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Upper bounds on Attractor dimensions for lattices of arbitrary size

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By using the link between extreme value theory and Poincaré recurrences, we compute the attractor dimension D - namely the effective number of degrees of freedom - of systems whose dynamics consist of n non-interacting particles with Gaussian underlying distributions. We also derive a theoretical expression for D in terms of finite size limits and test its validity and applicability with numerical experiments. We find that D in a non-interacting lattice is noticeably lower than the lattice size itself. We then estimate the attractor dimension of a collection of time-series issued from conceptual dynamical systems, finance and climate datasets. We find that spatial correlation within the particles reduce the attractor dimension. We also derive numerically the upper boundary for D of non-Gaussian systems, as their dimension can exceed the Gaussian theoretical limit.

The usual approach to analyse high dimensional complex systems is via statistical mechanics or statistical approaches. Mean Field Theories [1, 2], Principal Orthogonal Decomposition [3] or Empirical Orthogonal Functions[4] are commonly used to recover the stationary states of the system, to discriminate the anomalous configurations[5] and to study transitions between the different metastable states [6, 7]. Among the possible dynamical representations of a complex system, one generally looks for the most parsimonious model [8], namely that able to represent the switching between metastable states in terms of the least number of degrees of freedom, e.g. the Lorenz attractor [9] – a model of Rayleigh-Benard convection – the Pomeau-Manneville [10] maps – a conceptual model of turbulence intermittency – and the Ising model – a prototype of phase transitions [11]. In the context of geophysical flows, attempts have been made to develop low-dimensional models capturing the atmospheric dynamics [12, 13]. The limits of validity of these models have been repeatedly questioned [14–16] because of their simplicity: why should a model with few degrees of freedom be able to represent the dynamics of a complex system with a large (sometimes infinite) number of variables? In order to answer this question, one has to compute the minimum number of degrees of freedom needed to describe the dynamics. In dynamical systems, this is equivalent to

finding the dimension of the underlying attractors [17]. The attractor is a compact object which hosts all the trajectories of a system. In the 80s, it seemed possible to derive estimates of the attractor dimension D of complex systems from short time series of a variable measured at a point in space[18, 19]. It was soon realized that this approach - termed embedding method - was not feasible, because it required a time series whose length is exponential in the number of degrees of freedom of the system [20, 21]. However, data-driven approaches for the computation of D were also beset by numerical limitations when applied to complex systems [22, 23]. In the last decade there have been numerous theoretical advances in our understanding of the limiting distribution of Poincaré recurrences, summarised in [24]. The main finding is that, under suitable rescaling, recurrences of a state ζ_i at a time $i = 1, 2, \dots, M$ follow a Gumbel distribution whose scale parameter is the inverse of the local dimension $d(\zeta_i)$ [25]. Due to the universality of the Gumbel law, this result can be likened to a central limit theorem of Poincaré recurrences. The attractor dimension D can be obtained by averaging $d(\zeta_i)$ for several ζ_i belonging to the attractor. This theoretical result has recently been applied to the atmospheric circulation over the North Atlantic [26]. Here we generalize these findings, deriving an expression for the expected value of the attractor dimension D

as a function of the phase space dimension n . We consider systems such that the anomalies with respect to the reference field are Gaussian, spatially and serially uncorrelated. This does not imply an uncorrelated time dynamics in the process, but only that random resampling of the time series is a white noise. Our results are well-matched by numerical computations, analysis of financial and climate data and yield a theoretical upper bound on attractor dimension for systems with Gaussian anomalies and arbitrary correlation structure. We also study numerically systems with non-Gaussian anomaly distributions. We show that the flatness of the density function around the mean determines the behaviour of D , with an absolute upper limit given by the Uniform distribution. While larger than in the gaussian case, in the *uniform* case, D is still smaller than n .

