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# Estimates of solutions of elliptic equations with a source reaction term involving the product of the function and its gradient

Marie-Françoise Bidaut-Véron  
Marta Garcia-Huidobro  
Laurent Véron

**Abstract** We study local and global properties of positive solutions of  $-\Delta u = u^p |\nabla u|^q$  in a domain  $\Omega$  of  $\mathbb{R}^N$ , in the range  $p+q > 1$ ,  $p \geq 0$ ,  $0 \leq q < 2$ . We first prove local Harnack inequality and nonexistence of positive solutions in  $\mathbb{R}^N$  when  $p(N-2)+q(N-1) < N$  or in an exterior domain if  $p(N-2)+q(N-1) < N$  and  $0 \leq q < 1$ . Using a direct Bernstein method we obtain a first range of values of  $p$  and  $q$  in which  $u(x) \leq c(\text{dist}(x, \partial\Omega))^{\frac{q-2}{p+q-1}}$ . This holds in particular if  $p+q < 1 + \frac{4}{n-1}$ . Using an integral Bernstein method we obtain a wider range of values of  $p$  and  $q$  in which all the global solutions are constants. Our result contains Gidas and Spruck nonexistence result as a particular case. We also study solutions under the form  $u(x) = r^{\frac{q-2}{p+q-1}}\omega(\sigma)$ . We prove existence, nonexistence and rigidity of the spherical component  $\omega$  in some range of values of  $N$ ,  $p$  and  $q$ .

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*Key words.* elliptic equations; Bernstein methods; gradient estimates; global solutions; bifurcations.

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## 1 Introduction

The aim of this article is to study local and global properties of positive solutions of the following type of equations

$$-\Delta u = u^p |\nabla u|^q, \quad (1.1)$$

in  $\Omega \setminus \{0\}$  where  $\Omega$  is an open subset of  $\mathbb{R}^N$  containing 0,  $p$  and  $q$  are real exponents. In many cases we will assume the superlinearity of the right-hand side, i.e.  $p + q - 1 > 0$  and  $0 \leq q \leq 2$ . Equation (1.1) is invariant under the action of the transformations  $T_\sigma$  defined for  $\sigma > 0$  by

$$T_\sigma[u](x) = \sigma^{\frac{2-q}{p+q-1}} u(\sigma x). \quad (1.2)$$

If we look for radial positive solutions under the form  $u(x) = \Lambda |x|^{-\gamma}$  we find, if  $q < 2$  and  $p + q - 1 > 0$ ,  $\gamma := \gamma_{p,q} = \frac{2-q}{p+q-1}$  and

$$\Lambda := \Lambda_{N,p,q} = \gamma_{p,q}^{\frac{1-q}{p+q-1}} \left( N - \frac{2p+q}{p+q-1} \right)^{\frac{1}{p+q-1}}. \quad (1.3)$$

However this last quantity exists if and only if the exponents belong to the *supercritical range*, that is when

$$(N - 2)p + (N - 1)q > N. \quad (1.4)$$

In the *subcritical range* of exponents i.e. when

$$(N - 2)p + (N - 1)q < N, \quad (1.5)$$

we prove in Theorem 2.1 that Serrin's classical results (see [16], [17]) can be applied. We obtain a local Harnack inequality and an *a priori* estimate for positive solution  $u$  in  $B_R \setminus \{0\}$  under the form

$$u(x) + |x| |\nabla u(x)| \leq c |x|^{2-N} \quad \forall x \text{ s.t. } 0 < |x| \leq \frac{R}{2}, \quad (1.6)$$

with a constant  $c$  depending on  $u$ . We also prove the following surprising result:

**Theorem A** *Let  $N \geq 2$ ,  $p \geq 0$  and  $0 \leq q < 1$  satisfy (1.5). If  $u$  is a nonnegative  $C^1$  function in an exterior domain  $\Omega = B_R^c$ , where it satisfies*

$$-\Delta u \geq u^p |\nabla u|^q, \quad (1.7)$$

then there exists  $R' > R$  such that  $u$  is constant in  $B_{R'}^c$ .

Furthermore such solutions truly exist as the radial case shows it (see Theorem 2.3). When  $q = 0$ , it is known since the last millenium that any nonnegative solution of (1.7) is zero, however the proof in that case is straightforward since the spherical average of a solution satisfies the same inequality (see [9] in the radial case). If  $B_R^c$  is replaced by  $\mathbb{R}^N$  the fact that all nonnegative functions satisfying (1.7) are constant is due to Mitidieri and Pohozaev [15]. For the sake of completeness we give a slightly different proof which introduces the techniques we developed throughout our article.

Our main results deal with the supercritical range. We prove *a priori* estimates of positive solutions of (1.1) in a punctured domain and existence of ground states in  $\mathbb{R}^N$ . There are two approaches for obtaining these results. The *direct Bernstein method* and the *integral Bernstein method* popularized by Lions [14] and Gidas and Spruck in [10] respectively. Both methods are based upon differentiating the equation. The direct Bernstein method relies on obtaining pointwise estimates of the gradient through comparison principles via algebraic computations, an intensive use of Young's inequality and without any integration. Our main result in this framework is the following:

**Theorem B** *Let  $N \geq 2$ ,  $0 \leq q < 2$  and  $p \geq 0$  be such that  $p + q - 1 > 0$ . If  $u$  is a positive solution of (1.1) in  $B_R$  and one of the following assumptions is fulfilled,*

$$(i) \ p + q - 1 < \frac{4}{N-1},$$

$$(ii) \ 0 \leq p < 1 \text{ and } p + q - 1 < \frac{(p+1)^2}{p(N-1)}.$$

*Then there exist positive constants  $a = a(N, p, q)$  and  $c_1 = c_1(N, p, q)$  such that*

$$|\nabla u^a(0)| \leq c_1 R^{-1-a \frac{2-q}{p+q-1}}. \quad (1.8)$$

The value of the exponent  $a$  is not easy to compute, however, in several applications this difficulty can be bypassed. As a consequence of (1.8) there holds,

**Corollary B-1** *Under the assumptions on  $N$ ,  $p$  and  $q$  of Theorem B, any positive solution of (1.1) in  $\mathbb{R}^N$  is constant.*

Another consequence is the following,

**Corollary B-2** *Let  $\Omega$  be a smooth domain in  $\mathbb{R}^N$ ,  $N \geq 2$ , with a bounded boundary,  $0 \leq q < 2$  and  $p \geq 0$  such that  $p + q - 1 > 0$  and assume one of the assumptions (i)-(ii) of Theorem B holds. If  $u$  is a positive solution of (1.1) in  $\Omega$  there exists  $d_0 > 0$  depending on  $\Omega$  and  $c_2 = c_2(N, p, q) > 0$  such that*

$$u(x) \leq c_2 \left( (\text{dist}(x, \partial\Omega))^{-\frac{2-q}{p+q-1}} + \max\{u(z) : \text{dist}(z, \Omega) = d_0\} \right). \quad (1.9)$$

The aim of the integral Bernstein method is to obtain estimates of the  $L^r$ -norm of the gradient of the solutions in balls for  $r$  large enough. Combined with [16] this leads in [10] to Harnack inequality. Here we use these integral estimates to prove the non-existence of non-constant global solutions.

**Theorem C** Assume  $p \geq 0$ ,  $0 \leq q < 2$  and define the polynomial  $G$  by

$$G(p, q) = ((N - 1)^2 q + N - 2) p^2 + b(q) p - N q^2, \quad (1.10)$$

where  $b(q) = N(N - 1)q^2 - (N^2 + N - 1)q - N - 2$ .

If the couple  $(p, q)$  satisfies the inequality  $G(p, q) < 0$ , then all the positive solutions of (1.1) in  $\mathbb{R}^N$  are constant.

In the range of  $p$  and  $q$ , the condition  $G_N(p, q) < 0$  is equivalent to

$$0 \leq p < p_c(q) := \frac{-b(q) + \sqrt{b^2(q) + 4Nq^2((N - 1)^2 q + N - 2)}}{2((N - 1)^2 q + N - 2)}. \quad (1.11)$$

If  $q = 0$ , the above reads

$$0 \leq p < p_c(0) := \frac{N + 2}{N - 2}, \quad (1.12)$$

which is the well known condition obtained by Gidas and Spruck in [10]. Furthermore, it can be verified that the domain of  $(p, q)$  in which Theorem B applies is included into the set of  $(p, q)$  where  $G(p, q) < 0$ . Our proof is extremely technical and necessitates a long appendix in which many algebraic computations are carried out.

If we just look for radial solutions we present in Theorem 3.1 an optimal result namely:

**Theorem D** There exist non-constant radial positive solutions of (1.1) in  $\mathbb{R}^N$  if and only if  $p \geq 0$ ,  $0 \leq q < 1$  and

$$p(N - 2) + q(N - 1) \geq N + \frac{2 - q}{1 - q}. \quad (1.13)$$

If equality holds in (1.13), there exists an explicit one parameter family of positive radial solutions of (1.1) in  $\mathbb{R}^N$  under the form

$$u_c(r) = c \left( K c^{\frac{(2-q)^2}{(N-2)(1-q)}} + r^{\frac{2-q}{1-q}} \right)^{-\frac{(N-2)(1-q)}{2-q}}, \quad (1.14)$$

for any  $c > 0$  and some  $K = K(N, q) > 0$ .

In the last section we study the singular separable solutions of (1.1) written under the form

$$u(x) = u(r, \sigma) = r^{-\gamma_{p,q}} \omega(\sigma) \quad (r, \sigma) \in \mathbb{R}_+^* \times S^{N-1}. \quad (1.15)$$

Then  $\omega$  satisfies the following nonlinear equation on  $S^{N-1}$

$$-\Delta' \omega + \gamma_{p,q} \left( N - \frac{2p + q}{p + q - 1} \right) \omega - |\omega|^{p-1} \omega \left( \gamma_{p,q}^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{q}{2}} = 0, \quad (1.16)$$

where  $\nabla'$  and  $\Delta'$  are respectively the covariant gradient and the Laplace Beltrami operator on  $S^{N-1}$ . It is clear by integration that condition (1.4) is a necessary and sufficient condition for

the existence of solution, and the constant function  $\Lambda_{N,p,q}$  is such a solution. We introduce a more general equation on  $S^{N-1}$

$$-\Delta'\omega + \mu\omega - |\omega|^{p-1}\omega \left( \gamma^2\omega^2 + |\nabla'\omega|^2 \right)^{\frac{q}{2}} = 0, \quad (1.17)$$

where  $\mu$  and  $\gamma$  are positive real numbers. We have the following universal estimate.

**Theorem F** *Assume  $0 \leq q < 2$ ,  $p \in \mathbb{R}$  such that  $p + q - 1 > 0$ . If*

$$p(N - 3) + q(N - 2) < N - 1, \quad (1.18)$$

*holds, there exists  $c_4 = c_4(N, p, q) > 0$  and  $a = a(N, p, q) > 0$  such that for  $\gamma, \mu > 0$ , any solution  $\omega$  of (1.17) on  $S^{N-1}$  satisfies*

$$\|\omega\|_{L^\infty} \leq c_4 \mu^a \gamma^{-\frac{q}{p+q-1}}. \quad (1.19)$$

We also give a rigidity result which shows that the solutions which are not too far from being constant are indeed constant.

**Theorem F** *Assume  $p \geq 0$  and  $p + q - 1 > 0$ . Let  $\omega$  be a solution of (1.17) satisfying*

$$c_2^2 \leq \gamma^2\omega^2 + |\nabla'\omega|^2 \leq c_1^2, \quad (1.20)$$

*for some  $c_1 > c_2 > 0$  and set*

$$c_* = \begin{cases} c_1^{p+q-1} & \text{if } p \geq 1 \\ c_2^{p-1} c_1^q & \text{if } 0 \leq p < 1. \end{cases} \quad (1.21)$$

*If*

$$c_* \leq \frac{2(N-1+\mu)\gamma^p}{q\sqrt{N-1} + 2(p+q)\gamma}, \quad (1.22)$$

*then  $\omega$  is constant.*

In the Appendix we present many technical algebraic computations which leads to the delimitation of the regions of the  $(p, q)$ -plane in which Theorem C holds. Many computations can be easily verified by using Maple. Throughout the paper  $c$  denotes a generic constant depending on some parameters, specified in some cases, the value of which may change from one occurrence to another.

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## 2 Local estimates

### 2.1 The subcritical case: Proof of Theorem A

We show how the use of Serrin's result concerning Harnack inequality yields a blow-up estimate of any positive solution  $u$  of (1.1) in a punctured domain.

**Theorem 2.1** *Let  $\Omega \subset \mathbb{R}^N$  be a domain containing 0,  $N \geq 3$ ,  $p \geq 0$ ,  $0 \leq q \leq 2$  and (1.5) holds. If  $u \in C^2(\Omega \setminus \{0\})$  is a positive solution of (1.1) in  $\Omega \setminus \{0\}$ , then estimate (1.6) holds in a neighborhood of 0.*

*Proof.* Assume  $\overline{B_1} \subset \Omega$ . By Brezis-Lions's result [6] there holds

$$u \in M^{\frac{N}{N-2}}(B_1), \quad \nabla u \in M^{\frac{N}{N-1}}(B_1), \quad u^p |\nabla u|^q \in L^1(B_1), \quad (2.1)$$

where  $M^r = L^{r,\infty}$  denotes the Marcinkiewicz space or Lorentz space of index  $(r, \infty)$ , and there exists  $\alpha \geq 0$  such that

$$-\Delta u = u^p |\nabla u|^q + \alpha \delta_0 \quad \text{in } \mathcal{D}(B_1). \quad (2.2)$$

We assume first  $pq \neq 0$ . In order to fit with Serrin's formalism, we write  $u^p |\nabla u|^q = \mathcal{B}(u, \nabla u)$ . Hence  $\mathcal{B}$  satisfies the estimate

$$|\mathcal{B}(u, \nabla u)| \leq |u|^{p\theta} + |\nabla u|^{q\theta'} = c|u| + d|\nabla u|, \quad (2.3)$$

where  $\theta, \theta' \geq 1$ ,  $\frac{1}{\theta} + \frac{1}{\theta'} = 1$ ,  $c = |u|^{p\theta-1}$ ,  $d = |\nabla u|^{q\theta'-1}$ . If  $\theta > \max\{1, \frac{1}{p}\}$  and  $\theta' > \max\{1, \frac{1}{q}\}$ , then  $c \in M^{\frac{N}{(N-2)(p\theta-1)}}$  and  $d \in M^{\frac{N}{(N-1)(q\theta'-1)}}$ . We claim that we can choose  $\theta > 1$  such that

$$\frac{N}{(N-2)(p\theta-1)} > \frac{N}{2} \quad \text{and} \quad \frac{N}{(N-1)(q\theta'-1)} > N. \quad (2.4)$$

These inequalities are respectively equivalent to

$$\theta < \frac{N}{p(N-2)} \quad \text{and} \quad \theta' < \frac{N}{q(N-1)}, \quad (2.5)$$

which is clearly possible from (1.5) by taking  $\theta = \frac{N(1-\epsilon)}{p(N-2)}$  for  $\epsilon > 0$  small enough. Because  $M^r(B_1) \subset L^{r-\delta}(B_1)$  for any  $\delta > 0$ , we infer that  $c \in L^{\frac{N}{2}+\delta}(B_1)$  and  $d \in L^{N+\delta}(B_1)$  and  $u$  verifies Harnack inequality in  $B \setminus \{0\}$  by [16, Th 5]. This implies

$$\max_{|x|=r} u(x) \leq K \min_{|x|=r} u(x) \quad \forall r \in (0, \frac{1}{2}]. \quad (2.6)$$

The spherical average  $\bar{u}$  of  $u$  on  $\{x : |x| = r\}$  is superharmonic. Hence there exists some  $m \geq 0$  such that

$$\bar{u}(r) = mr^{2-N}. \quad (2.7)$$

Combined with (2.6) it yields  $u(x) \leq Km|x|^{2-N}$ . The estimate on the gradient is standard, see eg [19, Lemma 3.3.2].  $\square$

*Remark.* Estimate (1.6) is not universal since the constant  $K$  in (2.6) depends on the norms of  $c$  and  $d$  which could depend not only on  $N, p, q$ , but on the solution itself.

The following result is not new, except in the case  $p = 0, q = 1$ . It was proved in [15, Th 15.1] for  $p+q-1 > 0$  and extended to quasilinear operator by simulating the change of unknown  $u = v^b$ . It was used in order to derive a priori estimates [4]. Later on it was extended to more general operator in [8] where the new cases  $p+q-1 < 0$  and  $p+q-1 = 0$  with  $p > 0$  have been considered using a delicate proof. We give here a very general but general proof of all these results. Furthermore our method highlights the role of change of unknown function and which forshadows the method used in Theorem B, is also extendeable to very general quasilinear operators such as

$$-divA(x, u, \nabla u) \geq B(x, u, \nabla u) \quad \text{in } \mathbb{R}^N, \quad (2.8)$$

under the assumptions that  $\langle A(x, r, \xi), \xi \rangle \geq |\xi|^m$ ,  $B(x, r, \xi)r \leq c|r|^p|\xi|^q$ , and under the corresponding subcritical condition  $p(N-m) + q(N-1) < N(p-1)$ .

**Theorem 2.2** *Assume  $N \geq 3$ ,  $p$  and  $q$  are nonnegative and (1.5) holds. Then the only positive functions  $u \in C^1(\mathbb{R}^N)$  satisfying*

$$-\Delta u \geq u^p |\nabla u|^q, \quad (2.9)$$

in  $\mathbb{R}^N$  are the constants.

*Proof.* Assume  $u$  is such a solution. For  $p+q \neq 1$ , we set  $u = v^b$  with  $b(b-1) > 0$ .

$$-b\Delta v \geq b(b-1)\frac{|\nabla v|^2}{v} + |b|^q v^s |\nabla v|^q,$$

with

$$s = 1 - q + b(p+q-1).$$

If  $s > 0$ , then from Hölder's inequality,

$$|\nabla v|^{\frac{2s+q}{s+1}} = \left( \frac{|\nabla v|^2}{v} \right)^{\frac{s}{s+1}} v^{\frac{s}{s+1}} |\nabla v|^{\frac{q}{s+1}} \leq \delta^{\frac{s+1}{s}} \frac{|\nabla v|^2}{v} + \delta^{-1-s} v^s |\nabla v|^q,$$

for any  $\delta > 0$ . Hence, by chosing  $\delta$ , we see that there exists  $c > 0$  such that

$$-b\Delta v \geq c |\nabla v|^\omega \quad \text{where } \omega = \frac{2s+q}{s+1} = \frac{2-q+2b(p+q-1)}{2-q+b(p+q-1)}. \quad (2.10)$$

(i) In the case  $p+q-1 > 0$ , we take  $b = 1 + \epsilon$ , for  $\epsilon > 0$ . Then  $s = p + \epsilon(p+q-1) > 0$ , and  $s > 1 - q$ , thus  $\omega > 1$ . Form assumption (1.5) we can take  $\epsilon > 0$  small enough such that  $(N-2)s + (N-1)q < N$ , which is equivalent to  $\omega < \frac{N}{N-1}$ .

(ii) If  $p+q-1 < 0$ , hence  $0 \leq q < 1$ , we take  $b = -\epsilon$ , for  $\epsilon > 0$ . Then

$$-|b|\Delta v + c |\nabla v|^\omega \leq 0, \quad \text{where } c > 0. \quad (2.11)$$

and  $s = 1 - q - \epsilon(p+q-1) > 1 - q > 0$ . hence  $\omega = 1 - \frac{p+q-1}{2-q} > 1$  and we can choose  $\epsilon$  small enough such that  $\omega < \frac{N}{N-1}$ .



(iii) If  $p + q - 1 = 0$ , we set  $u = e^v$  (where  $v$  is a signed function) and derive that for any  $\tilde{\omega} \in (q, 2)$  one can find  $\tilde{c} > 0$  such that

$$-\Delta v \geq |\nabla v|^2 + e^{p+q-1} |\nabla v|^q \geq \tilde{c} |\nabla v|^{\tilde{\omega}}.$$

We can take in particular  $\tilde{\omega} < \frac{N}{N-1}$ .

