HAL
open science

The Ext algebra of a quantized cycle

Damien Calaque, Julien Grivaux

▶ **To cite this version:**

Damien Calaque, Julien Grivaux. The Ext algebra of a quantized cycle. Journal de l'École polytechnique - Mathématiques, 2019, 6, pp.31–77. 10.5802/jep.87 . hal-01649952

HAL Id: hal-01649952

https://hal.science/hal-01649952v1

Submitted on 18 Nov 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

THE EXT ALGEBRA OF A QUANTIZED CYCLE

DAMIEN CALAQUE AND JULIEN GRIVAUX

CONTENTS

1. Introduction 2
1.1. The diagonal embedding case 2
1.2. More general embeddings: tame quantized cycles 3
1.3. The results 4
1.4. Plan of the paper 4
2. Universal enveloping algebras and the categorical PBW theorem 5
2.1. Preliminary results of linear algebra 5
2.1.1. Partially antisymmetric tensors 5
2.1.2. Jacobi identity 6
2.2. Algebras satisfying the PBW isomorphism 6
2.2.1. The morphisms c_p^k 6
2.2.2. The Lie bracket 7
2.2.3. Induction formulæ for c_p^k 7
2.3. Universal algebras in the categorical setting 7
2.3.1. Reverse PBW theorem 7
2.3.2. Categorical construction of the product on $S(V)$ 8
2.3.3. Categorical PBW theorem 11
3. Distinguished elements in the universal enveloping algebra 12
3.1. The derivative of the multiplication map 12
3.1.1. Linear algebra computations 16
3.1.2. The trace identity 18
3.2. The Lie-theoretic cycle class 18
3.2.1. The Duflo element 19
3.2.2. Torsion morphisms 19
3.3. Tameness for pairs of Lie algebras 21
3.3.1. Setting 21
3.3.2. The envelopping algebra $\mathrm{U}(\mathfrak{n})$ 21
3.3.3. The induced representation 22
4. Generalities on formal neighbourhoods 23
4.1. Sheaves on split square zero extensions 23
4.1.1. Theta morphisms 24
4.1.2. Infinitesimal HKR isomorphism 24
4.1.3. Torsion and Atiyah class 25
4.2. Sheaves on a second order thickening 26
4.2.1. Setting and cohomological invariants 26
4.2.2. The second order HKR class 27
4.3. Quantized cycles 27
4.3.1. Setting, and tameness condition 28
4.3.2. Restriction of the Atiyah class 28
4.3.3. Quantized HKR isomorphism 29

Date: November 12, 2017.

5. Tame quantized cycles 29
5.1. Structure constants 29
5.1.1. Definition 29
5.1.2. The decomposition lemma 30
5.1.3. Main result 32
5.2. The results of Kapranov, Markarian, Ramadoss and Yu 34
5.2.1. The Lie algebra $\mathrm{T}_X[-1]$ 34
5.2.2. Big Chern classes 36
5.2.3. The quantized cycle class 36
6. The Ext algebras 37
6.1. Definitions 37
6.2. Enrichment 37
6.3. Structure theorem 38
6.4. The algebra $\mathcal{A}^{(2)}_{X/Y}$ 38
7. Conclusion and perspectives 39
7.1. State of the art in the tame case 39
7.2. Beyond the tame case 39
References 39

1. Introduction

1.1. **The diagonal embedding case.** Let be X a complex manifold or a smooth algebraic variety over a field of characteristic zero. Thanks to the celebrated result of Hochschild, Kostant and Rosenberg (*cf.* [10]), the Hochschild homology and cohomology groups of the sheaf $\mathcal{O}_X$ are given by $\mathrm{HH}_i(\mathcal{O}_X) = \Omega^i_X$ and $\mathrm{HH}^i(\mathcal{O}_X) = \Lambda^i \mathrm{T}_X$. These isomorphisms can be upgraded at the level of derived categories, and are called (geometric) HKR isomorphisms. For more history on this topic, we refer the reader to the paper [8], as well as references therein. In the present paper, we will be especially interested in the geometric HKR isomorphism involving Hochschild cohomology: this isomorphism is an additive sheaf isomorphism between the sheaf of polyvector fields $\mathrm{S}(\mathrm{T}_X[-1]) = \bigoplus_{i\geq 0} \Lambda^i \mathrm{T}_X[i]$ and the derived Hom sheaf of the diagonal $p_{1*}\mathcal{RH}om_{\mathcal{O}_{X\times X}}(\mathcal{O}_X, \mathcal{O}_X)$. Here we view X as the diagonal inside $X \times X$ (which is the geometric counterpart of looking at an algebra as a bimodule over itself) and p_1 is the first projection.

Both members of this isomorphism have multiplicative structure: the wedge product on polyvector fields, and the Yoneda product on the derived Hom complex. Taking the cohomology on both sides, for any non-negative integer p, the corresponding isomorphism between the algebras $\bigoplus_{i=0}^{p} \mathrm{H}^{p-i}(X, \Lambda^i \mathrm{T}_X)$ and $\mathrm{Ext}^i_{X\times X}(\mathcal{O}_X, \mathcal{O}_X)$ is not multiplicative in general (contrarily to the homological geometric HKR). It has been conjectured by Kontsevich [13] and later proved by Van den Bergh and the first author [2] that one can use the square root of the Todd class $\mathrm{Td}\,(X)$ on $\bigoplus_i \mathrm{H}^i(X, \Omega^i_X)$ of the tangent sheaf T_X to "twist" the global cohomological HKR isomorphism in order to get an isomorphism of algebras.

The Todd class of X is also intimately related to the geometry of the diagonal of X in another way: it is the correction term appearing on the Grothendieck-Riemann-Roch theorem, so thanks to the Lefschetz formalism it can be interpreted as a restriction of the Grothendieck cycle class to the diagonal itself. This has been hinted in the pioneering work of Toledo-Tong (*see* [17, §6]), and formalized using HKR isomorphisms in an unpublished manuscript of Kashiwara around 1992. Kashiwara's account can be found in [7], where the second author proves in fact that the Todd class is the Euler class of $\mathcal{O}_X$, proving a conjecture of Kaschiwara–Schapira [12].

The above results can be re-interpreted very naturally in Lie-theoretic terms after the works of Kapranov [11] and Markarian [14]:

– The shifted tangent sheaf $\mathrm{T}_X[-1]$ is a Lie algebra object in the derived category $\mathrm{D}^{\mathrm{b}}(X)$.

- Thanks to results of Markarian [14] and Ramadoss [15], the universal enveloping algebra of this Lie algebra object is indeed the derived Hom sheaf, and that the geometric HKR isomorphism can be re-interpreted as the Poincaré-Birkhoff-Witt (PBW) isomorphism.
- Every element $\mathcal{F}$ in $\mathrm{D}^{\mathrm{b}}(X)$ is naturally a representation of $\mathrm{T}_X[-1]$, and via the PBW isomorphism the character of this representation (which is a central function on $\mathrm{U}(\mathrm{T}_X[-1])$) can be identified with the Chern character of $\mathcal{F}$.
- The Todd class becomes the derivative of the multipication map in the universal enveloping algebra, and is therefore the Duflo element of $\mathrm{T}_X[-1]$.
- The isomorphism $\mathrm{HKR} \circ \iota_{\sqrt{\mathrm{Td}(X)}}$ from [13, 2] can be seen as a Duflo isomorphism (*see* [6]) for the Lie algebra object $\mathrm{T}_X[-1]$.

We refer to [4], [3] for further analogies between Lie theory and algebraic geometry.

1.2. **More general embeddings: tame quantized cycles.** In the present paper, we are interested in the more general situation where we replace the diagonal embedding $\Delta_X \hookrightarrow X \times X$ by an arbitrary closed immersion $X \hookrightarrow Y$, where X is a smooth closed subscheme of an ambiant smooth scheme Y.

In [1], Arinkin and Căldăraru gave a necessary and sufficient condition for an additive generalized geometric HKR isomorphism to exist between $\mathcal{RHom}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ and $\mathrm{S}(\mathrm{N}_{X/Y}[-1])$: the condition is that $\mathrm{N}_{X/Y}$ extends to a locally free sheaf on the first infinitesimal neighborhood of X in Y. The Lie theoretic interpretation of the first order neighborhood and of the above geometric condition has been given in [3] by Căldăraru, Tu, and the first author.

Earlier on, in Kashiwara's 1992 unpublished manuscript, a more restrictive condition is introduced: Kashiwara deals with subschemes with split conormal sequence, which means that the map from X to its first infinitesimal neiborhood in Y admits a global retraction (in this case, *any* locally free sheaf on X extends at order one in Y). On the Lie side, this corresponds to pairs $\mathfrak{h} \subset \mathfrak{g}$ that split as $\mathfrak{h}$-modules, these are usually called reductive pairs. In [8], the second author developed Kashiwara's construction in this framework. The data of a subscheme X of Y together with such a retraction σ is called a quantized cycle, and to such a cycle it is possible to associate geometric HKR isomorphisms, as well as a quantized cycle class $q_\sigma(X)$ living in $\bigoplus_{i\geq 0} \mathrm{H}^i(X, \Lambda^i \mathrm{N}^*_{X/Y})$ that generalizes the Todd class in the diagonal case. Recently, answering a question raised by the second author in the article [8], Yu has shown in [18] the following result: given a quantized cycle (X, σ) in Y such that $\sigma^* \mathrm{N}_{X/Y}$ extends to a locally free sheaf on the second infinitesimal neighborhood of X in Y,

- The quantized cycle $q_\sigma(X)$ class is completely determined by the geometry of the second infinitesimal neighborhood of X in Y.
- It can be expressed by an explicit formula similar to that of the usual Todd class.

Yu's proof is based on direct calculation using the dg Dolbeault complex as well as homological perturbation theory. In this paper we provide a Lie theoretic explanation of Yu's results. We introduce the notion of *tame quantized cycle* corresponding to Yu's condition: a quantized cycle (X, σ) is tame if $\sigma^* \mathrm{N}_{X/Y}$ extends to a locally free sheaf at the second order. We can list all the conditions that can be investigated on the cycle X (each condition being more restrictive than the previous one), and the corresponding conditions for Lie algebra pairs:

cycles $X \subset Y$	Lie pairs $\mathfrak{h} \subset \mathfrak{g}$
$\mathrm{N}^*_{X/Y}$ extends at the first order	delicate condition, *cf.* [4]
X admits a retraction at the first order in Y *i.e.* X can be quantized	$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ as $\mathfrak{h}$-modules
X is a tame quantized cycle	$[\pi_{\mathfrak{h}}([\mathfrak{m}, \mathfrak{m}]), \mathfrak{m}] = 0$
X admits a retraction at the second order in Y	$\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{m}$

If one of the two last conditions is satisfied, the object $\mathrm{N}_{X/Y}[-1]$ is naturally a Lie object in $\mathrm{D}^{\mathrm{b}}(X)$, but this is no longer the case if we drop the tameness assumption. In full generality (that is without any specific quantization conditions), the algebraic structure of $\mathrm{N}_{X/Y}[-1]$ has been investigated in [4]: it is a

derived Lie algebroid, whose anchor map is given by the extension class of the normal exact sequence of the pair (X, Y). Hence, our setting can be understood as the weaker universal hypotheses for which this derived algebroid is a true Lie object in the symmetric monoidal category $\mathrm{D}^{\mathrm{b}}(X)$.

1.3. **The results.** The two main geometric result we prove in this paper deals with tame quantized analytic cycle. Both rely heavily on abstract result on Lie algebras in symmetric monoidal categories, that we will discuss after the statements. The first result is the explicit computation of the enveloping algebra of the Lie algebra object $\mathrm{N}_{X/Y}[-1]$.

