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# EXISTENCE RESULTS FOR A LINEARIZED INTRINSIC ELASTO-PLASTICITY MODEL

SAMUEL AMSTUTZ AND NICOLAS VAN GOETHEM

ABSTRACT. A novel model of elasto-plasticity based on an intrinsic approach is proposed. The model variables are the linearized strain and its incompatibility. Elastic strain incompatibility accounts for the presence of dislocations in the microstructure, which are responsible for the plastic behaviour of solids. The functional analysis setting is built up, on which existence results are proved.

## 1. INTRODUCTION

1.1. **The intrinsic approach to elasticity.** An intrinsic approach to elasticity simply means that the main and primal variable is the strain, together with its derivatives, and that the displacement and rotation fields are possibly recovered in a second step, in case they are needed. This approach is most probably the first historically, since the strain was indeed used to measure deformation, that is, variation in length and in mutual orientations of infinitesimal fibers within a solid body. As a matter of fact, for the geometer the strain is a metric from which all other geometric concepts are retrieved. Specifically given a smooth strain tensor field  $\varepsilon$  the classical Volterra-Michell-Cesaro construction [3, 14, 23] (see also [13]) in linearized elasticity consists in

- introducing the Frank tensor<sup>1</sup>  $F = \text{Curl}^t \varepsilon$ ;
- defining the rotation field as  $\omega(x) = \omega(x_0) + \int_{x_0}^x F(\xi) dL(\xi)$ , on a smooth curve joining the endpoints  $x_0$  and  $x$ ;
- defining the displacement field as  $u(x) = u(x_0) + \int_{x_0}^x (\varepsilon - \epsilon(\omega))(\xi) dL(\xi)$ , where  $\epsilon(\omega)$  stands for the skew-symmetric rotation tensor constructed from  $\omega$ , namely  $(\epsilon(\omega))_{il} = \epsilon_{ilk} \omega_k$ .

In order for  $u$  and  $\omega$  to be well defined, i.e., to be path-independent in a simply connected domain, it is immediately seen that a sufficient and necessary condition be that

$$\text{inc } \varepsilon := \text{Curl } \text{Curl}^t \varepsilon = \text{Curl } F = 0,$$

introducing the strain incompatibility tensor  $\text{inc } \varepsilon$ , that is easily seen to be symmetric. Note that in case  $\text{inc } \varepsilon$  is not vanishing this means exactly that the rotation and/or displacement fields exhibit a jump around what is classically called a Burgers circuit (to this respect an important role is played by the choice of the origin  $x_0$  as shown in [21]). So far it appears clear that the important geometric quantities are  $\varepsilon$ ,  $\text{Curl}^t \varepsilon$  and  $\text{inc } \varepsilon$ . This is the seminal motivation for our model which is precisely designed from these variables. In particular, this choice makes the model of gradient-type. Note that it appears natural to consider the curl instead of the full gradient, and the  $\text{inc}$  instead of the full Hessian.

Strangely enough, though the incompatibility operator was regularly used in the engineering literature, the mathematical study of spaces of square integrable tensor-valued functions with square integrable incompatibility was not yet considered and our first step was hence to dedicate a paper to the subject [1].

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<sup>1</sup>This terminology was introduced in [21, 22] simply because its integral on a closed loop yields the so-called Frank tensor attached to a disclination singularity.

In finite elasticity the intrinsic approach was recently carried on in a systematic way by Ph. Ciarlet and co-authors [4–6]. In a first step, they proved that given a smooth enough metric  $C$  it can be written as  $C = \nabla^t \varphi \nabla \varphi$  for some smooth enough map  $\varphi$  provided the Riemannian curvature associated to  $C$  was vanishing. It turns out that the first-order terms of the Riemannian curvature is precisely  $\text{inc } \frac{1}{2}(C - \text{Id})$  (see [13]), that is, requiring vanishing strain incompatibility in linearized elasticity is the counterpart of requiring vanishing curvature in finite elasticity. In a second step (see [6] for instance), their aim was to rewrite classical boundary value problems of linearized elasticity in a intrinsic form, that is, with boundary conditions expressed in terms of the strain  $\varepsilon = \nabla^S u$  only, in place of the displacement. As a result, they showed that requiring vanishing displacement  $u$  on the boundary amounts to imposing vanishing tangential metric and second fundamental form (i.e., curvature associated with the metric). The purpose of [20] was to show in turn that these intrinsic boundary conditions were equivalent to requiring the tangential part of  $\varepsilon$  and  $\text{Curl}^t \varepsilon \times N$  to vanish on the boundary, where  $N$  stands for the outward unit normal.

**1.2. A critical view on traditional elasto-plasticity.** Traditional elasto-plasticity has been developed on three main postulates (see for instance [12]):

- (1) The decomposition of the strain into an elastic and a plastic part. In linearized elasto-plasticity the decomposition is additive:  $\varepsilon = \varepsilon^e + \varepsilon^p$ . In finite elasticity one decomposes the deformation gradient multiplicatively:  $F = F^e F^p$  (or arguably in the reverse order, see [8]).
- (2) The coupling of two evolution laws for the elastic and the plastic part, namely the balance equations in terms of the Cauchy stress, given by a constitutive function of some elastic kinematical quantity (i.e., related to  $\varepsilon^e$  or  $F^e$ ), and the postulated flow rules, expressed in terms of an appropriate plastic kinematic variable (i.e., related to  $\varepsilon^p$  or  $F^p$  and their time rates).
- (3) The existence of convex dissipation potentials required to define the time evolution of the plastic kinematical variables through the flow rules.

One usually justifies the partition hypothesis (1) by the different kinds of physical processes involved: while the elastic deformation models the change in interatomic distances, the plastic strain is a measure of the displacement of atoms with modification of interatomic bonds. Note that this partition is local, i.e.,  $\varepsilon(x) = \varepsilon^e(x) + \varepsilon^p(x)$  for any  $x \in \Omega$  and is purely of physical nature, that is, there is not any sort of mathematical structure that justifies the decomposition. In linearized elasto-plasticity the additive decomposition can be understood through the definition of the elastic strain from the Cauchy stress  $\sigma$  (a measurable quantity) as  $\varepsilon^e := \mathbb{A}^{-1} \sigma$  and the assumption that the plastic strain  $\varepsilon^p$  is the complement so that the total strain is compatible, that is, that there exists a displacement field  $u$  such that  $\varepsilon = \nabla^S u$ . This statement is not justified by any mathematical argument and the adoption of this hypothesis is made for simplicity. Indeed it automatically implies that the incompatibilities of elastic and plastic parts mutually compensate, without the need to let the flow rules comply with this property. The finite deformation case is more delicate since any kind of multiplicative decomposition (originally due to Nye [17]) relies on the generally postulated existence of an intermediate reference configuration from which elastic (or plastic) deformation applies. In the sequel we will not dwell on the finite deformation case, since our model is about linearized elasto-plasticity.

As for (2) it should be noted that the balance equation involves the Cauchy stress only. In a modelling step, the latter is related to the elastic strain by a constitutive law. In contrast, the knowledge of the plastic strain is obtained by the solution of the flow rules, which from a modelling standpoint, are unrelated to the balance equation. This mere superposition of conceptually distinct constructs (one originates from conservation principles, the other by arbitrary modelling choices), though leading to an extraordinary efficient model, might also appear as unsatisfactory from an intellectual viewpoint.

Lastly, assumption (3) cannot be given any rigorous justification. It allows one to use the powerful toolbox of convex calculus and again, as for (2) has led to establishing a model which

has proven to simulate accurately real world processes. One would like to understand plastic deformation without appealing to convexity, at least not in its premises.

Rigorous existence results in traditional plasticity can be found for instance in [7, 15, 18].

**1.3. Our model: from generalized elasticity to intrinsic elasto-plasticity.** The novel approach we propose has been introduced and discussed in [2]. In our model, neither of the three above postulates are considered. Our paradigm is radically different and our approach is based on the following rationales.

- (1) Strain rate is preferred to strain and is given its following, primordial definition. Identify three fibers at  $x$ , denoted by  $a_1, a_2, a_3$ , which at time  $t$  are oriented along the axes of a Cartesian coordinate system and of unit lengths. Then the deformation rate is defined at  $x$  as (see, e.g., [9])

$$d_{ij}(t) = \frac{1}{2} \left( \frac{\partial}{\partial t} (a_i \cdot a_j) \right)_t. \quad (1.1)$$

Having fixed an initial time  $t_0 = 0$ , the time integral of the objective tensor  $d$ , called the strain or deformation tensor reads  $\varepsilon(t) = \int_0^t d(s) ds$ . Note that (1.1) holds for infinitesimal as well as for finite strains and hence one is not forced to specify the quantitative nature of the deformations before they take place.

- (2) This strain defined in this fashion is neither elastic nor plastic, it simply has a compatible and an incompatible part, that are given by a structure theorem called Beltrami decomposition [13]:

$$\varepsilon = \nabla^S u + E^0. \quad (1.2)$$

As opposed to elasto-plastic partitions this decomposition is unique once boundary conditions for  $u$  are prescribed. Moreover, while  $\varepsilon$  is an objective field (in a general sense as discussed in [1]), neither  $\nabla^S u$  nor  $E^0$  are objective. Therefore the model will be constructed upon  $\varepsilon$  and its derivatives.

- (3) The governing equations should generalize classical linear elasticity in the sense that it must take into account the possible strain incompatibility. The idea behind is that the model should explicitly account for the physical cause of plasticity: the presence and motion of dislocations as microstructural perturbations.

Our model can be briefly described as follows (see [2] for details). One considers linearized gradient elasticity in the sense of Mindlin [16]. One assumes that the virtual strain rate  $\hat{d}$  and its gradient produce intrinsic work and by the virtual power principle we write

$$\int_{\Omega} (\sigma \cdot \hat{d} + \tau \cdot \nabla \hat{d}) dx = \int_{\Omega} \mathbb{K} \cdot \hat{d} dx,$$

where  $\sigma, \tau$  are the Cauchy stress and hyperstress tensors, respectively, and  $\mathbb{K}$  is a tensor representing external efforts. Our constitutive framework is that of linear gradient-elasticity, that is  $\sigma = \mathbb{A}\varepsilon$  and  $\tau = \mathbb{B}\nabla\varepsilon$ , where  $\mathbb{A}, \mathbb{B}$  are the Lamé and Mindlin tensors, respectively (see [16]). We require that the intrinsic power induced by the hyperstress  $\int_{\Omega} \nabla\varepsilon \cdot \tau dx$  vanishes as soon as the deformation is compatible, i.e., that it is only due to microstructural effects in the form of dislocations, since Kröner's formula links elastic strain incompatibility and dislocation density (see [11, 20, 21]). This yields the existence of a scalar  $\ell$  called incompatibility modulus, such that  $-\operatorname{div} \tau = \ell \operatorname{inc} \varepsilon$ . Therefore the VPP yields the weak form

$$\int_{\Omega} (\mathbb{A}\varepsilon + \ell \operatorname{inc} \varepsilon) \cdot \hat{d} dx = \int_{\Omega} \mathbb{K} \cdot \hat{d} dx, \quad \forall \hat{d} \in \mathcal{E}, \quad (1.3)$$

where  $\mathcal{E}$  is the set of virtual strain rates.

To see that this equation generalizes linearized elasticity, take  $\hat{d} = \nabla^S \hat{v}$  with  $\hat{v} = 0$  on  $\Gamma_D \subset \partial\Omega$  and take  $\mathbb{K}$  such that  $-\operatorname{div} \mathbb{K} = f$  in  $\Omega$  and  $\mathbb{K}N = g$  on  $\Gamma_N := \partial\Omega \setminus \Gamma_D$ . Then, plugging this into

(1.3) immediately yields

$$\begin{cases} -\operatorname{div}(\mathbb{A}\varepsilon + \ell \operatorname{inc} \varepsilon) &= f & \text{in } \Omega, \\ (\mathbb{A}\varepsilon + \ell \operatorname{inc} \varepsilon)N &= g & \text{on } \Gamma_N, \end{cases} \quad (1.4)$$

which is exactly the system of linearized elasticity in case of compatible strain, i.e. with  $\varepsilon = \nabla^S u$  with  $u = u_0$  on  $\Gamma_D$ , since for such strains  $\operatorname{inc} \varepsilon = 0$ .

More generally, we believe that our model is able to represent finite deformations through an incremental formulation. In this case,  $\varepsilon$  is a strain increment and  $(\mathbb{A}, \ell)$  are seen as tangent elasto-plastic moduli. Of course, their evolution should be driven by constitutive laws in order to account, for instance, for hardening phenomena. An energetic approach is to relate changes in these coefficients with mechanical dissipation. A sensitivity analysis of the dissipation functional with respect to a variation of  $\ell$  within a small inclusion has been conducted in [2] for a simplified model. The extension to the full model and the numerical implementation, for which the existence results of the present work were mandatory, is an ongoing work.

**1.4. Summary of our results.** Set  $\mathcal{E} = L^2(\Omega)$ . The main purpose of this work is to prove that (1.3), or equivalently the associated strong form  $\mathbb{A}\varepsilon + \ell \operatorname{inc} \varepsilon = \mathbb{K}$  in  $\Omega$  has a solution in the space of square integrable functions with square integrable incompatibility, with the additional condition on the dislocation flux at the boundary  $\operatorname{inc} \varepsilon N = 0$  on  $\partial\Omega$ . The main ingredients to achieve the proof are (i) orthogonal decompositions of  $L^2(\Omega)$  based on the Beltrami decomposition, and (ii) Fredholm's alternative. It is also to be stressed that our model has no variational structure in the sense that the solution is not minimum of an energy. Moreover we analyze the case  $|\ell| \rightarrow \infty$  which represents the elastic limit, i.e., with vanishing strain incompatibility. We conclude by two academic examples.

## 2. PRELIMINARY RESULTS

Let  $\Omega$  be a regular ( $C^\infty$ ) bounded domain of  $\mathbb{R}^3$ . We denote by  $\partial\Omega$  its boundary and by  $N$  its outward unit normal.

