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# EXISTENCE RESULTS FOR AN INTRINSIC LINEARIZED ELASTO-PLASTICITY MODEL

SAMUEL AMSTUTZ AND NICOLAS VAN GOETHEM

ABSTRACT. A novel model of elasto-plasticity is presented and analyzed, based on the linearized strain and its incompatibility tensor field. Elastic strain incompatibility accounts for the presence of dislocations, whose motion is ultimately responsible for the plastic behaviour of solids. The specific functional analysis setting is built up, on which existence results are proved. Our solution strategy is essentially based on the projection of the governing equations on appropriate subspaces in the spirit of Leray decomposition of square-integrable and solenoidal velocity fields as used in existence results in hydrodynamics. It is also strongly related with the Beltrami decomposition of symmetric tensor fields in the wake of previous works by the authors. Further, classical linearized elasticity is found as the limit case when the incompatibility modulus, the material coefficient measuring resistance to incompatibility, goes to infinity. Several examples are provided to illustrate this property.

## 1. INTRODUCTION

**1.1. The intrinsic approach to elasticity.** An intrinsic approach to elasticity simply means that the main and primal variable is the strain, together with its derivatives, and that the displacement and rotation fields are possibly recovered in a second step, in case they are needed. This approach is most probably the first historically, since the strain was indeed used to measure deformation, that is, variation in length and in mutual orientation of infinitesimal fibers within a solid body. As a matter of fact, for the geometer the strain is a metric from which all other geometric concepts are retrieved. Specifically, given a smooth strain tensor field  $\varepsilon$ , the classical Kirchhoff-Saint Venant construction (see [3] for an historical review, see also [31, 40]) in linearized elasticity basically consists in

- introducing the Frank tensor<sup>1</sup>  $\mathcal{F} = \text{Curl}^t \varepsilon$ , where  $\text{Curl}^t \varepsilon$  stands for the transpose of the curl of the symmetric tensor  $\varepsilon$  (computed row-wise);
- defining the rotation field as  $\omega(x) = \omega(x_0) + \int_{x_0}^x \mathcal{F}(\xi) dl(\xi)$ , on a smooth curve joining the endpoints  $x_0$  and  $x$ ;
- defining the displacement field as  $u(x) = u(x_0) + \int_{x_0}^x (\varepsilon - \epsilon(\omega))(\xi) dl(\xi)$ , where  $\epsilon(\omega)$  stands for the skew-symmetric rotation tensor constructed from  $\omega$ , namely  $(\epsilon(\omega))_{il} = \epsilon_{ilk} \omega_k$ .

Obviously such definitions are path-dependent. In order for  $u$  and  $\omega$  to be well-defined, i.e., to be path-independent in a simply connected domain, it is immediately seen that a sufficient and necessary condition be that

$$\text{inc } \varepsilon := \text{Curl } \text{Curl}^t \varepsilon = \text{Curl } \mathcal{F} = 0,$$

introducing the strain incompatibility tensor  $\text{inc } \varepsilon$ , that is easily seen to be symmetric. In this case, one retrieves the well-known expressions

$$\nabla^S u = \varepsilon \text{ and } \nabla u = \varepsilon - \epsilon(\omega).$$

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<sup>1</sup>This terminology was introduced in [47] simply because its integral on a closed loop yields the so-called Frank tensor attached to a disclination singularity.

On the contrary, if the tensor  $\text{inc } \varepsilon$  is not vanishing this means exactly that the rotation and/or displacement fields exhibit a jump around what is classically called a Burgers circuit (to this respect an important role is played by the choice of the origin  $x_0$  as shown in [47]). So far it appears clear that the important geometric quantities are

$$\varepsilon, \text{Curl}^t \varepsilon, \text{ and } \text{inc } \varepsilon.$$

This is the seminal motivation for our model which is precisely designed from these variables. In particular, this choice makes the model of gradient-type. Note that it appears natural to consider the curl instead of the full gradient, and the  $\text{inc}$  instead of the full Hessian.

Let us also stress that this approach, despite rarely seen today, has a long history: we date the origin of the intrinsic view to Riemann with ground-breaking applications in general relativity and later in mechanics (in particular see the Hodge and Prager approach in perfect plasticity [38]). In general, as explained in [3], Riemann's view is in contrast with Gauss' standpoint of immersions, that in our case is recognized as a displacement or velocity-based formulation. Furthermore, as far as dislocations are involved this geometric approach was very much developed and enhanced by the physicist E. Kröner in the second half of last century [29]. It should however be mentioned that in finite as well as in linearized elasticity the intrinsic approach was recently considered and developed during the last decades in a systematic way by Ph. Ciarlet and co-authors [10, 12, 13], but also by Amrouche and co-authors [1, 2] and Geymonat and co-authors [20–22] (see also [50] for a geometric approach). In particular, their aim was to write, for the elasticity system, the homogeneous boundary condition on the displacement in function of the elastic strain only.

Strangely enough, though the incompatibility operator was regularly used in the engineering literature, the mathematical study of spaces of square integrable tensor-valued functions with square integrable incompatibility was not yet considered and our first step was hence to dedicate a paper to the subject [4]. Based on these results, the model construction and its justification from a physical standpoint were developed in a second paper [5].

**1.2. Our model: from generalized elasticity to intrinsic elasto-plasticity.** Traditional elasto-plasticity has been developed on three main postulates (see for instance [30], see also [3]):

- (a) The decomposition of the strain into an elastic and a plastic part. In linearized elasto-plasticity the decomposition is additive:  $\varepsilon = \varepsilon^e + \varepsilon^p$ . In finite elasticity one decomposes the deformation gradient multiplicatively:  $F = F^e F^p$  (or arguably in the reverse order, see [16]).
- (b) The coupling of two evolution laws for the elastic and the plastic part, namely on the one hand the balance equations in terms of the Cauchy stress, given by a constitutive function of some elastic kinematical quantity (i.e., related to  $\varepsilon^e$  or  $F^e$ ), and on the other hand, the postulated flow rules, expressed in terms of an appropriate plastic kinematic variable (i.e., related to  $\varepsilon^p$  or  $F^p$  and their time rates).
- (c) The existence of convex dissipation potentials required to define the time evolution of the plastic kinematical variables through the flow rules.

Rigorous existence results in conventional plasticity has a long history as can be found for instance in [26, 27, 38, 39, 41] (see also the series of results [6, 14, 19, 34]).

The novel approach we propose has been first introduced in [5]. In our model, neither of the three above postulates are considered. For instance, as for (c), one would like to understand plastic deformation without appealing to convexity, at least not in its premises. For more objections and a discussion, we refer to [3]. Our paradigm is radically different, since our approach is based on the following rationales<sup>2</sup>.

- (1) Strain rate is preferred to strain and is given the following, primordial definition. The medium is considered as a collection of infinitesimal cells that deform smoothly, so that within each cell one can identify and follow fibers. Denote by  $a_1, a_2, a_3$  three such fibers,

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<sup>2</sup>”Puisque ces mystères nous dépassent, feignons d’en être l’organisateur” in *Les mariés de la tour Eiffel* de Jean Cocteau (1889-1963).

which at time  $t$  originate from point  $x$  and are oriented along the axes of a Cartesian coordinate system and scaled to be of unit lengths. Then the deformation rate is defined at  $x$  as (see, e.g., [18])

$$d_{ij}(t) = \frac{1}{2} \left( \frac{\partial}{\partial t} (a_i \cdot a_j) \right)_t. \quad (1.1)$$

Having fixed an initial time  $t_0 = 0$ , the time integral of the objective tensor  $d$ , called the strain or deformation tensor, reads  $\varepsilon(t) = \int_0^t d(s) ds$ . Note that (1.1) holds for infinitesimal as well as for finite strains and hence one is not forced to specify the quantitative nature of the deformations before they take place.

- (2) The strain defined in this fashion is neither elastic nor plastic, it simply has a compatible part and an incompatible part, that are given by a structure theorem called Beltrami decomposition [31]:

$$\varepsilon = \nabla^S u + E^0. \quad (1.2)$$

As opposed to elasto-plastic partitions this decomposition is unique once boundary conditions for  $u$  are prescribed. Moreover, while  $\varepsilon$  is an objective field by definition, neither  $\nabla^S u$  nor  $E^0$  are objective, see the discussion in [5]. Therefore the model will be constructed upon  $\varepsilon$  and its derivatives.

- (3) The governing equations should generalize classical linear elasticity in the sense that it must take into account the possible strain incompatibility. The idea behind this is that the model should account for the macroscopic effect of the presence and motion of dislocations as micro-structural perturbations.

Our model can be briefly described as follows (see [5] for details). One considers linearized gradient elasticity in the sense of Mindlin [35]. One assumes that the virtual strain rate  $\hat{d}$  and its gradient produce intrinsic work and by the virtual power principle we write

$$\int_{\Omega} (\sigma \cdot \hat{d} + \tau \cdot \nabla \hat{d}) dx = \int_{\Omega} \mathbb{K} \cdot \hat{d} dx,$$

where  $\sigma, \tau$  are the Cauchy stress and hyperstress tensors, respectively, and  $\mathbb{K}$  is a tensor representing external efforts. Constitutive relations are taken as  $\sigma = \mathbb{A}\varepsilon$  and  $\tau = \mathbb{B}\nabla\varepsilon$ , where  $\mathbb{A}, \mathbb{B}$  are the Lamé and Mindlin tensors, respectively (see [35]). We require that the intrinsic power induced by the hyperstress  $\int_{\Omega} \nabla\varepsilon \cdot \tau dx$  vanishes as soon as the deformation is compatible, i.e., that it is only due to micro-structural effects in the form of dislocations, since Kröner's formula connects elastic strain incompatibility and dislocation density (see [29, 44, 46, 47]). This yields the existence of a scalar  $\ell$  called incompatibility modulus, such that  $-\operatorname{div} \tau = \ell \operatorname{inc} \varepsilon$ . Therefore the virtual power principle yields the weak form

$$\int_{\Omega} (\mathbb{A}\varepsilon + \ell \operatorname{inc} \varepsilon) \cdot \hat{d} dx = \int_{\Omega} \mathbb{K} \cdot \hat{d} dx, \quad \forall \hat{d} \in \mathcal{E}, \quad (1.3)$$

where  $\mathcal{E}$  is the set of virtual strain rates.

To see that this equation generalizes linearized elasticity, take  $\hat{d} = \nabla^S \hat{v}$  with  $\hat{v} = 0$  on  $\Gamma_D \subset \partial\Omega$  and take  $\mathbb{K}$  such that  $-\operatorname{div} \mathbb{K} = f$  in  $\Omega$  and  $\mathbb{K}N = g$  on  $\Gamma_N := \partial\Omega \setminus \Gamma_D$ . Then, plugging this into (1.3) immediately yields

$$\begin{cases} -\operatorname{div} (\mathbb{A}\varepsilon + \ell \operatorname{inc} \varepsilon) & = f & \text{in } \Omega, \\ (\mathbb{A}\varepsilon + \ell \operatorname{inc} \varepsilon)N & = g & \text{on } \Gamma_N, \end{cases} \quad (1.4)$$

which is exactly the system of linearized elasticity in case of compatible strain, i.e. with  $\varepsilon = \nabla^S u$  with  $u = u_0$  on  $\Gamma_D$ , since for such strains  $\operatorname{inc} \varepsilon = 0$ .

More generally, we believe that our model of elasto-plasticity is able to represent finite deformations through an incremental formulation. Indeed, nonlinear problems in continuum mechanics are classically solved through the finite element method used in conjunction with an incremental solution procedure. In this way, nonlinear problems are reduced to a sequence of iterations consisting of linearized problems. In our case,  $\varepsilon$  is a strain increment and  $(\mathbb{A}, \ell)$  are seen as tangent elasto-plastic moduli. Of course, their evolution between increments should be driven by constitutive laws in

order to account, for instance, for hardening phenomena. A possible thermodynamic approach is to relate changes in these coefficients with dissipation. A sensitivity analysis of the dissipation functional with respect to a variation of  $\ell$  within a small inclusion has been conducted in [5] for a simplified model. The extension to the full model and the numerical implementation, for which the existence results of the present work were mandatory, is an ongoing work. For philosophical thoughts about modeling, we refer to [32].

**1.3. Summary of our results.** Set  $\mathcal{E} = L^2(\Omega, \mathbb{S}^3)$ . The main purpose of this work is to prove that (1.3), or equivalently the associated strong form

$$\mathbb{A}\varepsilon + \ell \operatorname{inc} \varepsilon = \mathbb{K} \quad \text{in } \Omega,$$

has a unique solution in the space of square integrable functions with square integrable incompatibility, with the additional condition on the dislocation flux at the boundary  $\operatorname{inc} \varepsilon N = h$  on  $\partial\Omega$ . The main ingredients to achieve the proof are (i) an orthogonal decompositions of  $L^2(\Omega, \mathbb{S}^3)$  related to the Beltrami decomposition, and (ii) Fredholm's alternative. It is also to be stressed that our model has no variational structure in the sense that the solution is not minimum of an energy. Moreover we analyze the limit case  $|\ell| \rightarrow \infty$ , showing that our model reduces to classical linearized elasticity. We conclude by three explicit computations to illustrate our approach.

## 2. GENERALITIES AND PRELIMINARY RESULTS

Let  $\Omega$  be a regular ( $C^\infty$ ) bounded domain of  $\mathbb{R}^3$ . We denote by  $\partial\Omega$  its boundary and by  $N$  its outward unit normal.

**2.1. The curvilinear frame.** For all  $x \in \partial\Omega$ , the system  $(\tau^A(x), \tau^B(x))$  is an orthonormal basis of the tangent plane to  $\partial\Omega$ , that can be naturally extended along  $N(x)$  in a tubular neighborhood  $W$  of  $\partial\Omega$  (see [4]). The curvatures along  $\tau^A$  and  $\tau^B$  are denoted by  $\kappa^A$  and  $\kappa^B$ , respectively. Define the normal derivative as  $\partial_N := N \cdot \nabla$  and the tangential derivatives as  $\partial_R := \tau^R \cdot \nabla$ , for  $R \in \{A, B\}$ . We will also use the notation  $R^* = B$  if  $R = A$ ,  $R^* = A$  if  $R = B$ . The following results are proved in [4].

**Theorem 2.1.** *There exist smooth scalar fields  $\xi, \gamma^A, \gamma^B$  in  $W$  such that*

$$\partial_N N = \partial_N \tau^R = 0, \quad (2.1)$$

$$\partial_R N = \kappa^R \tau^R + \xi \tau^{R^*}, \quad (2.2)$$

$$\partial_R \tau^R = -\kappa^R N - \gamma^{R^*} \tau^{R^*}, \quad (2.3)$$

$$\partial_{R^*} \tau^R = \gamma^R \tau^{R^*} - \xi N. \quad (2.4)$$

If  $(\tau^A(x), \tau^B(x))$  are oriented along the principal directions of curvature then  $\xi(x) = 0$ .

**Lemma 2.2.** *If  $f$  is twice differentiable in  $W$  then it holds*

$$\partial_R \partial_N f = \partial_N \partial_R f + \kappa^R \partial_R f + \xi \partial_{R^*} f. \quad (2.5)$$

**2.2. Basic function spaces.** Let  $\mathbb{M}^3$  be the set of  $3 \times 3$  real matrices and  $\mathbb{S}^3$  be the subset of symmetric matrices. We define

$$H^{\operatorname{curl}}(\Omega, \mathbb{M}^3) := \{E \in L^2(\Omega, \mathbb{M}^3) : \operatorname{Curl} E \in L^2(\Omega, \mathbb{M}^3)\},$$

$$H^{\operatorname{div}}(\Omega, \mathbb{S}^3) := \{E \in L^2(\Omega, \mathbb{S}^3) : \operatorname{div} E \in L^2(\Omega, \mathbb{R}^3)\},$$

$$H^{\operatorname{inc}}(\Omega, \mathbb{S}^3) := \{E \in L^2(\Omega, \mathbb{S}^3) : \operatorname{inc} E \in L^2(\Omega, \mathbb{S}^3)\}.$$

These spaces, endowed with the norms defined by  $\|E\|_{H^{\operatorname{curl}}}^2 = \|E\|_{L^2}^2 + \|\operatorname{Curl} E\|_{L^2}^2$ ,  $\|E\|_{H^{\operatorname{div}}}^2 = \|E\|_{L^2}^2 + \|\operatorname{div} E\|_{L^2}^2$ ,  $\|E\|_{H^{\operatorname{inc}}}^2 = \|E\|_{L^2}^2 + \|\operatorname{inc} E\|_{L^2}^2$ , respectively, and the corresponding inner products are obviously Hilbert spaces. Also, by classical regularization arguments (see e.g. [9, 15, 42]),  $C^\infty(\Omega, \mathbb{M}^3)$  [resp.  $C^\infty(\Omega, \mathbb{S}^3)$ ] is dense in each of these spaces. We also define

$$H_0^{\operatorname{inc}}(\Omega, \mathbb{S}^3) = \text{the closure of } \mathcal{D}(\Omega, \mathbb{S}^3) \text{ in } H^{\operatorname{inc}}(\Omega, \mathbb{S}^3),$$

as well as the trace space

$$\tilde{H}^{3/2}(\partial\Omega, \mathbb{S}^3) = \left\{ E \in H^{3/2}(\partial\Omega, \mathbb{S}^3) : \int_{\partial\Omega} ENdS(x) = 0 \right\}.$$

**Theorem 2.3** (Lifting [4]). *Let  $\mathbb{E} \in \tilde{H}^{3/2}(\partial\Omega, \mathbb{S}^3)$ , and  $\mathbb{G} \in H^{1/2}(\partial\Omega, \mathbb{S}^3)$ . There exists  $E \in H^2(\Omega, \mathbb{S}^3)$  such that*

$$\begin{cases} E = \mathbb{E} & \text{on } \partial\Omega, \\ (\partial_N E)_T = \mathbb{G}_T & \text{on } \partial\Omega, \\ \operatorname{div} E = 0 & \text{in } \Omega, \end{cases}$$

where the subscript  $T$  stands for the tangential part given by the components  $(\mathbb{G}_T)_{RR'} = \mathbb{G}_{\tau^R} \cdot \tau^{R'}$ ,  $R, R' \in \{A, B\}$ . In addition, such a lifting can be obtained through a linear continuous operator

$$\mathcal{L}_{\partial\Omega} : (\mathbb{E}, \mathbb{G}) \in \tilde{H}^{3/2}(\partial\Omega, \mathbb{S}^3) \times H^{1/2}(\partial\Omega, \mathbb{S}^3) \mapsto E \in H^2(\Omega, \mathbb{S}^3).$$

Define the subset of  $\mathcal{C}^\infty(\partial\Omega, \mathbb{S}^3)$

$$\mathcal{G} = \{V \odot N, V \in \mathbb{R}^3\},$$

with the notation  $U \odot V := (U \otimes V + V \otimes U)/2$ .