Theorem A. *Let (X, σ) be a tame quantized cycle in Y. The class α defines a Lie coalgebra structure on $\mathrm{N}^*_{X/Y}[1]$, hence a Lie algebra structure on $\mathrm{N}_{X/Y}[-1]$. Besides, the objects $\mathcal{RH}om^{\ell}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ and $\mathcal{RH}om^{r}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ are naturally algebra objects in the derived category $\mathrm{D}^{\mathrm{b}}(X)$, and there are commutative diagrams*

$$\begin{tikzcd}
\sigma_*\mathcal{RH}om_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X) \arrow[r] \arrow[dd, "\mathrm{HKR}"'] & \mathcal{RH}om^{\ell}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \arrow[d, "\mathrm{HKR}"] \\
& \mathrm{S}(\mathrm{N}_{X/Y}[-1]) \arrow[d, "\mathrm{PBW}"] \\
\mathrm{T}(\mathrm{N}_{X/Y}[-1]) \arrow[r] & \mathrm{U}(\mathrm{N}_{X/Y}[-1])
\end{tikzcd}$$

and

$$\begin{tikzcd}
\sigma_*\mathcal{RH}om_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X) \arrow[r] \arrow[dd, "\text{dual HKR}"'] & \mathcal{RH}om^{r}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \arrow[d, "\text{dual HKR}"] \\
& \mathrm{S}(\mathrm{N}_{X/Y}[-1]) \arrow[d, "\mathrm{PBW}"] \\
\mathrm{T}(\mathrm{N}_{X/Y}[-1]) \arrow[r] & \mathrm{U}(\mathrm{N}_{X/Y}[-1])
\end{tikzcd}$$

where all horizontal arrows are algebra morphisms, and all vertical arrows are isomorphisms.

1.4. **Plan of the paper.** The paper is organized as follows:

- §2 is devoted to some recollection about Lie algebra objects in a categorical setting. We claim no originality for this material, but we were not able to find the desired results in the form we needed in the literature. For instance, most references are written for abelian categories while we work in the slightly more general Karoubian framework.
- §3 deals with three different topics. §3.1 gives universal formulas for the multiplication map $\mathrm{U}(\mathfrak{g}) \otimes \mathfrak{g} \to \mathrm{U}(\mathfrak{g})$ via the PBW isomorphism. Up to our knowledge, this result has never been proved in this level of generality. In §3.2, we define an algebraic condition that characterizes uniquely the Duflo element of a Lie object in a symmetric monoidal category. This will be the key ingredient to our Lie-theoretic proof of Yu's result. In §3.3, we introduce the "tame condition" for pairs of Lie algebras, which is a Lie theoretic analog of Yu's condition.
- §4 recollects previous results on (first and second order) infinitesimal neighborhoods, HKR isomorphisms and quantized cycles. This is were we state the geometric tameness condition, a-k-a Yu's condition.
- in §5 we explain that the geometric tameness condition coincides with the Lie theoretic tameness condition for the pair $(\mathrm{T}_X[-1], \mathrm{T}_Y[-1]_{|X})$ of Lie algebra objects in $\mathrm{D}^{\mathrm{b}}(X)$. We get in particular that in the tame case, $\mathrm{N}_{X/Y}[-1]$ is a Lie algebra object in $\mathrm{D}^{\mathrm{b}}(X)$ and we describe its universal enveloping algebra in geometric terms. Using the Lie-theoretic results of §2-3, we are able to give enlightening and simple proofs of results of Ramadoss about Kapranov big Chern classes (diagonal case), and Yu's formula for the quantized cycle class.

– In §6 we use the above to get a description of the Ext algebra of a tame quantized cycle. We show in particular that it is completely determined by the second order infinitesimal neighborhood of X in Y.
– We conclude the paper with a few perspectives in §7.

2. Universal enveloping algebras and the categorical PBW theorem

2.1. Preliminary results of linear algebra.

2.1.1. *Partially antisymmetric tensors.* Let $\mathbf{k}$ be a field of characteristic zero and let C be a $\mathbf{k}$-linear symmetric monoidal category that is Karoubian (*i.e.* every idempotent splits[1]) and such that countable direct sums exist. We can assume without loss of generality that C is a strict monoidal category (*i.e.* it is harmless to drop the parenthesizations of iterated tensor products from the notation). For any non-negative integer n, the symmetric group $\mathfrak{S}_n$ acts naturally on $V^{\otimes n}$, where V is an object of C. Let π_n be the element $(n!)^{-1}\sum_{g\in\mathfrak{S}_n} g$, considered as an idempotent element of the group algebra $\mathbf{k}[\mathfrak{S}_n]$. It induces a natural idempotent[2] on $V^{\otimes n}$, whose kernel is denoted by $\widetilde{\Lambda}^n V$ and whose image is denoted by $\mathrm{S}^n V$. We therefore have a decomposition

$$V^{\otimes n} = \mathrm{S}^n V \oplus \widetilde{\Lambda}^n V\,.$$

Assume that n is at least two and let

$$\Psi_n\colon \bigoplus_{i=1}^{n-1}\left(V^{\otimes i-1}\otimes\Lambda^2 V\otimes V^{\otimes n-i-1}\right)\to V^{\otimes n}$$

be the map obtained by embedding each $\Lambda^2 V := \widetilde{\Lambda}^2 V$ in $V^{\otimes 2}$. We can provide a concrete description of $\widetilde{\Lambda}^n V$ via the map Ψ_n:

Lemma 2.1. *The image of Ψ_n exits and is canonically isomorphic to $\widetilde{\Lambda}^n V$.*

Proof. For i in $[\![1, n-1]\!]$, let τ_i be the transposition in the group $\mathfrak{S}_n$ that switches i and $i+1$. We first observe that $(\mathbf{1}+\tau_i)\pi_n = \pi_n(\mathbf{1}+\tau_i) = \pi_n$. Hence the kernel of $\mathbf{1}+\tau_i$ acting on $V^{\otimes n}$ is a (split) sub-object of the kernel of π_n acting on $V^{\otimes n}$:

$$V^{\otimes(i-1)}\otimes\Lambda^2 V\otimes V^{\otimes(n-i-1)}\subset\widetilde{\Lambda}^n V\,.$$

In other words, the map Ψ_n factors through $\widetilde{\Lambda}^n V$. In order to conclude, it is then sufficient to prove that $\bigoplus_{i=1}^{n-1}\left(V^{\otimes i-1}\otimes\Lambda^2 V\otimes V^{\otimes n-i-1}\right)\to\widetilde{\Lambda}^n V$ admits a section. We claim that the right ideal in $\mathbf{k}[\mathfrak{S}_n]$ generated by $\mathbf{1}-\tau_i$ $(1\le i\le n-1)$ contain all elements $\mathbf{1}-\tau$ for arbitrary τ in $\mathfrak{S}_n$. Indeed, for any elements $g_1,\dots,g_d$ in the group algebra $\mathbf{k}[\mathfrak{S}_n]$, we have

$$\mathbf{1}-\Pi_{i=1}^d g_i = (\mathbf{1}-\Pi_{i=1}^{d-1} g_i)g_d + (\mathbf{1}-g_d).$$

The claim follows again from the fact that the τ_i generate $\mathfrak{S}_n$. As a corollary, $\mathbf{1}-\pi$ sits in this ideal, so that we can choose elements $(a_i)_{1\le i\le n-1}$ in the group algebra such that

$$\sum_{i=1}^{n-1}(\mathbf{1}-\tau_i)\,a_i = 1-\pi_n$$

As a consequence we get that the map

$$\Phi_n := \prod_{i=1}^{n-1}(\mathbf{1}-\tau_i)\,a_i : V^{\otimes n}\longrightarrow\bigoplus_{i=1}^{n-1}V^{\otimes n}$$

factors through $\bigoplus_{i=1}^{n-1}\left(V^{\otimes i-1}\otimes\Lambda^2 V\otimes V^{\otimes n-i-1}\right)$ and $\Psi_n\circ\Phi_n$ is the projection onto the direct factor $\widetilde{\Lambda}^n V$ in $V^{\otimes n}$. □

[1]Therefore, every multiple of a projector has a kernel and an image; we will repeatedly use this property.

[2]Abusing notation, we will denote by the same symbol an element of $\mathbf{k}[\mathfrak{S}_n]$ and the induced endomorphism of $V^{\otimes n}$ in C.

Example 2.2. The simplest nontrivial case is $n = 3$. In this case we can take for instance

$$\begin{cases} 6a_1 = 3\mathbf{1} + \tau_2 + \tau_2\tau_1 \\ 3a_2 = \mathbf{1} + \tau_1 \end{cases}$$

2.1.2. *Jacobi identity.* Let us consider a morphism $\alpha\colon V^{\otimes 2} \to V$ in $\mathcal{C}$ which vanishes on $\mathrm{S}^2 V$ (*i.e.* such that $\alpha \circ \tau_1 = -\alpha$).

Proposition 2.3. *The pair (V, α) is a Lie algebra object if and only if there exists*

$$\beta\colon \widetilde{\Lambda}^3 V \to V$$

such that $\beta \circ \Psi_3 = (\alpha \circ (\alpha \otimes \mathrm{id}), \alpha \circ (\mathrm{id} \otimes \alpha))$.

Proof. Let $u = \alpha \circ (\alpha \otimes \mathrm{id})$. The Jacobi identity is equivalent to the identity

$$u - u \circ \tau_2 + u \circ \tau_2\tau_1 = 0\,.$$

Remark now that $\alpha \circ (\mathrm{id} \otimes \alpha) = -u \circ \tau_2\tau_1$, and that $u \circ \tau_1 = -u$. We now pre-compose $(u, -u \circ \tau_2\tau_1)$ with the right inverse of Ψ_3 given by Lemma 2.1. This gives a morphism $\beta\colon \widetilde{\Lambda}^3 V \to V$. Example 2.2 and a short calculation provide the following explicit formula:

$$\beta = \frac{1}{3}(3u - u \circ \tau_2 - u \circ \tau_2\tau_1)\,.$$

This morphism β is the only possible candidate to fulfil the desired condition $\beta \circ \Psi_3 = (u, -u \circ \tau_2\tau_1)$. Then we observe that

$$\begin{cases} \beta_{|\Lambda^2 V \otimes V} = u \\ \beta_{|V \otimes \Lambda^2 V} = -u \circ \tau_2\tau_1 + \dfrac{2}{3}(2u + u \circ \tau_2\tau_1) \end{cases}$$

Hence $\beta \circ \Psi_3 = (u, -u \circ \tau_2\tau_1)$ if and only if $2u + u \circ \tau_2\tau_1$ vanishes on $V \otimes \Lambda^2 V$. This condition is clearly implied by the Jacobi identity, but it is in fact equivalent to it. Indeed, $2u + u \circ \tau_2\tau_1$ vanishes on $V \otimes \Lambda^2 V$ if and only if $(2u + u \circ \tau_2\tau_1) \circ (\mathbf{1} - \tau_2) = 0$, and

$$\begin{aligned} (2u + u \circ \tau_2\tau_1) \circ (\mathbf{1} - \tau_2) &= 2u + u \circ \tau_2\tau_1 - 2u \circ \tau_2 - u \circ \tau_2\tau_1\tau_2 \\ &= 2u + u \circ \tau_2\tau_1 - 2u \circ \tau_2 + u \circ \tau_2\tau_1 \\ &= 2(u - u \circ \tau_2 + u \circ \tau_2\tau_1). \end{aligned}$$

□

2.2. **Algebras satisfying the PBW isomorphism.**

2.2.1. *The morphisms c_p^k.* Let V be an object in $\mathcal{C}$ and let $\mathcal{A}$ be a unital augmented algebra object in $\mathcal{C}$ together with an algebra morphism $\Delta\colon \mathrm{T}(V) \to \mathcal{A}$. From now on, we require

Assumption (A1) $\boxed{\text{The composition } \Delta_+\colon \mathrm{S}(V) \to \mathrm{T}(V) \to \mathcal{A} \text{ is an isomorphism}^3.}$

Here $\mathrm{T}(V) := \bigoplus_{n\geq 0} V^{\otimes n}$ is equipped with the concatenation product, $\mathrm{S}(V) := \bigoplus_{n\geq 0} \mathrm{S}^n V$ and the map $\mathrm{S}(V) \to \mathrm{T}(V)$ is the direct sum of direct factor embeddings $\mathrm{S}^n V \subset V^{\otimes n}$. Note that $\mathcal{A}$ carries a split increasing filtration: $\mathrm{F}^p\mathcal{A} := \Delta_+(\bigoplus_{i=0}^p \mathrm{S}^i V)$. Moreover, Δ is a filtered morphism for the obvious degree filtration on $\mathrm{T}(V)$. We now consider the restriction Δ^p of the filtered morphism $\Delta_+^{-1} \circ \Delta$ on each homogeneous component $V^{\otimes p}$; it decomposes as follows:

$$\Delta^p := \Delta_+^{-1} \circ \Delta_{|V^{\otimes p}} = \sum_{k=0}^{p} c_p^k\,,$$

where $c_p^k \in \mathrm{Hom}_{\mathcal{C}}(V^{\otimes p}, \mathrm{S}^k V)$. Since $\mathcal{A}$ is augmented, c_p^0 vanishes. We further make the following

[3] Only as objects of $\mathcal{C}$, not as algebras.

