^\infty(\Omega)$. Let c_P be the Poincaré constant of Proposition 5.6. If $\mathbb{C}$ is uniformly positive definite and either $\ell > c_P|\mathbb{C}|$ a.e. or $\ell < -c_P|\mathbb{C}|$ a.e., then there exists one and only one $E\in\mathcal{F}$ such that*

$$\mathbb{C}E + \ell \operatorname{inc} E = \mathbb{K}.$$

Moreover we have the a priori estimate

$$\|\operatorname{inc} E\|_{L^2} \le \frac{\|\ell^{-1}\mathbb{C}\|_{L^\infty}}{1 - c_P\|\ell^{-1}\mathbb{C}\|_{L^\infty}} \|\mathbb{C}^{-1}\mathbb{K}\|_{L^2}. \tag{8.1}$$

Proof. We write the problem as

$$E + \mathbb{B}\operatorname{inc} E = \mathbb{H} \tag{8.2}$$

with $\mathbb{B} := \ell\mathbb{C}^{-1}$ and $\mathbb{H} := \mathbb{C}^{-1}\mathbb{K}$. Note that $\mathbb{B}$ is uniformly positive definite if $\ell > 0$ and uniformly negative definite if $\ell < 0$. We will first prove uniqueness and then existence.
Step 1. Uniqueness. Let $E\in\mathcal{F}$ be such that

$$E + \mathbb{B}\operatorname{inc} E = 0. \tag{8.3}$$

Take the orthogonal decomposition $E = E_c + E_i$ with $E_c\in\mathcal{V}$ and $E_i\in\mathcal{Z}_0$. We have

$$E_c + E_i + \mathbb{B}\operatorname{inc} E_i = 0. \tag{8.4}$$

Take $\hat{F}\in\mathcal{Z}_0$. Then

$$\int_\Omega E_c\cdot\operatorname{inc}\hat{F}dx + \int_\Omega E_i\cdot\operatorname{inc}\hat{F}dx + \int_\Omega \mathbb{B}\operatorname{inc} E_i\cdot\operatorname{inc}\hat{F}dx = 0.$$

By $\operatorname{inc}\hat{F}N = 0$ on $\partial\Omega$ the first integral vanishes. Specifically, take $\hat{F} = E_i$. We obtain

$$\int_\Omega E_i\cdot\operatorname{inc} E_i dx + \int_\Omega \mathbb{B}\operatorname{inc} E_i\cdot\operatorname{inc} E_i dx = 0.$$

Set $\tilde{\mathbb{B}} = \mathbb{B}$ if $\ell > 0$, $\tilde{\mathbb{B}} = -\mathbb{B}$ if $\ell < 0$, so that $\tilde{\mathbb{B}}$ is always positive definite. We have

$$\begin{aligned}\|\operatorname{inc} E_i\|_{L^2}^2 &= \int_\Omega \operatorname{inc} E_i\cdot\operatorname{inc} E_i dx = \int_\Omega \tilde{\mathbb{B}}^{-1}(\tilde{\mathbb{B}}^{1/2}\operatorname{inc} E_i)\cdot(\tilde{\mathbb{B}}^{1/2}\operatorname{inc} E_i)dx\\ &\le \|\tilde{\mathbb{B}}^{-1}\|_{L^\infty}\int_\Omega \tilde{\mathbb{B}}\operatorname{inc} E_i\cdot\operatorname{inc} E_i dx = \|\mathbb{B}^{-1}\|_{L^\infty}\left|\int_\Omega \mathbb{B}\operatorname{inc} E_i\cdot\operatorname{inc} E_i dx\right|.\end{aligned} \tag{8.5}$$

By Proposition 5.6 we obtain

$$\begin{aligned}c_P\|\operatorname{inc} E_i\|_{L^2}^2 &\ge \|E_i\|_{L^2}\|\operatorname{inc} E_i\|_{L^2}\\ &\ge \left|\int_\Omega E_i\cdot\operatorname{inc} E_i dx\right| = \left|\int_\Omega \mathbb{B}\operatorname{inc} E_i\cdot\operatorname{inc} E_i\right| \ge \|\mathbb{B}^{-1}\|_{L^\infty}^{-1}\|\operatorname{inc} E_i\|_{L^2}^2,\end{aligned}$$

that is,

$$(c_P\|\mathbb{B}^{-1}\|_{L^\infty} - 1)\|\operatorname{inc} E_i\|_{L^2}^2 \ge 0.$$

If $\|\mathbb{B}^{-1}\|_{L^\infty} < c_P^{-1}$ we infer $\operatorname{inc} E_i = 0$ then $E_i = 0$, by Proposition 5.6. Thus (8.4) yields $E_c = 0$, and eventually $E = 0$.
Step 2. Existence. Let $E = E_c + E_i\in\mathcal{F}$, $E_c\in\mathcal{V}$, $E_i\in\mathcal{Z}_0$. Then (8.2) is equivalent to

$$\begin{cases}\int_\Omega (E_c + \mathbb{B}\operatorname{inc} E_i)\cdot\hat{E}_c dx = \int_\Omega \mathbb{H}\cdot\hat{E}_c dx,\ \forall\hat{E}_c\in\mathcal{V},\\ \int_\Omega (E_i + \mathbb{B}\operatorname{inc} E_i)\cdot\hat{E}_i dx = \int_\Omega \mathbb{H}\cdot\hat{E}_i dx,\ \forall\hat{E}_i\in\mathcal{W}^0_{\partial\Omega},\end{cases} \tag{8.6}$$

itself, by Proposition 5.7, equivalent to

$$\begin{cases}\int_\Omega (E_c + \mathbb{B}\operatorname{inc} E_i)\cdot\nabla^S\hat{v}dx = \int_\Omega \mathbb{H}\cdot\nabla^S\hat{v}dx\ \forall\hat{v}\in H^1(\Omega), & (a)\\ \int_\Omega (E_i + \mathbb{B}\operatorname{inc} E_i)\cdot\operatorname{inc}\hat{F}dx = \int_\Omega \mathbb{H}\cdot\operatorname{inc}\hat{F}dx\ \forall\hat{F}\in\mathcal{Z}_0. & (b)\end{cases} \tag{8.7}$$

Define the operators $L_{\mathbb{B}} : \mathcal{Z}_0 \to \mathcal{Z}_0'$ and $M : L^2(\Omega, \mathbb{S}^3) \to \mathcal{Z}_0'$ by

$$\langle L_{\mathbb{B}}\Psi, \Phi\rangle = \int_\Omega \mathbb{B}\operatorname{inc}\Psi \cdot \operatorname{inc}\Phi dx, \qquad \langle M\Psi, \Phi\rangle = \int_\Omega \Psi \cdot \operatorname{inc}\Phi dx.$$

Equation (8.7)(b) is equivalent to

$$(M + L_{\mathbb{B}})E_i = M\mathbb{H}. \tag{8.8}$$

By Lemma 5.8 (which obviously holds also true if $\mathbb{B}$ is negative definite and if $\mathcal{Z}$ is replaced by $\mathcal{Z}_0$), $L_{\mathbb{B}} : \mathcal{Z}_0 \to \mathcal{Z}_0'$ is invertible. Thus, (8.8) is equivalent to

$$(I + L_{\mathbb{B}}^{-1}M)E_i = L_{\mathbb{B}}^{-1}M\mathbb{H}. \tag{8.9}$$

The operator $L_{\mathbb{B}}^{-1}M : L^2(\Omega, \mathbb{S}^3) \to L^2(\Omega, \mathbb{S}^3)$ is compact, since it is continuous from $L^2(\Omega, \mathbb{S}^3)$ to $\mathcal{Z}_0$ and $\mathcal{Z}_0$ is compactly embedded in $L^2(\Omega, \mathbb{S}^3)$ as consequence of Proposition 5.6. Furthermore, under the condition $\|\mathbb{B}^{-1}\|_{L^\infty} < {c_P}^{-1}$, the operator $I + L_{\mathbb{B}}^{-1}M : L^2(\Omega, \mathbb{S}^3) \to L^2(\Omega, \mathbb{S}^3)$ is injective due to the uniqueness claim. Thus, Fredholm's alternative provides the existence of $E_i \in L^2(\Omega, \mathbb{S}^3)$ solution of (8.9). From $E_i = L_{\mathbb{B}}^{-1}M(\mathbb{H} - E_i)$ we infer $E_i \in \mathcal{Z}_0$. We have found $E_i \in \mathcal{Z}_0$ solution of (8.7)(b).

Let us turn to (8.7)(a). We have to find $E_c = \nabla^S v$, $v \in H^1(\Omega, \mathbb{R}^3)$ such that

$$\int_\Omega \nabla^S v \cdot \nabla^S \hat{v} dx = \int_\Omega (\mathbb{H} - \mathbb{B}\operatorname{inc} E_i) \cdot \nabla^S \hat{v} dx, \ \forall \hat{v} \in H^1(\Omega, \mathbb{R}^3). \tag{8.10}$$

This is a standard linear elasticity problem.

