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# Attractor dimensions for lattices of arbitrary size

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By using the link between extreme value theory and Poincaré recurrences, we compute the attractor dimension - namely the effective number of degrees of freedom - when the dynamics consist of  $n$  non-interacting particles with a stochastic behavior. We derive the theoretical expression in terms of finite size limits and test its validity and applicability with numerical experiments. We find that the effective number of degrees of freedom in a stochastic lattice is noticeably lower than the lattice size itself. We then estimate the attractor dimension of a collection of time-series issued from finance and climate, and find that they are below the theoretical stochastic limit.

The usual approach to analyse high dimensional complex systems such as turbulent flows, geophysical flows or complex networks is via statistical mechanics or statistical approaches. Mean Field Theories [1, 2], Principal Orthogonal Decomposition [3] or Empirical Orthogonal Functions[4] are commonly used to recover the stationary states of the system. This information is essential to understand what are the most probable configurations of the system and, by anthesis, to discriminate the anomalous configurations, i.e. the extreme events [5]. Unfortunately, all these techniques do not capture the dynamics of the system, that is the transitions between the different metastable states. Such dynamics can generally be represented by a non-unique set of differential equations that approximate the behavior of the system. Among the possible dynamical representations, one generally looks for the simplest and most parsimonious model [6], able to represent the switching between metastable states in terms of a small number of degrees of freedom. The most famous example is the Lorenz attractor [7], a system of three variables that can represent some features of Rayleigh-Benard convection. Other examples include the Pomeau-Manneville [8] maps, describing turbulence intermittency, and the Ising model, which is a prototype of phase transitions [9]. Recently, such low dimensional attractors have been found also for laboratory flows in highly turbulent regimes [10]. In the context of geophysical flows, effective reductions of atmospheric

dynamics have been also proposed [11, 12]

The limits of validity of these models have been repeatedly questioned [13–15] because of their simplicity: why should a model with few degrees of freedom be able to represent the dynamics of a complex system with a large (sometimes infinite) number of variables? In order to answer this question, one has to compute the minimum number of degrees of freedom needed to describe the dynamics. In dynamical systems, this is equivalent to finding the dimension of the underlying attractors [16]. The attractor is a compact object which hosts all the trajectories of a system. Its dimension is generally smaller than the phase space dimension  $n$ . In the 80s, it seemed possible to derive estimates of the attractor dimension  $D$  of complex systems (such as the atmosphere) from short time series of a variable measured at a point in space[17, 18]. However, it was soon realized that this approach - termed embedding method - was not feasible, because it required a time series whose length is exponential in the number of degrees of freedom of the system [19, 20]. Due to this limitation, data-driven approaches for the computation of attractor dimension progressed for several years, mostly pointing out the numerical limitations of the estimates [21, 22].

In the last decade there have been numerous theoretical advances in our understanding of the limiting distribution of Poincaré recurrences. Those are condensed in the book [23]. The main finding is that, under suitable

rescaling, recurrences of a state  $\zeta_i$  at a time  $i = 1, 2, \dots, M$  follow a Gumbel distribution whose scale parameter is the inverse of the local dimension  $d(\zeta_i)$ . The power of this result is due to the universality of the Gumbel law. In other words [24] have discovered a sort of central limit theorem of Poincaré recurrences. The attractor dimension  $D$  can be obtained by averaging  $d(\zeta_i)$  for several  $\zeta_i$  belonging to the attractor. The motivations for the theoretical studies presented in this work come from [25], where this technique has been applied to describe the atmospheric circulation over the North Atlantic, using daily sea level pressure fields from 1948 to 2016. The authors found that the active numbers of degrees of freedom was two orders of magnitude smaller than the number of grid points of the two dimensional spatial fields. This value was robust with respect to coarse graining of the phase space and the use of different datasets. Similar results were obtained when using geopotential height in a subsequent study [26]. One can ask whether such low dimensional estimates of  $D$  are related to a slow logarithmic convergence of the method towards the true value of  $D$ , or whether they are instead related to a more fundamental relationship between the dimensions of the attractor and of the phase space. In order to investigate this problem we compute here, theoretically and numerically, the attractor dimension as a function of the phase space dimension  $n$  when the dynamics is completely stochastic. We show that the theoretical curve is well-matched by numerical computations, and that in stochastic systems  $D$  is smaller than  $n$ . We then compare the theoretical stochastic curve with those obtained numerically for climate and finance datasets and find that additional correlation between the variables squeezes  $D$  to values

lower than those predicted for the stochastic behavior. A theoretical stochastic limit noticeably smaller than the system's phase space dimension would have potentially major implications for the optimization of computations and data storage when analyzing large datasets, as it would effectively provide a measure of the redundancy of the data.

