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A Discrete-Time State Observer Approach to Discovering Portfolio Holdings

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Abstract: This paper is devoted to the derivation of a new approach for discovering mutual fund holdings from returns only. The approach is based on the use of a discrete-time asymptotic observer with forgetting factor which allows to estimate on-line both the holdings and the holding changes in portfolios. A realistic example is also provided, which demonstrates the effectiveness of the proposed approach. To the best of our knowledge, such an approach has never been proposed before.

Keywords: Discrete-Time State Observers, Portfolio Holding Estimation, Social Systems, Finance.

1. INTRODUCTION

A portfolio is perfectly defined by its components at every moment. However, this information is not always available, either because it is not free or because the disclosure about the composition of the portfolio is not realized every day. Is there a method to guess the portfolio holdings with little data? In this paper, we assume to only know the portfolio returns and the asset universe. Using a nonlinear estimation approach, we are able by a reverse engineering technique to find the asset proportions in the portfolio. Why is it important to know precisely and constantly the portfolio holdings of mutual funds? The main reasons are related to the choice of a portfolio strategy and to the improvement of portfolio supervision and performance measurement. First, one can try to discover a portfolio strategy to imitate a winning manager by constructing a copycat portfolio that strictly duplicates the original portfolio holdings. For some funds who want to keep their holdings secret, the holdings are not available. For traditional mutual funds, the same ignorance occurs on periods between two disclosure dates. Second, to identify the portfolio holdings is a way to know if the assets in a fund are in accordance with the benchmark at any time, not only at disclosure dates. In other words, it provides information on compliance with commitments. Some funds exhibit "window dressing", a strategy consisting in changing the portfolio holdings just before disclosure dates, in order to improve the appearance of the fund. The constant knowledge of holdings will help to improve the fund performance measure by a better risk assessment. Third, the study of portfolio holding evolution permits to verify to what extent funds have an active strategy and to compare the fees to the level of activity.

Academic papers dealing with the topics of uncovering portfolio holdings are rather scarce. Practitioners are more concerned with this issue. A paper from di Bartolomeo (2007) is trying to analyze a portfolio with undisclosed holdings. They base their analysis on the observed returns

of the funds. They try to identify a proxy portfolio that represents the most likely set of underlying investments in the fund. They begin first with a returns based style analysis defining a set of market indices that is appropriate for a fund. A time series regression of the fund returns on the mimicking portfolios defines the mix of indices that best fits the fund returns. The second step consists in an optimization to choose inside the indices the assets such that the constructed portfolio volatility approximates the observed portfolio volatility and matches the benchmark in style.

There is a debate about the optimal frequency of mutual fund holding disclosure. A total transparency is not necessarily the best policy for everybody. Wermers (2001) explain the impact of more frequent portfolio disclosure for mutual funds. She fears a reduction of the returns for shareholders because of front running and free riding. Front running would increase mutual fund trading costs, outside investors being able to better anticipate fund trades and capture their price impact. Free riding consists in benefiting from fund investment research to implement winning strategies without incurring any cost.

Portfolio disclosure is mandatory for some funds at a given frequency. For example, in the United States, the Securities Exchange Act of 1934 plans mandatory disclosure of all mutual fund holdings (regardless of the funds size or the size of the positions held in individual securities) at a quarterly frequency with a delay of no longer than 60 days. This disclosure is devoted to increase transparency in the capital markets, the risk being that more transparency deters fund managers to collect and process information. Agarwal et al. (2015), by comparing two periods with different disclosure frequency (semiannual and quarterly) show that increasing the disclosure frequency can reduce the fund incentives to collect and process information.

Is there really a difference between the performance of a copycat portfolio based on mandatory disclosed holdings and the true underlying portfolio? Kacperczyk et al. (2008) compare the actual mutual fund return with the return of a hypothetical portfolio invested in the last disclosed fund holdings. The difference is called the return gap and the authors show that the decile portfolio of funds with the highest return gap earns abnormal positive returns. If investors can guess the holdings of these funds, they can use them to build winning strategies. Agarwal et al. (2013) study the confidential holdings of institutional investors, especially hedge funds. These funds can request the SEC (Securities and Exchange Commission) a delay before disclosure, necessary or appropriate in the public interest or for the protection of investors. The SEC can deny the request. Hedge funds account for 56% of all the confidential filings and allocate on average about one-third of their total portfolio value into confidentiality. To hide portfolio holdings allows funds to fully profit from a strategy and avoid front-runners. It is also useful to prevent speculators to anticipate further trades of the funds and profit from the temporary price impact. Finally, it is a way to avoid window dressing, consisting in modifying the portfolio at the disclosure date to hide risky or neglected firms investors would not appreciate. The study supports private information as the main motive for confidentiality. These results are confirmed by those of Aragon et al. (2013). They find that confidential positions earn positive and significant abnormal returns over the post-filing confidential period. It is consistent with managers seeking confidentiality to protect proprietary information.