Third step. A priori estimate. Equation (8.7)(b) entails

$$\int_\Omega E_i \cdot \operatorname{inc} E_i dx + \int_\Omega \mathbb{B}\operatorname{inc} E_i \cdot \operatorname{inc} E_i dx = \int_\Omega \mathbb{H} \cdot \operatorname{inc} E_i dx.$$

Using (8.5) we obtain

$$\|\operatorname{inc} E_i\|_{L^2}^2 \le \|\mathbb{B}^{-1}\|_{L^\infty} \left| \int_\Omega (\mathbb{H} - E_i) \cdot \operatorname{inc} E_i dx \right| \le \|\mathbb{B}^{-1}\|_{L^\infty} (\|\mathbb{H}\|_{L^2} + \|E_i\|_{L^2}) \|\operatorname{inc} E_i\|_{L^2}.$$

Proposition 5.6 yields

$$\|\operatorname{inc} E_i\|_{L^2} \le \|\mathbb{B}^{-1}\|_{L^\infty} (\|\mathbb{H}\|_{L^2} + c_P \|\operatorname{inc} E_i\|_{L^2}),$$

from which we arrive at (8.1). □

Remark 8.2. *The solution space $\mathcal{F}$ encompasses the transmission conditions stated in Lemma 3.3. In particular, no tangential slip along internal surfaces can occur. This is in contrast with classical formulations in perfect plasticity where spaces of bounded deformations are involved, see e.g. [19]. Moreover, the continuity of the incompatibility flux across internal surfaces stated in Lemma 3.2 shows that $\operatorname{inc} E\, N = 0$ at the interface between a region with incompatible strain and a purely compatible region. Prescribing $\operatorname{inc} E\, N = 0$ on $\partial\Omega$ can be interpreted as considering the exterior of Ω as filled with a purely compatible phase, with Ω and its exterior forming a continuum. Other assumptions, such as discussed in Sections 8.3 and 10.3, are needed in order to enhance incompatibilities near the boundary.*

Remark 8.3. *For ℓ constant, let E^ℓ be the solution of $\mathbb{C}E + \ell \operatorname{inc} E = \mathbb{K}$. Then (8.1) implies that $\mathcal{G}^\ell := \ell \operatorname{inc} E^\ell$ converges weakly in $L^2(\Omega, \mathbb{S}^3)$ to some $\mathcal{G}$ as $\ell \to \pm\infty$, up to a subsequence. More precise limiting results will be given in the next section. On the other side, the condition $|\ell| > c_P|\mathbb{C}|$ suggests that some degeneracy occurs when $|\ell|$ goes to 0, unless $\mathbb{C}$ also tends to 0. When $\ell \to 0$ the incompatibility is no longer controlled, in particular the transmission conditions recalled in Remark 8.2 do not hold any more. However one can let $\mathbb{C}$ go to 0 keeping ℓ constant, in order to represent a nearly void material. This will be considered in Section 10.3.*

8.3. Well-posedness with arbitrary incompatibility flux.

Theorem 8.2 (Second existence result). *Assume Ω is simply connected. Let $\mathbb{K} \in L^2(\Omega, \mathbb{S}^3)$, $\mathbb{C} \in L^\infty(\Omega, \mathbb{S}^{3\times 3})$, $\ell \in L^\infty(\Omega)$ and $h \in H^{-1/2}(\partial\Omega, \mathbb{R}^3)$ such that $\int_{\partial\Omega} h dS(x) = 0$. Let c_P be the Poincaré constant of Proposition 5.6. If $\mathbb{C}$ is uniformly positive definite and and either $\ell > c_P|\mathbb{C}|$ a.e. or $\ell < -c_P|\mathbb{C}|$ a.e. then there exists one and only one $E \in H^{\rm inc}(\Omega, \mathbb{S}^3)$ such that*

$$\begin{cases} \mathbb{C}E + \ell \operatorname{inc} E = \mathbb{K} \text{ in } \Omega \\ \operatorname{inc} E\, N = h \text{ on } \partial\Omega. \end{cases}$$

Moreover there exists constants c_1 and c_2 such that

$$\|\operatorname{inc} E\|_{L^2} \leq \frac{\|\ell^{-1}\mathbb{C}\|_{L^\infty}}{1 - c_P\|\ell^{-1}\mathbb{C}\|_{L^\infty}} \left(\|\mathbb{C}^{-1}\mathbb{K}\|_{L^2} + c_1(1 + \|\ell\mathbb{C}^{-1}\|_{L^\infty})\|h\|_{H^{-1/2}}\right) + c_2\|h\|_{H^{-1/2}}. \quad (8.11)$$

Proof. Let $w \in H^1(\Omega, \mathbb{R}^3)$ be solution of

$$\begin{cases} -\operatorname{div} \nabla^S w = 0 \text{ in } \Omega \\ \nabla^S w N = h \text{ on } \partial\Omega \end{cases}$$

and set $W = \nabla^S w$. By Corollary 4.3 there exists $H \in \mathcal{Z}$ such that $W = \operatorname{inc} H$. Let $\tilde{E} = E - H \in \mathcal{F}$, which has to solve

$$\mathbb{C}\tilde{E} + \ell \operatorname{inc} \tilde{E} = \mathbb{K} - \mathbb{C}H - \ell \operatorname{inc} H.$$

Existence and uniqueness follow from Theorem 8.1. The a priori estimate of Theorem 8.1 combined with Proposition 5.6 and standard elliptic regularity provide (8.11). □

9. Elastic limit

Proposition 9.1. *Consider a sequence $\mathbb{C}_k \in L^\infty(\Omega, \mathbb{S}^3)$ with $c_1|\xi|^2 \leq \mathbb{C}_k(x)\xi \cdot \xi \leq c_2|\xi|^2$ $\forall \xi \in \mathbb{R}^3$, a.e. $x \in \Omega$, $c_1, c_2 > 0$, and a sequence $\ell_k \in L^\infty(\Omega, \mathbb{R}^*_+)$ with $\inf_\Omega \ell_k \to +\infty$. Assume that $\mathbb{K} \in L^2(\Omega, \mathbb{S}^3)$, $E^k \in \mathcal{F}$, $\mathbb{C}_k E^k + \ell_k \operatorname{inc} E^k = \mathbb{K}$. Then $\|\operatorname{inc} E^k\|_{L^2} \to 0$.*

Proof. It is a straightforward consequence of (8.1), since $\|\ell_k^{-1}\mathbb{C}_k\|_{L^\infty} \to 0$. □

Obviously the same holds for a sequence $\ell_k \in L^\infty(\Omega, \mathbb{R}^*_-)$ with $\inf_\Omega |\ell_k| \to +\infty$.

Proposition 9.2. *If ℓ is constant, $\mathbb{K} \in L^2(\Omega, \mathbb{S}^3)$, and $E \in \mathcal{F}$ satisfies*

$$\mathbb{C}E + \ell \operatorname{inc} E = \mathbb{K} \text{ in } \Omega,$$

then

$$\int_\Omega \mathbb{C}E \cdot \hat{E} dx = \int_\Omega \mathbb{K} \cdot \hat{E} dx \qquad \forall \hat{E} \in \mathcal{V}.$$

Note that the above relation is similar to a linear elasticity system, nevertheless E may not be a symmetric gradient.

Proof. Take $\hat{E} \in \mathcal{V}$ and observe that due to the assumptions, one has

$$\int_\Omega \ell \operatorname{inc} E \cdot \hat{E} dx = 0.$$

□

Theorem 9.3 (Elastic limit: homogeneous flux). *Assume that $\mathbb{C} \in L^\infty(\Omega, \mathbb{S}^{3\times 3})$ uniformly positive definite and $\mathbb{K} \in L^2(\Omega, \mathbb{S}^{3\times 3})$ are fixed and that $\ell \neq 0$ is constant. Let $E^\ell \in \mathcal{F}$ be the unique solution of $\mathbb{C}E^\ell + \ell \operatorname{inc} E^\ell = \mathbb{K}$ in Ω and let $E^\infty \in \mathcal{V}$ be the unique solution of the linear elasticity problem*

$$\int_\Omega \mathbb{C}E^\infty \cdot \hat{E} dx = \int_\Omega \mathbb{K} \cdot \hat{E} dx \qquad \forall \hat{E} \in \mathcal{V}. \quad (9.1)$$

Then $\|E^\ell - E^\infty\|_{L^2} \to 0$ when either $\ell \to +\infty$ or $\ell \to -\infty$.

Note that here E^∞ is a symmetric gradient.

Proof. Existence and uniqueness for (9.1) is a consequence of the Riesz representation theorem in the Hilbert space $\mathcal{V}$ for the inner product $(E,\hat{E}) \mapsto \int_\Omega \mathbb{C}E\cdot\hat{E}dx$. Consider the decomposition $E^\ell = E^\ell_c + E^\ell_i \in \mathcal{V}\oplus\mathcal{Z}_0$. We have by Proposition 9.2

$$\int_\Omega \mathbb{C}(E^\ell_c + E^\ell_i)\cdot\hat{E}_c dx = \int_\Omega \mathbb{K}\cdot\hat{E}_c dx \qquad \forall \hat{E}_c \in \mathcal{V}.$$

Substracting (9.1), one has

$$\int_\Omega \mathbb{C}(E^\ell_c - E^\infty)\cdot\hat{E}_c dx = -\int_\Omega \mathbb{C}E^\ell_i\cdot\hat{E}_c dx \qquad \forall \hat{E}_c \in \mathcal{V}.$$

By Propositions 9.1 and 5.6 we have $\|E^\ell_i\|_{H^1}\to 0$. Since $E^\ell_c - E^\infty \in \mathcal{V}$ it follows from the above relation that $\|E^\ell_c - E^\infty\|_{L^2}\to 0$ hence $\|E^\ell - E^\infty\|_{L^2}\to 0$. □

Hence, as $|\ell|\to+\infty$, one retrieves the standard linear elasticity problem with Neumann boundary conditions. In case of non-vanishing incompatibility flux the following holds.

