Wythoff Wisdom

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Abstract. Six authors tell their stories from their encounters with the famous combinatorial game Wythoff Nim and its sequences, including a short survey on exactly covering systems.

1. A modification of the game of Nim

The game of Nim only preceded Wythoff's modification by a few years. By the famous theory of Sprague and Grundy some decades later, Nim drew a lot of attention. Wythoff Nim (here also called Wythoff's game), on the other hand, only became regularly revisited towards the end of the 20th century, but its winning strategy is related to Fibonacci's old discovery of the evolution of a rabbit population. In this survey we give a personal account of some recent work and we are proud to include sections of the current masters in the field: without the consistent revisits of Wythoff's game by Fraenkel, and also Wythoff's sequences and arrays by Kimberling, this survey would probably have been delayed several decades. Each approach in the survey is obtained by viewing the original game in a new personal perspective, and we hope there will be many yet to come.

Let us recall Wythoff's original definition of the game, given in item 1 in his paper [149].

> *The game is played by two persons. Two piles of counters are placed on the table, the number of each pile being arbitrary. The players play alternately and either take from one of the piles an arbitrary number of counters or from both piles an equal number. The player who takes up the last counter or counters, wins.*

Wythoff proceeds by designating the *safe* positions (P-positions in current jargon) of his game, first by noting that the heaps are unordered, which implies that (x, y) is safe if and only if so is (y, x), and where x and y denotes the respective number of counters in each pile. Then he proceeds to the nowadays celebrated *minimal exclusive algorithm* MEX (but he did not give it a name). Let U be a finite subset of the nonnegative integers. Then the minimal excludant of U, $\operatorname{mex} U$, is the smallest nonnegative integer not in U.

Theorem 1 (Recursive characterization of Wythoff Nim's P-positions). *Let $\{(A_n, B_n), (B_n, A_n) : n \geq 0\}$ be the set of P-positions of* Wythoff Nim*. Then, for all $n \geq 0$:*

$$A_n = \operatorname{mex}\{A_i, B_i : 0 \leq i < n\} \quad (1)$$

$$B_n = A_n + n. \quad (2)$$

We display the first few terms of the sequences.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A_n	0	1	3	4	6	8	9	11	12	14	16	17	19	21	22
B_n	0	2	5	7	10	13	15	18	20	23	26	28	31	34	36

TABLE 1. The first few terms of the A and B sequences.

This result implies an exponential-time winning strategy in the input size $\log(xy)$ of an arbitrary input position (x, y). Wythoff's game has become famous because of the algebraic characterization of the P-positions (item 6 of [149]), the solution via the floor function and the Golden ratio, which implies a poly-time winning strategy.

Theorem 2 (Algebraic characterization of WYTHOFF NIM's P-positions). *A combination of pile sizes of* WYTHOFF NIM *is a P-position if and only if it is of the form $\lfloor \varphi n \rfloor, \lfloor \varphi^2 n \rfloor$, for some nonnegative integer n, and where $\varphi = (1 + \sqrt{5})/2$ is the Golden ratio.*

It also became famous because of the difficulty to compute the non-zero Sprague-Grundy values, and the problem to find a generalization to several heaps that preserves properties of Wythoff Nim (see Section 4.3).

We will often encounter the more general concept of a *Beatty sequence* [7]. Let α denote a positive irrational. Then $(\lfloor \alpha n \rfloor)$ is a Beatty sequence, where n ranges over the positive integers. Two or more sets of positive integers are *complementary* if each positive integer occurs in precisely one of them. Such systems are also known under the name of *exactly covering systems*, or *splitting systems*. It is a well known result that the sets $\{\lfloor \alpha n \rfloor\}$ and $\{\lfloor \beta n \rfloor\}$, where n ranges over the positive integers, are complementary if and only if α, β are positive irrationals satisfying $\alpha^{-1} + \beta^{-1} = 1$. Such a pair of sequences $(\lfloor \alpha n \rfloor), (\lfloor \beta n \rfloor)$ is often called a pair of *complementary Beatty sequences*, although the result was discovered by Rayleigh in the book "The Theory of Sound" [131] (without giving a proof) and independently proved by Hyslop and Ostrowski, and Aitken [8]. WYTHOFF NIM's P-positions give a special case of this, where α is the Golden ratio.

In a survey paper [23] concerning the Golden ratio, phyllotaxis and Wythoff Nim in 1953, Coxeter sketches a simple proof of Theorem 2, recalling the proof of Hyslop and Ostrowski [8], using also Theorem 1. He omits to elaborate of the formula for B_n (a similar shortcoming appears in Wythoff's original proof). Namely, the mex-property for A_n holds by the (graph theory) kernel property of the P-positions of an impartial game. For the relation $B_n = A_n + n$, however, an inductive argument of a fill-rule property of diagonal parents of P-positions is also required. It is a geometric argument, and we display the idea in Figure 5, noting that if (x, y) is a P-position, then for example $(x + t, y + t)$ is an N-position for all $t > 0$. This *fill-rule property* is further generalized in a renormalization approach of generalized diagonal Wythoff Nim games (Linear Nimhoff); see Section 8.

Wythoff Nim has been considered in the context of phyllotaxis more recently in Chapter 17 in the book "Symmetry in Plants" [78] and here the *maximal Fibonacci representation* is used to represent Wythoff's sequences (Adamson's Wythoff Wheel [78, Chapter 17]), while we more often encounter the *minimal Fibonacci representation* (often called the Zeckendorff [150] representation although it was discovered by Ostrowski [129] and Lekkerkerker [117]) together with left and right shifts e.g. [136, 38].

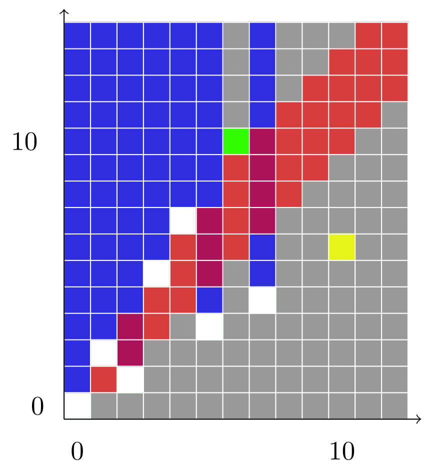


FIGURE 1. The P-position (6,10) (green) of Wythoff Nim have been computed via a fill rule algorithm, using the P-positions closer to the origin (light). We also indicated the symmetric P-position (10,6) (yellow). We colored vertical (blue) and diagonal (red) N-positions that have these smaller P-positons as options. View the old P-positions as sources of light, and the N-positions as colored light-beams: the green and yellow cells will next become light sources and invoke new colored beams, giving birth to new green and yellow cells, and so on. Note that the diagonal red beam is tightly packed, and this is part of the induction hypothesis, whereas the vertical beams leave a-periodic gaps, related to the famous rabbit-birth algorithm of Fibonacci: one baby rabbit if and only if we find two *upper* green cells (P-positions) within the same gap (so the picture illustrates an immature rabbit). We omitted the horizontal beams in the picture because they are not required in the algorithm. Neither are the diagonal beams below the line $y = x$ needed. The fill-rule idea gives a nice conjecture for a generalization of Wythoff Nim (see also Figure 5).

In the book Theory of Graphs, by C. Berge (1962), R. P. Isaac mentions the game of Wythoff Nim in Example 1 in Section 6, but played with a Queen of Chess; this variation of the game is often called 'Corner the Lady' [60]. Without reference, the winning strategy is mentioned, and for the first time, a nice illustration of the fill rule of the diagonal moves is presented. The picture also gives the initial Grundy values of Wythoff's game. The section concerns "Nim type games" and illustrates "the kernel of a graph" idea for positions of Grundy value 0.

In 1959, Connell [20] restricted Wythoff's game by requiring to take a multiple of $b \geq 1$ from a single pile. For the P-positions, he obtained b pairs of sequences, each pair consisting of two complementary non-homogenous Beatty sequences [36].

In 1968 Holladay [75] extended Wythoff's game, to a k-Wythoff Nim, by the extension of the diagonal rule "take from both piles, but do not take more than k more counters from one pile than from the other". This game gives a simple but elegant generalization of both the minimal exclusive description in Theorem 1, and

13	8	5	3	2	1	n	8	5	3	2	1
					1	1					1
				1	0	**2**				1	0
			1	0	0	3				1	1
			1	0	1	4			1	0	1
		1	0	0	0	**5**			1	1	0
		1	0	0	1	6			1	1	1
		1	0	1	0	**7**		1	0	1	0
	1	0	0	0	0	8		1	0	1	1
	1	0	0	0	1	9		1	1	0	1
	1	0	0	1	0	**10**		1	1	1	0
	1	0	1	0	0	11		1	1	1	1
	1	0	1	0	1	12	1	0	1	0	1
1	0	0	0	0	0	**13**	1	0	1	1	0

TABLE 2. The minimal (no two consecutive 1s) and maximal (no two consecutive 0s) Fibonacci representation of the first few integers, respectively. The bold numbers are the ones in the B sequence. These representations satisfy a number of interesting number theoretical properties. Note that the numbers in the A sequence have even number of rightmost "0"s in the minimal representation. In the maximal representation they are the ones ending in a "1". The so-called left-shift property holds for both representations: the number $B(n)$ is obtained by shifting the digits of $A(n)$ one step to the left and putting a "0" as the least significant bit.

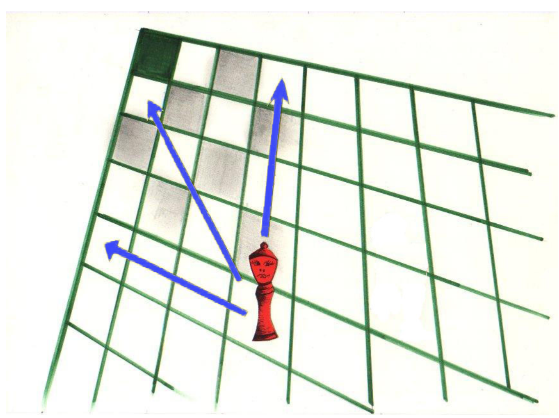


FIGURE 2. Wythoff Nim is often played with a single Queen of Chess on a semi infinite Chess board. By moving, the Queen must get closer to the single corner, labeled $(0, 0)$. Martin Gardner coined the other classical name for this game, "Corner the Lady", and attributed this variation to Rufus P. Isaacs.

also the algebraic description for the P-positions in Theorem 2. In fact, he defines four variations to these game rules with exactly the same set of P-positions. The k-Wythoff game was revisited by Fraenkel [38], where computational aspects and connections to continued fractions are emphasized. Holladay also studies related games where at most k counters may be removed from both piles, or any number from just one pile, but these variations do not invoke the fill-rule-property, and therefore does not generalize Wythoff's characterizations of the P-positions.

In 1973 Fraenkel and Borosh [49] generalized Wythoff's game in a way that does not include Connel's game, but includes Holladay's k-Wythoff Nim. In 2009 Larsson [102, 107] generalized Connell's and Holladay's approach in one and the same game (in fact three games are defined with the same set of P-positions).

Wythoff's game is closely connected to complementary and to disjoint integer, rational and irrational Beatty sequences; this is a concept that generalize standard arithmetic progression. Such sequences are considered in many papers (see the next section) with or without references to Wythoff's game, and often concerning exactly covering systems. In Section 2, we review some of this work, which has partly been inspired by Wythoff's sequences but also stem from diverse origins. Starting with E. Duchêne's vision in Section 3, the author's contributions are presented in alphabetical order[1].

2. Exactly Covering Sequences

Obvious exactly covering systems are arithmetic sequences, such as $2n-1$, $2n$, $n \geq 1$; or $4n-3$, $4n-1$, $4n$, $n \geq 1$. In these two examples, the two largest *moduli* (2 and 4 respectively) are the same. This is a general property of exactly covering arithmetic sequences: If all the moduli α_i are integers with $m \geq 2$ and $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_m$, then $\alpha_{m-1} = \alpha_m$. A proof using complex numbers and roots of unity was given by Mirsky, Newman, Davenport and Rado – see Erdös [33]. Since every math paper – even a survey paper! – should contain at least one proof, we present here a "proof from the book" of their result.

Let $\{na_i + b_i : 1 \leq i \leq m; n = 1, 2 \ldots\}$ be an exactly covering system of $m \geq 2$ arithmetic sequences, where $a_1 \leq \ldots \leq a_m$. Consider the generating function $z^{b_i}/(1-z^{a_i}) = \sum_{n\geq 1} z^{na_i+b_i}$. The fact that the system is exactly covering is expressed by the identity: $\sum_{i=1}^{m} z^{b_i}/(1-z^{a_i}) = z/(1-z)$. Let ξ be a primitive a_m-th root of unity and let $z \to \xi$. If $a_{m-1} < a_m$, then the only unbounded term in the identity is $z^{b_m}/(1-z^{a_m})$, a contradiction. Hence $a_{m-1} = a_m$.