Assumption (A2) For any non-negative integer p the morphism $V^{\otimes p} \to \mathrm{Gr}^p\mathcal{A} \simeq \mathrm{S}^p V$ is the canonical projection π_p.

Note that this morphism is nothing but Δ^p followed by the projection onto $\mathrm{S}^p V$. This second assumption on $\mathcal{A}$ ensures that the restriction of c_p^k to $\mathrm{S}^p V$ is zero if $k \leq p-1$, and c_p^p is the canonical projection π_p from $V^{\otimes p}$ to $\mathrm{S}^p V$. In particular, if $1 \leq k \leq p-1$, then c_p^k lives naturally in $\mathrm{Hom}_C(\widetilde{\Lambda}^p V, \mathrm{S}^k V)$.

2.2.2. *The Lie bracket.* We define $\alpha = c_2^1 \colon \Lambda^2 V \to V$. Since $\mathcal{A}$ is augmented, we can identify $\mathrm{F}^1\mathcal{A}$ with $\mathbb{1} \oplus V$ (*via* Δ_+), where $\mathbb{1}$ is the monoidal unit of C.

Lemma 2.4. *Let $m \colon \mathcal{A}^{\otimes 2} \to \mathcal{A}$ be the associative product. Then the morphism $V^{\otimes 2} \to \mathcal{A}$ defined by*

$$m \circ \Delta_{|V}^{\otimes 2} \circ (\mathbf{1} - \tau_1)$$

factors through $\Delta_+(V) \subset \mathrm{F}^1\mathcal{A}$, and coincides with $2\Delta_+ \circ \alpha$.

Proof. As Δ is an algebra morphism, we have that

$$m \circ \Delta_{|V}^{\otimes 2} \circ (\mathbf{1} - \tau_1) = \Delta_{|V^{\otimes 2}} \circ (\mathbf{1} - \tau_1) = \Delta_+ \circ (c_2^1 + \pi_2) \circ (\mathbf{1} - \tau_1) = 2\Delta_+ \circ c_2^1 = 2\Delta_+ \circ \alpha\,.$$

We are done. □

Corollary 2.5. *The morphism α defines a Lie structure on V.*

2.2.3. *Induction formulæ for c_p^k.* We can now provide explicit induction formulæ for the morphisms c_p^k. Recall that for $0 \leq k \leq p-1$, we consider c_p^k as an element of $\mathrm{Hom}_C(\widetilde{\Lambda}^p V, \mathrm{S}^k V)$.

Proposition 2.6. *Given a pair $(V, \mathcal{A})$ as above, the coefficients c_p^k are determined as follows:*

$$\begin{cases} c_p^p = \pi_p : V^{\otimes p} \to \mathrm{S}^p V. \\ c_p^k \circ \Psi_p = \{c_{p-1}^k \circ (\mathrm{id}_{V^{\otimes i-1}} \otimes \alpha \otimes \mathrm{id}_{V^{\otimes p-i-1}})\}_{1 \leq i \leq p-1} \ \ \textit{if} \ \ 1 \leq k \leq p-1. \end{cases}$$

Recall here that $\alpha = c_2^1$.

Proof. For $1 \leq i \leq p-1$ we have

$$\begin{aligned} \Delta_{|V^{\otimes p}} \circ \frac{1}{2}(\mathbf{1} - \tau_i) &= m^{(2)} \circ \Delta_{|V^{\otimes i-1}} \otimes (\Delta_{|V^{\otimes 2}} \circ \frac{1}{2}(\mathbf{1} - \tau_1)) \otimes \Delta_{|V^{\otimes p-i-1}} \\ &= m^{(2)} \circ \Delta_{|V^{\otimes i-1}} \otimes (\Delta \circ \alpha) \otimes \Delta_{|V^{\otimes p-i-1}} \qquad \text{(by Lemma 2.4)} \\ &= \Delta_{|V^{\otimes p-1}} \circ (\mathrm{id}_{V^{\otimes i-1}} \otimes \alpha \otimes \mathrm{id}_{V^{\otimes p-i-1}})\,, \end{aligned}$$

where $m^{(2)} := m \circ (m \otimes \mathrm{id})$. Applying Δ_+^{-1} followed by the projection on the direct factor $\mathrm{S}^k V$ we get

$$c_p^k \circ \frac{1}{2}(\mathbf{1} - \tau_i) = c_{p-1}^k \circ (\mathrm{id}_{V^{\otimes i-1}} \otimes \alpha \otimes \mathrm{id}_{V^{\otimes p-i-1}})\,.$$

Hence both members of the induction relation agree on $V^{\otimes i-1} \otimes \Lambda^2 V \otimes V^{\otimes p-i-1}$ for every $1 \leq i \leq p-1$. □

2.3. **Universal algebras in the categorical setting.**

2.3.1. *Reverse PBW theorem.* We can now prove our first main result: assuming that the algebra $\mathcal{A}$ satisfies the PBW theorem (*i.e.* Assumptions (A1) and (A2)), we prove that it is the universal enveloping algebra of V endowed with the Lie bracket $2c_2^1$.

Proposition 2.7 (**Reverse PBW**)**.** *If $\mathcal{A}$ satisfies Assumptions (A1-A2) of §2.2.1, then $\mathcal{A}$ is a universal enveloping algebra of the Lie algebra $(V, 2\alpha)$.*

Proof. For any associative algebra object B in C, with product $m_B : B^{\otimes 2} \to B$, we denote by $B_{\mathrm{Lie}} = (B, \mu_B)$ the Lie algebra object which is B endowed with the Lie bracket $\mu_B = m_B \circ (\mathbf{1} - \tau_1)$. Lemma 2.4 tells us that the direct factor inclusion $(\Delta_+)_{|V} : V \hookrightarrow \mathcal{A}$ is a morphism of Lie algebra objects from $(V, 2\alpha)$ to $\mathcal{A}_{\mathrm{Lie}}$ in C.

Assume now to be given a morphism $f : V \to B$. By the universal property of the tensor algebra, it defines an algebra morphism $\widetilde{f} \colon \mathrm{T}(V) \to B$.

Lemma 2.8. *If f is a Lie algebra morphism (from $(V, 2\alpha)$ to B_{Lie}) then $\widetilde{f}$ factors through a unique morphism $g : \mathcal{A} \to B$.*

Proof of the Lemma. It is sufficient to prove that $\widetilde{f} = \widetilde{f} \circ s \circ \Delta$, where s is a section of Δ. Indeed, the only possible choice is $g = \widetilde{f} \circ s$. Here we use the section s given by $\Delta_+^{-1} : \mathcal{A} \to \mathrm{S}(V) \subset \mathrm{T}(V)$.

We will prove by induction on p that $\widetilde{f}_{|V^{\otimes p}} = \widetilde{f} \circ \Delta_+^{-1} \circ \Delta_{|V^{\otimes p}}$. Note that this identity is obviously satisfied when restricted to $\mathrm{S}^p V \subset V^{\otimes p}$. The only thing left to prove is thus that $\widetilde{f}_{|\widetilde{\Lambda}^p V} = \widetilde{f} \circ s \circ \Delta_{|\widetilde{\Lambda}^p V}$.

- For $p \in \{0, 1\}$ the result is obvious.
- For $p = 2$, we have
$$\widetilde{f}_{|\Lambda^2 V} = \frac{1}{2}\mu_B \circ (f \otimes f)_{|\Lambda^2 V} = f \circ \alpha = f \circ c_2^1 = \widetilde{f} \circ \Delta_+^{-1} \circ \Delta_{|\Lambda^2 V}\,.$$
- Let us now assume that the required equality holds for a given $p \geq 2$. We compute
$$\begin{aligned}\widetilde{f} \circ s \circ \Delta_{|V^{\otimes i-1} \otimes \Lambda^2 V \otimes V^{\otimes p-i}} &= \widetilde{f} \circ \left(\sum_{k=1}^{p} c_{p+1}^k\right)_{|V^{\otimes i-1} \otimes \Lambda^2 V \otimes V^{\otimes p-i}} \\ &= \widetilde{f} \circ \left(\sum_{k=1}^{p} c_p^k\right) \circ (\mathrm{id}_{V^{\otimes i-1}} \otimes \alpha \otimes \mathrm{id}_{V^{\otimes p-i}}) \quad \text{(by Proposition 2.6)} \\ &= \widetilde{f} \circ (\mathrm{id}_{V^{\otimes i-1}} \otimes \alpha \otimes \mathrm{id}_{V^{\otimes p-i}}) \quad \text{(by induction)}\,.\end{aligned}$$
Finally one can prove that
$$\widetilde{f}_{|V^{\otimes i-1} \otimes \Lambda^2 V \otimes V^{\otimes p-i}} = \widetilde{f} \circ (\mathrm{id}_{V^{\otimes i-1}} \otimes \alpha \otimes \mathrm{id}_{V^{\otimes p-i}})$$
in the same way as for the case when $p = 2$.

□

End of the proof of Proposition 2.7. In order to conclude one has to prove that g is indeed a morphism of algebras:
$$\begin{aligned}g \circ m &= \widetilde{f} \circ s \circ m = \widetilde{f} \circ s \circ m \circ (\Delta \circ s)^{\otimes 2} \\ &= \widetilde{f} \circ s \circ \Delta \circ m \circ s^{\otimes 2} = \widetilde{f} \circ m \circ s^{\otimes 2} \\ &= m \circ (\widetilde{f} \circ s)^{\otimes 2} = m \circ g^{\otimes 2}\,.\end{aligned}$$

□

2.3.2. *Categorical construction of the product on $S(V)$.* In this section, we prove the following delicate result:

Theorem 2.9. *There exists an algebra $\mathcal{A}$ that satisfies Assumptions (A1-A2).*

Remark 2.10. In [5, Lemma 1.3.7.5], the authors provide an explicit multiplication law $m_\star$ on $\mathrm{S}(V)$ in the case when C is the category of graded vector spaces (or more generally any abelian category), and prove that it is associative. Then the algebra $\mathcal{A} = (\mathrm{S}(V), m_\star)$ satisfies all required properties. Their proof should carry on as well for a general C, but the formulas defining $m_\star$ are daunting to write down in the categorical setting. This is why we provide another approach, which is close in spirit of [5], but more adapted to the categorical framework.

The proof is long and technical. Before starting, let us introduce some notation that will be used constantly in the sequel:

- On $\mathrm{S}(V)$ we have an associative and commutative product m_0 defined as the composition
$$\mathrm{S}(V)^{\otimes 2} \subset \mathrm{T}(V)^{\otimes 2} \to \mathrm{T}(V) \twoheadrightarrow \mathrm{S}(V)\,.$$
- We write $\tau_{p,q}$ for the transposition (p, q) in the symmetric group, as well as for the corresponding action on $V^{\otimes n}$.

– We consider the morphism $\omega_{(V,\mu)} : \mathrm{S}^n V \otimes V \to \mathrm{S}^{n-1} V \otimes V$ defined as

$$\omega_{(V,\mu)} := (\mathrm{id}_{V^{\otimes n-1}} \otimes \mu) \circ \sum_{i=1}^{n} \tau_{i,n} = n\,(\mathrm{id}_{V^{\otimes n-1}} \otimes \mu)\,, \tag{1}$$

where $\mu := 2\alpha$. We leave it as an exercise to check that the image of $\omega_{(V,\mu)}$ indeed factors through $\mathrm{S}^{n-1} V \otimes V \subset V^{\otimes n}$. We often simply write ω as the choice of (V, μ) is clear from the context.

Proof of Theorem 2.9. Let (V, α) be a Lie algebra object in C. For any integer $n \geq 2$, we are going to construct a pair $(m_\star, \pi)$ that satisfies the seven following properties:

A For $i, j \leq 0$ with $i + j \leq n$, $m_\star : \mathrm{S}^i(V) \otimes \mathrm{S}^j(V) \to \mathrm{S}^{\leq i+j}(V)$.

B The product $m_\star$ is associative: if $i + j + k \leq n$ then $m_\star(m_\star \otimes \mathrm{id})$ and $m_\star(\mathrm{id} \otimes m_\star)$ agree on $\mathrm{S}^i(V) \otimes \mathrm{S}^j(V) \otimes \mathrm{S}^k(V)$. And its unit is given by the canonical direct factor embedding $\mathbf{1} \subset \mathrm{S}^{\leq n}(V)$.