Theorem 9.4 (Elastic limit: general flux)**.** *Assume that $\mathbb{C}\in L^\infty(\Omega,\mathbb{S}^{3\times 3})$ uniformly positive definite and $\mathbb{K}\in L^2(\Omega,\mathbb{S}^{3\times 3})$ are fixed and that $\ell\neq 0$ is constant. Suppose that $E^\ell \in H^{\rm inc}(\Omega,\mathbb{S}^3)$ satisfies $\mathbb{C}E^\ell + \ell\,\mathrm{inc}\,E^\ell = \mathbb{K}$ in Ω, $\mathrm{inc}\,E^\ell N = h^\ell$ on $\partial\Omega$, $h^\ell\in H^{-1/2}(\partial\Omega)$, $\ell h^\ell\to\bar{h}$ in $H^{-1/2}(\partial\Omega)$ when $\ell\to+\infty$. Then $\|E^\ell - E^\infty\|_{L^2}\to 0$ when $\ell\to+\infty$, where $E^\infty\in\mathcal{V}$ is the unique solution of*

$$\int_\Omega \mathbb{C}E^\infty\cdot\nabla^S\hat{v}dx = \int_\Omega \mathbb{K}\cdot\nabla^S\hat{v}dx - \int_{\partial\Omega}\bar{h}\cdot\hat{v}dS(x) \qquad \forall\hat{v}\in H^1(\Omega). \tag{9.2}$$

A similar result holds when $\ell\to-\infty$.

Proof. Existence and uniqueness for (9.2) follows from the Riesz representation theorem. For all $\hat{v}\in H^1(\Omega,\mathbb{R}^3)$ one has

$$\int_\Omega \mathbb{C}E^\ell\cdot\nabla^S\hat{v}dx + \int_{\partial\Omega}\ell h\cdot\hat{v}dS(x) = \int_\Omega \mathbb{K}\cdot\nabla^S\hat{v}dx.$$

Hence

$$\int_\Omega \mathbb{C}(E^\ell - E^\infty)\cdot\nabla^S\hat{v}dx = -\int_{\partial\Omega}(\ell h - \bar{h})\cdot\hat{v}dS(x).$$

Consider the decomposition $E^\ell = E^\ell_c + E^\ell_i \in \mathcal{V}\oplus\mathcal{Z}$, see Proposition 5.3. By (8.11), one infers $\|\mathrm{inc}\,E^\ell\|_{L^2}\to 0$, and subsequently, by Proposition 5.6, $\|E^\ell_i\|_{L^2}\to 0$. Finally,

$$\int_\Omega \mathbb{C}(E^\ell_c - E^\infty)\cdot\nabla^S\hat{v}dx = -\int_\Omega \mathbb{C}E^\ell_i\cdot\nabla^S\hat{v}dx - \int_{\partial\Omega}(\ell h-\bar{h})\cdot\hat{v}dS(x)$$

yields $\|E^\ell_c - E^\infty\|_{L^2}\to 0$, then $\|E^\ell - E^\infty\|_{L^2}\to 0$. □

10. Interpretation of the kinematical framework and external efforts

10.1. **External efforts.** Consider a virtual strain $\hat{E}\in L^2(\Omega,\mathbb{S}^3)$ decomposed as

$$\hat{E} = \nabla^S\hat{v} + \mathrm{inc}\,\hat{F}. \tag{10.1}$$

The work of the external efforts against $\hat{E}$ reads

$$\begin{aligned} W_{\rm ext}(\hat{E}) = \int_\Omega \mathbb{K}\cdot\hat{E}dx = &- \int_\Omega \mathrm{div}\,\mathbb{K}\cdot\hat{v}dx + \int_{\partial\Omega}\mathbb{K}N\cdot\hat{v}dS(x) + \int_\Omega \mathrm{inc}\,\mathbb{K}\cdot\hat{F}dx \\ &+ \int_{\partial\Omega}\Big(\mathcal{T}_1(\mathbb{K})\cdot\hat{F} + \mathcal{T}_0(\mathbb{K})\cdot\partial_N\hat{F}\Big)dS(x). \end{aligned} \tag{10.2}$$

The fields $-\mathrm{div}\,\mathbb{K}$ and $\mathbb{K}N$ are recognized as classical body and contact forces, while $\mathrm{inc}\,\mathbb{K}$ and $(\mathcal{T}_0(\mathbb{K}),\mathcal{T}_1(\mathbb{K}))$ are body and contact forces working against the divergence-free part of the virtual strain. The above fields are in principle known in the first place. The issue is then how and under which conditions it is possible to construct a corresponding tensor field $\mathbb{K}$. Formally the boundary forces $\mathbb{K}N$, $\mathcal{T}_0(\mathbb{K})$ and $\mathcal{T}_1(\mathbb{K})$ exhibit some coupling, as stressed in [6]. To address these points one must specify a kinematical framework ensuring the uniqueness of the decomposition (10.1).

10.2. **Kinematical framework.** Take $\hat{E} = \nabla^S \hat{v} + \operatorname{inc} \hat{F}$ with $\nabla^S \hat{v} \in \mathcal{V}^{00}_{\Gamma_1}$ and $\operatorname{inc} \hat{F} \in \mathcal{W}^0_{\Gamma_2}$ for some partition $\Gamma_1 \cup \Gamma_2$ of $\partial\Omega$, with Γ_1 relatively open in $\partial\Omega$. As said above, $f := -\operatorname{div} \mathbb{K}$ is identified with the body force, and $g := \mathbb{K}N$ is identified with a surface load on Γ_2. Now, if $\mathbb{K} \in \mathcal{V}^{00}_{\Gamma_1}$ the last two integrals of (10.2) vanish by virtue of Lemma 5.2. Then (10.2) rewrites in the classical form

$$\int_\Omega \mathbb{K} \cdot \hat{E} dx = \int_\Omega f \cdot \hat{v} dx + \int_{\partial\Omega} g \cdot \hat{v} dS(x). \tag{10.3}$$

More generally we have the following existence result.

Proposition 10.1. *Let $f \in L^2(\Omega, \mathbb{R}^3)$ and $g \in H^{-1/2}(\partial\Omega, \mathbb{R}^3)$ be such that $\int_\Omega f dx + \int_{\partial\Omega} g dS(x) = 0$. Consider $G \in L^2(\Omega, \mathbb{S}^3)$ such that $\operatorname{div} G = 0$ in Ω, $GN = 0$ on $\partial\Omega$, and $P \in H^{\operatorname{inc}}(\Omega, \mathbb{S}^3)$. There exists $\mathbb{K} \in L^2(\Omega, \mathbb{S}^3)$ such that*

$$\begin{cases} \operatorname{inc} \mathbb{K} = G & \text{in } \Omega, \\ -\operatorname{div} \mathbb{K} = f & \text{in } \Omega, \\ \mathbb{K}N = g & \text{on } \Gamma_2, \\ \mathcal{T}_i(\mathbb{K}) = \mathcal{T}_i(P) & \text{on } \Gamma_1 \qquad (i = 0, 1). \end{cases}$$

Proof. Let $F \in \mathcal{Z}$ be the solution of

$$\int_\Omega \operatorname{inc} F \cdot \operatorname{inc} \hat{F} dx = \int_\Omega G \cdot \hat{F} dx + \int_\Omega (P \cdot \operatorname{inc} \hat{F} - \operatorname{inc} P \cdot \hat{F}) dx \qquad \forall \hat{F} \in \mathcal{Z}.$$

Arguing as in Proposition 6.3 we derive the corresponding strong form

$$\begin{cases} \operatorname{inc} \operatorname{inc} F = G & \text{in } \Omega, \\ -\operatorname{div} F = 0 & \text{in } \Omega, \\ FN = 0 & \text{on } \partial\Omega, \\ \mathcal{T}_i(\operatorname{inc} F) = \mathcal{T}_i(P) & \text{on } \partial\Omega \qquad (i = 0, 1). \end{cases}$$

Then, let $v \in \mathcal{V}^{00}_{\Gamma_1}$ be a solution (unique if $|\Gamma_1| > 0$) of

$$\int_\Omega \nabla^S v \cdot \nabla^S \hat{v} dx = \int_\Omega f \cdot \hat{v} dx + \int_{\partial\Omega} g \cdot \hat{v} dS(x) - \int_\Omega \operatorname{inc} F \cdot \nabla^S \hat{v} dx \ \forall \hat{v} \in \mathcal{V}^{00}_{\Gamma_1}.$$

The standard strong form reads

$$\begin{cases} -\operatorname{div} \nabla^S v = f & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma_1, \\ \nabla^S v N = g - \operatorname{inc} F N & \text{on } \Gamma_2. \end{cases}$$

Setting $\mathbb{K} = \operatorname{inc} F + \nabla^S v$ fulfills all the required conditions (recall Corollary 3.6). □

Observe that one cannot prescribe $\mathcal{T}_0(\mathbb{K})$ and $\mathcal{T}_1(\mathbb{K})$ on Γ_2, although they might yield a contribution in the last integral of (10.2). In fact, if $G = 0$, one can write $\mathbb{K} = \nabla^S w$ for some $w \in H^1(\Omega, \mathbb{R}^3)$, and as in the proof of Lemma 5.2 one has for all $\hat{F} \in H^{\operatorname{inc}}(\Omega, \mathbb{S}^3)$

$$\int_{\partial\Omega} \left(\mathcal{T}_1(\mathbb{K}) \cdot \hat{F} + \mathcal{T}_0(\mathbb{K}) \cdot \partial_N \hat{F} \right) dS(x) = \int_{\partial\Omega} \operatorname{inc} \hat{F} N \cdot w dS(x). \tag{10.4}$$

In the framework under consideration where $\operatorname{inc} \hat{F} \in \mathcal{W}^0_{\Gamma_2}$, the integral at the right-hand side of (10.4) is localized on Γ_2, and specifically w appears as work-conjugate to the virtual incompatibility flux on Γ_2.