A first elementary proof of this result was given independently by Berger et.al. [14] and by Simpson [134]. *Beatty sequences* are normally associated with irrational *moduli* α, β. Recent studies deal with rational moduli α, β. Clearly if $a/b \neq g/h$ are rational, then the sequences $\{\lfloor na/b \rfloor\}$ and $\{\lfloor ng/h \rfloor\}$ cannot be complementary, since $kbg \times a/b = kha \times g/h = kag$ for all $k \geq 1$. Also the former sequence is missing the integers $ka-1$ and the latter $kg-1$, so both are missing the integers $kag-1$ for all $k \geq 1$. However, complementarity can be maintained for the *nonhomogeneous* case: In [36, 128], necessary and sufficient conditions on α, γ, β, δ are given so that the sequences $\{\lfloor n\alpha+\gamma \rfloor\}$ and $\{\lfloor n\beta+\delta \rfloor\}$ are complementary – for both irrational moduli and rational moduli. We are not aware of any previous work in this direction, except that in Bang [4] necessary and sufficient conditions are given for $\{\lfloor n\alpha \rfloor\} \supseteq \{\lfloor n\beta \rfloor\}$ to hold, both for the case α, β irrational and the case α, β rational. Results of this sort also appear in Niven [127], for the homogeneous case only. In Skolem [137] and Skolem [138] the homogeneous and nonhomogeneous cases are studied,

[1] U. Larsson wrote the introduction and edited the paper, assisted by E. Duchêne, A. S. Fraenkel and N. B. Ho.

but only for α and β irrational. Incidentally, Skolem set out from the point of view of Steiner systems [140] and discovered Wythoff's sequences, but without making the connection to Wythoff Nim.

Uspensky [146] demonstrated, by using a well-known theorem of Kronecker, that if we have $k > 1$ homogeneous Beatty sequences with real moduli $a, b, \ldots,$ that partition the positive integers, then $k = 2$, and $1/a + 1/b = 1$, and a and b are irrational. Graham [64] later demonstrated that $k = 2$, by elementary means in a one page proof.

These investigations spawned the following interesting conjecture [37], see also Erdös and Graham [34]: If the system $\cup_{i=1}^{m}\{\lfloor n\alpha_i + \gamma_i\rfloor\}$, $n = 1, 2, \ldots$ splits the positive integers with $m \geq 3$ and $\alpha_1 < \alpha_2 < \ldots < \alpha_m$, then

$$\alpha_i = (2^m - 1)/2^{m-i},\ i = 1, \ldots, m. \tag{3}$$

Graham [65] showed that if all the m moduli are real and $m \geq 3$, then two moduli are equal. Thus distinct integer moduli or irrational moduli cannot exist for $m \geq 2$ or $m \geq 3$ respectively in a splitting system.

The conjecture was proved for $m = 3$ by Morikawa [122], $m = 4$ by Altman et. al [3], for all $3 \leq m \leq 6$ by Tijdeman [145] and for $m = 7$ by Barát and Varjú [5] and was generalized by Graham and O'Bryant [67]. Other partial results were given by Morikawa [123], Simpson [135]. Many others have contributed partial results – see Tijdeman [144] for a detailed history. The conjecture has some applications in job scheduling and related industrial engineering areas, see e.g., Altman et. al [3], Brauner and Jost [17]; also in [119, 147]. However, the conjecture itself has not been settled.

> So, this is a problem that has been solved for the integers, has been solved for the irrationals, and is wide open for the rationals!

The conjecture provides a challenge to find game rules using the sequences as candidate sets of P-positions [48]. Thus, for example, for the rat (rat–rational) game, the P-positions are $\{(\lfloor 7n/4\rfloor, \lfloor 7n/2\rfloor, 7n-3),\ n = 1, 2, \ldots\} \cup \{0\}$. For the related mouse game on two piles, the use for an invariant analogue, a mouse trap [108], became apparent, which brings us to a modern trend in CGT. A typical interest in CGT is, given a finite rule set describing a game, find its P-positions, or also, when possible, its Sprague-Grundy function. A modern trend is to reverse this process: given a subsequence R of nonnegative vectors, is there a game whose set of P-positions is precisely R? Suppose we gave a family of games for which the moves and the outcomes have the same description, for example t-tuples of nonnegative integers (for Wythoff Nim $t = 2$. Any such game for which some move cannot be made from all game-positions (sometimes because it would be a move connecting two P-positions), is a *variant* game (e.g. the rat- and mouse games). Duchêne and Rigo [31] conjectured that if R is the set of numbers produced by a pair of complementary homogeneous Beatty sequences (with irrational moduli), then there is an *invariant* game whose set of P-positions is R, together with $(0, 0)$. Larsson et. al proved a generalization thereof [112]. Informally, a game is invariant if every move can be done from every position, provided only that the result is a game position. Much earlier Golumb [63] defined the notion of a *vector subtraction game*, which is an instance of the family of invariant games, including Wythoff Nim and many other impartial heap games. The *move-size dynamic* games Fibonacci Nim

[148, 114, 115] and IMITATION NIM [101], also have winning strategies related to the P-sequences of WYTHOFF NIM, although they are non-invariant in this sense.

Before we move on, one should note that Stolarsky has contributed some interesting papers, related to Beatty sequences, for example one with Fraenkel [56], where many curios identities involving φ (=golden section) are proved. For example, the reals $\{n\varphi\}$ are closed under ordinary multiplication, where $\{x\}$ is the fractional part of x. In fact, $\{m\varphi\}\{n\varphi\} = \{k\varphi\}$, where $k = mn - m\lfloor\varphi\rfloor - n\lfloor m\varphi\rfloor$. Another more recent contribution of Stolarsky and Kimberling [96] concerns interesting kinds of convergence. A sequence (x_n) *converges deviously to* L if, in addition to converging to L, it is true that for every real B, there exists $\ell \neq L$ such that $x_n = \ell$ for more than B numbers n. For example, let

$$g(n) = \frac{n}{\varphi\lfloor n/\varphi\rfloor)},$$

where $\varphi = (1+\sqrt{5})/2$. The sequence $(\lfloor n/\varphi\rfloor)$ is an example of a *slow Beatty sequence*, and $(g(n))$ converges deviously to 1. Next, suppose that sequences w_1 and w_2 partition the positive integers. Suppose further that (a_n) is a sequence such that $a_{w_1(n)} \to L_1$ and $a_{w_2(n)} \to L_2$, so that (a_n) converges if and only if $L_1 = L_2$. Otherwise, (a_n) *diverges partitionally on* w. Define

$$h(n) = n(g(n+1) - g(n)),$$

and for irrational $t > 1$, let $w_1(n) = \lfloor nt\rfloor$ and $w_2(n) = \lfloor nt/(t-1)\rfloor$, these being complementary Beatty sequences. Then $(h(n))$ is partitionally divergent, with $h(w_1(n)) \to 1 - t$ and $h(w_2(n)) \to 1$.[2]

[2] A. S. Fraenkel and U. Larsson wrote this section, and the last paragraph was composed by A. S. Fraenkel and C. Kimberling.

In this section, I have chosen to consider WYTHOFF NIM under two "visions":

- WYTHOFF NIM can be considered as an instance of the more general VECTOR SUBTRACTION GAMES introduced by Golomb in [63]. A large set of variants of WYTHOFF NIM found in the literature can be seen as instances of Golomb's game. In section 2.1, I will give a couple of personal results obtained for some particular cases of this game.
- My second vision is about the link between WYTHOFF NIM and Fibonacci, and more precisely the Fibonacci word. Variants of WYTHOFF NIM based on other words, such as the so-called *Tribonacci word* will be described.

3.1. WYTHOFF NIM **as a vector subtraction game.** In [63], Golomb introduced the notion of *t-vector subtraction games*. Given t piles of counters, a *position* of such a game is a t-tuple of non-negative integers, corresponding to the number of counters in each pile. A *move* is also a t-tuple of non-negative integers corresponding to the number of counters that are removed from each pile. Let $p = (p_1, \ldots, p_t)$ be a position and $m = (m_1, \ldots, m_t)$ be a non-zero move. The move m can be applied to the position p provided that $m \leq p$, i.e., for all i, $m_i \leq p_i$. The position resulting from the application of m is the t-tuple $p - m$. Given a set $\mathcal{M}$ of allowed moves and a starting position p, two players alternately apply a move from $\mathcal{M}$. The first player unable to move loses the game. Clearly, WYTHOFF NIM is an instance of the vector subtraction game with

$$\mathcal{M}_{WYT} = \{(0, i), (i, 0), (i, i) : i > 0\}$$

In the literature, several games can be seen as instances of the vector subtraction game. In particular for $t = 2$, some of them have a set $\mathcal{M}$ corresponding to a proper subset (restriction) or superset (extension) of WYTHOFF NIM. Some of these games will be detailed in the following section.

Remark 1. *Note that in* [31]*, such games are also called take-away* invariant *games, since the allowed moves do not depend on the initial position (i.e., if $m \in \mathcal{M}$, playing $p - m$ is allowed for any p provided $m \leq p$). If invariant take-away games and vector subtraction games are equivalent, the notion of invariance is devoted to be expanded in a more general context than the one of take-away games.*

2*-vector subtraction games.* Some instances of the 2-vector subtraction game are considered in the current paper. This is the case of k-WYTHOFF defined by Fraenkel [38] (see section 3.1), where

$$\mathcal{M}_{GW}(k) = \{(0, i), (i, 0) : i > 0\} \cup \{(i, j) : |i - j| < k,\ i, j > 0\}$$

The games NIM(a, b) [71] (see Section 4), MAHARAJA NIM [113] (see Section 7.4) are other invariant WYTHOFF NIM variations. One can also mention the recent game WYT(F) [58], where f is a given function $\mathbb{N} \to \mathbb{N}$, and defined as follows:

$$\mathcal{M}_{WYT}(f) = \{(0, i), (i, 0) : i > 0\} \cup \{(i, j) : 1 \leq i \leq j < f(k)\}$$

When the number of moves is finite, one can mention the game where the allowed vectors correspond to the moves of a knight in chess. In that case, the Grundy function was proved to be periodic [6]. The same kind of result is also true for the king and its powers [28] (in other words, this is WYTHOFF NIM where one can remove at most k counters per heap, for a given k).

In [28], Duchêne and Gravier have considered a restriction of Wythoff Nim, namely the $[a,b]$-vector game (a, b being two positive integers), which can be expressed as a particular family of 2-vector subtraction games:

$$\mathcal{M}(a,b) = \{(0,i),(i,0),(ia,ib) : i > 0\}$$

If $a \neq b$, it is proved that the P-positions of this game are exactly the set $\{(i,i) : i \geq 0\}$. The situation is more tricky when $a = b$. For $a = b = 1$, the game is equivalent to Wythoff Nim. In [28], an acceptable exponential algorithm is given to compute the P-positions of the $[2,2]$-game. Yet, it seems to us that a closed formula should be available, since it is conjectured that the P-positions follow the progression of $\lfloor \frac{n(3+\sqrt{(17)})}{4} \rfloor$. Note that the $[a,b]$-game can be naturally extended to n heaps, including the most natural extension of Wythoff Nim:

> *Given n piles of counters, both players alternately take either from one of the piles an arbitrary number of counters or from all piles an equal number. The player who takes up the last counter or counters, wins.*

When n is odd, it was shown in [28] that this game has the same set of P-positions as Nim. When n is even, the resolution remains open (except for $n = 2$, i.e., Wythoff Nim).

Given a subset K of $\mathbb{N}$, another natural restriction of Wythoff Nim, namely Wyt_K, is the following instance of the 2-vector subtraction game:

$$\mathcal{M}_{WYT}(K) = \{(0,i),(i,0) : i > 0\} \cup \{(k,k) : k \in K\}$$

The game Wyt_K with $K = \mathbb{N}$ is Wythoff Nim. In [25, 53], this game has been investigated for $|K| = 1$. In such a case, the P-positions are known. A full characterization of the $\mathcal{G}$-function is even proved for $K = \{2^k\}$ for some $k \geq 0$, and also for every subset of the powers of 2 including 1. In a certain manner, this characterization shows that these Wythoff Nim restrictions are "closer" to Nim than Wythoff Nim, since their Grundy functions behave like the one of Nim (i.e., a latin square with a strong regularity).

In [31], another set of 2-vector subtraction games is considered:

$$\mathcal{M}_{DR}(k) = \mathcal{M}_{WYT} \setminus \{(2i,2i) \mid 0 < i < k\} \cup \{(2k+1,2k+2),(2k+2,2k+1)\}$$

In other terms, this games can be described as follows:

> *Given a positive integer k, either take a positive number from a single pile, or (i,i) from both as in* Wythoff Nim*, or $2k+1$, $k > 0$ from one and $2k+2$ from the other, except that the* Wythoff Nim *moves of taking $(2i,2i), i < k$ from both are disallowed.*

For this set of games, it was proved [31] that the P-positions can be expressed as a pair of complementary Beatty sequences, as it is the case for Wythoff Nim. More precisely, we have:

Theorem 3. *The P-positions of the game $\mathcal{M}_{DR}(k)$ are of the form $(\lfloor n\alpha_k \rfloor, \lfloor n\beta_k \rfloor)_{n\geq 0}$, where α_k is the quadratic irrational number having $(1;\overline{1,k})$ as continued fraction expansion, and $1/\alpha_k + 1/\beta_k = 1$.*

Remark 2. *For $t > 2$, there are few instances of the t-vector subtraction that are considered in the literature. Some of them were mentioned in* [28].