C The action of the Lie algebra V on $\mathrm{S}^{\leq n}(V)$ respects the product $m_\star$: if $i + j \leq n$ then

$$\mathrm{ad}(\mathrm{id} \otimes m_\star) = m_\star(\mathrm{ad} \otimes \mathrm{id} + (\mathrm{id} \otimes \mathrm{ad})(\tau_{V,\mathrm{S}^i(V)} \otimes \mathrm{id}))$$

on $V \otimes \mathrm{S}^i(V) \otimes \mathrm{S}^j(V)$.

D The action of the Lie algebra V on $\mathrm{S}^{<n}(V)$ is the commutator for $m_\star$ with degree one elements:

$$\mathrm{ad} = m_\star(\mathrm{id}_{V \otimes \mathrm{S}^n(V)} - \tau_{V,\mathrm{S}^{<n}(V)}).$$

E π is a filtered morphism from $\mathrm{T}^{\leq n}(V)$ to $\mathrm{S}^{\leq n}(V)$ that is multiplicative: if $i + j \leq n$,

$$\pi m_\star = m_\star(\pi \otimes \pi)$$

on $\mathrm{S}^i(V) \otimes \mathrm{S}^j(V)$.

F The restriction of π to $\mathrm{S}^{\leq n}(V)$ is the identity map.

G On $\mathrm{S}^{n-1}(V) \otimes V$, the morphism $m_\star$ is given by the explicit formula

$$m_0 + \frac{1}{n(n-1)} \pi \left(\sum_{k=1}^{n-1} k \tau_{k,n-1} \omega \right).$$

To see where condition (G) comes from, we refer the reader to Lemma 3.2. We argue by induction on n.

Initial step. The first interesting case is $n = 2$. Then $m_\star$ is defined by $m_0 + \alpha$ and π is given by $m_\star$ itself on $V \otimes V$. It is easy to check that all properties are satisfied.

Inductive step: definitions. Assume that the induction hypotheses are satisfied at level n. We define the product

$$\text{“}m_\star\text{”} : \mathrm{S}^{n-i}(V) \otimes \mathrm{S}^{i+1}(V) \to \mathrm{S}^{\leq n+1}(V)$$

inductively for $1 \leq i \leq n - 1$ as follows:

– For $i = 1$, “$m_\star$” is defined on $\mathrm{S}^n(V) \otimes V$ as

$$\text{“}m_\star\text{”} = m_0 + \frac{1}{n(n+1)} \pi \left(\sum_{k=1}^{n} k \tau_{k,n} \omega \right).$$

– Then “$m_\star$” is defined as “$m_\star$”$(m_\star \otimes \mathrm{id}_V)$ on

$$\mathrm{S}^{n-i}(V) \otimes \mathrm{S}^{i+1}(V) \subset \mathrm{S}^{n-i}(V) \otimes \mathrm{S}^i(V) \otimes V\,.$$

– Lastly we define “$m_\star$” $= m_\star$ for lower degrees.

For the projection π, on $\mathrm{T}^{n+1}(V) = \mathrm{T}^n(V) \otimes V$, we put

$$\text{“}\pi\text{”} = \text{“}m_\star\text{”}\,(\pi \otimes \mathrm{id}),$$

and “π” $= \pi$ on $\mathrm{T}^{\leq n}(V)$.

Notation.

- Symbols in between quotes are the ones that we've just defined for the induction step. For these ones we are not yet allowed to use properties (B–G), which we in fact have to prove. Note that property (G) holds on the nose, as this is exactly how we defined “$m_\star$”.
- In the symmetric group $\mathfrak{S}_\ell$, we write $\tau_{i,\dots,j}$ for the permutation $\tau_{j-1}\dots\tau_{i+1}\tau_i$ $(i < j)$.

Property (F) is satisfied.

$$\boxed{\text{If } i + j \leq n + 1 \text{ then “}m_\star\text{”} = m_0 = \mathrm{id} \text{ on } \mathrm{S}^{\leq i+j}(V) \subset \mathrm{S}^{\leq i}(V) \otimes \mathrm{S}^{\leq j}(V).}$$

First of all, if $i + j \leq n$ then “$m_\star$” $= m_\star$ and this is true because π preserves the product and is the identity on $\mathrm{S}^{\leq n}(V)$ according to (F). Now, if $i + j = n + 1$, then it follows from the definition of “$m_\star$” and from what we've just observed that the following diagram commutes[4]:

$$\begin{tikzcd}
\mathrm{S}^{n+1}(V) \arrow[r] \arrow[d] & \mathrm{S}^i(V) \otimes \mathrm{S}^j(V) \arrow[d, "\text{“}m_\star\text{”}"] \\
\mathrm{S}^n(V) \otimes V \arrow[r, "\text{“}m_\star\text{”}"] & \mathrm{S}^{\leq n+1}(V)
\end{tikzcd}$$

It is therefore sufficient to prove that “$m_\star$” coincides with m_0 on $\mathrm{S}^{n+1}(V) \subset \mathrm{S}^n(V) \otimes V$. This follows from the fact that the restriction of ω to $\mathrm{S}^{n+1}(V) \subset \mathrm{S}^n(V) \otimes V$ vanishes.

A useful identity.

$$\boxed{\text{“}m_\star\text{”} - \text{“}m_\star\text{”}\,(m_\star \otimes \mathrm{id}_V)\tau_n = m_\star\,(\mathrm{id}_{\mathrm{S}^{n-1}(V)} \otimes \alpha) \text{ on } \mathrm{S}^n(V) \otimes V} \tag{2}$$

$$\begin{aligned}
\text{“}m_\star\text{”}\,(m_\star \otimes \mathrm{id}_V)\tau_n &= \text{“}m_\star\text{”}\,(m_\star \otimes \mathrm{id}_V - m_0 \otimes \mathrm{id}_V)\tau_n + \text{“}m_\star\text{”}\,(m_0 \otimes \mathrm{id}_V)\tau_n \\
&= m_\star\,(m_\star \otimes \mathrm{id}_V - m_0 \otimes \mathrm{id}_V)\tau_n + \text{“}m_\star\text{”}\,(m_0 \otimes \mathrm{id}_V)\tau_n \\
&= \frac{1}{n(n-1)}\, m_\star \left(\pi\left(\sum_{k=1}^{n-1} k\tau_{k,n-1}\omega\right) \otimes \mathrm{id}_V\right)\tau_n + \text{“}m_\star\text{”}\,(m_0 \otimes \mathrm{id}_V)\tau_n
\end{aligned}$$

We will transform the second term. The first observation is that $\omega(m_0 \otimes \mathrm{id}_V)\tau_n = -\frac{1}{n}\omega$ on $\mathrm{S}^n(V) \otimes V$, because the map

$$(m_0 \otimes \mathrm{id}_V)\tau_n + \frac{1}{n}\mathrm{id}_{\mathrm{S}^n(V)\otimes V} \colon \mathrm{S}^n(V) \otimes V \to \mathrm{S}^n(V) \otimes V$$

[4] Indeed, the vertical “$m_\star$” is defined as “$m_\star$”$(m_\star \otimes \mathrm{id}_V)$ on $\mathrm{S}^i(V) \otimes \mathrm{S}^j(V) \subset \mathrm{S}^i(V) \otimes \mathrm{S}^{j-1}(V) \otimes V$, and $m_\star = m_0$ on $\mathrm{S}^i(V) \otimes \mathrm{S}^{j-1}(V)$.

factors through $\mathrm{S}^{n+1}(V)$[5]. This gives

$$\begin{aligned}
\text{“}m_\star\text{”}\,(m_0\otimes \mathrm{id}_V)\tau_n &= m_0(m_0\otimes \mathrm{id}_V)\tau_n + \frac{1}{n(n+1)}\pi\left(\sum_{k=1}^{n} k\tau_{k,n}\omega(m_0\otimes \mathrm{id}_V)\tau_n\right)\\
&= m_0 - \frac{1}{n^2(n+1)}\pi\left(\sum_{k=1}^{n} k\tau_{k,n}\omega\right)\\
&= m_0 - \frac{1}{n}(\text{“}m_\star\text{”} - m_0)\\
&= \frac{n+1}{n}m_0 - \frac{1}{n}\text{“}m_\star\text{”}.
\end{aligned}$$

Hence we get

$$\begin{aligned}
&\text{“}m_\star\text{”} - \text{“}m_\star\text{”}\,(m_\star\otimes \mathrm{id}_V)\tau_n\\
&\quad= \text{“}m_\star\text{”} - \frac{1}{n(n-1)}\,m_\star\left(\pi\left(\sum_{k=1}^{n-1} k\tau_{k,n-1}\omega\right)\otimes \mathrm{id}_V\right)\tau_n - \text{“}m_\star\text{”}\,(m_0\otimes \mathrm{id}_V)\tau_n\\
&\quad= \frac{n+1}{n}(\text{“}m_\star\text{”} - m_0) - \frac{1}{n(n-1)}\,m_\star\left(\pi\left(\sum_{k=1}^{n-1} k\tau_{k,n-1}\omega\right)\otimes \mathrm{id}_V\right)\tau_n\\
&\quad= \frac{1}{n^2}\,\pi\left(\sum_{k=1}^{n} k\tau_{k,n}\omega\right) - \frac{1}{n(n-1)}\,m_\star\left(\pi\left(\sum_{k=1}^{n-1} k\tau_{k,n-1}\omega\right)\otimes \mathrm{id}_V\right)\tau_n.
\end{aligned}$$

We claim that for $1\le k\le n-1$, the two terms corresponding to the index k in the two sums cancel. This follows from the identity

$$\frac{1}{n}\tau_{k,n}\omega = \frac{1}{n-1}\left((\tau_{k,n-1}\omega)\otimes \mathrm{id}_V\right)\tau_n \tag{3}$$

on $\mathrm{S}^n(V)\otimes V$ that we prove as follows: ω acting on $\mathrm{S}^n(V)\otimes V$ is the restriction of $n\times \mathrm{id}_{\mathrm{S}^{n-1}(V)}\otimes\alpha$ to $\mathrm{S}^n(V)\otimes V$. Hence (3) is equivalent to

$$\tau_{k,n}(\mathrm{id}_{\mathrm{S}^{n-1}(V)}\otimes\alpha) = \left(\tau_{k,n-1}(\mathrm{id}_{\mathrm{S}^{n-2}(V)}\otimes\alpha)\otimes \mathrm{id}_V\right)\tau_n$$

on $\mathrm{S}^n(V)\otimes V$. Remark now that

$$(\mathrm{id}_{\mathrm{S}^{n-2}(V)}\otimes\alpha\otimes \mathrm{id}_V)\tau_n = \tau_n(\mathrm{id}_{\mathrm{S}^{n-1}(V)}\otimes\alpha)$$

on $\mathrm{S}^n(V)\otimes V$. Hence (3) is equivalent to the fact that $(\mathrm{id}_{\mathrm{S}^{n-1}(V)}\otimes\alpha)$ is invariant by post-composition with $\tau_{k,n}\tau_{k,n-1}\tau_n$. But this element is $\tau_{k,n-1}$, so it acts trivially on $\mathrm{S}^{n-1}(V)$. This gives (3). This being done, we obtain that

$$\text{“}m_\star\text{”} - \text{“}m_\star\text{”}\,(m_\star\otimes \mathrm{id}_V)\tau_n = \frac{1}{n}\pi\omega,$$

which is nothing but $m_\star\,(\mathrm{id}_{\mathrm{S}^{n-1}(V)}\otimes\alpha)$. This finishes the proof of the identity.

Property (D) is satisfied □

2.3.3. *Categorical PBW theorem.*

Theorem 2.11. *Let V be an object of $\mathcal{C}$ and α be an element of* $\mathrm{Hom}_{\mathcal{C}}(V,\Lambda^2V)$.

– *The system of equations*

$$\begin{cases} c_p^k \in \mathrm{Hom}_{\mathcal{C}}(\widetilde{\Lambda}^pV, \mathrm{S}^kV) \text{ for } 1\le k\le p-1\\ c_p^p = \pi_p : V^{\otimes p}\to \mathrm{S}^pV\\ c_p^k\circ\Psi_p = \{c_{p-1}^k\circ(\mathrm{id}_{V^{\otimes i-1}}\otimes\alpha\otimes \mathrm{id}_{V^{\otimes p-i-1}})\}_{1\le i\le p-1} \;\; \text{if } 1\le k\le p-1 \end{cases} \tag{4}$$

has a solution $(c_p^k)_{1\le k\le p}$ if and only if α is a Lie bracket on V. If it exists, this solution is unique.