**Lemma 2.4** (Dual trace space [4]). *Every  $\mathbb{E} \in H^{-3/2}(\partial\Omega, \mathbb{S}^3)/\mathcal{G}$  admits a unique representative  $\tilde{\mathbb{E}}$  such that*

$$\int_{\partial\Omega} \tilde{\mathbb{E}}NdS(x) = 0. \quad (2.6)$$

Moreover, the dual space of  $\tilde{H}^{3/2}(\partial\Omega, \mathbb{S}^3)$  is canonically identified with  $H^{-3/2}(\partial\Omega, \mathbb{S}^3)/\mathcal{G}$ .

**2.3. Green formula and applications.** Recall that the Green formula for the divergence allows to define, for any  $T \in H^{\operatorname{div}}(\Omega, \mathbb{S}^3)$ , its normal trace  $TN \in H^{-1/2}(\partial\Omega, \mathbb{R}^3)$  by

$$\int_{\partial\Omega} (TN) \cdot \varphi dS(x) := \int_{\Omega} \operatorname{div} T \cdot \tilde{\varphi} + T \cdot \nabla^S \tilde{\varphi} \quad \forall \varphi \in H^{1/2}(\partial\Omega, \mathbb{R}^3),$$

with  $\tilde{\varphi} \in H^1(\Omega, \mathbb{R}^3)$  an arbitrary lifting of  $\varphi$ , see e.g. [23, 42]. For the incompatibility operator one has the following counterpart.

**Lemma 2.5** (Green formula for the incompatibility [4]). *Suppose that  $T \in \mathcal{C}^2(\bar{\Omega}, \mathbb{S}^3)$  and  $\eta \in H^2(\Omega, \mathbb{S}^3)$ . Then*

$$\int_{\Omega} T \cdot \operatorname{inc} \eta dx = \int_{\Omega} \operatorname{inc} T \cdot \eta dx + \int_{\partial\Omega} \mathcal{T}_1(T) \cdot \eta dS(x) + \int_{\partial\Omega} \mathcal{T}_0(T) \cdot \partial_N \eta dS(x) \quad (2.7)$$

with the trace operators defined as

$$\mathcal{T}_0(T) := (T \times N)^t \times N, \quad (2.8)$$

$$\mathcal{T}_1(T) := (\operatorname{Curl} (T \times N)^t)^S + ((\partial_N + k)T \times N)^t \times N + (\operatorname{Curl}^t T \times N)^S, \quad (2.9)$$

where  $k := \kappa^A + \kappa^B$  is twice the mean curvature of  $\partial\Omega$ ,  $T^S := (T + T^t)/2$  is the symmetric part of  $T$ , and cross products are computed row-wise. In addition, it holds

$$\int_{\partial\Omega} \mathcal{T}_1(T)NdS(x) = 0. \quad (2.10)$$

Alternative expressions for  $\mathcal{T}_1(T)$  are given in [4], like

$$\begin{aligned} \mathcal{T}_1(T) = & - \sum_R \kappa^R (T \times \tau^R)^t \times \tau^R - \sum_R \xi (T \times \tau^R)^t \times \tau^{R*} + ((-\partial_N + k)T \times N)^t \times N \\ & - 2 \left( \sum_R (\partial_R T \times N)^t \times \tau^R \right)^S. \end{aligned} \quad (2.11)$$

For a general symmetric tensor  $T$ , with components  $T_{RR'} := T\tau^{R'} \cdot \tau^R$  in the curvilinear frame, one has:

$$T = \begin{pmatrix} T_{AA} & T_{AB} & T_{AN} \\ T_{BA} & T_{BB} & T_{BN} \\ T_{NA} & T_{NB} & T_{NN} \end{pmatrix}, \quad (T \times N)^t \times N = \begin{pmatrix} T_{BB} & -T_{AB} & 0 \\ -T_{AB} & T_{AA} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (2.12)$$

$$(T \times \tau^A)^t \times \tau^A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & T_{NN} & -T_{BN} \\ 0 & -T_{BN} & T_{BB} \end{pmatrix}, \quad (T \times \tau^B)^t \times \tau^B = \begin{pmatrix} T_{NN} & 0 & -T_{AN} \\ 0 & 0 & 0 \\ -T_{AN} & 0 & T_{AA} \end{pmatrix}, \quad (2.13)$$

$$(T \times N)^t \times \tau^A = \begin{pmatrix} 0 & T_{BN} & -T_{BB} \\ 0 & -T_{AN} & T_{AB} \\ 0 & 0 & 0 \end{pmatrix}, \quad (T \times N)^t \times \tau^B = \begin{pmatrix} -T_{BN} & 0 & T_{AB} \\ T_{AN} & 0 & -T_{AA} \\ 0 & 0 & 0 \end{pmatrix}. \quad (2.14)$$

As shown in [4], we can define the traces  $\mathcal{T}_0(T) \in H^{-1/2}(\partial\Omega, \mathbb{S}^3)$  and  $\mathcal{T}_1(T) \in H^{-3/2}(\partial\Omega, \mathbb{S}^3)/\mathcal{G}$  for every  $T \in H^{\text{inc}}(\Omega, \mathbb{S}^3)$  by

$$\langle \mathcal{T}_0(T), \varphi_0 \rangle = \int_{\Omega} T \cdot \text{inc } \eta_0 dx - \int_{\Omega} \text{inc } T \cdot \eta_0 dx, \quad \forall \varphi_0 \in H^{1/2}(\partial\Omega, \mathbb{S}^3),$$

$$\langle \mathcal{T}_1(T), \varphi_1 \rangle = \int_{\Omega} T \cdot \text{inc } \eta_1 dx - \int_{\Omega} \text{inc } T \cdot \eta_1 dx, \quad \forall \varphi_1 \in \tilde{H}^{3/2}(\partial\Omega, \mathbb{S}^3),$$

with  $\eta_0 = \mathcal{L}_{\partial\Omega}(0, \varphi_0)$  and  $\eta_1 = \mathcal{L}_{\partial\Omega}(\varphi_1, 0)$  (recall that  $\mathcal{L}_{\partial\Omega}$  is the lifting operator defined in Theorem 2.3, and observe that, by Lemma 2.5 and density of  $\mathcal{C}^\infty(\bar{\Omega}, \mathbb{S}^3)$  in  $H^{\text{inc}}(\Omega, \mathbb{S}^3)$ , these definitions are independent of the choices of liftings). In addition, by Lemma 2.4,  $\mathcal{T}_1(T)$  admits a unique representative satisfying (2.10). By linearity of  $\mathcal{L}_{\partial\Omega}$ , this extends formula (2.7) to any functions  $T \in H^{\text{inc}}(\Omega, \mathbb{S}^3)$  and  $\eta \in H^2(\Omega, \mathbb{S}^3)$ .

**Remark 2.1.** *We have defined  $\mathcal{T}_1(T)$  against test functions which admit divergence-free liftings, because spaces of divergence-free tensors arise naturally in problems involving the incompatibility, see the Beltrami decomposition and its consequences in the next sections. But we could also have defined  $\mathcal{T}_1(T) \in H^{-3/2}(\partial\Omega, \mathbb{S}^3)$  by using a classical lifting in  $H^2(\Omega, \mathbb{S}^3)$ . Upon adopting the convention that representatives in  $H^{-3/2}(\partial\Omega, \mathbb{S}^3)/\mathcal{G}$  satisfying the gauge condition (2.10) are chosen, the two definitions are equivalent.*

### 3. PROPERTIES OF THE TRACE OPERATORS IN $H^{\text{inc}}(\Omega, \mathbb{S}^3)$

In this section, homogeneous displacement-like boundary conditions are analyzed in terms of traces of the symmetric strain. These results should be put in perspective with previous results about this problem obtained by Ciarlet and co-authors by means of change-of-metric and change-of-curvature tensors (see [12, 13]). Though our characterization is different, the objective of expressing Dirichlet boundary conditions in terms of intrinsic quantities is the same.

To begin with, as particular cases of the two Green formulae recalled in the previous section, one readily obtains the following.

**Lemma 3.1.** *1. For all  $v \in H^1(\Omega, \mathbb{R}^3)$ , one has  $\text{inc } \nabla^S v = 0$  in the sense of distributions.  
2. For all  $E \in H^{\text{inc}}(\Omega, \mathbb{S}^3)$ , one has  $\text{div } \text{inc } E = 0$  in the sense of distributions.*

Consequently, if  $E \in H^{\text{inc}}(\Omega, \mathbb{S}^3)$ , then  $\text{inc } EN$  is defined in  $H^{-1/2}(\partial\Omega, \mathbb{R}^3)$  by

$$\int_{\partial\Omega} \text{inc } EN \cdot \varphi dS(x) = \int_{\Omega} \text{inc } E \cdot \nabla^S \varphi dx \quad \forall \varphi \in H^1(\Omega, \mathbb{R}^3).$$

Hereafter, if  $\omega \subset\subset \Omega$  and  $u$  is a vector or tensor field defined over  $\Omega$  with well-defined traces on each side of  $\partial\omega$ , we denote by  $\llbracket u \rrbracket$  the jump of  $u$  across  $\partial\omega$  with inner term counted positively.

**Lemma 3.2.** *If  $E \in H^{\text{inc}}(\Omega, \mathbb{S}^3)$  and  $\omega \subset\subset \Omega$ , then  $\llbracket \text{inc } EN \rrbracket = 0$  across  $\partial\omega$ .*

*Proof.* Let  $\varphi \in \mathcal{D}(\Omega, \mathbb{R}^3)$ . By definition and Lemma 3.1, one has

$$\int_{\partial\omega} [\text{inc } EN] \cdot \varphi dx = \int_{\Omega} \text{inc } E \cdot \nabla^S \varphi dx = 0.$$

By density this is also true for any  $\varphi \in H_0^1(\Omega, \mathbb{R}^3)$ , and subsequently for any trace of  $\varphi$  in  $H^{1/2}(\partial\omega, \mathbb{R}^3)$ .  $\square$

**Lemma 3.3.** *If  $E \in L^2(\Omega, \mathbb{S}^3)$ ,  $E|_{\omega} \in H^{\text{inc}}(\omega, \mathbb{S}^3)$ ,  $E|_{\Omega \setminus \bar{\omega}} \in H^{\text{inc}}(\Omega \setminus \bar{\omega}, \mathbb{S}^3)$  and  $[\mathcal{T}_0(E)] = [\mathcal{T}_1(E)] = 0$  across  $\partial\omega$ , then  $E \in H^{\text{inc}}(\Omega, \mathbb{S}^3)$ .*

*Proof.* Let  $\Phi \in \mathcal{D}(\Omega, \mathbb{S}^3)$ . The Green formula yields

$$\int_{\Omega} E \cdot \text{inc } \Phi dx = \int_{\omega} \text{inc } E \Phi dx + \int_{\Omega \setminus \bar{\omega}} \text{inc } E \Phi dx + \int_{\partial\omega} [\mathcal{T}_0(E)] \cdot \partial_N \Phi dS(x) + \int_{\partial\omega} [\mathcal{T}_1(E)] \cdot \Phi dS(x).$$

The last two integrals vanish by assumption, hence the distribution  $\text{inc } E \in \mathcal{D}'(\Omega, \mathbb{S}^3)$  is actually an  $L^2$  function.  $\square$

If  $E \in H^{\text{inc}}(\Omega, \mathbb{S}^3)$  and  $\mathcal{T}_0(E) = \mathcal{T}_1(E) = 0$  on  $\partial\Omega$ , then extending  $E$  by 0 and applying Lemmas 3.3 and 3.2 yields  $\text{inc } EN = 0$  on  $\partial\Omega$ . If  $v \in H_0^1(\Omega, \mathbb{R}^3)$ , then by density of  $\mathcal{D}(\Omega, \mathbb{R}^3)$  and continuity of the trace operators in  $H^{\text{inc}}(\Omega, \mathbb{S}^3)$ , it follows  $\mathcal{T}_0(\nabla^S v) = \mathcal{T}_1(\nabla^S v) = 0$  on  $\partial\Omega$ . These two remarks admit the following local versions.

Considering an open subset  $\Gamma$  of  $\partial\Omega$  and given  $E \in H^{\text{inc}}(\Omega, \mathbb{S}^3)$ , we say that  $\mathcal{T}_0(E) = \mathcal{T}_1(E) = 0$  on  $\Gamma$  if the corresponding distributions vanish on  $\Gamma$ , namely

$$\langle \mathcal{T}_0(E), \varphi_1 \rangle = \langle \mathcal{T}_1(E), \varphi_0 \rangle = 0 \quad \forall \varphi_0, \varphi_1 \in \mathcal{C}^\infty(\partial\Omega, \mathbb{S}^3), \text{ spt } \varphi_0 \subset \Gamma, \text{ spt } \varphi_1 \subset \Gamma, \quad (3.1)$$

and similarly that  $\text{inc } EN = 0$  on  $\Gamma$  if

$$\int_{\partial\Omega} \text{inc } EN \cdot \varphi dS(x) = 0 \quad \forall \varphi \in \mathcal{C}^\infty(\partial\Omega, \mathbb{R}^3), \text{ spt } \varphi \subset \Gamma. \quad (3.2)$$

**Lemma 3.4.** *If  $E \in H^{\text{inc}}(\Omega, \mathbb{S}^3)$  satisfies  $\mathcal{T}_0(E) = \mathcal{T}_1(E) = 0$  on  $\Gamma$  then  $\text{inc } EN = 0$  on  $\Gamma$ .*

*Proof.* Let  $z \in \Gamma$  and  $B$  be an open ball of center  $z$  such that  $\partial\Omega \cap B = \Gamma \cap B$  and  $\Omega \cap B$  is on one side of  $\Gamma \cap B$ . Let  $v \in \mathcal{C}^\infty(\Omega, \mathbb{R}^3)$  with  $\text{spt } v \subset B$ . We have by the Green formulae

$$\int_{\partial\Omega} \text{inc } EN \cdot v dS(x) = \int_{\Omega} \text{inc } E \cdot \nabla^S v dx = - \int_{\partial\Omega} (\mathcal{T}_0(E) \cdot \partial_N \nabla^S v + \mathcal{T}_1(E) \cdot \nabla^S v) dS(x) = 0.$$

By lifting, this holds true for any  $v \in \mathcal{C}^\infty(\partial\Omega, \mathbb{R}^3)$  with support in  $B$ . By linearity, covering and partition of unity this extends to any  $v \in \mathcal{C}^\infty(\partial\Omega, \mathbb{R}^3)$  with support in  $\Gamma$ .  $\square$

**Lemma 3.5.** *If  $v \in H^1(\Omega, \mathbb{R}^3)$  satisfies  $v = 0$  on  $\Gamma$  in the sense of traces, then  $\mathcal{T}_0(\nabla^S v) = \mathcal{T}_1(\nabla^S v) = 0$  on  $\Gamma$ .*

*Proof.* Let  $z \in \Gamma$  and  $B$  be an open ball of center  $z$  such that  $\partial\Omega \cap B = \Gamma \cap B$  and  $\Omega \cap B$  is on one side of  $\Gamma \cap B$ . Let  $\varphi \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{S}^3)$  with  $\text{spt } \varphi \subset B$ . We have

$$\langle \mathcal{T}_0(\nabla^S v), \partial_N \varphi \rangle + \langle \mathcal{T}_1(\nabla^S v), \varphi \rangle = \int_{\Omega} \nabla^S v \cdot \text{inc } \varphi dx = \int_{\partial\Omega} \text{inc } \varphi N \cdot v dS(x) = 0.$$

We conclude as in Lemma 3.4.  $\square$

**Corollary 3.6.** *Let  $v \in H^1(\Omega, \mathbb{R}^3)$  be such that  $v = r$  on  $\Gamma$  in the sense of traces, with  $r$  a rigid displacement field. Then  $\mathcal{T}_0(\nabla^S v) = \mathcal{T}_1(\nabla^S v) = 0$  on  $\Gamma$ .*

*Proof.* On  $\Gamma$  it holds

$$\mathcal{T}_i(\nabla^S v) = \mathcal{T}_i(\nabla^S(v - r)) = 0, \quad i = 0, 1,$$

by Lemma 3.5.  $\square$

**Lemma 3.7.** *Let  $T \in H^2(\Omega)$ . Then  $T = \mathcal{T}_1(T) = 0$  on  $\Gamma$  if and only if  $T = (\partial_N T \times N)^t \times N = \text{Curl}^t T \times N = 0$  on  $\Gamma$ .*