Remark 10.1. *If one prescribes $\mathcal{T}_0(\mathbb{K}) = \mathcal{T}_1(\mathbb{K}) = 0$ on Γ_1, then there is no virtual work associated with this boundary condition even if the virtual strain is allowed to have tangential components. Indeed, in the present model, both for finite ℓ and in the elastic limit, the values of $\mathcal{T}_0(E)$ and $\mathcal{T}_1(E)$ are unconstrained, hence allowing for slip at the boundary. But as $|\ell| \to \infty$, one may want to retrieve a Dirichlet boundary condition on Γ_1, including $\mathcal{T}_0(E^\infty) = \mathcal{T}_1(E^\infty) = 0$, a property which is not verified, see Theorem 9.3. To obtain such a condition in the elastic limit, one may either prescribe kinematical constraints in the space of solutions or incorporate some generalized*

force working against boundary slip. Let us recall that tangential slip along Dirichlet boundaries are present in classical models of plasticity, see e.g. [19] where it is related to concentrated (in a measure sense) plastic deformation. For instance, setting a boundary condition of the type $\mathcal{T}_i(\mathbb{K}) = \alpha \mathcal{T}_i(E)$ $(i = 0, 1)$ *(i.e., taking* $P = \alpha E$ *in Proposition 10.1), for some constant* α *and with* E *the sought solution itself, we are led to solve a coupled system for* $(\mathbb{K}, E)$*, for which the existence of solutions for* α *small enough can be inferred from the Banach fixed point theorem or the Neumann series. To see that this formulation is likely to impose a Dirichlet boundary condition in the elastic limit, consider a case where* $\partial\Omega = \Gamma_1$*,* $G = 0$ *and* $E = \nabla^S v$ *for some* $v \in H^1(\Omega, \mathbb{R}^3)$*. Then the condition* $\int_{\partial\Omega} (\mathcal{T}_1(E) \cdot \hat{F} + \mathcal{T}_0(E) \cdot \partial_N \hat{F}) dS(x) = 0$ *for all* $\hat{F} \in H^{\rm inc}(\Omega, \mathbb{S}^3)$ *is reached if and only if* $\mathcal{T}_0(E) = \mathcal{T}_1(E) = 0$ *on* $\partial\Omega$*, by Theorem 2.3, which is itself equivalent to* v *being a rigid displacement on* $\partial\Omega$*, by Proposition 4.5. This is expected if one lets* α *tend to infinity with* ℓ *in the governing system. However, the precise asymptotic analysis, together with the existence issue, are beyond the scope of the present work.*

10.3. **Alternative to the vanishing incompatibility flux condition.** The condition $\operatorname{inc} E\, N = h$ on $\partial\Omega$ is related to the flux of dislocations at the boundary of Ω. This can be specified using Kröner's formula $\operatorname{inc} E = \operatorname{Curl} \kappa$, with κ the contortion tensor related to the dislocation density tensor. Prescribing an a priori given h (possibly zero) may seem an artificial or ad-hoc condition. An alternative is to consider that the exterior of Ω is filled with a fictitious material that mimicks void, with transmission conditions representing the fact that the two phases constitute a continuum. Therefore we restrict ourselves to the pure Neumann type boundary condition, that is, a surface load without kinematical restriction. For a full space extension, $\mathbb{K} = \nabla^S w$ is defined by

$$\begin{cases} -\operatorname{div}(\nabla^S w) = f \text{ in } \mathbb{R}^3, \\ [\![\nabla^S w N]\!] = g \text{ on } \partial\Omega, \end{cases}$$

with f extended by 0 outside Ω. The equilibrium equation $\mathbb{C}E + \ell \operatorname{inc} E = \mathbb{K}$ is fulfilled over $\mathbb{R}^3$ with $(\mathbb{C}, \ell)$ extended by $(\mathbb{C}^{\rm ext}, \ell^{\rm ext})$ outside Ω. In order to approximate a Neumann condition one needs that $\mathbb{C}^{\rm ext}$ be chosen significantly smaller than $\mathbb{C}$ within Ω. Then on $\partial\Omega$ one has

$$\operatorname{inc} EN \approx \frac{1}{\ell^{\rm ext}} (\mathbb{K}N)_{\rm ext},$$

with $\mathbb{K} = \nabla^S w$ as found above. It is reasonable to assume that ℓ is continuous across $\partial\Omega$, in such a way that $\partial\Omega$ has a neutral effect on the transport of dislocations (transmission without reflection). Under this assumption we have

$$\begin{cases} \mathbb{C}E + \ell \operatorname{inc} E = \mathbb{K} \text{ in } \Omega \\ \operatorname{inc} EN = \frac{1}{\ell} (\mathbb{K}N)_{\rm ext} \text{ on } \partial\Omega. \end{cases}$$

Existence and uniqueness of a solution has been shown in Theorem 8.2. In addition we derive on $\partial\Omega$

$$\mathbb{C}EN = \mathbb{K}N - \ell \operatorname{inc} EN = \mathbb{K}N - (\mathbb{K}N)_{\rm ext} = g,$$

which is obviously the standard Neumann condition on the Cauchy stress.

Let us now examine the limit case. In view of Theorem 9.4, we have $E \to E^\infty \in \mathcal{V}$ as $|\ell| \to \infty$ where

$$\int_\Omega \mathbb{C}E^\infty \cdot \nabla^S \hat{v} dx = \int_\Omega \mathbb{K} \cdot \nabla \hat{v} dx - \int_{\partial\Omega} (\mathbb{K}N)_{\rm ext} \cdot \hat{v} dS(x) \qquad \forall \hat{v} \in H^1(\Omega).$$

This rewrites as

$$\int_\Omega \mathbb{C}E^\infty \cdot \nabla^S \hat{v} dx = \int_\Omega f \cdot \hat{v} dx + \int_{\partial\Omega} g \cdot \hat{v} dS(x) \qquad \forall \hat{v} \in H^1(\Omega).$$

The standard Neumann elasticity problem in Ω is retrieved.

11. Interpretation of the incompatibility modulus by numerical examples

The purpose of the following examples is to show how the incompatibility modulus ℓ can be related to dissipative processes. Of course, irreversible behaviors in the form of plasticity are most often localized within regions that typically grow when loading increases. Here, in order to obtain explicit expressions for all quantities and illustrate the limits stated in Theorems 9.3 and 9.4, we assume that ℓ is constant in space.

Since we have only set up linearized governing equations, we simply define the free energy of the solid body submitted to a given load as the external work that can be recovered by purely elastic unloading, keeping $\mathbb{C}$ invariant for simplicity. To be more precise, in the small strain framework, call $\mathbb{K}(t)$ the load at current time t, and $E(t)$ the associated total strain. In our model, $E(t)$ is typically obtained through $\mathbb{C}\dot{E} + \ell \operatorname{inc} \dot{E} = \dot{\mathbb{K}}$ over $[0,t]$, $E(0) = 0$ and a boundary condition on the incompatibility flux. For some $T > t$ consider a virtual extension of $\mathbb{K}$ over $[t,T]$ by a smooth function such that $\mathbb{K}(T) = 0$. The free energy is then defined by

$$\Psi(t) = -\int_t^T \int_\Omega \mathbb{K} \cdot \dot{E}_{\rm rev} dx ds, \tag{11.1}$$

where $E_{\rm rev}(s) = E(t) + \nabla^S u_{\rm rev}(s)$, $\nabla^S u_{\rm rev}(s) = \displaystyle\int_t^s \nabla^S \dot{u}_{\rm rev}(\tau) d\tau$, and for all $\tau \in [t,T]$, $\dot{u}_{\rm rev}(\tau) \in H^1(\Omega)$ satisfies

$$\int_\Omega \mathbb{C}\nabla^S \dot{u}_{\rm rev}(\tau) \cdot \nabla^S \hat{v} dx = \int_\Omega \dot{\mathbb{K}}(\tau) \cdot \nabla^S \hat{v} dx \qquad \forall \hat{v} \in H^1(\Omega). \tag{11.2}$$

Equation (11.1) ensures that $\dot{\Psi}(t) = \int_\Omega \mathbb{K}(t) \cdot \dot{E}_{\rm rev}(t) dx$, which is the external power expenditure in case of reversible transformation. Furthermore, on the one hand, we have $\Psi(t) = -\int_t^T \int_\Omega \mathbb{K} \cdot \nabla^S \dot{u}_{\rm rev} dx ds$, which after integration by parts and using (11.2) yields

$$\Psi(t) = \frac{1}{2} \int_\Omega \mathbb{C}\nabla^S u_{\rm rev}(T) \cdot \nabla^S u_{\rm rev}(T) dx.$$

On the other hand, integrating (11.2) in time leads to

$$\int_\Omega \mathbb{C}\nabla^S u_{\rm rev}(T) \cdot \nabla^S \hat{v} dx = -\int_\Omega \mathbb{K}(t) \cdot \nabla^S \hat{v} dx \qquad \forall \hat{v} \in H^1(\Omega).$$

The two above equations give a practical way to compute the free energy, i.e. $\Psi(t) = -\frac{1}{2}\int_\Omega \mathbb{K}(t) \cdot \nabla^S u_{\rm rev}(T) dx$, and show that this latter is independent of the path along which $\mathbb{K}$ is driven to 0.