We now derive the theoretical stochastic limit. Let $\zeta_i, Z_{j,i}$ be fields with $j = 1, \dots, M$, i.e. $\zeta = \{\zeta_i\}_{i=1}^n$ is a n -dimensional reference sample field and Z_1, Z_2, \dots, Z_M are n -dimensional samples from M realizations of the field. We assume that, at any time, a perturbation of the reference field can be expressed as $Z_{j,i} = \zeta_i + \varepsilon_{j,i}$, so that $\varepsilon_{j,i} = (Z_{j,i} - \zeta_i) \sim N(\mu, \sigma^2)$ and independent, and $\sigma^{-1}(Z_{j,i} - \zeta_i - \mu) \sim N(0, 1)$. With no loss of generality, we assume that $\varepsilon_{j,i} \sim N(0, 1)$, since the same results will hold for any system with Gaussian anomalies after standardization, even if the dynamics of each single particle is not i.i.d. We define $\delta^2(n) = \{\delta_j^2(n)\}_{j=1}^M$ as the M -dimensional vector of the squared Euclidean distances between each field and the reference field

$$\delta_j^2(n) = \sum_{i=1}^n (Z_{j,i} - \zeta_i)^2. \quad (1)$$

It is straightforward to notice that $\delta_j^2(n) = \sum_{i=1}^n (N(0, 1))^2 \sim \chi^2(n)$. We now consider the functional of the Euclidean distance $g(n) = -\ln \sqrt{\delta^2(n)}$. Then, the probability density function of $g(n)$ reads

$$f_g(g) = f_{\delta^2}(h^{-1}(\delta^2)) \left| \frac{d\delta^2}{dg} \right| = \frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})} \exp \left\{ -ng - \frac{1}{2}e^{-2g} \right\}, \quad (2)$$

where $h(\delta^2(n)) = -\frac{1}{2} \ln \delta^2(n)$ and $\left| \frac{d\delta^2}{dg} \right| = |-2e^{-2g}|$. Let q be an extreme quantile for g ; we consider the extremes in the $(1-q)$ right tail of the distribution, denoted $Y = \{Y_k\}_{k=1}^m$, shifted so that $\liminf\{Y\} = 0$, with expected sample size $m = (1-q)M$. From Extreme Value Theory [27], we know that $Y \sim \text{Exp}(\lambda)$,

so that $f_Y(y) = \lambda^{-1}e^{-y/\lambda}$. It is known from [28] that the mean and standard deviation of this distribution are $\mu_Y = \sigma_Y = \lambda$ and that the dimension is given by the precision of the distribution $D = \lambda^{-1}$. Notice that both μ_Y and σ_Y are functions of n , as will be detailed later in this section. Under such model specification, the unbiased maximum likelihood estimator for λ is the sample mean, $\hat{\lambda} = \frac{1}{m} \sum_{k=1}^m y_k = \bar{y}$, so that the dimension can be obtained as $\hat{D} = \bar{y}^{-1}$. The next step consists of finding a link between the expected value of the extremes λ , estimated by their sample mean, and the number of degrees of freedom n of the initial Gaussian random fields. This result can be obtained by writing the parameter of the extreme value distribution, λ , as a function of known moments of g . In order to do so, we observe that the median of the exponential distribution is given by $\tilde{\mu}_Y = \sqrt{2}\lambda$, and then $D = \frac{\sqrt{2}}{\tilde{\mu}_Y}$: this implies that the expectation of the extremes is proportional to a fixed quantile, corresponding to the quantile 0.5 of the shifted extreme distribution and to $q + 0.5(1-q)$ of the distribution of g . When applying peaks over threshold (POT) to the sample, the threshold quantile q corresponds to a certain sample order statistics $g_{r:M}$ and is estimated by $q' = r/M$. Therefore, it is possible to choose $r \in \mathbb{N}$ to approximate the quantile corresponding to λ , i.e. such that $g_{r:M} \simeq \lambda$. The position of such a quantile on the support of g changes as a function of μ_g and σ_g in a way dictated by the quantile function of g . Since the latter cannot be obtained in close form for the probability density function 2, we resort to more general considerations, based on tight bounds of order statistics. In particular, we refer to [29], which provides tight bounds for the expected value of the r -th order statistics $x_{r:M}$ from a given distribution with mean μ_X and standard deviation σ_X . We recall that we consider shifted extremes, so that their minimum is 0 and the effect due to μ_g is removed. Under such conditions, the bound for the expected r -th order statistics reads