**2.1. The curvilinear frame.** For all  $x \in \partial\Omega$ , the system  $(\tau^A(x), \tau^B(x), N(x))$  is an orthonormal basis of the tangent plane to  $\partial\Omega$ , that can be naturally extended along  $N$  in a neighborhood  $W$  of  $\partial\Omega$  (see [1]). The curvatures along  $\tau^A$  and  $\tau^B$  are denoted by  $\kappa^A$  and  $\kappa^B$ . Define the normal derivative as  $\partial_N := N \cdot \nabla$  and tangential derivative as  $\partial_R := \tau^R \cdot \nabla$ , for  $R \in \{A, B\}$ . We will also use the notation  $R^* = B$  if  $R = A$ ,  $R^* = A$  if  $R = B$ . The following results are proved in [1].

**Theorem 2.1.** *There exist smooth scalar fields  $\xi, \gamma^A, \gamma^B$  in  $W$  such that*

$$\partial_N N = \partial_N \tau^R = 0, \quad (2.1)$$

$$\partial_R N = \kappa^R \tau^R + \xi \tau^{R^*}, \quad (2.2)$$

$$\partial_R \tau^R = -\kappa^R N - \gamma^{R^*} \tau^{R^*}, \quad (2.3)$$

$$\partial_{R^*} \tau^R = \gamma^R \tau^{R^*} - \xi N. \quad (2.4)$$

If  $(\tau^A(x), \tau^B(x))$  are oriented along the principal directions of curvature then  $\xi(x) = 0$ .

**Lemma 2.2.** *If  $f$  is twice differentiable in  $W$  it holds*

$$\partial_R \partial_N f = \partial_N \partial_R f + \kappa^R \partial_R f + \xi \partial_{R^*} f. \quad (2.5)$$

**2.2. Basic function spaces.** Define

$$\begin{aligned} H^{\operatorname{curl}}(\Omega, \mathbb{M}^3) &:= \{E \in L^2(\Omega, \mathbb{M}^3) : \operatorname{Curl} E \in L^2(\Omega, \mathbb{M}^3)\}, \\ H^{\operatorname{div}}(\Omega, \mathbb{S}^3) &:= \{E \in L^2(\Omega, \mathbb{S}^3) : \operatorname{div} E \in L^2(\Omega, \mathbb{R}^3)\}, \\ H^{\operatorname{inc}}(\Omega) &:= \{E \in L^2(\Omega, \mathbb{S}^3) : \operatorname{inc} E \in L^2(\Omega, \mathbb{S}^3)\}. \end{aligned}$$

These spaces are endowed with the norms defined by  $\|E\|_{\tilde{H}^{\text{curl}}}^2 = \|E\|_{L^2}^2 + \|\text{Curl } E\|_{L^2}^2$ ,  $\|E\|_{\tilde{H}^{\text{div}}}^2 = \|E\|_{L^2}^2 + \|\text{div } E\|_{L^2}^2$ ,  $\|E\|_{\tilde{H}^{\text{inc}}}^2 = \|E\|_{L^2}^2 + \|\text{inc } E\|_{L^2}^2$ , respectively. We also define

$$H_0^{\text{inc}}(\Omega, \mathbb{S}^3) = \text{the closure of } \mathcal{D}(\Omega, \mathbb{S}^3) \text{ in } H^{\text{inc}}(\Omega, \mathbb{S}^3),$$

as well as the trace space

$$\tilde{H}^{3/2}(\partial\Omega, \mathbb{S}^3) = \left\{ E \in H^{3/2}(\partial\Omega, \mathbb{S}^3) : \int_{\partial\Omega} ENdS(x) = 0 \right\}.$$

**Theorem 2.3** (Lifting [1]). *Let  $\mathbb{E} \in \tilde{H}^{3/2}(\partial\Omega, \mathbb{S}^3)$ , and  $\mathbb{G} \in H^{1/2}(\partial\Omega, \mathbb{S}^3)$ . There exists  $E \in H^2(\Omega, \mathbb{S}^3)$  such that*

$$\begin{cases} E = \mathbb{E} & \text{on } \partial\Omega, \\ (\partial_N E)_T = \mathbb{G}_T & \text{on } \partial\Omega, \\ \text{div } E = 0 & \text{in } \Omega, \end{cases}$$

where the subscript  $T$  stands for the tangential part. In addition, such a lifting can be obtained through a linear continuous operator

$$\mathcal{L}_{\partial\Omega} : (\mathbb{E}, \mathbb{G}) \in \tilde{H}^{3/2}(\partial\Omega, \mathbb{S}^3) \times H^{1/2}(\partial\Omega, \mathbb{S}^3) \mapsto E \in H^2(\Omega, \mathbb{S}^3).$$

Define the subset of  $\mathcal{C}^\infty(\partial\Omega, \mathbb{S}^3)$

$$\mathcal{G} = \{V \odot N, V \in \mathbb{R}^3\},$$

with the notation  $U \odot V := (U \otimes V + V \otimes U)/2$ .

**Lemma 2.4** (Dual trace space [1]). *Every  $\mathbb{E} \in H^{-3/2}(\partial\Omega, \mathbb{S}^3)/\mathcal{G}$  admits a unique representative  $\tilde{\mathbb{E}}$  such that*

$$\int_{\partial\Omega} \tilde{\mathbb{E}}NdS(x) = 0. \quad (2.6)$$

Moreover, the dual space of  $\tilde{H}^{3/2}(\partial\Omega, \mathbb{S}^3)$  is canonically identified with  $H^{-3/2}(\partial\Omega, \mathbb{S}^3)/\mathcal{G}$ .

**2.3. Green formula and applications.** Recall that the Green formula for the divergence allows to define, for any  $T \in H^{\text{div}}(\Omega, \mathbb{S}^3)$ , its normal trace  $TN \in H^{-1/2}(\partial\Omega, \mathbb{R}^3)$  by

$$\int_{\partial\Omega} (TN) \cdot \varphi dS(x) := \int_{\Omega} \text{div } T \cdot \tilde{\varphi} + T \cdot \nabla^S \tilde{\varphi} \quad \forall \varphi \in H^{1/2}(\partial\Omega, \mathbb{R}^3),$$

with  $\tilde{\varphi} \in H^1(\Omega, \mathbb{R}^3)$  an arbitrary lifting of  $\varphi$ . For the incompatibility operator one has the following counterpart.

**Lemma 2.5** (Green formula for the incompatibility [1]). *Suppose that  $T \in \mathcal{C}^2(\bar{\Omega}, \mathbb{S}^3)$  and  $\eta \in H^2(\Omega, \mathbb{S}^3)$ . Then*

$$\int_{\Omega} T \cdot \text{inc } \eta dx = \int_{\Omega} \text{inc } T \cdot \eta dx + \int_{\partial\Omega} \mathcal{T}_1(T) \cdot \eta dS(x) + \int_{\partial\Omega} \mathcal{T}_0(T) \cdot \partial_N \eta dS(x) \quad (2.7)$$

with the trace operators defined as

$$\mathcal{T}_0(T) := (T \times N)^t \times N, \quad (2.8)$$

$$\mathcal{T}_1(T) := (\text{Curl } (T \times N)^t)^S + ((\partial_N + k)T \times N)^t \times N + (\text{Curl}^t T \times N)^S, \quad (2.9)$$

where  $k$  is twice the mean curvature on  $\partial\Omega$  and  $T^S = (T + T^t)/2$ . In addition, it holds

$$\int_{\partial\Omega} \mathcal{T}_1(T)NdS(x) = 0. \quad (2.10)$$

Alternative expressions for  $\mathcal{T}_1(T)$  are given in [1], like

$$\begin{aligned} \mathcal{T}_1(T) = & - \sum_R \kappa^R (T \times \tau^R)^t \times \tau^R - \sum_R \xi (T \times \tau^R)^t \times \tau^{R*} + ((-\partial_N + k)T \times N)^t \times N \\ & - 2 \left( \sum_R (\partial_R T \times N)^t \times \tau^R \right)^S. \end{aligned} \quad (2.11)$$

For a general symmetric tensor  $T$ , with components  $T_{RR'} := T \tau^{R'} \cdot \tau^R$  in the curvilinear frame, one has:

$$T = \begin{pmatrix} T_{AA} & T_{AB} & T_{AN} \\ T_{BA} & T_{BB} & T_{BN} \\ T_{NA} & T_{NB} & T_{NN} \end{pmatrix}, \quad (T \times N)^t \times N = \begin{pmatrix} T_{BB} & -T_{AB} & 0 \\ -T_{AB} & T_{AA} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (2.12)$$

$$(T \times \tau^A)^t \times \tau^A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & T_{NN} & -T_{BN} \\ 0 & -T_{BN} & T_{BB} \end{pmatrix}, \quad (T \times \tau^B)^t \times \tau^B = \begin{pmatrix} T_{NN} & 0 & -T_{AN} \\ 0 & 0 & 0 \\ -T_{AN} & 0 & T_{AA} \end{pmatrix}, \quad (2.13)$$

$$(T \times N)^t \times \tau^A = \begin{pmatrix} 0 & T_{BN} & -T_{BB} \\ 0 & -T_{AN} & T_{AB} \\ 0 & 0 & 0 \end{pmatrix}, \quad (T \times N)^t \times \tau^B = \begin{pmatrix} -T_{BN} & 0 & T_{AB} \\ T_{AN} & 0 & -T_{AA} \\ 0 & 0 & 0 \end{pmatrix}. \quad (2.14)$$

As shown in [1], we can define the traces  $\mathcal{T}_0(T) \in H^{-1/2}(\partial\Omega, \mathbb{S}^3)$  and  $\mathcal{T}_1(T) \in H^{-3/2}(\partial\Omega, \mathbb{S}^3)/\mathcal{G}$  for every  $T \in H^{\text{inc}}(\Omega, \mathbb{S}^3)$  by

$$\langle \mathcal{T}_0(T), \varphi_0 \rangle = \int_{\Omega} T \cdot \text{inc } \eta_0 dx - \int_{\Omega} \text{inc } T \cdot \eta_0 dx, \quad \forall \varphi_0 \in H^{1/2}(\partial\Omega, \mathbb{S}^3),$$

$$\langle \mathcal{T}_1(T), \varphi_1 \rangle = \int_{\Omega} T \cdot \text{inc } \eta_1 dx - \int_{\Omega} \text{inc } T \cdot \eta_1 dx, \quad \forall \varphi_1 \in \tilde{H}^{3/2}(\partial\Omega, \mathbb{S}^3),$$

with  $\eta_0 = \mathcal{L}_{\partial\Omega}(0, \varphi_0)$  and  $\eta_1 = \mathcal{L}_{\partial\Omega}(\varphi_1, 0)$  (recall that  $\mathcal{L}_{\partial\Omega}$  is the lifting operator defined in Theorem 2.3). In addition, by Lemma 2.4,  $\mathcal{T}_1(T)$  admits a unique representative satisfying (2.10). By linearity of  $\mathcal{L}_{\partial\Omega}$ , this extends formula (2.7) to any functions  $T \in H^{\text{inc}}(\Omega, \mathbb{S}^3)$  and  $\eta \in H^2(\Omega, \mathbb{S}^3)$ . Note that we could well have defined  $\mathcal{T}_1(T) \in H^{-3/2}(\partial\Omega, \mathbb{S}^3)$  by using a classical lifting in  $H^2(\Omega, \mathbb{S}^3)$ , but spaces of divergence-free tensors arise naturally in our problems.

From the two Green formulas recalled above one easily infers the following.

- Lemma 2.6.** *1. For all  $v \in H^1(\Omega, \mathbb{R}^3)$ , one has  $\text{inc } \nabla^S v = 0$  in the sense of distributions.  
2. For all  $E \in H^{\text{inc}}(\Omega, \mathbb{S}^3)$ , one has  $\text{div } \text{inc } E = 0$  in the sense of distributions.*

Consequently, if  $E \in H^{\text{inc}}(\Omega, \mathbb{S}^3)$ ,  $\text{inc } EN$  is defined in  $H^{-1/2}(\partial\Omega, \mathbb{R}^3)$  by

$$\int_{\partial\Omega} \text{inc } EN \cdot \varphi dx = \int_{\Omega} \text{inc } E \cdot \nabla^S \varphi dx \quad \forall \varphi \in H^1(\Omega, \mathbb{R}^3).$$

If  $\omega \subset\subset \Omega$  and  $u$  is a vector or tensor field defined over  $\Omega$  with well-defined traces on each side of  $\partial\omega$ , we denote by  $JuK$  the jump of  $u$  across  $\partial\omega$  with inner term counted positively.

**Lemma 2.7.** *If  $E \in H^{\text{inc}}(\Omega, \mathbb{S}^3)$  and  $\omega \subset\subset \Omega$ , then  $J \text{inc } ENK = 0$  across  $\partial\omega$ .*

*Proof.* Let  $\varphi \in \mathcal{D}(\Omega, \mathbb{R}^3)$ . By definition and Lemma 2.6,

$$\int_{\partial\omega} J \text{inc } ENK \cdot \varphi dx = \int_{\Omega} \text{inc } E \cdot \nabla^S \varphi dx = 0.$$

By density this is also true for any  $\varphi \in H_0^1(\Omega, \mathbb{R}^3)$ , and subsequently for any  $\varphi \in H^{1/2}(\partial\omega, \mathbb{R}^3)$ .  $\square$

**Lemma 2.8.** *If  $E \in L^2(\Omega, \mathbb{S}^3)$ ,  $E|_{\omega} \in H^{\text{inc}}(\omega, \mathbb{S}^3)$ ,  $E|_{\Omega \setminus \bar{\omega}} \in H^{\text{inc}}(\Omega \setminus \bar{\omega}, \mathbb{S}^3)$  and  $J\mathcal{T}_0(E)K = J\mathcal{T}_1(E)K = 0$  across  $\partial\omega$ , then  $E \in H^{\text{inc}}(\Omega, \mathbb{S}^3)$ .*

*Proof.* By the Green formula.  $\square$

**Corollary 2.9.** *If  $E \in H^{\text{inc}}(\Omega, \mathbb{S}^3)$  and  $\mathcal{T}_0(E) = \mathcal{T}_1(E) = 0$  on  $\partial\Omega$ , then  $\text{inc } EN = 0$  on  $\partial\Omega$ .*

*Proof.* Extend  $E$  by 0 and apply Lemmas 2.8 and 2.7.  $\square$

**Corollary 2.10.** *If  $v \in H_0^1(\Omega, \mathbb{R}^3)$ , then  $\mathcal{T}_0(\nabla^S v) = \mathcal{T}_1(\nabla^S v) = 0$  on  $\partial\Omega$ .*

*Proof.* Extend  $v$  by 0 and apply Lemma 2.7.  $\square$

Let  $\Gamma$  be a smooth subset of  $\partial\Omega$ . Corollary 2.10 extends as follows.

