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► **To cite this version:**

Didier Georges. Optimal Sensor Location and Mobile Sensor Crowd Modeling for Environmental Monitoring. IFAC WC 2017 - 20th IFAC World Congress, Jul 2017, Toulouse, France. pp.7076-7081, 10.1016/j.ifacol.2017.08.1355 . hal-01651631

HAL Id: hal-01651631

<https://hal.science/hal-01651631>

Submitted on 29 Nov 2017

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Optimal Sensor Location and Mobile Sensor Crowd Modeling for Environmental Monitoring

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Abstract: In this paper, the optimal sensor location problem is first discussed for environmental monitoring of physical phenomena governed by some advection-diffusion partial-differential equations. In particular, the exact derivation of the observability Gramian of an advection-diffusion PDE is investigated. Based on the optimality criteria derived from this analysis, a conservation law governing the behavior of a crowd of mobile sensors is proposed to ensure convergence of the sensor density towards an optimal location. The monitoring of pollution on a 2D domain is the case study used throughout the paper to illustrate the effectiveness of the proposed approach.

Keywords: Pollution monitoring, optimal sensor location, sensor crowds, collective behavior, conservation laws, approximate observability.

1. INTRODUCTION

Mobile sensors in networks (such as coordinated fleet of drones) represent an attractive way to ensure monitoring or tracking of time-varying spatially-distributed environmental phenomena (weather events, wildfires, air, soil, river or sea pollution ...). Presently monitoring systems are still mostly based on static networks of sensors, see Ghanem (2004) for instance. However the use of mobile sensors can potentially provide more flexibility in collecting distributed data when the conditions are changing.

The navigation of mobile sensors for environmental monitoring is classically based on concentration gradient and flow direction to track pollutant sources (Cortes (2004)). In Demetriou (2011); Ucinski (2005), the authors derive a stable distributed-parameter state observer by using measurements from some mobile sensors which have to be controlled to satisfy this goal. Following the same idea, in Georges (2013), a nonlinear conservation law is proposed to model the collective behavior of a mobile sensor continuum used for pollution monitoring purpose. The sensor density reaches an equilibrium corresponding to the necessary conditions for optimality (observability maximization or trade-off between observability maximization and obstacle avoidance in the presence of obstacles).

In the present paper the later approach is improved in the sense that the proposed conservation law derivation is now based on the exact infinite-dimensional observability gramian of an advection-diffusion PDE in 2D, which can be easily computed, rather than on an approximate finite-dimensional observability gramian derived from a finite-dimensional model obtained thanks to a Galerkin method. The application field still concerns all the physical phenomena governed by advection-diffusion PDEs. The

here-proposed approach can be closely related to the work by Privat et al (2015) for the wave equation.

The main objective remains to get an optimal configuration of the sensors suitable to enhance the performance of state observers designed for estimation or prediction of the distributed pollution dynamics, by mainly improving sensor measurement output sensitivity with respect to the initial state distribution.

The organization of the paper is now as follows: In section 2, the advection-diffusion PDE governing the pollution dispersion phenomena is recalled. Section 3 is devoted to both the exact derivation of both the infinite-dimensional Gramian of a linear advection-diffusion PDE and an observability criteria used for the optimal location of sensors. In section 4, a new nonlinear conservation law governing a crowd of mobile sensors is derived using the results of section 3. In section 5, a case study is investigated, which demonstrate the effectiveness of the methodology. Finally the paper ends with some conclusions and perspectives.