However, the previous results do not mean window dressing does not exist. Ortiz et al. (2013) test the potential manipulation of portfolios in mandatory reports by comparing the return of the fund portfolio holdings and the observed fund return. On a Spanish sample of domestic equity funds, they find that only a low percentage of reports may be classified as window-dressed portfolios. On Spanish bond funds, Ortiz et al. (2012) test window dressing using monthly undisclosed portfolios. They compare changes in public debt allocations around quarterly disclosures. They find window dressing changes with market conditions but is not widely spread in the Spanish bond fund industry. Morey & O'Neal (2006) examine US bond funds and find different results. Their analysis is centered on the detection of differences in the credit quality exhibited in disclosure versus non-disclosure periods. Bond funds on average hold significantly more government bonds during disclosure. Window dressing is evidenced, fund managers probably trying to present a safer portfolio to investors. As pointed out by Morey & O'Neal (2006), the detrimental effects of window dressing are two-fold: first, investors are misled about the sources of fund performance; second, the fund bears additional transaction costs to "build and dismantle cosmetic positions"

In this paper, we address the portfolio estimation problem by using estimation theory applied to nonlinear dynamical systems (see Besançon (2007) for a reference on that topic), and more precisely we investigate the opportunity to estimate the asset holdings of a portfolio whose composition is known a priori, and the use of an exact asymptotic

discrete-time observer. To the best of our knowledge, this is the first time that such an approach is used.

The paper is now organized as follows: Section 2 is devoted to the description of the dynamical behavior of asset holdings in a portfolio according to the decisions made by the investors and the returns of the assets. In section 3, both the observability analysis and observer design are presented. Section 4 deals with a case study which demonstrates the effectiveness of the proposed approach. Section 5 finally sums up some conclusions and perspectives.

2. DYNAMICAL MODEL FOR PORTFOLIO ASSET HOLDINGS

The dynamics of N -asset holdings in a portfolio over time can be classically expressed as a discrete-time system defined as follows (see the Appendix for the model derivation):

$$W_i(t+1) = \frac{(1 + R_i(t))W_i(t)}{1 + \sum_{j=1}^N R_j(t)W_j(t)} + U_i(t), \quad i = 1, \dots, N, (1)$$

where $W_i(t)$ is the holding of asset i at time sample t , $R_i(t)$ is the individual return of asset i at time sample t , which is supposed to be known for each t , and U_i is the decision of changing the holding of asset i at time sample t made by the investor.

The return of the N -asset portfolio is given by

$$R_p(t) = \sum_{j=1}^N R_j(t)W_j(t). \quad (2)$$

Furthermore, the dynamics is constrained by the fact that necessarily

$$\sum_{j=1}^N W_j(t) = 1, \quad (3)$$

that implies

$$\sum_{j=1}^N U_j(t) = 0. \quad (4)$$

This means that any of the N asset holdings (let say the N th one for instance) can be expressed as a linear combination of the other $N - 1$ holdings given by

$$W_N(t) = 1 - \sum_{j=1}^{N-1} W_j(t), \quad \forall t, \quad (5)$$

and the same holds for the U_i 's:

$$U_N(t) = - \sum_{j=1}^{N-1} U_j(t), \quad \forall t. \quad (6)$$

Consequently, system (1) with (2) can be reduced to $N - 1$ states, according to the following dynamics:

$$W_i(t+1) = \frac{(1 + R_i(t))W_i(t)}{1 + R_N(t) + \sum_{j=1}^{N-1} (R_j(t) - R_N(t))W_j(t)} + U_i(t), \quad i = 1, \dots, N-1, \quad (7)$$

$$R_p(t) = R_N(t) + \sum_{j=1}^{N-1} (R_j(t) - R_N(t))W_j(t). \quad (8)$$

3. A DISCRETE-TIME STATE OBSERVER FOR BOTH HOLDING AND HOLDING CHANGE ESTIMATION

Our main goal is to estimate in real time both holdings $W_i(t)$ and changes $U_i(t)$ in the composition of portfolios whose return R_p is known together with individual asset returns $R_i(t)$, $i = 1, \dots, N$. This problem is clearly a nonlinear estimation problem since dynamics (7)-(8) are nonlinear.