Finally, let  $R > 0$  and  $\zeta \in C_0^\infty(\mathbb{R}^N)$ , with values in  $[0, 1]$ , such that  $\zeta = 1$  on  $B_{\frac{R}{2}}$ ,  $\zeta = 0$  on  $B_R^c$  and  $|\nabla \zeta| \leq 2R^{-1}$ . Then, in each of the three cases, there exists  $c_1 > 0$  such that

$$\begin{aligned} \int_{B_R} \zeta^{\omega'} |\nabla v|^\omega dx &\leq c_1 \left| \int_{B_R} \langle \nabla v, \nabla \zeta \rangle \zeta^{\omega'-1} dx \right| \\ &\leq \frac{1}{2} \int_{B_R} \zeta^{\omega'} |\nabla v|^\omega dx + c_2 \int_{B_R} |\nabla \zeta|^{\omega'} dx, \end{aligned}$$

from which follows

$$\int_{B_R} |\nabla v|^\omega dx \leq 2C \int_{B_R} |\nabla \zeta|^{\omega'} dx \leq C' R^{N-\omega'}.$$

which implies the claim since  $\omega' > N$ .  $\square$

If the inequality (2.10) is considered in an exterior domain, the situation differs according  $0 \leq q > 1$  or  $1 \leq q < 2$ . The case  $p = 0$  is known for a long time.

**Theorem 2.3** *Let  $N \geq 3$ ,  $p \geq 0$ ,  $0 \leq q \leq 2$  and (1.5) holds in  $G := \overline{B}_1^c$ .*

(i) *If  $0 \leq q < 1$  any positive function  $u \in C^1(\overline{G})$  satisfying (2.9) in  $G$  is constant on  $B_R^c$  for some  $R \geq 1$ .*

(ii) *If  $q \geq 1$  there exist positive functions in  $C^1(G)$  satisfying (2.10) in  $G$  which are nonconstant on  $B_R^c$  for any  $R > 1$ .*

*Proof.* In the next two steps we prove (i).

*Step 1.* The function  $r \mapsto \min\{u(x) : |x| = r\}$  is nondecreasing and the following dichotomy holds:

(ii) either  $r \mapsto \min\{u(x) : |x| = r\}$  is increasing on  $[1, \infty)$ ,

(i) or there exists a minimal  $r_0 \geq 1$  such that  $u$  is constant in  $B_{r_0}^c$ .

For proving this claim we use the transformation  $u = v^b$  as in the proof of Theorem 2.2. For  $\tau > 0$  we set

$$\gamma_\tau(r) = \begin{cases} 1 & \text{if } r \geq 1 \\ 1 - \frac{(1-r)^2}{2\tau} & \text{if } 1 - \tau \leq r \leq 1 \\ r & \text{if } r \leq 1 - \tau, \end{cases} \quad (2.12)$$

and for  $k > 0$   $\gamma_{\tau,k}(r) = k\gamma_\tau(\frac{r}{k})$ . Then

$$\gamma'_{\tau,k}(r) = \begin{cases} 0 & \text{if } r \geq k \\ 1 - \frac{k - k\tau}{k\tau} & \text{if } k(1 - \tau) \leq r \leq k \\ 1 & \text{if } r \leq k(1 - \tau), \end{cases} \quad (2.13)$$

We put  $v_{\tau,k} = \gamma_{\tau,k}(v)$ . Since  $\gamma_{\tau,k}$  is concave,  $0 \leq \gamma'_{\tau,k} \leq 1$  and  $\omega > 1$ , we derive from (2.10),

$$-\Delta v_{\tau,k} \geq -\gamma'_{\tau,k}(v)\Delta v \geq c(\gamma'_{\tau,k}(v))^{1-\omega} |\nabla v_{\tau,k}|^\omega \geq c |\nabla v_{\tau,k}|^\omega, \quad (2.14)$$

in  $B_1^c$ . For  $R > 2\rho \geq 2$ , we consider a test function  $\zeta \in C_0^2(\mathbb{R}^N)$  with value in  $[0, 1]$  such that  $\zeta = 1$  on  $B_{\frac{R}{2}}$ ,  $\zeta = 0$  in  $B_R^c$  and  $|\nabla \zeta| \leq \frac{4}{R}$  and we obtain with the notations of (2.10),

$$\begin{aligned} c \int_{B_R \setminus B_\rho} \zeta^{\omega'} |\nabla v_{\tau,k}|^\omega dx &\leq \omega' \int_{B_R \setminus B_\rho} \langle \nabla v_{\tau,k}, \nabla \zeta \rangle \zeta^{\omega'-1} dx - \int_{\partial B_\rho} \frac{\partial v_{\tau,k}}{\partial \nu} dS \\ &\leq \frac{c}{2} \int_{B_R} \zeta^{\omega'} |\nabla v_{\tau,k}|^\omega dx + C \int_{B_R} |\nabla \zeta|^{\omega'} dx - \int_{\partial B_\rho} \frac{\partial v_{\tau,k}}{\partial \nu} dS, \end{aligned}$$

which implies

$$c \int_{B_{\frac{R}{2}} \setminus B_\rho} |\nabla v_{\tau,k}|^\omega dx \leq c(N)R^{N-\omega'} - 2c \int_{\partial B_\rho} \frac{\partial v_{\tau,k}}{\partial \nu} dS. \quad (2.15)$$

Taking  $b > 1$  such that  $\omega' > N$  as in the proof of Theorem 2.2, we derive by letting  $R \rightarrow \infty$ ,

$$c \int_{B_\rho^c} |\nabla v_{\tau,k}|^\omega dx \leq -2c \int_{\partial B_\rho} \frac{\partial v_{\tau,k}}{\partial \nu} dS. \quad (2.16)$$

Next we choose  $k < \min\{v(x) : |x| = \rho\}$ . Then there exists  $\tau_0 > 0$  such that  $v_{\tau,k}(x) \equiv k$  for  $\rho - \tau \leq |x| \leq \rho + \tau$  for any  $\tau \in (0, \tau_0]$  by continuity. Hence the right-hand side of (2.16) is zero, which implies that  $v_{\tau,k}$  is constant on  $B_\rho^c$  with value  $k$ , and this means  $\min\{v(x) : |x| \geq \rho\} > k$ . Letting  $\tau \downarrow 0$  and  $k \uparrow \min\{v(x) : |x| = \rho\}$ , we infer that

$$\inf\{v(x) : |x| \geq \rho\} = \min\{v(x) : |x| = \rho\} \implies \inf\{u(x) : |x| \geq \rho\} = \min\{u(x) : |x| = \rho\},$$

since  $z \mapsto z^{\frac{1}{b}}$  is monotone increasing on  $\mathbb{R}_+$ . As a consequence  $\rho \mapsto \min\{u(x) : |x| \geq \rho\}$  is nondecreasing, and if there exists  $r' > r > 1$  such that it is constant on  $(r, r')$ , then the minimum of  $u$  on  $\overline{B}_r^c$  is achieved at some point  $x_0 \in \overline{B}_r^c$ . Since  $u$  is superharmonic it implies that  $u$  is locally constant; hence it is constant by connectedness. In case (i),  $\rho \mapsto \min\{u(x) : |x| \geq \rho\}$  is increasing and if  $x_\rho$  is the point where  $\min\{u(x) : |x| = \rho\} = u(x_\rho)$ , then  $u$  is positive in  $\overline{B}_\rho^c$  and in achieves its minimum at  $x_\rho$ . By Hopf boundary lemma it implies

$$\frac{\partial u}{\partial n_{x_\rho}}(x_\rho) > 0 \quad \text{with } n_{x_\rho} = \rho^{-1}x_\rho. \quad (2.17)$$

In both cases there holds  $u(x) \geq \min_{|y|=1} u(y) > 0$  and up to a multiplication, we can consider that  $u$  is a nonnegative solution of

$$-\Delta u \geq |\nabla u|^q, \quad (2.18)$$

in  $B_1^c$  with  $(N-1)q < N$ .

*Step 2: Construction of radial solutions when  $0 < q < 1$ .* If  $v$  is a radial solution of

$$-\Delta u = |\nabla u|^q, \quad (2.19)$$

in  $B_1^c$  it is nondecreasing and satisfies  $-(r^{N-1}u')' = r^{N-1}(u')^q$  in  $(1, \infty)$ . This equation can be explicitly solved by setting  $w(r) = r^{N-1}u'(r)$  which solves  $-w' = r^{(N-1)(1-q)}w^q$  and we get

$$u'(r) = r^{1-N} \left( (u'(1))^{1-q} - \frac{1-q}{N-q(N-1)} \left( r^{N-q(N-1)} - 1 \right) \right)^{\frac{1}{1-q}}, \quad (2.20)$$

as long as  $r < r^*$  for some explicit  $r^*$ . Hence to any  $a > 0$  we denote by  $r_a$  the unique  $r > 1$  such that

$$a^{1-q} = \frac{1-q}{N-q(N-1)} \left( r^{N-q(N-1)} - 1 \right) \iff r_a = \left( \frac{N-q(N-1)}{1-q} a^{1-q} + 1 \right)^{\frac{1}{N-q(N-1)}}.$$

The mapping  $a \mapsto r_a$  is continuous on  $[0, \infty)$ ,  $C^\infty$  on  $(0, \infty)$ , increasing with limit 1 at  $a = 0$  and infinite when  $a \rightarrow \infty$ , and if we set

$$\mu_a = \int_1^{r_a} \left( a^{1-q} - \frac{1-q}{N-q(N-1)} \left( s^{N-q(N-1)} - 1 \right) \right)^{\frac{1}{1-q}} s^{1-N} ds, \quad (2.21)$$

$a \mapsto \mu_a$  defines a continuous increasing function on  $[0, \infty)$ ,  $C^\infty$  on  $(0, \infty)$  with limit 0 at 0 and  $\infty$  at  $\infty$ .

Next, given  $\mu > 0$  and  $R > 1$ , the radial solution of the Dirichlet problem

$$\begin{aligned} -\Delta u &= |\nabla u|^q && \text{in } B_R \setminus \overline{B}_1 \\ u &= 0 && \text{in } \partial B_1 \\ u &= \mu && \text{in } \partial B_R, \end{aligned} \quad (2.22)$$

can be constructed in the following way: Let  $a > 0$  be the unique solution of  $\mu_a = \mu$ . If  $r_a < R$  the solution  $u$  of (2.22) is expressed by

$$u(r) = \begin{cases} \int_1^r \left( a^{1-q} - \frac{1-q}{N-q(N-1)} \left( s^{N-q(N-1)} - 1 \right) \right)^{\frac{1}{1-q}} s^{1-N} ds & \text{if } 0 \leq r \leq r_a \\ \mu & \text{if } r_a < r \leq R. \end{cases} \quad (2.23)$$

If  $r_a \geq R$ , there holds

$$u(r) = \int_1^r \left( a^{1-q} - \frac{1-q}{N-q(N-1)} \left( s^{N-q(N-1)} - 1 \right) \right)^{\frac{1}{1-q}} s^{1-N} ds \quad \text{if } 0 \leq r \leq R. \quad (2.24)$$

We also notice that any radial solution  $v$  of (2.19) in  $B_1^c$  which satisfies  $v'(1) < 0$  is uniquely determined by Cauchy-Lipchitz theorem. It is decreasing and tends to  $-\infty$  at infinity since it is expressed by

$$u(r) = u(1) - \int_1^r \left( (-u'(1))^{1-q} + \frac{1-q}{N-q(N-1)} \left( r^{N-q(N-1)} - 1 \right) \right)^{\frac{1}{1-q}} s^{1-N} ds \quad \forall r > 1. \quad (2.25)$$

At en we prove that all the radial solutions of (2.19) in  $B_1^c$  which are bounded from below have the form (2.23) for some  $a > 0$  with  $R = \infty$ . By Step 1, such a solution  $v$  is nondecreasing.

Since  $u(r) \geq u(1) > 0$ , we set  $w = r^{N-1}u'$  and derive that there holds on the interval  $I$  where  $w > 0$ ,

$$-w' \geq r^{(N-1)(1-q)}u^p w^q \geq r^{(N-1)(1-q)}u^p(1)w^q \implies \left( \frac{w^{1-q}}{1-q} + \frac{r^{N-q(N-1)}u^p(1)}{N-q(N-1)} \right)' \leq 0.$$

If  $I = (1, \infty)$ , it yields a contradiction by letting  $r \rightarrow \infty$ . Hence  $I = (1, r_0)$  and  $u$  is constant on  $[r_0, \infty)$ .

*Step 3: We first prove (i). Construction of the minorant solution.* If  $u$  is not constant in  $B_R^c$  for some  $R > 0$  the sequence  $\{\mu_n\}$  defined by  $\mu_n = \min\{u(x) : |x| = n\}$  is increasing. For  $n \in \mathbb{N}$ ,  $n \geq 2$ , we consider the sequence  $\{u_n\}$  defined by

$$\begin{aligned} -\Delta u_n &= |\nabla u_n|^q && \text{in } \Gamma_{1,n} := B_n \setminus \overline{B_1} \\ u_n &= \mu_1 && \text{on } \partial B_1 \\ u_n &= \mu_n && \text{on } \partial B_{r_n}. \end{aligned} \quad (2.26)$$

Using Step 2, we obtain the following expression for  $u_n$

$$u_n(|x|) = \mu_1 + \int_1^{|x|} \left( (u'_n(1))^{1-q} - \frac{1-q}{N-q(N-1)} (s^{N-q(N-1)} - 1) \right)_+^{\frac{1}{1-q}} s^{1-N} ds. \quad (2.27)$$

This expression, combined with (2.23) and (2.24) shows that  $n \mapsto u_n(r)$  is eventually increasing for  $r$  fixed. Next we compare  $u$  and  $u_n$  in  $\Gamma_{1,n}$ . For  $\epsilon > 0$   $u_n - \epsilon$  is smaller than  $u$  on  $\partial\Gamma_{1,n}$ . If we assume that for any  $\epsilon > 0$  small enough there does not hold  $u \geq u_n - \epsilon$ , then there exists  $\epsilon_0 > \epsilon$  such that the graphs of  $u$  and  $u_n - \epsilon_0$  are tangent at some point  $x_\rho$  such that  $1 < |x_\rho| = \rho < n$ . Since  $u_n$  is radial, there holds

$$u(x_\rho) = \min\{u(x) : |x| = \rho\},$$

and by (2.17),

$$\nabla u(x_\rho) = \frac{\partial u}{\partial n_{x_\rho}}(x_\rho)n_{x_\rho} = u'_n(\rho)n_{x_\rho} \neq 0.$$

Hence we can linearize  $|\nabla u|^q - |\nabla u_n|^q$  near  $x_\rho$  and derive that the function  $w = u - u_n + \epsilon_0$  is nonnegative and satisfies

$$-\Delta w + \langle c(x), \nabla w \rangle \geq 0, \quad (2.28)$$

in  $B_\delta(x_\rho)$  for some  $\delta > 0$  where  $c(x)$  is some bounded vector field. Hence  $w$  is constant in this ball. By connectedness we derive that  $w = 0$  in  $\Gamma_{1,n}$ , contradiction. Letting  $\epsilon \rightarrow 0$  we obtain that  $u \geq u_n$  in  $\Gamma_{1,n}$ . Furthermore since  $u_n$  is positive and super-harmonic in  $\Gamma_{1,n}$  it satisfies for any  $n \geq 2$ ,

$$u_n(x) \geq u_{0,n}(x) := \frac{\mu_1 - \mu_n}{1 - n^{2-N}} |x|^{2-N} + \frac{\mu_n - n^{2-N}\mu_1}{1 - n^{2-N}} \quad \forall 1 \leq |x| \leq n. \quad (2.29)$$

Hence, letting  $n \rightarrow \infty$  we infer that  $u_n$  increases and converges locally uniformly as well as its gradient to a positive radial solution  $u_\infty$  of (2.19) in  $B_1^c$  which satisfies  $0 \leq u_\infty \leq u$ , is nondecreasing and satisfies

$$\lim_{r \rightarrow \infty} u_\infty(r) = \lim_{n \rightarrow \infty} \mu_n = \mu_\infty. \quad (2.30)$$

If  $\mu_\infty = \infty$ , it would imply that  $u_\infty(1) = \infty$  by formula (2.21) which is not compatible with  $\frac{\partial u}{\partial n_{x_\rho}}(x_\rho) \geq u'_\infty(1)$  since  $u$  is  $C^1$ . Hence  $\mu_\infty < \infty$ . Since by Step 2 any radial solution of (2.19) in  $B_1^c$  is constant is  $B_R^c$  for some  $R > 1$  it implies that  $u_\infty$  endows this property. Furthermore since  $u_n(n) = \mu_n$  and  $u_\infty$  is nondecreasing it yields  $u_\infty(r) = \mu_\infty$  for  $r \geq R$ . This contradicts the fact that for  $n > R$ ,

$$\mu_\infty = u_\infty(n) \leq u(n) = \mu_n < u(n+1) = \mu_{n+1} < \mu_\infty. \quad (2.31)$$

*Step 4.* Proof of (ii): The function  $u(x) = 1 - A|x|^{-\frac{2-q}{q-1}}$  is a positive radial and increasing supersolution of (1.1) in  $B_R^c$  for  $R$  large enough provided  $1 < q < \frac{N}{N-1}$  and

$$0 < A < \frac{1-q}{2-q} \left( \frac{q}{q-1} - N \right)^{\frac{1}{q-1}}. \quad (2.32)$$

When  $q = 1$  the function  $u(x) = 1 - Ae^{-\alpha|x|}$  is a supersolution provided  $\alpha > N - 1$  and  $A$  is small enough.  $\square$

## 2.2 Proof of Theorem B

The next result will be useful in the sequel.

**Lemma 2.4** *Let  $q > 1$  and  $a, R > 0$ . Assume  $v$  is continuous and nonnegative on  $\overline{B_R}$  and  $C^1$  on the set  $\mathcal{U}_+ = \{x \in B_R : v(x) > 0\}$ . If  $v$  satisfies*

$$-\Delta v + v^q \leq a \frac{|\nabla v|^2}{v} \quad (2.33)$$

on each connected component of  $\mathcal{U}_+$ , there holds

$$v(0) \leq c_{N,q,a} R^{-\frac{2}{q-1}}. \quad (2.34)$$

*Proof.* We can always suppose  $a > 0$  and set  $v = W^\alpha$  for some  $\alpha > 0$  to be defined. Then

$$-\Delta W + (1 - \alpha) \frac{|\nabla W|^2}{W} + \frac{1}{\alpha} W^{\alpha(q-1)+1} \leq a\alpha^2 \frac{|\nabla W|^2}{W}.$$

If we choose  $1 - \alpha \geq a\alpha^2$ , or equivalently  $0 < \alpha \leq \frac{1}{a+1}$  we derive

$$-\Delta W + \frac{1}{\alpha} W^{\alpha(q-1)+1} \leq 0.$$

on each connected component of  $\mathcal{U}_+$ . A standard computation shows that there exists  $c_{N,\alpha,q} > 0$  such that the function

$$x \mapsto \psi(x) := \frac{c_{N,\alpha,q} (R^2 \alpha)^{\frac{1}{\alpha(q-1)}}}{(R^2 - |x|^2)^{\frac{2}{\alpha(q-1)}}}$$

satisfies

$$-\Delta \psi + \frac{1}{\alpha} \psi^{\alpha(q-1)+1} \geq 0 \quad \text{in } B_R.$$

Assume that there exists a connected component  $G$  of  $\{x \in B_R : W(x) > \psi(x)\}$ , then  $G \subset \mathcal{U}_+$ . The function  $\phi = W - \psi$  is subharmonic and continuous in  $\overline{G}$  and vanishes on  $\partial G$ . Hence  $\phi \leq 0$ , contradiction. Hence  $G = \emptyset$  and  $W \leq \psi$  in  $\mathcal{U}_+$ . Since  $W \equiv 0$  in  $B_R \setminus \mathcal{U}_+$  we derive that  $W \leq \psi$  in  $B_R$ . Therefore

$$W(0) \leq \frac{c_{N,\alpha,q} \alpha^{\frac{1}{\alpha(q-1)}}}{R^{\frac{2}{\alpha(q-1)}}} \quad (2.35)$$

which leads to (2.34).  $\square$

*Proof of Theorem B:* In any open subset  $\mathcal{U}$  of  $B_R$  where  $|\nabla u| > 0$  the function  $u$  is  $C^\infty$  and the next computations are justified.