*Proof.* From  $T = 0$  on  $\Gamma$  one infers that  $\partial_R T = 0$  on  $\Gamma$ , with  $R = A, B$ . Moreover, using (2.11) and (2.12)-(2.14) one obtains the expression in the basis of principal curvatures for simplicity ( $\xi = 0$ ):

$$\mathcal{T}_1(T) = \begin{pmatrix} -\partial_N T_{BB} + 2(\partial_B T)_{BN} + kT_{BB} - \kappa^B T_{NN} & \partial_N T_{AB} - kT_{AB} - (\partial_B T)_{AN} - (\partial_A T)_{BN} \\ \partial_N T_{AB} - kT_{AB} - (\partial_B T)_{AN} - (\partial_A T)_{BN} & -\partial_N T_{AA} + 2(\partial_A T)_{AN} + kT_{AA} - \kappa^A T_{NN} \\ \kappa^B T_{AN} + (\partial_A T)_{BB} - (\partial_B T)_{AB} & \kappa^A T_{BN} - (\partial_A T)_{AB} + (\partial_B T)_{AA} \\ & \kappa^B T_{AN} + (\partial_A T)_{BB} - (\partial_B T)_{AB} \\ & \kappa^A T_{BN} - (\partial_A T)_{AB} + (\partial_B T)_{AA} \\ & -\kappa^A T_{BB} - \kappa^B T_{AA} \end{pmatrix}.$$

From the knowledge that  $T_{AA} = T_{AB} = T_{BB} = 0$ , one obtains

$$\mathcal{T}_1(T) = \begin{pmatrix} -\partial_N T_{BB} + 2(\partial_B T)_{BN} - \kappa^B T_{NN} & \partial_N T_{AB} - (\partial_B T)_{AN} - (\partial_A T)_{BN} \\ \partial_N T_{AB} - (\partial_B T)_{AN} - (\partial_A T)_{BN} & -\partial_N T_{AA} + 2(\partial_A T)_{AN} + kT_{AA} - \kappa^A T_{NN} \\ \kappa^B T_{AN} + (\partial_A T)_{BB} - (\partial_B T)_{AB} & \kappa^A T_{BN} - (\partial_A T)_{AB} + (\partial_B T)_{AA} \\ & \kappa^B T_{AN} + (\partial_A T)_{BB} - (\partial_B T)_{AB} \\ & \kappa^A T_{BN} - (\partial_A T)_{AB} + (\partial_B T)_{AA} \end{pmatrix}. \quad (3.3)$$

Thus, (3.3) rewrites as

$$0 = \begin{pmatrix} -\partial_N T_{BB} & \partial_N T_{AB} & 0 \\ \partial_N T_{AB} & -\partial_N T_{AA} & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\mathcal{T}_0(\partial_N T)$$

on  $\Gamma$ , achieving the proof of the direct implication, the last statement being proved in [4]. The converse is shown in the same manner.  $\square$

**Lemma 3.8.** *We have the characterization*

$$H_0^{\text{inc}}(\Omega, \mathbb{S}^3) = \{E \in H^{\text{inc}}(\Omega, \mathbb{S}^3) : \mathcal{T}_0(E) = \mathcal{T}_1(E) = 0 \text{ on } \partial\Omega\}.$$

*Proof.* Suppose  $E_n \in \mathcal{D}(\Omega, \mathbb{S}^3)$ ,  $E \in H^{\text{inc}}(\Omega, \mathbb{S}^3)$ ,  $E_n \rightarrow E$  in  $H^{\text{inc}}(\Omega, \mathbb{S}^3)$ . Of course,  $\mathcal{T}_0(E_n) = \mathcal{T}_1(E_n) = 0$  on  $\partial\Omega$ . Then by continuity  $\mathcal{T}_0(E) = \mathcal{T}_1(E) = 0$  on  $\partial\Omega$ .

Suppose now  $E \in H^{\text{inc}}(\Omega, \mathbb{S}^3)$  with  $\mathcal{T}_0(E) = \mathcal{T}_1(E) = 0$  on  $\partial\Omega$ . Extend  $E$  by 0 to get  $\tilde{E} \in H^{\text{inc}}(\mathbb{R}^3, \mathbb{S}^3)$ . By local charts, shifting and convolution with mollifiers, we can define through a standard construction  $E_n \in \mathcal{D}(\mathbb{R}^3, \mathbb{S}^3)$  such that  $E_n \rightarrow \tilde{E}$  in  $H^{\text{inc}}(\mathbb{R}^3, \mathbb{S}^3)$  and  $\text{spt } E_n \subset \Omega$ . Hence  $E_n \rightarrow E$  in  $H^{\text{inc}}(\Omega, \mathbb{S}^3)$ , which yields  $E \in H_0^{\text{inc}}(\Omega, \mathbb{S}^3)$ .  $\square$

#### 4. BELTRAMI DECOMPOSITION AND RELATED RESULTS

In this section we first recall the Beltrami decomposition of symmetric tensor fields, stated here in an  $L^p$  version for the sake of generality. A specific proof of the  $L^2$  version, which in fact is our main concern, can be found in, e.g., [21, 22]. This structure theorem is named after Eugenio Beltrami (1835-1900), an Italian physicist and mathematician known in particular for his works on elasticity, in particular by stating the equilibrium equations of a body in terms of the stress in place of the strain [8]<sup>3</sup>, but also in non-Euclidean geometries in the wake of Gauss and Riemann<sup>4</sup>.

We need to introduce first the so-called Saint-Venant-Beltrami condition, originally considered by Saint-Venant in [7], then extended by Donati in [17], Ting [43], Moreau [36] Ciarlet and Ciarlet in [11], Amrouche et al. [2], and eventually by Geymonat and Krasucki [20] in  $L^p$  for  $p \neq 2$ . Below we give the version found in [31] (originally from [1]).

<sup>3</sup>Here, Beltrami also showed a new proof of the conditions when six given functions are the components of an elastic deformation.

<sup>4</sup>Beltrami was indeed a friend of Riemann whom he met at Pisa university where he had a chair. Moreover, his chair of mathematical physics in Rome was later transmitted to Volterra in 1900. Vito Volterra (1860-1940) is presumably the first who gave a correct definition of dislocations and disclinations in [48].

**Theorem 4.1** (Saint-Venant-Beltrami compatibility conditions). *Assume that  $\Omega$  is simply-connected. Let  $p \in (1, +\infty)$  be a real number and let  $E \in L^p(\Omega, \mathbb{S}^3)$ . Then,*

$$\text{inc } E = 0 \text{ in } W^{-2,p}(\Omega, \mathbb{S}^3) \iff E = \nabla^S v$$

for some  $v \in W^{1,p}(\Omega, \mathbb{R}^3)$ . Moreover,  $u$  is unique up to rigid displacements.

Let us also refer to [25] and [37] for more details and references on this topic. The following decomposition will show crucial in our model. Pioneer version of this result can be found in [22] for  $p = 2$ .

**Theorem 4.2** (Beltrami decomposition [31]). *Assume that  $\Omega$  is simply-connected. Let  $p \in (1, +\infty)$  be a real number and let  $E \in L^p(\Omega, \mathbb{S}^3)$ . Then, for any  $v_0 \in W^{1/p,p}(\partial\Omega)$ , there exists a unique  $v \in W^{1,p}(\Omega, \mathbb{R}^3)$  with  $v = v_0$  on  $\partial\Omega$  and a unique  $F \in L^p(\Omega, \mathbb{S}^3)$  with  $\text{Curl } F \in L^p(\Omega, \mathbb{R}^{3 \times 3})$ ,  $\text{inc } F \in L^p(\Omega, \mathbb{S}^3)$ ,  $\text{div } F = 0$  and  $FN = 0$  on  $\partial\Omega$  such that*

$$E = \nabla^S v + \text{inc } F. \quad (4.1)$$

We call  $v$  and  $F$  the velocity and incompatibility fields, respectively, associated with  $E$ . The following result is the dual counterpart of Saint-Venant's conditions.

**Corollary 4.3** (Representation of solenoidal symmetric tensors). *Assume that  $\Omega$  is simply-connected. If  $E \in L^2(\Omega, \mathbb{S}^3)$  satisfies  $\text{div } E = 0$  in  $H^{-1}(\Omega)$ , then there exists a unique  $F \in L^2(\Omega, \mathbb{S}^3)$  with  $\text{Curl } F \in L^2(\Omega, \mathbb{S}^3)$ ,  $\text{div } F = 0$  and  $FN = 0$  on  $\partial\Omega$  such that  $E = \text{inc } F$ .*

*Proof.* Theorem 4.2 yields

$$E = \nabla^S v + \text{inc } F,$$

with the appropriate  $F$  and  $v \in H_0^1(\Omega, \mathbb{S}^3)$ . The condition  $0 = \text{div } E = \text{div } \nabla^S v$  entails  $v = 0$ .  $\square$

We now specialize Saint-Venant's decomposition in the case of homogeneous boundary conditions.

**Proposition 4.4** (Saint-Venant with boundary conditions). *Assume that  $\Omega$  is simply-connected. If  $E \in L^2(\Omega, \mathbb{S}^3)$  satisfies*

$$\begin{cases} \text{inc } E = 0 \text{ in } \Omega, \\ \mathcal{T}_0(E) = \mathcal{T}_1(E) = 0 \text{ on } \partial\Omega, \end{cases} \quad (4.2)$$

then there exists  $v \in H_0^1(\Omega, \mathbb{R}^3)$  such that  $\nabla^S v = E$ . Moreover, the map  $E \in L^2(\Omega, \mathbb{S}^3) \mapsto v \in H_0^1(\Omega, \mathbb{R}^3)$  is linear and continuous.

*Proof.* Let  $A : H^{-1}(\Omega, \mathbb{R}^3) \rightarrow L^2(\Omega, \mathbb{S}^3)$  be the linear map defined by  $A\varphi = \nabla^S u$  with

$$\begin{cases} -\text{div } \nabla^S u = \varphi \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Let  $A^* : L^2(\Omega, \mathbb{S}^3) \rightarrow H_0^1(\Omega, \mathbb{R}^3)$  be the adjoint operator of  $A$ . Let  $v = A^*E \in H_0^1(\Omega, \mathbb{R}^3)$ . Let  $\Phi \in \mathcal{D}(\Omega, \mathbb{S}^3)$ . By definition we have

$$-\int_{\Omega} A^*E \text{div } \Phi dx = -\int_{\Omega} E \cdot A(\text{div } \Phi) dx.$$

Set  $\Psi = A(\text{div } \Phi)$ . We have  $-\text{div } \Psi = \text{div } \Phi$ . By Corollary 4.3,  $\Psi = -\Phi + \text{inc } \zeta$  for some  $\zeta \in H^{\text{inc}}(\Omega, \mathbb{S}^3)$ . We obtain

$$-\int_{\Omega} A^*E \text{div } \Phi dx = \int_{\Omega} E \cdot \Phi dx - \int_{\Omega} E \cdot \text{inc } \zeta dx.$$

By the Green formula and the assumptions it holds

$$\int_{\Omega} E \cdot \text{inc } \zeta dx = 0.$$

We arrive at

$$-\int_{\Omega} A^*E \text{div } \Phi dx = \int_{\Omega} E \cdot \Phi dx,$$

thus

$$\nabla^S(A^*E) = E$$

in the sense of distributions.  $\square$

We can now state a converse to Lemma 3.5.

**Proposition 4.5.** *Assume that  $\Omega$  is simply connected. If  $v \in H^1(\Omega, \mathbb{R}^3)$  is such that  $\mathcal{T}_0(\nabla^S v) = \mathcal{T}_1(\nabla^S v) = 0$  on  $\partial\Omega$  then there exists a rigid displacement field  $r$  such that  $v = r$  on  $\partial\Omega$ .*

*Proof.* By Proposition 4.4, there exists  $w \in H_0^1(\Omega, \mathbb{R}^3)$  such that  $\nabla^S v = \nabla^S w$ . Hence there exists a rigid displacement field  $r$  such that  $v = w + r$ . On  $\partial\Omega$  this reduces to  $v = r$ .  $\square$

## 5. ORTHOGONAL DECOMPOSITIONS

We assume in this section that  $\Omega$  is simply-connected.

**5.1. Orthogonal decomposition of  $L^2(\Omega, \mathbb{S}^3)$ .** In this section we obtain a decomposition of  $L^2(\Omega, \mathbb{S}^3)$  into orthogonal subspaces, in the same spirit as in [22], but to account for more general boundary conditions. We define the spaces

$$\begin{aligned} \mathcal{V} &= \{E \in L^2(\Omega, \mathbb{S}^3) : \text{inc } E = 0\}, \\ \mathcal{W} &= \{E \in L^2(\Omega, \mathbb{S}^3) : \text{div } E = 0\}, \end{aligned}$$

and, given a subset  $\Gamma$  of  $\partial\Omega$ ,

$$\begin{aligned} \mathcal{V}_\Gamma^0 &= \{E \in \mathcal{V} : \mathcal{T}_0(E) = \mathcal{T}_1(E) = 0 \text{ on } \Gamma\}, \\ \mathcal{V}_\Gamma^{00} &= \{\nabla^S v : v \in H^1(\Omega, \mathbb{R}^3), v = 0 \text{ on } \Gamma\}, \\ \mathcal{W}_\Gamma^0 &= \{E \in \mathcal{W} : EN \equiv 0 \text{ on } \Gamma\}. \end{aligned}$$

Recall that  $\mathcal{V}_\Gamma^0$  is well-defined by (3.1) if  $\Gamma$  is an open subset of  $\partial\Omega$ . In the definition of  $\mathcal{W}_\Gamma^0$ ,  $EN \equiv 0$  on  $\Gamma$  means

$$\int_{\partial\Omega} EN \cdot \varphi dS(x) = 0 \quad \forall \varphi \in H^{1/2}(\partial\Omega, \mathbb{R}^3), \varphi_{\partial\Omega \setminus \Gamma} = 0.$$

This is usually stronger than vanishing in the sense of distributions, see e.g. [24] for density and extension properties in fractional Sobolev spaces.

**Remark 5.1.** *By Theorem 4.1 and Corollary 4.3 we have*

$$\mathcal{V} = \{\nabla^S v, v \in H^1(\Omega, \mathbb{R}^3)\}, \quad (5.1)$$

$$\mathcal{W} = \{\text{inc } F, F \in L^2(\Omega, \mathbb{S}^3), \text{Curl } F \in L^2(\Omega, \mathbb{S}^3), \text{div } F = 0 \text{ in } \Omega, FN = 0 \text{ on } \partial\Omega\}. \quad (5.2)$$

*Moreover, the velocity field  $v$  in (5.1) is unique up to a rigid displacement field. The incompatibility field  $F$  in (5.2) is unique.*

**Remark 5.2.** *If  $|\Gamma| > 0$ , the velocity field  $v$  in the definition of  $\mathcal{V}_\Gamma^{00}$  is unique.*

**Theorem 5.1** (Orthogonal decomposition of  $L^2$ ). *Assume that  $\partial\Omega$  admits the partition  $\partial\Omega = \Gamma_1 \cup \Gamma_2$  with  $\Gamma_1 \cap \Gamma_2 = \emptyset$  and  $\Gamma_1 = \partial\Omega \setminus \Gamma_2$ . We have the orthogonal decomposition*

$$L^2(\Omega, \mathbb{S}^3) = \mathcal{V}_{\Gamma_1}^{00} \oplus \mathcal{W}_{\Gamma_2}^0.$$

*Proof.* i) Let  $\hat{E} \in \mathcal{V}_{\Gamma_1}^{00}$ ,  $E \in \mathcal{W}_{\Gamma_2}^0$ . We have  $\hat{E} = \nabla^S \hat{v}$  for some  $\hat{v} \in H^1(\Omega)$ ,  $\hat{v} = 0$  on  $\Gamma_1$ . The Green formula entails

$$\int_{\Omega} \hat{E} \cdot E dx = \int_{\Omega} \nabla^S \hat{v} \cdot E dx = - \int_{\Omega} \hat{v} \cdot \text{div } E dx + \int_{\partial\Omega} \hat{v} \cdot EN dS(x) = 0.$$

ii) Let  $E \in L^2(\Omega, \mathbb{S}^3)$ . Write the Beltrami decomposition of Theorem 4.2 as  $E = \nabla^S v + \text{inc } F$  with  $v = 0$  on  $\partial\Omega$ . Let  $w \in H^1(\Omega, \mathbb{R}^3)$  be the solution of

$$\begin{cases} -\text{div } \nabla^S w = 0 \text{ in } \Omega, \\ w = 0 \text{ on } \Gamma_1, \\ \nabla^S w N = \text{inc } FN \text{ on } \Gamma_2, \end{cases}$$

that is,  $w \in H_{\Gamma_1}^1(\Omega, \mathbb{R}^3) := \{\varphi \in H^1(\Omega, \mathbb{R}^3) : \varphi|_{\Gamma_1} = 0\}$ ,

$$\int_{\Omega} \nabla^S w \cdot \nabla^S \varphi dx = \int_{\Omega} \text{inc } F \cdot \nabla^S \varphi dx \quad \forall \varphi \in H_{\Gamma_1}^1(\Omega, \mathbb{R}^3).$$

We have

$$E = \nabla^S(v + w) + (\text{inc } F - \nabla^S w) \in \mathcal{V}_{\Gamma_1}^{00} + \mathcal{W}_{\Gamma_2}^0,$$

since by definition

$$\int_{\partial\Omega} (\text{inc } F - \nabla^S w) N \cdot \varphi dS(x) = \int_{\Omega} (\text{inc } F - \nabla^S w) \cdot \nabla^S \varphi dx.$$

This completes the proof.  $\square$

**Remark 5.3.** By Lemma 3.5 we have

$$\mathcal{V}_{\Gamma}^{00} \subset \mathcal{V}_{\Gamma}^0,$$

whenever  $\Gamma$  is an open subset of  $\partial\Omega$ , and we infer from Proposition 4.4 that

$$\mathcal{V}_{\partial\Omega}^{00} = \mathcal{V}_{\partial\Omega}^0.$$

In this case the decomposition of Theorem 5.1 is the same as in [22, Theorem 2.1.].