**Lemma 2.11.** *If  $v \in H^1(\Omega, \mathbb{R}^3)$  satisfies  $v = 0$  on  $\Gamma$  in the sense of traces, then  $\mathcal{T}_0(\nabla^S v) = \mathcal{T}_1(\nabla^S v) = 0$  on  $\Gamma$ .*

*Proof.* Denote  $T = \nabla^S v$ . One obtains from Theorem 2.1:

$$\begin{aligned} T_{NN} &= (\nabla^S v)N \cdot N = \frac{1}{2}(\partial_j v_i + \partial_i v_j)N_i N_j = \partial_N v_N, \\ T_{RR} &= (\nabla^S v)\tau^R \cdot \tau^R = \frac{1}{2}(\partial_j v_i + \partial_i v_j)\tau_i^R \tau_j^R = \partial_R v_R + \kappa^R v_N + \gamma^{R*} v_{R*}, \\ T_{AB} &= (\nabla^S v)\tau^B \cdot \tau^A = \frac{1}{2}(\partial_B v_A + \partial_A v_B) - \frac{1}{2}(\gamma^A v_B + \gamma^B v_A) + \xi v_N, \\ T_{RN} &= \frac{1}{2}(\partial_j v_i + \partial_i v_j)\tau_i^R N_j = \frac{1}{2}(\partial_N v_R + \partial_R v_N - \kappa^R v_R - \xi v_{R*}). \end{aligned}$$

Therefore condition  $v = 0$  on  $\Gamma$  implies  $T_{AA} = T_{AB} = T_{BB} = 0$ , whereby  $\mathcal{T}_0(T) = 0$ .

Moreover, using (2.11) and (2.12)-(2.14) one obtains the expression in the basis of principal curvatures for simplicity ( $\xi = 0$ ):

$$\mathcal{T}_1(T) = \begin{pmatrix} -\partial_N T_{BB} + 2(\partial_B T)_{BN} + kT_{BB} - \kappa^B T_{NN} & \partial_N T_{AB} - kT_{AB} - (\partial_B T)_{AN} - (\partial_A T)_{BN} \\ \partial_N T_{AB} - kT_{AB} - (\partial_B T)_{AN} - (\partial_A T)_{BN} & -\partial_N T_{AA} + 2(\partial_A T)_{AN} + kT_{AA} - \kappa^A T_{NN} \\ \kappa^B T_{AN} + (\partial_A T)_{BB} - (\partial_B T)_{AB} & \kappa^A T_{BN} - (\partial_A T)_{AB} + (\partial_B T)_{AA} \\ & \kappa^B T_{AN} + (\partial_A T)_{BB} - (\partial_B T)_{AB} \\ & \kappa^A T_{BN} - (\partial_A T)_{AB} + (\partial_B T)_{AA} \\ & -\kappa^A T_{BB} - \kappa^B T_{AA} \end{pmatrix}.$$

From the knowledge that  $T_{AA} = T_{AB} = T_{BB} = 0$ , one obtains

$$\mathcal{T}_1(T) = \begin{pmatrix} -\partial_N T_{BB} + 2(\partial_B T)_{BN} - \kappa^B T_{NN} & \partial_N T_{AB} - (\partial_B T)_{AN} - (\partial_A T)_{BN} \\ \partial_N T_{AB} - (\partial_B T)_{AN} - (\partial_A T)_{BN} & -\partial_N T_{AA} + 2(\partial_A T)_{AN} + kT_{AA} - \kappa^A T_{NN} \\ \kappa^B T_{AN} + (\partial_A T)_{BB} - (\partial_B T)_{AB} & \kappa^A T_{BN} - (\partial_A T)_{AB} + (\partial_B T)_{AA} \\ & \kappa^B T_{AN} + (\partial_A T)_{BB} - (\partial_B T)_{AB} \\ & \kappa^A T_{BN} - (\partial_A T)_{AB} + (\partial_B T)_{AA} \\ & 0 \end{pmatrix}. \quad (2.15)$$

Yet one has in a general basis

$$(\partial_R T)_{RN} = \partial_R T_{RN} - \kappa^R T_{RR} + \gamma^{R*} T_{R*N} + \kappa^R T_{NN} - \xi T_{RR*},$$

$$(\partial_{R*} T)_{RN} = \partial_{R*} T_{RN} - \kappa^{R*} T_{RR*} - \gamma^R T_{R*N} + \xi T_{NN} - \xi T_{RR}.$$

Recalling that  $v = 0$  on  $\Gamma$  and by Lemma 2.2 the first diagonal term of  $\mathcal{T}_1(T)$  vanishes since

$$-\partial_N T_{BB} + 2(\partial_B T)_{BN} - \kappa^B T_{NN} \quad (2.16)$$

$$= -\partial_N T_{BB} + 2\partial_B T_{BN} + 2\gamma^A T_{AN} + \kappa^B T_{NN} \quad (2.17)$$

$$= -\partial_N(\partial_B v_B + \kappa^B v_N + \gamma^A v_A) + (\partial_{BN} v_B - \partial_{BB} v_B - \partial_B(\kappa^B v_B)) + \gamma^A(\partial_N v_A + \partial_A v_N - \kappa^A v^A)$$

$$- \partial_B(\xi v_A) - \gamma^A \xi v_B + \kappa^B \partial_N v_N$$

$$= \partial_{BN} v_B - \partial_{NB} v_B = \gamma^B \partial_B v_B = 0. \quad (2.18)$$

The same computation holds for the second diagonal term. For the  $AB$ -term one has

$$\partial_N T_{AB} - (\partial_B T)_{AN} - (\partial_A T)_{BN} = \partial_N T_{AB} - \partial_B T_{AN} - \partial_A T_{BN} + \gamma^A T_{BN} + \gamma^B T_{AN} - 2\xi T_{NN},$$



which vanishes whenever  $\xi = 0$  by the same arguments. Finally one has  $(\partial_R T)_{R^*R^*} = \partial_R T_{R^*R^*} - 2\gamma^{R^*} T_{RR^*} + 2\xi T_{R^*N}$  and  $(\partial_{R^*} T)_{RR^*} = \partial_{R^*} T_{RR^*} + \kappa^{R^*} T_{RN} + \gamma^R T_{RR} - \gamma^{R^*} T_{RR^*} + \xi T_{RN}$ , hence the  $AN$ - and  $BN$ -terms also vanish.  $\square$

**Corollary 2.12.** *Let  $v \in H^1(\Omega, \mathbb{R}^3)$  be such that  $v = r$  on  $\Gamma$  in the sense of traces, with  $r$  a rigid displacement field. Then  $\mathcal{T}_0(\nabla^S v) = \mathcal{T}_1(\nabla^S v) = 0$  on  $\Gamma$ .*

*Proof.* On  $\Gamma$  it holds

$$\mathcal{T}_i(\nabla^S v) = \mathcal{T}_i(\nabla^S(v - r)) = 0, \quad i = 0, 1,$$

by Lemma 2.11.  $\square$

**Lemma 2.13.** *Let  $E \in H^2(\Omega)$ . Then  $E = \mathcal{T}_1(E) = 0$  on  $\Gamma \subset \partial\Omega$  if and only if  $E = (\partial_N T \times N)^t \times N = \text{Curl}^t E \times N = 0$  on  $\Gamma$ .*

*Proof.* From  $E = 0$  on  $\Gamma$  one infers that  $\partial_R E = 0$  on  $\Gamma$ , with  $R = A, B$ . Thus, (2.15) rewrites as

$$0 = \begin{pmatrix} -\partial_N E_{BB} & \partial_N E_{AB} & 0 \\ \partial_N E_{AB} & -\partial_N E_{AA} & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\mathcal{T}_0(\partial_N E)$$

on  $\Gamma$ , achieving the proof, the last statement being proved in [1]. The converse is proven in the same manner.  $\square$

**Lemma 2.14.** *We have the characterization*

$$H_0^{\text{inc}}(\Omega, \mathbb{S}^3) = \{E \in H^{\text{inc}}(\Omega, \mathbb{S}^3) : \mathcal{T}_0(E) = \mathcal{T}_1(E) = 0 \text{ on } \partial\Omega\}.$$

*Proof.* Suppose  $E_n \in \mathcal{D}(\Omega, \mathbb{S}^3)$ ,  $E \in H^{\text{inc}}(\Omega, \mathbb{S}^3)$ ,  $E_n \rightarrow E$  in  $H^{\text{inc}}(\Omega, \mathbb{S}^3)$ . Of course,  $\mathcal{T}_0(E_n) = \mathcal{T}_1(E_n) = 0$  on  $\partial\Omega$ . Then by continuity  $\mathcal{T}_0(E) = \mathcal{T}_1(E) = 0$  on  $\partial\Omega$ .

Suppose now  $E \in H^{\text{inc}}(\Omega, \mathbb{S}^3)$  with  $\mathcal{T}_0(E) = \mathcal{T}_1(E) = 0$  on  $\partial\Omega$ . Extend  $E$  by 0 to get  $\tilde{E} \in H^{\text{inc}}(\mathbb{R}^3, \mathbb{S}^3)$ . By local charts, shifting and convolution with mollifiers, we can define through a standard construction  $E_n \in \mathcal{D}(\mathbb{R}^3, \mathbb{S}^3)$  such that  $E_n \rightarrow \tilde{E}$  in  $H^{\text{inc}}(\mathbb{R}^3, \mathbb{S}^3)$  and  $\text{spt } E_n \subset \Omega$ . Hence  $E_n \rightarrow E$  in  $H^{\text{inc}}(\Omega, \mathbb{S}^3)$ , which yields  $E \in H_0^{\text{inc}}(\Omega, \mathbb{S}^3)$ .  $\square$

#### 2.4. Beltrami decomposition and related results.

**Theorem 2.15** (Saint-Venant compatibility conditions [13]). *Assume that  $\Omega$  is simply-connected. Let  $p \in (1, +\infty)$  be a real number and let  $E \in L^p(\Omega, \mathbb{S}^3)$ . Then,*

$$\text{inc } E = 0 \text{ in } W^{-2,p}(\Omega, \mathbb{S}^3) \iff E = \nabla^S v$$

for some  $v \in W^{1,p}(\Omega, \mathbb{R}^3)$ . Moreover,  $u$  is unique up to rigid displacements.

**Theorem 2.16** (Beltrami decomposition [13]). *Assume that  $\Omega$  is simply-connected. Let  $p \in (1, +\infty)$  be a real number and let  $E \in L^p(\Omega, \mathbb{S}^3)$ . Then, for any  $v_0 \in W^{1/p,p}(\partial\Omega)$ , there exists a unique  $v \in W^{1,p}(\Omega, \mathbb{R}^3)$  with  $v = v_0$  on  $\partial\Omega$  and a unique  $F \in L^p(\Omega, \mathbb{S}^3)$  with  $\text{Curl } F \in L^p(\Omega, \mathbb{R}^{3 \times 3})$ ,  $\text{inc } F \in L^p(\Omega, \mathbb{S}^3)$ ,  $\text{div } F = 0$  and  $FN = 0$  on  $\partial\Omega$  such that*

$$E = \nabla^S v + \text{inc } F. \tag{2.19}$$

We call  $v$  and  $F$  the velocity and incompatibility fields, respectively, associated with  $E$ . The following result is the dual counterpart of Saint-Venant's decomposition.

**Corollary 2.17.** *Assume that  $\Omega$  is simply-connected. If  $E \in L^2(\Omega, \mathbb{S}^3)$  satisfies  $\text{div } E = 0$  in  $H^{-1}(\Omega)$ , then there exists a unique  $F \in L^2(\Omega, \mathbb{S}^3)$  with  $\text{Curl } F \in L^2(\Omega, \mathbb{S}^3)$ ,  $\text{div } F = 0$  and  $FN = 0$  on  $\partial\Omega$  such that  $E = \text{inc } F$ .*

*Proof.* Theorem 2.16 yields

$$E = \nabla^S v + \text{inc } F,$$

with the appropriate  $F$  and  $v \in H_0^1(\Omega, \mathbb{R}^3)$ . The condition  $0 = \text{div } E = \text{div } \nabla^S v$  entails  $v = 0$ .  $\square$

We now specialize Saint-Venants's decomposition in the case of boundary conditions.