For that purpose, an extended system is introduced to allow estimation of the holding changes in an asymptotic way:

$$W_i(t+1) = \frac{(1 + R_i(t))W_i(t)}{1 + R_p(t)} + U_i(t), \quad (9)$$

$$U_i(t+1) = U_i(t), \quad i = 1, \dots, N-1, \quad (10)$$

$$R_p(t) = R_N(t) + \sum_{j=1}^{N-1} (R_j(t) - R_N(t))W_j(t). \quad (11)$$

Introducing (10) is a classical way to allow the estimation of constant or slowly-time-varying unknown inputs and to reduce state estimation bias by inclusion of an integral action.

A key feature which will facilitate the design of an exact state observer is that system (9)-(11) belongs to the class of nonlinear time-varying systems which are linear in state and whose state matrix only depends on both output y and t :

$$x(t+1) = A(y(t), t)x(t), \quad (12)$$

$$y(t) = C(t)x(t), \quad (13)$$

where $x = (W_1, \dots, W_{N-1}, U_1, \dots, U_{N-1})^T$, $y = R_p - R_N$,

$$A(y(t), t) = \begin{pmatrix} \alpha_1(y(t), t) & \dots & 0 & 1 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & \alpha_{N-1}(y(t), t) & 0 & \dots & 1 \\ 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 \end{pmatrix} \quad (14)$$

with $\alpha_i(y(t), t) = \frac{1 + R_i(t)}{1 + R_N(t) + y(t)}$, $i = 1, \dots, N-1$, and

$$C(t) = (R_1(t) - R_N(t) \dots R_{N-1}(t) - R_N(t), 0, \dots, 0). \quad (15)$$

3.1 Observability Analysis

We consider the problem of determining state $x(t) \in \mathbb{R}^{2(N-1)}$ from available returns $y(t) = R_p(t) - R_N(t)$ and $R_i(t)$, $i = 1, \dots, N$ only. This problem consists in seeking a solution $x(t)$ to the following set of equations:

$$\begin{aligned} y(t) &= C(t)x(t) \\ y(t+1) &= C(t+1)A(y(t), t)x(t) \\ &\vdots \\ y(t+2N-3) &= C(t+2N-3) \\ &\times \prod_{i=1}^{2N-3} A(y(t+2N-3-i), t+2N-3-i)x(t) \end{aligned} \quad (16)$$

If matrix \mathcal{O} defined by

$$\begin{pmatrix} C(t) \\ C(t+1)\Phi(t+1, t) \\ \vdots \\ C(t+2N-3)\Phi(t+2N-3, t) \end{pmatrix}$$

where $\Phi(l, k) = A(y(l-1), l-1)A(y(l-2), l-2)\dots A(y(k), k)$, ($l \geq k$), $\Phi(k, k) = I_{2(N-1)}$, is invertible, then $x(t)$ is given by $x(t) = \mathcal{O}^{-1}Y$, where $Y = (y(t), y(t+1), \dots, y(t+2N-3))^T$.

Theorem 1. Provided that $\forall t$, $1 + R_N(t) + y(t) \neq 0$, and $R_i(t) \neq R_N(t)$, $R_i(t) \neq -1$, $i = 1, \dots, N-1$, the system is completely uniformly observable in the sense that $x(t)$ (the vector of the $N-1$ asset holdings and $N-1$ holding changes) is always uniquely defined in $2(N-1)$ time samples, by $x(t) = \mathcal{O}^{-1}Y$, where both \mathcal{O} and Y only depend on returns known on interval $[t, t+2N-3]$.