3.2. Wythoff Nim **as the "Fibonacci game".** In [29], Duchêne and Rigo introduced a new characterization of the P-positions of Wythoff Nim, which deals with the Fibonacci word. Given a two-letter alphabet $\{a, b\}$, take the morphism $\phi : \{a, b\} \to \{a, b\}^*$ defined as follows:

$$\begin{aligned} \phi(a) &= ab \\ \phi(b) &= a \end{aligned}$$

By iterating this morphism from a, we get the famous *Fibonacci word* $w_F = (w_n)_{n\geq 1} = \lim_{n\to+\infty} \phi^n(a)$,

$$w_F = abaababaabaababaababaabaababa\cdots$$

In this word, we will use the convention that the first letter has index 1. For $X = A, B$ (resp. $x = a, b$), we define the sets

$$X = \{X_1 < X_2 < \cdots\} = \{n \in \mathbb{N} \mid w_n = x\}$$

Roughly speaking, the indices of the letters a (resp. b) in w_F correspond to the sequence (A_n) (resp. (B_n)). In addition, we set $A_0 = B_0 = 0$. According to this definition, the P-positions of Wythoff Nim exactly correspond to the sequence (A_n, B_n).

Remark 3. *Note that a similar characterization has been obtained* [30] *for the P-positions of k-*Wythoff *using the morphism*

$$\begin{aligned} \phi'(a) &= a^k b \\ \phi'(b) &= a \end{aligned}$$

Since the P-positions of Wythoff Nim are correlated to the Fibonacci word, a natural question arose: does there exist a 3-heap game whose P-positions can be coded by the so-called *Tribonacci word* w_T, defined as the unique fixed-point of the morphism $\psi : \{a, b, c\} \to \{a, b, c\}^*$, starting from a:

$$\begin{aligned} \psi(a) &= ab \\ \psi(b) &= ac \\ \psi(c) &= a \end{aligned}$$

Hence w_T starts with:

$$w_T = abacabaabacababacabaabacabacabaabacababaca\cdots$$

The first values of the sequence (A_n, B_n, C_n) derived from the Tribonacci word are given in Table 3.

In [29], a 3-heap game is built, whose P-positions exactly correspond to the sequence (A_n, B_n, C_n) (with all their permutations adjoined). This game has been called the Tribonacci game:

> *Given* 3 *piles of counters, the rules are the following:*
>
> - *Any positive number of tokens from up to two piles can be removed.*

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\mathbf{A_n}$	0	1	3	5	7	8	10	12	14	16	18	20	21	23	25	27
$\mathbf{B_n}$	0	2	6	9	13	15	19	22	26	30	33	37	39	43	46	50
$\mathbf{C_n}$	0	4	11	17	24	28	35	41	48	55	61	68	72	79	85	92

TABLE 3. First values of the sequences $(A_n)_{n\geq 0}$, $(B_n)_{n\geq 0}$ and $(C_n)_{n\geq 0}$.

- *Let α, β, γ be three positive integers such that*
$$2\max\{\alpha, \beta, \gamma\} \leq \alpha + \beta + \gamma.$$
Then one can remove α (resp. β, γ) from the first (resp. second, third) pile.
- *Let $\beta > 2\alpha > 0$. From position (a, b, c) one can remove the same number α of counters from any two piles and β counters from the unchosen one with the following condition. If a' (resp. b', c') denotes the number of counter in the pile which contained a (resp. b, c) tokens before the move, then the configuration*
$$a' < c' < b'$$
is not allowed.

In [29, 62, 30, 31], several take-away games were deeply investigated with the use of such words. In many cases, deciding whether a given position is N or P is proved to be polynomial thanks to a numeration system derived from the underlying morphism.

Remark 4. *Note that the* TRIBONACCI GAME *is not invariant (i.e., it is not an instance of the* 3*-vector subtraction game). In* [32]*, one provide an algorithm which decides whether invariant rules could have been proposed to fit this set of P-positions.*

4. Aviezri S. Fraenkel

This section is a partial survey – *partial* in two senses: It is tailored to our own partial taste, not impartial; and it contains only a small part of the appetizing Wythoff Nim curiosities, not comprehensive. In contrast — and third sense of partiality — it contains only one study, in subsection 4.2, of partial games, more often termed partizan games; otherwise only impartial games are surveyed: occasionally we refer to properties of all games, not just Wythoff Nim. Then all *impartial* games are meant.

Notation 1. The set of all P-positions of a game is denoted $\mathcal{P}$; the set of all its N-positions is $\mathcal{N}$.

There are extensions and restrictions of Wythoff Nim.

4.1. Three 2-pile extensions.

(i) A Nim move restriction. Connell [20] restricted Wythoff's game by requiring to take a multiple of $b \geq 1$ from a single pile. For the P-positions, he obtained b pairs of sequences, each pair consisting of two complementary non-homogenous Beatty sequences [36].

(ii) Nim-move restriction and diagonal move extension/restriction. In [49] we analysed the following generalization of Connell's game, dubbed bt-Wythoff: remove a positive multiple of b tokens from a *single* pile, or, say, $k > 0$ from one and $\ell > 0$ from the other, provided that $|k - \ell| < bt$, $k - \ell \equiv 0 \pmod{b}$.

(iii) A diagonal move extension: t-Wythoff Nim.

> *The moves are of two types: remove any positive number from a single pile* (Nim *move*)*, or, say, $k > 0$ from one and $\ell > 0$ from the other, provided that $|k - \ell| < t$, where t is a fixed positive integer parameter* (*diagonal move*)*.*

Notice that $t = 1$ is classical Wythoff Nim, where the *same* amount has to be taken from both piles.

In [38] three strategies are presented for computing the P-positions.

- Recursive. $\mathcal{P} = \cup_{i=0}^{\infty}\{(A_i, B_i)\}$, where $A_n = \text{mex}\{A_i, B_i : 0 \leq i < n\}$, $B_n = A_n + tn$, $n \geq 0$.
- Algebraic. $\mathcal{P} = \cup_{n=0}^{\infty}\{(\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor)\}$, where $\alpha^{-1} + (\alpha + t)^{-1} = 1$, $\beta = \alpha + t$, so $\alpha = (2 - t + \sqrt{t^2 + 4})/2$, $\beta = (2 + t + \sqrt{t^2 + 4})/2$.
- Arithmetic. In the exotic numeration system whose basis elements are the numerators of the convergents of the simple continued fraction expansion $\alpha = [1, t, t, t, \ldots]$, the numbers A_i end in an even number of 0s; B_i is the "left shift" of A_i, that is, it is the representation A_i with a 0 adjoined at the end of the representation. For further details see [38].

We call this game t-Wythoff. The input size of any t-Wythoff-position (x, y) is $\log(xy)$. The computation for deciding whether $(x, y) \in \mathcal{P}$ is exponential for the first of the three strategies, but polynomial for the last two.

t-Wythoff is the special case $b = 1$ of 2-pile bt-Wythoff [49] with moves: remove a positive multiple of b tokens from a *single* pile, or, say, $k > 0$ from one and $\ell > 0$ from the other, provided that $|k - \ell| < bt$, $k - \ell \equiv 0 \pmod{b}$.

For $b > 1$, the first of these moves is a restriction of the Nim-move of Wythoff Nim.

4.2. **Misère t-Wythoff Nim.** In [39], t-Wythoff Nim in misère play was studied. As for normal play, recursive, algebraic and arithmetic strategies were given. Let S_1 and S_2 denote the P-positions for normal and misère play respectively. Curiously, for $t = 1$, $S_1 = S_2$ except for the first two P-positions, where $(A_0, B_0) = (2, 2)$, $(A_1, B_1) = (0, 1)$; and $S_1 \cap S_2 = \emptyset$ for all $t > 1$. Specifically,

- Recursive. For $t = 1$, $(A_0, B_0) = (2, 2)$, $A_n = \text{mex}\{A_i, B_i : 0 \le i < n\}$, $B_n = A_n + n$, $n \ge 1$. For $t > 1$, $A_n = \text{mex}\{A_i, B_i : 0 \le i < n\}$, $B_n = A_n + n + 1$, $n \ge 0$.
- Algebraic. For $t = 1$, $(A_0, B_0) = (2, 2)$, $(A_1, B_1) = (0, 1)$, $A_n = \lfloor n(1 + \sqrt{5})/2 \rfloor$, $B_n = \lfloor n(3 + \sqrt{5})/2 \rfloor$, $n \ge 2$. For $t > 1$, $A_n = \lfloor n\alpha + \gamma \rfloor$, $B_n = \lfloor n\beta + \delta \rfloor$, $n \ge 2$, where $\alpha = (2 - t + \sqrt{t^2 + 4})/2$, $\beta = \alpha + t$, $\gamma = \alpha^{-1}$, $\delta = \gamma + 1$.
- Arithmetic. For $t = 1$, $(A_0, B_0) = (2, 2)$, $(A_1, B_1) = (0, 1)$; for $n \ge 2$, A_n, B_n are the same as for normal play. For $t > 1$, The characterization is too long to state here – see [39].

Also a subset of binary trees, dubbed *cedar trees*, was constructed and used for conducting generalized searches and consolidating the three strategies of Wythoff Nim in both normal and misère play.

4.3. **Multi-pile Wythoff Nim.** When our interest in combinatorial games first arose, we noticed that Wythoff Nim seems to be rather more difficult than Nim in at least two aspects, though both are acyclic two-player games with perfect information and no chance moves:

(i) Computation of the Sprague-Grundy function g.
(ii) Generalization to more than two piles. No generalization seemed to preserve the properties $B_n = A_n + n$ for the P-positions of some two piles, and the role of φ in the strategy.

Study of the 1-values of g and related aspects was done in [15]. Some light was shed on the approximate distance from the nonzero g-values to the 0s in Nivasch [126]. In Dress, Flammenkamp, Pink [24], the additive periodicity of the Sprague-Grundy function of Wythoff Nim was established: the g-function of each row minus its saltus is periodic. A much simpler proof was given independently by Landman [99]. All of these studies attest to the difficulty of computing the g-function of Wythoff Nim.

We asked the experts for an explanation of this discrepancy. We were told that it is due to the non-disjunctive move of taking from both piles simultaneously. We tested this claim by replacing the diagonal move by taking k from one, ℓ from the other for any $k \ne \ell$. To our surprise we saw that the experts were wrong: the P-position strategy remained precisely that of Nim (though not necessarily the nonzero g-values). This was true for both $k \ne \ell$ fixed, say $(k, \ell) = (4, 7)$, or selecting any $k \ne \ell$ at each move.

Many authors attempted to generalize Wythoff Nim to multi-pile Wythoff Nim by taking the same number from couples or triples or all the piles, and many similar variations. None of those preserved the properties (ii) above.

It turns out that taking the same number of tokens from both piles in Wythoff Nim is a red herring! Rather, taking k from one and ℓ from the other pile such that $k \oplus \ell = 0$ (so $k = \ell$), where $\oplus$ denotes Nim-addition, is the key for understanding the nature of Wythoff Nim: The couples (k, k) are the P-positions of 2-pile Nim.

Adjoining them as moves, necessarily destroys the P-positions of Nim, since $\mathcal{P}$ of any game is an independent set. The independence follows from the fundamental properties of any acyclic game:

$$u \in \mathcal{P} \iff F(u) \subseteq \mathcal{N}, \quad u \in \mathcal{N} \iff F(u) \cap \mathcal{P} \neq \emptyset,$$

where $F(u)$ denotes the set of direct followers of position u in the game. More precise information in [16] and [52], sect. 5.1.

These observations led us to the conclusion that the proper set of diagonal moves for N-pile Wythoff Nim is the set of P-positions of N-pile Nim (only moves that leave nonnegative pile sizes). This, in turn, led to our conjectures stated below. Its budding is in [40] sect. 6. See Nowakowski and Guy [72] and [41].

Define an N-pile Wythoff Nim game as follows:

> *Given $N \geq 2$ piles of finitely many tokens, whose sizes are $p_1, \ldots, p_N$. The moves are to take any positive number of tokens from a single pile or to take $(a_1, \ldots, a_N) \in \mathbb{Z}^N_{\geq 0}$ from all the piles — a_i from the i-th pile — subject to the conditions:* (i) $a_i > 0$ for some i, (ii) $a_i \leq p_i$ for all i, (iii) $a_1 \oplus \cdots \oplus a_N = 0$. The player making the last move wins and the opponent loses.

Notice that Wythoff Nim is the case $N = 2$.

For $N \geq 3$, denote the P-positions for N-pile Wythoff Nim by

$$(A^1, \ldots, A^{N-2}, A^{N-1}_n, A^N_n), \quad A^{N-2} \leq A^{N-1}_n \leq A^N_n$$

and $A^{N-1}_n < A^{N-1}_{n+1}$ for all $n \geq 0$. The notation is intended to imply that $A^1, \ldots, A^{N-2}$ are fixed.

Conjecture 4. *There exists an integer m_1, depending only on $A^1, \ldots, A^{N-2}$, such that $A^{N-1}_n = \mathrm{mex}(\{A^{N-1}_i, A^N_i : 0 \leq i < n\} \cup T)$, $A^N_n = A^{N-1}_n + n$ for all $n \geq m_1$ where T is a (small) set of integers depending only on $A^1, \ldots, A^{N-2}$.*

Conjecture 5. *There exist integers m_2, a, such that $A^{N-1}_n = \lfloor n\varphi \rfloor + a + \epsilon_n$ and $A^N_n = A^{N-1}_n + n$, $-1 \leq \epsilon_n \leq 1$ for all $n \geq m_2$.*

Both conjectures were proved by Sun and Zeilberger [143] for the special case $N = 3$ and $1 \leq A^1 \leq 10$. In [52] it was shown, inter alia, that Conjecture 4 implies Conjecture 5. This was also proved, inter alia, by Sun [142], Corollary 4.6. (In his abstract – not in the paper itself – it is stated erroneously that the two conjectures are proven to be equivalent.) See also Coxeter [23], [54], [59] and [40].