**Proposition 2.18.** *Assume that  $\Omega$  is simply-connected. If  $E \in L^2(\Omega, \mathbb{S}^3)$  satisfies*

$$\begin{cases} \text{inc } E = 0 \text{ in } \Omega, \\ \mathcal{T}_0(E) = \mathcal{T}_1(E) = 0 \text{ on } \partial\Omega \end{cases} \quad (2.20)$$

*there exists  $v \in H_0^1(\Omega, \mathbb{R}^3)$  such that  $\nabla^S v = E$ . Moreover, the map  $E \in L^2(\Omega, \mathbb{S}^3) \mapsto v \in H_0^1(\Omega, \mathbb{R}^3)$  is linear and continuous.*

*Proof.* Up to passing to the limit of a sequence, we can without loss of generality assume that  $E \in \mathcal{D}(\Omega, \mathbb{S}^3)$ . Let  $A : H^{-1}(\Omega, \mathbb{R}^3) \rightarrow L^2(\Omega, \mathbb{S}^3)$  be the linear map defined by  $A\varphi = \nabla^S u$  with

$$\begin{cases} -\text{div } \nabla^S u = \varphi \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Let  $A^* : L^2(\Omega, \mathbb{S}^3) \rightarrow H_0^1(\Omega, \mathbb{R}^3)$  be the adjoint operator of  $A$ . Let  $E \in L^2(\Omega, \mathbb{S}^3)$  satisfy (2.20) and  $v = A^*E \in H_0^1(\Omega, \mathbb{R}^3)$ . Let  $\Phi \in \mathcal{D}(\Omega, \mathbb{S}^3)$ . By definition we have

$$-\int_{\Omega} A^*E \text{div } \Phi dx = -\int_{\Omega} E \cdot A(\text{div } \Phi) dx.$$

Set  $\Psi = A(\text{div } \Phi)$ . We have  $-\text{div } \Psi = \text{div } \Phi$ . By Corollary 2.17,  $\Psi = -\Phi + \text{inc } \zeta$  for some  $\zeta \in H^{\text{inc}}(\Omega, \mathbb{S}^3)$ . We obtain

$$-\int_{\Omega} \left( A^*E \text{div } \Phi = \int_{\Omega} E \cdot \Phi - \int_{\Omega} E \cdot \text{inc } \zeta \right) dx.$$

Since  $E \in \mathcal{D}(\Omega, \mathbb{S}^3)$  and  $\text{inc } E = 0$  it holds

$$\int_{\Omega} E \cdot \text{inc } \zeta dx = 0.$$

We arrive at

$$-\int_{\Omega} A^*E \text{div } \Phi dx = \int_{\Omega} E \cdot \Phi dx,$$

thus

$$\nabla^S(A^*E) = E$$

in the sense of distributions. □

We can now state a converse to Corollary 2.10.

**Proposition 2.19.** *Assume that  $\Omega$  is simply connected. If  $v \in H^1(\Omega, \mathbb{R}^3)$  is such that  $\mathcal{T}_0(\nabla^S v) = \mathcal{T}_1(\nabla^S v) = 0$  on  $\partial\Omega$  then there exists a rigid displacement field  $r$  such that  $v = r$  on  $\partial\Omega$ .*

*Proof.* By Proposition 2.18, there exists  $w \in H_0^1(\Omega, \mathbb{R}^3)$  such that  $\nabla^S v = \nabla^S w$ . Next, there exists a rigid displacement field  $r$  such that  $v = w + r$ . On  $\partial\Omega$  we have  $v = r$ . □

### 3. ORTHOGONAL DECOMPOSITIONS

We assume in this section that  $\Omega$  is simply-connected.

**3.1. Orthogonal decomposition of  $L^2(\Omega, \mathbb{S}^3)$ .** Let  $\Gamma$  be a smooth subset of  $\partial\Omega$ . We define the sets

$$\begin{aligned} \mathcal{V} &= \{E \in L^2(\Omega, \mathbb{S}^3) : \text{inc } E = 0\}, \\ \mathcal{V}_{\Gamma}^0 &= \{E \in \mathcal{V} : \mathcal{T}_0(E) = \mathcal{T}_1(E) = 0 \text{ on } \Gamma\}, \\ \mathcal{V}_{\Gamma}^{00} &= \{\nabla^S v : v \in H^1(\Omega), v = 0 \text{ on } \Gamma\}, \\ \mathcal{W} &= \{E \in L^2(\Omega, \mathbb{S}^3) : \text{div } E = 0\}, \\ \mathcal{W}_{\Gamma}^0 &= \{E \in \mathcal{W} : EN = 0 \text{ on } \Gamma\}. \end{aligned}$$

**Remark 3.1.** By Theorem 2.15 and Corollary 2.17 we have

$$\mathcal{V} = \{\nabla^S v, v \in H^1(\Omega, \mathbb{R}^3)\}, \quad (3.1)$$

$$\mathcal{W} = \{\text{inc } F, F \in L^2(\Omega, \mathbb{S}^3), \text{Curl } F \in L^2(\Omega, \mathbb{S}^3), \text{div } F = 0 \text{ in } \Omega, FN = 0 \text{ on } \partial\Omega\}. \quad (3.2)$$

Moreover, the velocity field  $v$  in (3.1) is unique up to a rigid displacement field. The incompatibility field  $F$  in (3.2) is unique.

**Remark 3.2.** If  $|\Gamma| > 0$ , the velocity field  $v$  in the definition of  $\mathcal{V}_\Gamma^{00}$  is unique.

**Remark 3.3.** By Lemma 2.11 we have

$$\mathcal{V}_\Gamma^{00} \subset \mathcal{V}_\Gamma^0,$$

and we infer from Proposition 2.18 that

$$\mathcal{V}_{\partial\Omega}^{00} = \mathcal{V}_{\partial\Omega}^0.$$

**Theorem 3.1.** Assume that  $\partial\Omega$  admits the partition  $\partial\Omega = \Gamma_1 \cup \Gamma_2$  with  $\Gamma_1 \cap \Gamma_2 = \emptyset$ . We have the orthogonal decomposition

$$L^2(\Omega, \mathbb{S}^3) = \mathcal{V}_{\Gamma_1}^{00} \oplus \mathcal{W}_{\Gamma_2}^0.$$

*Proof.* i) Let  $\hat{E} \in \mathcal{V}_{\Gamma_1}^{00}$ ,  $E \in \mathcal{W}_{\Gamma_2}^0$ . We have  $\hat{E} = \nabla^S \hat{v}$ ,  $\hat{v} \in H^1(\Omega)$ ,  $\hat{v} = 0$  on  $\Gamma_1$ . The Green formula entails

$$\int_{\Omega} \hat{E} \cdot E dx = \int_{\Omega} \nabla^S \hat{v} \cdot E dx = - \int_{\Omega} \hat{v} \cdot \text{div } E dx + \int_{\partial\Omega} \hat{v} \cdot EN dS(x) = 0.$$

ii) Let  $E \in L^2(\Omega)$ . Write the Beltrami decomposition of Theorem 2.16 as  $E = \nabla^S v + \text{inc } F$  with  $v = 0$  on  $\partial\Omega$ . Let  $w \in H^1(\Omega)$  be the solution of

$$\begin{cases} -\text{div } \nabla^S w = 0 \text{ in } \Omega, \\ w = 0 \text{ on } \Gamma_1, \\ \nabla^S w N = \text{inc } FN \text{ on } \Gamma_2. \end{cases}$$

We have

$$E = \nabla^S(v + w) + (\text{inc } F - \nabla^S w) \in \mathcal{V}_{\Gamma_1}^{00} + \mathcal{W}_{\Gamma_2}^0,$$

which completes the proof.  $\square$

We have the following additional property.

**Lemma 3.2.** If  $\mathbb{K} \in \mathcal{V}_{\Gamma_1}^{00}$  and  $\text{inc } \hat{F} \in \mathcal{W}_{\Gamma_2}^0$  it holds

$$\int_{\Gamma_2} \left( \mathcal{T}_1(\mathbb{K}) \cdot \hat{F} + \mathcal{T}_0(\mathbb{K}) \cdot \partial_N \hat{F} \right) dS(x) = 0.$$

*Proof.* First, as  $\mathcal{V}_{\Gamma_1}^{00} \subset \mathcal{V}_{\Gamma_1}^0$ , we have

$$\int_{\Gamma_2} \left( \mathcal{T}_1(\mathbb{K}) \cdot \hat{F} + \mathcal{T}_0(\mathbb{K}) \cdot \partial_N \hat{F} \right) dS(x) = \int_{\partial\Omega} \left( \mathcal{T}_1(\mathbb{K}) \cdot \hat{F} + \mathcal{T}_0(\mathbb{K}) \cdot \partial_N \hat{F} \right) dS(x).$$

By the Green formula, we obtain

$$\int_{\partial\Omega} \left( \mathcal{T}_1(\mathbb{K}) \cdot \hat{F} + \mathcal{T}_0(\mathbb{K}) \cdot \partial_N \hat{F} \right) dS(x) = \int_{\Omega} \left( \mathbb{K} \cdot \text{inc } \hat{F} - \text{inc } \mathbb{K} \cdot \hat{F} \right) dx = \int_{\Omega} \mathbb{K} \cdot \text{inc } \hat{F} dx.$$

Writing  $\mathbb{K} = \nabla^S \hat{v}$  and applying the Green formula yields

$$\int_{\partial\Omega} \left( \mathcal{T}_1(\mathbb{K}) \cdot \hat{F} + \mathcal{T}_0(\mathbb{K}) \cdot \partial_N \hat{F} \right) dS(x) = \int_{\partial\Omega} \text{inc } \hat{F} N \cdot \hat{v} dS(x).$$

However,  $\hat{v} = 0$  on  $\Gamma_1$  while  $\text{inc } \hat{F} N = 0$  on  $\Gamma_2$ , achieving the proof.  $\square$

**3.2. Orthogonal decomposition of  $H^{\text{inc}}(\Omega)$  and related results.** Define

$$\begin{aligned}\mathcal{Z} &= \{E \in H^{\text{inc}}(\Omega) : \operatorname{div} E = 0 \text{ in } \Omega, EN = 0 \text{ on } \partial\Omega\}, \\ \mathcal{Z}_0 &= \{E \in \mathcal{Z} : \operatorname{inc} EN = 0 \text{ on } \partial\Omega\}, \\ \mathcal{F} &= \{E \in H^{\text{inc}}(\Omega) : \operatorname{inc} EN = 0 \text{ on } \partial\Omega\}.\end{aligned}\tag{3.3}$$

By virtue of Theorem 3.1 we infer the following decompositions.

**Proposition 3.3.** *We have the orthogonal decompositions*

$$\begin{aligned}H^{\text{inc}}(\Omega) &= \mathcal{Z} \oplus \mathcal{V}, \\ \mathcal{F} &= \mathcal{Z}_0 \oplus \mathcal{V}.\end{aligned}\tag{3.4}$$

We now gather some properties of the spaces  $\mathcal{Z}$  and  $\mathcal{Z}_0$ .

**Proposition 3.4.** *If  $E \in \mathcal{Z}$  then  $\operatorname{Curl} E \in L^2$ . Moreover there exists  $c > 0$  such that*

$$\|E\|_{L^2} + \|\operatorname{Curl} E\|_{L^2} \leq c \|\operatorname{inc} E\|_{L^2} \quad \forall E \in \mathcal{Z}.$$

*Proof.* Let

$$\begin{aligned}X &= \{F \in L^2(\Omega, \mathbb{S}^3), \operatorname{Curl} F \in L^2, \operatorname{inc} F \in L^2, \operatorname{div} F = 0, FN = 0 \text{ on } \partial\Omega\}, \\ Y &= \{F \in L^2(\Omega, \mathbb{S}^3), \operatorname{div} F = 0\}\end{aligned}$$

and define the linear map  $\Phi : X \rightarrow Y$  by  $\Phi(E) = \operatorname{inc} E$ . Equip  $X$  and  $Y$  with the norms

$$\begin{aligned}\|F\|_X &= \|F\|_{L^2} + \|\operatorname{Curl} F\|_{L^2} + \|\operatorname{inc} F\|_{L^2}, \\ \|F\|_Y &= \|F\|_{L^2}.\end{aligned}$$

Clearly,  $X$  and  $Y$  are Banach spaces and  $\Phi$  is continuous. By Corollary 2.17,  $\Phi$  is bijective. The open mapping theorem entails that  $\Phi^{-1}$  is continuous. Hence there exists  $c > 0$  such that

$$\|\Phi^{-1}(\operatorname{inc} E)\|_X \leq c \|\operatorname{inc} E\|_{L^2}.$$

Let  $E \in \mathcal{Z}$ . Then  $\operatorname{inc} E \in L^2$ . Set  $F = \Phi^{-1}(\operatorname{inc} E)$ . From  $\operatorname{inc} F = \operatorname{inc} E$ ,  $\operatorname{div} F = \operatorname{div} E = 0$  and  $FN = EN = 0$  on  $\partial\Omega$  we infer  $F = E$ . Therefore

$$\|E\|_{L^2} + \|\operatorname{Curl} E\|_{L^2} + \|\operatorname{inc} E\|_{L^2} = \|E\|_X = \|\Phi^{-1}(\operatorname{inc} E)\|_X \leq c \|\operatorname{inc} E\|_{L^2}\tag{3.5}$$

and the results follow.  $\square$

**Theorem 3.5** (Kozono-Yanagisawa [10] and von Wahl [24]). *There exists a constant  $c > 0$  such that*

$$\|\nabla u\|_{L^2} \leq c(\|\operatorname{div} u\|_{L^2} + \|\operatorname{Curl} u\|_{L^2})$$

for all

$$u \in \{v \in L^2(\Omega, \mathbb{R}^3), \operatorname{div} v \in L^2, \operatorname{Curl} v \in L^2, v.N = 0 \text{ on } \partial\Omega\}.$$

**Proposition 3.6.** *There exists  $C > 0$  such that for all  $E \in \mathcal{Z}$*

$$\|E\|_{H^1} \leq C \|\operatorname{inc} E\|_{L^2}.$$

*Proof.* Let  $E \in \mathcal{Z}$ . By Proposition 3.4 we already have

$$\|E\|_{L^2} + \|\operatorname{Curl} E\|_{L^2} \leq c \|\operatorname{inc} E\|_{L^2}.$$

Then Theorem 3.5 yields

$$\|\nabla E\|_{L^2} \leq c \|\operatorname{Curl} E\|_{L^2}$$

for some other constant  $c$ . This completes the proof.  $\square$

We infer in particular that  $\mathcal{Z}$  is imbedded in  $H^1(\Omega, \mathbb{S}^3)$  and compactly imbedded in  $L^2(\Omega, \mathbb{S}^3)$ .

**Proposition 3.7.** *We have the representation*

$$\mathcal{W}_{\partial\Omega}^0 = \{\operatorname{inc} F, F \in \mathcal{Z}_0\}.$$

*Proof.* Of course, if  $F \in \mathcal{Z}_0$ , then  $\text{inc } F \in \mathcal{W}_{\partial\Omega}^0$ .