*Step 1: Transformation of the equation.* Set  $u = v^{-\beta}$  where  $\beta$  is a nonzero real number to be chosen. Then

$$\begin{aligned} \Delta v &= (1 + \beta) \frac{|\nabla v|^2}{v} + |\beta|^{q-2} \beta v^{1-q-\beta(p+q-1)} |\nabla v|^q \\ &= (1 + \beta) \frac{z}{v} + |\beta|^{q-2} \beta v^s z^{\frac{q}{2}}, \end{aligned} \quad (2.36)$$

if we denote  $z = |\nabla v|^2$  and  $s = 1 - q - \beta(p + q - 1)$ . We recall that

$$\frac{1}{2} \Delta |\nabla v|^2 = |D^2 v|^2 + \langle \nabla \Delta v, \nabla v \rangle.$$

Since

$$|D^2 v|^2 \geq \frac{1}{N} (\Delta v)^2 = \frac{1}{N} \left( (1 + \beta)^2 \frac{z^2}{v^2} + \beta^{2(q-1)} v^{2s} z^q + 2(1 + \beta) |\beta|^{q-2} \beta v^{s-1} z^{\frac{q}{2}+1} \right),$$

we get

$$\begin{aligned} \frac{1}{2} \Delta z &\geq \frac{1}{N} \left( (1 + \beta)^2 \frac{z^2}{v^2} + \beta^{2(q-1)} v^{2s} z^q + 2(1 + \beta) |\beta|^{q-2} \beta v^{s-1} z^{\frac{q}{2}+1} \right) - (1 + \beta) \frac{z^2}{v^2} \\ &\quad + s |\beta|^{q-2} \beta v^{s-1} z^{\frac{q}{2}+1} + (1 + \beta) \frac{\langle \nabla z, \nabla v \rangle}{v} + \frac{q}{2} |\beta|^{q-2} \beta v^s z^{\frac{q}{2}-1} \langle \nabla z, \nabla v \rangle. \end{aligned} \quad (2.37)$$

Expanding the expression we obtain

$$\begin{aligned} \frac{1}{2} \Delta z &\geq \frac{1}{N} \left( (1 + \beta)^2 \frac{z^2}{v^2} + \beta^{2(q-1)} v^{2s} z^q + 2(1 + \beta) |\beta|^{q-2} \beta v^{s-1} z^{\frac{q}{2}+1} \right) - (1 + \beta) \frac{z^2}{v^2} \\ &\quad + s |\beta|^{q-2} \beta v^{s-1} z^{\frac{q}{2}+1} + (1 + \beta) \frac{\langle \nabla z, \nabla v \rangle}{v} + \frac{q}{2} |\beta|^{q-2} \beta v^s z^{\frac{q}{2}-1} \langle \nabla z, \nabla v \rangle, \end{aligned}$$

which can be re-written as

$$\begin{aligned} -\frac{1}{2} \Delta z &+ \left( \frac{(1 + \beta)^2}{N} - (1 + \beta) \right) \frac{z^2}{v^2} + \frac{1}{N} \beta^{2(q-1)} v^{2s} z^q + \left( \frac{2(1 + \beta)}{N} + s \right) |\beta|^{q-2} \beta v^{s-1} z^{\frac{q}{2}+1} \\ &\quad + (1 + \beta) \frac{\langle \nabla z, \nabla v \rangle}{v} + \frac{q}{2} |\beta|^{q-2} \beta v^s z^{\frac{q}{2}-1} \langle \nabla z, \nabla v \rangle \leq 0. \end{aligned} \quad (2.38)$$

Next we set  $z = v^{-\lambda}Y$  for some parameter  $\lambda$ , then

$$-\Delta z = \lambda v^{-\lambda-1}Y \Delta v - \lambda(\lambda + 1)v^{-2\lambda-2}Y^2 + 2\lambda v^{-\lambda-1}\langle \nabla Y, \nabla v \rangle - v^{-\lambda}\Delta Y.$$

Since  $\Delta v = (1 + \beta)\frac{z}{v} + |\beta|^{q-2}\beta v^s z^{\frac{q}{2}} = (1 + \beta)v^{-\lambda-1}Y + |\beta|^{q-2}\beta v^{s-\frac{\lambda q}{2}}Y^{\frac{q}{2}}$  we use

$$\frac{z^2}{v^2} = v^{-2\lambda-2}Y^2, \quad v^{2s}z^q = v^{2s-\lambda q}Y^q \quad \text{and} \quad v^{s-1}z^{1+\frac{q}{2}} = v^{s-1-\lambda-\frac{\lambda q}{2}}Y^{\frac{q}{2}+1},$$

and get

$$-\Delta z = \lambda(\beta - \lambda)v^{-2-2\lambda}Y^2 + \lambda|\beta|^{q-2}\beta v^{s-1-\lambda-\frac{\lambda q}{2}}Y^{\frac{q}{2}+1} + 2\lambda v^{-\lambda-1}\langle \nabla Y, \nabla v \rangle - v^{-\lambda}\Delta Y.$$

Reporting into (2.38) yields

$$\begin{aligned} 0 &\geq \frac{\lambda}{2}(\beta - \lambda)v^{-2-2\lambda}Y^2 + \frac{\lambda}{2}|\beta|^{q-2}\beta v^{s-1-\lambda-\frac{\lambda q}{2}}Y^{\frac{q}{2}+1} + \lambda v^{-\lambda-1}\langle \nabla Y, \nabla v \rangle - \frac{1}{2}v^{-\lambda}\Delta Y \\ &\quad + \left( \frac{(1 + \beta)^2}{N} - (1 + \beta) \right) v^{-2\lambda-2}Y^2 + \left( \frac{2(1 + \beta)}{N} + s \right) |\beta|^{q-2}\beta v^{s-1-\lambda-\frac{\lambda q}{2}}Y^{\frac{q}{2}+1} \\ &\quad + \frac{1}{N}\beta^{2(q-1)}v^{2s-\lambda q}Y^q + (1 + \beta)(v^{-\lambda-1}\langle \nabla Y, \nabla v \rangle - \lambda v^{-2\lambda-2}Y^2) \\ &\quad + \frac{q}{2}|\beta|^{q-2}\beta \left( v^{s-\frac{\lambda q}{2}}Y^{\frac{q}{2}-1}\langle \nabla Y, \nabla v \rangle - \lambda v^{s-1-\lambda-\frac{\lambda q}{2}}Y^{1+\frac{q}{2}} \right), \end{aligned}$$

which can be re-written under the form

$$\begin{aligned} &-\frac{1}{2}v^{-\lambda}\Delta Y + \left( \frac{(1 + \beta)^2}{N} - (1 + \beta) - \frac{\lambda}{2}(\lambda + \beta + 2) \right) v^{-2\lambda-2}Y^2 \\ &\quad + \left( \frac{2(1 + \beta)}{N} + s - \frac{\lambda(q-1)}{2} \right) |\beta|^{q-2}\beta v^{s-1-\lambda-\frac{\lambda q}{2}}Y^{\frac{q}{2}+1} + \frac{1}{N}\beta^{2(q-1)}v^{2s-\lambda q}Y^q \\ &\leq - \left( \frac{q}{2}|\beta|^{q-2}\beta v^{s-\frac{\lambda q}{2}}Y^{\frac{q}{2}-1} + (\lambda + 1)v^{-\lambda-1} \right) \langle \nabla Y, \nabla v \rangle. \end{aligned}$$

Multiplying this relation by  $v^\lambda$  yields

$$\begin{aligned} &-\frac{1}{2}\Delta Y + \left( \frac{(1 + \beta)^2}{N} - (1 + \beta) - \frac{\lambda}{2}(\lambda + \beta + 2) \right) v^{-\lambda-2}Y^2 \\ &\quad + \left( \frac{2(1 + \beta)}{N} + s - \frac{\lambda(q-1)}{2} \right) |\beta|^{q-2}\beta v^{s-1-\frac{\lambda q}{2}}Y^{\frac{q}{2}+1} + \frac{1}{N}\beta^{2(q-1)}v^{2s-\lambda q+\lambda}Y^q \\ &\leq - \left( \frac{q}{2}|\beta|^{q-2}\beta v^{s-\frac{\lambda q}{2}+\lambda}Y^{\frac{q}{2}-1} + \frac{\lambda + 1}{v} \right) \langle \nabla Y, \nabla v \rangle. \end{aligned}$$

*Step 2: Estimate on  $Y$ .* Let  $\epsilon_0 \in (0, 1)$ . For any  $\epsilon > 0$  one has

$$\left| \frac{\langle \nabla Y, \nabla v \rangle}{v} \right| \leq \frac{1}{4\epsilon} \frac{|\nabla Y|^2}{Y} + \epsilon v^{-\lambda-2}Y^2.$$

Taking  $\epsilon = \frac{\epsilon_0}{|\lambda+1|}$ , we get

$$\left| (\lambda+1) \frac{\langle \nabla Y, \nabla v \rangle}{v} \right| \leq \frac{(\lambda+1)^2}{4\epsilon_0} \frac{|\nabla Y|^2}{Y} + \epsilon_0 v^{-\lambda-2} Y^2.$$

In the same way, with  $\epsilon = \frac{2\epsilon_0}{q|\beta|^{q-1}}$ ,

$$\left| v^{s-\frac{\lambda q}{2}+\lambda} Y^{\frac{q}{2}-1} \langle \nabla Y, \nabla v \rangle \right| = v^{s-\frac{\lambda q}{2}+\lambda} Y^{\frac{q}{2}-1} Y^{\frac{q-1}{2}} \frac{|\nabla Y|}{\sqrt{Y}} \leq \epsilon' v^{2s-\lambda q+\lambda} Y^q + \frac{1}{4\epsilon'} \frac{|\nabla Y|^2}{Y},$$

and

$$-\frac{q}{2} |\beta|^{q-2} \beta v^{s-\frac{\lambda q}{2}+\lambda} Y^{\frac{q}{2}-1} \langle \nabla Y, \nabla v \rangle \leq \frac{\epsilon' q |\beta|^{q-1}}{2} v^{2s-\lambda q+\lambda} Y^q + \frac{q |\beta|^{q-1}}{8\epsilon'} \frac{|\nabla Y|^2}{Y}.$$

We infer

$$\begin{aligned} -\frac{1}{2} \Delta Y + \left( \frac{(1+\beta)^2}{N} - (1+\beta) - \frac{\lambda(\beta+\lambda+2-\epsilon_0)}{2} \right) v^{-\lambda-2} Y^2 \\ + \left( \frac{2(1+\beta)}{N} + s - \frac{\lambda(q-1)}{2} \right) |\beta|^{q-2} \beta v^{s-1-\frac{\lambda q}{2}} Y^{1+\frac{q}{2}} \\ + \left( \frac{\beta^{2(q-1)}}{N} - \epsilon_0 \right) v^{2s-\lambda q+\lambda} Y^q \leq C(\epsilon_0) \frac{|\nabla Y|^2}{Y}, \end{aligned} \quad (2.39)$$

with  $C(\epsilon_0) = \left( \frac{(\lambda+1)^2}{4} + \frac{q\beta^{2(q-1)}}{16} \right) \frac{1}{\epsilon_0}$ . Next we put

$$\begin{aligned} H = \left( \frac{(1+\beta)^2}{N} - (1+\beta) - \frac{\lambda(\beta+\lambda+2-\epsilon_0)}{2} \right) v^{-\lambda-2} Y^2 \\ + \left( \frac{2(1+\beta)}{N} + s - \frac{\lambda(q-1)}{2} \right) |\beta|^{q-2} \beta v^{s-1-\frac{\lambda q}{2}} Y^{1+\frac{q}{2}} + \left( \frac{\beta^{2(q-1)}}{N} - \epsilon_0 \right) v^{2s-\lambda q+\lambda} Y^q, \end{aligned}$$

and consider the trinom

$$\begin{aligned} \mathbf{T}_{\epsilon_0}(t) = \left( \frac{\beta^{2(q-1)}}{N} - \epsilon_0 \right) t^2 + \left( \frac{2(1+\beta)}{N} + s - \frac{\lambda(q-1)}{2} \right) |\beta|^{q-2} \beta t \\ + \left( \frac{(1+\beta)^2}{N} - (1+\beta) - \frac{\lambda(\beta+\lambda+2-\epsilon_0)}{2} \right). \end{aligned}$$

If its discriminant is negative there exists  $\alpha = \alpha(N, p, q, \beta, \lambda, \epsilon_0) > 0$  such that  $\mathbf{T}_{\epsilon_0}(t) \geq \alpha(t^2+1)$ , hence

$$H \geq \alpha \left( v^{-\lambda-2} Y^2 + v^{2s-\lambda q+\lambda} Y^q \right). \quad (2.40)$$

Assuming  $\lambda \neq -2$ , we introduce

$$S = \frac{2s-\lambda q+\lambda}{\lambda+2} = 1-q - \frac{2\beta(p+q-1)}{\lambda+2}, \quad (2.41)$$



then, if  $S > \max\{0, 1 - q\}$ , we have  $\frac{2S+q}{S+1} > 1$  and

$$Y^{\frac{2S+q}{S+1}} = \left( \frac{Y^2}{v^{\lambda+2}} \right)^{\frac{S}{S+1}} v^{\frac{(\lambda+2)S}{S+1}} Y^{\frac{q}{S+1}} \leq \frac{Y^2}{v^{\lambda+2}} + v^{(\lambda+2)S} Y^q = \frac{Y^2}{v^{\lambda+2}} + v^{2s-\lambda q+\lambda} Y^q.$$

From this we infer the key inequality

$$-\Delta Y + 2\alpha Y^{\frac{2S+q}{S+1}} \leq 2C(\epsilon_0) \frac{|\nabla Y|^2}{Y}. \quad (2.42)$$

Using Lemma 2.4, we derive

$$Y(0) \leq cR^{-\frac{2(S+1)}{S+q-1}} = cR^{-\frac{2(s+1)-\lambda(q-1)}{s+q-1}} = cR^{-2+\frac{(2+\lambda)(2-q)}{\beta(p+q-1)}},$$

from which follows

$$\left| \nabla u^{-\frac{2+\lambda}{2\beta}}(0) \right| \leq \frac{|2+\lambda|\sqrt{c}}{2} R^{-1+\frac{(2+\lambda)(2-q)}{2\beta(p+q-1)}}, \quad (2.43)$$

from which it stems inequality (1.8) in Theorem B with  $a = -\frac{\lambda+2}{2\beta} > 0$  from (2.41). In what follows we shall see that under the assumptions of Theorem B we can always choose such  $\beta, \lambda$ .

*Step 3: Study of the trinom  $\mathbf{T}_{\epsilon_0}$ .* The discriminant of the trinom  $\mathbf{T}_{\epsilon_0}$  is a polynomial in its coefficients. Hence it is sufficient to prove that the discriminant of  $\mathbf{T}_0$  is negative to derive that the same property holds for  $\mathbf{T}_{\epsilon_0}$  for  $\epsilon_0$  small enough. If

$$\begin{aligned} \mathbf{T}_0(t) = & \frac{\beta^{2(q-1)}}{N} t^2 + \left( \frac{2(1+\beta)}{N} + s - \frac{\lambda(q-1)}{2} \right) |\beta|^{q-2} \beta t \\ & + \left( \frac{(1+\beta)^2}{N} - (1+\beta) - \frac{\lambda(\beta+\lambda+2)}{2} \right), \end{aligned}$$

its discriminant  $D$  verifies

$$\beta^{2(1-q)} D = \left( \frac{2(1+\beta)}{N} + s - \frac{\lambda(q-1)}{2} \right)^2 - \frac{4}{N} \left( \frac{(1+\beta)^2}{N} - (1+\beta) - \frac{\lambda(\beta+\lambda+2)}{2} \right).$$

Using  $\beta + 1 = \frac{p-s}{p+q-1}$  we obtain

$$\beta^{2(1-q)} D = \left( s - \lambda \frac{q-1}{2} \right)^2 + \frac{4(p-s)}{N(p+q-1)} \left( s + 1 + \lambda \frac{2-q}{2} \right) + \frac{2\lambda(\lambda+1)}{N}.$$

Since  $S = \frac{2s+\lambda(1-q)}{\lambda+2}$ ,  $s - \lambda \frac{q-1}{2} = \frac{(\lambda+2)S}{2}$ , hence

$$\begin{aligned} \beta^{2(1-q)} D = & \frac{(\lambda+2)^2 S^2}{4} + \frac{4}{N(p+q-1)} \left[ -\frac{(\lambda+2)^2 S^2}{4} + \left( p-1 - \frac{\lambda q}{2} \right) \frac{(\lambda+2)S}{2} \right. \\ & \left. + \left( p - \frac{\lambda(q-1)}{2} \right) \left( \frac{\lambda+2}{2} \right) \right] + \frac{2\lambda(\lambda+1)}{N}. \end{aligned}$$

Set  $Q = p + q - 1$  and  $D_1 = NQ\beta^{2'1-q}D$ , then

$$D_1 = (\lambda + 2)^2 \left( \frac{NQ}{4} - 1 \right) S^2 + 2 \left( p - 1 - \frac{\lambda q}{2} \right) (\lambda + 2) + \tilde{L},$$

where miraculously,

$$\begin{aligned} \tilde{L} &= (2p + \lambda(1 - q))(\lambda + 2) + 2\lambda(\lambda + 1)Q \\ &= Q\lambda^2 + p(\lambda + 2)^2 > 0. \end{aligned}$$

So we require that  $\lambda + 2 \neq 0$  and set  $\ell = \frac{\lambda}{\lambda+2}$ ; equivalently  $\ell \neq 1$  and  $\lambda + 2 = \frac{2}{1-\ell}$ . We obtain

$$D_2(S, \ell) := \frac{D_1}{(\lambda + 2)^2} = \left( \frac{NQ}{4} - 1 \right) S^2 + (p - 1 - Q\ell)S + Q\ell^2 + p. \quad (2.44)$$

So we are led to find  $\ell \neq 1$  and  $S > (1 - q)_+$  such that  $D_2(S, \ell) < 0$ . Indeed,  $S$  being fixed,  $D_2$  is a trinomial in  $\ell$ :

$$\mathcal{T}_S(\ell) := D_2(S, \ell) = Q\ell^2 - QS\ell + \left( \frac{NQ}{4} - 1 \right) S^2 - (1 - p)S + p.$$

In order that  $\mathcal{T}_S(\ell) < 0$ , we require that its discriminant is positive, which reduces to the condition

$$\mathcal{H}(S) := \left( 1 - \frac{(N - 1)Q}{4} \right) S^2 + (1 - p)S - p > 0. \quad (2.45)$$

(i) We first assume  $Q < \frac{4}{N-1}$ . Then we choose  $S$  large enough such that  $\mathcal{H}(S) > 0$  and  $S > 2 > (1 - q)_+$ . Then we fix  $\ell = \frac{S}{2}$  so that  $\ell > 1$  and all the conditions are fulfilled.

(ii) Next we assume  $p < 1$ . Note that our assumption  $Q < \frac{(p+1)^2}{(N-1)p}$  is equivalent to

$$d = (p + 1)^2 - p((N - 1)Q - 4) > 0 \quad (2.46)$$

where  $d$  is the discriminant of  $\mathcal{H}$ . In particular it holds if  $Q \leq \frac{4}{N-1}$ . Next we assume  $Q > \frac{4}{N-1}$ . Hence the coefficient of  $S^2$  in  $\mathcal{H}(S)$  is negative, and  $\mathcal{H}$  has two positive roots. Then we take  $S$  as the half sum of the roots, i.e.

$$S = S_0 := \frac{2(1 - p)}{(N - 1)Q - 4}. \quad (2.47)$$

We have still to check that  $S_0 > (1 - q)_+$ . This is clear if  $q \geq 1$ . If  $q < 1$  we get from (2.46)

$$S_0 + q - 1 = \frac{2(1 - p)}{(N - 1)Q - 4} + q - 1 > \frac{2p}{1 - p} + q - 1 = \frac{p(2 - q) + Q}{1 - p} > 0.$$

At end we choose  $\ell \neq 1$  close enough to  $\frac{S_0}{2}$  so that all the conditions are satisfied (always with  $a = -\frac{\lambda+2}{2\beta} > 0$ ).  $\square$

*Remark.* In the case  $Q < \frac{4}{N-1}$ , we have fixed some  $\ell > 1$  so that  $\lambda + 2 < 0$  and then  $\beta > 0$ . If  $Q < \frac{4}{N}$ , a much simpler possible choice is  $\lambda = \ell = 0$  so that  $\beta < 0$ .