4.4. **Bridges between** Nim **and** Wythoff Nim**.** Motivated by the difficulty to compute the Sprague-Grundy function g for Wythoff Nim and the ease to do the same for Nim, we attempted to bridge these two games with in-between games. In Nimhoff (hybrid of Nim and Wythoff Nim) [53], the diagonal move is restricted in various ways. A closed formula for the Sprague-Grundy function g is given for most of these games. A generalized Nim-sum is given to guarantee the polynomiality of g for these games. A second bridge between the two games is established in [25], which continues the above first bridge. The diagonal moves are restricted by taking k from both piles only if k belongs to a predetermined given set K. The P-positions are computed; it is determined under what conditions on K is $(a_j, a_j + j) \in \mathcal{P}$; the g-function is computed, and regularity properties of g are studied.

4.5. **The game of** End-Wythoff**.** Motivated by the game End-Nim of Albert and Nowakowski [1], we studied End-Wythoff in normal play [57].

> *A position in* End-Wythoff *is a vector of finitely many piles of finitely many tokens. Two players alternate in taking a positive number of tokens from either end-pile ("burning-the-candle-at-both-ends"), or taking the same positive number of tokens from both ends.*

A recursive characterization of the P-positions (a_i, K, b_i) is presented. For special cases of the vector K of middle-piles, the recursive characterization can be improved. It is also shown that $b_i - a_i = i$ for sufficiently large i (which holds for all i in Wythoff Nim). Further, it is shown that if K is a P-position, then (a, K, b) is a P-position if and only if (a, b) is a P-position of Wythoff Nim. Finally, a polynomial algorithm is given for computing the P-positions (a_i, K, b_i).

4.6. **Extensions, restrictions of** Wythoff Nim **preserving its P-positions.** We show that no strict subset of rules of Wythoff Nim is the ruleset of a game having the same set of P-positions as Wythoff Nim [27]. On the other hand, we characterize all moves that can be adjoined while preserving the set of P-postions of Wythoff Nim. Testing if a move belongs to such an extended set of rules is shown to be doable in polynomial time.

Many arguments rely on the infinite Fibonacci word, automatic sequences and the corresponding numeration system. With these tools, we provide new two-dimensional morphisms generating an infinite picture encoding P-positions of Wythoff Nim and moves that can be adjoined.

4.7. **Rat games.** The general rat game considered here is played on $m \geq 2$ piles. The k-th component of its P-positions has the form,

$$\left\lfloor \frac{2^m - 1}{2^{m-k}} n \right\rfloor - 2^{k-1} + 1, \quad k = 1, \dots, m; \quad n = 1, 2, \dots \,. \tag{4}$$

The m terms $\lfloor (2^m - 1)/2^{m-k} \rfloor$, $k = 1, \dots, m$ are called the *moduli* of the system (4).

Rat games (Rat – Rational) are studied in [48]. For $m = 3$, the rat game is played on 3 piles of tokens. Positions are denoted (x, y, z) with $0 \leq x \leq y \leq z$, and moves $(x, y, z) \to (u, v, w)$, where also $0 \leq u \leq v \leq w$. The following is the essence of the game rules:

(I) *Take any positive number of tokens from up to 2 piles.*

(II) *Take $\ell > 0$ from the x pile, $k > 0$ from the y pile, and an arbitrary positive number from the z pile, subject to the constraint $|k - \ell| < a$, where*

$$a = \begin{cases} 1 \ \textit{if}\ y - x \not\equiv 0 \pmod 7 \\ 2 \ \textit{if}\ y - x \equiv 0 \pmod 7). \end{cases}$$

(III) *Take $\ell > 0$ from the x pile, $k > 0$ from the z pile, and an arbitrary positive number from the y pile, subject to the constraint $|k - \ell| < b$, where $b = 3$ if $w = u$; otherwise,*

$$b = \begin{cases} 5 \ \textit{if}\ w - u \not\equiv 4 \pmod 7 \\ 6 \ \textit{if}\ w - u \equiv 4 \pmod 7). \end{cases}$$

Also for $m = 2$ (the "mouse" game), game rules were given there. But for $m \geq 4$, we didn't find "nice" game rules.

I have shown [37] that for every $m \geq 2$, the m sequences of the form (4) split the positive integers into m nonintersecting complementary sets. I further conjectured that this splitting is unique: it is the only system that splits the positive integers with *distinct* moduli [41], Erdös and Graham [34]. The motivation for this study is thus 4-fold: (i) a games approach might help to settle the conjecture, (ii) demonstrate existence of a take-away game whose P-positions depend on *rational* numbers, (iii) find another analyzable non-Nim take-away game played on more than 2 piles, (iv) present another challenge of finding "nice" game rules, given the game's P-positions. (Such a challenge is implicit in Duchêne and Rigo [31] and Larsson et. al [112].)

Since "nice" game rules based on a given set of P-positions were mentioned, I feel that a tentative definition thereof should be given here, though this is a survey: The game rules are *nice* if they depend on at most a finite number of the P-positions or range values of functions thereof. Notice that according to this definition, the game rules given above for $m = 3$ are nice. (Nice game rules may, perhaps, be called *invariant* game rules.)

4.8. Ratwyt**.** This is another game played with rational numbers (Rat–rational, Wyt–Wythoff). Given a rational number p/q in lowest terms, a *step* is defined by

$$\frac{p}{q} \to \frac{p-q}{q},$$

if $p/q \geq 1$, otherwise

$$\frac{p}{q} \to \frac{p}{q-p}.$$

> Ratwyt [47] *is played on a pair of reduced rational numbers* $(p_1/q_1, p_2/q_2)$. *A move consists of either* (i) doing any positive number of steps to precisely one of the rationals, or (ii) doing the same number of steps to both. The first player unable to play (because both numerators are 0) loses.

A winning strategy using the Calkin Wilf tree [18] is given.

4.9. **Games played by Boole and Galois.** In [42] we proved:

Theorem 6. *Let* $S = \cup_{i\geq 0}(a_i, b_i)$, *where for all* $n \geq 0$,

$$a_n = \text{mex}\{a_i, b_i : 0 \leq i < n\},$$

$b_0 = 0$, *and for all* $n > 0$,

$$b_n = f(a_{n-1}, b_{n-1}, a_n) + b_{n-1} + a_n - a_{n-1}.$$

If f is positive, monotone and semi-additive (*defined there in sect.* 4), *then S is the set of P-positions of a general* 2*-pile subtraction game with* **constraint function** f, *and the sequences* $A = \{a_i\}_{i\geq 0}$, $B = \{b_i\}_{i\geq 0}$ *have the following properties:* (i) they partition the positive integers; (ii) $b_{n+1} - b_n \geq 2$ for all $n \geq 0$; (iii) $a_{n+1} - a_n \in \{1, 2\}$ for all $n \geq 0$.

The case $f = t$ is t-Wythoff Nim considered above.

In [42] we illustrated Theorem 6 with a collection of games based on sequences A and B, known ones such as Prouhet-Thue-Morse, Hofstadter sequence, and mainly on new sequences. In [44] we gave an assortment of games based on constraint functions over Boolean variables or GF(2) (Galois).

4.10. Wythoff**-like games.** We introduced a class of variants of Wythoff Nim whose diagonal move is constrained by a function f [58].

> *Three types of functions f are considered: f a constant, f strictly increasing and superadditive, $f(k) = \sum_{i=0}^{s} a_i k^i$ a polynomial of degree $s > 1$ with nonnegative integer coefficients and $a_0 > 0$.*

A function from the nonnegative integers to the nonnegative integers is *superadditive* if it satisfies $f(k) \geq k$ and $f(k+\ell) \geq f(k) + f(\ell)$ for all $k, \ell \geq 0$. The P-positions are pairs (A_n, B_n), $n \geq 0$, where A_n is computed by the mex function, and B_n is a function of A_n.

4.11. **Harnessing the unwieldy MEX function.** A pair of integer sequences that split the positive integers is often – especially in the context of combinatorial game theory such as Wythoff Nim like games – defined recursively by $a_n = \text{mex}\{a_i, b_i : 0 \leq i < n\}$, $b_n = a_n + c_n$ $(n \geq 0)$. A typical problem is given integers $0 \leq x \leq y$, decide whether $x = a_n$, $y = b_n$. For general functions c_n, the best known algorithm for this decision problem is exponential in the input size $|\Omega(\log x + \log y)|$.

In [55] we produced a poly-time algorithm for solving this problem for the case of approximately linear functions c_n. We call the sequence $\cup_{i \geq 0} c_i$ *approximately linear* if there exist real constants α, u_1, u_2 such that $u_1 \leq c_n - n\alpha \leq u_2$ for all $n \geq 0$.

This result solves constructively and efficiently the complexity question of a number of previously analyzed take-away games of various authors.

4.12. **Translations of** Wythoff Nim**'s P-positions.** In [51], the translation phenomenon of the P-positions of Wythoff Nim was studied. The question was whether there exists a variant of Wythoff Nim whose P-positions, except for a finite number, are translations of those of Wythoff Nim, forming the set

$$S \cup \{(\lfloor \varphi n \rfloor + k, \lfloor \varphi^2 n \rfloor + k) : n \geq n_0\},$$

where $k \neq 0$, $n_0 \geq 0$ and S is a finite set of pairs of integers. Two variants of Wythoff Nim that answer the question for all positive integers k were established.

> *Given $k \geq 1$, in the variant called $\mathcal{W}_k$, each move is either removing a number of tokens from a single pile or removing an equal number of tokens from both piles, provided that none of the resulting piles has size less than k: The move from (a,b) to $(a-i, b-i)$ with $\min(a-i, b-i) < k$ is not allowed.*

The P-positions of $\mathcal{W}_k$ form the set

$$\{(i,i) : 0 \leq i < k\} \cup \{(\lfloor \varphi n \rfloor + k, \lfloor \varphi^2 n \rfloor + k) : n \geq 0\}.$$

A variant of $\mathcal{W}_k$ that also exhibits the translation phenomenon is then introduced. Let $0 \leq j \leq k$.

In the variant $\mathcal{W}_{j,k}$, each move from a position (a, b) with $a \leq b$ is either removing a number of tokens from a single pile or removing an equal number $i > 0$ of tokens from both piles provided that the resulting position $(a - i, b - i)$ satisfies both $a - i \geq j$ and $b - i \geq k$.

Notice that the two games $\mathcal{W}_{k,k}$ and $\mathcal{W}_k$ are identical. The set $\mathcal{P}$ of $\mathcal{W}_{j,k}$ is identical to that of $\mathcal{W}_k$ for all $j \leq k$. It is important to note that the set $\mathcal{P}$ of $\mathcal{W}_{j,k}$ depends only on k.

The other variant of Wythoff Nim, called $\mathcal{T}_k$, is as follows:

from a position (a, b) with $a \leq b$, one can either

(i) *remove a positive number of tokens from a single pile, or*

(ii) *remove an equal positive number, say s, of tokens from both piles provided that $a - s > 0$ and*

$$\left| \left\lfloor \frac{b-s}{a-s} \right\rfloor - \left\lfloor \frac{b}{a} \right\rfloor \right| \leq k.$$

Note that the diagonal move (ii) is a restriction of the diagonal move of Wythoff Nim. In this move, the condition $a - s > 0$ guarantees that the ratio $(b - s)/(a - s)$ is defined. Thus, when making a diagonal move in $\mathcal{T}_k$, one must ensure that the difference between the ratios of the bigger entry over the smaller entry before and after the move must not exceed k.

Consider the special case $k = \infty$. The game $\mathcal{T}_\infty$ is the variant of Wythoff Nim in which the only restriction is that the diagonal move cannot make any pile empty.

The following general question has been proposed: Does there exist a variant of Wythoff Nim whose P-positions, except possibly a finite number, are $(A_n + k, B_n + l)$ for some fixed integers $k \neq l$? [51]

4.13. **Wythoff Nim and Euclid.** The game Euclid, like Wythoff Nim, is played on two piles of tokens, though we usually play it, equivalently, on a pair of positive integers. Unlike Wythoff Nim, the integers remain positive throughout; a move consists of decreasing the larger number by any positive multiple of the smaller, as long as the result remains positive. The player first unable to move loses, see Lengyel [118].

Two exotic characterizations of the Sprague-Grundy function (g-function) values of Euclid's game, in terms of the winning strategy of t-Wythoff Nim, are given in [43]. A novel polynomial-time algorithm for computing the g-function for Euclid is given in Nivasch [125].

5. Vladimir Gurvich

The Game NIM(a, b); recursive solution, asymptotic, and polynomial algorithm based on the Perron-Frobenius theory

For any positive integer a and b, a game NIM(a, b) was introduced in [71] as follows.