Take  $E \in \mathcal{W}_{\partial\Omega}^0$ . By Corollary 2.17 there exists  $F \in H^{\text{inc}}(\Omega)$  with  $\text{div } F = 0$  and  $FN = 0$  on  $\partial\Omega$  such that  $E = \text{inc } F$ . The condition  $EN = 0$  on  $\partial\Omega$  yields  $F \in \mathcal{Z}_0$ .  $\square$

**Lemma 3.8.** *Given a symmetric uniformly positive definite fourth order tensor field  $\mathbb{B}$  (i.e.  $\mathbb{B}(x)T.T > \alpha|T|^2 \forall T \in \mathbb{S}^3$  for some  $\alpha > 0$  independent of  $x$ ) define the linear map  $L_{\mathbb{B}} : \mathcal{Z} \rightarrow \mathcal{Z}'$  by*

$$\langle L_{\mathbb{B}}E, \Phi \rangle = \int_{\Omega} \mathbb{B} \text{inc } E \cdot \text{inc } \Phi dx \quad \forall E, \Phi \in \mathcal{Z}.$$

*Then  $L_{\mathbb{B}}$  is an isomorphism from  $\mathcal{Z}$  into  $\mathcal{Z}'$ .*

*Proof.* By Proposition 3.4,  $\langle L_{\mathbb{B}}E, E \rangle$  defines a norm in  $\mathcal{Z}$  equivalent to the  $H^{\text{inc}}$ -norm. Let  $\mathcal{T} \in \mathcal{Z}'$ . By Riesz theorem, there exists  $T \in \mathcal{Z}$  such that  $\langle \mathcal{T}, E \rangle = \langle L_{\mathbb{B}}T, E \rangle$ .  $\square$

Therefore we can define the inverse map  $L_{\mathbb{B}}^{-1} : \mathcal{Z}' \rightarrow \mathcal{Z}$ , that is continuous by Banach's continuous inverse theorem. Since  $\mathcal{Z} \subset L^2(\Omega) \subset \mathcal{Z}'$ , the restriction  $L_{\mathbb{B}}^{-1} : L^2(\Omega) \rightarrow L^2(\Omega)$  is also well-defined.

**Lemma 3.9.** *The operator  $L_{\mathbb{B}}^{-1} : L^2(\Omega) \rightarrow L^2(\Omega)$  is self-adjoint positive definite and compact.*

*Proof.* The compactness stems from the compact embedding  $\mathcal{Z} \hookrightarrow L^2(\Omega)$  of Proposition 3.6. One has for all  $E, F \in L^2(\Omega)$

$$\int_{\Omega} L_{\mathbb{B}}^{-1}E \cdot F dx = \int_{\Omega} L_{\mathbb{B}}^{-1}E \cdot L_{\mathbb{B}}L_{\mathbb{B}}^{-1}F dx = \int_{\Omega} \mathbb{B} \text{inc } (L_{\mathbb{B}}^{-1}E) \cdot \text{inc } (L_{\mathbb{B}}^{-1}F) dx.$$

It follows that  $L_{\mathbb{B}}^{-1}F$  is self-adjoint and positive definite, achieving the proof.  $\square$

**3.3. Two elliptic boundary value problems for the incompatibility.** Lemmas 3.8 and 3.9 yield the following two propositions.

**Proposition 3.10.** *Let  $\mathbb{K} \in L^2(\Omega, \mathbb{S}^3)$  and  $\mathbb{B}$  a symmetric uniformly positive definite fourth order tensor field. There exists a unique  $E \in \mathcal{Z}$  such that*

$$\int_{\Omega} \mathbb{B} \text{inc } E \cdot \text{inc } \hat{E} dx = \int_{\Omega} \mathbb{K} \cdot \hat{E} dx \quad \forall \hat{E} \in \mathcal{Z}. \quad (3.6)$$

*Moreover, the solution map  $\Phi : \mathbb{K} \in L^2(\Omega, \mathbb{S}^3) \rightarrow E \in L^2(\Omega, \mathbb{S}^3)$  is linear and compact.*

**Proposition 3.11.** *Let  $\mathbb{K} \in L^2(\Omega, \mathbb{S}^3)$  and  $B \in L^{\infty}(\Omega, \mathbb{S}^3)$  uniformly positive definite. There exists a unique  $E \in \mathcal{Z}_0$  such that*

$$\int_{\Omega} B \text{inc } E \cdot \text{inc } \hat{E} dx = \int_{\Omega} \mathbb{K} \cdot \hat{E} dx \quad \forall \hat{E} \in \mathcal{Z}_0. \quad (3.7)$$

*Moreover, the solution map  $\Phi_0 : \mathbb{K} \in L^2(\Omega, \mathbb{S}^3) \rightarrow E \in L^2(\Omega, \mathbb{S}^3)$  is linear and compact.*

**Proposition 3.12.** *Let  $\mathbb{K}$  be such that  $\text{div } \mathbb{K} = 0$  in  $\Omega$  and  $\mathbb{K}N = 0$  on  $\partial\Omega$ . Then, the strong form of (3.6) reads*

$$\left\{ \begin{array}{ll} \text{inc } (B \text{inc } E) = \mathbb{K} & \text{in } \Omega \\ \text{div } E = 0 & \text{in } \Omega \\ EN = 0 & \text{on } \partial\Omega \\ \mathcal{T}_0(B \text{inc } E) = \mathcal{T}_1(B \text{inc } E) = 0 & \text{on } \partial\Omega \end{array} \right. \quad (3.8)$$

*whose solution coincide with to the solution of the weak form.*

*Proof.* Eq. (3.6) holds actually true for all  $\hat{E} \in \mathcal{Z} + \mathcal{V} = H^{\text{inc}}(\Omega)$ . In particular for  $\hat{E} \in \mathcal{D}(\Omega)$  and for  $\hat{E}$  with arbitrary traces  $\mathcal{T}_0(\partial_N \hat{E})$  and  $\hat{E}$  on  $\partial\Omega$ , by Theorem 2.3. Then the Green formula provides the strong form, which is seen to be equivalent to the weak form.  $\square$

**Remark 3.4.** *The solution of (3.7) satisfies the strong form*

$$\begin{cases} \operatorname{inc} (B \operatorname{inc} E) = \mathbb{K} & \text{in } \Omega \\ \operatorname{div} E = 0 & \text{in } \Omega \\ EN = 0 & \text{on } \partial\Omega \\ \operatorname{inc} EN = 0 & \text{on } \partial\Omega \end{cases} \quad (3.9)$$

In fact, taking any test function  $\hat{E} \in \mathcal{D}(\Omega) \subset \mathcal{F}$ . Since  $\mathcal{F} = \mathcal{V} + \mathcal{Z}_0$ , one obtains the strong form in  $\Omega$ . The boundary conditions are given by the essential condition of the space.

#### 4. ELASTO-PLASTICITY MODEL

**4.1. Power of internal efforts.** We recall the main features of the model introduced in [2].

*Assumption 1.* The power of the internal efforts against the virtual strain rate  $\hat{E} \in \mathcal{D}(\Omega, \mathbb{S}^3)$  is of form

$$W_{\text{int}}(\hat{E}) = \int_{\Omega} (\sigma \cdot \hat{E} + \tau \cdot \nabla \hat{E}) dx.$$

The tensor fields  $\sigma$  and  $\tau$  are called the stress and hyperstress tensors, respectively.

*Assumption 2.* The power of the internal efforts is a continuous linear functional of  $\hat{E} \in L^2(\Omega, \mathbb{S}^3)$ . We infer  $\sigma - \operatorname{div} \tau \in L^2(\Omega, \mathbb{S}^3)$  and

$$W_{\text{int}}(\hat{E}) = \int_{\Omega} (\sigma - \operatorname{div} \tau) \cdot \hat{E} dx \quad \forall \hat{E} \in L^2(\Omega, \mathbb{S}^3).$$

*Assumption 3.* The material is piecewise linear homogeneous isotropic: there exists a partition of  $\Omega$  as  $\Omega = \bigcup \Omega_p$  such that, in each  $\Omega_p$ ,

$$\sigma = \mathbb{A}_p E, \quad \tau = \mathbb{B}_p \nabla E \quad (4.1)$$

where  $E$  is the strain,  $\mathbb{A}_p$  is the standard Hooke tensor and  $\mathbb{B}_p$  is the Mindlin tensor. They read componentwise

$$\sigma_{ij} = \lambda \delta_{ij} E_{kk} + 2\mu E_{ij}, \quad (4.2)$$

$$\begin{aligned} \tau_{ijk} &= c_1 (\delta_{ki} \partial_l E_{lj} + \delta_{kj} \partial_l E_{li}) + \frac{c_2}{2} (\delta_{ki} \partial_j E_{ll} + \delta_{kj} \partial_i E_{ll} + 2\delta_{ij} \partial_l E_{lk}) + 2c_3 \delta_{ij} \partial_k E_{ll} \\ &+ 2c_4 \partial_k E_{ij} + c_5 (\partial_i E_{jk} + \partial_j E_{ik}), \end{aligned} \quad (4.3)$$

where  $\lambda, \mu, c_1, \dots, c_5$  are constants assigned in each  $\Omega_p$  (index  $p$  is dropped for readability).

*Assumption 4.* The hyperstress  $\tau$  does not produce any virtual intrinsic power as soon as the strain  $E$  is compatible. This means

$$\operatorname{inc} E = 0 \Rightarrow \int_{\Omega} \tau \cdot \nabla \hat{E} dx = 0 \quad \forall \hat{E} \in \mathcal{D}(\Omega, \mathbb{S}^3),$$

or equivalently  $\operatorname{inc} E = 0 \Rightarrow -\operatorname{div} \tau = 0$  in  $\Omega$ . From expression (4.3) we derive the existence within each  $\Omega_p$  of a constant  $\ell_p$  such that  $-\operatorname{div} \tau = \ell_p \operatorname{inc} E$ .

*Conclusion.* We denote  $\ell = \sum \ell_p \chi_{\Omega_p}$  and  $\mathbb{A} = \sum \mathbb{A}_p \chi_{\Omega_p}$ , whereby  $\sigma = \mathbb{A} E$  and  $-\operatorname{div} \tau = \ell \operatorname{inc} E$  in  $\Omega$ . The expression of the internal virtual power is

$$W_{\text{int}}(\hat{E}) = \int_{\Omega} (\mathbb{A} E + \ell \operatorname{inc} E) \cdot \hat{E} dx \quad \forall \hat{E} \in L^2(\Omega, \mathbb{S}^3).$$

**4.2. Power of external efforts.** The power of external efforts is assumed to be a linear functional on  $L^2(\Omega, \mathbb{S}^3)$ . By Riesz representation, there exists  $\mathbb{K} \in L^2(\Omega, \mathbb{S}^3)$  such that

$$W_{\text{ext}}(\hat{E}) = \int_{\Omega} \mathbb{K} \cdot \hat{E} dx.$$

We emphasize that the power of external efforts may be at first expressed in terms of the non-objective fields  $\hat{v}$  and  $\hat{F}$  of the Beltrami decomposition of  $\hat{E}$ . However, provided attention is paid to the uniqueness of the decomposition, these fields are themselves linear functions of  $\hat{E}$ . This will be specified in Section 7.1.

**4.3. Virtual power principle.** The virtual power principle in the absence of inertia reads

$$W_{\text{int}}(\hat{E}) = W_{\text{ext}}(\hat{E}),$$

that is

$$\int_{\Omega} (\mathbb{A}E + \ell \operatorname{inc} E) \cdot \hat{E} dx = \int_{\Omega} \mathbb{K} \cdot \hat{E} dx, \quad (4.4)$$

for all  $\hat{E} \in L^2(\Omega, \mathbb{S}^3)$  satisfying the kinematical constraints. Of course, in the absence of kinematical constraints, (4.4) is equivalent to

$$\mathbb{A}E + \ell \operatorname{inc} E = \mathbb{K},$$

but this strong formulation requires that  $\mathbb{K}$  be known (see for instance the example of Section 8.1).

**4.4. Time-evolution of a nonlinear elasto-plasticity model.** Within an incremental formulation,  $\mathbb{A}$  and  $\ell$  are tangent elasto-plastic moduli. They need to be updated at each increment as soon as plasticity phenomena occur. The stress-strain relation is therefore piecewise linear. Typically, in a region with plastic deformations, the Lamé coefficients and the incompatibility modulus  $\ell$  are expected to be less than in purely elastic regions. The way these coefficients evolve is driven by nonlinear constitutive laws that substitute to flow rules and hardening models. We emphasize that dislocations may be created / moved without creation of incompatibility, as shown by Kröner's relation  $\operatorname{inc} E = \operatorname{Curl} \Lambda$  with  $\Lambda$  the dislocation density tensor [11, 20]. Therefore varying  $\ell$  alone is not sufficient to describe plastic effects.

## 5. SOLUTION OF ELASTO-PLASTICITY EQUATIONS WITH NATURAL BOUNDARY CONDITION

The main problem we address is the following: given  $\mathbb{K} \in L^2(\Omega, \mathbb{S}^3)$ , find  $E$  solution of (4.4).

**5.1. Kinematical setting.** We will see that the absence of kinematical constraints leads to nonunique solutions. We will assume that  $\operatorname{inc} EN = 0$  on  $\partial\Omega$ . In view of Lemma 2.7, this is consistent with assuming that the exterior of  $\Omega$  is filled with a purely elastic phase ( $|\ell_{\text{ext}}| \rightarrow +\infty$ ). A more general setting would be to consider a boundary condition of form  $\operatorname{inc} EN = \Phi(\mathcal{T}_0(E), \mathcal{T}_1(E))$  with an appropriate linear operator  $\Phi$ .