Elements of proof: The proof relies on an observability condition of time-varying linear systems with a block-triangular state matrix A : System $x(t+1) = A(t)x(t)$, $y(t) = C(t)x(t)$ is completely uniformly observable¹, where $A(t)$, $C(t)$ are bounded functions of t , if output matrix $C(t)$ does not contain any column equal to zero (here $R_i(t) \neq R_N(t)$, $i = 1, \dots, N-1$, $\forall t$), provided that the $A(t)$'s are uniformly regular (that is guaranteed here if $R_i(t) \neq -1$, $i = 1, \dots, N-1$, $\forall t$). $1 + R_N(t) + y(t) \neq 0$ is also needed to avoid singularities of $A(t)$. Under the assumptions of theorem 1, it can be easily proved that the

observability Gramian $\sum_{l=k-\sigma}^k \Phi(l, k)^T C_l^T C_l \Phi(l, k)$ is always positive definite.

Remarks: The assumption that the returns are not equal to -1 is not restrictive, since, in normal operations, such situations do not occur. If a return is equal to -1 , the asset value decreases to 0 meaning for example for a stock that the firm makes default. Furthermore, if $R_i(t) = R_N(t)$, it means that assets i and N are not distinguishable; it can exceptionally occur at isolated dates. In such situations, the loss of observability is not likely to append.

¹ $\exists \sigma, \alpha_1, \alpha_2$ such that $0 < \alpha_1 I_N \leq \sum_{l=k-\sigma}^k \Phi(l, k)^T C_l^T C_l \Phi(l, k) \leq \alpha_2 I_N$, where $\Phi(l, k) = A(l-1)A(l-2)\dots A(k)$, ($l > k$), $\Phi(k, k) = I_d$.

3.2 Observer Design

Rather than using a direct inversion technique based on system (16) at each time sample t , which would be computationally demanding for large portfolios, an asymptotic estimation approach based on a deterministic exponential forgetting factor observer in discrete time is proposed in this paper (see Ticlea & Besançon (2013)). First we observe that the asset holding dynamics is linear with respect to the state vector (since state matrix A depends only on $y(t)$ and t). This renders possible the use of an asymptotic linear state observer.

The proposed approach consists in solving the following deterministic optimization problem at each time sample $t > 0$:

$$\begin{aligned} \min_{\hat{x}_0^t} & \lambda^t (\hat{x}_0^t - \hat{x}_0)^T P_0^{-1} (\hat{x}_0^t - \hat{x}_0) \\ & \sum_{l=0}^t \lambda^{t-l} (y(l) - C(l)\Phi(l,0)\hat{x}_0^t)^T \\ & \quad \times R_t^{-1} (y(l) - C(l)\Phi(l,0)\hat{x}_0^t) \end{aligned} \quad (17)$$

with $\lambda \in (0, 1)$, $P_0 = P_0^T > 0$, $R_t = R_t^T > 0$, and where \hat{x}_0^t and \hat{x}_0 denote the state estimate at time sample t and an initial guess of the state, respectively,

subject to

$$\hat{x}_t = \Phi(t, 0)\hat{x}_0^t. \quad (18)$$

Similarly to the classical discrete-time Kalman filter, the solution of this optimal linear-quadratic inverse problem can be easily computed in a recursive way (see Ticlea & Besançon (2013) for a detailed derivation of the result), according to the following update equations:

$$\hat{x}_{t|t} = \hat{x}_{t|t-1} - K_t(C(t)\hat{x}_{t|t-1} - y_t), \quad \hat{x}_{0|-1} = \hat{x}_0, \quad (19)$$

$$P_{t|t} = P_{t|t-1} - K_t C(t) P_{t|t-1}, \quad P_{0|-1} = P_0, \quad (20)$$

$$(21)$$

with

$$K_t = P_{t|t-1} C(t)^T (C(t) P_{t|t-1} C(t)^T + R_t)^{-1}, \quad (22)$$

and time update equations

$$\hat{x}_{t+1|t} = A(y(t), t)\hat{x}_{t|t}, \quad (23)$$

$$P_{t+1|t} = \lambda^{-1} A(y(t), t) P_{t|t} A(y(t), t)^T. \quad (24)$$

4. AN ILLUSTRATIVE EXAMPLE

4.1 Case Study Description

As an illustration, we choose to work on a portfolio invested in the French stock market through sector indices. We could have chosen to directly work on a large number of stocks but the graphs would not have been easily readable. The sector indices are the 10 CAC sector indices based on the first level of the Industry Classification Benchmark (ICB 1). The ten industries are: oil and gas, basic materials, industrials, consumer goods, health

care, consumer services, telecommunications, utilities, financials, technology. The data have been extracted from <https://finance.yahoo.com>. Returns are calculated on a daily basis as the difference between two successive close prices divided by the first price. The simulations use 150 successive daily returns for each index. Each day, the investor can modify the holdings in his portfolio by selling and buying assets.