> *Two piles contain x and y matches. Two players alternate turns. By one move, it is allowed to take x' and y' matches from these two piles such that*

(5) $0 \leq x' \leq x,\ 0 \leq y' \leq y,\ 0 < x' + y',$ *and either* $|x' - y'| < a$ *or* $\min(x', y') < b.$

In other words, a player can take "approximately equal" (differing by at most $a-1$) numbers of matches from both piles or any number of matches from one pile but at most $b-1$ from the other. This game, NIM(a, b), extends further the game NIM(a) = NIM$(a, 1)$ considered by Fraenkel (1982, 1984) [38, 39], which, in its turn, is a generalization of the classic game NIM$(1, 1)$ introduced by Wythoff (1907) [149], see also [23].

A position of NIM(a, b) is a non-negative integer pair (x, y). Due to obvious symmetry, positions (x, y) and (y, x) are equivalent. By default, we will assume that $x \leq y$.

Obviously, $(0, 0)$ is a unique terminal position. By definition, the player entering this position is the winner in the *normal* version of the game and (s)he is the loser in its *misére* version.

The normal version of NIM(a, b) was solved in [71]: It was shown that the P-positions (x_n, y_n) are characterized by the recursion:

$$x_n = \mathrm{mex}_b(\{x_i, y_i \ \mid \ 0 \leq i < n\}),\ y_n = x_n + an;\ n \geq 0, \tag{6}$$

where $x_n \leq y_n$ and the function mex_b is defined as follows:

Given a finite non-empty subset $S \subset \mathbb{Z}_+$ of m non-negative integers, let us order S and extend it by $s_{m+1} = \infty$ and by $s_0 = -b$, to get the sequence $s_0 < s_1 < \cdots < s_m < s_{m+1}$. Obviously there is a unique minimum i such that $s_{i+1} - s_i > b$. By definition, let us set $\mathrm{mex}_b(S) = s_i + b$; in particular, $\mathrm{mex}_b(\emptyset) = 0$.

It is easily seen that mex_b is well-defined and for $b = 1$ it is exactly the classic minimum excludant mex, which assigns to S the (unique) minimum non-negative integer missing in S. Thus, $\mathrm{mex}_1 = \mathrm{mex}$ and (6) turns into the recursive solution of NIM$(a, 1)$ given by Fraenkel [38, 39].

The first ten P-positions (with $x < y$) of the games NIM$(1, 1)$ and NIM$(2, 1)$ are given in Table 4.

Furthermore, Fraenkel solved the recursion for NIM$(a, 1)$ and got the following explicit formula for (x_n, y_n): Let $\alpha_a = \frac{1}{2}(2 - a + \sqrt{a^2 + 4})$ be the (unique) positive root of the quadratic equation $\widehat{z}^2 + (a-2)\widehat{z} - a = 0$, or equivalently, $\frac{1}{\widehat{z}} + \frac{1}{\widehat{z}+a} = 1$. In particular, $\alpha_1 = \frac{1}{2}(1 + \sqrt{5})$ is the *golden section* and $\alpha_2 = \sqrt{2}$. Then, it follows that for all $n \in \mathbb{Z}_+$ we have

$$x_n = \lfloor \alpha_a n \rfloor \ \text{ and } \ y_n = x_n + an \equiv \lfloor n(\alpha_a + a) \rfloor. \tag{7}$$

n	x_n	y_n
0	0	0
1	1	2
2	3	5
3	4	7
4	6	10
5	8	13
6	9	15
7	11	18
8	12	20
9	14	23

n	x_n	y_n
0	0	0
1	1	3
2	2	6
3	4	10
4	5	13
5	7	17
6	8	20
7	9	23
8	11	27
9	12	30

TABLE 4. $(a = b = 1)$ and $(a = 2, b = 1)$

This recursion implies the asymptotic $\lim\limits_{n\to\infty} \dfrac{x_n(a)}{n} = \alpha_a$ and $\lim\limits_{n\to\infty} \dfrac{y_n(a)}{n} = \alpha_a + a$.

As it was mentioned in [38], the explicit formula (7) solves the game in linear time, in contrast to recursion (6), providing only an exponential algorithm. Yet, it looks too difficult to solve (6) explicitly when $b > 1$, because of the following bounds from [71]:

$$b \le x_{n+1} - x_n \le 2b \quad \text{and} \quad b + a \le y_{n+1} - y_n \le 2b + a. \tag{8}$$

For $b = 1$ the difference $x_{n+1} - x_n$ is either 1 or 2, and thus, $\alpha_a n$ is a good approximation of x_n. When $b > 1$, it seems harder to find a similar estimate, since the bound of (8) for $x_{n+1} - x_n$ is looser. Although no closed form expressions for x_n and y_n are known in case $b > 1$, yet, in [11], these values were computed (and thus, NIM(a, b) solved) by a polynomial time algorithm based on the Perron-Frobenius theory.

The first ten P-positions (with $x \le y$) are given in Table 5 for $(a = 1,\ b = 2)$ and $(a = 2,\ b = 3)$.

n	x_n	y_n
0	0	0
1	2	3
2	5	7
3	9	12
4	11	15
5	14	19
6	17	23
7	21	28
8	25	33
9	27	36

n	x_n	y_n
0	0	0
1	3	5
2	8	12
3	11	17
4	15	23
5	20	30
6	26	38
7	29	43
8	33	49
9	36	54

TABLE 5. $(a = 1,\ b = 2)$ and $(a = 2,\ b = 3)$.

The linear asymptotic still holds, not only for $b = 1$ but for $b > 1$, as well. In [71] it was conjectured that the limits $\ell(a, b) = \lim\limits_{n\to\infty} x_n(a, b)/n$ exist for all positive integer a, b and are irrational algebraic numbers. This conjecture was proven in

[11]; moreover, the following explicit formula for the limiting values was obtained: The limit $\ell(a,b)$ exists for all positive integer a, b and, when they are co-prime, $\gcd(a,b) = 1$, it is given by the fraction $\ell(a,b) = \frac{a}{r-1}$, where $r > 1$ is a unique positive real root of the polynomial

$$P(z) = z^{b+1} - z - 1 - \sum_{i=1}^{a-1} z^{\lceil ib/a \rceil}, \tag{9}$$

which is the characteristic polynomial of a non-negative $(b+1) \times (b+1)$ integer matrix $M_{i,j}$ associated to game NIM(a,b) and depending only on parameters a and b as follows: $M_{i,j} =$

$$\begin{cases} \alpha = \lfloor \frac{a+j-1}{b} \rfloor & \text{if } i = 0 \text{ and } 0 \le j \le b - \beta \\ \alpha + 1 = \lfloor \frac{a+j-1}{b} \rfloor & \text{if } i = 0 \text{ and } b - \beta < j \le b \\ 1 & \text{if } i > 0 \text{ and } (j + a - i \bmod b) = 0 \\ 0 & \text{if } i > 0 \text{ and } (j + a - i \bmod b) \neq 0 \end{cases} \tag{10}$$

	0	1	$\cdots$	$b-\beta-1$	$b-\beta$	$b-\beta+1$	$\cdots$	$b-1$	b
0	α	α	$\cdots$	α	α	$\alpha+1$	$\cdots$	$\alpha+1$	$\alpha+1$
1	0	0	$\cdots$	0	0	1	$\cdots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$\beta-1$	0	0	$\cdots$	0	0	0	$\cdots$	1	0
β	1	0	$\cdots$	0	0	0	$\cdots$	0	1
$\beta+1$	0	1	$\cdots$	0	0	0	$\cdots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$b-1$	0	0	$\cdots$	1	0	0	$\cdots$	0	0
b	0	0	$\cdots$	0	1	0	$\cdots$	0	0

By the Perron-Frobenius theorem, we have $|r'| < r$ for any other root r' of $P(z)$.

The case $\gcd(a,b) > 1$ is easily reduced to the case $\gcd(a,b) = 1$ considered above, since, as it was shown in [71], $x_n(a,b)$ (and, hence, $y_n(a,b)$ and $\ell(a,b)$ as well) are uniform functions of a and b, that is,

$$x_n(ka,kb) = kx_n(a,b), \; y_n(ka,kb) = ky_n(a,b), \quad \text{and} \quad \ell(ka,kb) = k\ell(a,b). \tag{11}$$

The main results of [11], were derived with help of the Perron-Frobenius theorem and the Collatz-Wielandt formula for the non-negative matrices; see Chapter 8 of the textbook [120]. Alternatively, these results can be derived from the Cauchy-Ostrovsky theorem; see theorems 1.1.3 and 1.1.4 in the textbook [130] and verify that our polynomial $P(z)$ satisfies all condition of the latter.

For the joint consideration of the normal and misère versions of an impartial game we refer the reader to the books [6] and [19] Chapter 12. This approach was applied to NIM$(a,1)$ in [39] and to NIM(a,b) in [70, 71].

However, the results differ in cases $a = 1$ and $a > 1$. In case $a = 1$ (for any $b \ge 1$) the set of P-positions P^N and P^M (of the normal and misère versions, respectively) "almost coincide". More precisely, their symmetric difference consists of only six positions:

$P^N \setminus P^M = \{(0,0),(b,b+1),(b+1,b)\}$, while $P^M \setminus P^N = \{(0,1),(1,0),(b+1,b+1)\}$.

This result was obtained in [39] for $b = 1$ and extended for any positive integer b in [71].

Also in [70] the following three properties were shown for $a = 1$:

(i) from any position of $P^M \setminus P^N$ there is a move to $P^N \setminus P^M$;

(ii) from any non-terminal position of $P^N \setminus P^M$, that is, from $(b, b+1)$ or $(b+1, b)$, there is a move to $P^N \setminus P^M$;

(iii) from any position $(x,y) \notin P^N \cup P^M$, either both sets P^N and P^M can be reached in one move from, or none of them.

In case $a > 1$ (for any $b \geq 1$) the kernel of the misère version is defined by the recursion

$$\tilde{x}_n = \text{mex}_b(\{\tilde{x}_i, \tilde{y}_i \mid 0 \leq i < n\}), \;\; \tilde{y}_n = \tilde{x}_n + an + 1; \;\; n \in \mathbb{Z}_+. \tag{12}$$

This formula was proven in [39] for $b = 1$ and extended to any positive integer b in [71]. Let us notice that formulas (6) and (12) differ just slightly. Comparing them we immediately conclude that for any integer $a > 1$ and $b \geq 1$ the sets of P-positions of the normal and misère versions are disjoint, in contrast to the case $a = 1$; see [70] for more details and, in particular, for the cases $a = 0$ or $b = 0$. According to terminology of [68, 70], NIM(a,b) is *strongly miserable* game when $a = 1$ and it is *miserable* (but not strongly) when $a > 1$.

Some conclusions and open problems:

Two main recursions (6) and (12) are deterministic, yet, their solutions (the kernels, or equivalently, the P-positions of the normal and misère versions of NIM(a,b)) behave in a "pseudo-chaotic way" when $b > 1$. For which other combinatorial games, their kernels demonstrate such behavior? It seems that the four parametric game NIM$(a,b;p,q)$, introduced in [69], is a good candidate. This game is a generalization of NIM(a,b) and Larsson's NIM(a,p) from [73, 107]; see also [101, 102]). Yet, the class in question might be much larger.

Both recursions (6) and (12) can be solved by a polynomial algorithm based on the Perron-Frobenius theorem. Which other recursions can be solved in such a way?

For $b = 1$ the solutions of both recursions are given by closed formulas, while for $b > 1$ this is unlikely.

Cases $a = 1$ and $a > 1$ also differ substantially. In the first case, the symmetric difference $P^N \Delta\, P^M$ consists of only six positions, while in the second case these two sets are disjoint, $P^N \cap P^M = \emptyset$. In [70] such two types of games are named miserable and strongly miserable, respectively, and simple characterizations for both classes are obtained.

May recursions (6) and (12) or similar ones have other applications, perhaps, beyond game theory?

6. Nhan Bao Ho

We split the section into two, Section 6.1 by N. B. Ho and U. Larsson, and Section 6.2 by N. B. Ho.

6.1. **Additive periodicity of Wythoff Nim's Sprague-Grundy function.**

We give a quick overview of Landman's FSM based proof of arithmetic periodicity of Sprague-Grundy values of Wythoff Nim [99]. Since the argument is of a geometric nature, we include a figure to illustrate the important concepts.

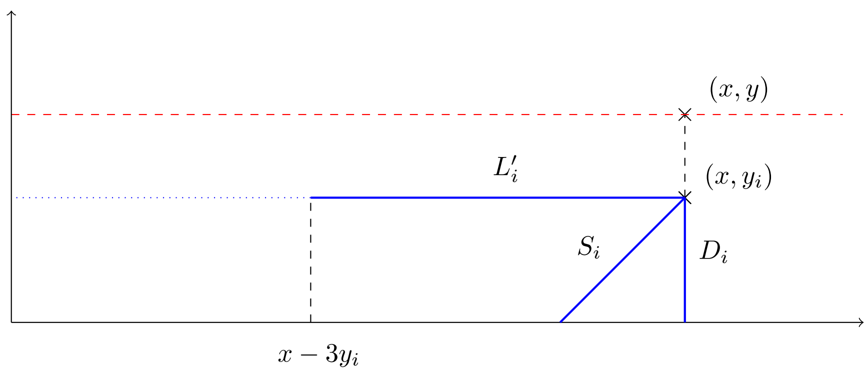


Figure 3. An FSM-analysis of Grundy values of Wythoff Nim.