**Remark 5.1.** *A particular kinematical setting is to require  $\mathbb{K} \in \mathcal{V}$ , and a very special case occurs when  $\mathbb{K} = \nabla^S v$  with  $\operatorname{div} v = \operatorname{tr} \mathbb{K}$  constant. Then for  $\mathbb{A}$  constant a solution to  $\mathbb{A}E + \ell \operatorname{inc} E = \mathbb{K}$  is  $E = \mathbb{A}^{-1}\mathbb{K}$ . Indeed by the structure of  $\mathbb{A}^{-1}$  one has  $E$  proportional to  $\mathbb{K}$  plus a constant and hence  $\operatorname{inc} E = 0$ .*

**5.2. Well-posedness.** Our main result is the following.

**Theorem 5.1.** *Assume  $\Omega$  is simply connected. Let  $\mathbb{K} \in L^2(\Omega, \mathbb{S}^3)$ . Let  $C$  be the Poincaré constant of Proposition 3.6. If  $\mathbb{A}$  is uniformly positive definite and  $|\ell| > C|\mathbb{A}|$  a.e., then there exists one and only one  $E \in \mathcal{F}$  such that*

$$\mathbb{A}E + \ell \operatorname{inc} E = \mathbb{K}.$$

Moreover we have the a priori estimate

$$\|\operatorname{inc} E\|_{L^2} \leq \frac{\|\ell^{-1}\mathbb{A}\|_{L^\infty}}{1 - C\|\ell^{-1}\mathbb{A}\|_{L^\infty}} \|\mathbb{A}^{-1}\mathbb{K}\|_{L^2}. \quad (5.1)$$

*Proof.* We assume that  $\ell > 0$ . The other case is deduced considering  $\tilde{\ell} = -\ell$ ,  $\tilde{\mathbb{A}} = -\mathbb{A}$ ,  $\tilde{\mathbb{K}} = -\mathbb{K}$ . We write the problem as

$$E + \mathbb{B} \operatorname{inc} E = \mathbb{H} \quad (5.2)$$

with  $\mathbb{B} := \ell\mathbb{A}^{-1}$  and  $\mathbb{H} := \mathbb{A}^{-1}\mathbb{K}$ . We will first prove uniqueness and then existence of a solution.

*Step 1. Uniqueness.* Let  $E \in \mathcal{F}$  be such that

$$E + \mathbb{B} \operatorname{inc} E = 0. \quad (5.3)$$

Take the orthogonal decomposition  $E = E_c + E_i$  with  $E_c \in \mathcal{V}$  and  $E_i \in \mathcal{Z}_0$ . We have

$$E_c + E_i + \mathbb{B} \operatorname{inc} E_i = 0. \quad (5.4)$$

Take  $\hat{F} \in \mathcal{Z}_0$ . Then

$$\int_{\Omega} E_c \cdot \text{inc } \hat{F} dx + \int_{\Omega} E_i \cdot \text{inc } \hat{F} dx + \int_{\Omega} \mathbb{B} \text{inc } E_i \cdot \text{inc } \hat{F} dx = 0.$$

By  $\text{inc } \hat{F} N = 0$  on  $\partial\Omega$  the first integral vanishes. Specifically, take  $\hat{F} = E_i$ . We obtain

$$\int_{\Omega} E_i \cdot \text{inc } E_i dx + \int_{\Omega} \mathbb{B} \text{inc } E_i \cdot \text{inc } E_i dx = 0.$$

We have

$$\begin{aligned} \|\text{inc } E_i\|_{L^2}^2 &= \int_{\Omega} \text{inc } E_i \cdot \text{inc } E_i dx = \int_{\Omega} \mathbb{B}^{-1}(\mathbb{B}^{1/2} \text{inc } E_i) \cdot (\mathbb{B}^{1/2} \text{inc } E_i) dx \\ &\leq \|\mathbb{B}^{-1}\|_{L^\infty} \int_{\Omega} \mathbb{B} \text{inc } E_i \cdot \text{inc } E_i dx. \end{aligned} \quad (5.5)$$

By Proposition 3.6 we obtain

$$\begin{aligned} C \|\text{inc } E_i\|_{L^2}^2 &\geq \|E_i\|_{L^2} \|\text{inc } E_i\|_{L^2} \\ &\geq \left| \int_{\Omega} E_i \cdot \text{inc } E_i dx \right| = \int_{\Omega} \mathbb{B} \text{inc } E_i \cdot \text{inc } E_i \geq \|\mathbb{B}^{-1}\|_{L^\infty}^{-1} \|\text{inc } E_i\|_{L^2}^2, \end{aligned}$$

that is,

$$(C\|\mathbb{B}^{-1}\|_{L^\infty} - 1) \|\text{inc } E_i\|_{L^2}^2 \geq 0.$$

If  $\|\mathbb{B}^{-1}\|_{L^\infty} < C^{-1}$  we infer  $\text{inc } E_i = 0$  then  $E_i = 0$ , by Proposition 3.6. Thus (5.4) yields  $E_c = 0$ , and eventually  $E = 0$ .

*Step 2. Existence.* Let  $E = E_c + E_i \in \mathcal{F}$ ,  $E_c \in \mathcal{V}$ ,  $E_i \in \mathcal{Z}_0$ . Then (5.2) is equivalent to

$$\begin{cases} \int_{\Omega} (E_c + \mathbb{B} \text{inc } E_i) \cdot \hat{E}_c dx = \int_{\Omega} \mathbb{H} \cdot \hat{E}_c dx, \quad \forall \hat{E}_c \in \mathcal{V}, \\ \int_{\Omega} (E_i + \mathbb{B} \text{inc } E_i) \cdot \hat{E}_i dx = \int_{\Omega} \mathbb{H} \cdot \hat{E}_i dx, \quad \forall \hat{E}_i \in \mathcal{W}_{\partial\Omega}^0, \end{cases} \quad (5.6)$$

itself, by Proposition 3.7, equivalent to

$$\begin{cases} \int_{\Omega} (E_c + \mathbb{B} \text{inc } E_i) \cdot \nabla^S \hat{v} dx = \int_{\Omega} \mathbb{H} \cdot \nabla^S \hat{v} dx \quad \forall \hat{v} \in H^1(\Omega), & (a) \\ \int_{\Omega} (E_i + \mathbb{B} \text{inc } E_i) \cdot \text{inc } \hat{F} dx = \int_{\Omega} \mathbb{H} \cdot \text{inc } \hat{F} dx \quad \forall \hat{F} \in \mathcal{Z}_0. & (b) \end{cases} \quad (5.7)$$

Define the operators  $L_{\mathbb{B}} : \mathcal{Z}_0 \rightarrow \mathcal{Z}'_0$  and  $M : L^2(\Omega, \mathbb{S}^3) \rightarrow \mathcal{Z}'_0$  by

$$\langle L_{\mathbb{B}} \Psi, \Phi \rangle = \int_{\Omega} \mathbb{B} \text{inc } \Psi \cdot \text{inc } \Phi dx, \quad \langle M \Psi, \Phi \rangle = \int_{\Omega} \Psi \cdot \text{inc } \Phi dx.$$

Equation (5.7)(b) is equivalent to

$$(M + L_{\mathbb{B}}) E_i = M \mathbb{H}. \quad (5.8)$$

By Lemma 3.8,  $L_{\mathbb{B}} : \mathcal{Z}_0 \rightarrow \mathcal{Z}'_0$  is invertible. Thus, (5.8) is equivalent to

$$(I + L_{\mathbb{B}}^{-1} M) E_i = L_{\mathbb{B}}^{-1} M \mathbb{H}. \quad (5.9)$$

The operator  $L_{\mathbb{B}}^{-1} M : L^2(\Omega, \mathbb{S}^3) \rightarrow L^2(\Omega, \mathbb{S}^3)$  is compact, since it is continuous from  $L^2(\Omega, \mathbb{S}^3)$  to  $\mathcal{Z}_0$  and  $\mathcal{Z}_0$  is compactly embedded in  $L^2(\Omega, \mathbb{S}^3)$  by Proposition 3.6. Furthermore, under the condition  $\|\mathbb{B}^{-1}\|_{L^\infty} < C^{-1}$ , the operator  $I + L_{\mathbb{B}}^{-1} M : L^2(\Omega, \mathbb{S}^3) \rightarrow L^2(\Omega, \mathbb{S}^3)$  is injective due to the uniqueness claim. Thus, Fredholm's alternative provides the existence of  $E_i \in L^2(\Omega)$  solution of (5.9). From  $E_i = L_{\mathbb{B}}^{-1} M(\mathbb{H} - E_i)$  we infer  $E_i \in \mathcal{Z}_0$ . We have found  $E_i \in \mathcal{Z}_0$  solution of (5.7)(b).

Let us turn to (5.7)(a). We have to find  $E_c = \nabla^S v$ ,  $v \in H^1(\Omega, \mathbb{R}^3)$  such that

$$\int_{\Omega} \nabla^S v \cdot \nabla^S \hat{v} dx = \int_{\Omega} (\mathbb{H} - \mathbb{B} \text{inc } E_i) \cdot \nabla^S \hat{v} dx, \quad \forall \hat{v} \in H^1(\Omega, \mathbb{R}^3). \quad (5.10)$$

This is a standard linear elasticity problem.

*Third step. A priori estimate.* Equation (5.7)(b) entails

$$\int_{\Omega} E_i \cdot \text{inc } E_i dx + \int_{\Omega} \mathbb{B} \text{inc } E_i \cdot \text{inc } E_i dx = \int_{\Omega} \mathbb{H} \cdot \text{inc } E_i dx.$$



Using (5.5) we obtain

$$\|\text{inc } E_i\|_{L^2}^2 \leq \|\mathbb{B}^{-1}\|_{L^\infty} \int_{\Omega} (\mathbb{H} - E_i) \cdot \text{inc } E_i dx \leq \|\mathbb{B}^{-1}\|_{L^\infty} (\|\mathbb{H}\|_{L^2} + \|E_i\|_{L^2}) \|\text{inc } E_i\|_{L^2}.$$

Proposition 3.6 yields

$$\|\text{inc } E_i\|_{L^2} \leq \|\mathbb{B}^{-1}\|_{L^\infty} (\|\mathbb{H}\|_{L^2} + C \|\text{inc } E_i\|_{L^2}),$$

from which we arrive at (5.1).  $\square$

**Remark 5.2** (Dislocation-induced stress). *Because of Kröner's formula we call  $\mathcal{G} := \ell \text{inc } E$  the dislocation-induced stress tensor. For  $\ell$  constant, let  $E^\ell$  be the solution of  $\mathbb{A}E + \ell \text{inc } E = \mathbb{K}$ . Then (5.1) implies that  $\mathcal{G}^\ell := \ell \text{inc } E^\ell$  converges weakly in  $L^2(\Omega, \mathbb{S}^3)$  to some  $\mathcal{G}$  as  $\ell \rightarrow \infty$ . More precise limiting results will be given in the next section.*

*Conversely, the condition  $|\ell| > C|\mathbb{A}|$  prevents  $|\ell|$  from going to 0 without assuming that  $\mathbb{A}$  also tends to 0. Such limit cases are left for future work.*

## 6. ELASTIC LIMIT

**Proposition 6.1.** *Consider a sequence  $\mathbb{A}_k \in L^\infty(\Omega, \mathbb{S}^3)$  with  $c_1|\xi|^2 \leq \mathbb{A}_k(x)\xi \cdot \xi \leq c_2|\xi|^2 \forall \xi \in \mathbb{R}^3$ , a.e.  $x \in \Omega$ ,  $c_1, c_2 > 0$ , and a sequence  $\ell_k \in L^\infty(\Omega, \mathbb{R}_*^+)$  with  $\inf_{\Omega} \ell_k \rightarrow +\infty$ . Assume that  $\mathbb{K} \in L^2(\Omega, \mathbb{S}^3)$ ,  $E^k \in \mathcal{F}$ ,  $\mathbb{A}_k E^k + \ell_k \text{inc } E^k = \mathbb{K}$ . Then  $\|\text{inc } E^k\|_{L^2} \rightarrow 0$ .*

*Proof.* It is a straightforward consequence of (5.1), since  $\|\ell_k^{-1} \mathbb{A}_k\|_{L^\infty} \rightarrow 0$ .  $\square$

Obviously the same holds for a sequence  $\ell_k \in L^\infty(\Omega, \mathbb{R}_*^-)$  with  $\inf_{\Omega} |\ell_k| \rightarrow +\infty$ .

**Proposition 6.2.** *If  $\ell$  is constant,  $\mathbb{K} \in L^2(\Omega, \mathbb{S}^3)$ ,  $E \in \mathcal{F}$ ,  $\mathbb{A}E + \ell \text{inc } E = \mathbb{K}$  in  $\Omega$  then*

$$\int_{\Omega} \mathbb{A}E \cdot \hat{E} dx = \int_{\Omega} \mathbb{K} \cdot \hat{E} dx \quad \forall \hat{E} \in \mathcal{V}.$$

*Proof.* Take  $\hat{E} \in \mathcal{V}$  and observe that due to the assumptions, one has

$$\int_{\Omega} \ell \text{inc } E \cdot \hat{E} dx = 0.$$

$\square$

**Theorem 6.3.** *Assume that  $\mathbb{A}$ ,  $\mathbb{K}$  are fixed,  $\ell$  is constant,  $E^\ell \in \mathcal{F}$ ,  $\mathbb{A}E^\ell + \ell \text{inc } E^\ell = \mathbb{K}$  in  $\Omega$ . There exists a unique  $E^\infty \in \mathcal{V}$  such that*

$$\int_{\Omega} \mathbb{A}E^\infty \cdot \hat{E} dx = \int_{\Omega} \mathbb{K} \cdot \hat{E} dx \quad \forall \hat{E} \in \mathcal{V}. \quad (6.1)$$

*Moreover  $\|E^\ell - E^\infty\|_{L^2} \rightarrow 0$  when  $|\ell| \rightarrow +\infty$ .*

*Proof.* Existence and uniqueness for (6.1) is a consequence of the Riesz representation theorem in the Hilbert space  $\mathcal{V}$  for the inner product  $(E, \hat{E}) \mapsto \int_{\Omega} \mathbb{A}E \cdot \hat{E} dx$ . Consider the decomposition  $E^\ell = E_c^\ell + E_i^\ell \in \mathcal{V} \oplus \mathcal{Z}_0$ . We have by Proposition 6.2

$$\int_{\Omega} \mathbb{A}(E_c^\ell + E_i^\ell) \cdot \hat{E}_c dx = \int_{\Omega} \mathbb{K} \cdot \hat{E}_c dx \quad \forall \hat{E}_c \in \mathcal{V}.$$

Substracting (6.1), one has

$$\int_{\Omega} \mathbb{A}(E_c^\ell - E^\infty) \cdot \hat{E}_c dx = - \int_{\Omega} \mathbb{A}E_i^\ell \cdot \hat{E}_c dx \quad \forall \hat{E}_c \in \mathcal{V}.$$

By Propositions 6.1 and 3.6 we have  $\|E_i^\ell\|_{H^1} \rightarrow 0$ . It follows  $\|E_c^\ell - E^\infty\|_{L^2} \rightarrow 0$  hence  $\|E^\ell - E^\infty\|_{L^2} \rightarrow 0$ .  $\square$

Hence, as  $|\ell| \rightarrow +\infty$ , one retrieves the standard linear elasticity problem with Neumann boundary conditions. The next theorem addresses how to obtain Dirichlet conditions.