Let  $\zeta_i, Z_{j,i}$  be random fields with  $j = 1, \dots, M$ , i.e.  $\zeta = \{\zeta_i\}_{i=1}^n$  is a  $n$ -dimensional reference sample field and  $Z_1, Z_2, \dots, Z_M$  are  $n$ -dimensional samples drawn from  $M$  realizations of the field. We assume that the field at any time is a white noise perturbation of the reference field, i.e.  $Z_{j,i} = \zeta_i + \varepsilon_{j,i}$ , so that  $\varepsilon_{j,i} = (Z_{j,i} - \zeta_i) \sim N(\mu, \sigma^2)$  and independent, and  $\sigma^{-1}(Z_{j,i} - \zeta_i - \mu) \sim N(0, 1)$ . With no loss of generality, we assume that  $\varepsilon_{j,i} \sim N(0, 1)$ , since the same results will hold for any system Gaussian i.i.d. anomalies, after standardization.

We define  $\delta^2(n) = \{\delta_j^2(n)\}_{j=1}^M$  as the  $M$ -dimensional vector of the squared Euclidean distances between each field and the reference field

$$\delta_j^2(n) = \sum_{i=1}^n (Z_{j,i} - \zeta_i)^2. \quad (1)$$

It is straightforward to notice that  $\delta_j^2(n) = \sum_{i=1}^n (N(0, 1))^2 \sim \chi^2(n)$ . We now consider the functional of the Euclidean distance  $g(n) = -\ln \sqrt{\delta^2(n)}$ . Notice that  $g(n) \sim -\ln \sqrt{\chi^2(n)} = -\frac{1}{2} \ln \chi^2(n)$ . We now proceed to derive the probability distribution of  $g(n)$ . First, consider  $X = h(\delta^2(n)) = \ln \delta^2(n) \sim \ln \chi^2(n)$ , so that

$$f_X(x) = f_{\delta^2}(h^{-1}(x)) \left| \frac{d\delta^2}{dX} \right| = \frac{1}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} \exp \left\{ \frac{n}{2}x - \frac{1}{2}e^x \right\} \sim \log\text{-}\chi^2(n). \quad (2)$$

Now,  $g = -\frac{1}{2}X$  and  $\left| \frac{dX}{dg} \right| = |-2|$ , leading to the expression of the density

$$f_g(g) = 2f_X(x = -2g) = \frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})} \exp \left\{ -ng - \frac{1}{2}e^{-2g} \right\}. \quad (3)$$

Let  $q$  be an extreme quantile for  $g$ ; we consider the extremes in the  $(1 - q)$  right tail of the distribution, denoted  $Y = \{Y_k\}_{k=1}^m$ , shifted so that  $\liminf\{Y\} = 0$ , with expected sample size  $m = (1 - q)M$ . From the Extreme Value Theory[27], we know that  $Y \sim \text{Exp}(\lambda)$ , so that  $f_Y(y) = \lambda^{-1}e^{-y/\lambda}$ . It is known from [28] that the mean and standard deviation of this distribution

are  $\mu_Y = \sigma_Y = \lambda$  and that the dimension is given by the precision of the distribution  $D = \lambda^{-1}$ . Notice that both  $\mu_Y$  and  $\sigma_Y$  are function of  $n$ , as will be detailed later in this section. Under such model specification, the unbiased maximum likelihood estimator for  $\lambda$  the sample mean,  $\hat{\lambda} = \frac{1}{m} \sum_{k=1}^m y_k = \bar{y}$ , so that the dimension can be obtained as  $\hat{D} = \bar{y}^{-1}$ .

The next required step consists of finding a link between the expected value of the extremes  $\lambda$ , estimated by their sample mean, and the number of degrees of freedom  $n$  of the initial Gaussian random fields. This result can be obtained writing  $\lambda$  as a function of  $\mu_g$  and  $\sigma_g$ .