We have the following additional property.

**Lemma 5.2.** If  $\mathbb{K} \in \mathcal{V}_{\Gamma_1}^{00}$  and  $\text{inc } \hat{F} \in \mathcal{W}_{\Gamma_2}^0$  it holds

$$\int_{\partial\Omega} \left( \mathcal{T}_1(\mathbb{K}) \cdot \hat{F} + \mathcal{T}_0(\mathbb{K}) \cdot \partial_N \hat{F} \right) dS(x) = 0.$$

*Proof.* By the Green formula, we obtain

$$\int_{\partial\Omega} \left( \mathcal{T}_1(\mathbb{K}) \cdot \hat{F} + \mathcal{T}_0(\mathbb{K}) \cdot \partial_N \hat{F} \right) dS(x) = \int_{\Omega} \left( \mathbb{K} \cdot \text{inc } \hat{F} - \text{inc } \mathbb{K} \cdot \hat{F} \right) dx = \int_{\Omega} \mathbb{K} \cdot \text{inc } \hat{F} dx.$$

Writing  $\mathbb{K} = \nabla^S w$  and applying the Green formula for the divergence yields

$$\int_{\partial\Omega} \left( \mathcal{T}_1(\mathbb{K}) \cdot \hat{F} + \mathcal{T}_0(\mathbb{K}) \cdot \partial_N \hat{F} \right) dS(x) = \int_{\partial\Omega} \text{inc } \hat{F} N \cdot w dS(x).$$

However, we have  $w = 0$  on  $\Gamma_1$  while  $\text{inc } \hat{F} N = 0$  on  $\Gamma_2$ , achieving the proof.  $\square$

**5.2. Orthogonal decomposition of  $H^{\text{inc}}(\Omega, \mathbb{S}^3)$  and related results.** Define

$$\begin{aligned} \mathcal{Z} &= \{E \in H^{\text{inc}}(\Omega, \mathbb{S}^3) : \text{div } E = 0 \text{ in } \Omega, EN = 0 \text{ on } \partial\Omega\}, \\ \mathcal{Z}_0 &= \{E \in \mathcal{Z} : \text{inc } EN = 0 \text{ on } \partial\Omega\}, \\ \mathcal{F} &= \{E \in H^{\text{inc}}(\Omega, \mathbb{S}^3) : \text{inc } EN = 0 \text{ on } \partial\Omega\}. \end{aligned} \tag{5.3}$$

By virtue of Theorem 5.1 we infer the following decompositions.

**Proposition 5.3** (Orthogonal decomposition of  $H^{\text{inc}}$ ). *We have the orthogonal decompositions*

$$\begin{aligned} H^{\text{inc}}(\Omega, \mathbb{S}^3) &= \mathcal{Z} \oplus \mathcal{V}, \\ \mathcal{F} &= \mathcal{Z}_0 \oplus \mathcal{V}. \end{aligned} \tag{5.4}$$

We now gather some properties of the spaces  $\mathcal{Z}$  and  $\mathcal{Z}_0$ .

**Proposition 5.4.** *If  $E \in \mathcal{Z}$  then  $\text{Curl } E \in L^2(\Omega, \mathbb{S}^3)$ . Moreover there exists  $c > 0$  such that*

$$\|E\|_{L^2} + \|\text{Curl } E\|_{L^2} \leq c \|\text{inc } E\|_{L^2} \quad \forall E \in \mathcal{Z}.$$

*Proof.* Let

$$X = \{F \in L^2(\Omega, \mathbb{S}^3), \text{Curl } F \in L^2, \text{inc } F \in L^2, \text{div } F = 0, FN = 0 \text{ on } \partial\Omega\},$$

$$Y = \{F \in L^2(\Omega, \mathbb{S}^3), \text{div } F = 0\}$$

and define the linear map  $\Phi : X \rightarrow Y$  by  $\Phi(E) = \text{inc } E$ . Equip  $X$  and  $Y$  with the norms

$$\|F\|_X = \|F\|_{L^2} + \|\text{Curl } F\|_{L^2} + \|\text{inc } F\|_{L^2},$$

$$\|F\|_Y = \|F\|_{L^2}.$$

Clearly,  $X$  and  $Y$  are Banach spaces and  $\Phi$  is continuous. If  $E \in X$  and  $\Phi(E) = 0$  then  $\text{inc } E = 0$ ,  $\text{div } E = 0$  and  $EN = 0$  on  $\partial\Omega$ , whereby  $E = 0$  since it is a symmetric gradient by Theorem 4.1. Hence  $\Phi$  is injective. By Corollary 4.3,  $\Phi$  is also surjective. The open mapping theorem entails that  $\Phi^{-1}$  is continuous. Hence there exists  $c > 0$  such that

$$\|\Phi^{-1}(F)\|_X \leq c\|F\|_{L^2} \quad \forall F \in Y.$$

Let  $E \in \mathcal{Z}$ . From  $E = \Phi^{-1} \circ \Phi(E)$  we obtain

$$\|E\|_{L^2} + \|\text{Curl } E\|_{L^2} + \|\text{inc } E\|_{L^2} = \|E\|_X = \|\Phi^{-1}(\text{inc } E)\|_X \leq c\|\text{inc } E\|_{L^2} \quad (5.5)$$

and the result follows.  $\square$

The following result is proved in [23, Theorem 3.8.], see [28, 49] for  $L^p$  versions and extensions to non simply connected domains.

**Theorem 5.5.** *There exists a constant  $c > 0$  such that*

$$\|u\|_{H^1} \leq c(\|\text{div } u\|_{L^2} + \|\text{Curl } u\|_{L^2})$$

for all  $u \in L^2(\Omega, \mathbb{R}^3)$  such that  $\text{div } u \in L^2$ ,  $\text{Curl } u \in L^2$  and  $u \cdot N = 0$  on  $\partial\Omega$ .

**Proposition 5.6** (Poincaré's inequality in  $\mathcal{Z}$ ). *There exists  $C > 0$  such that for all  $E \in \mathcal{Z}$*

$$\|E\|_{H^1} \leq C\|\text{inc } E\|_{L^2}.$$

*Proof.* Let  $E \in \mathcal{Z}$ . By Proposition 5.4 we already have

$$\|E\|_{L^2} + \|\text{Curl } E\|_{L^2} \leq c\|\text{inc } E\|_{L^2}.$$

Then Theorem 5.5 yields

$$\|\nabla E\|_{L^2} \leq c\|\text{Curl } E\|_{L^2}$$

for some other constant  $c$ . This completes the proof.  $\square$

We infer in particular that  $\mathcal{Z}$  is imbedded in  $H^1(\Omega, \mathbb{S}^3)$  and compactly imbedded in  $L^2(\Omega, \mathbb{S}^3)$ .

**Proposition 5.7.** *We have the representation*

$$\mathcal{W}_{\partial\Omega}^0 = \text{inc } \mathcal{Z}_0.$$

*Proof.* Of course, if  $F \in \mathcal{Z}_0$ , then  $\text{inc } F \in \mathcal{W}_{\partial\Omega}^0$ . Take  $E \in \mathcal{W}_{\partial\Omega}^0$ . By Corollary 4.3 there exists  $F \in H^{\text{inc}}(\Omega, \mathbb{S}^3)$  with  $\text{div } F = 0$  and  $FN = 0$  on  $\partial\Omega$  such that  $E = \text{inc } F$ . The condition  $EN = 0$  on  $\partial\Omega$  yields  $F \in \mathcal{Z}_0$ .  $\square$

**Lemma 5.8.** *Given a symmetric uniformly positive definite fourth order tensor field  $\mathbb{B}$  (i.e.  $\mathbb{B}(x)T \cdot T > \alpha|T|^2 \forall T \in \mathbb{S}^3$  for some  $\alpha > 0$  independent of  $x$ ) define the linear map  $L_{\mathbb{B}} : \mathcal{Z} \rightarrow \mathcal{Z}'$  by*

$$\langle L_{\mathbb{B}}E, \Phi \rangle = \int_{\Omega} \mathbb{B} \text{inc } E \cdot \text{inc } \Phi dx \quad \forall E, \Phi \in \mathcal{Z}.$$

Then  $L_{\mathbb{B}}$  is an isomorphism from  $\mathcal{Z}$  into  $\mathcal{Z}'$ .

*Proof.* By Proposition 5.4,  $\langle L_{\mathbb{B}}E, E \rangle$  defines a norm in  $\mathcal{Z}$  equivalent to the  $H^{\text{inc}}$ -norm. Let  $\mathcal{T} \in \mathcal{Z}'$ . By the Riesz representation theorem, there exists  $T \in \mathcal{Z}$  such that  $\langle \mathcal{T}, \Phi \rangle = \langle L_{\mathbb{B}}T, \Phi \rangle$  for all  $\Phi \in \mathcal{Z}$ . Therefore  $L_{\mathbb{B}}$  is an isomorphism.  $\square$

We define the inverse map  $L_{\mathbb{B}}^{-1} : \mathcal{Z}' \rightarrow \mathcal{Z}$ , that is continuous by Banach's continuous inverse theorem. Since  $\mathcal{Z} \subset L^2(\Omega, \mathbb{S}^3) \subset \mathcal{Z}'$ , the restriction  $L_{\mathbb{B}}^{-1} : L^2(\Omega, \mathbb{S}^3) \rightarrow L^2(\Omega, \mathbb{S}^3)$  is also well-defined.

**Lemma 5.9.** *The operator  $L_{\mathbb{B}}^{-1} : L^2(\Omega, \mathbb{S}^3) \rightarrow L^2(\Omega, \mathbb{S}^3)$  is self-adjoint positive definite and compact.*

*Proof.* The compactness stems from the compact embedding  $\mathcal{Z} \hookrightarrow L^2(\Omega, \mathbb{S}^3)$  of Proposition 5.6. One has for all  $E, F \in L^2(\Omega, \mathbb{S}^3)$

$$\int_{\Omega} L_{\mathbb{B}}^{-1} E \cdot F dx = \int_{\Omega} L_{\mathbb{B}}^{-1} E \cdot L_{\mathbb{B}} L_{\mathbb{B}}^{-1} F dx = \int_{\Omega} \mathbb{B} \operatorname{inc} (L_{\mathbb{B}}^{-1} E) \cdot \operatorname{inc} (L_{\mathbb{B}}^{-1} F) dx.$$

It follows that  $L_{\mathbb{B}}^{-1} F$  is self-adjoint and positive definite, achieving the proof.  $\square$

**5.3. Two elliptic boundary value problems for the incompatibility.** Lemmas 5.8 and 5.9 yield the following proposition.

**Proposition 5.10** (Weak form in  $\mathcal{Z}$ ). *Let  $\mathbb{K} \in L^2(\Omega, \mathbb{S}^3)$  and  $\mathbb{B}$  a symmetric uniformly positive definite fourth order tensor field. There exists a unique  $E \in \mathcal{Z}$  such that*

$$\int_{\Omega} \mathbb{B} \operatorname{inc} E \cdot \operatorname{inc} \hat{E} dx = \int_{\Omega} \mathbb{K} \cdot \hat{E} dx \quad \forall \hat{E} \in \mathcal{Z}. \quad (5.6)$$

Moreover, the solution map  $\Phi : \mathbb{K} \in L^2(\Omega, \mathbb{S}^3) \rightarrow E \in L^2(\Omega, \mathbb{S}^3)$  is linear and compact.

Similarly we have the following.

**Proposition 5.11** (Weak form in  $\mathcal{Z}_0$ ). *Let  $\mathbb{K} \in L^2(\Omega, \mathbb{S}^3)$  and  $\mathbb{B}$  a symmetric uniformly positive definite fourth order tensor field. There exists a unique  $E \in \mathcal{Z}_0$  such that*

$$\int_{\Omega} \mathbb{B} \operatorname{inc} E \cdot \operatorname{inc} \hat{E} dx = \int_{\Omega} \mathbb{K} \cdot \hat{E} dx \quad \forall \hat{E} \in \mathcal{Z}_0. \quad (5.7)$$

Moreover, the solution map  $\Phi_0 : \mathbb{K} \in L^2(\Omega, \mathbb{S}^3) \rightarrow E \in L^2(\Omega, \mathbb{S}^3)$  is linear and compact.

**Proposition 5.12** (Strong form in  $\mathcal{Z}$ ). *Let  $\mathbb{K}$  be such that  $\operatorname{div} \mathbb{K} = 0$  in  $\Omega$  and  $\mathbb{K}N = 0$  on  $\partial\Omega$ . Then, the strong form of (5.6) reads*

$$\left\{ \begin{array}{ll} \operatorname{inc} (\mathbb{B} \operatorname{inc} E) = \mathbb{K} & \text{in } \Omega, \\ \operatorname{div} E = 0 & \text{in } \Omega, \\ EN = 0 & \text{on } \partial\Omega, \\ \mathcal{T}_0(\mathbb{B} \operatorname{inc} E) = \mathcal{T}_1(\mathbb{B} \operatorname{inc} E) = 0 & \text{on } \partial\Omega, \end{array} \right. \quad (5.8)$$

whose solution coincide with to the solution of the weak form.

*Proof.* Eq. (5.6) holds actually true for all  $\hat{E} \in \mathcal{Z} + \mathcal{V} = H^{\operatorname{inc}}(\Omega, \mathbb{S}^3)$ , in particular for  $\hat{E} \in \mathcal{D}(\Omega, \mathbb{S}^3)$  and for  $\hat{E}$  with arbitrary traces  $\mathcal{T}_0(\partial_N \hat{E})$  and  $\hat{E}$  on  $\partial\Omega$ , by Theorem 2.3. Then the Green formula provides the strong form, which is seen to be equivalent to the weak form.  $\square$

**Remark 5.4** (Strong form in  $\mathcal{Z}_0$ ). *The solution of (5.7) satisfies the strong form*

$$\left\{ \begin{array}{ll} \operatorname{inc} (\mathbb{B} \operatorname{inc} E) = \mathbb{K} & \text{in } \Omega, \\ \operatorname{div} E = 0 & \text{in } \Omega, \\ EN = 0 & \text{on } \partial\Omega, \\ \operatorname{inc} EN = 0 & \text{on } \partial\Omega. \end{array} \right. \quad (5.9)$$

In fact, one can take any test function  $\hat{E} \in \mathcal{D}(\Omega, \mathbb{S}^3) \subset \mathcal{F} = \mathcal{V} + \mathcal{Z}_0$ . One obtains the strong form in  $\Omega$ . The boundary conditions are given by the essential condition of the space.

## 6. ELASTO-PLASTICITY MODEL

**6.1. Power of internal efforts.** We recall the main features of the model introduced in [5].

*Assumption 1.* The power of the internal efforts against the virtual strain rate  $\hat{E} \in \mathcal{D}(\Omega, \mathbb{S}^3)$  is of form

$$W_{\text{int}}(\hat{E}) = \int_{\Omega} (\sigma \cdot \hat{E} + \tau \cdot \nabla \hat{E}) dx.$$

The second and third order tensor fields  $\sigma$  and  $\tau$  are called the stress and hyperstress tensors, respectively.

*Assumption 2.* The power of the internal efforts is a continuous linear functional of  $\hat{E} \in L^2(\Omega, \mathbb{S}^3)$ . We infer  $\sigma - \text{div } \tau \in L^2(\Omega, \mathbb{S}^3)$  and

$$W_{\text{int}}(\hat{E}) = \int_{\Omega} (\sigma - \text{div } \tau) \cdot \hat{E} dx \quad \forall \hat{E} \in L^2(\Omega, \mathbb{S}^3).$$

*Assumption 3.* The material is piecewise linear homogeneous isotropic: there exists a partition of  $\Omega$  as  $\Omega = \bigcup \Omega_p$  such that, in each  $\Omega_p$ ,

$$\sigma = \mathbb{A}_p E, \quad \tau = \mathbb{B}_p \nabla E \quad (6.1)$$

where  $E$  is the strain,  $\mathbb{A}_p$  is the standard Hooke tensor and  $\mathbb{B}_p$  is the Mindlin tensor [35]. They read componentwise

$$\sigma_{ij} = \lambda \delta_{ij} E_{kk} + 2\mu E_{ij}, \quad (6.2)$$

$$\begin{aligned} \tau_{ijk} &= c_1 (\delta_{ki} \partial_l E_{lj} + \delta_{kj} \partial_l E_{li}) + \frac{c_2}{2} (\delta_{ki} \partial_j E_{ll} + \delta_{kj} \partial_i E_{ll} + 2\delta_{ij} \partial_l E_{lk}) + 2c_3 \delta_{ij} \partial_k E_{ll} \\ &+ 2c_4 \partial_k E_{ij} + c_5 (\partial_i E_{jk} + \partial_j E_{ik}), \end{aligned} \quad (6.3)$$

where  $\lambda, \mu, c_1, \dots, c_5$  are constants assigned to each  $\Omega_p$  (index  $p$  is dropped for readability).