**Theorem 6.4.** *Assume that  $\ell$  is constant,  $\mathbb{A}_\ell(x) = \mathbb{A}$  if  $x \in \Omega \setminus \omega$  and  $\mathbb{A}_\ell(x) = \alpha_\ell \mathbb{A}$  if  $x \in \omega$ , with  $\omega \subset \subset \Omega$ ,  $\mathbb{A}$  a symmetric positive definite tensor,  $0 < \alpha_\ell < |\ell|/(C|\mathbb{A}|)$ ,  $\lim_{|\ell| \rightarrow +\infty} \alpha_\ell = +\infty$ ,  $\alpha_\ell = O(\sqrt{|\ell|})$ . Set*

$$\mathcal{V}_\omega = \{E \in \mathcal{V}, E|_\omega = 0\}.$$

*Assume that  $\mathbb{K}$  is fixed,  $E^\ell \in \mathcal{F}$ ,  $\mathbb{A}_\ell E^\ell + \ell \operatorname{inc} E^\ell = \mathbb{K}$  in  $\Omega$ . There exists a unique  $E^\infty \in \mathcal{V}_\omega$  such that*

$$\int_\Omega \mathbb{A} E^\infty \cdot \hat{E} dx = \int_\Omega \mathbb{K} \cdot \hat{E} dx \quad \forall \hat{E} \in \mathcal{V}_\omega. \quad (6.2)$$

*Moreover  $\|E^\ell - E^\infty\|_{L^2} \rightarrow 0$  when  $|\ell| \rightarrow +\infty$ .*

*Proof.* Problem (6.2) is a standard linear elasticity problem with Dirichlet boundary condition on  $\partial\omega$  and Neumann boundary condition on  $\partial\Omega$ . Existence and uniqueness are standard.

Decompose  $E^\ell = E_c^\ell + E_i^\ell \in \mathcal{V} \oplus \mathcal{Z}_0$ . By Theorem 5.1 and Proposition 3.6 we have  $\|E_i^\ell\|_{H^1} = O(|\ell|^{-1} \alpha_\ell) = O(|\ell|^{-1/2})$ . We have by Proposition 6.2

$$\int_\Omega \mathbb{A}_\ell E_c^\ell \cdot \hat{E}_c dx = \int_\Omega \mathbb{K} \cdot \hat{E}_c dx - \int_\Omega \mathbb{A}_\ell E_i^\ell \cdot \hat{E}_c dx \quad \forall \hat{E}_c \in \mathcal{V}.$$

In view of the assumptions this shows that  $\|E_c^\ell\|_{L^2(\Omega)}$  is bounded. Choosing  $\hat{E}_c = E_c^\ell$  we infer

$$\int_\omega \mathbb{A} E_c^\ell \cdot E_c^\ell dx \leq \alpha_\ell^{-1} \left| \int_\Omega \mathbb{K} \cdot E_c^\ell dx - \int_\Omega \mathbb{A}_\ell E_i^\ell \cdot E_c^\ell dx \right|.$$

This yields

$$\|E_c^\ell\|_{L^2(\omega)} \rightarrow 0. \quad (6.3)$$

We have by difference

$$\int_\Omega \mathbb{A}_\ell (E_c^\ell - E^\infty) \cdot \hat{E}_c dx = - \int_\Omega (\mathbb{A}_\ell - \mathbb{A}) E^\infty \cdot \hat{E}_c dx - \int_\Omega \mathbb{A}_\ell E_i^\ell \cdot \hat{E}_c dx \quad \forall \hat{E}_c \in \mathcal{V}_\omega,$$

that is

$$\int_{\Omega \setminus \omega} \mathbb{A} (E_c^\ell - E^\infty) \cdot \hat{E}_c dx = - \int_{\Omega \setminus \omega} \mathbb{A} E_i^\ell \cdot \hat{E}_c dx \quad \forall \hat{E}_c \in \mathcal{V}_\omega.$$

Let  $v^\ell \in H^1(\Omega)$  be such that  $E_c^\ell = \nabla^S v^\ell$  and  $\|v^\ell\|_{H^1(\Omega)} \leq c_1 \|E_c^\ell\|_{L^2(\Omega)}$  for some geometrical constant  $c_1$ . Let  $w^\ell \in H^1(\Omega)$  be an extension on  $(v^\ell)|_\omega$  such that  $\|w^\ell\|_{H^1(\Omega)} \leq c_2 \|v^\ell\|_{H^1(\omega)}$  for some other geometrical constant  $c_2$ . Then  $R_c^\ell := \nabla^S w^\ell \in \mathcal{V}$  is an extension of  $(E_c^\ell)|_\omega$  such that  $\|R_c^\ell\|_{L^2(\Omega)} \leq c_1 c_2 \|E_c^\ell\|_{L^2(\omega)}$ . Moreover,  $E_c^\ell - E^\infty - R_c^\ell \in \mathcal{V}_\omega$  and hence we have

$$\begin{aligned} & \int_{\Omega \setminus \omega} \mathbb{A} (E_c^\ell - E^\infty) \cdot (E_c^\ell - E^\infty) dx \\ &= \int_{\Omega \setminus \omega} \mathbb{A} (E_c^\ell - E^\infty) \cdot (E_c^\ell - E^\infty - R_c^\ell) dx + \int_{\Omega \setminus \omega} \mathbb{A} (E_c^\ell - E^\infty) \cdot R_c^\ell dx \\ &= - \int_{\Omega \setminus \omega} \mathbb{A} E_i^\ell \cdot (E_c^\ell - E^\infty - R_c^\ell) dx + \int_{\Omega \setminus \omega} \mathbb{A} (E_c^\ell - E^\infty) \cdot R_c^\ell dx. \end{aligned}$$

This goes to 0 as  $|\ell| \rightarrow +\infty$ , since  $\|E_i^\ell\|_{L^2(\Omega)} \rightarrow 0$  and  $\|R_c^\ell\|_{L^2(\Omega)} \rightarrow 0$  by (6.3), while  $E_c^\ell - E^\infty - R_c^\ell$  and  $E_c^\ell - E^\infty$  are uniformly bounded. We infer that  $\|E_c^\ell - E^\infty\|_{L^2(\Omega \setminus \omega)} \rightarrow 0$ . This implies  $\|E_c^\ell - E^\infty\|_{L^2(\Omega)} \rightarrow 0$  and eventually  $\|E^\ell - E^\infty\|_{L^2(\Omega)} \rightarrow 0$ .  $\square$

## 7. INTERPRETATION OF THE KINEMATICAL FRAMEWORK AND EXTERNAL EFFORTS

7.1. **External efforts.** Consider a virtual strain  $\hat{E} \in L^2(\Omega, \mathbb{S}^3)$  decomposed as

$$\hat{E} = \nabla^S \hat{v} + \text{inc } \hat{F}. \quad (7.1)$$

The work of the external efforts against  $\hat{E}$  reads

$$\begin{aligned} W_{\text{ext}}(\hat{E}) &= \int_{\Omega} \mathbb{K} \cdot \hat{E} dx = - \int_{\Omega} \text{div } \mathbb{K} \cdot \hat{v} dx + \int_{\partial\Omega} \mathbb{K} N \cdot \hat{v} dS(x) + \int_{\Omega} \text{inc } \mathbb{K} \cdot \hat{F} dx \\ &\quad + \int_{\partial\Omega} \left( \mathcal{T}_1(\mathbb{K}) \cdot \hat{F} + \mathcal{T}_0(\mathbb{K}) \cdot \partial_N \hat{F} \right) dS(x). \end{aligned} \quad (7.2)$$

The fields  $-\text{div } \mathbb{K}$  and  $\mathbb{K}N$  are recognized as classical body and contact forces. The fields  $\text{inc } \mathbb{K}$ ,  $\mathcal{T}_0(\mathbb{K})$ ,  $\mathcal{T}_1(\mathbb{K})$  are body and contact forces working on the incompatible part of the virtual strain. The above fields are in principle known in the first place. The issue is then how and under which conditions it is possible to construct a corresponding  $\mathbb{K}$ . Formally the boundary forces  $\mathbb{K}N$ ,  $\mathcal{T}_0(\mathbb{K})$  and  $\mathcal{T}_1(\mathbb{K})$  exhibit some coupling, as stressed in [2]. To address these points one must specify a kinematical framework ensuring the uniqueness of the decomposition (7.1).

7.2. **Kinematical framework.** Take  $\hat{E} = \nabla^S \hat{v} + \text{inc } \hat{F}$  with  $\nabla^S \hat{v} \in \mathcal{V}_{\Gamma_1}^{00}$  and  $\text{inc } \hat{F} \in \mathcal{W}_{\Gamma_2}^0$  for some partition  $\Gamma_1 \cup \Gamma_2$  of  $\partial\Omega$ . As said above,  $f := -\text{div } \mathbb{K}$  is identified with the body force, and  $g := \mathbb{K}N$  is identified with a surface load on  $\Gamma_2$ . Now, if  $\mathbb{K} \in \mathcal{V}_{\Gamma_1}^{00}$  the last two integrals of (7.2) vanish by virtue of Lemma 3.2. Then (7.2) rewrites as

$$\int_{\Omega} \mathbb{K} \cdot \hat{E} dx = \int_{\Omega} f \cdot \hat{v} dx + \int_{\Gamma_2} g \cdot \hat{v} dS(x). \quad (7.3)$$

To sum up, given  $f \in L^2(\Omega, \mathbb{R}^3)$  and  $g \in H^{-1/2}(\Omega, \mathbb{R}^3)$  such that  $\int_{\Omega} f dx + \int_{\partial\Omega} g dS(x) = 0$ , one wants to determine  $\mathbb{K} = \nabla^S w \in L^2(\Omega, \mathbb{S}^3)$  such that

$$\begin{cases} -\text{div } \nabla^S w = f & \text{in } \Omega, \\ w = 0 & \text{on } \Gamma_1, \\ \nabla^S w N = g & \text{on } \Gamma_2. \end{cases} \quad (7.4)$$

It is well-known that this problem admits a unique solution  $w \in H^1(\Omega)$ .

**Proposition 7.1.** *Assume that  $\Omega$  is simply-connected. Let  $\mathbb{K} \in L^2(\Omega, \mathbb{S}^3)$  be such that  $\mathbb{K} = \nabla^S w$  satisfies (7.4). Then the weak form*

$$\int_{\Omega} (\mathbb{A}E + \ell \text{inc } E) \cdot (\nabla^S \hat{v} + \hat{E}^0) dx = \int_{\Omega} f \cdot \hat{v} dx + \int_{\Gamma_2} g \cdot \hat{v} dS(x), \quad \forall (\nabla^S \hat{v}, \hat{E}^0) \in \mathcal{V}_{\Gamma_1}^{00} \times \mathcal{W}_{\Gamma_2}^0 \quad (7.5)$$

admits a unique solution  $E \in \mathcal{F}$  that corresponds to the solution of the strong form

$$\begin{cases} -\text{div } (\mathbb{A}E + \ell \text{inc } E) = f & \text{in } \Omega \\ \text{inc } (\mathbb{A}E + \ell \text{inc } E) = 0 & \text{in } \Omega \\ (\mathbb{A}E + \ell \text{inc } E)N = (\mathbb{A}E)N = g & \text{on } \Gamma_2 \\ \text{inc } EN = 0 & \text{on } \partial\Omega \\ \mathcal{T}_0(\mathbb{A}E + \ell \text{inc } E) = \mathcal{T}_1(\mathbb{A}E + \ell \text{inc } E) = 0 & \text{on } \Gamma_1 \\ \mathcal{T}_0(\mathbb{A}E + \ell \text{inc } E) = \mathcal{T}_0(\mathbb{K}) & \text{on } \Gamma_2 \\ \mathcal{T}_1(\mathbb{A}E + \ell \text{inc } E) = \mathcal{T}_0(\mathbb{K}) & \text{on } \Gamma_2. \end{cases} \quad (7.6)$$

*Proof.* Eq. (7.5) is equivalent to  $\mathbb{A}E + \ell \text{inc } E = \mathbb{K}$ , itself equivalent to

$$\begin{cases} \text{div } (\mathbb{A}E + \ell \text{inc } E) = \text{div } \mathbb{K} \text{ and } \text{inc } (\mathbb{A}E + \ell \text{inc } E) = \text{inc } \mathbb{K} \text{ in } \Omega, \\ (\mathbb{A}E + \ell \text{inc } E)N = \mathbb{K}N \text{ and } \mathcal{T}_0(\mathbb{A}E + \ell \text{inc } E) = \mathcal{T}_0(\mathbb{K}) \text{ and } \mathcal{T}_1(\mathbb{A}E + \ell \text{inc } E) = \mathcal{T}_1(\mathbb{K}) \text{ on } \partial\Omega. \end{cases}$$

The assumptions and the essential conditions complete the system.  $\square$

**Remark 7.1.** *We remark that the normal component of the Cauchy stress  $(\mathbb{A}E)N$  can be prescribed on  $\Gamma_2$ , whereas the tangential part of the full stress  $\mathcal{T}_0(\mathbb{A}E + \ell \text{inc } E)$  is fixed to zero on  $\Gamma_1$ .*