2. THE ADVECTION-DIFFUSION PDE (ADPDE)

The pollution dispersion phenomena may be well modeled (see Zannetti (1990) for instance), in the case of air pollution) on a domain Ω by the following advection-diffusion partial differential equation:

$$\frac{\partial u}{\partial t}(x, t) + V(x, t) \cdot \nabla u(x, t) = k\Delta u(x, t) - \beta u(x, t) + S(x, t) \quad (1)$$

where $x \in \Omega \subset \mathbb{R}^N$, with $N = 1, 2$ or 3 , $u(x, t)$ is the concentration of a chemical species (the pollutant), $V(x, t)$ is a vector of flow velocities which is supposed to be known (through measurements or computation of other PDEs,

such as the Saint-Venant equations governing the dynamics of open-channel hydraulic systems or meteorological models in the case of air pollution), $k > 0$ is a constant diffusion coefficient which is supposed to be known, $\beta > 0$ is a reaction coefficient, and source term $S(x, t)$ acts in the domain Ω . ∇ and Δ stand for the gradient and the Laplacian respectively. " \cdot " denotes the standard scalar product.

In this paper, only the 2D case will be studied. This means that there is no concentration gradient according to the vertical coordinate. It should be however pointed out that the methodology proposed here can be easily extended to the 3D case.

A rectangular domain $\Omega = \{(x, y) : 0 \leq x \leq L, 0 \leq y \leq H\}$ is introduced, together with the following initial and boundary conditions¹

$$u(x, y, t = 0) = u_0(x, y), \quad (2)$$

and

$$u(0, y, t) = u(L, y, t) = 0, \quad (3)$$

$$u(x, 0, t) = u(x, H, t) = 0. \quad (4)$$

Now we use the following assumptions in what follows:

Assumption 1: Velocity field $V(x, y, t)$ is supposed to be uniform over domain Ω : $V(x, y, t) = V(t)$.

Assumption 2: The source term $S(x, y, t)$ is known.

Assumption 3: The time-varying velocity $V(t)$ is replaced by a mean velocity \bar{V} defined over a finite time interval $[0, T]$:

$$\bar{V} = \frac{1}{T} \int_0^T V(t) dt. \quad (5)$$

This assumption means that the velocity field $V(t)$ is available through measurements or predictive computation over $[0, T]$.

Even if the attention will be paid to 2D air pollution in the paper, many other applications are covered by the methodology proposed here: Monitoring of pollution advection-diffusion in 1D or 2D shallow water systems (rivers, lakes, estuaries or seas), underseas pollution or groundwater monitoring.

3. OBSERVABILITY ANALYSIS OF THE ADPDE

3.1 Background on Observability Gramian of Linear PDEs

For linear time-invariant finite-dimensional systems:

$$\dot{x} = Ax \quad (6)$$

$$y = Cx \quad (7)$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$, the so-called "transient observability function" is defined as (see Brockett (1970) for instance)

¹ Neumann Boundary Conditions can also be used without restriction.

$$L_o(X, T) = \frac{1}{2} \int_0^T \|y(t)\|^2 dt, \quad x(0) = X, \quad (8)$$

that is the output energy generated by any initial state $X \in \mathbb{R}^n$ in the time interval $[0, T]$. L_o may be rewritten as

$$L_o(X, T) = \frac{1}{2} X^T W(T) X, \quad (9)$$

with

$$W(T) = \int_0^T e^{A^T t} C^T C e^{At} dt. \quad (10)$$

$W(T)$ is the so-called "transient observability Gramian" matrix.

A necessary and sufficient condition for observability (resp. detectability) of the pair (C, A) is that there exists $\forall t \in [0, T]$, a positive definite (resp. non negative definite) symmetric matrix $W(t)$, solution to the following differential Lyapunov equation:

$$-\dot{W}(t) + A^T W(t) + W(t) A = -C^T C, \quad (11)$$

$$W(0) = 0.$$

The computation of this Lyapunov equation provides the observability Gramian at time T .