The dissipation rate is then classically defined as $\mathcal{D} := \mathcal{P} - \dot{\Psi}$, where $\mathcal{P}(t) := \int_\Omega \mathbb{K}(t) \cdot \dot{E}(t) dx$ is the external power expenditure. We arrive at

$$\mathcal{D}(t) = \int_\Omega \mathbb{K}(t) \cdot \Big(\dot{E}(t) - \dot{E}_{\rm rev}(t) \Big) dx.$$

In the following examples we will consider finite time increments in which $\mathbb{K}$ is constant and compute the dissipated energy. As said above, we will limit ourselves to spatially constant incompatibility moduli. Such configurations will be thermodynamically consistent under the condition that the dissipation be positive, which we expect.

11.1. 1D case: uniaxial traction.

We consider the domain $\Omega = \mathbb{R}^3$. We assume a uniform traction of density $g = 1$ on the planes $\{z = \pm h\}$. Hence the tensor

$$\mathbb{K} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k \end{pmatrix},$$

with $k = \chi_{\{|z|<h\}}$, provides the virtual power $\int_{\mathbb{R}^3} \mathbb{K} \cdot (\nabla^S \hat{v} + \operatorname{inc} \hat{F}) dx = \int_{\{z=h\}} e_z.\hat{v} dS(x) - \int_{\{z=-h\}} e_z.\hat{v} dS(x)$. We search for a strain field of form

$$E = \begin{pmatrix} \varphi & 0 & 0 \\ 0 & \varphi & 0 \\ 0 & 0 & \psi \end{pmatrix},$$

where φ, ψ are functions of the z variable. In this case one has

$$\mathbb{C}E = \begin{pmatrix} 2(\lambda+\mu)\varphi + \lambda\psi & 0 & 0 \\ 0 & 2(\lambda+\mu)\varphi + \lambda\psi & 0 \\ 0 & 0 & 2\lambda\varphi + (\lambda+2\mu)\psi \end{pmatrix},$$

$$\operatorname{inc} E = \begin{pmatrix} \varphi'' & 0 & 0 \\ 0 & \varphi'' & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{11.3}$$

Hence $\mathbb{C}E + \ell \operatorname{inc} E = \mathbb{K}$ if and only if

$$\begin{cases} 2(\lambda+\mu)\varphi + \lambda\psi + \ell\varphi'' = 0 \\ 2\lambda\varphi + (\lambda+2\mu)\psi = k. \end{cases} \tag{11.4}$$

Elementary algebra leads to

$$\psi = \frac{1}{\lambda+2\mu}(k - 2\lambda\varphi),$$

$$2\mu(3\lambda+2\mu)\varphi + \ell(\lambda+2\mu)\varphi'' = -\lambda k.$$

This ordinary differential equation leads us to assume that $\ell < 0$, since in the other case the solutions do not decay when $|z| \to \infty$. We denote

$$\omega = \sqrt{\frac{2\mu(3\lambda+2\mu)}{\lambda+2\mu}}.$$

We obtain:

(1) For $|z| < h$,

$$\varphi(z) = \frac{-\lambda}{2\mu(3\lambda+2\mu)} \left[1 - \exp\left(-\frac{\omega h}{\sqrt{|\ell|}} \right) \cosh\left(\frac{\omega z}{\sqrt{|\ell|}} \right) \right],$$

$$\psi(z) = \frac{1}{\lambda+2\mu} \left\{ 1 + \frac{\lambda^2}{\mu(3\lambda+2\mu)} \left[1 - \exp\left(-\frac{\omega h}{\sqrt{|\ell|}} \right) \cosh\left(\frac{\omega z}{\sqrt{|\ell|}} \right) \right] \right\}.$$

(2) For $|z| > h$,

$$\varphi(z) = \frac{-\lambda}{2\mu(3\lambda+2\mu)} \sinh\left(\frac{\omega h}{\sqrt{|\ell|}} \right) \exp\left(-\frac{\omega |z|}{\sqrt{|\ell|}} \right),$$

$$\psi(z) = \frac{\lambda^2}{\mu(\lambda+2\mu)(3\lambda+2\mu)} \sinh\left(\frac{\omega h}{\sqrt{|\ell|}} \right) \exp\left(-\frac{\omega |z|}{\sqrt{|\ell|}} \right).$$

Observe that

$$\lim_{|\ell|\to\infty} \varphi(z) = 0, \qquad \lim_{|\ell|\to\infty} \psi(z) = \begin{cases} \dfrac{1}{\lambda+2\mu} \text{ if } |z| < h, \\ 0 \text{ if } |z| > h, \end{cases}$$

which is the classical elastic solution with uniaxial strain.

Let $\nabla^S U + E_0$ be the Beltrami decomposition of E in the domain $\Omega_a := \{|z| < a\}$ such that $E_0 \in \mathcal{W}^0_{\partial\Omega_a}$ (see Theorem 5.1), i.e., $\operatorname{div} E_0 = 0$ in Ω_a and $E_0 N = 0$ on $\partial\Omega_a$. One has

$$\begin{cases} \operatorname{div} \nabla^S U = \operatorname{div} E & \text{in } \Omega_a, \\ \nabla^S U N = EN & \text{on } \partial\Omega_a. \end{cases}$$

Denoting by u the z component of U we obtain $u'' = \psi'$, $u'(a) = \psi(a)$, $u'(-a) = \psi(-a)$. Thus $u' = \psi$ and, setting $u(0) = 0$,

$$u(z) = \int_0^z \psi(s)ds.$$

We obtain in particular

$$u(h) = \frac{1}{\lambda + 2\mu} \left\{ h + \frac{\lambda^2}{\mu(3\lambda + 2\mu)} \left[h - \frac{\sqrt{|\ell|}}{2\omega} \left(1 - \exp\left(-\frac{2\omega h}{\sqrt{|\ell|}} \right) \right) \right] \right\}.$$

The functions φ, ψ are plotted in Figure 1 for $h = 1$, Young modulus $Y = 10$ and Poisson ratio $\nu = 1/3$. The value of $u(h)$ as a function of ℓ is also depicted. As expected, we observe an increase of elongation as $|\ell|$ decreases. Therefore, if we consider a quasi-static small-strain experiment consisting of a first increment with ℓ finite and a further purely elastic increment with opposite load (unloading), a residual positive elongation remains. This shows that a portion of the external work has been dissipated.

One of the main outcomes of this example is that physically acceptable solutions are obtained on the condition that $\ell < 0$. Henceforth this condition will be assumed.

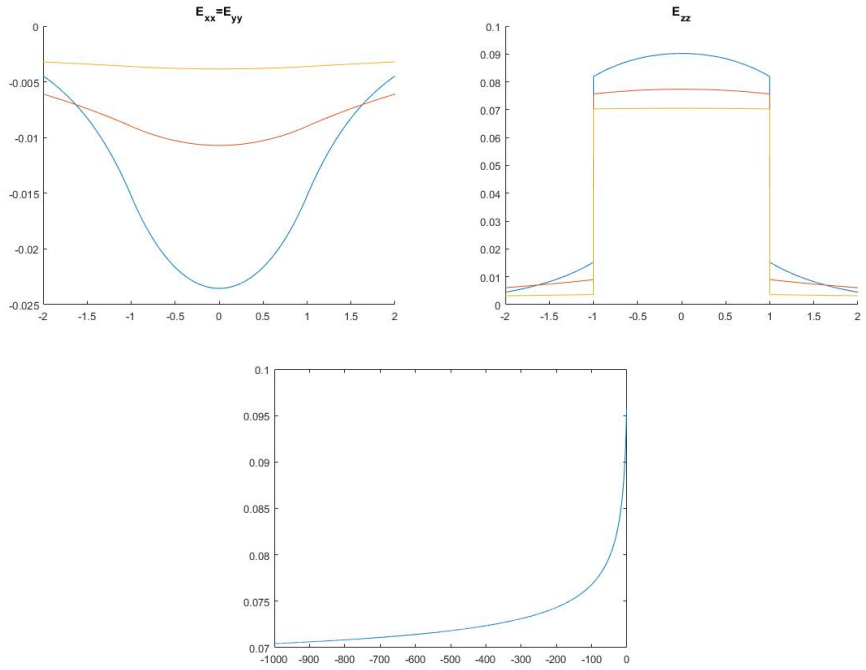

FIGURE 1. In-plane strain ($\varphi(z)$, top left) and vertical strain ($\psi(z)$, top right) with z on the horizontal axis, for $\ell = -10$ (blue), $\ell = -100$ (red), $\ell = -1000$ (yellow). Value of $u(h)$ as a function of ℓ (bottom right)