$$0 < E(g_{r:M}) \leq \sigma_g \sqrt{\frac{r-1}{M-r+1}}. \quad (3)$$

In the same paper, it is shown that the upper bound on the expectation of the sample maximum is $E(g_{M:M}) \leq \sigma_g \sqrt{M-1}$, which shows that a linear relation between the sample quantile and σ holds also in the limit $q \rightarrow 1$.

We now focus on σ_g , which is the relevant factor in the equation above, given that it contains all the dependence from n . The moment generating function of g is

$$M_g(t) = E[e^{tg}] = \frac{2^{1-n/2}}{\Gamma(\frac{n}{2})} \int_{-\infty}^{+\infty} \exp \left\{ tg - ng - \frac{1}{2}e^{-2g} \right\} dg = \frac{2^{-t/2} \Gamma(\frac{n-t}{2})}{\Gamma(\frac{n}{2})}. \quad (4)$$

and then the first two moments can be obtained as

$$\begin{aligned} E(g) &= M'_g(t=0) = -\frac{1}{2} \left[\ln 2 + \psi\left(\frac{n}{2}\right) \right] \\ E(g^2) &= M''_g(t=0) = \frac{1}{4}(\ln 2)^2 + \frac{1}{2}(\ln 2)\psi\left(\frac{n}{2}\right) + \frac{1}{4}\psi\left(\frac{n}{2}\right)^2 + \frac{1}{4}\psi^1\left(\frac{n}{2}\right) \end{aligned} \quad (5)$$

where $\psi(z) = \Gamma'(z)/\Gamma(z) = \frac{d}{dz} \ln \Gamma(z)$ is the digamma function and $\psi^r(z) = \frac{d^r}{dz^r} \psi(z) = \frac{d^{r+1}}{dz^{r+1}} \ln \Gamma(z)$ is the polygamma function of order r . It is now straightforward to obtain the standard deviation as

$$\sigma_g = \sqrt{E(g^2) - E(g)^2} = \frac{1}{2} \sqrt{\psi^1\left(\frac{n}{2}\right)}. \quad (6)$$

Combining Equations 3 and 6, we obtain an expression for the upper bound of $E(\lambda)$ as a function of n . Since $D = \lambda^{-1}$, the upper bound for $E(D)$ is $+\infty$. However, it is known [30] that the increase of the attractor dimension with the number of degrees of freedom is bounded by n itself, so that we can write

$$\left(\sigma_g(n) \sqrt{\frac{r-1}{M-r+1}} \right)^{-1} \leq E(D(n)) \leq n. \quad (7)$$

Numerical experiments suggest that $D(n) = n$ is never observed, and that the actual attractor dimension is always a concave function of n . Eq. 7 provides tight bounds for a general order statistics of a system of Gaussian particles with noninteracting dynamics. However we argue that, by considering extreme quantiles in the computation of D , the two bounds should have the same shape in other dynamical systems, constraining the expected value to be a linear function of σ_g . Then, the dimension $D = \lambda^{-1}$, will be a function of σ_g^{-1} :

$$E(D) \simeq \alpha(M, r, q) + \beta(M, r, q) \left[\psi^1\left(\frac{n}{2}\right) \right]^{-\frac{1}{2}}. \quad (8)$$

Below, we tackle this hypothesis via numerical simulation. Notice that, assuming that Eq. 8 holds, $E(D)$ is known up to $\alpha(M, r, q)$ and $\beta(M, r, q)$, that are constant with respect to n , but are possibly functions of the chosen extreme quantile. In particular, from the expression given above for the tight bounds of order statistics, we may expect that $\beta(q)$ is function of $q/(1-q)$, that is the limiting version of $(r-1)/(M-r+1)$.