We are concerned with the Sprague-Grundy function, G, of Wythoff Nim along a fixed y-coordinate, indicated with the horizontal dashed red line in the figure, and we want to show arithmetic periodicity. It suffices to show that a finite state machine (FSM), with no input, can compute a function $H(x, y) = G(x, y) - x + 2y$. Then H must be periodic and hence G arithmetic (or additive) periodic. The definition of H is chosen because one can prove that G is bounded, namely $x - 2y \leqslant G(x, y) \leqslant x + y$, and so $0 \leqslant H(x, y) \leqslant 3y$. An FSM has finite memory, and hence this bound is necessary, and also sufficient to establish periodicity of H, as we now establish. For each $0 \leqslant i \leqslant y$, it suffices to keep track of the H values of the sets L'_i (bounded left), D_i (down) and S_i (slant). These can be stored in three bit strings of length $3y_i + 1$, where a "1" at index j indicates membership of the jth H-value. Then the NOR operator finds the smallest index for which there is no "1" and via the analogue to the MEX-function, $H(x, y_i)$ and so also $G(x, y_i)$ is computed for each i. It is easy to update the bit strings for computation of the $H(x+1, y_i)$s. This establishes that an FSM can simulate the H-function along a given y-coordinate, and so H must be ultimately periodic. Indeed there are only finitely many possible states; the cardinality is $O(y^2)$. Hence the Sprague-Grundy function is ultimately arithmetic periodic along any given y-coordinate (or x-coordinate, by symmetry of Wythoff Nim).

Note that a key ingredient for Landman's method is the establishment of the bounds for $G(x, y)$. Using similar bounds, one can apply this technique for other 2-pile Nim-like games. One such example was analyzed in [77].

6.2. **Two variants that preserve P-positions of Wythoff Nim.**

In [27], Duchêne et al. characterize modifications of Wythoff Nim that preserves its P-positions (also see Section 3.6). In the context of that work, modifications of WYTHOFF NIM are invariant in the sense that if the move $(a,b) \to (a-i,b-j)$ is allowed then the move $(a,b) \to (a-j,b-i)$ is also allowed, provided that $a \geq j$ and $b \geq i$.

In [74], the author studies two non-invariant modifications, one extension and one restriction, of WYTHOFF NIM preserving its P-positions.

> *In the restriction called $\mathcal{R}$-*WYTHOFF*, the constraint is that removing tokens from the smaller pile is not allowed.*

In other words, from a $\mathcal{R}$-WYTHOFF position (a,b) with $a \leq b$, one can move either $(a,b) \to (a,b-i)$ or $(a,b) \to (a-i,b-i)$. The author also proves that there is no restriction of $\mathcal{R}$-WYTHOFF that preserves $\mathcal{R}$-WYTHOFF's P-positions.

> *In the extension called $\mathcal{E}$-*WYTHOFF*, along with original moves of* WYTHOFF NIM*, there exists an extra move of the form $(a,b) \to (a-k,b-l)$ in which $a \leq b$ and $l < k$.*

The author also establishes positions whose Sprague-Grundy values are 1 of both games as follows:

$$\{(2,2),(4,6),(\lfloor \phi n \rfloor - 1, \lfloor \phi n \rfloor + n - 1) | n \geq 1, n \neq 2\} \qquad \text{for } \mathcal{R}\text{-WYTHOFF};$$
$$\{(\lfloor \phi n \rfloor - 1, \lfloor \phi n \rfloor + n - 1) | n \geq 1\} \qquad \text{for } \mathcal{E}\text{-WYTHOFF}.$$

Note that these two sets differ only three positions at the beginning. Moreover, the formulas for these positions show that they are remarkably close to P-positions of WYTHOFF NIM, being determining by a translation except for only three positions.

The author also proves the following property of Sprague-Grundy functions for both games: for any nonnegative integer a and Sprague-Grundy value g, there exists b such that $\mathcal{G}(a,b) = g$. Actually, this property is equivalent to the following feature of the sequence of positions whose Sprague-Grundy values are g. Let $((a_n,b_n))_{n \geq 0}$ be the sequence of positions whose Sprague-Grundy values are g, in which $a_i < a_j$ if $i < j$. Then the set $\{a_n, b_n | n \geq 0\}$ contains every nonnegative integer.

Another feature of Sprague-Grundy values of both $\mathcal{R}$-WYTHOFF and $\mathcal{E}$-WYTHOFF NIM is the additive periodicity of the sequence $(\mathcal{G}(a,n))_{n \geq 0}$ in the sense that there exist n_0 and $p > 0$ such that $\mathcal{G}(a,n+p) = \mathcal{G}(a,n) + p$ for all $n \geq n_0$. This type of periodicity of WYTHOFF NIM is discussed in Section 6.1.

7. Clark Kimberling

In response to an invitation, the author surveys the Wythoff array and its many associates, concentrating on his own contributions and those of others.

7.1. **The Wythoff Array.** During the first year of existence of the Fibonacci Association, one of the founders published a short article [2] in the first volume of *The Fibonacci Quarterly*. There, Brother Alfred Brousseau discusses an ordering of the set of *all* Fibonacci sequences of positive integers. He concludes with these words: "The above approach in representing Fibonacci sequences and ordering them is all by way of suggestion. There are doubtless other ways of achieving the same objective. It would be very helpful if additional proposals were aired before a final standard is adopted."

A second ordering of the positive Fibonacci sequences appears in Kenneth Stolarsky's one-page article [141], in which he introduces an ordering in a form now known as a *Stolarsky array*. Three years later, David Morrison, then a student at Harvard University, published another ordering. If—to borrow Brother Alfred's words—there is a "final standard", this must be it. In Morrison's ordering, the Fibonacci sequences appear as rows of the array shown here:

1	2	3	5	8	13	21	34	55	89	144	⋯
4	7	11	18	29	47	76	123	199	322	521	⋯
6	10	16	26	42	68	110	178	288	466	754	⋯
9	15	24	39	63	102	165	267	432	699	1131	⋯
12	20	32	52	84	136	220	356	576	932	1508	⋯
14	23	37	60	97	157	254	411	665	1076	1741	⋯
17	28	45	73	118	191	309	500	809	1309	2118	⋯
19	31	50	81	131	212	343	555	898	1453	2351	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Table 6. The Wythoff Array

Morrison named this array after Wythoff because the rows consist of Wythoff pairs—these being the winning pairs for Wythoff's game. For example, in row 1, the Wythoff pairs are $(1,2)$, $(3,5)$, $(8,13),\ldots$; in row 2, they are $(4,7)$, $(11,18)$, $(29,47),\ldots$; and so on. The Wythoff pairs are given by $(\lfloor n\alpha\rfloor, \lfloor n\alpha^2\rfloor)$, where $\alpha = (1+\sqrt{5})/2$, the golden ratio. Properties of the Wythoff array, W, include the following:

(1) Every row is a Fibonacci sequence; i.e., the recurrence $x_n = x_{n-1} + x_{n-2}$ holds.

(2) The rows extend indefinitely to the left by "precursion" ($x_{n-2} = x_n - x_{n-1}$), resulting in an array that contains every Fibonacci sequence of integers.

The array W is an interspersion and a dispersion. The first of these means, briefly, that every row is interspersed by every other row, and the second means that the first column can be used to disperse its complement using certain iterated compositions. More precise definitions [80] follow:

An array $A = A(i,j)$ of positive integers is an *interspersion* if

(I1) every positive integer occurs exactly once in A;

(I2) every row of A is an increasing sequence;

(I3) every column of A is an increasing sequence;

(I4) if (u_i) and (v_j) are distinct rows of A, and if i and h are indices for which $u_i < v_h < u_{i+1}$, then $u_{i+1} < v_{h+1} < u_{i+2}$.

To define *dispersion,* suppose that s is an increasing sequence of positive integers, that the complement t of s is infinite, and that $t(1) = 1$. The dispersion of s is the array whose nth row is

$$t(n), s(t(n)), s(s(t(n))), s(s(s(t(n)))), \ldots$$

The Wythoff array is the dispersion of its first column; indeed, the first column is the lower Wythoff sequence, and the rest of W consists of the numbers in the upper Wythoff sequence. The main theorem on interspersions and dispersions is that they are equivalent [80].

Another property of W is observed when, for each n, we write the number of the row of W that contains n, resulting in this sequence:

$$f = (1, 1, 1, 2, 1, 3, 2, 1, 4, 3, 2, 5, 1, 6, 4, 3, 7, 2, 8, 5, 1, \ldots).$$

Deleting the first occurrence of each positive integer leaves the same sequence, f. Because this lower-trim operation can be repeated indefinitely and always returns f, this sequence, and any other that arises similarly from an interspersion is called a *fractal sequence* [85]. The upper-trim of f, obtained by deleting all the 0's from $f - 1$, is also a fractal sequence.

There are several modifications of the Wythoff array, W, that appear in the Online Encyclopedia of Integer Sequences (OEIS) [132]. Aside from W itself, which is indexed as $A033513$ in OEIS, the left-justified Wythoff array, $A165357$, formed from W by left extending each row to a pair (a, b) such that $a > b$, has the following property: every (a, b) satisfying $a > b \geq 0$ occurs exactly once, and every (c, d) satisfying $0 \leq c \leq d$ occurs exactly once.

A second array obtained from W is the Wythoff difference array, D ([89] and $A080164$), formed by differences between Wythoff pairs in W:

1	2	5	13	34	89	233	610	1597	$\cdots$
3	7	18	47	123	322	843	2207	5778	$\cdots$
4	10	26	68	178	466	1220	3194	8362	$\cdots$
6	15	39	102	267	699	1830	4791	12543	$\cdots$
8	20	52	136	356	932	2440	6388	16724	$\cdots$
9	23	60	157	411	1076	2817	7375	19308	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

TABLE 7. The Wythoff Difference Array

Properties of D include the following:

(1) The difference between adjacent terms in every column is a Fibonacci number.

(2) Every term of column 1 of W is in column 1 of D.

(3) Every term in a row of D, except the first, is in the corresponding row of W.

(4) D is an interspersion.

$n:$	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
$L(n):$	1	3	4	6	8	9	11	12	14	16	17	19	$\cdots$
$U(n):$	2	5	7	10	13	15	18	20	23	26	28	31	$\cdots$

Table 8. n, L, and U

(5) D is the dispersion of the upper Wythoff sequence, $A001950$, whereas W is the dispersion of the lower Wythoff sequence, $A000201$.

7.2. **Zeckendorf Arrays and the Wythoff Array.** Every positive integer n is a sum of Fibonacci numbers, no two of which are consecutive. This unique sum is known as the Zeckendorf representation of n. The *Zeckendorf array,* $Z = Z(i, j)$, is defined [83] as follows: column j of Z is the increasing sequence of all n in whose Zeckendorf representation the least term is F_{j+1}. For example, row 1 of Z is given by

$$z(1,1) = 1 = F_2, \qquad z(1,2) = 2 = F_3,\ \ldots,\ z(1,j) = F_{j+1},\ \ldots\ .$$

When working with Zeckendorf representations, it is often helpful to refer to a shift function, s, defined from a Zeckendorf representation as follows:

$$n = \sum_{i=1}^{\infty} c_i F_{i+1} \implies s(n) = \sum_{i=1}^{\infty} c_i F_{i+2}.$$

Using s, it is proved [83] that the Zeckendorf array is identical to the Wythoff array. Related developments, including higher-order Zeckendorf arrays and Zeckendorf/Wythoff trees, are found in Lang [100]; Bicknell-Johnson [10]; a paper on Fibonacci Phyllotaxis [133] by Spears, Bicknell-Johnson, and Yan; Cooper [22]; and Ericksen and Anderson [35]. Other related arrays are discussed in Hegarty and Larsson [73].

7.3. **Lower and Upper Wythoff Sequences.** The winning solutions of Wythoff's game are the previously mentioned pairs $(\lfloor n\alpha \rfloor, \lfloor n\alpha^2 \rfloor)$. Separating the components gives the lower Wythoff sequence, $L = (\lfloor n\alpha \rfloor) = A000201$, and the upper Wythoff sequence, $U = (\lfloor n\alpha^2 \rfloor) = A001950$. Clearly, the terms of L fill the odd numbered columns of the Wythoff array, and those of U, the even numbered columns.

It is easy to write out terms of L and U without reference to an irrational number or Wythoff's game. Consider the rows in Table 8:

The first step is to write row 1 and then to place 1 below the 1 in row 1. Add the two numbers to get $U(1) = 2$. Thereafter, take each $L(n)$ to be the least positive integer not yet in rows 2 and 3, and take $U(n) = n + L(n)$.

A related procedure, from a comment by Roland Schroeder at $A000201$, produces L from a Mancala-type game as follows: n stacks of chips are aligned and numbered from left to right as #1, #2, #3, etc, with stack #n consisting initially of n chips. One step in the game consists of transferring from the leftmost stack all of its chips so that the stacks to the right each gain 1 chip until one of two things happens: either there are no more chips, or otherwise, the leftover chips are used to create new stacks, one chip per stack, lined up to the right of the stacks already present. The game continues until there are n stacks, no two of which have the same number of chips. The number of steps for the whole game is $L(n)$.

A variant of the Schroeder-Mancala game is described in a proposal by Ron Knott, with a solution by Sam Northshield: "As an infinite Mancala game, suppose a line of pots contains pebbles, 1 in the first, 2 in the second, and n in the nth, without end. The pebbles are taken from the leftmost non-empty pot and added, one per pot, to the pots to the right. Prove that the number of pebbles in pot n as it is emptied is $\lfloor n\varphi \rfloor$, where φ is the golden ratio, $(1+\sqrt{5})/2$."