*Proof of Corollary B-2.* Since  $\partial\Omega$  is smooth, there exists  $d_0 > 0$  such that for any  $z \in \Omega$  verifying  $\text{dist}(z, \partial\Omega) \leq d_0$ , there exists a unique  $\zeta_z \in \partial\Omega$  such that  $\text{dist}(z, \partial\Omega) = |z - \zeta_z|$ . If  $\text{dist}(z_0, \partial\Omega) = d_0$ , we denote by  $\mathbf{n}_{\zeta_{z_0}}$  the normal inward unit vector to  $\partial\Omega$  at  $\zeta_{z_0}$  and we set  $x_t = t\mathbf{n}_{\zeta_{z_0}}$ ,  $0 < t \leq d_0$ . By (1.8),

$$|\nabla u^a(x_t)| \leq c_1 t^{-1-a\frac{2-q}{p+q-1}},$$

hence

$$|u^a(x_t) - u^a(z_0)| \leq c_1 \int_t^{d_0} s^{-1-a\frac{2-q}{p+q-1}} ds.$$

This implies

$$u^a(x_t) \leq u^a(z_0) + \frac{c_1(p+q-1)}{a(2-q)} t^{-a\frac{2-q}{p+q-1}} = u^a(z_0) + \frac{c_1(p+q-1)}{a(2-q)} (\text{dist}(x_t, \partial\Omega))^{-a\frac{2-q}{p+q-1}}.$$

If  $a \geq 1$  it yields

$$u(x_t) \leq u(z_0) + \frac{c_1(p+q-1)}{(2-q)} (\text{dist}(x_t, \partial\Omega))^{-\frac{2-q}{p+q-1}},$$

while, if  $0 < a < 1$  we can only obtain

$$u(x_t) \leq c_2 \left( u(z_0) + \frac{c_1(p+q-1)}{(2-q)} (\text{dist}(x_t, \partial\Omega))^{-\frac{2-q}{p+q-1}} \right).$$

In any case we derive (1.9). □

## 3 Global solutions

### 3.1 Radial solutions

In the next theorem we characterize all the positive global radial solutions of (1.1) in  $\mathbb{R}^N$ . We can always assume that the solutions are radial with respect to 0.

**Theorem 3.1** *Assume  $p, q \geq 0$  such that  $p+q-1 > 0$ . If  $q \geq 1$  the only positive global radial solutions of (1.1) are the constants. If  $0 \leq q < 1$  there exists positive global radial solutions if and only if*

$$p(N-2) + q(N-1) \geq N + \frac{2-q}{1-q}. \quad (3.1)$$

Furthermore, if  $p(N-2) + q(N-1) = N + \frac{2-q}{1-q}$  there exists a one parameter of solutions under the form

$$u_c(r) = c \left( K c^{\frac{(2-q)^2}{(N-2)(1-q)}} + r^{\frac{2-q}{1-q}} \right)^{-\frac{(N-2)(1-q)}{2-q}} \quad \text{with } K = \frac{(N-2)^{q-1}}{N - (N-1)q}. \quad (3.2)$$

*Proof.* The radial form of (1.1) is the following

$$-u_{rr} - \frac{N-1}{r}u_r = u^p |u_r|^q, \quad (3.3)$$

and  $u'(0) = 0$  since any solution is  $C^2$ . Thus  $u$  can be written under the form

$$u(r) = u(0) + \int_0^r s^{1-N} \int_0^s u^p(t) |u_r(t)|^q t^{N-1} dt \quad \forall r > 0. \quad (3.4)$$

If  $q \geq 1$ , the solution satisfying  $u(0) = a > 0$  is the unique fixed point of the mapping  $v \mapsto \mathcal{T}[v]$  defined in the set of functions in  $C([0, r_0])$  with value  $a$  for  $r = 0$  by

$$\mathcal{T}[v](r) := a + \int_0^r s^{1-N} \int_0^s v^p(t) |v_r(t)|^q t^{N-1} dt \quad \forall r > 0.$$

Clearly  $\mathcal{T}$  is a strict contraction if  $r_0 > 0$  is small enough. Since  $u \equiv a$  is a solution in  $\mathbb{R}^N$  it is the unique one.

Hereafter we assume  $0 \leq q < 1$ . As we have noticed it in the proof of Theorem 2.3 we can write (3.3) under the form

$$\Delta_m^\nu u + (1-q)u^p = 0, \quad (3.5)$$

where  $\Delta_m^\nu$  is a  $m$ -Laplacian in dimension  $\nu$  applied to radial functions. An important critical value of  $p$  is the following

$$p_c := \frac{\nu(m-1) + m}{\nu - m} = \frac{(N - (N-1)q)(1-q) + 2 - q}{(N-2)(1-q)}. \quad (3.6)$$

For this specific value there exists an explicit family of ground states

$$u_c(r) = c \left( K_c^2 + r^{\frac{m}{m-1}} \right)^{\frac{m-\nu}{m+\nu}} = c \left( K_c^2 + r^{\frac{2-q}{1-q}} \right)^{-\frac{(N-2)(1-q)}{2-q}}, \quad (3.7)$$

where  $c > 0$  and

$$u_c(r) = c \left( K c^{\frac{m^2}{\nu-m}} + r^{\frac{m}{m-1}} \right)^{\frac{m-\nu}{m}} = c \left( K c^{\frac{(2-q)^2}{N-2-q(N+2)}} + r^{\frac{2-q}{1-q}} \right)^{\frac{(N-2)(q-1)}{2-q}}, \quad (3.8)$$

with

$$K = \frac{1}{\nu} \left( \frac{\nu - m}{m - 1} \right)^{1-m}. \quad (3.9)$$

Set

$$\begin{aligned} F_u(r) &= r^\nu \left( \frac{|u_r|^m}{m} + \frac{u^{p+1}}{p+1} + \frac{\nu - m}{m} \frac{|u_r|^{m-2} u_r}{r} \right) \\ &= r^{N-(N-1)q} \left( \frac{(1-q)|u_r|^{2-q}}{2-q} + \frac{u^{p+1}}{p+1} + \frac{(N-2)(1-q)}{2-q} \frac{|u_r|^{-q} u_r}{r} \right). \end{aligned} \quad (3.10)$$

Then

$$F'_u(r) = r^{\nu-1} \left( \frac{\nu}{p+1} - \frac{\nu-m}{m} \right) u^{p+1}. \quad (3.11)$$

We notice that  $F'_u \equiv 0$  if and only if  $p = p_c$ ,  $F'_u > 0$  (resp.  $F'_u < 0$ ) if and only if  $p > p_c$  (resp.  $p < p_c$ ). By [2, Th 5.2, 5.3], if  $p < p_c$  all the solutions of (3.5) which have a finite limit at  $r = 0$  oscillates around 0 when  $r \rightarrow \infty$ . Hence there exists no ground state. By [2, Th 5.1], if  $p > p_c$ , for any  $\alpha > 0$  there exist a positive solution  $u$  of (3.5) in  $\mathbb{R}^N$  satisfying  $u(0) = \alpha$ .  $\square$

### 3.2 Proof of Theorem C

*Step 1: Integral inequalities.* The aim of this paragraph is to prove that under the assumptions (1.10) the gradient of any nonnegative solution  $u$  of (1.1) in whole  $\mathbb{R}^N$  is null. The method is an extension of the one developed in [10], [5], in the sense that we still set  $u = v^{-\beta}$  and  $v$  satisfies (2-1), and  $z = |\nabla v|^2$ . The main novelty is that we multiply the equation satisfied by  $z$  by  $v^\lambda z^e$  where  $e > 0$  and  $\lambda$  are two real parameters (in [10] and [5] they have chosen  $e = 0$ ). The algebraic computation is heavy and we present a very technical part of it in Appendix. Furthermore, since the exponent  $e$  is going to be smaller than 1, we have to replace  $z^e$  by  $f(z)$  where  $f$  is a smooth approximation and to consider many equations in the weak sense since  $u$ , and hence  $v$  is merely  $C^{2,q}$  if  $0 < q < 1$ . We start with the following Weintzenböck inequality already used in the proof of Theorem B, but taken here in the weak sense,

$$\int_{\mathbb{R}^N} \left( \frac{1}{2} \langle \nabla z, \nabla \phi \rangle + \frac{(\Delta v)^2}{N} \phi - \Delta v (\langle \nabla v, \nabla \phi \rangle + \phi \Delta v) \right) dx \leq 0,$$

for all  $\phi \in C_0^1(\mathbb{R}^N)$ ,  $\phi \geq 0$ , hence

$$\int_{\mathbb{R}^N} \left( \frac{1}{2} \langle \nabla z, \nabla \phi \rangle - \frac{N-1}{N} \phi (\Delta v)^2 - \Delta v \langle \nabla v, \nabla \phi \rangle \right) dx \leq 0. \quad (3.12)$$

We choose  $\phi = v^\lambda f(z) \eta$  where  $\eta \in C_0^3(\mathbb{R}^N)$ ,  $\eta \geq 0$  and  $f \in C^1([0, \infty))$ ,  $f \geq 0$  and get

$$\begin{aligned} & \frac{\lambda}{2} \int_{\mathbb{R}^N} v^{\lambda-1} f(z) \eta \langle \nabla v, \nabla z \rangle dx + \frac{1}{2} \int_{\mathbb{R}^N} v^\lambda f'(z) |\nabla z|^2 \eta dx - \frac{N-1}{N} \int_{\mathbb{R}^N} (\Delta v)^2 v^\lambda f(z) \eta dx \\ & - \lambda \int_{\mathbb{R}^N} v^{\lambda-1} (\Delta v) f(z) z \eta dx - \int_{\mathbb{R}^N} v^\lambda (\Delta v) f'(z) \eta \langle \nabla v, \nabla z \rangle dx \\ & \leq -\frac{1}{2} \int_{\mathbb{R}^N} v^\lambda f(z) \langle \nabla z, \nabla \eta \rangle dx + \int_{\mathbb{R}^N} v^\lambda (\Delta v) f(z) \eta \langle \nabla v, \nabla z \rangle dx. \end{aligned} \quad (3.13)$$

This inequality proved with regular functions  $v$  and  $f$  is extendable by density to  $v \in C^2(\mathbb{R}^N)$  and  $f$  locally Lipschitz continuous. We apply this relation to the function  $v$  which satisfies (2.36) in the range  $0 \leq q < 2$ ,  $\beta \in \mathbb{R} \setminus \{1\}$  and where  $s = 1 - q - \beta(p + q - 1)$ . We consider the different terms appearing in (3.13) with the help of (2.36).

$$\Sigma = \int_{\mathbb{R}^N} (\Delta v) v^\lambda f'(z) \eta \langle \nabla v, \nabla z \rangle dx = (1 + \beta) \Sigma_1 + |\beta|^{q-2} \beta \Sigma_2$$

with

$$\Sigma_1 = - \int_{\mathbb{R}^N} z f'(z) v^{\lambda-1} \eta \langle \nabla v, \nabla \eta \rangle dx \quad \text{and} \quad \Sigma_2 = - \int_{\mathbb{R}^N} z^{\frac{q}{2}} f'(z) v^{\lambda+s} \eta \langle \nabla v, \nabla \eta \rangle dx.$$

Then

$$\begin{aligned} \Sigma_1 &= - \int_{\mathbb{R}^N} \eta \langle v^{\lambda-1} \nabla v, z f'(z) \nabla z \rangle dx \\ &= - \int_{\mathbb{R}^N} \eta \langle v^{\lambda-1} \nabla v, \nabla (g(z)) \rangle dx \quad \text{where} \quad g(t) = \int_0^t s f'(s) ds \\ &= - \int_{\mathbb{R}^N} \langle v^{\lambda-1} \nabla v, \nabla (\eta g(z)) \rangle dx + \int_{\mathbb{R}^N} v^{\lambda-1} g(z) \langle \nabla v, \nabla \eta \rangle dx \\ &= \int_{\mathbb{R}^N} \eta g(z) \nabla \cdot (v^{\lambda-1} \nabla v) dx + \int_{\mathbb{R}^N} v^{\lambda-1} g(z) \langle \nabla v, \nabla \eta \rangle dx \\ &= (\lambda - 1) \int_{\mathbb{R}^N} v^{\lambda-2} \eta g(z) z dx + \int_{\mathbb{R}^N} \eta g(z) v^{\lambda-1} (\Delta v) dx + \int_{\mathbb{R}^N} v^{\lambda-1} g(z) \langle \nabla v, \nabla \eta \rangle dx \\ &= (\lambda - 1) \int_{\mathbb{R}^N} v^{\lambda-2} \eta g(z) z dx + (1 + \beta) \int_{\mathbb{R}^N} \eta g(z) z v^{\lambda-2} dx + \\ &\quad |\beta|^{q-2} \beta \int_{\mathbb{R}^N} \eta g(z) z^{\frac{q}{2}} v^{\lambda-1+s} dx + \int_{\mathbb{R}^N} v^{\lambda-1} g(z) \langle \nabla v, \nabla \eta \rangle dx. \end{aligned}$$

Similarly,

$$\begin{aligned} \Sigma_2 &= - \int_{\mathbb{R}^N} \langle v^{\lambda+s} \nabla v, z^{\frac{q}{2}} f'(z) \nabla z \rangle \eta dx \\ &= - \int_{\mathbb{R}^N} \langle v^{\lambda+s} \nabla v, \nabla h(z) \rangle \eta dx \quad \text{where} \quad h(t) = \int_0^t s^{\frac{q}{2}} f'(s) ds \\ &= - \int_{\mathbb{R}^N} \langle v^{\lambda+s} \nabla v, \nabla (\eta h(z)) \rangle dx + \int_{\mathbb{R}^N} v^{\lambda+s} h(z) \langle \nabla v, \nabla \eta \rangle dx \\ &= (\lambda + s) \int_{\mathbb{R}^N} v^{\lambda+s-1} z h(z) \eta dx + \int_{\mathbb{R}^N} v^{\lambda+s} (\Delta v) h(z) \eta dx + \int_{\mathbb{R}^N} v^{\lambda+s} h(z) \langle \nabla v, \nabla \eta \rangle dx \\ &= (\lambda + s + \beta + 1) \int_{\mathbb{R}^N} v^{\lambda+s-1} z h(z) \eta dx + |\beta|^{q-2} \beta \int_{\mathbb{R}^N} v^{\lambda+2s} z^{\frac{q}{2}} h(z) \eta dx \\ &\quad + \int_{\mathbb{R}^N} v^{\lambda+s} h(z) \langle \nabla v, \nabla \eta \rangle dx. \end{aligned}$$

Next we compute the term

$$\begin{aligned}
\Theta_1 &= \frac{\lambda}{2} \int_{\mathbb{R}^N} v^{\lambda-1} f(z) \eta \langle \nabla v, \nabla z \rangle dx \\
&= \frac{\lambda}{2} \int_{\mathbb{R}^N} v^{\lambda-1} \eta \langle \nabla v, \nabla j(z) \rangle dx \quad \text{where } j(t) = \int_0^t f(s) ds \\
&= \frac{\lambda}{2} \int_{\mathbb{R}^N} v^{\lambda-1} \langle \nabla v, \nabla (j(z) \eta) \rangle dx - \frac{\lambda}{2} \int_{\mathbb{R}^N} v^{\lambda-1} j(z) \langle \nabla v, \nabla \eta \rangle dx \\
&= -\frac{\lambda}{2} \int_{\mathbb{R}^N} j(z) \eta (v^{\lambda-1} \Delta v + (\lambda-1) v^{\lambda-2} z) dx - \frac{\lambda}{2} \int_{\mathbb{R}^N} v^{\lambda-1} j(z) \langle \nabla v, \nabla \eta \rangle dx.
\end{aligned}$$

Finally we compute

$$\begin{aligned}
\Theta_2 &= \frac{1}{2} \int_{\mathbb{R}^N} v^\lambda f(z) \langle \nabla z, \nabla \eta \rangle dx = \frac{1}{2} \int_{\mathbb{R}^N} v^\lambda \langle \nabla (j(z)), \nabla \eta \rangle dx \\
&= -\frac{\lambda}{2} \int_{\mathbb{R}^N} v^{\lambda-1} j(z) \langle \nabla v, \nabla \eta \rangle dx - \frac{1}{2} \int_{\mathbb{R}^N} v^\lambda j(z) (\Delta \eta) dx.
\end{aligned}$$

Carrying forward these estimates into (3.13) yields

$$\begin{aligned}
& -\frac{\lambda(\lambda-1)}{2} \int_{\mathbb{R}^N} v^{\lambda-2} j(z) z \eta dx - \frac{\lambda}{2} \int_{\mathbb{R}^N} (\Delta v) v^{\lambda-1} j(z) \eta dx + \frac{1}{2} \int_{\mathbb{R}^N} v^\lambda f'(z) |\nabla z|^2 \eta dx \\
& - \frac{N-1}{N} \int_{\mathbb{R}^N} (\Delta v)^2 v^\lambda f(z) \eta dx - \lambda \int_{\mathbb{R}^N} (\Delta v) v^{\lambda-1} z f(z) \eta dx \\
& - (1+\beta)(\lambda+\beta) \int_{\mathbb{R}^N} v^{\lambda-2} z g(z) \eta dx + (1+\beta) |\beta|^{q-2} \beta \int_{\mathbb{R}^N} v^{\lambda-1+s} z^{\frac{q}{2}} g(z) \eta dx \\
& + |\beta|^{q-2} \beta (\beta+1+\lambda+s) \int_{\mathbb{R}^N} v^{\lambda+s-1} z h(z) \eta dx + \beta^{2(q-1)} \int_{\mathbb{R}^N} v^{\lambda+2} z^{\frac{q}{2}} h(z) \eta dx \\
& \leq \int_{\mathbb{R}^N} \left( \lambda j(z) + f(z) v \Delta v - (1+\beta) g(z) - |\beta|^{q-2} \beta v^{1+s} h(z) \right) v^{\lambda-1} \langle \nabla v, \nabla \eta \rangle dx \\
& \quad - \int_{\mathbb{R}^N} v^\lambda j(z) \Delta \eta dx.
\end{aligned} \tag{3.14}$$

Next we fix  $e \geq 0$  and choose  $f(t) = t^e$  if  $e \geq 1$  and, for  $0 < \epsilon < 1$ ,  $f(t) = f_\epsilon(t) = \min\{t^e, \epsilon^{e-1} t\}$  if  $0 \leq e < 1$ . Then  $f_\epsilon$  is locally Lipschitz continuous. In order to let  $\epsilon \rightarrow 0$  in (3.14), replacing  $f, j, g$  and  $h$  respectively by  $f_\epsilon, j_\epsilon, g_\epsilon$  and  $h_\epsilon$  we notice that

$$f_\epsilon(z) \uparrow z^e, \quad g_\epsilon(z) \uparrow \frac{e}{1+e} z^{1+e}, \quad h_\epsilon(z) \uparrow \frac{2e}{q+2e} z^{\frac{q}{2}+e} \quad \text{and} \quad j_\epsilon(z) \uparrow \frac{1}{1+e} z^{1+e}.$$