We test the robustness of the intuition summarized in Eq. 8 by performing a Monte Carlo (MC) experiment. The outcomes are reported in Fig. 1. We consider Gaussian random fields $\sim N(0, 1)$ with degrees of freedom $n \in [2, 500]$ and an interval of extreme quantiles $q \in [0.980, 0.999]$ partitioned with a grid step $\delta q = 0.0001$. We fix the number of MC replicates for the extremes to $m = 1000$, so that the number of generated fields varies

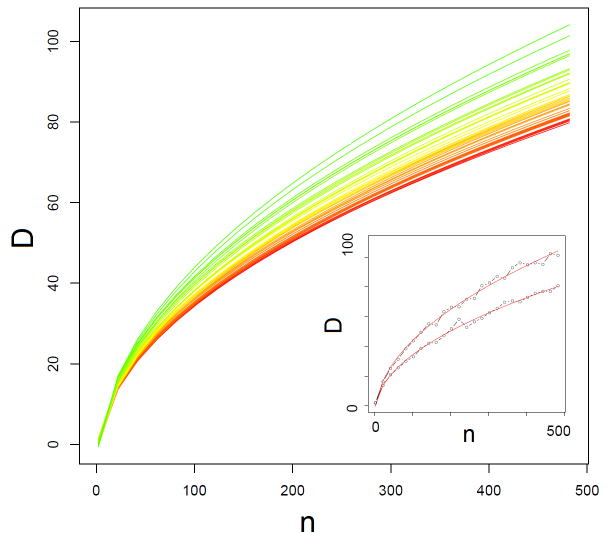


FIG. 1: Predicted values of the dimension D from the linear regression resulting from Eq. 8, ranging from $q=0.98$ (lower red line) to $q = 0.999$ (upper green line). Inset: comparison between observed and predicted dimension using quantiles $q=0.98$ (lower) and $q=0.999$ (upper) as thresholds. Circles: estimated D . Red line: linear regression from Eq. 8.

with the quantile and is given by $M = m(1-q)^{-1}$. For all the quantiles considered, we estimate the regression in Eq. 8, which provides an accurate prediction of the dimension, with linear determination coefficients all larger than 0.989. The p-values associated to the β coefficient in Eq. 8 are all smaller than 10^{-23} for every n .

First of all, let us underline that $\alpha(q)$ lacks a physical meaning, since it is the predicted dimension for a system with $n = 0$. On the other hand, we know that $D(1) = 1$ and $D(2) \simeq 2$, and we can assume these, in particular the former, as constraints derived from the observed behavior in case of a very small number of degrees of freedom, so that our results hold for $n \geq 3$.

We investigate the functional form of both $\alpha(q)$ and $\beta(q)$ in the considered range of quantiles. From Figure 1, we observe that a realistic model for the intercept is $\alpha(q) = \alpha_0 + \alpha_1 q + \alpha_2 q^2$, while for the slope $\beta(q) = \beta_0 + \beta_1 \ln\left(\frac{q}{1-q}\right)$. The estimated coefficients are $\alpha_0 = -7649$, $\alpha_1 = 15561$, $\alpha_2 = -7918$, $\beta_0 = 1.59$, $\beta_1 = 0.29$, while the goodness of the two fits is $R^2 = 0.46$ and $R^2 = 0.963$ for $\alpha(q)$ and $\beta(q)$, respectively. Fits are shown in Fig. 2.