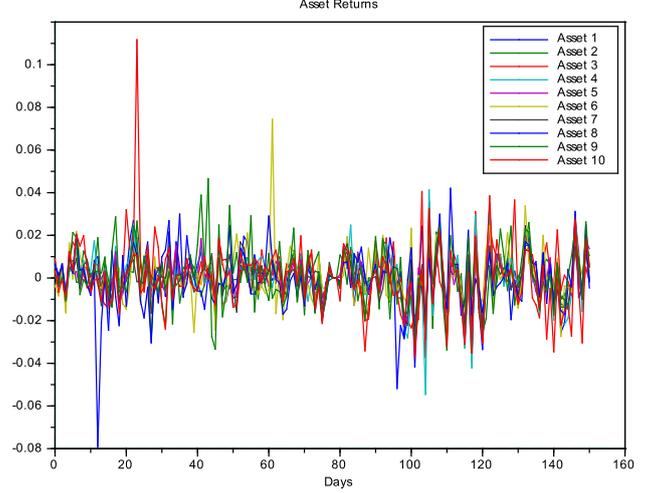


Fig. 1. Asset returns over 150 days.

All the simulations have been performed using Scilab Software (<http://www.scilab.org/fr>) with the observer parameters depicted in Table 1. In all the simulations, an equally weighted portfolio is used as initial estimate.

Table 1. Observer parameters

λ	R_t	P_0
0.65	1	$\begin{pmatrix} 1e^2 \times I_{N-1} & \mathcal{O}_{N-1 \times N-1} \\ \mathcal{O}_{N-1 \times N-1} & 1e^{-3} \times I_{N-1} \end{pmatrix}$

4.2 Determination of the components of a portfolio

In this subsection, the ability of the observer to determine the composition of the portfolio is investigated. For clarity of presentation and without restriction of the approach, the estimation of the composition of a portfolio containing only 5 assets among 10 potential ones is simulated without holding changes decided by the portfolio manager and by only considering changes due to the asset returns. Assets 1, 4, 6, 8, and 9 are supposed to be not included in the portfolio. Fig. 2 illustrates the observer convergence, while Fig. 3 demonstrates the effectiveness of the portfolio identification, since the estimated holdings \hat{W}_1 , \hat{W}_4 , \hat{W}_6 , \hat{W}_8 , and \hat{W}_9 converge towards zero. As predicted by the theory and with the parameters given in Table 1, the convergence is ensured in about 20 time samples (we recall that the maximum attainable rate of convergence corresponds to $2(N-1)$ time samples).

4.3 Estimation in Presence of some One-day Changes

In order to facilitate interpretation, we consider an estimation problem with only 4 assets. Here we investigate

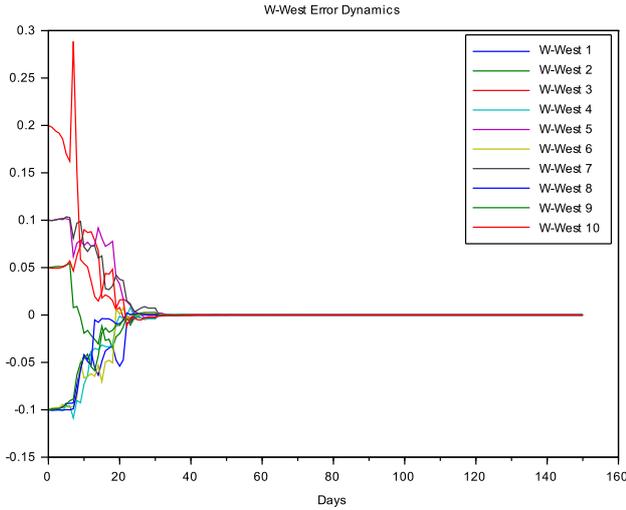


Fig. 2. Estimation error dynamics.