Since  $v^\lambda f'_\epsilon(z) |\nabla z|^2 \eta$  converges a.e. to  $v^\lambda z^{e-1} |\nabla z|^2 \eta$  we derive by Fatou's lemma

$$\int_{\mathbb{R}^N} v^\lambda z^{e-1} |\nabla z|^2 \eta dx \leq \liminf_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} v^\lambda f'_\epsilon(z) |\nabla z|^2 \eta dx,$$

which may not be finite. All the other terms in (3.14) converge by Lebesgue's theorem, hence

$$\begin{aligned}
& -\frac{\lambda(\lambda-1)}{2(1+e)} \int_{\mathbb{R}^N} v^{\lambda-2} z^{2+e} \eta dx - \lambda \left( \frac{1}{2(1+e)} + 1 \right) \int_{\mathbb{R}^N} (\Delta v) v^{\lambda-1} z^{1+e} \eta dx \\
& + \frac{e}{2} \int_{\mathbb{R}^N} v^\lambda z^{e-1} |\nabla z|^2 \eta dx - \frac{N-1}{N} \int_{\mathbb{R}^N} (\Delta v)^2 v^\lambda z^e \eta dx + \frac{e(1+\beta)(\lambda+\beta)}{1+e} \int_{\mathbb{R}^N} v^{\lambda-2} z^{e+2} \eta dx \\
& + |\beta|^{q-2} \beta \left( \frac{(\beta+1)e}{1+e} + \frac{2(\beta+1+\lambda+s)e}{q+2e} \right) \int_{\mathbb{R}^N} v^{\lambda+s-1} z^{\frac{q}{2}+1+e} \eta dx \\
& + \frac{2e\beta^{2(q-1)}}{q+2e} \int_{\mathbb{R}^N} v^{\lambda+2s} z^{q+e} \eta dx \leq M_0,
\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}
M_0 &= \frac{2\lambda - (1+\beta)e}{1+e} \int_{\mathbb{R}^N} v^{\lambda-1} z^{1+e} \langle \nabla v, \nabla \eta \rangle dx - \frac{1}{1+e} \int_{\mathbb{R}^N} v^\lambda z^{1+e} \Delta \eta dx \\
& + \int_{\mathbb{R}^N} (\Delta v) v^\lambda z^e \langle \nabla v, \nabla \eta \rangle dx - \frac{2e|\beta|^{q-2}\beta}{q+2e} \int_{\mathbb{R}^N} v^{\lambda+s} z \langle \nabla v, \nabla \eta \rangle dx \\
& \leq c(M + L + R),
\end{aligned} \tag{3.16}$$

with

$$\begin{aligned}
M &= \int_{\mathbb{R}^N} v^\lambda z^{1+e} |\Delta \eta| dx, \quad L = \int_{\mathbb{R}^N} v^{\lambda-1} z^{\frac{3}{2}+e} |\nabla \eta| dx \\
R &= \int_{\mathbb{R}^N} v^{\lambda+s} z^{\frac{q+1+2e}{2}} |\nabla \eta| dx;
\end{aligned} \tag{3.17}$$

Here we use  $|\langle \nabla v, \nabla \eta \rangle| \leq z^{\frac{1}{2}} |\nabla \eta|$  and the value of  $\Delta v$  given by (2.36) in order to have

$$\int_{\mathbb{R}^N} (\Delta v) v^\lambda z^e \langle \nabla v, \nabla \eta \rangle dx \leq (1+\beta)L + |\beta|^q R.$$

Set

$$\begin{aligned}
G &= \int_{\mathbb{R}^N} v^\lambda z^{-1+e} |\nabla z|^2 \eta dx, \quad P = \int_{\mathbb{R}^N} v^{\lambda-1+s} z^{1+\frac{q}{2}+e} \eta dx \\
F &= \int_{\mathbb{R}^N} v^{\lambda-2} z^{2+e} \eta dx, \quad U = \int_{\mathbb{R}^N} v^{\lambda+2s} z^{q+e} \eta dx.
\end{aligned} \tag{3.18}$$

We replace again  $\Delta v$  and  $(\Delta v)^2$  in the left-hand side of (3.15).

$$\begin{aligned}
& \int_{\mathbb{R}^N} (\Delta v) v^{\lambda-1} z^{1+e} \eta dx = (1+\beta)F + |\beta|^{q-2} \beta P, \\
& \int_{\mathbb{R}^N} (\Delta v)^2 v^\lambda z^e \eta dx = (1+\beta)^2 F + \beta^{2(q-1)} U + 2|\beta|^{q-2} (1+\beta)P.
\end{aligned}$$

replacing these terms into the left-hand side of (3.15), we obtain

$$\frac{e}{2} G + AF + |\beta|^q BP + CU \leq M_0 \tag{3.19}$$



with

$$\begin{aligned}
A &= -\frac{\lambda(\lambda + \beta)}{2(1 + e)} - \lambda(1 + \beta) - \frac{(N - 1)(1 + \beta)^2}{N} + \frac{e(1 + \beta)(\lambda + \beta)}{1 + e} \\
&= -\frac{\lambda(\lambda + \beta)}{2(1 + e)} - (1 + \beta) \left( 1 - \frac{1 + \beta}{N} + \frac{\lambda + \beta}{1 + e} \right), \\
\beta B &= -\lambda \left( 1 + \frac{1}{2(1 + e)} \right) - \frac{2(N - 1)(1 + \beta)}{N} + \frac{e(1 + \beta)}{1 + e} + \frac{2e(\beta + 1 + \lambda + s)}{q + 2e}.
\end{aligned}$$

and

$$C = \beta^{2(q-1)} \left( \frac{2e}{q + 2e} - \frac{N - 1}{N} \right).$$

Next we take  $e = \frac{(N-1)q}{2}$  so that  $C = 0$ , if  $q > 0$  and  $e > 0$  arbitrarily small so that  $C = \frac{\beta^{2(q-1)}}{N}$  if  $q = 0$ , i.e.  $\frac{2e}{q+2e} = 1$  and

$$\beta B = -\lambda \left( \frac{1}{N} + \frac{1}{2(1 + e)} \right) + \frac{e(1 + \beta)}{1 + e} - \frac{(N - 1)(\beta p + (\beta + 1)q)}{N}$$

by replacing  $s$  by  $1 - q - \beta(p + q - 1)$ . Next we introduce  $\delta = -\frac{\lambda}{\beta}$  and  $y = \frac{1+\beta}{\beta}$ , hence

$$B = \delta \left( \frac{1}{N} + \frac{1}{2(1 + e)} \right) + y \left( \frac{e}{1 + e} - \frac{q(N - 1)}{N} \right) - \frac{p(N - 1)}{N}.$$

*Step 2: Study of the coefficients A and B.* Our method is to choose the real parameters  $\delta$  and  $y$  in order to ensure  $A$  and  $B$  to be positive. We set

$$m = \delta - (2 + (N - 1)q) y. \quad (3.20)$$

In the sequel we keep the parameters  $y$  and  $m$  as variables and eliminate  $\delta$ .

(i) *Condition  $A > 0$ .* We define

$$\begin{aligned}
\mathcal{E}(m, y) &= -(2 + (N - 1)q)A_0 \\
&= \frac{(N - 1)(Nq + 1)(2 + (N - 1)q)}{N} y^2 + 2(1 + (N - 1)q)(m - 1)y + m(m - 1).
\end{aligned}$$

In the  $(m, y)$ -plane the set of points such that  $\mathcal{E}(m, y) = 0$  is a conic. Its points at infinity in the associated projective space  $\mathbb{P}^3(\mathbb{R}) = \{(\tilde{y}, \tilde{m}, \tilde{t})\}$  satisfy, with  $\tilde{t} = 0$ ,

$$(2 + (N - 1)q) \left( (N - 1)q + \frac{N - 1}{N} \right) \tilde{y}^2 + 2(1 + (N - 1)q) \tilde{m} \tilde{y} + \tilde{m}^2 = 0.$$

The discriminant of this quadratic form is

$$\tilde{\Delta} = (1 + (N - 1)q)^2 - (2 + (N - 1)q) \left( (N - 1)q + \frac{N - 1}{N} \right),$$

which is always negative since  $N \geq 2$ . Hence  $\mathcal{E}(m, y) = 0$  is the equation of an ellipse, and it is easy to check that

$$\{(m, y) \in \mathbb{R}^2 : \mathcal{E}(m, y) < 0\} \subset \left\{ (m, y) \in \mathbb{R}^2 : -\frac{N(1 + (N-1)q)^2}{N-2 + (N-1)q} < m < 1 \right\}. \quad (3.21)$$

(ii) *Condition  $B > 0$ .* We have

$$\begin{aligned} \mathcal{D}_p(m, y) &:= N(2 + (N-1)q)B \\ &= (N+2 + (N-1)q)m + 2(N+2 + (N-1)^2q)y - (N-1)(2 + (N-1)q)p \end{aligned}$$

The condition  $\mathcal{D}_p(m, y) > 0$  means that  $(m, y)$  belong to the upper half plane defined by the line  $\mathcal{D}_p$  with equation

$$y = -am + bp \quad (3.22)$$

where

$$a = \frac{N+2 + (N-1)q}{N+2 + (N^2-1)q} \quad \text{and} \quad b = \frac{(N-1)(2 + (N-1)q)}{N+2 + (N^2-1)q}. \quad (3.23)$$

The problem is reduced to find the variable  $m$  so that the set  $\mathcal{E}(m, y) < 0$  intersects the set  $y + am - bp > 0$ . This means that the second degree equation  $\mathcal{E}(m, -am + bp) = 0$  has two real zeroes. Hence

$$\begin{aligned} 0 &= \frac{(N-1)(Nq+1)(2 + (N-1)q)}{N} (am - bp)^2 \\ &\quad - 2(1 + (N-1)q)(m-1)(am - bp) + m^2 - m \end{aligned} \quad (3.24)$$

where  $a$  and  $b$  are given by (3.23). Its discriminant is given by  $\mathcal{D} = -\frac{b^2}{N}G(p, q)$  where  $G(p, q)$  is defined in (1.10). The condition reads  $G(p, q) < 0$ . If  $q = 0$  we obtain  $p < \frac{N+2}{N-2}$ , which the optimal condition obtained in [10]. More generally the condition on  $p$  is

$$p < p_c(q) := \frac{-b(p) + \sqrt{b^2(q) + 4Nq^2((N-1)^2q + N-2)}}{2((N-1)^2q + N-2)}. \quad (3.25)$$

*Step 3: Elimination of the right-hand side.* Since  $e < 1$ , in order to estimate  $U$  in (3.18) we set  $\gamma_\epsilon(z) = \min\{z^{\frac{q}{2}+e}, \epsilon^{\frac{q}{2}+e-1}z\}$  if  $\frac{q}{2} + e < 1$  and  $\gamma_\epsilon(z) = z^{\frac{q}{2}+e}$  if  $\frac{q}{2} + e \geq 1$ . We multiply (2.36) by  $v^{\lambda+s}\gamma_\epsilon(z)\eta$  and get

$$\begin{aligned} |\beta|^{q-1} \int_{\mathbb{R}^N} v^{2s+\lambda} z^{\frac{q}{2}} \gamma_\epsilon(z) \eta dx &\leq |\lambda + s - \beta - 1| \int_{\mathbb{R}^N} v^{s+\lambda-1} z \gamma_\epsilon(z) \eta dx + \left| \int_{\mathbb{R}^N} \eta v^{\lambda+s} \gamma'_\epsilon(z) \langle \nabla v, \nabla z \rangle dx \right| \\ &\quad + \left| \int_{\mathbb{R}^N} v^{\lambda+s} \gamma_\epsilon(z) \langle \nabla v, \nabla \eta \rangle dx \right|. \end{aligned}$$

There holds by dominated convergence

$$\lim_{\epsilon \rightarrow 0} \left| \int_{\mathbb{R}^N} v^{\lambda+s} \gamma_\epsilon(z) \langle \nabla v, \nabla \eta \rangle dx \right| \leq \int_{\mathbb{R}^N} v^{\lambda+s} z^{\frac{q+1}{2}+1} |\nabla \eta| dx = R.$$

We recall that  $f_\epsilon(z) = z^e$  if  $e \geq 1$  and  $f_\epsilon(z) = \min\{z^e, \epsilon^{e-1}z\}$  if  $e < 1$ . By Cauchy-Schwarz inequality

$$\begin{aligned} \left| \int_{\mathbb{R}^N} \eta v^{\lambda+s} \gamma'_\epsilon(z) \langle \nabla v, \nabla z \rangle dx \right| &\leq \int_{\mathbb{R}^N} v^{\frac{\lambda}{2}} |\nabla z| \sqrt{f'_\epsilon(z)} \frac{v^{\frac{\lambda}{2}+s} \gamma'_\epsilon(z) \sqrt{z}}{\sqrt{f'_\epsilon(z)}} \eta dx \\ &\leq \alpha \int_{\mathbb{R}^N} \frac{v^{\lambda+2s} (\gamma'_\epsilon(z))^2 z}{f'_\epsilon(z)} \eta dx + \frac{4}{\alpha} \int_{\mathbb{R}^N} v^\lambda |\nabla z|^2 f'_\epsilon(z) \eta dx. \end{aligned}$$

We have already seen that there exists

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} v^\lambda |\nabla z|^2 f'_\epsilon(z) \eta dx = e \int_{\mathbb{R}^N} v^\lambda |\nabla z|^2 z^{e-1} \eta dx = eG < \infty.$$

Considering separately the cases  $0 < e < \frac{q}{2} + e < 1$ ,  $0 < e < 1 \leq \frac{q}{2} + e$  and  $1 \leq e < \frac{q}{2} + e$ , we obtain, after some computations,

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} \frac{v^{\lambda+2s} (\gamma'_\epsilon(z))^2 z}{f'_\epsilon(z)} \eta dx = \frac{(q+2e)^2}{4e} \int_{\mathbb{R}^N} v^{\lambda+2s} z^{q+e} \eta dx = \frac{(q+2e)^2}{4e} U,$$

This yields

$$|\beta|^{q-1} U \leq |\lambda + s - \beta - 1| P + R + \frac{\alpha(q+2e)^2}{4e} U + \frac{4e}{\alpha} G. \quad (3.26)$$

Choosing  $\alpha > 0$  small enough we infer

$$U \leq c(P + R + G), \quad (3.27)$$

for some  $c > 0$  depending on the parameters.

From now we assume that the conditions on  $N$ ,  $p$  and  $q$  which ensure the positivity of  $A_0$  and  $B_0$  are fulfilled, and that in this range of values we can find  $m$  such that

$$-2 - (N-1)q < m < 0. \quad (3.28)$$

Combining (3.27) with (3.16) and (3.19) we derive,

$$G + P + F + U \leq c(M + L + R), \quad (3.29)$$

for some  $c > 0$  depending on  $N$ ,  $p$ ,  $\beta$ ,  $\lambda$ ,  $\delta$  and  $q$ .

The method is now to absorb the terms  $M^*$ ,  $L^*$  and  $R^*$  by  $F$ ,  $P$  and  $U$  by a repeated use of Hölder's inequality. Following the method developed in [10] and [5] it is simpler to return to the original function  $u$  and the original exponents, and for homogeneity reason we set  $\eta = \xi^\kappa$  ( $\kappa > 0$ ). Hence (3.29) yields

$$G + \overline{F} + \overline{P} + \overline{U} \leq c_1(\overline{M} + \overline{L} + \overline{R}) \quad (3.30)$$

where

$$\begin{aligned} (i) \quad \overline{F} &= \int_{\mathbb{R}^N} u^{m-2} |\nabla u|^{4+2e} \xi^\kappa dx = \int_{\mathbb{R}^N} u^{m-2} |\nabla u|^{4+(N-1)q} \xi^\kappa dx \\ (ii) \quad \overline{P} &= \int_{\mathbb{R}^N} u^{m+p-1} |\nabla u|^{q+2+2e} \xi^\kappa dx = \int_{\mathbb{R}^N} u^{m+p-1} |\nabla u|^{2+Nq} \xi^\kappa dx \\ (iii) \quad \overline{U} &= \int_{\mathbb{R}^N} u^{m+2p} |\nabla u|^{2q+2e} \xi^\kappa dx = \int_{\mathbb{R}^N} u^{m+2p} |\nabla u|^{(N+1)q} \xi^\kappa dx \end{aligned} \quad (3.31)$$

and

$$\begin{aligned}
(i) \quad \bar{M} &= \int_{\mathbb{R}^N} u^m |\nabla u|^{2+2e} \left( \xi |\Delta \xi| + |\nabla \xi|^2 \right) \xi^{\kappa-2} dx \\
(ii) \quad \bar{L} &= \int_{\mathbb{R}^N} u^{m-1} |\nabla u|^{3+2e} |\nabla \xi| \xi^{\kappa-1} dx \\
(iii) \quad \bar{R} &= \int_{\mathbb{R}^N} u^{m+p} |\nabla u|^{q+1+2e} |\nabla \xi| \xi^{\kappa-1} dx.
\end{aligned} \tag{3.32}$$

*Absorption of  $\bar{L}$ .* Using Hölder's inequality we have

$$\begin{aligned}
\bar{L} &= \int_{\mathbb{R}^N} \left( \xi^\alpha u^{m-1-A} |\nabla u|^B \right) \left( \xi^\gamma u^A |\nabla u|^{3+2e-B} \right) \left( \xi^{\kappa-1-\alpha-\gamma} |\nabla \xi| \right) dx \\
&\leq \int_{\mathbb{R}^N} \xi^{\theta\alpha} u^{\theta(m-1-A)} |\nabla u|^{\theta B} dx + \int_{\mathbb{R}^N} \xi^{t\gamma} u^{tA} |\nabla u|^{t(3+2e-B)} dx \\
&\quad + \int_{\mathbb{R}^N} \xi^{2\sigma(\kappa-1-\alpha-\gamma)} |\nabla \xi|^{2\sigma} dx,
\end{aligned}$$

with

$$\frac{1}{\theta} + \frac{1}{t} + \frac{1}{2\sigma} = 1. \tag{3.33}$$

We choose the unknown exponents so that  $\bar{L} \leq \bar{F} + \bar{P}$  terms in  $\xi$ . We find

$$A = \frac{m+p-1}{t}, \quad B = \frac{4+2e}{\theta},$$

for the exponents of  $|\nabla u|$ ,

$$\alpha = \frac{\kappa}{\theta}, \quad \gamma = \frac{\kappa}{t} \quad \text{and} \quad \kappa = 1 + \frac{\kappa}{\theta} + \frac{\kappa}{t} = 2\sigma,$$

for the ones of  $\xi$  and

$$\theta(m-1-A) = m-2 \quad \text{and} \quad t(3+2e-B) = q+2e+2,$$

for the ones of  $u$ . Eliminating  $A$  and  $B$  leads to a linear system in  $t$  and  $\theta$ ,

$$\begin{aligned}
\frac{m+p-1}{t} + \frac{m-2}{\theta} &= m-1 \\
\frac{q+2+2e}{t} + \frac{4+2e}{\theta} &= 3+2e.
\end{aligned} \tag{3.34}$$

The direct computation shows that

$$\frac{1}{t} = \frac{m+2e+2}{(2-q)m+2((2+e)p+q+e)}, \quad \frac{1}{\theta} = \frac{(1-m)(q-1)+(3+2e)p}{(2-q)m+2((2+e)p+q+e)}, \tag{3.35}$$

hence

$$2\sigma = \frac{(2-q)m+2((2+e)p+q+e)}{p+q-1}. \tag{3.36}$$

We set  $Y = 2\sigma(p + q - 1) = \theta X$ , hence

$$Y = ((2 - q)m + (N - 1)pq + 4p + (N + 1)q)$$

and

$$X = (1 - m)(q - 1) + (N - 1)pq + 3p.$$

First we check that under (3.28)  $Y > 0$ . Indeed we have

$$\begin{aligned} Y - 2(p + q - 1) &= (2 - q)m + (N - 1)pq + 2p + (N - 1)q + 2 \\ &\geq (2 - q)m + (N - 1)pq + 2p - m \\ &= (1 - q)m + p(2 + (N - 1)q), \end{aligned} \quad (3.37)$$

from (3.28). If  $0 \leq q < 1$  the right-hand side of (3.37) is larger than  $(p + q - 1)(2 + (N - 1)q)$  which is positive, while if  $q \geq 1$  and  $m \leq 0$ , the right-hand side is larger than  $m(1 - p - q)$  which is nonnegative. Next we check that  $X > 0$ : if  $q \geq 1$  this is clear by (3.28). If  $0 \leq q < 1$  we have, also by (3.28),

$$X \geq -(3 + (N - 1)q)(1 - q) + ((N - 1)q + 3)p = (3 + (N - 1)q)(p + q - 1) > 0.$$

As a by-product we derive that  $\theta$  and  $t$  are positive and therefore larger than 1 because of (3.33).