[5]This map is nothing but the morphism $m_0\colon \mathrm{S}^n(V)\otimes V\to \mathrm{S}^{n+1}(V)$.

– *Given a Lie algebra object* (V,α) *in* C*, let* $(c_p^k)_{1\leq k\leq p}$ *be coefficients satisfying* (4)*. If we define a product* $m_\star$ *on* $\mathrm{S}(V)$ *by the formula*

$$(m_\star)_{|\mathrm{S}^pV\otimes\mathrm{S}^qV} := \left(\sum_{k=1}^{p+q} c_{p+q}^k\right)_{|\mathrm{S}^pV\otimes\mathrm{S}^qV},$$

then $m_\star$ *is associative and* $\mathcal{A} = (\mathrm{S}(V), m_\star)$ *is a universal enveloping algebra of* $(V, 2\alpha)$*. Besides, the PBW theorem holds.*

Proof. Assume that α is a Lie bracket. Theorem 2.9 gives an algebra $\mathcal{A}$ satisfying Assumptions (A1-A2) of §2.2.1. Hence Proposition 2.6 provides the existence of the c_p^k. Uniqueness is clear. Conversely, if the equations (4) are satisfied, then we have

$$c_3^1 \circ \Psi_3 = (\alpha\circ(\alpha\otimes\mathrm{id}), \alpha\circ(\mathrm{id}\otimes\alpha)).$$

Thanks to Proposition 2.3, α satisfies the Jacobi identity.

Now assume that α is a Lie bracket, and let $\mathcal{A}$ be an algebra satisfying Assumptions (A1-A2) of §2.2.1. Then we can transport the algebra structure on $\mathrm{S}(V)$ via the isomorphism Δ_+: we have

$$\begin{aligned}(m_\star)_{|\mathrm{S}^pV\otimes\mathrm{S}^qV} &= \Delta_+^{-1}\circ m\circ(\Delta_{|\mathrm{S}^pV}\otimes\Delta_{|\mathrm{S}^qV})\\ &= \Delta_+^{-1}\circ\Delta_{|\mathrm{S}^pV\otimes\mathrm{S}^qV}\\ &= \left(\sum_{k=1}^{p+q} c_{p+q}^k\right)_{|\mathrm{S}^pV\otimes\mathrm{S}^qV}.\end{aligned}$$

This gives the result. □

3. Distinguished elements in the universal enveloping algebra

3.1. **The derivative of the multiplication map.** We borrow the notation from the previous Section: (V,α) is a Lie algebra object in C and $m_\star$ is the associative product on $\mathrm{S}(V)$ from Proposition 2.11. Our aim is to give a closed formula for the restriction φ of $m_\star$ to $\mathrm{S}(V)\otimes V\subset\mathrm{S}(V)\otimes\mathrm{S}(V)$. We use the notation introduced in the beginning of §2.3.2, especially the morphism ω defined in (1).

We consider the morphism $\varpi_V : \mathrm{S}^nV\otimes V\to\mathrm{S}^{n-1}V\otimes\Lambda^2V\subset V^{\otimes n+1}$ defined as the restriction of $\omega_{\mathrm{T}(V)_{\mathrm{Lie}}}$ to $\mathrm{S}^nV\otimes V$. In other words, since $\tau_{i,n}=\mathrm{id}$ on $\mathrm{S}^nV\otimes V$,

$$\varpi_V = (\mathbf{1}-\tau_n)\circ\sum_{i=1}^n\tau_{i,n} = n\,(\mathbf{1}-\tau_n).$$

We often simply write ϖ as the choice of V is clear from the context.

Recall that the collection $(\mathfrak{S}_n)_{n\geq 0}$ is a simplicial group: for every non-decreasing map $f:\{1,\dots,n\}\to\{1,\dots,m\}$ we have the corresponding morphism $(-)^f:\mathfrak{S}_m\to\mathfrak{S}_n$ defined as follows: for any i in $\{1,\dots,m\}$, let us write $f^{-1}(i)=I_i$. Then σ_f is entirely characterized by the following properties:

$$\begin{cases}\sigma^f \text{ acts by translation on each block } I_k\\ \sigma^f(I_{\sigma^{-1}(1)})<\sigma^f(I_{\sigma^{-1}(2)})<\ldots<\sigma^f(I_{\sigma^{-1}(m)})\end{cases}$$

We denote by $f=(\mathbf{i})$ the only surjection $\{1,\dots,n+1\}\to\{1,\dots,n\}$ that "hits" i twice.

Theorem 3.1. *The map* $\varphi = m_{\star|\mathrm{S}(V)\otimes V}$ *is the composition of* $\dfrac{\omega}{1-\exp(-\omega)}$ *with the multiplication morphism* m_0 *of the symmetric algebra* $\mathrm{S}(V)$*.*

We first need an intermediate result of categorical linear algebra.

Lemma 3.2. *The restriction of* $\tau_{p+1,n+1}$ *to* $\mathrm{S}^nV\otimes V$ *followed by the projection* $V^{\otimes n+1}\to\widetilde{\Lambda}^{n+1}V$ *equals:*

$$\frac{1}{n(n+1)}\left(\sum_{i=1}^p i\tau_{i,n}^{(\mathbf{n})} - \sum_{j=1}^{n-p} j\tau_{n+1-j,n}^{(\mathbf{n})}\right)\varpi$$

Note that $\tau_{i,i} = \mathbf{1}$ by convention. Also remark that, if $p \in \{0, n\}$, then one of the two sums in the above expression is empty and thus vanishes.

Proof. First recall that the projection $V^{\otimes n+1} \to \widetilde{\Lambda}^{n+1} V$ is given by $\mathbf{1} - \pi_{n+1}$. Remark that the subgroup $\mathfrak{S}_n$ of $\mathfrak{S}_{n+1}$ corresponds to elements in $\mathfrak{S}_{n+1}$ fixing $p+1$ act trivially on $\tau_{p+1,n+1}(\mathrm{S}^n V \otimes V)$. Besides, $\mathfrak{S}_{n+1}/\mathfrak{S}_n$ consists of the $(n+1)$ left classes $(\tau_{i,p+1}\mathfrak{S}_n)_{1\le i\le n+1}$. Hence, restricted to $\tau_{p+1,n+1}(\mathrm{S}^n V \otimes V)$, we have the identity

$$\begin{aligned}
\mathbf{1} - \pi_{n+1} &= \mathbf{1} - \frac{1}{(n+1)!}\sum_{\sigma\in\mathfrak{S}_{n+1}} \sigma \\
&= \mathbf{1} - \frac{1}{(n+1)!}\sum_{\sigma\in\mathfrak{S}_{n+1}/\mathfrak{S}_n} n!\,\sigma \\
&= \mathbf{1} - \frac{1}{n+1}\sum_{i=1}^{n+1} \tau_{i,\,p+1}\,.
\end{aligned}$$

Then observe[6] that on $\mathrm{S}^n V \otimes V \subset V^{\otimes n+1}$, we have the following identities:

$$\tau^{(\mathbf{n})}_{i,n} = \tau^{(\mathbf{n})}_{i,n}\tau_{i,n} = \tau_{i+1,n+1} \qquad \text{and similarly} \qquad \tau^{(\mathbf{n})}_{i,n}\tau_n = \tau_{i,n+1}.$$

Using this, we compute:

$$\begin{aligned}
&\frac{1}{n}\left(\sum_{i=1}^{p} i\tau^{(\mathbf{n})}_{i,n} - \sum_{j=1}^{n-p} j\tau^{(\mathbf{n})}_{n+1-j,n}\right)\varpi \\
&= \sum_{i=1}^{p} i(\tau_{i+1,n+1} - \tau_{i,n+1}) - \sum_{j=1}^{n-p} j(\tau_{n+2-j,n+1} - \tau_{n+1-j,n+1}) \\
&= \sum_{i=1}^{p} i\tau_{i+1,n+1} - \sum_{i=0}^{p-1}(i+1)\tau_{i+1,n+1} - \sum_{j=0}^{n-p-1}(j+1)\tau_{n+1-j,n+1} + \sum_{j=1}^{n-p} j\tau_{n+1-j,n+1} \\
&= -\sum_{i=0}^{p-1}\tau_{i+1,n+1} + p\tau_{p+1,n+1} - \sum_{j=0}^{n-p-1}\tau_{n+1-j,n+1} + (n-p)\tau_{p+1,n+1} \\
&= -\sum_{\substack{i=0\\ i\neq p}}^{n}\tau_{i+1,n+1} + n\tau_{p+1,n+1} \\
&= -\sum_{i=0}^{n}\tau_{i+1,n+1} + (n+1)\tau_{p+1,n+1} \\
&= -\sum_{i=0}^{n}\tau_{i+1,\,p+1}\tau_{p+1,n+1} + (n+1)\tau_{p+1,n+1} \qquad (\text{since } \tau_{i+1,n+1} = \tau_{i+1,\,p+1}\tau_{p+1,n+1} \text{ on } \mathrm{S}^n V\otimes V)
\end{aligned}$$

which is indeed $(n+1)$ times the desired map. □

Corollary 3.3. *Let n, ℓ be non-negative integers with $0 \le \ell \le n-1$. For any $p \in \{1, \dots, n\}$, on $\mathrm{S}^{n-1}V \otimes V$ we have*

$$c_n^{n-\ell} \circ \tau_{p,n} = \lambda(n,\ell,p)\, m_0 \circ \omega^{\circ\ell}$$

where $\lambda(n,\ell,p)$ are rational numbers satisfying the induction relations

$$\lambda(n+1,\ell,p+1) = \frac{1}{n(n+1)}\left(\sum_{i=1}^{p} i\lambda(n,\ell-1,i) - \sum_{j=1}^{n-p} j\lambda(n,\ell-1,n+1-j)\right) \quad \textit{if } 1 \le p \le n-1$$

and $\lambda(n,0,p) = 1$.

[6] It suffices to check that these permutations agree on the value $n+1$.

Proof. We proceed by induction. The case $\ell = 0$ is easy: this amounts to observe that $c_n^n \circ \tau_{p,n} = \pi_n = m_0$ on $\mathrm{S}^{n-1}V \otimes V$. For the general case, on $\mathrm{S}^n V \otimes V$ we have

$$\begin{aligned} c_{n+1}^{n+1-\ell} \circ \tau_{p+1,n+1} &= \frac{1}{n(n+1)} c_{n+1}^{n+1-\ell} \circ \left(\sum_{i=1}^{p} i\tau_{i,n}^{(\mathbf{n})} - \sum_{j=1}^{n-p} j\tau_{n+1-j,n}^{(\mathbf{n})} \right) \varpi \\ &= \frac{1}{n(n+1)} c_n^{n-(\ell+1)} \circ \left(\sum_{i=1}^{p} i\tau_{i,n} - \sum_{j=1}^{n-p} j\tau_{n+1-j,n} \right) \omega \\ &= \frac{1}{n(n+1)} \left(\sum_{i=1}^{p} i\lambda(n,\ell-1,i) - \sum_{j=1}^{n-p} j\lambda(n,\ell-1,j) \right) m_0 \circ \omega^{\circ \ell}, \end{aligned}$$

where the first equality follows from Lemma 3.2, the second one from Corollary **??**, and the last one from the induction hypothesis. □

Example 3.4. We have

$$\begin{aligned} \lambda(n+1,1,p+1) &= \frac{1}{n(n+1)} \left(\sum_{i=1}^{p} i - \sum_{j=1}^{n-p} j \right) \\ &= \frac{1}{n(n+1)} \left(\frac{p(p+1)}{2} - \frac{(n-p)(n-p+1)}{2} \right) \\ &= -\frac{1}{2} + \frac{p}{n}. \end{aligned}$$

In particular, $\lambda(n,1,n) = \dfrac{1}{2}$.

In order to prove Theorem 3.1, it remains to show the following:

Lemma 3.5. *For* $0 \le \ell \le n$, $\lambda(n+1,\ell,n+1) = \dfrac{B_\ell}{\ell!}$.