In order to do so, we observe that the median of the exponential distribution is given by  $\tilde{\mu}_Y = \sqrt{2}\lambda$ , and then  $D = \frac{\sqrt{2}}{\tilde{\mu}_Y}$ : this implies that the expectation of the extremes is proportional to a fixed quantile, corresponding to the quantile 0.5 of the shifted extreme distribution and to  $q + 0.5(1 - q)$  of the distribution of  $g$ . We notice that, when applying POTs to the sample, the threshold quantile  $q$  corresponds to a certain sample order statistics  $g_{r:M}$  and is estimated by  $q' = r/M$ . Therefore, it is possible to choose  $r \in \mathbb{N}$  to approximate the quantile corresponding to  $\lambda$ , i.e. such that  $g_{r:M} \simeq \lambda$ . Clearly, the position of such a quantile on the support of  $g$  changes as a function of  $\mu_g$  and  $\sigma_g$ ; in order to understand how, we refer to [29], which provides tight bounds for the expected value of the order statistics  $x_{r:M}$  from a given distribution with mean  $\mu$  and standard deviation  $\sigma$ . Such bounds read  $\mu_g - \sigma_g \sqrt{\frac{M-r}{r}} \leq g_{r:M} \leq \mu_g + \sigma_g \sqrt{\frac{r-1}{M-r-1}}$ . However, the effect due to  $\mu_g$  is removed by the shifting of the extreme sample, so that our tight bound actually reads

$$-\sigma_g \sqrt{\frac{M-r}{r}} \leq g_{r:M} \leq \sigma_g \sqrt{\frac{r-1}{M-r-1}}.$$

This result extends the one obtained by [30] for the upper bound on the expectation of the sample maximum:

$$g_{M:M} \leq \sigma_g \sqrt{M-1},$$

which shows that a linear relation between the sample quantile and  $\sigma$  is maintained also in the limit  $q \rightarrow 1$ . While this bound is tight in a statistical sense, the inferior bound (which would be an upper bound for the dimension) is meaningless in the physical sense, since it can assume negative values, being formulated for a general quantile, rather than for an extreme one. However, the matter of interest in our case in that these bounds imply a linear relation between the expectation of the extremes and the standard deviation of  $g$  through functions of  $M, r$  and then of  $q$ , but constant with respect to  $n$ :

$$\lambda = \alpha'(q) + \beta'(q)\sigma_g. \quad (4)$$

We now focus on  $\sigma_g$ , which is the relevant factor in the equation above, given that it contains all the dependence from  $n$ . The moment generating function of  $g$  is

$$M_g(t) = E[e^{tg}] = \frac{2^{1-n/2}}{\Gamma(\frac{n}{2})} \int_{-\infty}^{+\infty} \exp\left\{tg - ng - \frac{1}{2}e^{-2g}\right\} dg = \frac{2^{-t/2}\Gamma(\frac{n-t}{2})}{\Gamma(\frac{n}{2})}. \quad (5)$$

and then the first two moments can be obtained

$$\begin{aligned} E(g) &= M'_g(t=0) = -\frac{1}{2} \left[ \ln 2 + \psi\left(\frac{n}{2}\right) \right] \\ E(g^2) &= M''_g(t=0) = \frac{1}{4}(\ln 2)^2 + \frac{1}{2}(\ln 2)\psi\left(\frac{n}{2}\right) + \frac{1}{4}\psi\left(\frac{n}{2}\right)^2 + \frac{1}{4}\psi^1\left(\frac{n}{2}\right) \end{aligned} \quad (6)$$

where  $\psi(z) = \Gamma'(z)/\Gamma(z) = \frac{d}{dz} \ln \Gamma(z)$  is the digamma function and  $\psi^r(z) = \frac{d^r}{dz^r} \psi(z) = \frac{d^{r+1}}{dz^{r+1}} \ln \Gamma(z)$  is the polygamma function of order  $r$ . It is now straightforward to obtain the standard deviation as

$$\sigma_g = \sqrt{E(g^2) - E(g)^2} = \frac{1}{2} \sqrt{\psi^1\left(\frac{n}{2}\right)}. \quad (7)$$

Finally, in analogy with equation 4, we argue that the following linear relation holds for the dimension  $D = \lambda^{-1}$ ,

$$D = \alpha(q) + \beta(q) \left[ \psi^1\left(\frac{n}{2}\right) \right]^{-\frac{1}{2}}. \quad (8)$$

Thus, the dimension of the considered random system is known up to  $\alpha(q)$  and  $\beta(q)$ , that are constant respect to  $n$ , but are possibly functions of the chosen extreme quantile. In particular, from the expression given above for the tight bounds of order statistics, we may expect that  $\beta(q)$  is function of  $q/(1-q)$ , that is the continuous

version of  $(r-1)/(M-r-1)$ . At our present knowledge, these relations must be investigated empirically through numerical simulation.