*Assumption 4.* The hyperstress  $\tau$  does not produce any virtual intrinsic power as soon as the strain  $E$  is compatible. This means

$$\text{inc } E = 0 \Rightarrow \int_{\Omega} \tau \cdot \nabla \hat{E} dx = 0 \quad \forall \hat{E} \in \mathcal{D}(\Omega, \mathbb{S}^3),$$

or equivalently  $\text{inc } E = 0 \Rightarrow -\text{div } \tau = 0$  in  $\Omega$ . From expression (6.3) we derive the existence within each  $\Omega_p$  of a constant  $\ell_p$  such that  $-\text{div } \tau = \ell_p \text{inc } E$ .

*Conclusion.* We denote  $\ell = \sum \ell_p \chi_{\Omega_p}$  and  $\mathbb{A} = \sum \mathbb{A}_p \chi_{\Omega_p}$ , whereby  $\sigma = \mathbb{A}E$  and  $-\text{div } \tau = \ell \text{inc } E$  in  $\Omega$ . The expression of the internal virtual power is

$$W_{\text{int}}(\hat{E}) = \int_{\Omega} (\mathbb{A}E + \ell \text{inc } E) \cdot \hat{E} dx \quad \forall \hat{E} \in L^2(\Omega, \mathbb{S}^3).$$

The scalar field  $\ell$  is called incompatibility modulus, as it expresses the resistance of the material to incompatible deformations.

**6.2. Power of external efforts.** The power of external efforts is assumed to be a linear functional on  $L^2(\Omega, \mathbb{S}^3)$ . By Riesz representation, there exists  $\mathbb{K} \in L^2(\Omega, \mathbb{S}^3)$  such that

$$W_{\text{ext}}(\hat{E}) = \int_{\Omega} \mathbb{K} \cdot \hat{E} dx.$$

We emphasize that the power of external efforts may be at first expressed in terms of the non-objective fields  $\hat{v}$  and  $\hat{F}$  of the Beltrami decomposition of  $\hat{E}$ . However, provided attention is paid to the uniqueness of the decomposition, these fields are themselves linear functions of  $\hat{E}$ . This will be specified in Section 9.1.

**6.3. Virtual power principle.** The virtual power principle in the absence of inertia reads

$$W_{\text{int}}(\hat{E}) = W_{\text{ext}}(\hat{E}),$$

that is

$$\int_{\Omega} (\mathbb{A}E + \ell \text{inc } E) \cdot \hat{E} dx = \int_{\Omega} \mathbb{K} \cdot \hat{E} dx, \quad (6.4)$$

for all  $\hat{E} \in L^2(\Omega, \mathbb{S}^3)$  satisfying possible kinematical constraints. In the absence of kinematical constraints, (6.4) is obviously equivalent to

$$\mathbb{A}E + \ell \text{inc } E = \mathbb{K}. \quad (6.5)$$

**6.4. Time-evolution of a nonlinear elasto-plasticity model.** Within an incremental formulation,  $\mathbb{A}$  and  $\ell$  are tangent elasto-plastic moduli. They need to be updated at each increment as soon as plasticity phenomena occur. The stress-strain relation is therefore piecewise linear. Typically, in a region with plastic deformations, the Lamé coefficients and the incompatibility modulus  $\ell$  are expected to be less than in purely elastic regions. The way these coefficients evolve is driven by nonlinear constitutive laws that substitute to flow rules and hardening models. We emphasize that dislocations may be created / moved without creation of incompatibility, as shown by Kröner's relation  $\text{inc } E = \text{Curl } \Lambda$  with  $\Lambda$  the dislocation density tensor [29, 46]. Therefore varying  $\ell$  alone is not sufficient to describe plastic effects.

## 7. SOLUTION OF ELASTO-PLASTICITY EQUATIONS WITH NATURAL BOUNDARY CONDITION

The main problem we address is the following: given  $\mathbb{K} \in L^2(\Omega, \mathbb{S}^3)$ , find  $E$  solution of (6.4).

**7.1. Kinematical setting.** We will limit ourselves to the case where no kinematical constraint is assumed on the virtual strain  $\hat{E}$ . Therefore, the problem reduces to (6.5). However, we will see that the absence of constraint on  $E$  leads to nonunique solutions, and that uniqueness is obtained by prescribing the incompatibility flux  $\text{inc } EN$  on  $\partial\Omega$ . The homogeneous case  $\text{inc } EN = 0$  is studied first. However prescribing a given value, either 0 or not, for the incompatibility flux may seem artificial from a modeling point of view. This issue is related to the characterization of the behavior of dislocations at interfaces, whose difficulty is emphasized in [33] in the case of grain boundaries for polycrystals. An attempt to determine the incompatibility flux through a domain extension technique is proposed in Section 9.3.

**Remark 7.1.** *A particular kinematical setting is to require  $\mathbb{K} \in \mathcal{V}$ , and a very special case occurs when  $\mathbb{K} = \nabla^S v$  with  $\text{div } v = \text{tr } \mathbb{K}$  constant. Then for  $\mathbb{A}$  constant a solution to  $\mathbb{A}E + \ell \text{inc } E = \mathbb{K}$  is  $E = \mathbb{A}^{-1}\mathbb{K}$ . Indeed by the structure of  $\mathbb{A}^{-1}$  one has  $E$  proportional to  $\mathbb{K}$  plus a constant tensor hence  $\text{inc } E = 0$ .*

**7.2. Well-posedness with vanishing incompatibility flux.** Our main result is the following.

**Theorem 7.1** (First existence result). *Assume  $\Omega$  is simply connected. Let  $\mathbb{K} \in L^2(\Omega, \mathbb{S}^3)$ . Let  $C$  be the Poincaré constant of Proposition 5.6. If  $\mathbb{A}$  is uniformly positive definite and  $|\ell| > C|\mathbb{A}|$  a.e., then there exists one and only one  $E \in \mathcal{F}$  such that*

$$\mathbb{A}E + \ell \text{inc } E = \mathbb{K}.$$

Moreover we have the a priori estimate

$$\|\text{inc } E\|_{L^2} \leq \frac{\|\ell^{-1}\mathbb{A}\|_{L^\infty}}{1 - C\|\ell^{-1}\mathbb{A}\|_{L^\infty}} \|\mathbb{A}^{-1}\mathbb{K}\|_{L^2}. \quad (7.1)$$

*Proof.* We assume that  $\ell > 0$ . The other case is deduced considering  $\tilde{\ell} = -\ell$ ,  $\tilde{\mathbb{A}} = -\mathbb{A}$ ,  $\tilde{\mathbb{K}} = -\mathbb{K}$ . We write the problem as

$$E + \mathbb{B} \text{inc } E = \mathbb{H} \quad (7.2)$$

with  $\mathbb{B} := \ell\mathbb{A}^{-1}$  and  $\mathbb{H} := \mathbb{A}^{-1}\mathbb{K}$ . We will first prove uniqueness and then existence.



*Step 1. Uniqueness.* Let  $E \in \mathcal{F}$  be such that

$$E + \mathbb{B} \operatorname{inc} E = 0. \quad (7.3)$$

Take the orthogonal decomposition  $E = E_c + E_i$  with  $E_c \in \mathcal{V}$  and  $E_i \in \mathcal{Z}_0$ . We have

$$E_c + E_i + \mathbb{B} \operatorname{inc} E_i = 0. \quad (7.4)$$

Take  $\hat{F} \in \mathcal{Z}_0$ . Then

$$\int_{\Omega} E_c \cdot \operatorname{inc} \hat{F} dx + \int_{\Omega} E_i \cdot \operatorname{inc} \hat{F} dx + \int_{\Omega} \mathbb{B} \operatorname{inc} E_i \cdot \operatorname{inc} \hat{F} dx = 0.$$

By  $\operatorname{inc} \hat{F} N = 0$  on  $\partial\Omega$  the first integral vanishes. Specifically, take  $\hat{F} = E_i$ . We obtain

$$\int_{\Omega} E_i \cdot \operatorname{inc} E_i dx + \int_{\Omega} \mathbb{B} \operatorname{inc} E_i \cdot \operatorname{inc} E_i dx = 0.$$

We have

$$\begin{aligned} \|\operatorname{inc} E_i\|_{L^2}^2 &= \int_{\Omega} \operatorname{inc} E_i \cdot \operatorname{inc} E_i dx = \int_{\Omega} \mathbb{B}^{-1}(\mathbb{B}^{1/2} \operatorname{inc} E_i) \cdot (\mathbb{B}^{1/2} \operatorname{inc} E_i) dx \\ &\leq \|\mathbb{B}^{-1}\|_{L^\infty} \int_{\Omega} \mathbb{B} \operatorname{inc} E_i \cdot \operatorname{inc} E_i dx. \end{aligned} \quad (7.5)$$

By Proposition 5.6 we obtain

$$\begin{aligned} C \|\operatorname{inc} E_i\|_{L^2}^2 &\geq \|E_i\|_{L^2} \|\operatorname{inc} E_i\|_{L^2} \\ &\geq \left| \int_{\Omega} E_i \cdot \operatorname{inc} E_i dx \right| = \int_{\Omega} \mathbb{B} \operatorname{inc} E_i \cdot \operatorname{inc} E_i \geq \|\mathbb{B}^{-1}\|_{L^\infty}^{-1} \|\operatorname{inc} E_i\|_{L^2}^2, \end{aligned}$$

that is,

$$(C \|\mathbb{B}^{-1}\|_{L^\infty} - 1) \|\operatorname{inc} E_i\|_{L^2}^2 \geq 0.$$

If  $\|\mathbb{B}^{-1}\|_{L^\infty} < C^{-1}$  we infer  $\operatorname{inc} E_i = 0$  then  $E_i = 0$ , by Proposition 5.6. Thus (7.4) yields  $E_c = 0$ , and eventually  $E = 0$ .

*Step 2. Existence.* Let  $E = E_c + E_i \in \mathcal{F}$ ,  $E_c \in \mathcal{V}$ ,  $E_i \in \mathcal{Z}_0$ . Then (7.2) is equivalent to

$$\begin{cases} \int_{\Omega} (E_c + \mathbb{B} \operatorname{inc} E_i) \cdot \hat{E}_c dx = \int_{\Omega} \mathbb{H} \cdot \hat{E}_c dx, \quad \forall \hat{E}_c \in \mathcal{V}, \\ \int_{\Omega} (E_i + \mathbb{B} \operatorname{inc} E_i) \cdot \hat{E}_i dx = \int_{\Omega} \mathbb{H} \cdot \hat{E}_i dx, \quad \forall \hat{E}_i \in \mathcal{W}_{\partial\Omega}^0, \end{cases} \quad (7.6)$$

itself, by Proposition 5.7, equivalent to

$$\begin{cases} \int_{\Omega} (E_c + \mathbb{B} \operatorname{inc} E_i) \cdot \nabla^S \hat{v} dx = \int_{\Omega} \mathbb{H} \cdot \nabla^S \hat{v} dx \quad \forall \hat{v} \in H^1(\Omega), & (a) \\ \int_{\Omega} (E_i + \mathbb{B} \operatorname{inc} E_i) \cdot \operatorname{inc} \hat{F} dx = \int_{\Omega} \mathbb{H} \cdot \operatorname{inc} \hat{F} dx \quad \forall \hat{F} \in \mathcal{Z}_0. & (b) \end{cases} \quad (7.7)$$

Define the operators  $L_{\mathbb{B}} : \mathcal{Z}_0 \rightarrow \mathcal{Z}'_0$  and  $M : L^2(\Omega, \mathbb{S}^3) \rightarrow \mathcal{Z}'_0$  by

$$\langle L_{\mathbb{B}} \Psi, \Phi \rangle = \int_{\Omega} \mathbb{B} \operatorname{inc} \Psi \cdot \operatorname{inc} \Phi dx, \quad \langle M \Psi, \Phi \rangle = \int_{\Omega} \Psi \cdot \operatorname{inc} \Phi dx.$$

Equation (7.7)(b) is equivalent to

$$(M + L_{\mathbb{B}}) E_i = M \mathbb{H}. \quad (7.8)$$

By Lemma 5.8,  $L_{\mathbb{B}} : \mathcal{Z}_0 \rightarrow \mathcal{Z}'_0$  is invertible. Thus, (7.8) is equivalent to

$$(I + L_{\mathbb{B}}^{-1} M) E_i = L_{\mathbb{B}}^{-1} M \mathbb{H}. \quad (7.9)$$

The operator  $L_{\mathbb{B}}^{-1} M : L^2(\Omega, \mathbb{S}^3) \rightarrow L^2(\Omega, \mathbb{S}^3)$  is compact, since it is continuous from  $L^2(\Omega, \mathbb{S}^3)$  to  $\mathcal{Z}_0$  and  $\mathcal{Z}_0$  is compactly embedded in  $L^2(\Omega, \mathbb{S}^3)$  by Proposition 5.6. Furthermore, under the condition  $\|\mathbb{B}^{-1}\|_{L^\infty} < C^{-1}$ , the operator  $I + L_{\mathbb{B}}^{-1} M : L^2(\Omega, \mathbb{S}^3) \rightarrow L^2(\Omega, \mathbb{S}^3)$  is injective due to the uniqueness claim. Thus, Fredholm's alternative provides the existence of  $E_i \in L^2(\Omega, \mathbb{S}^3)$  solution of (7.9). From  $E_i = L_{\mathbb{B}}^{-1} M(\mathbb{H} - E_i)$  we infer  $E_i \in \mathcal{Z}_0$ . We have found  $E_i \in \mathcal{Z}_0$  solution of (7.7)(b).

Let us turn to (7.7)(a). We have to find  $E_c = \nabla^S v$ ,  $v \in H^1(\Omega, \mathbb{R}^3)$  such that

$$\int_{\Omega} \nabla^S v \cdot \nabla^S \hat{v} dx = \int_{\Omega} (\mathbb{H} - \mathbb{B} \operatorname{inc} E_i) \cdot \nabla^S \hat{v} dx, \quad \forall \hat{v} \in H^1(\Omega, \mathbb{R}^3). \quad (7.10)$$

This is a standard linear elasticity problem.

*Third step. A priori estimate.* Equation (7.7)(b) entails

$$\int_{\Omega} E_i \cdot \operatorname{inc} E_i dx + \int_{\Omega} \mathbb{B} \operatorname{inc} E_i \cdot \operatorname{inc} E_i dx = \int_{\Omega} \mathbb{H} \cdot \operatorname{inc} E_i dx.$$

Using (7.5) we obtain

$$\|\operatorname{inc} E_i\|_{L^2}^2 \leq \|\mathbb{B}^{-1}\|_{L^\infty} \int_{\Omega} (\mathbb{H} - E_i) \cdot \operatorname{inc} E_i dx \leq \|\mathbb{B}^{-1}\|_{L^\infty} (\|\mathbb{H}\|_{L^2} + \|E_i\|_{L^2}) \|\operatorname{inc} E_i\|_{L^2}.$$

Proposition 5.6 yields

$$\|\operatorname{inc} E_i\|_{L^2} \leq \|\mathbb{B}^{-1}\|_{L^\infty} (\|\mathbb{H}\|_{L^2} + C \|\operatorname{inc} E_i\|_{L^2}),$$

from which we arrive at (7.1).  $\square$

**Remark 7.2** (Dislocation-induced stress). *Because of Kröner's formula we call  $\mathcal{G} := \ell \operatorname{inc} E$  the dislocation-induced stress tensor. For  $\ell$  constant, let  $E^\ell$  be the solution of  $\mathbb{A}E + \ell \operatorname{inc} E = \mathbb{K}$ . Then (7.1) implies that  $\mathcal{G}^\ell := \ell \operatorname{inc} E^\ell$  converges weakly in  $L^2(\Omega, \mathbb{S}^3)$  to some  $\mathcal{G}$  as  $\ell \rightarrow \pm\infty$ . More precise limiting results will be given in the next section.*

*Conversely, the condition  $|\ell| > C|\mathbb{A}|$  prevents  $|\ell|$  from going to 0 without assuming that  $\mathbb{A}$  also tends to 0. Such limit cases are left for future work.*

### 7.3. Well-posedness with arbitrary incompatibility flux.

**Theorem 7.2** (Second existence result). *Assume  $\Omega$  is simply connected. Let  $\mathbb{K} \in L^2(\Omega, \mathbb{S}^3)$  and  $h \in H^{-1/2}(\partial\Omega, \mathbb{R}^3)$  such that  $\int_{\partial\Omega} h dS(x) = 0$ . Let  $C$  be the Poincaré constant of Proposition 5.6. If  $\mathbb{A}$  is uniformly positive definite and  $|\ell| > C|\mathbb{A}|$  a.e., then there exists one and only one  $E \in H^{\operatorname{inc}}(\Omega, \mathbb{S}^3)$  such that*

$$\begin{cases} \mathbb{A}E + \ell \operatorname{inc} E = \mathbb{K} & \text{in } \Omega \\ \operatorname{inc} EN = h & \text{on } \partial\Omega. \end{cases}$$

Moreover there exists constants  $c_1$  and  $c_2$  such that

$$\|\operatorname{inc} E\|_{L^2} \leq \frac{\|\ell^{-1}\mathbb{A}\|_{L^\infty}}{1 - C\|\ell^{-1}\mathbb{A}\|_{L^\infty}} (\|\mathbb{A}^{-1}\mathbb{K}\|_{L^2} + c_1(1 + \|\ell\mathbb{A}^{-1}\|_{L^\infty})\|h\|_{H^{-1/2}}) + c_2\|h\|_{H^{-1/2}}. \quad (7.11)$$

*Proof.* Let  $w \in H^1(\Omega, \mathbb{R}^3)$  be solution of

$$\begin{cases} -\operatorname{div} \nabla^S w = 0 & \text{in } \Omega \\ \nabla^S w N = h & \text{on } \partial\Omega \end{cases}$$

and set  $W = \nabla^S w$ . By Corollary 4.3 there exists  $H \in \mathcal{Z}$  such that  $W = \operatorname{inc} H$ . Let  $\tilde{E} = E - H \in \mathcal{F}$ , which has to solve

$$\mathbb{A}\tilde{E} + \ell \operatorname{inc} \tilde{E} = \mathbb{K} - \mathbb{A}H - \ell \operatorname{inc} H.$$

Existence and uniqueness follow from Theorem 7.1. The a priori estimate of Corollary 4.3 combined with Proposition 5.6 and standard elliptic regularity provide (7.11).  $\square$

## 8. ELASTIC LIMIT

**Proposition 8.1.** *Consider a sequence  $\mathbb{A}_k \in L^\infty(\Omega, \mathbb{S}^3)$  with  $c_1|\xi|^2 \leq \mathbb{A}_k(x)\xi \cdot \xi \leq c_2|\xi|^2 \forall \xi \in \mathbb{R}^3$ , a.e.  $x \in \Omega$ ,  $c_1, c_2 > 0$ , and a sequence  $\ell_k \in L^\infty(\Omega, \mathbb{R}_+^*)$  with  $\inf_{\Omega} \ell_k \rightarrow +\infty$ . Assume that  $\mathbb{K} \in L^2(\Omega, \mathbb{S}^3)$ ,  $E^k \in \mathcal{F}$ ,  $\mathbb{A}_k E^k + \ell_k \operatorname{inc} E^k = \mathbb{K}$ . Then  $\|\operatorname{inc} E^k\|_{L^2} \rightarrow 0$ .*

*Proof.* It is a straightforward consequence of (7.1), since  $\|\ell_k^{-1}\mathbb{A}_k\|_{L^\infty} \rightarrow 0$ .  $\square$

Obviously the same holds for a sequence  $\ell_k \in L^\infty(\Omega, \mathbb{R}_+^*)$  with  $\inf_{\Omega} |\ell_k| \rightarrow +\infty$ .