Proof. We start by simplifying a little bit the induction relation given in Corollary 3.3. To do so, we claim that for $\ell \ge 1$, $\sum_{k=1}^{n} \lambda(n,\ell,k)$ vanishes. This follows from a direct calculation:

$$\begin{aligned} \sum_{k=1}^{n-1} \lambda(n+1,\ell,k+1) &= \frac{1}{n(n+1)} \sum_{k=1}^{n-1} \left(\sum_{i=1}^{k} i\lambda(n,\ell-1,i) - \sum_{j=1}^{n-k} j\lambda(n,\ell-1,n+1-j) \right) \\ &= \frac{1}{n(n+1)} \sum_{k=1}^{n-1} \left(\sum_{i=1}^{k} i\lambda(n,\ell-1,i) - \sum_{j=k+1}^{n} (n+1-j)\lambda(n,\ell-1,j) \right) \\ &= \frac{1}{n(n+1)} \left(\sum_{i=1}^{n-1} \sum_{k=i}^{n-1} i\lambda(n,\ell-1,i) - \sum_{j=2}^{n} \sum_{k=1}^{j-1} (n+1-j)\lambda(n,\ell-1,j) \right) \\ &= \frac{1}{n(n+1)} \left(\sum_{i=1}^{n-1} (n-i)i\lambda(n,\ell-1,i) - \sum_{j=2}^{n} (j-1)(n+1-j)\lambda(n,\ell-1,j) \right) \\ &= \frac{1}{n(n+1)} \left(\sum_{i=1}^{n-1} (n+1-2i)\lambda(n,\ell-1,i) - (n-1)\lambda(n,\ell-1,n) \right) \\ &= \frac{1}{n(n+1)} \left(-\sum_{i=1}^{n} i\lambda(n,\ell-1,i) + \sum_{i=1}^{n} (n+1-i)\lambda(n,\ell-1,i) \right) \\ &= -\lambda(n+1,\ell,n+1) - \lambda(n+1,\ell,1). \end{aligned}$$

Hence for $\ell \geq 2$, we see that

$$\begin{aligned}\lambda(n+1,\ell,1) &= -\frac{1}{n(n+1)}\sum_{j=1}^{n} j\lambda(n,\ell-1,n+1-j)\\ &= -\frac{1}{n(n+1)}\sum_{i=1}^{n}(n+1-i)\lambda(n,\ell-1,i)\\ &= \frac{1}{n(n+1)}\sum_{i=1}^{n}\lambda(n,\ell-1,i)\\ &= \lambda(n+1,\ell,n+1).\end{aligned}$$

Besides,

$$\begin{aligned}\lambda(n+1,\ell,p+1) &= \frac{1}{n(n+1)}\left(\sum_{i=1}^{p} i\lambda(n,\ell-1,i) - \sum_{j=1}^{n-p} j\lambda(n,\ell-1,n+1-j)\right)\\ &= \frac{1}{n(n+1)}\left(\sum_{i=1}^{p} i\lambda(n,\ell-1,i) - \sum_{j=p+1}^{n}(n+1-j)\lambda(n,\ell-1,j)\right)\\ &= \frac{1}{n(n+1)}\sum_{i=1}^{n} i\lambda(n,\ell-1,i) - \frac{1}{n}\sum_{j=p+1}^{n}\lambda(n,\ell-1,n+1-j)\\ &= \lambda(n+1,\ell,1) + \frac{1}{n}\sum_{i=1}^{p}\lambda(n,\ell-1,i).\end{aligned}$$

Remark that this relation is also valid for $\ell = 1$.

Let us now show by induction on ℓ that $\lambda(n,\ell,n+1)$ doesn't depend on n, and equals to $\frac{B_\ell}{\ell!}\cdot$ We have already seen that it is true for $\ell = 1$. We now assume that the induction hypothesis is true at rank ℓ. For any i, we put $c_i = \frac{B_i}{i!}$ if $i \geq 2$ and $c_1 = -\frac{1}{2}\cdot$ We define modified power sums $T_k(n)$ as follows:

$$T_k(n) = \#\{(i_1,\ldots,i_k) \text{ such that } 1 \leq i_k < i_{k-1} < \ldots < i_1 \leq n\}$$

Then $T_k(n)$ is the number of subsets of cardinality k in $\{1,\ldots,n\}$, which is $\binom{n}{k}$. We claim that for $1 \leq s \leq \ell-1$ and $2 \leq p \leq n$,

$$\begin{aligned}\lambda(n+1,\ell,p+1) = \lambda(n+1,\ell,1) &+ \sum_{k=1}^{s}\frac{\{T_{k-1}(p-1)+T_k(p-1)\}\,c_{\ell-k}}{n(n-1)\ldots(n-k+1)}\\ &+ \frac{1}{n(n-1)\ldots(n-s)}\sum_{1\leq i_s\leq i_{s-1}<i_{s-2}<\ldots<i_2<i_1\leq p-1}\lambda(n-s,\ell-s-1,i_s)\end{aligned}$$

The proof of the claim is by induction on s. For $s = 1$, we have

$$\begin{aligned}\lambda(n+1,\ell,p+1) &= \lambda(n+1,\ell,1) + \frac{1}{n}\sum_{i=1}^{p}\lambda(n,\ell-1,i)\\ &= \lambda(n+1,\ell,1) + \frac{c_{\ell-1}}{n} + \frac{1}{n}\sum_{i=1}^{p-1}\lambda(n,\ell-1,i+1)\\ &= \lambda(n+1,\ell,1) + \frac{(1+T_1(p-1))c_{\ell-1}}{n} + \frac{1}{n(n-1)}\sum_{i=1}^{p-1}\sum_{j=1}^{i}\lambda(n-1,\ell-2,j)\end{aligned}$$

The induction step follows from the fact that

$$\begin{aligned}
&\sum_{1\le i_s\le i_{s-1}<i_{s-2}<\ldots<i_2<i_1\le p-1} \lambda(n-s,\ell-s-1,i_s)\\
&= \sum_{0\le i_s<i_{s-1}<i_{s-2}<\ldots<i_2<i_1\le p-1} \lambda(n-s,\ell-s-1,i_s+1)\\
&= \sum_{1\le i_{s-1}<i_{s-2}<\ldots<i_2<i_1\le p-1} c_{\ell-s-1,1} + \sum_{1\le i_s<i_{s-1}<i_{s-2}<\ldots<i_2<i_1\le p-1} \lambda(n-s,\ell-s-1,i_s+1)\\
&= T_{s-1}(p-1)c_{\ell-s-1}\\
&\qquad + \sum_{1\le i_s<i_{s-1}<i_{s-2}<\ldots<i_2<i_1\le p-1} \left(c_{\ell-s-1} + \frac{1}{n-s-1}\sum_{i_{s+1}=1}^{i_s} \lambda(n-s-1,\ell-s-2,i_{s+1})\right)\\
&= (T_{s-1}(p-1)+T_s(p-1))c_{\ell-s-1}\\
&\qquad + \frac{1}{n-s-1}\sum_{1\le i_{s+1}\le i_s<i_{s-1}<i_{s-2}<\ldots<i_2<i_1\le p-1} \lambda(n-s-1,\ell-s-2,i_{s+1})
\end{aligned}$$

We take $s=\ell-1$. We have

$$T_{k-1}(p-1)+T_k(p-1) = \binom{p-1}{k-1}+\binom{p-1}{k} = \binom{p}{k}$$

Hence

$$\lambda(n+1,\ell,p+1) = \lambda(n+1,\ell,1) + \sum_{k=1}^{\ell} \frac{\binom{p}{k}c_{\ell-k}}{n(n-1)\ldots(n-k+1)}$$

Now we have the formula

$$\sum_{p=0}^{n}\binom{p}{k} = \frac{(n+1)(n)(n-1)\ldots(n-k+1)}{(k+1)!}$$

so that

$$\sum_{p=0}^{n}\lambda(n+1,\ell,p+1) = (n+1)\lambda(n+1,\ell,1) + \sum_{k=1}^{\ell}\frac{(n+1)(n)(n-1)\ldots(n-k+1)}{(k+1)!n(n-1)\ldots(n-k+1)}c_{\ell-k}$$

and therefore $\lambda(n+1,\ell,1) = -\sum_{k=1}^{\ell}\frac{c_{\ell-k}}{(k+1)!}$. This proves that $\lambda(n+1,\ell,1)$ is independant of n. To conclude, it suffices to prove the relation

$$\sum_{k=0}^{\ell}\frac{c_{\ell-k}}{(k+1)!} = 0$$

which is straightforward, since this coefficient is the coefficient in x^ℓ in the power series

$$\left(\sum_{n=0}^{+\infty} c_n x^n\right)\times\frac{e^x-1}{x},$$

which is x. □

3.1.1. *Linear algebra computations.* All along this Subsection, we assume that V is a dualizable object in $\mathcal{C}$ and we denote by V^* its dual[7]: in particular, we have a coevaluation map $\epsilon : \mathbf{1} \to V^*\otimes V$ and an evaluation map $\delta : V\otimes V^* \to \mathbf{1}$ that satisfy the "snake" identity

$$(\delta\otimes \mathrm{id}_V)\circ(\mathrm{id}_V\otimes\epsilon) = \mathrm{id}_V\,.$$

[7]The category being symmetric monoidal, every left dual is a right dual as well, so that we allow ourselves to simply speak about duals.

– One shows easily that the restriction of $\mathrm{id}_{V^{\otimes n-1}} \otimes \delta$ to $V^{\otimes n} \otimes V^*$ to $\mathrm{S}^n V \otimes V^*$ factors through $\mathrm{S}^{n-1}V$. As a consequence the direct factor, inclusion $\mathrm{S}^n V \to \mathrm{S}^{n-1}V \otimes V$ can be re-written as the restriction of

$$(\mathrm{id}_{V^{\otimes n-1}} \otimes \delta \otimes \mathrm{id}_V) \circ (\mathrm{id}_{V^{\otimes n}} \otimes \epsilon)\,.$$

to $\mathrm{S}^n V$. Therefore[8] $\omega : \mathrm{S}^n V \otimes V \to \mathrm{S}^{n-1}V \otimes V$ equals n times

$$(\mathrm{id}_{V^{\otimes n-1}} \otimes (\mu \circ (\delta \otimes \mathrm{id}_{V^{\otimes 2}}))) \circ (\mathrm{id}_{V^{\otimes n}} \otimes \epsilon \otimes \mathrm{id}_V)\,,$$

which also equals

$$(\mathrm{id}_{V^{\otimes n-1}} \otimes ((\delta \otimes \mathrm{id}_V) \circ (\mathrm{id}_{V \otimes V^*} \otimes \mu))) \circ (\mathrm{id}_{V^{\otimes n}} \otimes \epsilon \otimes \mathrm{id}_V)\,.$$

– We have an adjunction between the functor $V \otimes -$ and the functor $V^* \otimes -$: for any two objects Y, Z in C,

$$\mathrm{Hom}_C(V \otimes Y, Z) \cong \mathrm{Hom}_C(Y, V^* \otimes Z)\,,$$

where the bijection is given by sending $\phi \in \mathrm{Hom}_C(V \otimes Y, Z)$ to $\phi^* := (\mathrm{id}_{V^*} \otimes \phi) \circ (\epsilon \otimes \mathrm{id}_Y)$. The inverse bijection sends $\psi \in \mathrm{Hom}_C(Y, V^* \otimes Z)$ to $(\delta \otimes \mathrm{id}_Z) \circ (\mathrm{id}_V \otimes \psi)$. For instance, the element μ^* in $\mathrm{Hom}_C(V, V^* \otimes V)$ is understood as the adjoint action, and we have

$$\omega = n \times \mathrm{id}_{V^{\otimes n-1}} \otimes ((\delta \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes \mu^*))\,. \tag{5}$$

– We also have an adjunction in the reverse way between the functor $- \otimes V$ and the functor $- \otimes V^*$:

$$\mathrm{Hom}_C(Y \otimes V^*, Z) \cong \mathrm{Hom}_C(Y, Z \otimes V)\,,$$

where the bijection is given by sending $\phi \in \mathrm{Hom}_C(Y \otimes V^*, Z)$ to $\phi^* := (\phi \otimes \mathrm{id}_V) \circ (\mathrm{id}_Y \otimes \epsilon)$.