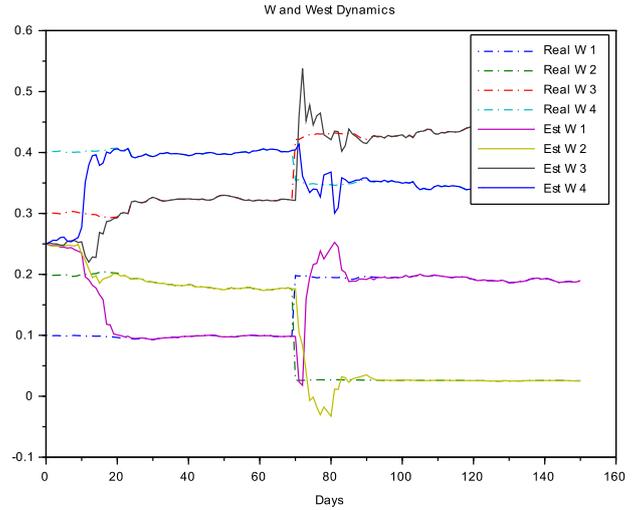


Fig. 4. Estimation convergence with day-70 changes.

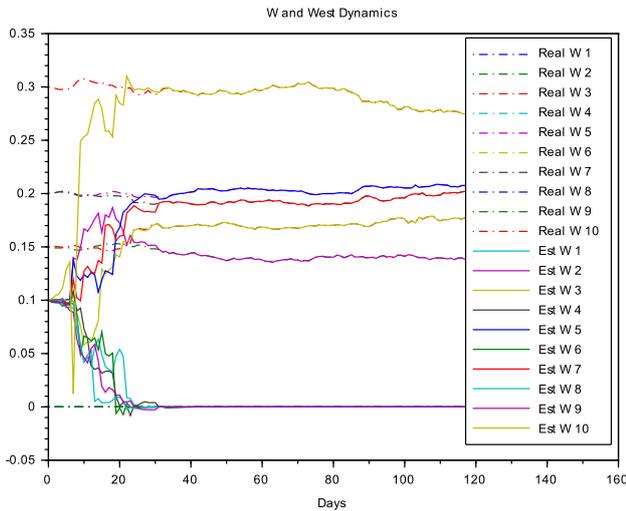


Fig. 3. Detection of the portfolio composition.

the ability of the observer to recover the asset holdings of a 4-asset portfolio based on the same data set, when $U_1(t) = 0.1$, $U_2(t) = -0.15$, and $U_3(t) = 0.1$, and $U_4(t) = -0.05$, at day $t = 70$. Fig. 4 and 5 show that the observer is able to identify the average dynamical behaviour of the holdings.

4.4 Estimation of Quasi Piecewise Constant Holding Changes

In this example, the occurrence of 3 unknown daily holding changes ($U_1(t) = 0.0015 + d_1(t)$, $U_2(t) = -0.002 + d_2(t)$, and $U_3(t) = 0.0005 + d_3(t)$), where the $d_i(t)$'s are some uniform random processes on interval $[-4e^{-4}, +4e^{-4}]$, applied on time period $[20, 50]$. This investigates the ability of the observer to estimate the average of daily changes. Fig. 6 shows that the observer is able to recover the average holding changes of the portfolio. Fig. 7 shows that the

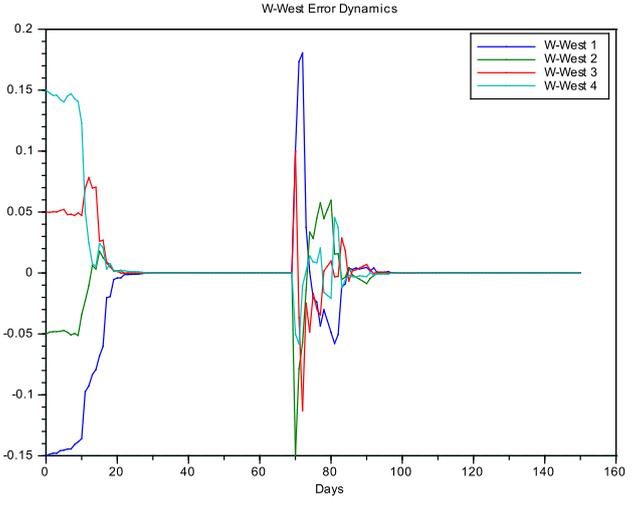


Fig. 5. Estimation error dynamics with day-70 changes.

observer is able to produce a nice estimation of the asset holdings.

5. CONCLUSIONS AND PERSPECTIVES

In this paper, the important issue of estimating components and changes in portfolios has been addressed by using a discrete-time asymptotic observer design.