In the case of asymptotically stable observable (resp. detectable) linear systems defined by the pair (C, A) , L_o is finite when $T \rightarrow +\infty$ and $\lim_{T \rightarrow +\infty} W(T) = W^\infty$. W^∞ is obtained as the unique positive definite (resp. non negative definite) solution to the Lyapunov problem:

$$A^T W + W A = -C^T C. \quad (12)$$

It follows that $W(T)$ or W^∞ can be used as a measure of the observability degree of the system, since the eigenvalues of W or W^∞ represent the sensitivity of output y with respect to each component of any initial state $x(0) = X$. Indeed, if the sensitivity of y to the initial state (denoted $\partial_{x_0} y^T(t)$) is given by

$$\frac{d}{dt}(\partial_{x_0} x^T(t)) = A \partial_{x_0} x^T(t), \quad \partial_{x_0} x^T(0) = I_d, \quad (13)$$

$$\partial_{x_0} y^T(t) = C \partial_{x_0} x^T(t), \quad (14)$$

where $\partial_{x_0} x^T$ denotes the sensitivity matrix of x with respect to initial state x_0 , then the observability Gramian can be recovered as a Fisher Information Matrix (FIM):

$$\partial_{x_0} y^T(t) = C e^{At} \quad (15)$$

$$\int_0^T \partial_{x_0} y(t) \partial_{x_0} y^T(t) dt = \int_0^T e^{A^T t} C^T C e^{At} dt = W(T). \quad (16)$$

It is interesting to notice that such a sensitivity analysis can be useful to compute the FIM of the sensitivity of y to an unknown constant or slowly time-varying input (source) u for the state-space system:

$$\dot{x}(t) = Ax(t) + Bu, \quad x \in \mathbb{R}^N, \quad u \in \mathbb{R}^m, \quad x(0) = x_0$$

$$y(t) = Cx(t), \quad y \in \mathbb{R}^p.$$

Applying the previous sensitivity computation leads to:

$$\partial_u x^T(t) = \int_0^t e^{A(t-\tau)} B d\tau, \quad (17)$$

$$\partial_u y^T(t) = CA^{-1}(e^{At} - I_d)B, \quad (18)$$

$$W_u(T) = \int_0^T \partial_u y(t) \partial_u y^T(t) dt$$

$$= \int_0^T B^T (e^{A^T t} - I_d) A^{-T} C^T C A^{-1} (e^{At} - I_d) B dt. \quad (19)$$

Since our goal is to maximize the observability of a physical phenomenon, we seek to maximize a norm of the observability Gramian; for instance, the trace of the Gramian that is the sum of the eigenvalues of the Gramian:

$$I_o(0, T) = \text{trace}(W(T)). \quad (20)$$

This index allows to define the best location of sensors in average, i.e. guarantees only detectability for a asymptotically stable system. Other criteria are possible such the minimal eigenvalue or the determinant of $W(T)$ (see for instance Van de Wal and de Jager (2001) for an analysis for finite-dimensional systems). However the use of such criteria may be limited due to the bad conditioning of $W(T)$ for some large-scale systems. The extension of the Gramian notion to infinite-dimensional systems has been performed (see El Jai and Pritchard (1988) and Curtain and Zwart (1995), p. 154-156) for linear infinite-dimensional operators, acting as infinitesimal generators of a C_0 -semigroup. If we consider the the system defined in abstract form by

$$\dot{z}(t) = Az(t) \quad (21)$$

$$y(t) = Cz(t), \quad (22)$$

where A is the infinitesimal generator of a C_0 -semigroup $T(t)$ on a Hilbert space Z , and C is an output linear and bounded operator from Z to a Hilbert space Y , the pair (A, C) is said to be *approximately observable* on $[0, T]$ (for some finite $T > 0$), if the so-called observability map of (A, C) on $[0, T]$, which is the bounded linear map $C^T : Z \rightarrow L_2([0, T]; Y)$ defined by

$$C^T z = CT(\cdot)z \quad (23)$$

is such that $\ker C^T = \{0\}$. Furthermore the observability Gramian of (A, C) , which is defined by the following linear self-adjoint operator

$$W^T = C^{T*} C^T, \quad (24)$$

is such that $W^T > 0$.