– Taking into account that $\mathrm{S}^p(V^*)$ is canonically isomorphic to $\mathrm{S}^p(V)^*$, we get a canonical "contraction map" element $\mathfrak{c}_p \in \mathrm{Hom}_C(\mathrm{S}^n V \otimes \mathrm{S}^p V^*, \mathrm{S}^{n-p}V)$ given as $\dfrac{n!}{(n-p)!}$ times the adjoint to the direct factor inclusion $\mathrm{S}^n V \hookrightarrow \mathrm{S}^{n-p}V \otimes \mathrm{S}^p V$. More explicitly, on $\mathrm{S}^n V \otimes \mathrm{S}^p V^*$

$$\mathfrak{c}_p = \frac{n!}{(n-p)!} \times (\mathrm{id}_{V^{\otimes n-p}} \otimes \delta) \circ \cdots \circ (\mathrm{id}_{V^{\otimes n-1}} \otimes \delta \otimes \mathrm{id}_{(V^*)^{\otimes p-1}})\,.$$

– Notice that $\mathfrak{c}_p = \mathfrak{c}_1 \circ (\mathfrak{c}_1 \otimes \mathrm{id}_{V^*}) \circ \cdots \circ (\mathfrak{c}_1 \otimes \mathrm{id}_{(V^*)^{\otimes p-1}})$. One the other hand, using (5), we obtain that ω equals $(\mathfrak{c}_1 \otimes \mathrm{id}_V) \circ (\mathrm{id}_{V^{\otimes n}} \otimes \mu^*)$ and thus

$$\omega^{\circ p} = (\mathfrak{c}_1 \otimes \mathrm{id}_V) \circ (\mathrm{id}_{V^{\otimes n-p+1}} \otimes \mu^*) \circ \cdots \circ (\mathfrak{c}_1 \otimes \mathrm{id}_V) \circ (\mathrm{id}_{V^{\otimes n}} \otimes \mu^*)\,. \tag{6}$$

We now introduce a convenient notation. Let X, Y, Z be three objects in C and let (A, m_A) be an associative algebra object in C. We then have a $\mathbf{k}$-linear associative composition product

$$- \bullet - : \mathrm{Hom}_C(Y, A \otimes Z) \times \mathrm{Hom}_C(X, A \otimes Y) \to \mathrm{Hom}_C(X, A \otimes Z)$$

defined as follows: for every $\phi : Y \to A \otimes Z$ and every $\psi : X \to A \otimes Y$ we set

$$\phi \bullet \psi := (m_A \otimes \mathrm{id}_Z) \circ (\mathrm{id}_A \otimes \phi) \circ \psi\,.$$

In particular, if $X = Y = Z$ then we get that $\bullet$ turns $\mathrm{Hom}_C(Y, A \otimes Y)$ into a $\mathbf{k}$-linear associative algebra. We are interested in the case $(A, m_A) = (\mathrm{S}(V^*), m_0)$.

Lemma 3.6. *Let $\phi \in \mathrm{Hom}_C(Y, \mathrm{S}^p V^* \otimes Z)$ and $\psi \in \mathrm{Hom}_C(X, \mathrm{S}^q V^* \otimes Y)$. Then*

$$(\mathfrak{c}_p \otimes \mathrm{id}_Z) \circ (\mathfrak{c}_q \otimes \phi) \circ (\mathrm{id}_{\mathrm{S}^n V} \otimes \psi) = (\mathfrak{c}_{p+q} \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{\mathrm{S}^n V} \otimes (\phi \bullet \psi))\,,$$

as morphisms from $\mathrm{S}^n V \otimes X$ to $\mathrm{S}^{n-p-q}V \otimes Z$.

[8]The morphism ω has been defined at the beginning of §3.1.

Proof. First of all, in view of the expression for $\mathfrak{c}_p$ in terms of $\mathfrak{c}_1$'s, it is sufficient to prove the Lemma for $p = 1$. Then, in view of the definition of the product $\bullet$, we have that the r.h.s.

$$(\mathfrak{c}_{1+q} \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{\mathrm{S}^n V} \otimes (\phi \bullet \psi))$$

equals

$$(\mathfrak{c}_{1+q} \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{\mathrm{S}^n V} \otimes m_0 \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{\mathrm{S}^n V \otimes \mathrm{S}^q V^*} \otimes \phi) \circ (\mathrm{id}_{\mathrm{S}^n V} \otimes \psi)\,.$$

Hence it is sufficient to show the following identity

$$(\mathfrak{c}_1 \otimes \mathrm{id}_Z) \circ (\mathfrak{c}_q \otimes \phi) = (\mathfrak{c}_{1+q} \otimes \mathrm{id}_{V^* \otimes Z}) \circ (\mathrm{id}_{\mathrm{S}^n V} \otimes m_0 \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{\mathrm{S}^n V \otimes \mathrm{S}^q V^*} \otimes \phi)$$

in $\mathrm{Hom}_C(\mathrm{S}^n V \otimes \mathrm{S}^q V^* \otimes Y, \mathrm{S}^{n-q} V \otimes Z)$. Then observe that, from the very definition $\mathfrak{c}_{1+q}$, we have that

$$\mathfrak{c}_{1+q} \circ (\mathrm{id}_{\mathrm{S}^n V} \otimes m_0) = \mathfrak{c}_1 \circ (\mathfrak{c}_q \otimes \mathrm{id}_{V^*})$$

in $\mathrm{Hom}_C(\mathrm{S}^n V \otimes \mathrm{S}^q V^* \otimes V^*, \mathrm{S}^{n-q} V)$. As a consequence, it is sufficient to have that

$$\mathfrak{c}_q \otimes \phi = (\mathfrak{c}_q \otimes \mathrm{id}_{V^* \otimes Z}) \circ (\mathrm{id}_{\mathrm{S}^n V \otimes \mathrm{S}^q V^*} \otimes \phi)\,,$$

which is obvious. □

Corollary 3.7. *If we consider μ^* in $\mathrm{Hom}_C(V, V^* \otimes V)$, we have $\omega^{\circ p} = (\mathfrak{c}_p \otimes \mathrm{id}_V) \circ (\mathrm{id}_{\mathrm{S}^n V} \circ (\mu^*)^{\bullet p})$.*

Proof. Setting $Y = V$, we get a family of maps $(\mu^*)^{\bullet p} \in \mathrm{Hom}_C(V, \mathrm{S}^p V^* \otimes V)$. Lemma 3.6 gives the result. □

Lemma 3.8. *For every $p \geq 1$, $(\mu^*)^{\bullet p} \bullet \epsilon = 0$.*

Proof. First observe that it is sufficient to prove it for $p = 1$ (using the associativity of $\bullet$). Then note that since $\mu \colon \Lambda^2 V \to V$ then

$$(\mathrm{id}_{V^*} \otimes \mu^*) \circ \epsilon = (\mu^*)^*$$

lies in $\mathrm{Hom}_C(\mathbf{1}, \wedge^2 V^* \otimes V)$. Therefore

$$\mu^* \bullet \epsilon = (\tau_1 \otimes \mathrm{id}_V) \circ (\mu^* \bullet \epsilon) = \big((m_0 \circ \tau_1) \otimes \mathrm{id}_V\big) \otimes (\mu^*)^* = -(m_0 \otimes \mathrm{id}_V) \otimes (\mu^*)^* = -\mu^* \bullet \epsilon\,.$$

Therefore $\mu^* \bullet \epsilon = 0$, and we are done. □

3.1.2. *The trace identity.* For every two objects Y, Z in C one has a linear map

$$\mathrm{Tr} : \mathrm{Hom}_C(Y \otimes V, Z \otimes V) \to \mathrm{Hom}_C(Y, Z)$$

defined as follows:

$$\mathrm{Tr}(\phi) = (\mathrm{id}_Z \otimes \delta) \circ (\phi \otimes \mathrm{id}_{V^*}) \circ (\mathrm{id}_Y \otimes \bar{\epsilon})\,,$$

where $\bar{\epsilon}$ is given by ϵ followed by the symmetry morphism $V^* \otimes V \to V \otimes V^*$.

Proposition 3.9. *For every p, we have that*

$$\mathfrak{c}_1 \circ \big((m_0 \circ \omega^{\circ p}) \otimes \mathrm{id}_{V^*}\big) \circ (\mathrm{id}_{\mathrm{S}^n V} \otimes \bar{\epsilon}) = \mathfrak{c}_p \circ (\mathrm{id}_{\mathrm{S}^n V} \otimes \mathrm{Tr}((\mu^*)^{\bullet p}))\,.$$

Proof. Using the fact that $\mathfrak{c}_1$ defines an action of V^* on $\mathrm{S}(V)$ by derivations, we get that the r.h.s. is

$$m_0 \circ \big((\mathfrak{c}_1 \otimes \mathrm{id}_V) \circ (\omega^{\circ p})^* + \mathrm{Tr}(\omega^{\circ p})\big)\,.$$

Then observe that we have

$$m_0 \circ (\mathfrak{c}_1 \otimes \mathrm{id}_V) \circ (\omega^{\circ p})^* = (\mathfrak{c}_{p+1} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes (\mu^*)^{\bullet p} \bullet \epsilon) = 0\,.$$

Finally,

$$m_0 \circ \mathrm{Tr}(\omega^{\circ p}) = \mathrm{Tr}(\omega^{\circ p}) = \mathfrak{c}_p \circ (\mathrm{id}_{\mathrm{S}^n V} \otimes \mathrm{Tr}((\mu^*)^{\bullet p}))\,.$$

We are done. □

3.2. **The Lie-theoretic cycle class.**

3.2.1. *The Duflo element.* We borrow the notation from the previous paragraphs and introduce the *Duflo element* $\mathfrak{d} = \sum_{n=0}^{+\infty} \mathfrak{d}_n \in \prod_n \mathrm{Hom}_C(\mathbf{1}, \mathrm{S}^n V^*)$:

$$\mathfrak{d} := \det\left(\frac{\mu^*}{1-\exp(-\mu^*)}\right).$$

This has to be understood as a formal expression in terms of the "invariant polynomials" $\nu_k := \mathrm{Tr}\left((\mu^*)^{\bullet k}\right)$ that live in $\mathrm{Hom}_C(\mathbf{1}, \mathrm{S}^k V^*)$. For instance,

$$\mathfrak{d} = 1 + \frac{\nu_1}{2} + \left(\frac{3\nu_1^{\bullet 2} - \nu_2}{24}\right) + \frac{\nu_1^{\bullet 3} - \nu_1 \bullet \nu_2}{48} + \cdots$$

More formally, for any $N \geq 0$, we write formally

$$\prod_{n=1}^{N} \frac{x_i}{1-\exp(-x_i)} = \sum_{i \geq 0} P_i(y_1, y_2, \ldots, y_i)$$

where $y_i = \sum_{n=0}^{N} x_n^i$, and P_i is a polynomial independant from N, and of total degree i if each variable y_k has degree k. For instance,

$$\begin{cases} P_0 = 1 \\ P_1(y_1) = \dfrac{y_1}{2} \\ P_2(y_1, y_2) = \dfrac{3y_1^2 - y_2}{24} \\ P_3(y_1, y_2, y_3) = \dfrac{y_1^3 - y_1 y_2}{48} \end{cases}$$

Then for any $p \geq 0$, we have

$$\mathfrak{d}_p = P_p(\nu_1, \ldots, \nu_p).$$

3.2.2. *Torsion morphisms.* Let $\ell \in \mathbb{N}$ and let a be a morphism from an arbitrary object X to $\mathrm{S}^{\leq \ell} V$ and $(a_n)_{0 \leq n \leq \ell}$ be the graded components of a.

Definition 3.10. We say that such a morphism a is an ℓ-torsion morphism if $m_\star \circ (a \otimes \mathrm{id}_V)$ factors through $\mathrm{S}^{\ell+1} V \subset \mathrm{S}^{\leq \ell+1} V$.

Our main result is:

Theorem 3.11. *If a is an ℓ-torsion morphism, then $a = \mathfrak{c}(\mathfrak{d} \otimes a_\ell)$, where $\mathfrak{c}(\mathfrak{d} \otimes -)$ means $\sum_p \mathfrak{c}_p(\mathfrak{d}_p \otimes -)$, the sum being in fact automatically finite.*

Remark that this theorem tells nothing about the *existence* of ℓ-torsion morphisms.

Proof. Thanks to Theorem 3.1, a is an ℓ-torsion morphism if and only if the system of equations

$$\sum_{i=0}^{\ell} \frac{\mathrm{B}_i}{i!} \left\{m_0 \circ \omega^{\circ i}\right\} (a_{\ell-k+i} \otimes \mathrm{id}_V) = 0 \tag{7}$$

holds for $1 \leq k \leq \ell$. Each condition corresponds to the vanishing of the $\ell - k + 1^{\text{th}}$ graded piece of the element $m_\star \circ (a \otimes \mathrm{id}_V)$. Thanks to Proposition 3.9, we get

$$k a_{\ell-k} + \sum_{i=1}^{k} (-1)^i \frac{\mathrm{B}_i}{i!} \times \mathfrak{c}_i(a_{\ell-k+i} \otimes \nu_i) = 0.$$

Let us explain how it works on the first terms.