**Proposition 8.2.** *If  $\ell$  is constant,  $\mathbb{K} \in L^2(\Omega, \mathbb{S}^3)$ ,  $E \in \mathcal{F}$ ,  $\mathbb{A}E + \ell \operatorname{inc} E = \mathbb{K}$  in  $\Omega$  then*

$$\int_{\Omega} \mathbb{A}E \cdot \hat{E} dx = \int_{\Omega} \mathbb{K} \cdot \hat{E} dx \quad \forall \hat{E} \in \mathcal{V}.$$

*Proof.* Take  $\hat{E} \in \mathcal{V}$  and observe that due to the assumptions, one has

$$\int_{\Omega} \ell \operatorname{inc} E \cdot \hat{E} dx = 0.$$

□

**Theorem 8.3** (Elastic limit: homogeneous flux). *Assume that  $\mathbb{A}$ ,  $\mathbb{K}$  are fixed,  $\ell$  is constant,  $E^\ell \in \mathcal{F}$ ,  $\mathbb{A}E^\ell + \ell \operatorname{inc} E^\ell = \mathbb{K}$  in  $\Omega$ . There exists a unique  $E^\infty \in \mathcal{V}$  such that*

$$\int_{\Omega} \mathbb{A}E^\infty \cdot \hat{E} dx = \int_{\Omega} \mathbb{K} \cdot \hat{E} dx \quad \forall \hat{E} \in \mathcal{V}. \quad (8.1)$$

Moreover  $\|E^\ell - E^\infty\|_{L^2} \rightarrow 0$  when  $|\ell| \rightarrow +\infty$ .

*Proof.* Existence and uniqueness for (8.1) is a consequence of the Riesz representation theorem in the Hilbert space  $\mathcal{V}$  for the inner product  $(E, \hat{E}) \mapsto \int_{\Omega} \mathbb{A}E \cdot \hat{E} dx$ . Consider the decomposition  $E^\ell = E_c^\ell + E_i^\ell \in \mathcal{V} \oplus \mathcal{Z}_0$ . We have by Proposition 8.2

$$\int_{\Omega} \mathbb{A}(E_c^\ell + E_i^\ell) \cdot \hat{E}_c dx = \int_{\Omega} \mathbb{K} \cdot \hat{E}_c dx \quad \forall \hat{E}_c \in \mathcal{V}.$$

Substracting (8.1), one has

$$\int_{\Omega} \mathbb{A}(E_c^\ell - E^\infty) \cdot \hat{E}_c dx = - \int_{\Omega} \mathbb{A}E_i^\ell \cdot \hat{E}_c dx \quad \forall \hat{E}_c \in \mathcal{V}.$$

By Propositions 8.1 and 5.6 we have  $\|E_i^\ell\|_{H^1} \rightarrow 0$ . It follows  $\|E_c^\ell - E^\infty\|_{L^2} \rightarrow 0$  hence  $\|E^\ell - E^\infty\|_{L^2} \rightarrow 0$ . □

Hence, as  $|\ell| \rightarrow +\infty$ , one retrieves the standard linear elasticity problem with Neumann boundary conditions.

In case of non-vanishing incompatibility flux the following holds.

**Theorem 8.4** (Elastic limit: general flux). *Assume that  $\mathbb{A}$ ,  $\mathbb{K}$  are fixed,  $\ell$  is constant,  $E^\ell \in H^{\operatorname{inc}}(\Omega, \mathbb{S}^3)$ ,  $\mathbb{A}E^\ell + \ell \operatorname{inc} E^\ell = \mathbb{K}$  in  $\Omega$ ,  $\operatorname{inc} E^\ell N = h^\ell$  on  $\partial\Omega$ ,  $h^\ell \in H^{-1/2}(\partial\Omega)$ ,  $\ell h^\ell \rightarrow \bar{h}$  in  $H^{-1/2}(\partial\Omega)$  when  $|\ell| \rightarrow +\infty$ . There exists a unique  $E^\infty \in \mathcal{V}$  such that*

$$\int_{\Omega} \mathbb{A}E^\infty \cdot \nabla^S \hat{v} dx = \int_{\Omega} \mathbb{K} \cdot \nabla \hat{v} dx - \int_{\partial\Omega} \bar{h} \cdot \hat{v} dS(x) \quad \forall \hat{v} \in H^1(\Omega). \quad (8.2)$$

Moreover  $\|E^\ell - E^\infty\|_{L^2} \rightarrow 0$  when  $|\ell| \rightarrow +\infty$ .

*Proof.* For all  $\hat{v} \in H^1(\Omega, \mathbb{R}^3)$  one has

$$\int_{\Omega} \mathbb{A}E^\ell \cdot \nabla^S \hat{v} dx + \int_{\partial\Omega} \ell h \cdot \hat{v} dS(x) = \int_{\Omega} \mathbb{K} \cdot \nabla^S \hat{v} dx.$$

Hence

$$\int_{\Omega} \mathbb{A}(E^\ell - E^\infty) \cdot \nabla^S \hat{v} dx = - \int_{\partial\Omega} (\ell h - \bar{h}) \cdot \hat{v} dS(x).$$

Consider the decomposition  $E^\ell = E_c^\ell + E_i^\ell \in \mathcal{V} \oplus \mathcal{Z}$ , see Proposition 5.3. By (7.11), one infers  $\|\operatorname{inc} E^\ell\|_{L^2} \rightarrow 0$ , and subsequently by Proposition 5.6  $\|E_i^\ell\|_{L^2} \rightarrow 0$ . Finally,

$$\int_{\Omega} \mathbb{A}(E_c^\ell - E^\infty) \cdot \nabla^S \hat{v} dx = - \int_{\Omega} \mathbb{A}E_i^\ell \cdot \nabla^S \hat{v} dx - \int_{\partial\Omega} (\ell h - \bar{h}) \cdot \hat{v} dS(x)$$

yields  $\|E_c^\ell - E^\infty\|_{L^2} \rightarrow 0$ , then  $\|E^\ell - E^\infty\|_{L^2} \rightarrow 0$ . □

## 9. INTERPRETATION OF THE KINEMATICAL FRAMEWORK AND EXTERNAL EFFORTS

9.1. **External efforts.** Consider a virtual strain  $\hat{E} \in L^2(\Omega, \mathbb{S}^3)$  decomposed as

$$\hat{E} = \nabla^S \hat{v} + \text{inc } \hat{F}. \quad (9.1)$$

The work of the external efforts against  $\hat{E}$  reads

$$\begin{aligned} W_{\text{ext}}(\hat{E}) = \int_{\Omega} \mathbb{K} \cdot \hat{E} dx = & - \int_{\Omega} \text{div } \mathbb{K} \cdot \hat{v} dx + \int_{\partial\Omega} \mathbb{K} N \cdot \hat{v} dS(x) + \int_{\Omega} \text{inc } \mathbb{K} \cdot \hat{F} dx \\ & + \int_{\partial\Omega} \left( \mathcal{T}_1(\mathbb{K}) \cdot \hat{F} + \mathcal{T}_0(\mathbb{K}) \cdot \partial_N \hat{F} \right) dS(x). \end{aligned} \quad (9.2)$$

The fields  $-\text{div } \mathbb{K}$  and  $\mathbb{K}N$  are recognized as classical body and contact forces, while  $\text{inc } \mathbb{K}$  and  $(\mathcal{T}_0(\mathbb{K}), \mathcal{T}_1(\mathbb{K}))$  are body and contact forces working against the incompatible part of the virtual strain. The above fields are in principle known in the first place. The issue is then how and under which conditions it is possible to construct a corresponding  $\mathbb{K}$ . Formally the boundary forces  $\mathbb{K}N$ ,  $\mathcal{T}_0(\mathbb{K})$  and  $\mathcal{T}_1(\mathbb{K})$  exhibit some coupling, as stressed in [5]. To address these points one must specify a kinematical framework ensuring the uniqueness of the decomposition (9.1).

9.2. **Kinematical framework.** Take  $\hat{E} = \nabla^S \hat{v} + \text{inc } \hat{F}$  with  $\nabla^S \hat{v} \in \mathcal{V}_{\Gamma_1}^{00}$  and  $\text{inc } \hat{F} \in \mathcal{W}_{\Gamma_2}^0$  for some partition  $\Gamma_1 \cup \Gamma_2$  of  $\partial\Omega$ . As said above,  $f := -\text{div } \mathbb{K}$  is identified with the body force, and  $g := \mathbb{K}N$  is identified with a surface load on  $\Gamma_2$ . Now, if  $\mathbb{K} \in \mathcal{V}_{\Gamma_1}^{00}$  the last two integrals of (9.2) vanish by virtue of Lemma 5.2. Then (9.2) rewrites as

$$\int_{\Omega} \mathbb{K} \cdot \hat{E} dx = \int_{\Omega} f \cdot \hat{v} dx + \int_{\Gamma_2} g \cdot \hat{v} dS(x). \quad (9.3)$$

To sum up, given  $f \in L^2(\Omega, \mathbb{R}^3)$  and  $g \in H^{-1/2}(\Omega, \mathbb{R}^3)$  one wants to determine  $\mathbb{K} = \nabla^S w \in L^2(\Omega, \mathbb{S}^3)$  such that

$$\begin{cases} -\text{div } \nabla^S w = f & \text{in } \Omega, \\ w = 0 & \text{on } \Gamma_1, \\ \nabla^S w N = g & \text{on } \Gamma_2. \end{cases} \quad (9.4)$$

It is well-known that this problem admits a unique solution  $w \in H^1(\Omega, \mathbb{R}^3)$ , provided  $\int_{\Omega} f dx + \int_{\partial\Omega} g dS(x) = 0$  if  $|\Gamma_1| = 0$ .

**Proposition 9.1** (Generalized strong form for intrinsic elasto-plasticity). *Assume that  $\Omega$  is simply-connected. Let  $\mathbb{K} \in L^2(\Omega, \mathbb{S}^3)$  be such that  $\mathbb{K} = \nabla^S w$ , where  $w$  satisfies (9.4). Then the weak form*

$$\int_{\Omega} (\mathbb{A}E + \ell \text{inc } E) \cdot (\nabla^S \hat{v} + \hat{E}^0) dx = \int_{\Omega} f \cdot \hat{v} dx + \int_{\Gamma_2} g \cdot \hat{v} dS(x), \quad \forall (\nabla^S \hat{v}, \hat{E}^0) \in \mathcal{V}_{\Gamma_1}^{00} \times \mathcal{W}_{\Gamma_2}^0 \quad (9.5)$$

admits a unique solution  $E \in \mathcal{F}$  that corresponds to the solution of the strong form

$$\left\{ \begin{array}{ll} -\text{div } (\mathbb{A}E + \ell \text{inc } E) = f & \text{in } \Omega \\ \text{inc } (\mathbb{A}E + \ell \text{inc } E) = 0 & \text{in } \Omega \\ (\mathbb{A}E + \ell \text{inc } E)N = (\mathbb{A}E)N = g & \text{on } \Gamma_2 \\ \text{inc } EN = 0 & \text{on } \partial\Omega \\ \mathcal{T}_0(\mathbb{A}E + \ell \text{inc } E) = \mathcal{T}_1(\mathbb{A}E + \ell \text{inc } E) = 0 & \text{on } \Gamma_1 \\ \mathcal{T}_0(\mathbb{A}E + \ell \text{inc } E) = \mathcal{T}_0(\mathbb{K}) & \text{on } \Gamma_2 \\ \mathcal{T}_1(\mathbb{A}E + \ell \text{inc } E) = \mathcal{T}_1(\mathbb{K}) & \text{on } \Gamma_2. \end{array} \right. \quad (9.6)$$

*Proof.* Eq. (9.5) is equivalent to  $\mathbb{A}E + \ell \text{inc } E = \mathbb{K}$ , itself equivalent to

$$\left\{ \begin{array}{l} \text{div } (\mathbb{A}E + \ell \text{inc } E) = \text{div } \mathbb{K} \text{ and } \text{inc } (\mathbb{A}E + \ell \text{inc } E) = \text{inc } \mathbb{K} \text{ in } \Omega, \\ (\mathbb{A}E + \ell \text{inc } E)N = \mathbb{K}N \text{ and } \mathcal{T}_0(\mathbb{A}E + \ell \text{inc } E) = \mathcal{T}_0(\mathbb{K}) \text{ and } \mathcal{T}_1(\mathbb{A}E + \ell \text{inc } E) = \mathcal{T}_1(\mathbb{K}) \text{ on } \partial\Omega. \end{array} \right.$$

The assumptions and the essential conditions complete the system.  $\square$

**Remark 9.1.** *We remark that the normal component of the Cauchy stress  $(\mathbb{A}E)N$  can be prescribed on  $\Gamma_2$ , whereas the tangential part of the full stress  $\mathcal{T}_0(\mathbb{A}E + \ell \text{inc } E)$  is fixed to zero on  $\Gamma_1$ .*

**Remark 9.2.** *In this framework, there is no virtual work associated with the boundary condition on  $\Gamma_1$ , even if the virtual strain is allowed to have tangential components. This does not model a clamped condition, and when  $|\ell| \rightarrow \infty$  one does not retrieve  $\mathcal{T}_0(E^\infty) = \mathcal{T}_1(E^\infty) = 0$ , see Theorem 8.3. To do so one must prescribe kinematical constraints.*

**9.3. Alternative to the vanishing incompatibility flux condition.** As already said, the condition  $\text{inc } EN = 0$  included in the space  $\mathcal{F}$  may seem unphysical, as well as prescribing  $\text{inc } EN = h$  for an a priori given  $h$ . An alternative is to consider that the exterior of  $\Omega$  is filled with a fictitious material that mimicks void, with transmission conditions representing the fact that the two phases constitute a continuum. Therefore we restrict ourselves to the pure "Neumann" boundary condition, that is, a surface load without kinematical restriction. For a full space extension,  $\mathbb{K} = \nabla^S w$  is defined by

$$\begin{cases} -\text{div}(\nabla^S w) = f \text{ in } \mathbb{R}^3, \\ \llbracket \nabla^S w N \rrbracket = g \text{ on } \partial\Omega, \end{cases}$$

with  $f$  extended by 0 outside  $\Omega$ . The equilibrium equation  $\mathbb{A}E + \ell \text{inc } E = \mathbb{K}$  is fulfilled over  $\mathbb{R}^3$  with  $(\mathbb{A}, \ell)$  extended by  $(\mathbb{A}^{\text{ext}}, \ell^{\text{ext}})$  outside  $\Omega$ . In order to approximate a Neumann condition one needs that  $\mathbb{A}^{\text{ext}}$  be chosen significantly smaller than  $\mathbb{A}$  within  $\Omega$ . Then on  $\partial\Omega$  one has

$$\text{inc } EN \approx \frac{1}{\ell^{\text{ext}}} (\mathbb{K}N)_{\text{ext}}.$$

It is reasonable to assume that  $\ell$  is continuous across  $\partial\Omega$  in a way that  $\partial\Omega$  has a neutral effect on the transport of dislocations (transmission without reflection). Under this assumption we have

$$\begin{cases} \mathbb{A}E + \ell \text{inc } E = \mathbb{K} \text{ in } \Omega \\ \text{inc } EN = \frac{1}{\ell} (\mathbb{K}N)_{\text{ext}} \text{ on } \partial\Omega. \end{cases}$$

Existence and uniqueness of a solution has been shown in Theorem 7.2. In addition we derive on  $\partial\Omega$

$$\mathbb{A}EN = \mathbb{K}N - \ell \text{inc } EN = \mathbb{K}N - (\mathbb{K}N)_{\text{ext}} = g,$$

which is obviously the standard Neumann condition on the Cauchy stress.