Finding an explicit solution is not possible in general, except for Riesz-spectral operators, whose eigenvalues and eigenfunctions are known explicitly (for instance, the heat equation, see Liu (2010)).

3.2 Analytical Observability Gramian of the ADPDE

In Georges (2013), a Galerkin weighted-residual method was proposed with the goal of deriving a reduced-order finite-dimensional model of the ADPDE. Based on this reduced model, an finite-dimensional approximate observability Gramian is computed and then used for optimal

sensor location. Unlike the Galerkin approach which may lead to spillover phenomena (see Privat et al (2013)), the analytical derivation of the observability Gramian of the ADPDE is derived in this paper, based on the exact spectral analysis and a direct connection to approximate observability of the operator, according to the following procedure (see Dautray and Lions (2000) for instance):

- (1) Transform the ADPDE into an equivalent purely diffusive equation (the heat equation) thanks to an appropriate change of coordinates.
- (2) Compute the solutions to the related Sturm-Liouville problem in the two dimensions to get the exact eigenvalues and eigenfunctions of the equivalent heat equation.
- (3) Use the inverse transform to compute the observability map of the original ADPDE and then the observability Gramian.

In order to transform ADPDE (1) into the following diffusion-reaction equation with a source term S'

$$\frac{\partial v}{\partial t}(x, y, t) = k \frac{\partial^2 v}{\partial x^2}(x, y, t) + k \frac{\partial^2 v}{\partial y^2}(x, y, t) + S'(x, y, t) - qv(x, y, t) \quad (25)$$

with $q = \beta + \frac{1}{4}(\frac{v_x^2}{k} + \frac{v_y^2}{k})$, initial and boundary conditions

$$v(x, y, t = 0) = v_0(x, y), \quad (26)$$

and

$$v(0, y, t) = v(L, y, t) = 0, \quad (27)$$

$$v(x, 0, t) = v(x, H, t) = 0. \quad (28)$$

the following change of coordinates (Helmholtz transformation) is used:

$$u(x, y, t) = v(x, y, t)e^{p_1 x + p_2 y}, \quad (29)$$

where p_1 , and p_2 are some real coefficients to be defined.

After quite lengthy but simple calculations, equation (25) is obtained by choosing

$$p_1 = \frac{v_x}{2k}, \quad p_2 = \frac{v_y}{2k},$$

where v_x and v_y are the two components of velocity \bar{V} , with $S'(x, y, t) = S(x, y, t)e^{-(p_1 x + p_2 y)}$.

The problem now reduces to solving the eigenvalue problem of equation (25), using a classical variable separation:

$$v(x, y, t) = \phi(x, y)\psi(t)$$

which leads to

$$\frac{\psi'(t)}{k\psi(t)} = \frac{\Delta\phi(x, y)}{\phi(x, y)} = -(\lambda + q/k), \quad (30)$$

where λ is a constant.

A new variable separation $\phi(x, y) = \phi_x(x)\phi_y(y)$ leads to solving the following set of differential equations with two-point boundary conditions:

$$\phi_x'' + \mu\phi_x = 0, \quad \phi_x(0) = \phi_x(L) = 0, \quad (31)$$

$$\phi_y'' + \nu\phi_y = 0, \quad \phi_y(0) = \phi_y(H) = 0. \quad (32)$$

The eigenvalues and eigenfunctions are given by

$$\mu_n = \frac{\pi^2 n^2}{L^2}, \quad \phi_x^n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad (33)$$

$$\nu_n = \frac{\pi^2 n^2}{H^2}, \quad \phi_y^n(y) = \sin\left(\frac{n\pi y}{H}\right). \quad (34)$$

It follows that the general solution is given by

$$\phi_{nm}(x, y) = \sin\left(\frac{n\pi x}{L}\right) \sin\left(m\frac{\pi y}{H}\right). \quad (35)$$