11.2. **2D case: cylinder under uniform radial traction.** We consider a two-dimensional model of the variables (x, y) and we assume that

$$E = \begin{pmatrix} u & w & 0 \\ w & v & 0 \\ 0 & 0 & h \end{pmatrix}, \qquad \mathbb{K} = \begin{pmatrix} p & s & 0 \\ s & q & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

One has in Cartesian coordinates

$$\mathbb{C}E = \begin{pmatrix} (\lambda+2\mu)u+\lambda(v+h) & 2\mu w & 0 \\ 2\mu w & (\lambda+2\mu)v+\lambda(u+h) & 0 \\ 0 & 0 & \lambda(u+v)+(\lambda+2\mu)h \end{pmatrix},$$

$$\operatorname{inc} E = \begin{pmatrix} \partial_{yy}h & -\partial_{xy}h & 0 \\ -\partial_{xy}h & \partial_{xx}h & 0 \\ 0 & 0 & \partial_{xx}v - 2\partial_{xy}w + \partial_{yy}u \end{pmatrix}. \tag{11.5}$$

Hence $\mathbb{C}E + \ell \operatorname{inc} E = \mathbb{K}$ if and only if

$$\begin{cases} (\lambda+2\mu)u+\lambda(v+h)+\ell\partial_{yy}h = p \\ (\lambda+2\mu)v+\lambda(u+h)+\ell\partial_{xx}h = q \\ 2\mu w - \ell\partial_{xy}h = s \\ \lambda(u+v)+(\lambda+2\mu)h+\ell(\partial_{xx}v - 2\partial_{xy}w + \partial_{yy}u) = 0. \end{cases} \tag{11.6}$$

Elementary algebra leads to

$$\begin{aligned} u &= \frac{1}{4\mu(\lambda+\mu)}\left((\lambda+2\mu)p - \lambda q - 2\lambda\mu h + \ell\lambda\partial_{xx}h - \ell(\lambda+2\mu)\partial_{yy}h\right) \\ v &= \frac{1}{4\mu(\lambda+\mu)}\left(-\lambda p + (\lambda+2\mu)q - 2\lambda\mu h - \ell(\lambda+2\mu)\partial_{xx}h + \ell\lambda\partial_{yy}h\right) \\ w &= \frac{1}{2\mu}\left(s + \ell\partial_{xy}h\right). \end{aligned}$$

Within regions where $\operatorname{inc}\mathbb{K} = 0$, $\operatorname{div}\mathbb{K} = 0$ and λ, μ, ℓ are constant, substituting the above relations in the last equation of (11.6) entails

$$\ell^2(\lambda+2\mu)\Delta^2 h + 4\ell\lambda\mu\Delta h - 4\mu^2(3\lambda+2\mu)h = 2\lambda\mu(p+q). \tag{11.7}$$

This can be factorized as

$$\big(\ell(\lambda+2\mu)\Delta + 2\mu(3\lambda+2\mu)\big)\big(\ell\Delta - 2\mu\big)h = 2\lambda\mu(p+q). \tag{11.8}$$

We consider an infinite cylinder with cross section $B = B(0,1)$ under uniform radial traction on its boundary. However we take as working domain the whole space in order to circumvent the modeling of boundary conditions. Therefore we search for $\mathbb{K} = \nabla^S w$ such that $-\operatorname{div}\mathbb{K} = \delta_{\partial B} N$, with $\delta_{\partial B}$ the Dirac measure on ∂B and N the outer unit normal. A standard calculation using Airy stress functions yields in polar coordinates

$$\mathbb{K} = \chi_B \frac{1}{2}(e_r \otimes e_r + e_\theta \otimes e_\theta) + (1-\chi_B)\frac{1}{2r^2}(-e_r \otimes e_r + e_\theta \otimes e_\theta).$$

We choose the elastic and incompatibility moduli as

$$(\lambda, \mu, \ell) = \begin{cases} (\lambda_i, \mu_i, \ell_i) & \text{in } B \\ (\lambda_e, \mu_e, \ell_e) & \text{in } \mathbb{R}^2 \setminus B, \end{cases}$$

with $\lambda_i/\mu_i = \lambda_e/\mu_e$. On ∂B one has the transmission conditions $[\![\mathcal{T}_0(E)]\!] = [\![\mathcal{T}_1(E)]\!] = 0$. Let us place ourselves in polar coordinates and, due to symmetry, search for $h = h(r)$. The condition $[\![\mathcal{T}_0(E)]\!] = 0$ implies $[\![h]\!] = 0$ and $[\![E_{\theta\theta}]\!] = 0$. Using $\mathbb{C}E = \mathbb{K} - \ell \operatorname{inc} E$ one obtains the planar components of $\mathbb{C}E$ as

$$\begin{aligned} (\mathbb{C}E)_{\text{plan}} = \chi_B &\left[\left(\frac{1}{2} - \frac{\ell_i}{r}h'\right) e_r \otimes e_r + \left(\frac{1}{2} - \ell_i h''\right) e_\theta \otimes e_\theta\right] \\ &+ (1-\chi_B)\left[\left(-\frac{1}{2r^2} - \frac{\ell_e}{r}h'\right) e_r \otimes e_r + \left(\frac{1}{2r^2} - \ell_e h''\right) e_\theta \otimes e_\theta\right]. \end{aligned}$$

Then, Hooke's law together with $E_{rr}+E_{\theta\theta}=u+v=(1-2\lambda h-\ell\Delta h)/(2(\lambda+\mu))$ yield

$$(\mathbb{C}E)_{zz}=\chi_B\left[\frac{\lambda_i}{2(\lambda_i+\mu_i)}\left(1-\frac{\ell_i}{r}h'-\ell_i h''\right)+\frac{\mu_i(3\lambda_i+2\mu_i)}{\lambda_i+\mu_i}h\right]$$
$$+(1-\chi_B)\left[\frac{\lambda_e}{2(\lambda_e+\mu_e)}\left(-\frac{\ell_e}{r}h'-\ell_e h''\right)+\frac{\mu_e(3\lambda_e+2\mu_e)}{\lambda_e+\mu_e}h\right],$$

and

$$E_{\theta\theta}=\chi_B\frac{1}{4\mu_i(\lambda_i+\mu_i)}\left[\mu_i-\ell_i(\lambda_i+2\mu_i)h''-\lambda_i\left(-\frac{\ell_i}{r}h'+2\mu_i h\right)\right]$$
$$+(1-\chi_B)\frac{1}{4\mu_e(\lambda_e+\mu_e)}\left[\frac{\lambda_e+\mu_e}{r^2}-\ell_e(\lambda_e+2\mu_e)h''-\lambda_e\left(-\frac{\ell_e}{r}h'+2\mu_e h\right)\right].$$

The condition $[\![E_{\theta\theta}]\!]=0$ rewrites as

$$\left[\!\!\left[\frac{\ell}{\mu}\frac{\lambda+2\mu}{\lambda+\mu}h''-\frac{\ell}{\mu}\frac{\lambda}{\lambda+\mu}h'\right]\!\!\right]=\frac{1}{\lambda_i+\mu_i}-\frac{1}{\mu_e}.$$

Next, from $\mathcal{T}_1(E)=-he_r\otimes e_r+(h-h')e_\theta\otimes e_\theta$ we infer $[\![h']\!]=0$.

Coming back to (11.7) one looks for

$$h=\frac{-\lambda_i}{2\mu_i(3\lambda_i+2\mu_i)}\chi_B+\tilde{h}$$

with $\tilde{h}$ solution of the homogeneous equation in B and $\mathbb{R}^2\setminus\bar{B}$. In view of (11.8) and recalling that $\ell<0$, $\tilde{h}$ is spanned by the Bessel-type functions $J_0(k^+r)$, $Y_0(k^+r)$, $I_0(k^-r)$ and $K_0(k^-r)$ with

$$k^+=\sqrt{\frac{2\mu}{-\ell}},\qquad k^-=\sqrt{\frac{2\mu}{-\ell}\frac{3\lambda+2\mu}{\lambda+2\mu}}.$$

Due to boundedness and decay at infinity, it remains

$$\tilde{h}(r)=\begin{cases}aJ_0(k^+r)+bI_0(k^-r) & \text{if } r<1\\ cK_0(k^-r) & \text{if } r>1.\end{cases}$$

The three transmission conditions fix a, b and c through a linear system.

In the following computations we take a material inside B with Young modulus $Y=10$ and Poisson ratio $\nu=1/3$, and a nearly void exterior phase such that $\mathbb{C}_e=10^{-5}\mathbb{C}_i$. The incompatibility modulus is taken constant over $\mathbb{R}^2$. The plots of the strain are given in Figure 2, and compared with the classical plane strain elastic strain. The external work

$$W=\int_{\mathbb{R}^2}\mathbb{K}\cdot E dx$$

is indicated in Table 1. This shows that decreasing the incompatibility modulus (in absolute value) increases the external work for a given load, thus increases dissipation.

ℓ	-1000	-100	-20
W	0.3006	0.4616	1.0620

TABLE 1. External work

11.3. **3D case: ball under uniform traction.** Consider the unit ball $\Omega=\{x\in\mathbb{R}^3,|x|<1\}$. We assume a uniform unit radial traction on $\partial\Omega$. We treat the two kinematical frameworks addressed in this paper, namely the case of vanishing incompatibility flux and the case of free incompatibility flux through domain extension described in Section 10.3.