Let us now examine the limit case. In view of Theorem 8.4, we have  $E \rightarrow E^\infty \in \mathcal{V}$  as  $|\ell| \rightarrow \infty$  where

$$\int_{\Omega} \mathbb{A}E^\infty \cdot \nabla^S \hat{v} dx = \int_{\Omega} \mathbb{K} \cdot \nabla \hat{v} dx - \int_{\partial\Omega} (\mathbb{K}N)_{\text{ext}} \cdot \hat{v} dS(x) \quad \forall \hat{v} \in H^1(\Omega).$$

This rewrites as

$$\int_{\Omega} \mathbb{A}E^\infty \cdot \nabla^S \hat{v} dx = \int_{\Omega} f \cdot \hat{v} dx + \int_{\partial\Omega} g \cdot \hat{v} dS(x) \quad \forall \hat{v} \in H^1(\Omega).$$

The standard Neumann elasticity problem in  $\Omega$  is retrieved.

## 10. EXAMPLES

**10.1. 1D case: uniaxial traction.** We consider the domain  $\Omega = \mathbb{R}^3$ . We assume a uniform traction of density  $g = 1$  on the planes  $\{z = \pm h\}$ . Hence the tensor

$$\mathbb{K} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k \end{pmatrix},$$

with  $k = \chi_{\{|z| < h\}}$ , provides the virtual power  $\int_{\mathbb{R}^3} \mathbb{K} \cdot (\nabla^S \hat{v} + \text{inc } \hat{F}) dx = \int_{\{z=h\}} e_z \cdot \hat{v} dS(x) - \int_{\{z=-h\}} e_z \cdot \hat{v} dS(x)$ . We search for a strain field of form

$$E = \begin{pmatrix} \varphi & 0 & 0 \\ 0 & \varphi & 0 \\ 0 & 0 & \psi \end{pmatrix},$$

where  $\varphi, \psi$  are functions of the  $z$  variable. In this case one has

$$\mathbb{A}E = \begin{pmatrix} 2(\lambda + \mu)\varphi + \lambda\psi & 0 & 0 \\ 0 & 2(\lambda + \mu)\varphi + \lambda\psi & 0 \\ 0 & 0 & 2\lambda\varphi + (\lambda + 2\mu)\psi \end{pmatrix},$$

$$\text{inc } E = \begin{pmatrix} \varphi'' & 0 & 0 \\ 0 & \varphi'' & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (10.1)$$

Hence  $\mathbb{A}E + \ell \text{inc } E = \mathbb{K}$  if and only if

$$\begin{cases} 2(\lambda + \mu)\varphi + \lambda\psi + \ell\varphi'' = 0 \\ 2\lambda\varphi + (\lambda + 2\mu)\psi = k. \end{cases} \quad (10.2)$$

Elementary algebra leads to

$$\psi = \frac{1}{\lambda + 2\mu}(k - 2\lambda\varphi),$$

$$2\mu(3\lambda + 2\mu)\varphi + \ell(\lambda + 2\mu)\varphi'' = -\lambda k.$$

This ordinary differential equation leads us to assume that  $\ell < 0$ , since in the other case the solutions do not decay when  $|z| \rightarrow \infty$ . We denote

$$\omega = \sqrt{\frac{2\mu(3\lambda + 2\mu)}{\lambda + 2\mu}}.$$

We obtain:

(1) For  $|z| < h$ ,

$$\varphi(z) = \frac{-\lambda}{2\mu(3\lambda + 2\mu)} \left[ 1 - \exp\left(-\frac{\omega h}{\sqrt{|\ell|}}\right) \cosh\left(\frac{\omega z}{\sqrt{|\ell|}}\right) \right],$$

$$\psi(z) = \frac{1}{\lambda + 2\mu} \left\{ 1 + \frac{\lambda^2}{\mu(3\lambda + 2\mu)} \left[ 1 - \exp\left(-\frac{\omega h}{\sqrt{|\ell|}}\right) \cosh\left(\frac{\omega z}{\sqrt{|\ell|}}\right) \right] \right\}.$$

(2) For  $|z| > h$ ,

$$\varphi(z) = \frac{-\lambda}{2\mu(3\lambda + 2\mu)} \sinh\left(\frac{\omega h}{\sqrt{|\ell|}}\right) \exp\left(-\frac{\omega|z|}{\sqrt{|\ell|}}\right),$$

$$\psi(z) = \frac{\lambda^2}{\mu(\lambda + 2\mu)(3\lambda + 2\mu)} \sinh\left(\frac{\omega h}{\sqrt{|\ell|}}\right) \exp\left(-\frac{\omega|z|}{\sqrt{|\ell|}}\right).$$

Observe that

$$\lim_{|\ell| \rightarrow \infty} \varphi(z) = 0, \quad \lim_{|\ell| \rightarrow \infty} \psi(z) = \begin{cases} \frac{1}{\lambda + 2\mu} & \text{if } |z| < h, \\ 0 & \text{if } |z| > h, \end{cases}$$

which is the classical elastic solution with uniaxial strain.

Let  $\nabla^S U + E_0$  be the Beltrami decomposition of  $E$  in the domain  $\Omega_t := \{|z| < t\}$  such that  $E_0 \in \mathcal{W}_{\partial\Omega_t}^0$  (see Theorem 5.1), i.e.,  $\text{div } E_0 = 0$  in  $\Omega_t$  and  $E_0 N = 0$  on  $\partial\Omega_t$ . One has

$$\begin{cases} \text{div } \nabla^S U = \text{div } E & \text{in } \Omega_t, \\ \nabla^S U N = E N & \text{on } \partial\Omega_t. \end{cases}$$

Denoting by  $u$  the  $z$  component of  $U$  we obtain  $u'' = \psi'$ ,  $u'(t) = \psi(t)$ ,  $u'(-t) = \psi(-t)$ . Thus  $u' = \psi$  and, setting  $u(0) = 0$ ,

$$u(z) = \int_0^z \psi(s) ds.$$

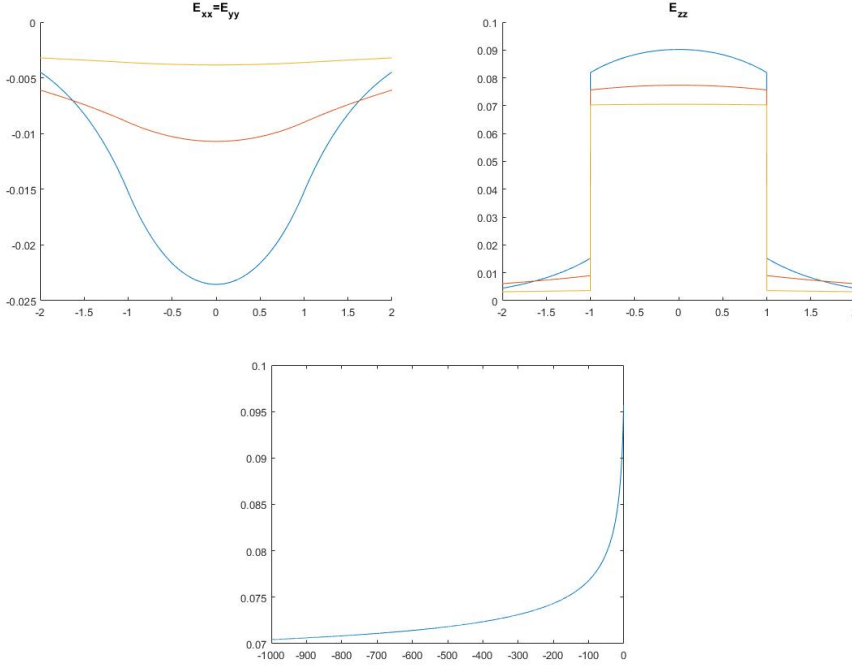


FIGURE 1. Functions  $\varphi$  (top left) and  $\psi$  (top right), for  $\ell = -10$  (blue),  $\ell = -100$  (red),  $\ell = -1000$  (yellow). Value of  $u(h)$  in function of  $\ell$  (bottom right)

We obtain in particular

$$u(h) = \frac{1}{\lambda + 2\mu} \left\{ h + \frac{\lambda^2}{\mu(3\lambda + 2\mu)} \left[ h - \frac{\sqrt{|\ell|}}{2\omega} \left( 1 - \exp\left(-\frac{2\omega h}{\sqrt{|\ell|}}\right) \right) \right] \right\}.$$

The functions  $\varphi$ ,  $\psi$  are plotted in Figure 1 for  $h = 1$ , Young's modulus  $Y = 10$  and Poisson ratio  $\nu = 1/3$ . The value of  $u(h)$  as a function of  $\ell$  is also depicted. As expected, we observe the increase of elongation as  $|\ell|$  decreases.

**10.2. 2D case: cylinder under uniform radial traction.** We consider a two-dimensional model of the variables  $(x, y)$  and we assume that

$$E = \begin{pmatrix} u & w & 0 \\ w & v & 0 \\ 0 & 0 & h \end{pmatrix}, \quad \mathbb{K} = \begin{pmatrix} p & s & 0 \\ s & q & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

One has in Cartesian coordinates

$$\begin{aligned} \mathbb{A}E &= \begin{pmatrix} (\lambda + 2\mu)u + \lambda(v + h) & 2\mu w & 0 \\ 2\mu w & (\lambda + 2\mu)v + \lambda(u + h) & 0 \\ 0 & 0 & \lambda(u + v) + (\lambda + 2\mu)h \end{pmatrix}, \\ \text{inc } E &= \begin{pmatrix} \partial_{yy}h & -\partial_{xy}h & 0 \\ -\partial_{xy}h & \partial_{xx}h & 0 \\ 0 & 0 & \partial_{xx}v - 2\partial_{xy}w + \partial_{yy}u \end{pmatrix}. \end{aligned} \quad (10.3)$$

Hence  $\mathbb{A}E + \ell \text{inc } E = \mathbb{K}$  if and only if

$$\begin{cases} (\lambda + 2\mu)u + \lambda(v + h) + \ell \partial_{yy}h = p \\ (\lambda + 2\mu)v + \lambda(u + h) + \ell \partial_{xx}h = q \\ 2\mu w - \ell \partial_{xy}h = s \\ \lambda(u + v) + (\lambda + 2\mu)h + \ell(\partial_{xx}v - 2\partial_{xy}w + \partial_{yy}u) = 0. \end{cases} \quad (10.4)$$

Elementary algebra leads to

$$\begin{aligned} u &= \frac{1}{4\mu(\lambda + \mu)} ((\lambda + 2\mu)p - \lambda q - 2\lambda\mu h + \ell\lambda\partial_{xx}h - \ell(\lambda + 2\mu)\partial_{yy}h) \\ v &= \frac{1}{4\mu(\lambda + \mu)} (-\lambda p + (\lambda + 2\mu)q - 2\lambda\mu h - \ell(\lambda + 2\mu)\partial_{xx}h + \ell\lambda\partial_{yy}h) \\ w &= \frac{1}{2\mu} (s + \ell\partial_{xy}h). \end{aligned}$$

Within regions where  $\text{inc } \mathbb{K} = 0$ ,  $\text{div } \mathbb{K} = 0$  and  $\lambda, \mu, \ell$  are constant, substituting the above relations in the last equation of (10.4) entails

$$\ell^2(\lambda + 2\mu)\Delta^2 h + 4\ell\lambda\mu\Delta h - 4\mu^2(3\lambda + 2\mu)h = 2\lambda\mu(p + q). \quad (10.5)$$

This can be factorized as

$$(\ell(\lambda + 2\mu)\Delta + 2\mu(3\lambda + 2\mu))(\ell\Delta - 2\mu)h = 2\lambda\mu(p + q). \quad (10.6)$$

We consider an infinite cylinder  $B = B(0, 1)$  under uniform radial traction on its boundary. Therefore we search for  $\mathbb{K} = \nabla^S w$  such that  $-\text{div } \mathbb{K} = \delta_{\partial B} N$ . A standard calculation using Airy stress functions yields in polar coordinates

$$\mathbb{K} = \chi_B \frac{1}{2} (e_r \otimes e_r + e_\theta \otimes e_\theta) + (1 - \chi_B) \frac{1}{2r^2} (e_r \otimes e_r - e_\theta \otimes e_\theta).$$

We choose the elastic and incompatibility moduli as

$$(\lambda, \mu, \ell) = \begin{cases} (\lambda_i, \mu_i, \ell_i) & \text{in } B \\ (\lambda_e, \mu_e, \ell_e) & \text{in } \mathbb{R}^2 \setminus B, \end{cases}$$

with  $\lambda_i/\mu_i = \lambda_e/\mu_e$ . On  $\partial B$  one has the transmission conditions  $[[\mathcal{T}_0(E)]] = [[\mathcal{T}_1(E)]] = 0$ . Let us place ourselves in polar coordinates and, due to symmetry, search for  $h = h(r)$ . The condition  $[[\mathcal{T}_0(E)]] = 0$  implies  $[[h]] = 0$  and  $[[E_{\theta\theta}]] = 0$ . Using  $\mathbb{A}E = \mathbb{K} - \ell \text{inc } E$  one obtains the planar components of  $\mathbb{A}E$  as

$$\begin{aligned} (\mathbb{A}E)_{\text{plan}} &= \chi_B \left[ \left( \frac{1}{2} - \frac{\ell_i}{r} h' \right) e_r \otimes e_r + \left( \frac{1}{2} - \ell_i h'' \right) e_\theta \otimes e_\theta \right] \\ &\quad + (1 - \chi_B) \left[ \left( -\frac{1}{2r^2} - \frac{\ell_e}{r} h' \right) e_r \otimes e_r + \left( \frac{1}{2r^2} - \ell_e h'' \right) e_\theta \otimes e_\theta \right]. \end{aligned}$$

Then, Hooke's law together with  $E_{rr} + E_{\theta\theta} = u + v = (1 - 2\lambda h - \ell\Delta h)/(2(\lambda + \mu))$  yield

$$\begin{aligned} (\mathbb{A}E)_{zz} &= \chi_B \left[ \frac{\lambda_i}{2(\lambda_i + \mu_i)} \left( 1 - \frac{\ell_i}{r} h' - \ell_i h'' \right) + \frac{\mu_i(3\lambda_i + 2\mu_i)}{\lambda_i + \mu_i} h \right] \\ &\quad + (1 - \chi_B) \left[ \frac{\lambda_e}{2(\lambda_e + \mu_e)} \left( -\frac{\ell_e}{r} h' - \ell_e h'' \right) + \frac{\mu_e(3\lambda_e + 2\mu_e)}{\lambda_e + \mu_e} h \right], \end{aligned}$$

and

$$\begin{aligned} E_{\theta\theta} &= \chi_B \frac{1}{4\mu_i(\lambda_i + \mu_i)} \left[ \mu_i - \ell_i(\lambda_i + 2\mu_i)h'' - \lambda_i \left( -\frac{\ell_i}{r} h' + 2\mu_i h \right) \right] \\ &\quad + (1 - \chi_B) \frac{1}{4\mu_e(\lambda_e + \mu_e)} \left[ \lambda_e + \mu_e - \ell_e(\lambda_e + 2\mu_e)h'' - \lambda_e \left( -\frac{\ell_e}{r} h' + 2\mu_e h \right) \right]. \end{aligned}$$

The condition  $[[E_{\theta\theta}]] = 0$  rewrites as

$$\left[ \frac{\ell}{\mu} \frac{\lambda + 2\mu}{\lambda + \mu} h'' - \frac{\ell}{\mu} \frac{\lambda}{\lambda + \mu} h' \right] = \frac{1}{\lambda_i + \mu_i} - \frac{1}{\mu_e}.$$

Next, from  $\mathcal{T}_1(E) = -he_r \otimes e_r + (h - h')e_\theta \otimes e_\theta$  we infer  $[[h']] = 0$ .

Coming back to (10.5) one looks for

$$h = \frac{-\lambda_i}{2\mu_i(3\lambda_i + 2\mu_i)} \chi_B + \tilde{h}$$



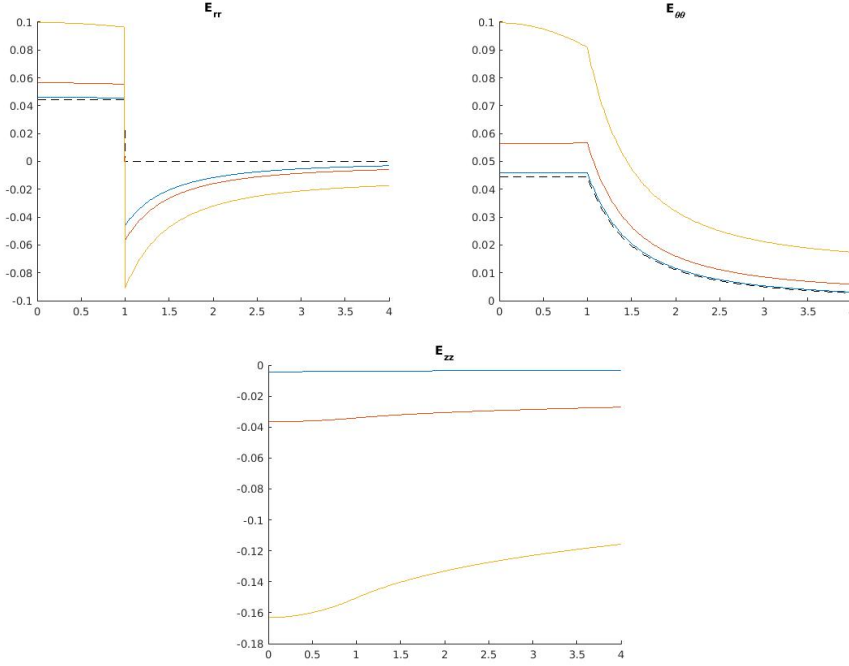


FIGURE 2. Strain components in cylindrical coordinates for  $\ell = -1000$  (blue),  $\ell = -100$  (red),  $\ell = -20$  (yellow), classical plane strain elastic solution (dashed)

$\ell$	-1000	-100	-20
$W$	0.2850	0.3498	0.6016

TABLE 1. External work

with  $\tilde{h}$  solution of the homogeneous equation in  $B$  and  $\mathbb{R}^2 \setminus \bar{B}$ . In view of (10.5) and assuming  $\ell < 0$ ,  $\tilde{h}$  is spanned by the Bessel-type functions  $J_0(k^+r)$ ,  $Y_0(k^+r)$ ,  $I_0(k^-r)$  and  $K_0(k^-r)$  with

$$k^+ = \sqrt{\frac{2\mu}{-\ell}}, \quad k^- = \sqrt{\frac{2\mu}{-\ell} \frac{3\lambda + 2\mu}{\lambda + 2\mu}}.$$

Due to boundedness and decay at infinity, it remains

$$\tilde{h}(r) = \begin{cases} aJ_0(k^+r) + bI_0(k^-r) & \text{if } r < 1 \\ cK_0(k^-r) & \text{if } r > 1. \end{cases}$$

The three transmission conditions fix  $a$ ,  $b$  and  $c$  through a linear system.