From (30), the solution for $\psi(t)$ is given by

$$\psi(t) = e^{-(k(\mu_n + \nu_m) + q)t} \quad (36)$$

Since the operator is Riesz-spectral, the general solution of eqn (25) is finally given by

$$v(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm} e^{-(k(\mu_n + \nu_m) + q)t} \phi_{nm}(x, y), \quad (37)$$

where the c_{nm} 's are imposed by initial conditions $u(x, y, 0) = u_0(x, y)$. Thanks to the orthogonality property of the eigenfunctions and the fact the operator is self-adjoint, they are given by

$$\begin{aligned} c_{nm} &= \frac{4}{LH} \int_0^L \int_0^H v_0(x, y) \phi_{nm}(x, y) dx dy \\ &= \frac{4}{LH} \int_0^L \int_0^H u_0(x, y) e^{-(p_1 x + p_2 y)} \phi_{nm}(x, y) dx dy. \end{aligned} \quad (38)$$

Finally, the solution of ADPDE (1) is given by

$$\begin{aligned} u(x, y, t) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm} e^{(-q - k(\mu_n + \nu_m) + p_1 x + p_2 y)t} \phi_{nm}(x, y) \\ &+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} e^{p_1 x + p_2 y} \phi_{nm}(x, y) \\ &\times \int_0^t e^{(-q - k(\mu_n + \nu_m))(t - \tau)} \\ &\times \left[\int_0^L \int_0^H \phi_{nm}(x', y') \times S'(x', y', \tau) dx' dy' \right] d\tau. \end{aligned} \quad (39)$$

If the output operator C is given by

$$y = Cu = \int_0^L \int_0^H \Delta_s(x - x_l, y - y_l) u(x, y, t) dx dy, \quad (40)$$

where $\Delta_s(x - x_l, y - y_l)$ is a shaping function of a sensor l located at position (x_l, y_l) , the output map \mathcal{C}^T is given by

$$\begin{aligned} \mathcal{C}^T u &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm} \int_0^L \int_0^H \Delta_s(x - x_l, y - y_l) \\ &\times \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) e^{p_1 x + p_2 y} dx dy \\ &\times e^{(-q - k(\mu_n + \nu_m))t}. \end{aligned} \quad (41)$$

The observability Gramian is then given by

$$W^T = \mathcal{C}^{T*} \mathcal{C}^T. \quad (42)$$

According to Theorem 4.2.3 in Curtain and Zwart (1995), ADPDE (1) is approximately observable if and only if

$$\begin{aligned} \mathcal{C}_{nm}(X_l) &= \int_0^L \int_0^H \Delta_s(x - x_l, y - y_l) \sin\left(\frac{n\pi x}{L}\right) \\ &\times \sin\left(\frac{m\pi y}{H}\right) e^{p_1 x + p_2 y} dx dy \neq 0, \quad \forall n, m, \end{aligned} \quad (43)$$

where $X_l = (x_l, y_l)$ is the coordinate vector of a sensor l .

3.3 Observability Index and Optimal Sensor Location

if N_s sensors are located at position $X_l = (x_l, y_l) \in \Omega$, $l = 1, \dots, N_s$, In the coordinates $c_{nm}(u_0)$ in the Riesz basis, the approximate observability of any initial u_0 can be investigated by considering the trace of the Riesz-basis Gramian on horizon T :

$$\begin{aligned} \sum_{l=1}^{N_s} \int_0^T &\begin{pmatrix} \mathcal{C}_{11}(X_l) e^{(-q - k(\mu_1 + \nu_1))t} \\ \vdots \\ \mathcal{C}_{\infty\infty}(X_l) e^{(-q - k(\mu_\infty + \nu_\infty))t} \end{pmatrix} \\ &\times (\mathcal{C}_{11}(X_l) e^{(-q - k(\mu_1 + \nu_1))t} \dots \\ &\mathcal{C}_{\infty\infty}(X_l) e^{(-q - k(\mu_\infty + \nu_\infty))t}) dt, \end{aligned} \quad (44)$$

that is

$$I_o^T = \sum_{l=1}^{N_s} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \int_0^T \mathcal{C}_{nm}^2(X_l) e^{2(-q - k(\mu_n + \nu_m))t} dt. \quad (45)$$

(45) may be interpreted as the deterministic equivalent to the so-called randomized observability constant in Privat et al (2015). In practice, the sums in n and m have to be truncated with large enough $n = N$ and $m = M$, respectively.