Another method for generating both L and U is to decree that $L(1) = 1$, that $a(n+1) = a(n)+2$ if $a(n)$ is already determined, and that $a(n+1) = a(n)+1$ otherwise. This procedure can be generalized (e.g., [90], A184117) to produce Beatty sequences other than L and U.

Quite a different approach to L and U is to arrange in increasing order all the numbers j/α and k/α^2 (or equivalently, $j\alpha$ and k), so that the list begins with

$$\frac{1}{\alpha^2}, \frac{1}{\alpha}, \frac{2}{\alpha^2}, \frac{3}{\alpha^2}, \frac{2}{\alpha}, \frac{4}{\alpha^2}, \frac{3}{\alpha}, \frac{5}{\alpha^2}, \frac{6}{\alpha^2}, \frac{4}{\alpha}, \frac{7}{\alpha^2}. \frac{8}{\alpha^2}, \frac{5}{\alpha}, \frac{9}{\alpha^2}, \frac{5}{\alpha}.$$

Here, for every n, the position of n/α is $\lfloor n\alpha^2 \rfloor$, and that of n/α^2 is $\lfloor n\alpha \rfloor$. The same method works for many pairs of irrational numbers: if $r > 1$ and $1/r + 1/s = 1$, then the positions of n/r and n/s in the joint ranking of all j/r and k/s are $\lfloor ns \rfloor$ and $\lfloor nr \rfloor$, respectively. This may be the shortest route for introducing pairs of Beatty sequences and proving that they partition the positive integers. The method extends to more than two sequences; e.g., Paul Hanna's three-way splitting of $\mathbb{N}$ using a zero γ of $\gamma^3 = \gamma^2 + \gamma + 1$, at A184820-A184822. See also A187950.

The lower and upper Wythoff sequences both occur in connection with both the greedy and lazy Fibonacci representations of positive integers. We have already discussed the "greedy" case, since the greedy algorithm simply finds the Zeckendorf representation, for which the numbers in L are those whose representation ends with an even number of 0s, and, of course, those in U end with an odd number of 0s.

Another way to find the Zeckendorf representation of n is to replace every run of 1s in $(n)_{\text{base 2}}$ by a single 1, thereby obtaining an ordered list of 1s and 0s which are the coefficients in $(n)_{\text{Zeckendorf}}$. On the other hand, if every run of 0s in $(n)_{\text{base 2}}$ is replaced by a single 0; the resulting list of 1s and 0s are the coefficients in $(n)_{\text{lazy.}}$ It can be shown that

$$(\#\text{ terms in } (n)_{\text{Zeckendorf}}) \leq (\#\text{ terms in } (n)_{\text{lazy}}),$$

as in A095792; indeed, the Zeckendorf representation is often called the minimial Fibonacci representation, and the lazy, the maximal Fibonacci representation.

An interesting way to present lazy Fibonacci representations is as a graph consisting of two components, L^* and U^*, each being a binary tree. The tree L^* is rooted in 1, of which the children are $1+2$ and $1+3$. The children of $1+2$ are $1+2+3$ and $1+2+5$, and the children of $1+3$ are $1+3+5$ and $1+3+8$; in general, the children of each

$$m = F_{i_1} + F_{i_2} + \cdots + F_{i_k}, \quad \text{where } i_1 < i_2 < \cdots < i_k,$$

are $m + F_{i_k+1}$ and $m + F_{i_k+2}$. The same rule of generation applies starting with the root 2 of U^*. The numbers in L^* and U^*, taken in order as generated, form the sequences A255773 and A255774, which are permutations of L and U, respectively; see also A095903.

Among the zero-one sequences known as the infinite Fibonacci word is

$$A003849 = (0, 1, 0, 0, 1, 0, 1, \ldots),$$

definable as the fixed point of the morphism $0 \to 01$, $1 \to 0$, starting with 0. The lower Wythoff sequence, L, tells the positions of 0 in $A003849$, and U, the positions of 1. Let S denote the infinite Fibonacci word $A003849$, and let $S(n) = (s(1), s(2), \ldots, s(n))$ be the initial segment of S that has length n. Then $S(n)$ occurs infinitely many times in S. We ask where each appearance starts and answer as follows: for $n = 1$, the segment consisting solely of 0, starts at positions given by L; for $n = 2$ and $n = 3$, the segment starts at positions given by the Wythoff AA numbers, $A003622$; for $n = 4, 5, 6$, the segment starts at the Wythoff AAA numbers, $A134859$; for the next F_5 numbers ($n = 7, 8, 9, 10, 11$), the segments start at positions given by the Wythoff AAAA numbers, $A151915$, and so on. See $A246354$.

Other appearances of U and L are as solutions to complementary equations, as introduced in [87]; that is, equations that can be put into the form $f(a, b) = 0$ where a and b are complementary sequences of positive integers. Following are complementary equations for which the initial condition is $a(1) = 1$ and the unique solution is $a = L$ (or equivalently, $b = U$):
(1) $a(a(n)) = b(n) - 1$
(2) $a(b(n)) = a(n) + b(n)$
(3) $b(a(n)) = a(n) + b(n) - 1$
(4) $b(b(n)) = a(n) + 2b(n)$

These four equations are used as lemmas for developing more elaborate complementary equations in which the columns of the Wythoff array play a central role ([50, 87, 88, 29, 46, 45, 91]). In particular, Fraenkel introduces the game of *flora* in [45].

7.4. **Wythoff-related trees.** Let T_1 be the tree generated by these rules: the root is 1, and for each node x, the children are $\lfloor nx \rfloor$ and $\lfloor nx^2 \rfloor$. The first four generations of T_1 are given by

$$\{1\}, \quad \{2\}, \quad \{3, 5\}, \quad \{4, 7, 8, 13\}, \quad \{6, 10, 11, 18, 12, 20, 21, 34\}.$$

In T_1, every Wythoff pair except $(1, 2)$ occurs as a pair of children. Taken in order of appearance in T_1, the numbers comprise $A074049$.

Next, let T_2 be the tree, essentially $A052499$, generated ([86]) as follows: $1 \in T_2$, and if $x \in T_2$, then $2x \in T_2$ and $4x - 1 \in T_2$. When all the terms of T_2 are arranged in increasing order and the initial 1 is removed, the remaining even numbers are in positions $1, 3, 4, 6, 8, \ldots$; i.e., the lower Wythoff sequence, L, and the odd numbers are in positions given by U.

The next tree, T_3, contains every integer exactly once. Here, $0 \in T_3$, and if $x \in T_3$, then $2x \in T_3$ and $1 - x \in T_3$, and duplicates are deleted as they occur. As in $A232723$, the numbers in order of generation are

$$0, 1, 2, 4, -1, 8, -3, -2, 16, -7, -6, -4, 3, 32, \ldots\ .$$

The even integers occupy the positions given by L, and the odds, by U.

Another tree, T_4, gives an ordering of the positive rational numbers. To generate T_4, start with 1, and if $x \in T_4$, then $x + 1 \in T_4$ and $1/x \in T_4$, and duplicates are deleted as they occur. The first few fractions in T_4 are

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{1}{2}, \frac{4}{1}, \frac{1}{3}, \frac{3}{2}, \frac{5}{1}, \frac{1}{4}, \frac{4}{3}, \frac{5}{2}, \frac{2}{3}, \frac{6}{1}.$$

Here, the positions of the positive integers comprise row 1 of the Wythoff array, W, and the positions of the numbers $n+1/2$ comprise row 2. In general, the positions of denominators ($A226080$) congruent to r mod n, where $0 < r < n$ and $\gcd(n, r) = 1$, comprise a row of W.

In T_4, taken as a sequence, the fractions ≤ 1 occupy positions given by $U - 1$, and those > 1, by $L - 1$. Other trees, including trees consisting of all the rational numbers, all the Gaussian integers, and all the Gaussian rational numbers, are introduced in [95]. See $A226080$ for an overview of such trees.

8. Urban Larsson

It has been very rewarding to study variations of Wythoff's game interacting with number theory, computer science and more. We may ask questions about rearranging game rules, or we may begin by looking at sequences or recurrences and ask for appropriate game rules. We generate new theory by altering old patterns, keeping some properties and shifting some, not too may at a time, just enough to be able to recognize some features and being able to see it as a generalization of the old.

8.1. Imitation Nim and Wythoff's sequences.

This paper concerns a variation of the classical game of Nim on two piles.

> *Suppose that the previous player removed x tokens from the smaller heap (any heap if they have equal size). Then the next player may not remove x tokens from the larger heap.*

We call this game Imitation Nim (although, as remarked by Aviezri Fraenkel the first time I met him, *L*imitation Nim would also have been an appropriate name). Notice that by this move restriction, the winning strategy of 2-pile Nim is altered. For example, the player who moves from the position $(1, 1)$ will lose in Nim, but win in Imitation Nim. It turns out that the P-positions correspond to those of Wythoff Nim. The game generalizes nicely. Suppose that $m - 1$ consecutive imitations from one and the same player are allowed, but not the mth one. For example, with $m = 2$ and $0 < x \leq y$, suppose that the three most recent moves were $(x, y) \to (x - z, y) \to (x - z, y - z) \to (x - z - w, y - z)$, alternating between the two players. Then precisely the move to $(x - z - w, y - z - w)$ is prohibited. The P-positions of this generalization of Imitation Nim correspond to those of a variation of Wythoff Nim with a so-called *blocking maneuver* on the diagonal options [73].

8.2. Three variations of Blocking Wythoff Nim.

In this section we study three so-called blocking variations of Wythoff Nim.

> *Let k be a positive integer. The first game is as* Wythoff Nim*, with one exception: the previous player may, before the next player moves, block off $k-1$ of the diagonal type options and declare them forbidden. When a player has moved, any blocking maneuver is forgotten.*

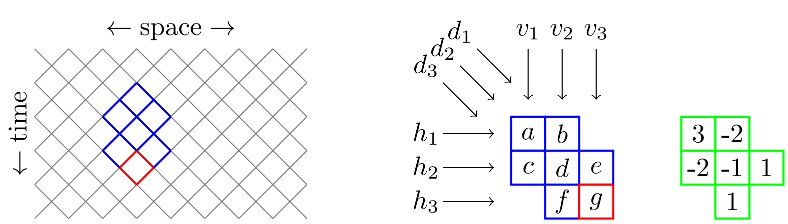
It turns out that the P-positions of this generalization have similar structure as that of Wythoff Nim. The sequences of their coordinates approximate very closely discrete half-lines with irrational slopes. Namely they can be approximated by homogeneous Beatty sequences $(\lfloor n\alpha \rfloor)$, where α is a positive irrational and n ranges over the positive integers. (This was proved independently by Hegarty in [101, Appendix] and Fraenkel, Peled in [55].)

Combinatorial games with a blocking maneuver, or so-called Muller Twist, were proposed via the game Quarto in "Mensa Best Mind games Award" in 1993. Later the idea appeared in the literature [76, 139, 61].

Having observed, in [73], that a blocking maneuver on the diagonal type options gives rise to interesting sequences of integers, I set out to study two more variations of Wythoff Nim with a blocking maneuver. In my second study [107] blocking is allowed exclusively on the Nim-type options, whereas in [103] blocking is allowed on any options of Wythoff Nim. In the latter paper, an exact formula for the P-positions is presented for the case where at most one option may be blocked. For this game, the *upper* P-positions have *split* into two sequences of P-positions, one with slope ϕ, similar to the Beatty type formula for Wythoff Nim, and the other with slope 2. A position (x, y) is *upper* if $y \geq x$. We also give a closed formula expression for the P-positions for the game with at most two blocked options and state precise conjectures for some greater blocking parameters. The general games seems harder. (In contrast, the P-positions of the games in [107] can be described via Beatty sequences for all blocking parameters, generalizing Holladay's classical m-Wythoff Nim [38].) In [21] we prove that there is a diamond shaped one-dimensional cellular automaton that emulates the full blocking variation of Wythoff Nim, k-Blocking Wythoff Nim; see also Figure 8.2. From the abstract:

"As k becomes large, parts of the pattern of winning positions converge to recurring chaotic patterns that are independent of k. The patterns for large k display a surprising amount of self-organization at many scales."

This self-organization is shown in figure 4. We sketch the definition of the cellular automaton that computes the "palace numbers - k" (the palace numbers represent the number of P-position that Wythoff's Queen sees from any given position). It updates diamond shaped cells in parallel, and time is running South-East.



(For references to colors, see the online arXiv-version of the survey.) The red cell's (containing the g) value is computed according to the formula

$$g = a - b - c + e + f + p.$$

The green squares (to the right) correspond to the blue cells (in the middle) and show the palace compensation terms. For any blue cell containing a negative value, the corresponding palace compensation term must be added. In the formula, p represents the total contribution of these palace compensation terms.