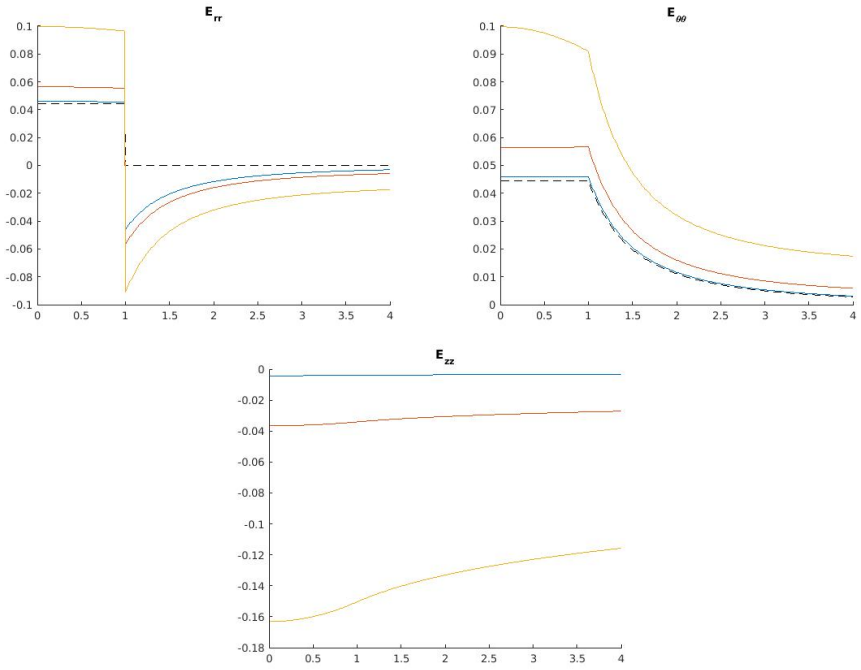

FIGURE 2. Strain components in cylindrical coordinates, as functions of r, for $\ell = -1000$ (blue), $\ell = -100$ (red), $\ell = -20$ (yellow), classical plane strain elastic solution (dashed)

11.3.1. *Case 1: vanishing incompatibility flux.* We assume in this case the condition $\operatorname{inc} E\, N = 0$ on $\partial\Omega$. We have $\mathbb{K} = \nabla^S w$ with

$$\begin{cases} -\operatorname{div} \nabla^S w = 0 \text{ in } \Omega, \\ \nabla^S w N = N \text{ on } \partial\Omega. \end{cases}$$

The solution is immediately found as $w = re_r$ and $\mathbb{K} = I$. Considering the form

$$E = \varphi(r) I + \psi(r) e_r \otimes e_r$$

we have

$$\mathbb{C}E = ((3\lambda + 2\mu)\varphi + \lambda\psi)\, I + 2\mu\psi e_r \otimes e_r,$$

and (see [40])

$$\operatorname{inc} E = \left(\varphi'' + \frac{\varphi'}{r} - \frac{\psi'}{r}\right) I + \left(-\varphi'' + \frac{\varphi'}{r} + \frac{\psi'}{r} - \frac{2\psi}{r^2}\right) e_r \otimes e_r.$$

Hence $\mathbb{C}E + \ell \operatorname{inc} E = \mathbb{K}$ if and only if

$$\begin{cases} (3\lambda + 2\mu)\varphi + \lambda\psi + \ell\left(\varphi'' + \dfrac{\varphi'}{r} - \dfrac{\psi'}{r}\right) = 1 \\ 2\mu\psi + \ell\left(-\varphi'' + \dfrac{\varphi'}{r} + \dfrac{\psi'}{r} - \dfrac{2\psi}{r^2}\right) = 0. \end{cases}$$

The condition $\operatorname{inc} E\, N = 0$ on $\partial\Omega$ entails $\varphi'(1) = \psi(1)$. The solution of the system is the classical elastic solution given by

$$\varphi = \frac{1}{3\lambda + 2\mu}, \qquad \psi = 0.$$

There is no strain incompatibility in this case. This is a consequence of $\operatorname{tr} \mathbb{K}$ being constant, as explained in Remark 8.1.

11.3.2. *Case 2: free incompatibility flux.* In this case we define $\mathbb{K} = \nabla^S w$ over the full space $\mathbb{R}^3$ by

$$\begin{cases} -\operatorname{div} \nabla^S w = 0 \text{ in } \Omega \cup (\mathbb{R}^3 \setminus \bar{\Omega}), \\ [\![\nabla^S w N]\!] = N \text{ on } \partial\Omega. \end{cases}$$

For $w = w(r)e_r$ one has

$$\nabla^S w = w' e_r \otimes e_r + \frac{w}{r}(e_\theta \otimes e_\theta + e_\phi \otimes e_\phi),$$

$$\operatorname{div} \nabla^S w = \left(w'' + 2\frac{w'}{r} - 2\frac{w}{r^2} \right) e_r = \left(\frac{1}{r^2}(r^2 w)' \right)' e_r.$$

Therefore

$$\operatorname{div} \nabla^S w = 0 \iff w = ar + \frac{b}{r^2}$$

for some constants a, b. The transmission condition yields

$$w(r) = \frac{r}{3}\chi_{\{r<1\}} + \frac{1}{3r^2}\chi_{\{r>1\}},$$

$$\mathbb{K} = \frac{1}{3} I \chi_{\{r<1\}} + \frac{1}{r^3}\chi_{\{r>1\}} \left(\frac{1}{3} I - e_r \otimes e_r \right).$$

We still search E of the form

$$E = \varphi(r) I + \psi(r) e_r \otimes e_r.$$

Outside Ω we assume that the Lamé coefficients are vanishing whereas the incompatibility modulus is kept the same as within Ω, as explained in Section 10.3. Hence $\mathbb{C}E + \ell \operatorname{inc} E = \mathbb{K}$ if and only if

$$\begin{cases} \chi_{\{r<1\}} \left((3\lambda + 2\mu)\varphi + \lambda\psi \right) + \ell \left(\varphi'' + \dfrac{\varphi'}{r} - \dfrac{\psi'}{r} \right) = \dfrac{1}{3}\chi_{\{r<1\}} + \dfrac{1}{3r^3}\chi_{\{r>1\}} \\ \chi_{\{r<1\}} 2\mu\psi + \ell \left(-\varphi'' + \dfrac{\varphi'}{r} + \dfrac{\psi'}{r} - \dfrac{2\psi}{r^2} \right) = -\dfrac{1}{r^3}\chi_{\{r>1\}}. \end{cases}$$

Note that we have taken exactly vanishing Lamé coefficients outside Ω for technical simplicity. This leads in principle to nonunique solutions, however this issue will be overcome later.

Let us first solve the system in Ω. Set $\rho = \psi - r\varphi'$. Substituting for ψ yields

$$\begin{cases} (3\lambda + 2\mu)\varphi + \lambda(r\varphi' + \rho) - \ell\dfrac{\rho'}{r} = \dfrac{1}{3} \\ 2\mu(r\varphi' + \rho) + \ell \left(\dfrac{\rho'}{r} - 2\dfrac{\rho}{r^2} \right) = 0. \end{cases}$$

We infer after some algebra

$$\varphi = \frac{1}{3\lambda + 2\mu} \left[\frac{1}{3} + \ell \left(1 + \frac{\lambda}{2\mu} \right) \frac{\rho'}{r} - \frac{\lambda\ell}{\mu} \frac{\rho}{r^2} \right], \tag{11.9}$$

$$\rho'' - \left(m + \frac{2}{r^2} \right) \rho = 0, \qquad \text{with } m = -\frac{2\mu}{\ell} \frac{3\lambda + 2\mu}{\lambda + 2\mu} > 0.$$

Setting $\rho = pr^2$ we obtain

$$p'' + \frac{4}{r} p' - mp = 0,$$

then, with $p(r) = q(r\sqrt{m})$ and $s = r\sqrt{m}$,

$$q'' + \frac{4}{s} q' - q = 0.$$

The last change of unknown $q(s) = \xi(s)/s$ provides the spherical Bessel equation

$$s^2 \xi'' + 2s\xi' - (s^2 + 2)\xi = 0.$$

Bounded solutions are spanned by the spherical Bessel function

$$i_1(s) := \frac{d}{ds} \left(\frac{\sinh(s)}{s} \right) = \frac{\cosh(s)}{s} - \frac{\sinh(s)}{s^2}. \tag{11.10}$$

Therefore, setting

$$h_0(s) = \frac{i_1(s)}{s},$$

we obtain for some constant a,

$$\rho(r) = ar^2 h_0(r\sqrt{m}).$$

On $\partial\Omega$ we have the condition on the incompatibility flux $\ell(\operatorname{inc} E)N = (\mathbb{K}N)_{\text{ext}}$, which reads

$$2\ell\,(\varphi'(1) - \psi(1))\,e_r = -\frac{2}{3}e_r.$$

It provides $\rho(1) = 1/3\ell$, hence

$$\rho(r) = \frac{1}{3\ell}r^2\frac{h_0(r\sqrt{m})}{h_0(\sqrt{m})}. \tag{11.11}$$

We obtain φ from (11.9), then ψ from $\psi = \rho + r\varphi'$.