*Absorption of  $\bar{R}$ .* We introduce new parameters  $A, B, t, \theta$  in order to absorb  $\bar{R}$  by  $\bar{P} + \bar{U} +$  term in  $\xi$ .

$$\begin{aligned} \bar{R} &= \int_{\mathbb{R}^N} \left( \xi^\alpha u^{m+p-A} |\nabla u|^B \right) \left( \xi^\gamma u^A |\nabla u|^{q+1+2e-B} \right) \left( \xi^{\kappa-1-\alpha-\gamma} |\nabla \xi| \right) dx \\ &\leq \int_{\mathbb{R}^N} \xi^{\theta\alpha} u^{\theta(m+p-A)} |\nabla u|^{\theta B} dx + \int_{\mathbb{R}^N} \xi^{t\gamma} u^{tA} |\nabla u|^{t(q+1+2e-B)} dx \\ &\quad + \int_{\mathbb{R}^N} \xi^{2\sigma(\kappa-1-\alpha-\gamma)} |\nabla \xi|^{2\sigma} dx, \end{aligned}$$

with  $t, \theta$  and  $\sigma$  satisfying (3.33). Hence

$$\begin{aligned} (m + p - A)\theta &= m + p - 1 \\ B\theta &= q + 2 + 2e \\ At &= m + 2p \\ (q + 1 + 2e - B)t &= 2q + e. \end{aligned} \quad (3.38)$$

Thus

$$\begin{aligned} \frac{m + 2p}{t} + \frac{m + p - 1}{\theta} &= m + p \\ \frac{2q + e}{t} + \frac{q + 2 + 2e}{\theta} &= q + 1 + 2e \end{aligned}$$

*Mutatis mutandis* it yields, always with  $\sigma = 2\kappa$  given by (3.36),

$$\frac{1}{\theta} = \frac{m(1 - q) + 2(e + 1)p}{Y}, \quad \frac{1}{t} = \frac{m + p + q + 1 + 2e}{Y}, \quad (3.39)$$

where  $Y$  is unchanged. Condition (3.28) implies  $m + p + q + 1 + 2e \geq p + q - 1 > 0$ , hence  $t > 0$ . Furthermore

$$m(1 - q) + 2(e + 1)p > (m + 2 + 2e)(1 - q) = (m + 2 + (N - 1)q)(1 - q) > 0.$$

which implies  $\theta > 0$  if  $q \leq 1$ . If  $q > 1$  and since  $m < 0$ , then

$$m(1 - q) + 2(e + 1)p > 2(e + 1)p > 0,$$

which again yields  $\theta > 0$ .

*Absorption of  $\overline{M}$ .* We set

$$\begin{aligned} \overline{M}_1 &= \int_{\mathbb{R}^N} u^m |\nabla u|^{2+2e} |\Delta \xi| \xi^{\kappa-1} dx \\ &= \int_{\mathbb{R}^N} \left( \xi^\alpha u^{m-A} |\nabla u|^B \right) \left( \xi^\gamma u^A |\nabla u|^{2+2e-B} \right) \left( \xi^{\kappa-1-\alpha-\gamma} |\Delta \xi| \right) dx \\ &\leq \int_{\mathbb{R}^N} \xi^{\theta\alpha} u^{\theta(m-A)} |\nabla u|^{\theta B} dx + \int_{\mathbb{R}^N} \xi^{t\gamma} u^{tA} |\nabla u|^{t(2+2e-B)} dx \\ &\quad + \int_{\mathbb{R}^N} \xi^{2\sigma(\kappa-1-\alpha-\gamma)} |\Delta \xi|^\sigma dx, \end{aligned}$$

with

$$\frac{1}{\theta} + \frac{1}{t} + \frac{1}{\sigma} = 1. \quad (3.40)$$

If we try to absorb  $\overline{M}_1$  by  $\overline{F} + \overline{U}$  term in  $\xi$ , we obtain

$$\begin{aligned} (m - A)\theta &= m - 2 \\ B\theta &= 4 + 2e \\ At &= m + 2p \\ (2 + 2e - B)t &= 2q + 2e. \end{aligned} \quad (3.41)$$

We find

$$\frac{1}{\theta} = \frac{m(1 - q) + 2(e + 1)p}{Y}, \quad \frac{1}{t} = \frac{m + 2 + 2e}{Y}. \quad (3.42)$$

Clearly (3.40) holds and conditions  $m(1 - q) + 2(e + 1)p > 0$  and  $m + 2 + 2e > 0$  are satisfied under the same condition as for the treatment of  $\overline{R}$ . The same proof works for

$$\overline{M}_2 = \int_{\mathbb{R}^N} u^m |\nabla u|^{2+2e} |\nabla \xi|^2 \xi^{\kappa-2} dx.$$

*Step 4: End of the proof.* It follows from Step 3 that there holds for any nonnegative  $\xi \in C_0^\infty(\mathbb{R}^N)$ :

$$\overline{L} + \overline{F} + \overline{U} \leq c_3 \int_{\mathbb{R}^N} \left( |\Delta \xi|^\sigma + |\nabla \xi|^{2\sigma} \right) dx. \quad (3.43)$$

Assuming  $\xi$  has support in  $B_1$  and applying (3.43) to  $\xi_R : x \mapsto \xi(\frac{x}{R})$ , we derive

$$\overline{L} + \overline{F} + \overline{U} \leq c_3 R^{N-2\sigma} \int_{B_1} \left( |\Delta \xi|^\sigma + |\nabla \xi|^{2\sigma} \right) dx. \quad (3.44)$$

Hence, if  $2\sigma > N$  we infer  $\overline{L} + \overline{F} + \overline{U} = 0$  by letting  $R \rightarrow \infty$ . It remains to prove that such an estimate holds. The condition  $2\sigma > N$  is equivalent to

$$m(2 - pq + (N - 1)pq + N + q) > (N - 4)p, \quad (3.45)$$

or, equivalently,

$$(2 - q)(m + 2 + (N - 1)q) > (p + q - 1)(N - 4 - (N - 1)q). \quad (3.46)$$

Since the left-hand side is positive by (3.28), this inequality holds at least when

$$N = 3, 4 \text{ or } N \geq 5 \text{ and } q \geq \frac{N - 4}{N - 1}. \quad (3.47)$$

The general proof of (3.28), (3.46) is technical and given in Appendix.  $\square$

## 4 Separable solutions

In the sequel we set  $n = N - 1$ , and consider a more general equation on  $S^n$ ,

$$-\Delta' \omega + \mu \omega = \omega^p (\gamma^2 \omega^2 + |\nabla' \omega|^2)^{\frac{q}{2}}, \quad (4.1)$$

where  $\gamma > 0$  and  $\mu$  are parameters and  $\Delta'$  and  $\nabla'$  are respectively the Laplace-Beltrami operator and the covariant gradient, which can be assimilated to the tangential gradient on  $S^n$ . Notice that if  $\mu > 0$  there exists a constant solution  $\omega_\mu$  to (4.1) given by

$$\omega_\mu = \left( \frac{\mu}{\gamma^q} \right)^{\frac{1}{p+q-1}}. \quad (4.2)$$

### 4.1 Uniform bounds: Proof of Theorem D

We set  $a \vee b = \max\{a, b\}$  and  $a \wedge b = \min\{a, b\}$ . By integration on  $S^n$  and Hölder's inequality there holds

$$\mu |S^n|^{\frac{p+q-1}{p+q}} \left( \int_{S^n} \omega^{p+q} dS \right)^{\frac{1}{p+q}} \geq \mu \int_{S^n} \omega dS = \int_{S^n} \omega^p (\gamma^2 \omega^2 + |\nabla' \omega|^2)^{\frac{q}{2}} dS \geq \gamma^q \int_{S^n} \omega^{p+q} dS.$$

Hence

$$\|\omega\|_{L^{p+q}} \leq \left( \frac{\mu}{\gamma^q} \right)^{\frac{1}{p+q-1}} |S^n|^{\frac{1}{p+q}}. \quad (4.3)$$

Therefore

$$\mu \int_{S^n} \omega dS = \int_{S^n} \omega^p (\gamma^2 \omega^2 + |\nabla' \omega|^2)^{\frac{q}{2}} dS \leq \left( \frac{\mu^{p+q}}{\gamma^q} \right)^{\frac{1}{p+q-1}} |S^n|. \quad (4.4)$$

For  $\alpha > 0$ , we also have

$$\begin{aligned}
\int_{S^n} \omega^{p+\alpha} \left( \gamma^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{q}{2}} dS &= \int_{S^n} \left( \alpha \omega^{\alpha-1} |\nabla' \omega|^2 + \mu \omega^{\alpha+1} \right) dS \\
&\geq \mu \wedge \frac{4\alpha}{(\alpha+1)^2} \int_{S^n} \left( \left| \nabla' \omega^{\frac{\alpha+1}{2}} \right|^2 + \left( \omega^{\frac{\alpha+1}{2}} \right)^2 \right) dS \\
&\geq C_1 \left( \mu \wedge \frac{4\alpha}{(\alpha+1)^2} \right) \|\omega\|_{L^{\frac{n(\alpha+1)}{n-2}}}^{\alpha+1},
\end{aligned} \tag{4.5}$$

using Sobolev inequality in  $H^1(S^n)$ . Furthermore, by Hölder's inequality,

$$\begin{aligned}
\int_{S^n} \omega^{p+\alpha} \left( \gamma^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{q}{2}} dS &= \int_{S^n} \omega^{p+\alpha+q\frac{1-\alpha}{2}} \left( \gamma^2 \left( \omega^{\frac{\alpha+1}{2}} \right)^2 + \frac{4}{(\alpha+1)^2} \left| \nabla' \omega^{\frac{\alpha+1}{2}} \right|^2 \right)^{\frac{q}{2}} dS \\
&\leq \gamma^q \vee \left( \frac{2}{1+\alpha} \right)^q \left( \int_{S^n} \omega^{\frac{2p+q}{2-q} + \alpha} dS \right)^{\frac{2-q}{2}} \left( \int_{S^n} \left( \left| \nabla' \omega^{\frac{\alpha+1}{2}} \right|^2 + \left( \omega^{\frac{\alpha+1}{2}} \right)^2 \right) dS \right)^{\frac{q}{2}}.
\end{aligned} \tag{4.6}$$

It implies

$$\int_{S^n} \left( \left| \nabla' \omega^{\frac{\alpha+1}{2}} \right|^2 + \left( \omega^{\frac{\alpha+1}{2}} \right)^2 \right) dS \leq \left( \gamma \vee \frac{2}{1+\alpha} \right)^{\frac{2q}{2-q}} \int_{S^n} \omega^{\frac{2p+q}{2-q} + \alpha} dS. \tag{4.7}$$

Jointly with (4.5) it yields

$$\|\omega\|_{L^{\frac{n(\alpha+1)}{n-2}}}^{\alpha+1} \leq C_2 \frac{(\gamma \vee (1+\alpha)^{-1})^{\frac{2q}{2-q}}}{\mu \wedge \alpha(\alpha+1)^{-2}} \int_{S^n} \omega^{\frac{2p+q}{2-q} + \alpha} dS. \tag{4.8}$$

We define the sequence  $\{\alpha_k\}$  by

$$\frac{2p+q}{2-q} + \alpha_k = \frac{n(\alpha_{k-1} + 1)}{n-2} \iff \alpha_k + 1 = \frac{n(\alpha_{k-1} + 1)}{n-2} - 2\frac{p+q-1}{2-q}. \tag{4.9}$$

The value of  $\alpha_0 > 0$  will be made precise later on. The value of  $\alpha_k$  is explicit:

$$\alpha_k + 1 = \left( \frac{n}{n-2} \right)^k \left( \alpha_0 + 1 - \frac{(p+q-1)(n-2)}{2-q} \right) + \frac{(p+q-1)(n-2)}{2-q}. \tag{4.10}$$

Notice that since  $(n-2)p + (n-1)q < n$ , then  $1 - \frac{(p+q-1)(n-2)}{2-q} > 0$ . Asymptotically

$$\alpha_k + 1 = A\ell^k + O(1) \quad \text{with } \ell = \frac{n}{n-2} > 1. \tag{4.11}$$

We set  $X_k = \|\omega\|_{L^{\frac{n(\alpha_k+1)}{n-2}}}$ , hence (4.8) reads

$$X_k \leq \left( C_2 \frac{(\gamma \vee (1+\alpha_k)^{-1})^{\frac{2q}{2-q}}}{\mu \wedge \alpha_k(\alpha_k+1)^{-2}} \right)^{\frac{1}{1+\alpha_k}} X_{k-1}^{1+2\frac{p+q-1}{(2-q)(\alpha_k+1)}}. \tag{4.12}$$



Because of (4.11),

$$\begin{aligned} C_2 \frac{(\gamma \vee (1 + \alpha_k)^{-1})^{\frac{2q}{2-q}}}{\mu \wedge \alpha_k (\alpha_k + 1)^{-2}} &\leq C_3 \alpha_k^{-1} \gamma^{\frac{2q}{2-q}} \\ \implies \left( C_2 \frac{(\gamma \vee (1 + \alpha_k)^{-1})^{\frac{2q}{2-q}}}{\mu \wedge \alpha_k (\alpha_k + 1)^{-2}} \right)^{\frac{1}{1+\alpha_k}} &\leq C_4^{\frac{1}{1+\alpha_k}} \gamma^{\frac{2q}{(1+\alpha_k)(2-q)}} \end{aligned}$$

provided  $\alpha_0 \geq \epsilon_0 > 0$ , hence

$$X_k \leq C_4^{\frac{1}{1+\alpha_k}} \gamma^{\frac{2q}{(1+\alpha_k)(2-q)}} X_{k-1}^{1+2\frac{p+q-1}{(2-q)(\alpha_k+1)}}. \quad (4.13)$$

Next we construct by induction an increasing sequence  $\Gamma_k$  such that

$$X_{k-1} \leq \Gamma_{k-1} \gamma^{-\frac{q}{p+q-1}}. \quad (4.14)$$

Then (4.14) holds at the order  $k$  with

$$\Gamma_k = C_4^{\frac{1}{1+\alpha_k}} \Gamma_{k-1}^{1+2\frac{p+q-1}{(2-q)(\alpha_k+1)}}, \quad (4.15)$$

and  $\Gamma_0$  will be fixed later on. We can assume that  $C_4 \geq 1$ , therefore  $\{\Gamma_k\}$  is increasing. If we put  $\theta_k = \ln \Gamma_k$ , then

$$\theta_k = \frac{1}{1+\alpha_k} \ln C_4 + \left( 1 + 2\frac{p+q-1}{(2-q)(\alpha_k+1)} \right) \theta_{k-1} = \frac{A}{1+\alpha_k} + \left( 1 + \frac{B}{1+\alpha_k} \right) \theta_{k-1}$$

Put  $\tilde{\theta}_k = \theta_k + \frac{A}{B}$ , then

$$\tilde{\theta}_k = \left( 1 + \frac{B}{1+\alpha_k} \right) \tilde{\theta}_{k-1} \implies \tilde{\theta}_k = \tilde{\theta}_0 \prod_{j=1}^k \left( 1 + \frac{B}{1+\alpha_j} \right)$$

Finally

$$\theta_k = \left( \theta_0 + \frac{A}{B} \right) \prod_{j=1}^k \left( 1 + \frac{B}{1+\alpha_j} \right) - \frac{A}{B}, \quad (4.16)$$

This infer that there exists

$$\Gamma^* = \lim_{k \rightarrow \infty} \Gamma_k. \quad (4.17)$$

By standard linear elliptic regularity theory with  $L^1$  data and (4.4),

$$\begin{aligned} \|\omega\|_{L^{\frac{n}{n-2}, \infty}} + \|\nabla' \omega\|_{L^{\frac{n}{n-1}, \infty}} &\leq C_5 \int_{S^n} \left( \omega^p (\gamma^2 \omega^2 + |\nabla' \omega|^2)^{\frac{q}{2}} + \mu \omega \right) dS \\ &\leq 2C_5 \left( \frac{\mu^{p+q}}{\gamma^q} \right)^{\frac{1}{p+q-1}} |S^n|, \end{aligned} \quad (4.18)$$

where in  $L^{r,\infty}$  denotes the usual Marcinkiewicz spaces (or Lorentz spaces). For any  $1 < \tau < \frac{n}{n-1}$ , there exists  $C_6 = C(n, \tau)$  such that

$$\|\omega\|_{W^{1,\tau}} \leq C_6 \left( \|\omega\|_{L^{\frac{n}{n-2},\infty}} + \|\nabla' \omega\|_{L^{\frac{n}{n-1},\infty}} \right),$$

and by Sobolev inequality

$$\|\omega\|_{L^{\tau^*}} \leq C_7 \|\omega\|_{W^{1,\tau}},$$

where  $\frac{1}{\tau^*} = \frac{1}{\tau} - \frac{1}{n}$ . Since  $n(p-2) + q(n-1) < n$  is equivalent to  $\frac{2p+q}{2-q} < \frac{n}{n-2}$ , we can take  $\frac{1}{\tau} = \left( \frac{2-q}{2p+q} + \alpha_1 \right)^{-1} + \frac{1}{n}$  for some  $\alpha_1 > 0$ . Using (4.9) we define the initial data of  $\{\alpha_k\}$  by

$$\frac{2p+q}{2-q} + \alpha_1 = \frac{n(\alpha_0 + 1)}{n-2}$$

and we derive

$$X_0 = \|\omega\|_{L^{\frac{n(\alpha_0+1)}{n-2}}} = \|\omega\|_{L^{\frac{2p+q}{2-q} + \alpha_1}} \leq C_8 \left( \frac{\mu^{p+q}}{\gamma^q} \right)^{\frac{1}{p+q-1}}. \quad (4.19)$$

Finally, we can fix

$$e^{\theta_0} = \Gamma_0 = C_8 \mu^{\frac{p+q}{p+q-1}},$$

and derive from (4.16)

$$\Gamma^* = C_9 e^{\theta_0 \frac{A}{B} \prod_{j=0}^{\infty} (1 + \frac{B}{1+\alpha_j})} = c_{10} \mu^{\frac{p+q}{p+q-1} \frac{A}{B} \prod_{j=0}^{\infty} (1 + \frac{B}{1+\alpha_j})} \quad (4.20)$$

with

$$C_9 = e^{\frac{A}{B} \prod_{j=0}^{\infty} (1 + \frac{B}{1+\alpha_j}) - \frac{A}{B}} \quad \text{and} \quad c_{10} = C_9 C_8^{\frac{A}{B} \prod_{j=0}^{\infty} (1 + \frac{B}{1+\alpha_j})}.$$

Since

$$\|\omega\|_{L^\infty} \leq \gamma^{-\frac{q}{p+q-1}} \Gamma^*,$$

the estimate follows.  $\square$

*Remark.* We conjecture that the best exponent  $a$  is equal to 1 and

$$\|\omega\|_{L^\infty} \leq C_{11} \left( \frac{\mu}{\gamma^q} \right)^{\frac{1}{p+q-1}}. \quad (4.21)$$

Notice that there always holds

$$\min_{S^n} \omega \leq \left( \frac{\mu}{\gamma^q} \right)^{\frac{1}{p+q-1}} \leq \max_{S^n} \omega. \quad (4.22)$$

## 4.2 Rigidity and symmetry

**Theorem 4.1** *Assume  $\gamma, \mu > 0, p + q - 1 > 0$  and  $\omega$  is a solution of (4.1) on  $S^n$  such that*

$$\begin{aligned} (i) \quad & \gamma^2\omega^2 + |\nabla'\omega|^2 \leq c_1^2 && \text{if } p \geq 1, \\ (ii) \quad & c_2^2 \leq \gamma^2\omega^2 + |\nabla'\omega|^2 \leq c_1^2 && \text{if } 0 \leq p < 1, \end{aligned} \quad (4.23)$$

for some  $c_1, c_2 > 0$  and set

$$c_* = \begin{cases} c_1 & \text{if } p \geq 1, \\ c_2^{\frac{p-1}{p+q-1}} c_1^{\frac{q}{p+q-1}} & \text{if } 0 \leq p < 1. \end{cases} \quad (4.24)$$

If  $c^*$  satisfies

$$c_*^{p+q-1} \leq \frac{2(n+\mu)}{q\gamma^{-p}\sqrt{n} + 2(p+q)\gamma^{1-p}}, \quad (4.25)$$

then  $\omega$  is constant.