Since the operator is exponentially stable, a truncated $I_o^\infty(X_1, \dots, X_{N_s})$, when $T \rightarrow +\infty$ is well defined as

$$I_o^\infty = \sum_{l=1}^{N_s} \sum_{n=1}^N \sum_{m=1}^M \frac{\mathcal{C}_{nm}^2(X_l)}{2(k(\mu_n + \nu_m) + q)} \quad (46)$$

Finally, the extension of this approach to the 3D case is possible and would follow the same derivations by using the change of coordinates

$$u(x, y, z, t) = v(x, y, z, t) e^{p_1 x + p_2 y + p_3 z}.$$

Finding an optimal sensor configuration will consists in solving the following maximization problem:

$$\max_{X_1, \dots, X_{N_s}} I_o^\infty(X_1, \dots, X_{N_s}). \quad (47)$$

subject to sensor density constraints (to avoid high sensor density and/or to ensure communications between the sensors (not too low density) for instance).

Here an observer can be designed by using transformation (29) as follows

$$\begin{aligned}
\frac{\partial \hat{v}}{\partial t}(x, y, t) &= k \frac{\partial^2 \hat{v}}{\partial x^2}(x, y, t) + k \frac{\partial^2 \hat{v}}{\partial y^2}(x, y, t) + S'(x, y, t) \\
&\quad - q \hat{v}(x, y, t) + \sum_{l=1}^{N_s} G_l(x, y)(y_l - C_l' \hat{v}(x, y, t)), \\
\hat{u}(x, y, t) &= \hat{v}(x, y, t) e^{p_1 x + p_2 y}, \\
\hat{v}(0, y, t) &= \hat{v}(L, y, t) = \hat{v}(x, 0, t) = \hat{v}(x, H, t) = 0,
\end{aligned} \tag{48}$$

where the output operator is given, for each of the N_s sensors, by

$$y_l = C_l' v = \int_0^L \int_0^H \Delta_s(x - x_l, y - y_l) e^{p_1 x + p_2 y} \times v(x, y, t) dx dy, \quad l = 1, \dots, N_s, \tag{49}$$

and gain operators $G_l(x, y) = \sum_{n=1}^N \sum_{m=1}^M G_{nm}^l \phi_{nm}(x, y)$ are used to tune the first $N \times M$ eigenvalues of the estimation error operator. Due to the paper length limitation, the computation of the G_{nm}^l (for instance via a pole placement technique applied to finite-dimensional operator $\mathcal{A} - \mathcal{G}\mathcal{C}$, where $\mathcal{A} = \text{diag}\{k(\lambda_n + \mu_m) - q\}_{n=1, \dots, N, m=1, \dots, M}$, ($N \times M, N_s$) matrix $\mathcal{G} = \{G_{nm}^l\}$, and ($N_s, N \times M$) matrix $\mathcal{C} = \{C_{nm}^l\}$, where $C_{nm}^l = \int_0^L \int_0^H \Delta_s(x - x_l, y - y_l) e^{p_1 x + p_2 y} \phi_{nm}(x, y) dx dy$), is not detailed here.