Let us now turn to the exterior solution, which is needed to find the displacement field by Beltrami decomposition. Recall the equations in $\mathbb{R}^3 \setminus \bar{\Omega}$

$$\begin{cases} \varphi'' + \dfrac{\varphi'}{r} - \dfrac{\psi'}{r} = \dfrac{1}{3\ell r^3} \\ -\varphi'' + \dfrac{\varphi'}{r} + \dfrac{\psi'}{r} - \dfrac{2\psi}{r^2} = -\dfrac{1}{\ell r^3}. \end{cases}$$

The general solution is obtained as

$$\varphi(r) = \left(\frac{1}{3\ell} - \beta\right)\frac{1}{r} + \frac{\alpha}{r^3}, \qquad \psi(r) = \frac{\beta}{r} - \frac{3\alpha}{r^3}$$

for some constants α, β. Denote by $\mathbb{C}^*$ the Hooke tensor of the weak phase outside Ω. We assume that $\mathbb{C}^* = \gamma\mathbb{C}$ for some constant $\gamma \to 0$. The equation $\mathbb{C}^*E + \ell \operatorname{inc} E = \mathbb{K}$ gives $\operatorname{div}\mathbb{C}^*E = \operatorname{div}\mathbb{K} = 0$, whereby $\operatorname{div}\mathbb{C}E = 0$. This entails

$$\beta = \frac{3\lambda + 2\mu}{6\ell(\lambda + 2\mu)}.$$

It remains to fix α through the transmission conditions $[\![\mathcal{T}_0(E)]\!] = [\![\mathcal{T}_1(E)]\!] = 0$ on $\partial\Omega$. The first condition is obviously equivalent to $[\![\varphi]\!] = 0$. From (2.11) we obtain $[\![\mathcal{T}_1(E)]\!] = -2[\![\varphi]\!]e_r \otimes e_r + [\![\psi - \varphi']\!](e_\theta \otimes e_\theta + e_\phi \otimes e_\phi)$, then given the first condition, the second one is fulfilled if and only if $[\![\psi - \varphi']\!] = 0$. Observe that this is also exactly the expression of $[\![\operatorname{inc} E\, N]\!] = 0$. Yet we have on the exterior side $\psi(1) - \varphi'(1) = 1/3\ell$, which turns out to be equal to $\rho(1) = \psi(1) - \varphi'(1)$ on the interior side. In fact, only the continuity of φ fixes α by

$$\alpha = \frac{1}{3\lambda + 2\mu}\left[\frac{1}{3} + \ell\left(1 + \frac{\lambda}{2\mu}\right)\rho'(1) - \frac{\lambda\ell}{\mu}\rho(1)\right] + \frac{\lambda - 2\mu}{6\ell(\lambda + 2\mu)}.$$

The exterior deformation field is completely determined.

We now compute the displacement U such that $E = \nabla^S U + E^0$, $\operatorname{div} E^0 = 0$, $E^0 N \to 0$ at infinity. Therefore U solves

$$\begin{cases} \operatorname{div}\nabla^S U = \operatorname{div} E \text{ in } \mathbb{R}^3 \\ \nabla^S U N \to EN \text{ at } \infty. \end{cases} \tag{11.12}$$

For $U = u(r)e_r$, the first equation reads when $r \neq 1$

$$u'' + 2\frac{u'}{r} - 2\frac{u}{r^2} = \varphi' + \psi' + 2\frac{\psi}{r}.$$

For $r > 1$, rewriting the left hand side and computing the right hand side provides

$$\left(\frac{1}{r^2}(r^2u)'\right)' = \frac{2\lambda}{3\ell(\lambda + 2\mu)}\frac{1}{r^2},$$

whereby for some constants c and d,

$$u(r) = \frac{-\lambda}{3\ell(\lambda + 2\mu)} + cr + \frac{d}{r^2}.$$

Now, the condition $\nabla^S UN \to EN$ at infinity yields $c = 0$, since $Ee_r = (\varphi + \psi)e_r \to 0$.

For $r < 1$ we find

$$\varphi' + \psi' + 2\frac{\psi}{r} = 4\varphi' + \rho' + r\varphi'' + \frac{2}{r}\rho = \frac{2\lambda}{\lambda + 2\mu}\frac{\rho}{r}.$$

Substituting (11.11) leads to the equation

$$\left(\frac{1}{r^2}(r^2 u)'\right)' = \frac{2\lambda}{3\ell(\lambda + 2\mu)}\frac{i_1(r\sqrt{m})}{i_1(\sqrt{m})}.$$

Using (11.10) we arrive at

$$u(r) = \frac{-\lambda}{3\mu(3\lambda + 2\mu)}\frac{i_1(r\sqrt{m})}{i_1(\sqrt{m})} + er$$

for some constant e. On $\partial\Omega$, (11.12) amounts to $[\![\nabla^S UN]\!] = [\![EN]\!]$, that is $[\![u' + 2u]\!] = [\![\psi]\!]$, hence we have the jump relations $[\![u]\!] = 0$ and $[\![u']\!] = [\![\psi]\!]$. This fixes the constants d and e through a linear system.

A Taylor expansion provides $i_1(s) = s/3 + o(s^2)$ as $s \to 0$. Then for $\ell \to -\infty$ it is easily checked that the elastic solution is retrieved, namely

$$\varphi^\infty(r) = \frac{1}{3\lambda + 2\mu}\chi_{\{r<1\}} + \frac{1}{3\lambda + 2\mu}\frac{1}{r^3}\chi_{\{r>1\}}, \qquad \psi^\infty(r) = \frac{-3}{3\lambda + 2\mu}\frac{1}{r^3}\chi_{\{r>1\}}$$

$$u^\infty(r) = \frac{1}{3\lambda + 2\mu}r\chi_{\{r<1\}} + \frac{1}{3\lambda + 2\mu}\frac{1}{r^2}\chi_{\{r>1\}}.$$

The functions φ, ψ and u are plotted on Figure 3 for different values of ℓ. The curves of displacement show an increase of dilation due to inelastic deformation. This again highlights a dissipative process, since the work $\int_{\mathbb{R}^3} \mathbb{K} \cdot E dx$ is proportional to $u(1)$.

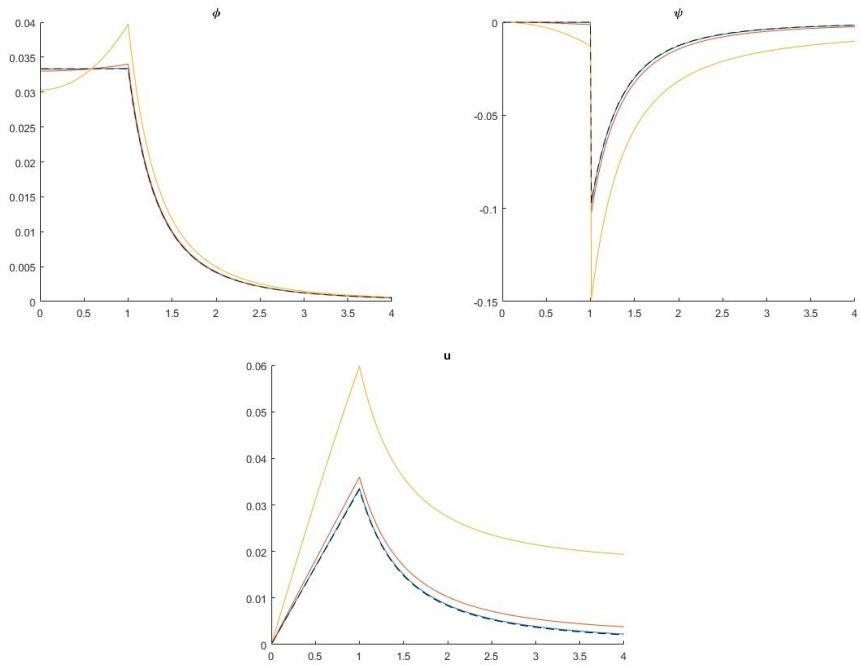

FIGURE 3. Functions φ, ψ and u for $\ell = -1000$ (blue), $\ell = -100$ (red), $\ell = -10$ (yellow), elastic solution (dashed).

12. Concluding remarks

The main goal of this work was to give a rigorous mathematical basis to the model presented in [6], as well as to complete and refine the analysis of the incompatibility operator conducted in [5]. Our study showed that a general model of small-strain generalized elasticity could be considered to account for the strain incompatibility and hence for the presence of dislocations at the micro-scale. Moreover, classical infinitesimal elasticity is recovered as a limit case as the incompatibility modulus tends to minus infinity. Further, it was shown on examples that this model was able to represent dissipative processes through residual deformations after unloading. Our next step is twofold: first to devise a computational algorithm, based on shape / topological sensitivity analysis, in the spirit of [1, 46], to simulate the time evolution of nonlinear irreversible effects, and second to understand the relation between this model and accepted models of elasto-plasticity. This challenge will require theoretical as well as computational efforts, expected in future works.

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Université d'Avignon, Laboratoire de Mathématiques d'Avignon, 301 rue Baruch de Spinoza, 84916 Avignon, France

E-mail address: samuel.amstutz@univ-avignon.fr

Universidade de Lisboa, Faculdade de Ciências, Departamento de Matemática, CMAFcIO, Alameda da Universidade, C6, 1749-016 Lisboa, Portugal

E-mail address: vangoeth@fc.ul.pt