To overcome the issue of dealing with a large set of potential assets, optimal linear-quadratic estimation approaches such as (17)-(18) may be not suitable due to the bottleneck of large matrix $P_{t|t}$ computation. A research perspective would be to investigate the use of Ensemble Kalman Filtering techniques (Evensen (2009)) or some similar approaches such as Particle Filters (see Del Moral (1996) for instance). This work contributes to improving transparency on the markets by revealing actions of fund managers. The SEC is now changing the rules, increasing

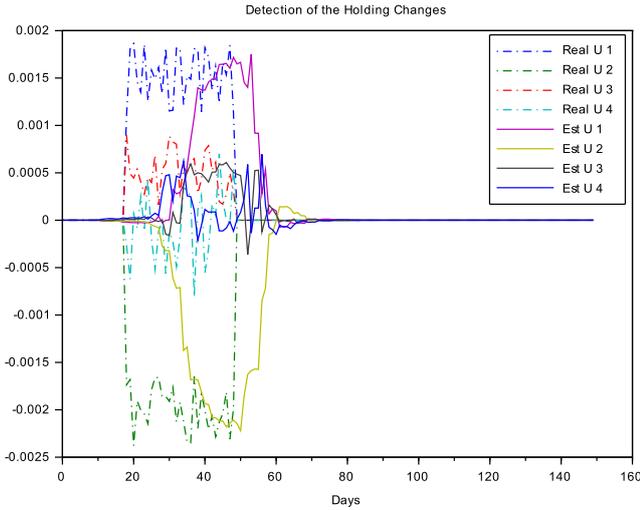


Fig. 6. Detection of the holding changes.

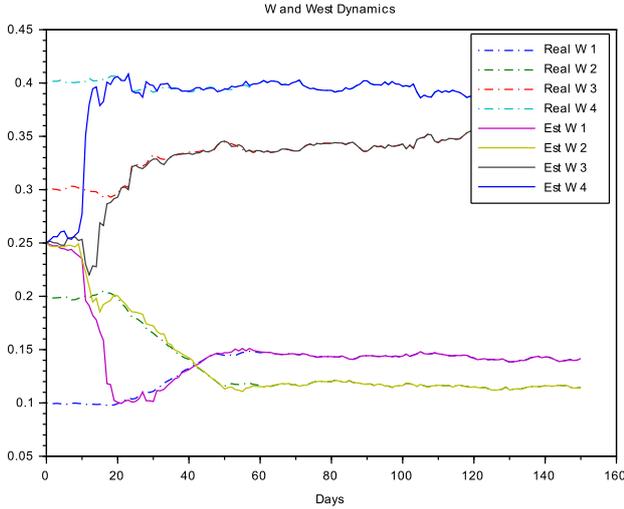


Fig. 7. Illustration of the estimation convergence.

the frequency of mutual fund disclosure from quarterly to monthly reporting, showing how crucial this question is.

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Appendix A. DERIVATION OF THE HOLDING DYNAMICS

Let $V_i(t)$ denote the amount invested in asset i at time t and $V(t)$, the portfolio value at same time t . The part invested in asset i at time t , denoted as $W_i(t)$ is given by

$$W_i(t) = \frac{V_i(t)}{V(t)}. \quad (\text{A.1})$$

If $R_i(t)$ denotes the rate of return of asset i between t and $t+1$, it follows that

$$V_i(t+1) = (1 + R_i(t))V_i(t), \quad (\text{A.2})$$

$$V(t+1) = \sum_{j=1}^N (1 + R_j(t))V_j(t). \quad (\text{A.3})$$

Consequently, if the asset manager does not buy or sell the asset, the input-free model is given by

$$\begin{aligned} W_i(t+1) &= \frac{(1 + R_i(t))V_i(t)}{\sum_{j=1}^N (1 + R_j(t))V_j(t)} = \frac{(1 + R_i(t))\frac{V_i(t)}{V(t)}}{\sum_{j=1}^N (1 + R_j(t))\frac{V_j(t)}{V(t)}} \\ &= \frac{(1 + R_i(t))W_i(t)}{\sum_{j=1}^N (1 + R_j(t))W_j(t)}. \end{aligned} \quad (\text{A.4})$$

In case of holding changes $U_i(t)$ between t and $t+1$, we finally get

$$W_i(t+1) = \frac{(1 + R_i(t))W_i(t)}{\sum_{j=1}^N (1 + R_j(t))W_j(t)} + U_i(t) \quad (\text{A.5})$$