*Proof.* If  $w$  is a function defined on  $S^n$ , we put

$$\bar{w} = \frac{1}{|S^n|} \int_{S^n} w(\sigma) dS.$$

If

$$-\Delta'\omega + \mu\omega - |\omega|^{p-1}\omega(\gamma^2\omega^2 + |\nabla'\omega|^2)^{\frac{q}{2}} = 0, \quad (4.26)$$

we have

$$-\Delta'\bar{\omega} + \mu\bar{\omega} - \overline{|\omega|^{p-1}\omega(\gamma^2\omega^2 + |\nabla'\omega|^2)^{\frac{q}{2}}} = 0.$$

Since  $\overline{\omega - \bar{\omega}} = 0$ ,  $\bar{\omega}$  in the orthogonal projection in  $L^2(S^n)$  of  $w$  on  $\ker(-\Delta')$  and  $n$  is the first nonzero eigenvalue, we have

$$\begin{aligned} & \int_{S^n} \left( \omega^p(\gamma^2\omega + |\nabla'\omega|^2)^{\frac{q}{2}} - \overline{\omega^p(\gamma^2\omega^2 + |\nabla'\omega|^2)^{\frac{q}{2}}} \right) (\omega - \bar{\omega}) dS \\ &= \int_{S^n} \left( \omega^p(\gamma^2\omega^2 + |\nabla'\omega|^2)^{\frac{q}{2}} - \bar{\omega}^p(\gamma^2\bar{\omega}^2 + |\nabla'\bar{\omega}|^2)^{\frac{q}{2}} \right) (\omega - \bar{\omega}) dS \end{aligned} \quad (4.27)$$

and

$$\begin{aligned} \int_{S^n} (-\Delta'(\omega - \bar{\omega} + \mu(\omega - \bar{\omega}))) (\omega - \bar{\omega}) dS &= \int_{S^n} (|\nabla'(\omega - \bar{\omega})|^2 + \mu(\omega - \bar{\omega})^2) dS \\ &\geq (\mu + n) \int_{S^n} (\omega - \bar{\omega})^2 dS \end{aligned} \quad (4.28)$$

Set  $F(X, Y) = |X|^{p-1} X(\gamma^2 X^2 + |Y|^2)^{\frac{q}{2}}$  and

$$G := \{(X, Y) \in \mathbb{R} \times \mathbb{R}^n : c_2^2 \leq \gamma^2 X^2 + |Y|^2 \leq c_1^2\}.$$

Then

$$D_1F(X, Y) = |X|^{p-1} \left( \gamma^2 X^2 + |Y|^2 \right)^{\frac{q}{2}-1} \left( \gamma^2(p+q)|X|^2 + p|Y|^2 \right)$$

$$D_2F(X, Y) = q|X|^{p-1} X \left( \gamma^2 X^2 + |Y|^2 \right)^{\frac{q}{2}-1} Y$$

If we assume that  $(\omega, \nabla' \omega) \in G$ , we have, with  $\xi = |\omega - \bar{\omega}|$  and  $\eta = |\nabla'(\omega - \bar{\omega})|$ ,

$$\sup\{|D_1F(\omega, \nabla' \omega)| : (\omega, \nabla' \omega) \in G\} \leq \begin{cases} (p+q)\gamma^{1-p}c_1^{p+q-1} & \text{if } p \geq 1, \\ (p+q)\gamma^{1-p}c_2^{p-1}c_1^q & \text{if } 0 \leq p < 1. \end{cases}$$

and, since  $|\omega||\nabla' \omega| \leq \frac{1}{2\gamma}(\gamma^2\omega^2 + |\nabla' \omega|^2)$ ,

$$\sup\{|D_2F(\omega, \nabla' \omega)| : (\omega, \nabla' \omega) \in G\} \leq \begin{cases} \frac{q}{2}\gamma^{-p}c_1^{p+q-1} & \text{if } p \geq 1, \\ \frac{q}{2}\gamma^{-p}c_2^{p-1}c_1^q & \text{if } 0 \leq p < 1. \end{cases}$$

$$\int_{S^n} (\eta^2 + \mu\xi^2) dS \leq \gamma^{1-p}c_*^{p+q-1} \int_{S^n} ((p+q)\xi^2 + \frac{q}{2}\gamma\eta\xi) dS. \quad (4.29)$$

Set

$$\Xi := \left( \int_{S^n} \xi^2 dS \right)^{\frac{1}{2}} \text{ and } H := \left( \int_{S^n} \eta^2 dS \right)^{\frac{1}{2}},$$

and recall that  $c_*$  is defined in (4.24). Then  $\sqrt{n}\Xi \leq H$ . We define the polynomials

$$P(\Xi, H) = H^2 - \frac{q}{2}\gamma^{-p}c_*^{p+q-1}H\Xi + (\mu - (p+q)\gamma^{1-p}c_*^{p+q-1})\Xi^2.$$

and, putting  $T = \frac{H}{\Xi}$  when  $\Xi > 0$ , we have

$$\mathcal{P}(T) = T^2 - \frac{q}{2}\gamma^{-p}c_*^{p+q-1}T + \mu - (p+q)\gamma^{1-p}c_*^{p+q-1}. \quad (4.30)$$

Then  $T \geq \sqrt{n}$  since  $n$  is the first nonzero eigenvalue of  $-\Delta'$  in  $H^1(S^n)$ .

Next we suppose that  $\mathcal{P}(\sqrt{n}) \geq 0$ . This means

$$n - \frac{q}{2}\gamma^{-p}c_*^{p+q-1}\sqrt{n} + \mu - (p+q)\gamma^{1-p}c_*^{p+q-1} \geq 0, \quad (4.31)$$

and it is equivalent to

$$\left( \frac{q}{2}\gamma^{-p}\sqrt{n} + (p+q)\gamma^{1-p} \right) c_*^{p+q-1} \leq n + \mu. \quad (4.32)$$

We have three possibilities:

(i) either  $\Xi > 0$  and

$$\frac{q^2}{4}\gamma^{-2p}c_*^{2(p+q-1)} + 4(p+q)\gamma^{1-p}c_*^{p+q-1} \geq 4\mu, \quad (4.33)$$

then the polynomial  $\mathcal{P}$  admits two real roots  $T_1 \leq T_2$  and  $\mathcal{P}(T) \leq 0$ . Jointly with the constraint on  $T$  it means

$$T_1 \leq T \leq T_2 \leq \sqrt{n} \leq T.$$

Then  $T = \sqrt{n}T$ , which implies  $\omega - \bar{\omega} = \tau\phi_1$  for some  $\tau \in \mathbb{R}^*$ . This is not compatible with the fact that  $\omega$  solves (4.26 ).

(ii) either  $\Xi > 0$  and

$$\frac{q^2}{4}\gamma^{-2p}c_*^2(p+q-1) + (p+q)\gamma^{1-p}c_1^{p+q-1} < 4\mu. \quad (4.34)$$

Then  $\mathcal{P}$  remains positive, which is impossible because of (4.29 ),

(iii) or  $\Xi = 0$ . In such a case  $\omega = \bar{\omega}$ ,  $\omega$  is a constant and  $\nabla'\omega = 0$ . Therefore, if (4.32) holds  $\omega = \bar{\omega}$  which ends the proof.  $\square$

*Remark.* We notice that if we suppose  $q = 0$  in (4.32 ) we find back condition (2.53) in [13, Th 2.2].

### 4.3 Bifurcation

In this paragraph we are interested in solution of (4.1) which bifurcate from the constant solution  $\omega_{\mu_*}$  defined by (4.2 ) with  $\mu_* = \frac{n}{p+q-1}$ .

**Theorem 4.2** *Assume  $\gamma > 0$ ,  $p+q-1 > 0$ , and set  $\mu_* = \frac{n}{p+q-1}$ . Then there exists a neighborhood  $\mathcal{O}$  of  $(\mu_*, \omega_{\mu_*})$  in  $\mathbb{R} \times C^1(S^n)$  such that if  $\omega$  is a solution of (4.1) in  $S^n$  such that  $(\mu, \omega) \in \mathcal{O}$ , there holds either  $(\mu, \omega) = (\mu, 0)$  or  $\mu > 0$ ,  $(\mu, \omega) = (\mu_* + \epsilon(s), \omega_{\mu_*} + s(\phi_1 + \phi(s)))$  where  $s \mapsto \epsilon(s)$  is a  $C^1$  positive function defined on  $[0, \tau]$ , vanishing at  $s = 0$  and  $s \mapsto \phi(s)$  is a  $C^1$  function defined on  $[0, \tau]$ , vanishing at  $s = 0$*

*Proof.* We set

$$\mathcal{L}(\omega) = -\Delta'\omega + \mu\omega - \omega^p(\gamma^2\omega^2 + |\nabla'\omega|^2)^{\frac{q}{2}}.$$

We look for solutions under the form  $\omega = \omega_{\mu_*} + \phi$  with  $\phi$  small. Then

$$\begin{aligned} \omega^p(\gamma^2\omega^2 + |\nabla'\omega|^2)^{\frac{q}{2}} &= (\omega_{\mu_*} + \phi)^p \left( \gamma^2(\omega_{\mu_*} + \phi)^2 + |\nabla'\phi|^2 \right)^{\frac{q}{2}} \\ &= \gamma^q \omega_{\mu_*}^{p+q} \left( 1 + \frac{1}{\omega_{\mu_*}} \phi \right)^p \left( 1 + \frac{2}{\omega_{\mu_*}} \phi + \frac{1}{\omega_{\mu_*}^2} \phi^2 + \frac{1}{\gamma^2 \omega_{\mu_*}^2} |\nabla'\phi|^2 \right)^{\frac{q}{2}} \\ &= \gamma^q \omega_{\mu_*}^{p+q} \left( 1 + \frac{p}{\omega_{\mu_*}} \phi + \frac{p(p-1)}{2\omega_{\mu_*}^2} \phi^2 \right) \left( 1 + \frac{q}{\omega_{\mu_*}} \phi + \frac{q(q-1)}{2\omega_{\mu_*}^2} \phi^2 + \frac{q}{2\gamma^2 \omega_{\mu_*}^2} |\nabla'\phi|^2 \right) \\ &\quad + O(\phi^3 + \phi |\nabla'\phi|^2) \\ &= \gamma^q \omega_{\mu_*}^{p+q} \left( 1 + \frac{p+q}{\omega_{\mu_*}} \phi + \frac{(p+q)(p+q-1)}{2\omega_{\mu_*}^2} \phi^2 + \frac{q}{2\gamma^2 \omega_{\mu_*}^2} |\nabla'\phi|^2 \right) \\ &\quad + O(\phi^3 + \phi |\nabla'\phi|^2). \end{aligned}$$

Then, because of the value of  $\omega_\mu$  and  $\gamma$ ,

$$\begin{aligned}\mathcal{L}(\omega_\mu + \phi) &= -\Delta' \phi + \mu \omega_\mu + \mu \phi - \gamma^q \omega_\mu^{p+q} \left( 1 + \frac{p+q}{\omega_\mu} \phi + \frac{(p+q)(p+q-1)}{2\omega_\mu^2} \phi^2 + \frac{q}{2\gamma^2 \omega_\mu^2} |\nabla' \phi|^2 \right) \\ &\quad + O(\phi^3 + \phi |\nabla' \phi|^2) \\ &= -\Delta' \phi + \left( \mu - (p+q) \gamma^q \omega_\mu^{p+q-1} \right) \phi \\ &\quad - \frac{(p+q)(p+q-1)}{2} \gamma^q \omega_\mu^{p+q-2} \phi^2 - \frac{q}{2} \gamma^{q-2} \omega_\mu^{p+q-2} |\nabla' \phi|^2 \\ &\quad + O(\phi^3 + \phi |\nabla' \phi|^2).\end{aligned}$$

Since  $\gamma^q \omega_\mu^{p+q-1} = \mu$

$$\begin{aligned}\mathcal{L}(\omega_\mu + \phi) &= -\Delta' \phi - \mu \phi - \frac{(p+q)(p+q-1)}{2} \gamma^q \omega_\mu^{p+q-2} \phi^2 - \frac{q}{2} \gamma^{q-2} \omega_\mu^{p+q-2} |\nabla' \phi|^2 \\ &\quad + O(\phi^3 + \phi |\nabla' \phi|^2).\end{aligned}\tag{4.35}$$

If

$$\mu(p+q-1) = n \iff \mu := \mu_* = \frac{n}{p+q-1},\tag{4.36}$$

we can take  $\phi = \epsilon \phi_1$  where  $\epsilon$  is small and  $\phi_1$  is the first non-zero eigenfunction (with corresponding eigenvalue  $n$ ). Then

$$\mathcal{L}(\omega_\mu + \epsilon \phi_1) = -\epsilon^2 \omega_\mu^{p+q-2} \gamma^{q-2} \left( \frac{(p+q)(p+q-1)}{2} \gamma^2 \phi_1^2 + \frac{q}{2} |\nabla' \phi_1|^2 \right) + O(\epsilon^3).\tag{4.37}$$

We want to apply [18, Th 13.4, 13.5, Ex 2 p. 174] and we consider solutions of

$$-\Delta' \omega + \mu \omega - |\omega|^{p-1} \omega (\gamma^2 \omega^2 + |\nabla' \omega|^2)^{\frac{q}{2}} = 0\tag{4.38}$$

depending only on the azimuthal angle  $\theta_n := \theta \in (0, \pi)$ , which means

$$-\omega_{\theta\theta} - (n-1) \cot \theta \omega_\theta + \mu \omega - |\omega|^{p-1} \omega (\gamma^2 \omega^2 + \omega_\theta^2)^{\frac{q}{2}}.\tag{4.39}$$

and denote by  $C_{rad}^{2,\delta}(S^n)$  ( $\delta \in (0, 1)$ ) the space of  $C^{2,\delta}$  functions depending only on the angle  $\theta$  (and thus radial with respect to the other variables  $(\theta_1, \dots, \theta_{n-1})$ ). The critical constant solution  $\omega_\mu$  is expressed by

$$\omega_{\mu_*} = (\gamma^{-q} \mu_*)^{\frac{1}{p+q-1}}.$$

Since the bifurcation point in [18] are taken at  $(\mu_*, \omega_{\mu_*})$ , we put  $\omega = \omega_{\mu_*} + w$  and

$$f(\mu, w) = -\Delta' w + \mu(\omega_{\mu_*} + w) - |\omega_{\mu_*} + w|^{p-1} (\omega_{\mu_*} + w) (\gamma^2 (\omega_{\mu_*} + w)^2 + |\nabla' w|^2)^{\frac{q}{2}}.$$

Then

$$D_2 f(\mu, 0) = -\Delta' - (p+q-1) \mu I.$$

If  $(p+q-1)\mu = n \iff \mu = \mu_*$ , then  $\ker D_2 f(\mu_*, 0)$  is spanned by  $\phi_1 : \theta \mapsto \cos \theta$  and

$$\mathbf{R}(D_2 f(\mu_*, 0)) = \left\{ \psi \in C_{rad}^\delta(S^n) : \int_{S^n} \psi \phi_1 dS = 0 \right\}.$$

Finally  $D_1D_2f(\mu_*, 0)(\mu, v) = -(p + q - 1)\mu v$  thus  $D_1D_2f(\mu_*, 0)(\mu_*, \phi_1)$  does not belong to  $\mathbb{R}(D_2f(\mu_*, 0))$ . Therefore the bifurcation theorem applies and there exists  $\epsilon > 0$  a  $C^1$  curve  $s \mapsto (\mu(s), \phi(s))$  defined on  $[-\epsilon, \epsilon]$  with value in  $\mathbb{R} \times \mathbb{R}(D_2f(\mu_*, \omega_{\mu_*}))$  such that

$$\begin{aligned} (i) \quad & \mu(0) = \mu_* \\ (ii) \quad & \phi(0) = 0 \\ (iii) \quad & f(\mu(s), s(\phi_1 + \phi(s))) = 0 \end{aligned} \tag{4.40}$$

Furthermore, there exists a neighborhood  $\mathcal{O}$  of  $(\mu_*, \omega_{\mu_*})$  in which any solution of  $f(\mu, 0) = 0$  is either  $(\mu, 0)$  or under the form  $(\mu(s), s(\phi_1 + \phi(s)))$ . Equivalently, any solution of (4.39) is either  $(\mu, \omega_{\mu_*})$  or is of the form  $(\mu(s), \omega_{\mu_*} + s(\phi_1 + \phi(s)))$ .

This is this last statement which applies in our case.

In order to see in what direction the bifurcation occurs we set  $\mu(s) = \mu_* + \epsilon s$  and  $\omega = \omega_{\mu_*} + s(\phi_1 + \phi)$  where  $\epsilon = \epsilon(s)$  and  $\phi = \phi(s)$ . Then, from (4.35) with  $\mu$  replaced by  $\mu_* + \epsilon s$ , we get

$$\begin{aligned} & -s\Delta'(\phi_1 + \phi) + \mu_*\omega_{\mu_*} + \epsilon\omega_{\mu_*} + s\mu_*(\phi_1 + \phi) \\ & \quad + s\epsilon(\phi_1 + \phi) - \gamma^q\omega_{\mu_*}^{p+q} - s\gamma^q\omega_{\mu_*}^{p+q-1}(\phi_1 + \phi) \\ & = \frac{\omega_{\mu_*}^{p+q-2}s^2\gamma^q}{2} \left( (p+q)(p+q-1)(\phi_1 + \phi)^2 + \gamma^{-2}q|\nabla'(\phi_1 + \phi)|^2 \right) + o(s^2), \end{aligned} \tag{4.41}$$

which reduces to

$$\begin{aligned} & \epsilon\omega_{\mu_*} + s\epsilon(\phi_1 + \phi) + s(-\Delta'\phi - (p+q-1)\mu_*\phi) \\ & = \frac{\omega_{\mu_*}^{p+q-2}s^2\gamma^q}{2} \left( (p+q)(p+q-1)(\phi_1 + \phi)^2 + \gamma^{-2}q|\nabla'(\phi_1 + \phi)|^2 \right) + o(s^2). \end{aligned} \tag{4.42}$$

Since  $\phi \in \mathbb{R}(D_2f(\mu_*, \omega_{\mu_*}))$

$$\int_{S_+^n} \phi\phi_1 dS = 0,$$

thus

$$\begin{aligned} & \epsilon \int_{S_+^n} (\omega_{\mu_*}\phi_1 + s\phi_1^2) dS \\ & = \frac{\omega_{\mu_*}^{p+q-2}s^2\gamma^q}{2} \int_{S_+^n} \left( (p+q)(p+q-1)(\phi_1 + \phi)^2 + \gamma^{-2}q|\nabla'(\phi_1 + \phi)|^2 \right) \phi_1 dS + o(s^2). \end{aligned} \tag{4.43}$$

Therefore  $\epsilon > 0$  for  $|s|$  small enough and the bifurcation goes in the direction of the increasing  $\mu$ .  $\square$

## 5 Appendix

### 5.1 Position of the problem

We set for simplicity

$$h = 2e = (N - 1)q \in [0, 2(N - 1)].$$

The conditions to be satisfied by the parameters  $(m, y)$  are:

$$A_0 > 0, \quad B_0 > 0, \quad (5.1)$$

with  $y \neq 1$  and with  $y \neq 1$  and

$$m < 0, \quad m + 2 + h > 0 \quad (5.2)$$

and

$$(m + h + 2)(2N - 2 - h) > (N - 4 - h)((N - 1)p + h + 1 - N). \quad (5.3)$$

We denote by  $\mathcal{E}$  the ellipse of equation  $\mathcal{E}(m, y) = 0$ , where

$$\mathcal{E}(m, y) := Ky^2 + 2(h + 1)(m - 1)y - 2(h + 1)y + m(m - 1)$$

with

$$K = \frac{2 + h}{N}(N - 1 + Nh).$$

Let  $\mathcal{D}_p$  be the line defined by the equation  $\mathcal{D}_p(m, y) = 0$ , where

$$\mathcal{D}_p(m, y) = y + am - bp$$

with

$$a = \frac{N + 2 + h}{2((N + 1)h + N + 2)}, \quad b = \frac{(N - 1)(h + 2)}{2((N + 1)h + N + 2)}$$