These results complete Eq. 8, so that we know the value of $E(D)$, for such a stochastic system, up to the val-

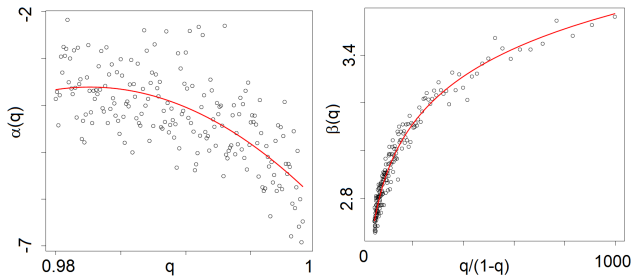


FIG. 2: Observed and predicted values of $\alpha(q) = \alpha_0 + \alpha_1 q + \alpha_2 q^2$ (left panel) and $\beta(q) = \beta_0 + \beta_1 \ln\left(\frac{q}{1-q}\right)$ (right panel), with q ranging from 0.98 to 0.999.

ues of the parameter vector $(\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1)$. Clearly the effective number of degrees of freedom in a stochastic lattice is noticeably smaller than the dimension of the phase-space itself. We now apply the above results to datasets issued from finance and climate. Any spatial or cross-sectional correlation should reduce D and, as for the stochastic case, we should get $D < n$. The details of the 22 financial time series analysed here are provided in table 1; we compute their log-return to remove non-stationarity [31]. As climate variable, we use 22 sea-level pressure (SLP) time series chosen from the roughly 10000 grid-points constituting the NCEP/NCAR 1948-2015 re-analysis dataset [32]. Both the SLP and the log-return of financial datasets have a Gaussian marginal distribution.

For each dataset we extract randomly 30 sequences of 22 time series. The extraction determines the order in which we add a new time series to construct different databases for $1 < n < 22$. We fix the quantile to be $q = 0.98$. When $n = 1$, we obtain $D = 1$ for all the choices. As soon as $n > 5$, the correlations in the data result in an effective dimension D well below both the phase space dimension and the dimension obtained in the stochastic case. For financial time series we have $D \simeq 10$ for $n = 22$, a slightly higher dimension than the SLP. This implies that the information contained in all the finance datasets can be extracted by using only about 10 well-chosen time series. A similar conclusion can be drawn about the climate data.

We now discuss the case of systems whose anomalies do not have a Gaussian distribution, reminding that in order for the reference field to be a point on the attractor, the anomalies should have a global maximum at zero [25]. There are therefore two extreme cases: i) particle systems whose anomalies are uniformly distributed in a fixed interval so that the global maximum is at all the points of the interval ii) particle systems with a sharp maximum at zero. The results for different probability density functions are displayed in Fig. 4. When the maximum in zero is sharper than the gaussian distribution (anomalies distributed with t-Student distribution with $\nu = 2$ or $\nu = 4$ degrees of freedom), the limiting curve is lower than the theoretical curve for gaussian anoma-

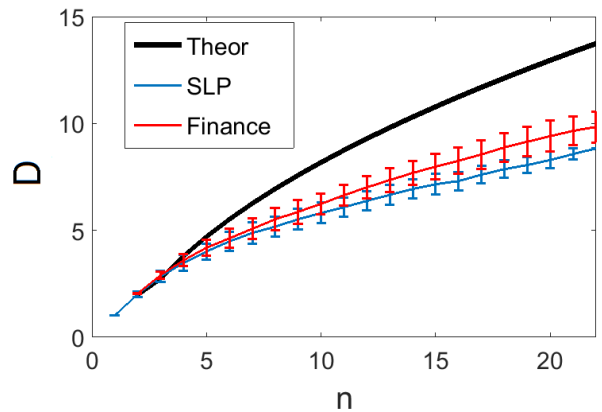


FIG. 3: Theoretical stochastic limit for $E(D)$ from Eq. 8 (black) vs observed D for 30 random combinations of daily sea level pressure (SLP) and financial data. The error bars represent one standard deviation of the mean.