**Remark 7.2.** *In this framework, there is no virtual work associated with the boundary condition on  $\Gamma_1$ , even if the virtual strain is allowed to have tangential components. This may not well model a clamped condition, and when  $|\ell| \rightarrow \infty$  one do not retrieve  $\mathcal{T}_0(E^\infty) = \mathcal{T}_1(E^\infty) = 0$ , see Theorem 6.3. To do so, one has to set  $\mathbb{K}N = \mathbb{A}E^{\text{el}}N$  on the whole  $\partial\Omega$ , where  $E^{\text{el}}$  is the strain associated with the purely elastic solution, i.e.,  $E^{\text{el}} = \nabla^S v^{\text{el}}$  with*

$$\begin{cases} -\operatorname{div}(\mathbb{A}\nabla^S v^{\text{el}}) = f & \text{in } \Omega, \\ v^{\text{el}} = 0 & \text{on } \Gamma_1, \\ \mathbb{A}\nabla^S v^{\text{el}}N = g & \text{on } \Gamma_2. \end{cases}$$

## 8. EXAMPLES

**8.1. Uniaxial traction.** We consider the domain  $\Omega = \mathbb{R}^3$ . We assume a uniform traction of density  $g = 1$  on the planes  $z = \pm h$ . Hence

$$\mathbb{K} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k \end{pmatrix},$$

with  $k = \chi_{\{|z| < h\}}$ , provides the virtual power  $\int_{\mathbb{R}^3} \mathbb{K} \cdot (\nabla^S \hat{v} + \operatorname{inc} \hat{F}) dx = \int_{\{z=h\}} e_z \cdot \hat{v} dS(x) - \int_{\{z=-h\}} e_z \cdot \hat{v} dS(x)$ . We search for a strain field of form

$$E = \begin{pmatrix} \varphi & 0 & 0 \\ 0 & \varphi & 0 \\ 0 & 0 & \psi \end{pmatrix},$$

where  $\varphi, \psi$  are functions of the  $z$  variable. In this case one has

$$\begin{aligned} \mathbb{A}E &= \begin{pmatrix} 2(\lambda + \mu)\varphi + \lambda\psi & 0 & 0 \\ 0 & 2(\lambda + \mu)\varphi + \lambda\psi & 0 \\ 0 & 0 & 2\lambda\varphi + (\lambda + 2\mu)\psi \end{pmatrix}, \\ \operatorname{inc} E &= \begin{pmatrix} \varphi'' & 0 & 0 \\ 0 & \varphi'' & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{8.1}$$

Hence  $\mathbb{A}E + \ell \operatorname{inc} E = \mathbb{K}$  if and only if

$$\begin{cases} 2(\lambda + \mu)\varphi + \lambda\psi + \ell\varphi'' = 0 \\ 2\lambda\varphi + (\lambda + 2\mu)\psi = k. \end{cases} \tag{8.2}$$

Elementary algebra leads to

$$\psi = \frac{1}{\lambda + 2\mu}(k - 2\lambda\varphi),$$

$$2\mu(3\lambda + 2\mu)\varphi + \ell(\lambda + 2\mu)\varphi'' = -\lambda k.$$

We are led to consider that  $\ell < 0$ , since in the other case the solution does not decay when  $|z| \rightarrow \infty$ . We denote

$$\omega = \sqrt{\frac{2\mu(3\lambda + 2\mu)}{\lambda + 2\mu}}.$$

We obtain:

(1) For  $|z| < h$

$$\begin{aligned} \varphi(z) &= \frac{-\lambda}{2\mu(3\lambda + 2\mu)} \left[ 1 - \exp\left(-\frac{\omega h}{\sqrt{|\ell|}}\right) \cosh\left(\frac{\omega z}{\sqrt{|\ell|}}\right) \right], \\ \psi(z) &= \frac{1}{\lambda + 2\mu} \left\{ 1 + \frac{\lambda^2}{\mu(3\lambda + 2\mu)} \left[ 1 - \exp\left(-\frac{\omega h}{\sqrt{|\ell|}}\right) \cosh\left(\frac{\omega z}{\sqrt{|\ell|}}\right) \right] \right\}. \end{aligned}$$

(2) For  $|z| > h$

$$\begin{aligned}\varphi(z) &= \frac{-\lambda}{2\mu(3\lambda + 2\mu)} \sinh\left(\frac{\omega h}{\sqrt{|\ell|}}\right) \exp\left(-\frac{\omega|z|}{\sqrt{|\ell|}}\right), \\ \psi(z) &= \frac{\lambda^2}{\mu(\lambda + 2\mu)(3\lambda + 2\mu)} \sinh\left(\frac{\omega h}{\sqrt{|\ell|}}\right) \exp\left(-\frac{\omega|z|}{\sqrt{|\ell|}}\right).\end{aligned}$$

Observe that

$$\lim_{|\ell| \rightarrow \infty} \varphi(z) = 0, \quad \lim_{|\ell| \rightarrow \infty} \psi(z) = \begin{cases} \frac{1}{\lambda + 2\mu} & \text{if } |z| < h, \\ 0 & \text{if } |z| > h, \end{cases}$$

which is the classical elastic solution with uniaxial strain. The von Mises stress is found as  $\sigma_M = 2\mu|\varphi - \psi|$ .

Let  $\nabla^S U + E_0$  be the Beltrami decomposition of  $E$  such that  $\nabla^S U \in \mathcal{V}$  and  $E_0 \in \mathcal{W}_{\partial\Omega}^0$  (see Theorem 3.1), i.e.,  $\operatorname{div} E_0 = 0$  in  $\Omega$  and  $E_0 N = 0$  on  $\partial\Omega$ . One has

$$\begin{cases} \operatorname{div} \nabla^S U = \operatorname{div} E & \text{in } \Omega, \\ \nabla^S U N = E N & \text{on } \partial\Omega. \end{cases}$$

This means  $u'' = \psi'$ ,  $u'(h) = \psi(h)$ ,  $u'(-h) = \psi(-h)$ . Thus  $u' = \psi$  and, setting  $u(0) = 0$ ,

$$u(z) = \int_0^z \psi(s) ds.$$

We obtain in particular

$$u(h) = \frac{1}{\lambda + 2\mu} \left\{ h + \frac{\lambda^2}{\mu(3\lambda + 2\mu)} \left[ h - \frac{\sqrt{|\ell|}}{2\omega} \left( 1 - \exp\left(-\frac{2\omega h}{\sqrt{|\ell|}}\right) \right) \right] \right\}.$$

The functions  $\varphi$ ,  $\psi$  are plotted in Figure 1 for  $h = 1$ , Young's modulus  $Y = 10$ , Poisson ratio  $\nu = 1/3$ . The value of  $u(h)$  as a function of  $\ell$  is also depicted.

**Remark 8.1.** Suppose the Poisson ratio  $\nu$  and the load  $\mathbb{K}$  are fixed, and the Young and incompatibility moduli  $Y$  and  $\ell$  are allowed to vary. In view of the equation  $\mathbb{A}E + \ell \operatorname{inc} E = \mathbb{K}$ , with  $\mathbb{A} = Y\mathbb{A}_0$  we get  $\mathbb{A}_0(YE) + (\ell/Y) \operatorname{inc}(YE) = \mathbb{K}$  hence there exists a function  $\Phi$  such that the strain  $E$  and Cauchy stress  $\sigma$  satisfy

$$E = \frac{1}{Y} \Phi\left(\frac{Y}{\ell}\right), \quad \sigma = \mathbb{A}_0 \Phi\left(\frac{Y}{\ell}\right).$$

In the previous example, at each point  $\sigma_M$  is an increasing function of  $Y/|\ell|$ . Therefore, if  $Y$  and  $|\ell|$  are decreased in such a way that the ratio  $Y/|\ell|$  is decreased, then the von Mises stress is alleviated, while the strain is likely to increase.

**8.2. Spherical shell under uniform pressure.** Consider the domain  $\Omega = \{x \in \mathbb{R}^3, R < |x| < 1\}$ . Call  $\Gamma_R$  and  $\Gamma_1$  the internal and external boundaries, respectively. We assume a uniform pressure  $p$  on  $\Gamma_R$  and free boundary on  $\Gamma_1$ . Hence one has  $\mathbb{K} = \nabla^S w$  with

$$\begin{cases} -\operatorname{div} \nabla^S w = 0 & \text{in } \Omega, \\ \nabla^S w N = 0 & \text{on } \Gamma_1, \\ \nabla^S w N = -pN & \text{on } \Gamma_R. \end{cases}$$

For  $w = w(r)e_r \otimes e_r$  one has

$$\nabla^S w = w' e_r \otimes e_r + \frac{w}{r} (e_\theta \otimes e_\theta + e_\phi \otimes e_\phi),$$

$$\operatorname{div} \nabla^S w = \left( w'' + 2\frac{w'}{r} - 2\frac{w}{r^2} \right) e_r.$$

Therefore

$$\operatorname{div} \nabla^S w = 0 \iff w = ar + \frac{b}{r^2}$$

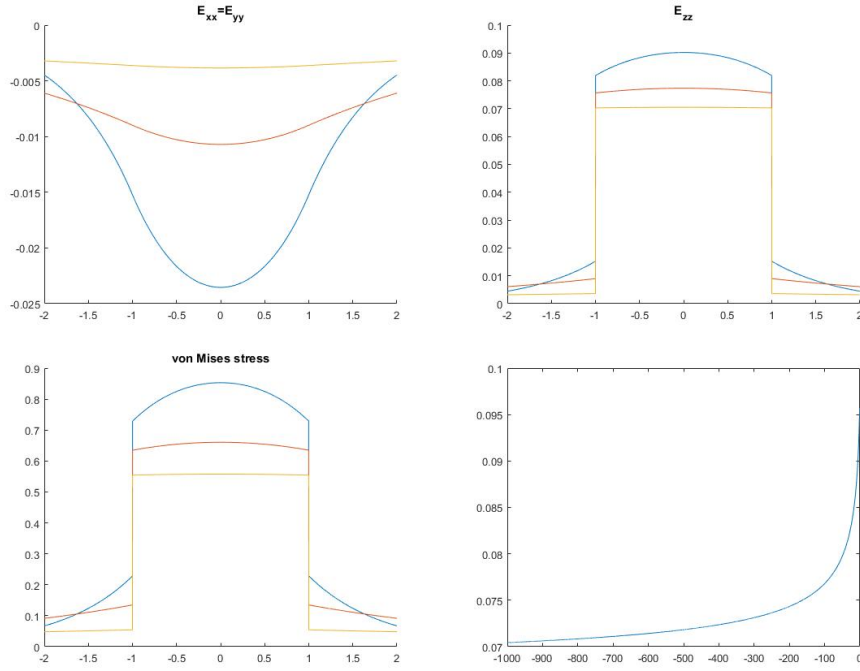


FIGURE 1. Functions  $\varphi$  (top left),  $\psi$  (top right),  $\sigma_M$  (bottom left) for  $\ell = -10$  (blue),  $\ell = -100$  (red),  $\ell = -1000$  (yellow). Value of  $u(h)$  in function of  $\ell$  (bottom right)

for some constants  $a, b$ . The conditions  $w'(1) = 0$  and  $w'(R) = -p$  yield

$$w = \frac{pR^3}{1-R^3} \left( r + \frac{1}{2r^2} \right), \quad w' = \frac{pR^3}{1-R^3} \left( 1 - \frac{1}{r^3} \right).$$

We arrive at

$$\mathbb{K} = \frac{pR^3}{1-R^3} \left[ \left( 1 + \frac{1}{2r^3} \right) I - \frac{3}{2r^3} e_r \otimes e_r \right].$$

Considering the guess

$$E = \varphi(r)I + \psi(r)e_r \otimes e_r$$

we have

$$\mathbb{A}E = ((3\lambda + 2\mu)\varphi + \lambda\psi)I + 2\mu\psi e_r \otimes e_r,$$

On the other hand (see [19])

$$\text{inc } E = \left( \varphi'' + \frac{\varphi'}{r} - \frac{\psi'}{r} \right) I + \left( -\varphi'' + \frac{\varphi'}{r} + \frac{\psi'}{r} - \frac{2\psi}{r^2} \right) e_r \otimes e_r.$$

Hence  $\mathbb{A}E + \ell \text{inc } E = \mathbb{K}$  if and only if

$$\begin{cases} (3\lambda + 2\mu)\varphi + \lambda\psi + \ell \left( \varphi'' + \frac{\varphi'}{r} - \frac{\psi'}{r} \right) = \frac{pR^3}{1-R^3} \left( 1 + \frac{1}{2r^3} \right) \\ 2\mu\psi + \ell \left( -\varphi'' + \frac{\varphi'}{r} + \frac{\psi'}{r} - \frac{2\psi}{r^2} \right) = -\frac{pR^3}{1-R^3} \frac{3}{2r^3}. \end{cases}$$

Moreover, the condition  $\text{inc } EN = 0$  on  $\Gamma_1$  and  $\Gamma_R$  read

$$\varphi'(1) = \psi(1), \quad \varphi'(R) = \frac{\psi(R)}{R}.$$

The solution of the system is the elastic solution:

$$\varphi = \frac{pR^3}{1-R^3} \left( \frac{1}{3\lambda+2\mu} + \frac{1}{4\mu r^3} \right), \quad \psi = \frac{-pR^3}{1-R^3} \frac{3}{4\mu r^3}.$$

There is no strain incompatibility here. This is a consequence of  $\operatorname{tr} \mathbb{K}$  being constant, see Remark 5.1.

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UNIVERSITÉ D’AVIGNON, LABORATOIRE DE MATHÉMATIQUES D’AVIGNON, 301 RUE BARUCH DE SPINOZA, 84916 AVIGNON, FRANCE

*E-mail address:* samuel.amstutz@univ-avignon.fr

UNIVERSIDADE DE LISBOA, FACULDADE DE CIÊNCIAS, DEPARTAMENTO DE MATEMÁTICA, CMAF+CIO, ALAMEDA DA UNIVERSIDADE, C6, 1749-016 LISBOA, PORTUGAL

*E-mail address:* vangoeth@fc.ul.pt