So the conditions (5.1) are equivalent to

$$\mathcal{E}(m, y) < 0 \quad \text{and} \quad \mathcal{D}_p(m, y) > 0,$$

which means that the line  $\mathcal{D}_p$  intersect the ellipse  $\mathcal{E}$ , and  $(m, y)$  lies inside  $\mathcal{E}$  and above  $\mathcal{D}_p$  where  $p > 0$ . First we write that  $\mathcal{D}$  intersects  $\mathcal{E}$ , that means the equation  $\mathcal{E}(m, -am + bp) = 0$  has at least one root. We obtain the equation  $\mathcal{T}(m) = 0$  with

$$\mathcal{T}(m) = \left( Ka^2 - 2\frac{bh}{N-1} \right) m^2 - 2 \left( bp(Ka - 1 - h) - \frac{bh}{N-1} \right) m + bp(Kbp - 2(1 + h)).$$

It is needed that its discriminant  $\mathcal{J}$  be nonnegative and it is convenient to express it in terms of  $t = bp$ , hence

$$\mathcal{J} = \left( tKa - \frac{bh}{N-1} - t(1 + h) \right)^2 - (Kt^2 - 2(1 + h)t) \left( Ka^2 - 2\frac{bh}{N-1} \right).$$

We notice that

$$2a(1 + h) - 1 = \frac{2bh}{N-1}, \quad (5.4)$$



therefore

$$\begin{aligned}
\mathcal{J} &= K^2 t^2 a^2 + \left( \frac{bh}{N-1} + t(1+h) \right)^2 - 2tKa \left( \frac{bh}{N-1} + t(1+h) \right) - K^2 t^2 a^2 \\
&\quad + 2(1+h)t \left( Ka^2 - 2\frac{bh}{N-1} \right) - 2\frac{bh}{N-1} Kt^2 \\
&= \left( (1+h)^2 - 2aK(1+h) - 2\frac{bh}{N-1}K \right) t^2 + 2b \left( -\frac{h}{N-1}(1+h) - \frac{ahK}{N-1} \right. \\
&\quad \left. + (1+h)K\frac{a^2}{b} \right) t + \frac{b^2 h^2}{(N-1)^2}.
\end{aligned}$$

Hence  $\mathcal{J} = -\frac{b^2}{N}\tilde{G}(p, h)$  where

$$\begin{aligned}
\tilde{G}(p, h) &= (K - (h+1)^2)p^2 + 2 \left( \frac{h(1+h)}{N-1} - \frac{a^2}{2b} \right) p + \frac{h^2}{(N-1)^2} \\
&= (N-1)((N-1)h + N - 2)p^2 \\
&\quad + (Nh^2 - (N^2 + N - 1)h - N^2 - N + 2)p - \frac{N}{N-1}h^2.
\end{aligned} \tag{5.5}$$

So we find precisely that  $\tilde{G}(p, (N-1)q) = G(p, q)$  where the function  $G$  is given in Theorem C. The equation  $\tilde{G}(p, h) = 0$  has two roots with opposite sign in  $p$ , that we call  $p_0(q) > 0 \geq p_1(q)$ . Both correspond to the fact that the lines  $\mathcal{D}_{p_i(q)}$  are tangent to the ellipse  $\mathcal{E}$ . The region  $\mathcal{D}_{p_1(q)}(m, y) > 0$  contains the whole region  $\mathcal{E}(m, y) < 0$  (the interior of  $\mathcal{E}$ ) while  $\mathcal{D}_{p_0(q)}(m, y) > 0$  has an empty intersection with the region  $\mathcal{E}(m, y) < 0$ . Hence for  $p > 0$ , the line  $\mathcal{D}_p$  intersect the ellipse if and only if  $p < p_0(q)$ . If  $q = 0$ , then we find that  $p_0(0) = \frac{N+2}{N-2}$ , which was the precise optimal value obtained for the Emden-Fowler equation  $-\Delta u = u^p$ . Now for  $p = p_0(q)$ , the line  $\mathcal{D}_{p_0(q)}$  is tangent to the ellipse at some point  $(m_0, y_0) = (m_0(q), y_0(q))$  in the upper part of  $\mathcal{E}$ , given by

$$y_0 = \frac{1}{K} \left( (1-m_0)(h+1) + \sqrt{(1-m_0)((1-m_0)(h+1)^2 + m_0K)} \right) > 0.$$

Suppose that we have proved that  $m_0(q)$  satisfies the conditions (5.2) and (5.3), then for a given  $p < p_0(q)$  any couple  $(y, m)$  with  $0 < y < y_0(q)$ ,  $y \neq 1$ , and  $m = m_0(q)$  will satisfy all the required conditions. Therefore it is sufficient to prove (5.2) and (5.3) in case  $m = m_0(q)$ .

*Remark.* If it happens that  $m_0 + 2 + h = 0$ , then we can take  $m = m_0 + \epsilon$  with  $\epsilon > 0$  small enough such that  $(m, y)$  stays in  $\mathcal{E}$ . We know from [5] that it happens precisely when  $N = 3$  and  $h = 0$ . We will see below that it is the only case.

Next we compute  $m_0$ . We note here that the discriminant of  $p \mapsto \tilde{G}(p, h)$  is

$$\mathcal{H} = (Nh^2 - (N^2 + N - 1)h - N^2 - N + 2)^2 + 4Nh^2((N-1)h + N - 2) > 0$$

and it can be written under the form  $\mathcal{H} = (Nh + N - 1)\mathcal{M}$ , where

$$\mathcal{M} = \mathcal{M}(h) = Nh^3 - (2N^2 - N + 1)h^2 + (N^3 + 2N^2 - 2N - 4)h + (N-1)(N+2)^2.$$

Then  $p_0 = p_0(q)$  is given by

$$p_0 = \frac{-(Nh^2 - (N^2 + N - 1)h - N^2 - N + 2) + \sqrt{(Nh + N - 1)\mathcal{M}}}{2(N - 1)((N - 1)h + N - 2)}. \quad (5.6)$$

Since  $m_0$  is also the minimizer of the trinom  $m \mapsto \mathcal{T}(m)$ , it is expressed by

$$m_0 = b \frac{(N - 1)(Ka - 1 - h)p_0 - h}{(N - 1)Ka^2 - 2bh},$$

and we obtain, after some computation,

$$m_0 = \frac{Q_1(h) + (N - 1)Q_2(h)p_0}{\mathcal{M}},$$

where

$$Q_1(h) = -2Nh((N + 1)h + N + 2), \quad (5.7)$$

and

$$Q_2(h) = Nh^3 - (N^2 - 3N + 1)h^2 - (N^2 - N + 4)h - 2(N - 2). \quad (5.8)$$

Replacing  $p_0$  by its value given in (5.6), we deduce after some simplifications that

$$2((N - 1)h + N - 2)m_0\mathcal{M} = -(Nh^2 + (N + 1)h + 2)\mathcal{M} + Q_2\sqrt{(Nh + N - 1)\mathcal{M}}.$$

Hence

$$2((N - 1)h + N - 2)m_0 = -P_1 + Q_2\sqrt{\frac{Nh + N - 1}{\mathcal{M}}} \quad (5.9)$$

with

$$P_1(h) = Nh^2 + (N + 1)h + 2.$$

Note that  $m_0$  can be also obtained equivalently by expressing the fact that  $(m_0, y_0)$  belongs in the upper part of  $\mathcal{E}$  and the slope of its tangent here has value  $a$ .

*Remark.* When  $q = 0$ , we rediscover the values given in [5],

$$p_0 = \frac{N + 2}{N - 2}, \quad m_0 = -\frac{2}{N - 2}, \quad y_0 = \frac{N}{N - 2}. \quad (5.10)$$

## 5.2 Proof that $m_0 < 0$ for $h \in [0, 2(N - 1)]$

We present a proof which avoids the heavy computation of  $m_0$ . The point  $(m_0, y_0)$  belongs to the upper part of  $\mathcal{E}$ , where by concavity the slope  $m \mapsto y'(m)$  is a decreasing function. Hence, the claim will follow provided  $y'(m_0) = -a > y'(0)$ . We obtain directly  $y(0) = \frac{2(1+h)}{K}$  and

$$y'(0) [2Ky(0) - 2(1 + h)] + 2(1 + h)y(0) - 1 = 0,$$

$$|y'(0)| = \frac{1}{2(1 + h)} \left( \frac{4(1 + h)^2}{K} - 1 \right);$$

hence, from (5.4),

$$\begin{aligned} (1+h)(|y'(0)| - a) &= \frac{2(1+h)^2}{K} - \frac{1}{2} - a(1+h) = \frac{2(1+h)^2}{K} - \left(1 - \frac{bh}{N-1}\right) \\ &= \frac{2(1+h)^2}{K} - \frac{N+2+(N-1)h-h^2}{N+2+(N+1)h}. \end{aligned}$$

It is therefore required that

$$\begin{aligned} &2N(1+h)^2(N+2+(N+1)h) - (2+h)(Nh+N-1)(N+2+(N-1)h-h^2) \\ &= Nh^4 + (N^2+6N-1)h^3 + (2N^2+12N-3)h^2 + (N^2+9N)h + 2(N+2) > 0 \end{aligned}$$

which clearly holds.

### 5.3 Proof that $m_0 + 2 + h > 0$ for $h \in (0, 2(N-1)]$

From (5.9) the value of  $m_0 + 2 + h$  is given by

$$2((N-1)h + N - 2)(m_0 + 2 + h) = P_2(h) + Q_2(h)\sqrt{\frac{Nh + N - 1}{\mathcal{M}}}, \quad (5.11)$$

with

$$P_2(h) = ((N-2)h^2 + (5N-9)h + 4N - 10), \quad (5.12)$$

and where we recall that

$$Q_2(h) = Nh^3 - (N^2 - 3N + 1)h^2 - (N^2 - N + 4)h - 2N - 4.$$

Note that  $P_2(h) > 0$  for any  $h \in [0, 2N-2]$ . Then  $m_0 + 2 + h > 0$  as soon as  $Q_2(h) > 0$ . Since  $Q_2$  can be written under the form

$$Q_2(h) = (h+2)(Nh - N - 1)(h - N + 2) - 2N^2,$$

it is an increasing function of  $h$  on  $[N-2, 2N-2]$ .

*The case  $N \geq 4$ .* Here  $Q_2(N-1) > 0$ , thus  $Q_2(h) > 0$  on  $[N-1, 2N-2]$ . Therefore it is sufficient to prove the assertion when  $h \in [0, N-1]$ . Our aim is to prove that there holds

$$P_2\sqrt{R} + Q_2 > 0 \text{ where } R = \frac{\mathcal{M}}{Nh + N - 1}$$

in this interval. By division we obtain, since  $Nh + N - 1 < N^2 - 1$ ,

$$\begin{aligned} R &= (N-h)^2 + 6N + 2 - \frac{2(N+2)(N+1)}{N} + \frac{4(N-1)(2N+1)}{N(Nh+N-1)} \\ &\geq (N-h)^2 + 6N + 2 - \frac{2(N+2)(N+1)}{N} + \frac{4(2N+1)}{N(N+1)} \\ &= (N-h)^2 + 4(N-1) + \frac{4}{N+1}. \end{aligned} \quad (5.13)$$

In particular  $\sqrt{R} \geq N - h$ , thus  $P_1\sqrt{R} + Q_2 \geq S$  with

$$S(h) = (N - h)P_2 + Q_2 = 2h^3 - 4(N - 2)h^2 + (4N^2 - 12N + 6)h + 4(N^2 - 3N - 1).$$

We see that  $S(0) > 0$  because  $N \geq 4$ , and since

$$S'(h) = 6h^2 - 2(4N - 8)h + 4N^2 - 12N + 6 > 0$$

is positive for any  $h \in \mathbb{R}$ ,  $S$  is an increasing function of  $h$ . This yields  $S(h) > 0$  on  $h \in [0, N - 1]$  and completes the proof in this case.

*The case  $N = 3$ .* Here we cannot use the minorization of  $R$  since equality holds for  $h = 0$ , as it was noticed above. Also, we observe that the cubic polynomial  $Q_2(h) = 3h^3 - h^2 - 10h - 10$  has its largest root  $h_0$  in the interval  $(2, 3)$ , since  $Q_2(2) = -10$  and  $Q_2(3) = 32$  and  $Q_2'(h) > 0$  on  $[2, \infty)$ . Hence we need only to prove the inequality when  $h \in (0, 3)$ . For this aim, it is sufficient that  $P_2\sqrt{R} + Q_2 > 0$ , which will be ensured provided  $\mathcal{M}P_2^2 - (3h + 2)Q_2^2 > 0$ . After some computation it reduces to prove that

$$h(-6h^6 + 5h^5 + 38h^4 + 35h^3 + 337h^2 + 484h + 160) > 0.$$

This inequality is clearly true, since  $6h^4 - 38h^2 - 337 < 0$  on  $(0, 3)$ . So finally  $m_0 + 2 + h > 0$  for any  $h \in (0, 4]$ .

#### 5.4 Proof that $\sigma > \frac{N}{2}$ for $N \geq 3$

We have to prove that for any  $h \in (0, 2(N - 1)]$  there holds,

$$(m_0 + h + 2)(2N - 2 - h) > (N - 4 - h)((N - 1)(p_0 - 1) + h). \quad (5.14)$$

In the preceding step we have already shown that the left hand side is positive and that  $(N - 1)(p_0 - 1) + h = (N - 1)(p + q - 1) > 0$ , inequality (5.14) is valid for  $N = 3, 4$  or  $N \geq 5$  and  $h \in [N - 4, 2(N - 1)]$ . Henceforth we assume  $N \geq 5$  and  $h \in (0, N - 4)$ . By replacing the value of  $m_0(h) + h + 2$  given the preceding section and  $p_0$  by its value, given in (5.6), we obtain after some computation that the relation can be expressed under the form

$$Q_3\sqrt{R} + Q_4 > 0,$$

where

$$Q_3(h) = (2N - 2)h^2 + (N^3 - 2N^2 + 10N - 6)h + N^3 - 3N^2 + 6N - 4,$$

and

$$Q_4(h) = (N^3 + 2N^2 - 6N - 2)h^2 - (N^4 - N^3 - 17N^2 + 12N + 8)h - (N - 1)(N^3 - 8N - 8).$$

From (5.13), we get

$$\begin{aligned} R &\geq (N - h)^2 + 4(N - 1) = (N - h)^2 \left(1 + \frac{4(N - 1)}{(N - h)^2}\right) \\ &\geq (N - h)^2 \left(1 + \frac{4(N - 1)}{N^2}\right). \end{aligned}$$

Setting  $\tau := \sqrt{1 + \frac{4(N-1)}{N^2}}$ , it is therefore sufficient that

$$Q_5 := \tau(N-h)Q_3 + Q_4 > 0.$$

An explicit computation yields

$$\begin{aligned} Q_5(h) &= -2(N-1)\tau h^3 + [(1-\tau)N^3 + (4\tau+2)N^2 - 6(2\tau+1)N + 6\tau - 2]h^2 \\ &\quad + [(\tau-1)N^4 + (1-3\tau)N^3 + (13\tau+17)N^2 - 12(\tau+1)N + 4(\tau-2)]h \\ &\quad + (\tau-1)N^4 + (1-3\tau)N^3 + (6\tau+8)N^2 - 4\tau N - 8 \end{aligned}$$

The function  $Q_5$  is a cubic with a negative leading coefficient. We claim that it is positive for  $h = 0$  and for  $h = N - 4$  and increasing near 0. Indeed

$$\begin{aligned} Q_5(0) &= (\tau-1)N^4 + (1-3\tau)N^3 + (6\tau+8)N^2 - 4\tau N - 8 \\ &= (N-1)(\tau N(N^2 - 2N + 4) - (N^3 - 8N - 8)). \end{aligned}$$

Since  $\tau^2 = \frac{N^2+4N-4}{N^2}$ , the inequality  $Q_5(0) > 0$  is equivalent to

$$(N^2 + 4N - 4)(N^2 - 2N + 4)^2 - (N^3 - 8N - 8)^2 > 0,$$

which can be easily verified since  $N \geq 5$ . Next

$$\begin{aligned} Q_5(N-4) &= \tau(4N^4 - 12N^3 - 12N^2 + 32N - 48) - 2N^4 + 8N^3 + 6N^2 - 40N - 8 \\ &> 2N^4 - 4N^3 - 6N^2 - 8N - 56 > 0 \end{aligned}$$

For the derivative, we compute

$$\begin{aligned} Q_5'(0) &= (\tau-1)N^4 + (1-3\tau)N^3 + (13\tau+17)N^2 - 12(\tau+1)N + 4(\tau-2) \\ &= \tau(N^4 - 3N^3 + 13N^2 - 12N + 4) - (N^4 - N^3 - 17N^2 + 12N + 8). \end{aligned}$$

Replacing  $\tau$  by its value, the sign of  $Q_5'(0)$  is the same as the one of

$$(N^2 + 4N - 4)(N^4 - 3N^3 + 13N^2 - 12N + 4)^2 - N^2(N^4 - N^3 - 17N^2 + 12N + 8)^2$$

which is equal to

$$40N^8 + 4N^7 - 580N^6 + 1492N^5 - 1964N^4 + 2432N^3 - 1424N^2 + 448N - 64$$

and is clearly positive since  $N \geq 5$ . At end  $Q_5$  stays positive on  $[0, N - 4]$  and the proof is achieved.

## 5.5 Comparison of the regions of Theorem B and Theorem C

In the variable  $h := (N-1)q$  these curves are given by

$$(N-1)p + h = N + 3 \quad \text{for } p > 1, \tag{5.15}$$

$$(N-1)p + h = N - 1 + \frac{(p+1)^2}{p} \quad \text{for } p \in [0, 1]. \tag{5.16}$$

In order to show that the curve  $\tilde{G}(p, h) = 0$ , where  $\tilde{G}$  is defined at (5.5), is above them, we only need to show that  $\tilde{G}(p, h) < 0$  on the curves. For the first curve defined by (5.15), using that  $(h - 2)(h - 3) \geq -\frac{1}{4}$  for all  $h$ , we obtain that for  $h \in [0, 2(N - 1)]$ ,

$$\begin{aligned} (N - 1)G\left(\frac{N + 3 - h}{N - 1}, h\right) &= -h^3 + 3h^2 - 4N + 4h - 12 \\ &= -(h + 2)(h - 2)(h - 3) - 4N \\ &\leq \frac{h + 2}{4} - 4N \\ &\leq \frac{N}{2} - 4N < 0. \end{aligned}$$

As for the second curve, we check that

$$\begin{aligned} (N - 1)\frac{p^2}{(p + 1)^2}\tilde{G}\left(p, N - 1 + \frac{(p + 1)^2}{p} - (N - 1)p\right) \\ = -p(p - 1)^2N^2 + (3p^3 - 2p^2 - p - 1)N - p^2(2p + 1) < 0 \end{aligned}$$

for  $p \in [0, 1]$ , because  $3p^3 - 2p^2 - p - 1)N - p^2(2p + 1) \leq p^2 - p - 1 \leq -1$ .

## 5.6 Final remark about the parameter $\beta$

When we set  $u = v^{-\beta}$  then  $y = \frac{\beta+1}{\beta}$  a natural question is about the sign of  $\beta$ . We have seen that  $y_0 > 0$ , and when  $p < p_0$  we have chosen  $y = y_0$  if  $y_0 \neq 1$ , or  $y = y_0 - \epsilon$  in case  $y_0 = 1$  (or in the special case  $N = 3$ , where  $m_0 = -2$ , so we have taken  $m = -2 + \epsilon$ ). So either  $y_0 > 1$ ,  $\beta > 0$ ; or  $y_0 \leq 1$ ,  $\beta < 0$ . We can remark that for  $q = 0$ , we have  $\beta > 0$ , since  $y_0 = \frac{N}{N-2}$  from (5.10). But for  $q = 2$  we find

$$p_0 = \frac{4}{2N - 3} = -m_0, \quad y_0 = \frac{2}{2N - 3}$$

hence  $y_0 < 1$ , thus  $\beta < 0$ .

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