lies. Conversely, when the maximum is flatter (anomalies with uniform distribution or Beta distribution with $\alpha' = \beta' = 2$ re-centered in zero), the limiting curve is higher than the reference Gaussian curve. In other words, the most efficient way to fill up the phase space is having a uniform distribution mass concentration and the least optimal way is to have the mass concentrated on the points of the attractor. In both cases, the curves maintain the same functional form as for the Gaussian limit. In Figure 4 we also analyse the role of spatial correlation in decreasing the attractor dimension. We consider the Bernoulli Shift coupled map:

$$x_{n+1}^{(i)} = (1 - \epsilon)f(x_n^{(i)}) + \frac{\epsilon}{n} \sum_j^n f(x_n^{(i \neq j)}) \quad (9)$$

where $f(x) = 3x \bmod 1$ and ϵ is the strength of the coupling. As shown in the figure, by increasing ϵ the dimension decreases as spatial degrees of freedom are progressively frozen. Moreover, for all couplings, the upper limiting curve is higher than the Gaussian theoretical curve but lower than that of the uniform random anomalies. The latter is then to be interpreted as a general upper bound for the dimension of a dynamical system, as no dynamics can be more efficient in covering the phase space than the one that covers it uniformly. We also show that, when the dynamics of particles consist of an auto-regressive process (AR(1)) $x_{n+1}^{(i)} = \phi x_n^{(i)} + \eta_n^{(i)}$, the values of D follow the theoretical curve ($\phi = 0.8$, $\eta \sim N(0, 1)$ i.i.d.).

Our results open the possibility to address a number of important challenges including: i) estimating the degrees of freedom needed to describe a purely stochastic process as a function of the phase space dimension and ii) introducing an algorithmic procedure to measure this quantity in any time-series data-set. In many fields, such as

TABLE I: International codes of the financial time series (daily data for the period 01-Jan-1998 to 03-May-2017).

IEUCT01	MOVEIDX	USEURSP	USDOLLR	JAPAYE\$	S&PCOMP	DJEURST	FTSEMIB
FRCAC40	DAXINDX	JAPDOWA	ER00(ML:OAS)	HE00(ML:OAS)	C0A0(ML:OAS)	H0A0(ML:OAS)	CBOEVIX
BMUS02Y(RY)	BMBD02Y(RY)	BMIT02Y(RY)	BMFR02Y(RY)	BMUK02Y(RY)	BMJP02Y(RY)		

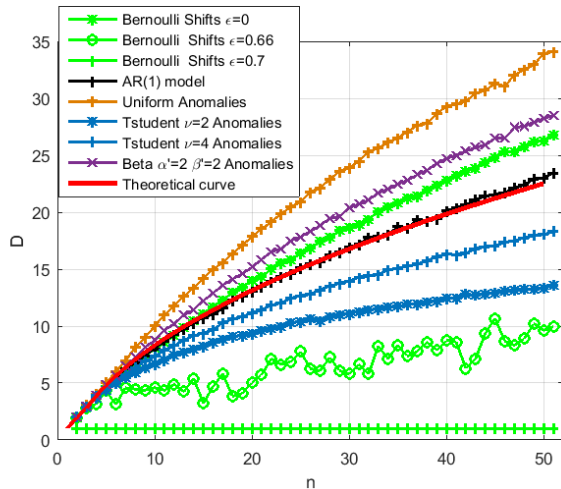


FIG. 4: Dimension D for systems with different underlying anomaly distributions and for the Bernoulli shift Eq.9 with $\epsilon = 0$ (uncoupled) $\epsilon = 0.66$ (transition to global coupled behavior) $\epsilon = 0.7$ (completely coupled). For each case, the displayed point is the average among 300 realizations and for series 10^6 length.

climate science, computational limitations on data processing and storage pose a major limit to researchers [33]. The finding that, even for systems with a uniform underlying anomaly distribution, the number of effective degrees of freedom is smaller than the phase space dimension is therefore crucial. This, coupled with the ability to inexpensively identify the effective number of degrees of freedom of a system, effectively allows to identify redundancies in the data. Our results thus have major implications and a wide-ranging significance.

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