4. A CONSERVATION LAW FOR THE OPTIMAL LOCATION OF A SENSOR CROWD

Rather than solving optimal sensor location problem (47) by using a gradient-based navigation scheme for a limited number of N_s sensors, and as in Georges (2013), a conservation law governing the behavior of a mobile sensor crowd towards an optimal sensor configuration is now derived, which only requires a rather simple numerical simulation. Following the same approach as in Canizo (2012) or Colombo (2012) for crowd modeling, the following 2D conservation law is proposed, if the conservation of sensor density denoted $\rho(X, t)$, $X \in \Omega$, w.r.t. time is assumed:

$$\frac{\partial \rho}{\partial t}(X, t) + \text{div}_X(\rho(X, t) V_\rho(\rho(X, t), X)) = 0, \tag{50}$$

with

$$\begin{aligned}
V_\rho(\rho(X, t), X) &= (1 - \rho(X, t)/\rho_{max})(\delta \nabla_X I_o^\infty(X) \\
&\quad + \mu \Psi(X)),
\end{aligned} \tag{51}$$

where $I_o^\infty(X) = \sum_{n=1}^N \sum_{m=1}^M \frac{C_{nm}^2(X)}{2(k(\mu_n + \nu_m) + q)}$ as defined

by (43) and (46), and $\Psi(X) = \frac{X}{1 + \|X\|^2}$ can be used to induce a radial sensor dispersion and controls the sensor density until convergence to an equilibrium (which maximizes the observability index). Mainly under the rather mild assumption $\nabla_X I_o^\infty(X)$ is smooth (that is the case here), the existence and uniqueness of the solution in $L^1(\mathbb{R}^2, [0, \rho_{max}])$ to Cauchy problem

$$\frac{\partial \rho}{\partial t}(X, t) + \text{div}_X(\rho(X, t) V_\rho(\rho(X, t), X)) = 0, \tag{52}$$

$$\rho(X, 0) = \rho_0(X) \tag{53}$$

is ensured (see Colombo (2012) for instance). As in Georges (2013) (see section 4.3), obstacle avoidance can be also introduced by considering repulsive gradients in (51).

5. A NUMERICAL EXAMPLE

The optimal navigation of a sensor crowd towards the equilibrium sensor distribution corresponding to an optimal sensor location, on domain $\Omega = [0km, 2km]^2$, is now considered. The ADPDE and conservation law parameters used in the simulation are given in table 1. The truncated steady-state ADPDE solution is computed from (39), with initial conditions equal to zero and source term $S(x, y, t) = \frac{1}{4\epsilon^2} 1_{[x_s - \epsilon, x_s + \epsilon] \times [y_s - \epsilon, y_s + \epsilon]} \bar{S}$ with source location $X_s = (x_s, y_s) = (0.4km, 0.5km)$, and where $1_{[\alpha, \beta] \times [\alpha', \beta']}$ = $\begin{cases} 1, & \alpha \leq x \leq \beta \text{ and } \alpha' \leq y \leq \beta' \\ 0, & \text{elsewhere} \end{cases}$, and with constant source magnitude $\bar{S} = 1$, as

$$\begin{aligned}
u_\epsilon(x, y) &= \sum_{n=1}^N \sum_{m=1}^M e^{p_1 x + p_2 y} \times \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) \\
&\quad \times \left[\int_0^L \int_0^H \sin\left(\frac{n\pi x'}{L}\right) \sin\left(\frac{m\pi y'}{H}\right) \right. \\
&\quad \times \left. \frac{1}{4\epsilon^2} 1_{[X_s - \epsilon, X_s + \epsilon]} \bar{S} e^{-(p_1 x' + p_2 y')} dx' dy' \right] \\
&\quad / (k(\mu_n + \nu_m) + q).
\end{aligned} \tag{54}$$