In the following simulations we take a material inside  $B$  with Young modulus  $Y = 10$  and Poisson ration  $\nu = 1/3$ , and a nearly void exterior phase such that  $\mathbb{A}_i = 10^{-5}\mathbb{A}_e$ . The compatibility modulus is taken constant over  $\mathbb{R}^2$ . The plots of the strain are given in Figure 2, and compared with the classical plane strain elastic strain. The external work

$$W = \int_{\mathbb{R}^2} \mathbb{K} \cdot E dx$$

is indicated in Table 1. This shows that decreasing the incompatibility modulus (in absolute value) increases dissipation.

**10.3. 3D case: ball under uniform traction.** Consider the domain  $\Omega = \{x \in \mathbb{R}^3, |x| < 1\}$ . We assume a uniform unit radial traction on  $\partial\Omega$ . We treat the two kinematical frameworks addressed

in this paper, namely the case of vanishing incompatibility flux and the case of free incompatibility flux through domain extension described in Section 9.3.

10.3.1. *Case 1: vanishing incompatibility flux.* We assume in this case the condition  $\text{inc } EN = 0$  on  $\partial\Omega$ . We have  $\mathbb{K} = \nabla^S w$  with

$$\begin{cases} -\text{div } \nabla^S w = 0 \text{ in } \Omega, \\ \nabla^S w N = N \text{ on } \partial\Omega. \end{cases}$$

The solution is immediately found as  $w = r e_r$  and  $\mathbb{K} = I$ . Considering the form

$$E = \varphi(r)I + \psi(r)e_r \otimes e_r$$

we have

$$\mathbb{A}E = ((3\lambda + 2\mu)\varphi + \lambda\psi)I + 2\mu\psi e_r \otimes e_r,$$

and (see [45])

$$\text{inc } E = \left( \varphi'' + \frac{\varphi'}{r} - \frac{\psi'}{r} \right) I + \left( -\varphi'' + \frac{\varphi'}{r} + \frac{\psi'}{r} - \frac{2\psi}{r^2} \right) e_r \otimes e_r.$$

Hence  $\mathbb{A}E + \ell \text{inc } E = \mathbb{K}$  if and only if

$$\begin{cases} (3\lambda + 2\mu)\varphi + \lambda\psi + \ell \left( \varphi'' + \frac{\varphi'}{r} - \frac{\psi'}{r} \right) = 1 \\ 2\mu\psi + \ell \left( -\varphi'' + \frac{\varphi'}{r} + \frac{\psi'}{r} - \frac{2\psi}{r^2} \right) = 0. \end{cases}$$

The condition  $\text{inc } EN = 0$  on  $\partial\Omega$  entails  $\varphi'(1) = \psi(1)$ . The solution of the system is the classical elastic solution given by

$$\varphi = \frac{1}{3\lambda + 2\mu}, \quad \psi = 0.$$

There is no strain incompatibility in this case. This is a consequence of  $\text{tr } \mathbb{K}$  being constant, as explained in Remark 7.1.

10.3.2. *Case 2: free incompatibility flux.* In this case we define  $\mathbb{K} = \nabla^S w$  over  $\mathbb{R}^3$  by

$$\begin{cases} -\text{div } \nabla^S w = 0 \text{ in } \Omega \cup (\mathbb{R}^3 \setminus \bar{\Omega}), \\ \llbracket \nabla^S w N \rrbracket = N \text{ on } \partial\Omega. \end{cases}$$

For  $w = w(r)e_r$  one has

$$\begin{aligned} \nabla^S w &= w' e_r \otimes e_r + \frac{w}{r} (e_\theta \otimes e_\theta + e_\phi \otimes e_\phi), \\ \text{div } \nabla^S w &= \left( w'' + 2\frac{w'}{r} - 2\frac{w}{r^2} \right) e_r = \left( \frac{1}{r^2} (r^2 w)' \right)' e_r. \end{aligned}$$

Therefore

$$\text{div } \nabla^S w = 0 \iff w = ar + \frac{b}{r^2}$$

for some constants  $a, b$ . The transmission condition yields

$$\begin{aligned} w(r) &= \frac{r}{3} \chi_{\{r < 1\}} + \frac{1}{3r^2} \chi_{\{r > 1\}}, \\ \mathbb{K} &= \frac{1}{3} I \chi_{\{r < 1\}} + \frac{1}{r^3} \chi_{\{r > 1\}} \left( \frac{1}{3} I - e_r \otimes e_r \right). \end{aligned}$$

We still search  $E$  of the form

$$E = \varphi(r)I + \psi(r)e_r \otimes e_r.$$

Outside  $\Omega$  we assume that the Lamé coefficients are vanishing whereas the incompatibility modulus is kept the same as within  $\Omega$ , as explained in Section 9.3. Hence  $\mathbb{A}E + \ell \operatorname{inc} E = \mathbb{K}$  if and only if

$$\begin{cases} \chi_{\{r<1\}} ((3\lambda + 2\mu)\varphi + \lambda\psi) + \ell \left( \varphi'' + \frac{\varphi'}{r} - \frac{\psi'}{r} \right) = \frac{1}{3}\chi_{\{r<1\}} + \frac{1}{3r^3}\chi_{\{r>1\}} \\ \chi_{\{r<1\}} 2\mu\psi + \ell \left( -\varphi'' + \frac{\varphi'}{r} + \frac{\psi'}{r} - \frac{2\psi}{r^2} \right) = -\frac{1}{r^3}\chi_{\{r>1\}}. \end{cases}$$

Note that we have taken exactly vanishing Lamé coefficients outside  $\Omega$  for technical simplicity. This leads in principle to nonunique solutions, however this issue will be overcome later.

Let us first solve the system in  $\Omega$ . Set  $\rho = \psi - r\varphi'$ . Substituting for  $\psi$  yields

$$\begin{cases} (3\lambda + 2\mu)\varphi + \lambda(r\varphi' + \rho) - \ell \frac{\rho'}{r} = \frac{1}{3} \\ 2\mu(r\varphi' + \rho) + \ell \left( \frac{\rho'}{r} - 2\frac{\rho}{r^2} \right) = 0. \end{cases}$$

We infer after some algebra

$$\begin{aligned} \varphi &= \frac{1}{3\lambda + 2\mu} \left[ \frac{1}{3} + \ell \left( 1 + \frac{\lambda}{2\mu} \right) \frac{\rho'}{r} - \frac{\lambda\ell}{\mu} \frac{\rho}{r^2} \right], \\ \rho'' - \left( m + \frac{2}{r^2} \right) \rho &= 0, \quad \text{with } m = -\frac{2\mu}{\ell} \frac{3\lambda + 2\mu}{\lambda + 2\mu} > 0. \end{aligned} \tag{10.7}$$

Setting  $\rho = pr^2$  we obtain

$$p'' + \frac{4}{r}p' - mp = 0,$$

then, with  $p(r) = q(r\sqrt{m})$  and  $s = r\sqrt{m}$ ,

$$q'' + \frac{4}{s}q' - q = 0.$$

The last change of unknown  $q(s) = \xi(s)/s$  provides the spherical Bessel equation

$$s^2\xi'' + 2s\xi' - (s^2 + 2)\xi = 0.$$

Bounded solutions are spanned by the spherical Bessel function

$$i_1(s) := \frac{d}{ds} \left( \frac{\sinh(s)}{s} \right) = \frac{\cosh(s)}{s} - \frac{\sinh(s)}{s^2}. \tag{10.8}$$

Therefore, setting

$$h_0(s) = \frac{i_1(s)}{s},$$

we obtain for some constant  $a$ ,

$$\rho(r) = ar^2 h_0(r\sqrt{m}).$$

On  $\partial\Omega$  we have the condition on the incompatibility flux  $\ell \operatorname{inc} EN = (\mathbb{K}N)_{\text{ext}}$ , which reads

$$2\ell(\varphi'(1) - \psi(1))e_r = -\frac{2}{3}e_r.$$

It provides  $\rho(1) = 1/3\ell$ , hence

$$\rho(r) = \frac{1}{3\ell} r^2 \frac{h_0(r\sqrt{m})}{h_0(\sqrt{m})}. \tag{10.9}$$

We obtain  $\varphi$  from (10.7), then  $\psi$  from  $\psi = \rho + r\varphi'$ .

Let us now turn to the exterior solution, which is needed to find the displacement field by Beltrami decomposition. Recall the equations in  $\mathbb{R}^3 \setminus \bar{\Omega}$

$$\begin{cases} \varphi'' + \frac{\varphi'}{r} - \frac{\psi'}{r} = \frac{1}{3\ell r^3} \\ -\varphi'' + \frac{\varphi'}{r} + \frac{\psi'}{r} - \frac{2\psi}{r^2} = -\frac{1}{\ell r^3}. \end{cases}$$

The general solution is obtained as

$$\varphi(r) = \left( \frac{1}{3\ell} - \beta \right) \frac{1}{r} + \frac{\alpha}{r^3}, \quad \psi(r) = \frac{\beta}{r} - \frac{3\alpha}{r^3}$$

for some constants  $\alpha, \beta$ . Denote by  $\mathbb{A}^*$  the Hooke tensor of the weak phase outside  $\Omega$ . We assume that  $\mathbb{A}^* = \gamma \mathbb{A}$  for some constant  $\gamma \rightarrow 0$ . The equation  $\mathbb{A}^* E + \ell \operatorname{inc} E = \mathbb{K}$  gives  $\operatorname{div} \mathbb{A}^* E = \operatorname{div} \mathbb{K} = 0$ , whereby  $\operatorname{div} \mathbb{A} E = 0$ . This entails

$$\beta = \frac{3\lambda + 2\mu}{6\ell(\lambda + 2\mu)}.$$

It remains to fix  $\alpha$  through the transmission conditions  $[\mathcal{T}_0(E)] = [\mathcal{T}_1(E)] = 0$  on  $\partial\Omega$ . The first condition is obviously equivalent to  $[\varphi] = 0$ . From (2.11) we obtain  $[\mathcal{T}_1(E)] = -2[\varphi]e_r \otimes e_r + [\psi - \varphi'](e_\theta \otimes e_\theta + e_\phi \otimes e_\phi)$ , then given the first condition, the second one is fulfilled if and only if  $[\psi - \varphi'] = 0$ . Observe that this also exactly the expression of  $[\operatorname{inc} EN] = 0$ . Yet we have on the exterior side  $\psi(1) - \varphi'(1) = 1/3\ell$ , which turns to be equal to  $\rho(1) = \psi(1) - \varphi'(1)$  on the interior side. In fact, only the continuity of  $\varphi$  fixes  $\alpha$  by

$$\alpha = \frac{1}{3\lambda + 2\mu} \left[ \frac{1}{3} + \ell \left( 1 + \frac{\lambda}{2\mu} \right) \rho'(1) - \frac{\lambda\ell}{\mu} \rho(1) \right] + \frac{\lambda - 2\mu}{6\ell(\lambda + 2\mu)}.$$

The exterior deformation field is completely determined.

We now compute the displacement  $U$  such that  $E = \nabla^S U + E^0$ ,  $\operatorname{div} E^0 = 0$ ,  $E^0 N \rightarrow 0$  at infinity. Therefore  $U$  solves

$$\begin{cases} \operatorname{div} \nabla^S U = \operatorname{div} E \text{ in } \mathbb{R}^3 \\ \nabla^S U N \rightarrow EN \text{ at } \infty. \end{cases} \quad (10.10)$$

For  $U = u(r)e_r$ , the first equation reads when  $r \neq 1$

$$u'' + 2\frac{u'}{r} - 2\frac{u}{r^2} = \varphi' + \psi' + 2\frac{\psi}{r}.$$

For  $r > 1$ , rewriting the left hand side and computing the right hand side provides

$$\left( \frac{1}{r^2} (r^2 u)' \right)' = \frac{2\lambda}{3\ell(\lambda + 2\mu)} \frac{1}{r^2},$$

whereby for some constants  $c$  and  $d$ ,

$$u(r) = \frac{-\lambda}{3\ell(\lambda + 2\mu)} + cr + \frac{d}{r^2}.$$

Now, the condition  $\nabla^S U N \rightarrow EN$  at infinity yields  $c = 0$ , since  $Ee_r = (\varphi + \psi)e_r \rightarrow 0$ .

For  $r < 1$  we find

$$\varphi' + \psi' + 2\frac{\psi}{r} = 4\varphi' + \rho' + r\varphi'' + \frac{2}{r}\rho = \frac{2\lambda}{\lambda + 2\mu} \frac{\rho}{r}.$$

Substituting (10.9) leads to the equation

$$\left( \frac{1}{r^2} (r^2 u)' \right)' = \frac{2\lambda}{3\ell(\lambda + 2\mu)} \frac{i_1(r\sqrt{m})}{i_1(\sqrt{m})}.$$

Using (10.8) we arrive at

$$u(r) = \frac{-\lambda}{3\mu(3\lambda + 2\mu)} \frac{i_1(r\sqrt{m})}{i_1(\sqrt{m})} + er$$

for some constant  $e$ . On  $\partial\Omega$ , (10.10) amounts to  $[\nabla^S U N] = [EN]$ , that is  $[u' + 2u] = [\psi]$ , hence we have the jump relations  $[u] = 0$  and  $[u'] = [\psi]$ . This fixes the constants  $d$  and  $e$  through a linear system.

A Taylor expansion provides  $i_1(s) = s/3 + o(s^2)$  as  $s \rightarrow 0$ . Then for  $\ell \rightarrow -\infty$  it is easily checked that the elastic solution is retrieved, namely

$$\varphi^\infty(r) = \frac{1}{3\lambda + 2\mu} \chi_{\{r < 1\}} + \frac{1}{3\lambda + 2\mu} \frac{1}{r^3} \chi_{\{r > 1\}}, \quad \psi^\infty(r) = \frac{-3}{3\lambda + 2\mu} \frac{1}{r^3} \chi_{\{r > 1\}}$$

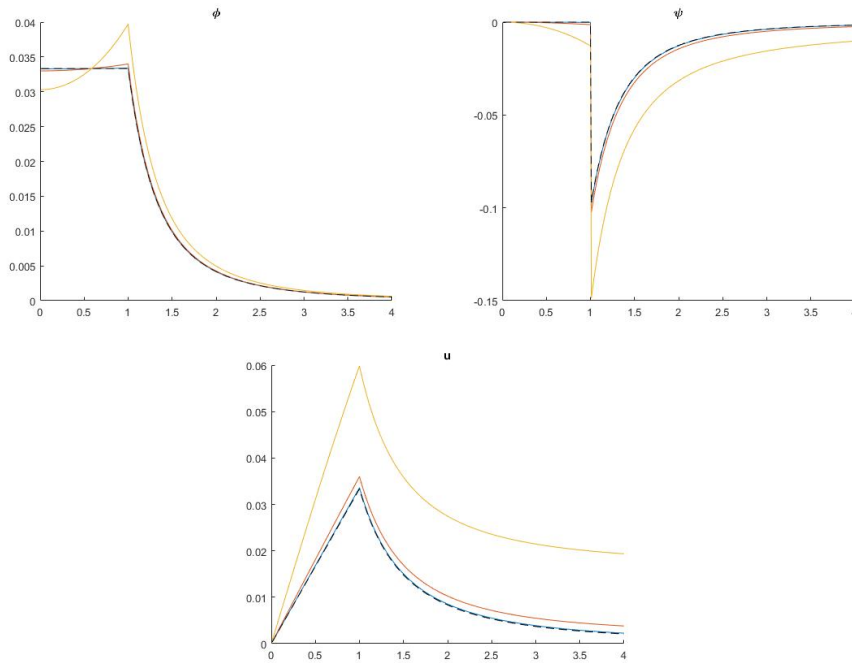


FIGURE 3. Functions  $\varphi$ ,  $\psi$  and  $u$  for  $\ell = -1000$  (blue),  $\ell = -100$  (red),  $\ell = -10$  (yellow), elastic solution (dashed).

$$u^\infty(r) = \frac{1}{3\lambda + 2\mu} r \chi_{\{r < 1\}} + \frac{1}{3\lambda + 2\mu} \frac{1}{r^2} \chi_{\{r > 1\}}.$$

The functions  $\varphi$ ,  $\psi$  and  $u$  are plotted on Figure 3 for different values of  $\ell$ . The curves of displacement show an increase of dilation due to inelastic deformation.

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