Theorem 7. *The k-Blocking Wythoff Nim position (x, y) is a P-position if and only if the CA gives a negative value at that position, when the CA is started from an initial condition defined by*

$$CA(x, y) = \begin{cases} k & x < 0 \text{ and } y < 0 \\ 0 & x < 0 \text{ and } y \geq 0 \\ 0 & x \geq 0 \text{ and } y < 0 \end{cases}$$

8.3. **A Generalized Diagonal Wythoff Nim and splitting beams of P-positions.** A *splitting* of sequences of P-positions into two sequences of distinct slopes is discussed in two papers. The P-positions of Nim lie on the single beam

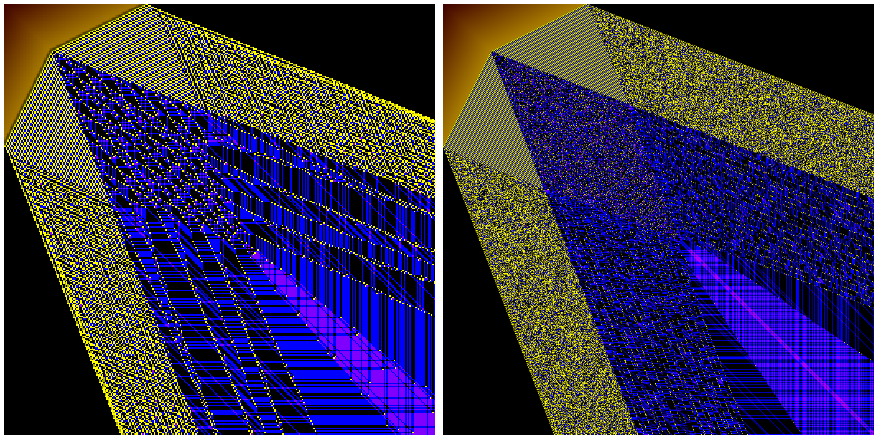


FIGURE 4. Self-organized regions in 100-Blocking Wythoff Nim (left, 300×300 region) and 1000-Blocking Wythoff Nim (right, 3000×3000 region). Here, the upper left corner is the $(0,0)$-position.

of slope 1, whereas those of WYTHOFF NIM lie on the beams of slopes ϕ and ϕ^{-1}. Therefore, going from NIM to WYTHOFF NIM has split the single *beam* of P-positions in NIM into two new P-beams for WYTHOFF NIM of distinct slopes. Let p, q be positive integers. If we adjoin, to the game of WYTHOFF NIM, new moves of the form (pt, qt) and (qt, pt), for all positive integers t, will the upper P-positions of the new game, denoted (p, q)-GDWN, split once again into two new distinct slopes? In the first paper on GDWN games, it was proved that the ratio of the coordinates of the upper P-positions of this game do not have a unique accumulation point if $p = 1$ and $q = 2$. Via experimental results it was conjecture that the upper P-positions of (p, q)-GDWN split if and only if (p, q) is either a Wythoff pair or a dual Wythoff pair, that is of the form $(p, q) = (\lfloor \phi n \rfloor, \lfloor n\phi^2 \rfloor)$ or $(\lceil \phi n \rceil, \lceil n\phi^2 \rceil)$, for n a positive integer. In the second paper [106], it is proved that $(1, 2)$-GDWN splits. Two new discoveries made this possible.

Theorem 8. *Suppose that the sequences (a_i) and (b_i) satisfy Property W. Then, for $n \in \mathcal{N}$,*

$$\frac{\#\{i > 0 \mid a_i < n\}}{n} \geq \phi^{-1} - o(1) \tag{13}$$

and

$$\frac{\#\{i > 0 \mid b_i < n\}}{n} \leq \phi^{-2} + o(1). \tag{14}$$

In particular the result holds for $\{(a_i, b_i)\}$ representing the upper P-positions of any WYTHOFF NIM *extension.*

Lemma 9. *If there is a positive lower asymptotic density of x-coordinates of P-positions above the line $y = 2x$, then the upper P-positions $\{(a_n, b_n)\}$ of $(1, 2)$-GDWN split.*

The conjecture is that there are precisely two accumulation points for the upper P-beams, namely to the ratio of coordinates $1.477\ldots$ and $2.247\ldots$ respectively; see Figure 6, the rightmost picture.

The idea of a 'fill-rule property' of the moves of Wythoff Nim has been used to generalize the GDWN games in another paper in this publication: "Geometric Analysis of a Generalized Wythoff Game". Central to the hypothesis built in that paper is a renormalization idea from physics: Figure 5 shows some of its behavior; here the N-beams are colored black (and the P-positions are not visible) but we can see their impact on the horizontal, (1,1)-diagonal and (3,5)-diagonal N-positions respectively. Further details are given in that paper.

FIGURE 5. The fill-rule properties of the rules $(0,t),(t,t)$ and $(3t,5t)$ respectively, computed for the game (3,5)-GDWN. Apart from the striking dynamics of the fill rules, there is also a quasi-log-periodicity visible in the two left most pictures, with a 'scale invariance' of factor $1.478\ldots$.

8.4. MAHARAJA NIM **and a dictionary process.** In this paper, coauthored with J. Wästlund, we study an *extension* of WYTHOFF NIM,

> *where the Queen and Knight of Chess are combined in one and the same piece, the* Maharaja (*no coordinate increases by moving*). *The game is called* MAHARAJA NIM.

One can also view this game as a *restriction* of $(1,2)$-GDWN. It is clear that the P-positions of WYTHOFF NIM will be altered for this game. Namely, the "smallest" non-zero P-positions of WYTHOFF NIM are $(1,2)$ and $(2,1)$, corresponding precisely to the new move options introduced for MAHARAJA NIM. However, we have succeeded in proving that the P-positions remain within a bounded distance of the half-lines of slopes ϕ and ϕ^{-1} respectively. To obtain such a result we have used an unconventional method in this field, namely, relating the upper P-positions to a certain dictionary process on binary words, a process that we also prove is in general undecidable. We also give a short proof for a generalization of an already very nice result in [55], concerning complementary sequences, to a "Central Lemma".

Lemma 10. *Suppose (x_n) and (y_n) are complementary sequences of positive integers with (x_n) increasing. Suppose further that there is a positive real constant, δ,* *such that, for all n,*

$$y_n - x_n = \delta n + O(1). \tag{15}$$

Then there are constants, $1 < \alpha < 2 < \beta$, such that, for all n,

$$x_n - \alpha n = O(1) \tag{16}$$

and

$$y_n - \beta n = O(1). \tag{17}$$

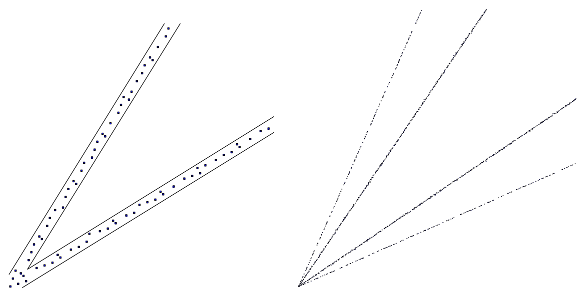


FIGURE 6. Some initial P-positions have been computed for MAHARAJA NIM and $(1, 2)$-GDWN respectively. The leftmost picture indicates that we are able to capture the behavior of MAHARAJA NIM's upper P-positions within a narrow stripe of slope ϕ. On the other hand, we have proved that the P-positions of GDWN to the right will eventually depart from any such stripe, however wide we make it. Extensive computations make us believe that perhaps the upper pair of P-beams' slopes will converge to the accumulation points $1.478\ldots$ and $2.248\ldots$ respectively.

8.5. **The $\star$-operator and Wythoff Nim.** The game of WYTHOFF NIM lead to the definition of the $\star$-operator [112]. Via a double application of $\star$ (obtaining the $\star\star$-operator), a P-equivalent game of WYTHOFF NIM that is neither a restriction nor an extension, is defined.

In [31] it was conjectured that, given a pair of complementary Beatty sequences (a_i) and (b_i) (as described in the Introduction), there is an invariant subtraction game for which the P-positions constitute precisely all the pairs (a_i, b_i) and (b_i, a_i), together with the terminal position $(0, 0)$.

Take the description of the candidate P-positions (without $(0, 0)$) as *moves* in another invariant subtraction game G. Then the non-zero P-positions of the new game $\mathcal{P}(G) \setminus \{(0, 0)\}$ correspond precisely to the moves of another invariant subtraction game $G^\star$. This game has the original candidate set of P-positions as its set of P-positions, that is $G = (G^\star)^\star$.

In Figure 8 the game which is P-equivalent to WYTHOFF NIM is displayed.

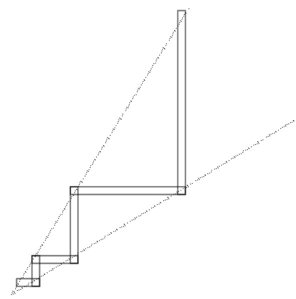


FIGURE 7. An open question: is it possible to decide in polynomial time, whether a given position is in P for MAHARAJA NIM? A "telescope" with focus $O(1)$ and reflectors along the lines ϕn and n/ϕ attempts to determine the outcome (P or N) of some position, (x, y) at the top of the picture. The method is successful for a similar game called $(2, 3)$-MAHARAJA NIM [109]. (It gives the correct value for all extensions of WYTHOFF NIM with a finite non-terminating converging dictionary). The focus is kept sufficiently wide (a constant) to provide correct translations in each step. The number of steps is linear in $\log(xy)$.

8.6. **Some new partizan direction.** In the preprint [121] we study an infinite subtraction game on just one heap. The players follow different rules, Left removes a number $A(n)$ of tokens and Right removes a number $B(n)$ of tokens, for some positive integer n. This is one of the rare encounters of Wythoff's game or sequences with Conway's famous theory for partizan games. We prove that each heap size is either a number or a reduced canonical form switch, accompanied by remarkable studies of the Fibonacci words and sequences.

8.7. **Other variations.** A paper in this publication, joint with Mike Weimerskirch, studies Continued Fraction extensions of WYTHOFF NIM, similar to those described in Sections 3 and 4. That is, candidate P-positions are assumed to be generated via certain periodic Continued Fraction sequences and an invariant game is sought for. The general question was asked at a meeting in BIRS, by Fraenkel. "When is there a game with nice game rules such that the P-positions can be described by complementary Beatty sequences?" Two more proposed answers to this question appear in these proceedings, one joint with Aviezri Fraenkel, and another joint with Michael Fisher.

In conclusion, the number of branches related to the discovery of W. A. Wythoff in the beginning of the last century is great and the diversions of its theories is even greater. But recall that WYTHOFF NIM was originally thought of as a variation of NIM, a game that, via the discoveries of Sprague and Grundy in the 1930s, naturally led authors towards the theory of *disjunctive sum* of games, later generalized by Milnor, Conway, Siegel and many more. By the removal of the independence of the heaps, I feel that Wythoff implicitly suggests that in general, the disjunctive sum of games is too restrictive to reveal a complete picture. Not all games satisfy such independence of move options: 'move in exactly one component at a time'. In fact, popular play games, such as GO and CHESS, only rarely show total independence of game components, and mostly only end-positions can be analyzed in this respect.

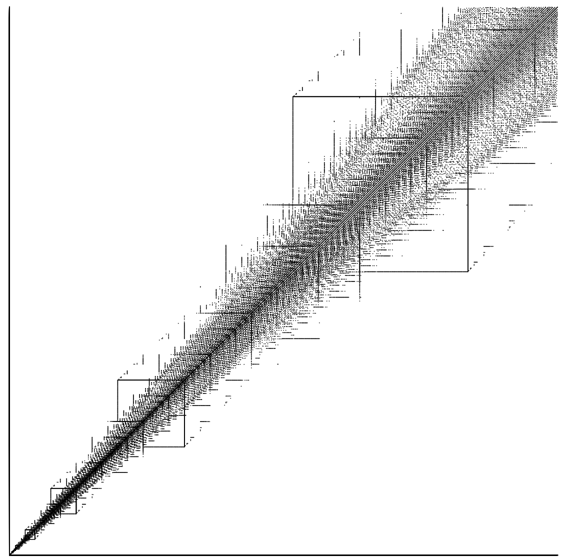


FIGURE 8. The picture illustrates the initial P-positions of the game (WYTHOFF NIM)*, or equivalently $(0,0)$ (the lower left corner) together with the moves of the game (WYTHOFF NIM)$^{\star\star} \neq$ WYTHOFF NIM (!). The picture shows all coordinates less than 5000, but we have made computations to 12000 obtaining a similar behavior. We understand some of its behavior, but the overall pattern remains a mystery, although it is contained between half lines from the origin of slopes ϕ^{-1} and ϕ. In fact, a characterization of infinitely many log-periodic positions has been obtained [105]. (WYTHOFF NIM)** is a very complicated game to play intelligently, although it has precisely the same set of P-positions as WYTHOFF NIM. But, the former game has a very nice property, which is absent in WYTHOFF NIM, namely it is reflexive, that is (WYTHOFF NIM)**=(WYTHOFF NIM)^{2k*} for all $k \geq 1$. Thus, 'simplest' rules do not always give the 'nicest' game properties.

WYTHOFF NIM is the first important step in the other direction, and in the future we will hopefully see many more approaches for how independent games become dependent, thereby distorting or/and enriching previous outcome patterns: from Xor to renormalization in physics and self-organization in phyllotaxis, enriching the classical self similarity of Fibonacci's rabbit sequences. Truly, there is some significant wisdom[3] in these games and sequences.

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[3]Coincidentally abbreviating Prof Fraenkel's institution: Weizman Institute of Science Department Of Mathemtics

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