The simulations of the conservation law were performed by using a 2D Lax-Friedrichs integration scheme. The initial density is given by $\rho_0(x) = e^{-\|x - x_I\|^2 / (2\sigma^2)}$, with $x_I = (1km, 1km)$ and $\sigma = 0.05$. The results are given by Fig. 1 and 2. Fig. 1 provides the distribution of the observability index, while Fig. 2 depicts the asymptotic sensor density distribution provided by conservation law (50), which strongly depends on velocity field $V(x, y)$ and diffusion coefficient k . In particular, a change of velocity field will induce a change of the sensor distribution, which can be easily recomputed via integration of (50). On Fig. 2, the level curves with values correspond to the pollution density levels at equilibrium, whereas the warmer colors represent the higher sensor densities. Not surprisingly, the sensors cluster at the higher values of the observability index. Knowing the optimal sensor density distribution, ensuring the actual location of the sensors has to be investigated. A comparison of the here-proposed observability index with the one proposed in Georges (2013), and the influence of the number of considered modes ($N \times M$) have also to be investigated in some future works.

Table 1. Simulation parameters

V_x	V_y	k	β	$N = M$	δ	μ	ρ_{max}
50km/h	5km/h	20	0	25	0.04	0	1

6. CONCLUSIONS AND PERSPECTIVES

The analysis of the approximate observability of the advection-diffusion PDE has been performed in this pa-

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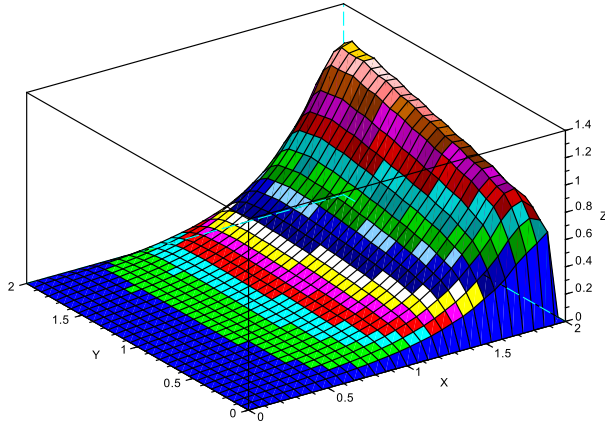


Fig. 1. Observability index $I_o^\infty(x, y)$ with $\Delta_s(x', y') = \delta_{x-x'} \times \delta_{y-y'}$.

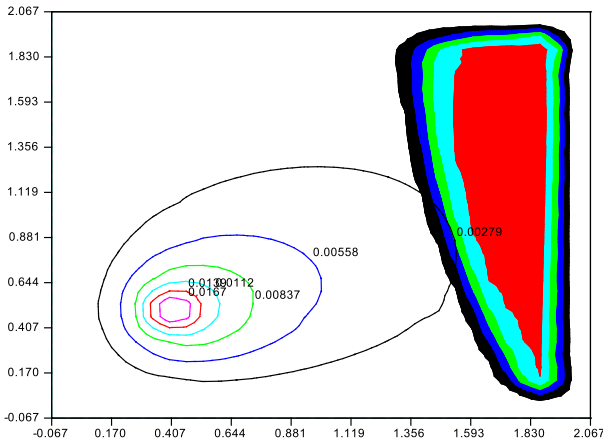


Fig. 2. Asymptotic sensor density distribution.

per. This analysis leads to the derivation of a Riesz-spectral observability index which allows to solve the optimal sensor location problem in 2D. On the basis of the observability index, a nonlinear conservation law has been proposed to model the collective behavior of a mobile sensor crowd used for monitoring purpose, via some convenient observers, such as the one shortly described in the paper. The sensor density reaches an equilibrium which maximizes the observability index. The derivation of a Riesz-spectral index for source estimation is a reachable perspective by considering extension of (19) to the infinite-dimensional case should be investigated. The derivation of an optimal sensor crowd location approach based on the optimal tuning of an observer (as in Demetriou (2011); Ucinski (2005)) is also an interesting perspective. Future works will be also devoted to similar observability index and conservation law derivations based on the infinite-dimensional observability gramian for wave PDEs. Indeed many applications are available, which are not limited to the case of advection-diffusion PDEs: seismic monitoring, or monitoring of large infrastructures in civil engineering (dams for instance).