We test the robustness of the result given by Eq. 8 by performing a simple Monte Carlo (MC) experiment. The outcomes are reported in Fig. 1. We consider Gaussian random fields  $\sim N(0,1)$  with degrees of freedom  $n \in [2, 500]$  and an interval of extreme quantiles  $q \in [0.980, 0.999]$  partitioned with a grid step  $\delta q = 0.0001$ . Since the distribution of the squared Euclidean distance between two i.i.d. Gaussian random fields is known exactly, we sample directly from the corresponding  $\chi^2(n)$ , rather than generating the random field.

We fix the number of MC replicates for the extremes to  $m = 1000$ , so that the number of generated fields varies with the quantile and is given by  $M = m(1-q)^{-1}$ . For all the quantiles considered, we estimate the regression in Eq. 8, which provides an accurate prediction of the

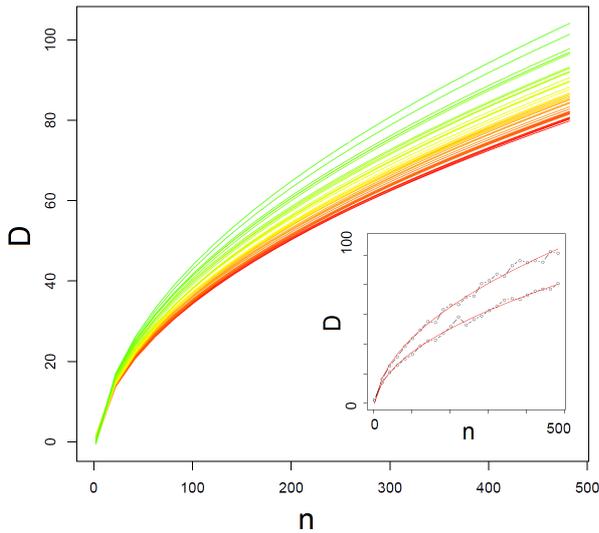


FIG. 1: Predicted values of the dimension  $D$  from the linear regression resulting from Eq. 8, ranging from  $q=0.98$  (lower red line) to  $q = 0.999$  (upper green line). Inset: comparison between observed and predicted dimension using quantiles  $q=0.98$  (lower) and  $q=0.999$  (upper) as thresholds. Circles: estimated  $D$ . Red line: linear regression resulting from Eq. 8.

dimension, with linear determination coefficients all larger than 0.989.

As previously mentioned, we have no prior knowledge about  $\alpha(q)$  and  $\beta(q)$ , but they can be estimated in the regression procedure. From the inspection of the plots as a function of the quantile  $q$ , we can make a functional hypothesis based on empirical observation. First of all, let us underline that  $\alpha(q)$  lacks a physical meaning, since it is the predicted dimension for a system with  $n = 0$ . On the other hand, we know that  $D(1) = 1$  and  $D(2) \simeq 2$ , and we can assume these, in particular the former, as constraints derived from the observed behavior in case of a very small number of degrees of freedom, so that our results hold for  $n \geq 3$ .

We investigate the functional form of both  $\alpha(q)$  and  $\beta(q)$  in the considered range of quantiles. From the plots, we observe that a realistic model for the intercept is  $\alpha(q) = \alpha_0 + \alpha_1 q + \alpha_2 q^2$ , while for the slope  $\beta(q) = \beta_0 + \beta_1 \ln\left(\frac{q}{1-q}\right)$ . The estimated coefficients are  $\alpha_0 = -7649$ ,  $\alpha_1 = 15561$ ,  $\alpha_2 = -7918$ ,  $\beta_0 = 1.59$ ,  $\beta_1 = 0.29$ , while the goodness of the two fits is  $R^2 = 0.46$  and  $R^2 = 0.963$  for  $\alpha(q)$  and  $\beta(q)$ , respectively. Fitted values are shown in Fig. 2.

These results complete Eq. 8, so that we know the value of  $d$ , for such a stochastic system, up to the values of the parameter vector  $(\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1)$ . As discussed above, the effective number of degrees of

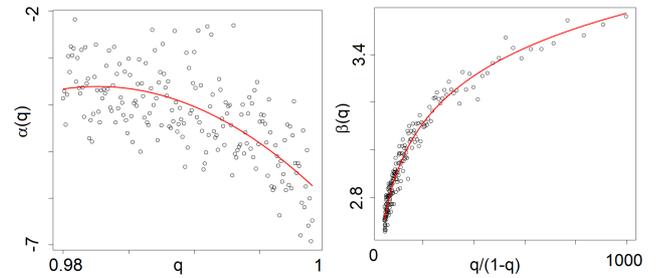


FIG. 2: Observed and predicted values of  $\alpha(q) = \alpha_0 + \alpha_1 q + \alpha_2 q^2$  (left panel) and  $\beta(q) = \beta_0 + \beta_1 \ln\left(\frac{q}{1-q}\right)$  (right panel), with  $q$  ranging from 0.98 to 0.999.

freedom in a stochastic lattice is noticeably smaller than the dimension of the phase-space itself. Monte Carlo estimation is the only way to retrieve the value of the above parameters, so that we must interpret them as universal constants, as it happens, for example, for the Kolmogorov constant  $C_0$  in the theory of turbulence.

We now apply the above results to datasets issued from finance and climate and estimate their effective dimensions. Any mutual correlations between the time series or autocorrelations should reduce the effective dimension and, as for the stochastic case, we should get attractor dimensions smaller than the phase space dimension. The details of the 22 financial time series analysed here are provided in table 1; we compute their log-return to remove non-stationarity (see, for example, [31]). As climate variable, we use 22 sea-level pressure (SLP) time series chosen from the roughly 10000 grid-points constituting the NCEP/NCAR 1948-2015 reanalysis dataset [32]. Both the SLP and the log-return of financial datasets have a Gaussian marginal distribution.

For each dataset we then extract randomly 30 sequences of 22 time series. The sequences determine the order in which we add a new time series to construct different databases for  $1 < n < 22$ . For the computation we fix a quantile  $q = 0.98$ . When  $n = 1$ , we obtain  $D = 1$  for practically all the choices. As soon as  $n > 5$ , the correlations in the data result in an effective dimension  $D$  well below both the phase space dimension and the dimension obtained in the stochastic case. Financial time series have a slightly higher dimension than the SLP. If we interpret the value of  $D$  as the number of effective degrees of freedom, for financial data with  $n = 22$ , we have  $D \simeq 10$ , i.e. the information contained in all the finance datasets can be extracted by using only about 10 well-chosen time series. A similar conclusion can be drawn about the climate data. This value could also be used to determine the number of principal components to be used when analysing a dataset or to quantify the degree of connectivity of a network.

The lack of a feasible methodology to compute the dimension of the attractor of complex systems has limited

TABLE I: International codes of the financial time series (daily data for the period 01-Jan-1998 to 03-May-2017).

IEUCT01	MOVEIDX	USEURSP	USDOLLR	JAPAYE\$	S&PCOMP	DJEURST	FTSEMIB
FRCAC40	DAXINDX	JAPDOWA	ER00(ML:OAS)	HE00(ML:OAS)	C0A0(ML:OAS)	H0A0(ML:OAS)	CBOEVIX
BMUS02Y(RY)	BMBD02Y(RY)	BMIT02Y(RY)	BMFR02Y(RY)	BMUK02Y(RY)	BMJP02Y(RY)		

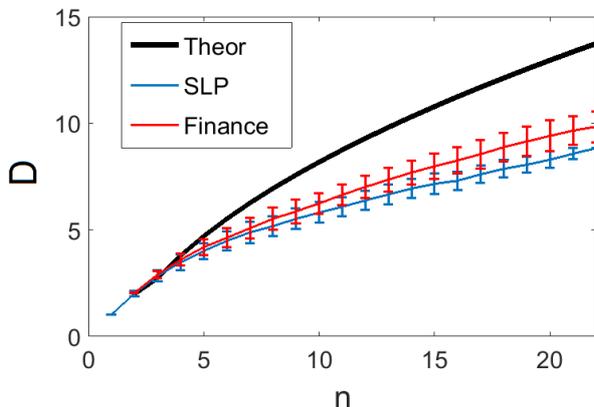


FIG. 3: Predicted values of the dimension  $D$  from the linear regression resulting from Eq. 8 (black) vs the dimension observed for 30 random combinations of daily sea level pressure (SLP) and financial data. The error bars represent one standard deviation of the mean.

the applicability of dynamical systems techniques to real-world problems. Recent advances [23] have overcome this limitation, allowing to inexpensively compute the attractor dimension for large datasets. This opens the possibility to address a number of important challenges including: i) estimating the degrees of freedom needed to describe a purely stochastic process as a function of the phase space dimension and ii) introducing an algorithmic procedure to measure this quantity in any time-series data-set. As a caveat, we note that the theoretical limit we compute should not be seen as an upper limit for the dimension of any system within a given phase space, as specific dynamical systems could have a higher dimension than the purely stochastic case considered here.

In many fields, such as climate science, computational limitations on data processing and storage pose a major limit to researchers [33]. The ability to inexpensively identify the effective number of degrees of freedom of a system, and hence any redundancies in the chosen data, therefore has major implications. Our results thus have a wide-ranging significance and can be fruitfully applied to a variety of research fields.

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