– For $k = 1$, the first equation is

$$a_{\ell-1} - \frac{1}{2}\mathfrak{c}_1(a_\ell \otimes \nu_1) = 0,$$

so

$$a_{\ell-1} = \mathfrak{c}_1\left(a_\ell \otimes \frac{\nu_1}{2}\right).$$

– For $k = 2$, the second equation is

$$2a_{\ell-2} - \frac{1}{2}\mathfrak{c}_1(a_{\ell-1} \otimes \nu_1) + \frac{1}{12}\mathfrak{c}_2(a_\ell \otimes \nu_2) = 0$$

and we get

$$\begin{aligned} a_{\ell-2} &= \frac{1}{4}\mathfrak{c}_1(a_{\ell-1} \otimes \nu_1) - \frac{1}{24}\mathfrak{c}_2(a_\ell \otimes \nu_2) \\ &= \frac{1}{8}\mathfrak{c}_2(a_\ell \otimes \nu_1^{\bullet 2}) - \frac{1}{24}\mathfrak{c}_2(a_\ell \otimes \nu_2) \\ &= \mathfrak{c}_2\left(a_\ell \otimes \frac{3\nu_1^{\bullet 2} - \nu_2}{24}\right). \end{aligned}$$

– For $k = 3$, the third equation is

$$3a_{\ell-3} - \frac{1}{2}\mathfrak{c}_1(a_{\ell-2} \otimes \nu_1) + \frac{1}{12}\mathfrak{c}_2(a_{\ell-1} \otimes \nu_2) = 0$$

Hence

$$\begin{aligned} a_{\ell-3} &= \frac{1}{3}\left(\frac{1}{2}\mathfrak{c}_1\left(\mathfrak{c}_2\left(\frac{1}{8}a_\ell \otimes \nu_1^{\bullet 2} - \frac{1}{24}a_r \otimes \nu_2\right) \otimes \nu_1\right) - \frac{1}{12}\mathfrak{c}_2\left(\mathfrak{c}_1\left(\frac{1}{2}a_\ell \otimes \nu_1\right)\right) \otimes \nu_2\right) \\ &= \mathfrak{c}_3\left(a_\ell \otimes \frac{\nu_1^{\bullet 3} - \nu_1 \bullet \nu_2}{48}\right) \end{aligned}$$

To conclude, it suffices to prove the induction relation

$$kP_k(y_1 \ldots, y_k) + \sum_{i=1}^{k}(-1)^i \frac{\mathrm{B}_i}{i!} \times y_i P_{k-i}(y_1 \ldots, y_{k-i}) = 0.$$

involving the polynomials P_i. For this we fix the variables $x_1, \ldots, x_k$, and put $y_i = \sum_{n=0}^{k} x_n^i$. Since $y_0 = k$, the identity is equivalent to

$$\sum_{i=0}^{k}(-1)^i \frac{\mathrm{B}_i}{i!} \times y_i P_{k-i}(y_1 \ldots, y_{k-i}) = 0$$

The left-hand side is the homogeneous term of degree k (in the variables x_i) in the product

$$\sum_{i\geq 0}(-1)^i \frac{\mathrm{B}_i}{i!} y_i \times \sum_{i\geq 0} P_i(y_1, \ldots, y_i),$$

which is

$$\sum_{p=1}^{k} \frac{x_p}{\exp(x_p) - 1} \times \prod_{q=1}^{k} \frac{x_q}{1 - \exp(-x_q)}$$

Let $\phi(x) = \dfrac{x}{1 - \exp(-x)}\cdot$ Then $\phi(-x)\phi(x) = \phi(x) - x\phi'(x)$ so that for any p with $1 \leq p \leq k$, the homogeneous coefficient of degree k in $\dfrac{x_p}{\exp(x_p) - 1} \times \prod_{q=1}^{k} \dfrac{x_q}{1 - \exp(-x_q)}$ is

$$\sum_{\alpha_1+\ldots+\alpha_k=k} \frac{B_{\alpha_1}}{\alpha_1!} \times \ldots \times \frac{B_{\alpha_{p-1}}}{\alpha_{p-1}!} \times \frac{B_{\alpha_p}(1-\alpha_p)}{\alpha_p!} \times \frac{B_{\alpha_{p+1}}}{\alpha_{p+1}!} \times \ldots \times \frac{B_{\alpha_k}}{\alpha_k!}\cdot$$

Taking the sum in p, we get

$$\sum_{\alpha_1+\ldots+\alpha_k=k}\left(\frac{B_{\alpha_1}}{\alpha_1!}\times\ldots\times\frac{B_{\alpha_k}}{\alpha_k!}\times\sum_{p=1}^{k}(1-\alpha_p)\right)$$

which is zero. □

3.3. **Tameness for pairs of Lie algebras.**

3.3.1. *Setting.* Assume to be given a triplet $(\mathfrak{g},\mathfrak{h},\mathfrak{n})$, where $\mathfrak{g}$ is a Lie algebra, $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, and $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{n}$ as $\mathfrak{h}$-module. This makes perfect sense in any abstract $\mathbf{k}$-linear symmetric monoidal category C. We denote by $\mu_{\mathfrak{g}}$ and $\mu_{\mathfrak{h}}$ the Lie brackets on $\mathfrak{g}$ and $\mathfrak{h}$ respectively; and by $\pi_{\mathfrak{h}}$ and $\pi_{\mathfrak{n}}$ the two projections from $\mathfrak{g}$ to $\mathfrak{h}$ and $\mathfrak{n}$ respectively. We also define α and β in $\mathrm{Hom}_C(\mathfrak{n}^{\otimes 2},\mathfrak{n})$ and $\mathrm{Hom}_C(\mathfrak{n}^{\otimes 2},\mathfrak{h})$ respectively by the formulas $\alpha=\pi_{\mathfrak{n}}\circ\mu_{\mathfrak{g}|\mathfrak{n}^{\otimes 2}}$ and $\beta=\pi_{\mathfrak{h}}\circ\mu_{\mathfrak{g}|\mathfrak{n}^{\otimes 2}}$.

Definition 3.12. The triplet $(\mathfrak{g},\mathfrak{h},\mathfrak{n})$ is called *tame* if the morphism

$$\mu_{\mathfrak{g}}\circ(\beta\otimes\mathrm{id}_{\mathfrak{n}})\colon\mathfrak{n}^{\otimes 3}\to\mathfrak{g}$$

vanishes.

Lemma 3.13. *Given a tame triplet* $(\mathfrak{g},\mathfrak{h},\mathfrak{n})$*, the morphism* α *defines a Lie structure on* $\mathfrak{n}$*. Besides,* $\mathfrak{n}$ *becomes a Lie object in the symmetric monoidal category of* $\mathfrak{h}$*-modules.*

Proof. We claim that we have $\pi_{\mathfrak{n}}\circ(\mu_{\mathfrak{g}}\circ(\mu_{\mathfrak{g}|\mathfrak{n}^{\otimes 2}}\otimes\mathrm{id}_{\mathfrak{n}}))=\alpha\circ(\alpha\otimes\mathrm{id}_{\mathfrak{n}})$. Indeed,

$$\begin{aligned}\pi_{\mathfrak{n}}\circ\mu_{\mathfrak{g}}\circ(\mu_{\mathfrak{g}|\mathfrak{n}^{\otimes 2}}\otimes\mathrm{id}_{\mathfrak{n}})&=\pi_{\mathfrak{n}}\circ\mu_{\mathfrak{g}}\circ(\alpha\otimes\mathrm{id}_{\mathfrak{n}})+\pi_{\mathfrak{n}}\circ\underbrace{\mu_{\mathfrak{g}}\circ(\beta\otimes\mathrm{id}_{\mathfrak{n}})}_{0}\\&=\pi_{\mathfrak{n}}\circ\mu_{\mathfrak{g}}\circ(\alpha\otimes\mathrm{id}_{\mathfrak{n}})\\&=\alpha\circ(\alpha\otimes\mathrm{id}_{\mathfrak{n}})\,.\end{aligned}$$

Using this, the Jacobi identity for $(\mathfrak{g},\mu_{\mathfrak{g}})$ restricted to $\mathfrak{n}^{\otimes 3}$ at the source, and projected to $\mathfrak{n}$ at the target, gives the Jacobi identity for $(\mathfrak{n},\alpha)$. □

3.3.2. *The envelopping algebra* $\mathrm{U}(\mathfrak{n})$. Since $\mathfrak{n}$ is a Lie algebra object in the category of $\mathfrak{h}$-modules in C, the algebra object $\mathrm{U}(\mathfrak{n})$ is naturally endowed with an action by derivation of $\mathfrak{h}$. This action is simply induced by the adjoint action of $\mathfrak{h}$ on the tensor algebra $\mathrm{T}(\mathfrak{n})$. We define a morphism $\mathfrak{g}\otimes\mathrm{U}(\mathfrak{n})\to\mathrm{U}(\mathfrak{n})$ componentwise as follows:

- The morphism $p\colon\mathfrak{n}\otimes\mathrm{U}(\mathfrak{n})\to\mathrm{U}(\mathfrak{n})$ is the multiplication in $\mathrm{U}(\mathfrak{n})$.
- The morphism $q\colon\mathfrak{h}\otimes\mathrm{U}(\mathfrak{n})\to\mathrm{U}(\mathfrak{n})$ is the action of $\mathfrak{h}$ on $\mathrm{U}(\mathfrak{n})$.

Lemma 3.14. *Given a tame triplet* $(\mathfrak{g},\mathfrak{h},\mathfrak{n})$*, the above morphism endows* $\mathrm{U}(\mathfrak{n})$ *with a* $\mathfrak{g}$*-module structure.*

Proof. For any elements x and y in C, be denote by τ the symmetry isomorphism $x\otimes y\xrightarrow{\sim}y\otimes x$. We check componentwise (that is on $\mathfrak{n}\otimes\mathfrak{n}$, $\mathfrak{n}\otimes\mathfrak{h}$ and on $\mathfrak{h}\otimes\mathfrak{h}$) that the map (p,q) defines a $\mathfrak{g}$-action.

- We have on $\mathfrak{n}\otimes\mathfrak{n}\otimes\mathrm{U}(\mathfrak{n})$
$$\underbrace{p\circ(\mathrm{id}_{\mathfrak{n}}\otimes p)-p\circ(\mathrm{id}_{\mathfrak{n}}\otimes p)\circ(\tau\otimes\mathrm{id}_{\mathrm{U}(\mathfrak{n})})-p\circ(\alpha\circ\mathrm{id}_{\mathrm{U}(\mathfrak{n})})}_{0\text{ since }\mathfrak{n}\text{ acts on }\mathrm{U}(\mathfrak{n})}-\underbrace{q\circ(\beta\circ\mathrm{id}_{\mathrm{U}(\mathfrak{n})})}_{0\text{ by tameness}}=0\,.$$
- Since $\mathfrak{h}$ acts by derivation on $\mathrm{U}(\mathfrak{n})$, we have on $\mathfrak{n}\otimes\mathfrak{h}\otimes\mathrm{U}(\mathfrak{n})$ the equality
$$q\circ((\mathrm{id}_{\mathfrak{h}}\otimes p)\circ(\tau\otimes\mathrm{id}_{\mathrm{U}(\mathfrak{n})}))=q\circ(\mu_{\mathfrak{g}}\circ\mathrm{id}_{\mathrm{U}(\mathfrak{n})})+p\circ(\mathrm{id}_{\mathfrak{n}}\otimes q),$$
that is
$$p\circ(\mathrm{id}_{\mathfrak{n}}\otimes q)-q\circ(\mathrm{id}_{\mathfrak{h}}\otimes p)\circ(\tau\otimes\mathrm{id}_{\mathrm{U}(\mathfrak{n})})-q\circ(\mu_{\mathfrak{g}}\circ\mathrm{id}_{\mathrm{U}(\mathfrak{n})})=0\,.$$
- Lastly, on $\mathfrak{h}\otimes\mathfrak{h}\otimes\mathrm{U}(\mathfrak{n})$, we have
$$q\circ(\mathrm{id}_{\mathfrak{h}}\otimes q)-q\circ(\mathrm{id}_{\mathfrak{h}}\otimes q)\circ(\tau\otimes\mathrm{id}_{\mathrm{U}(\mathfrak{n})})-q\circ(\mu_{\mathfrak{g}}\circ\mathrm{id}_{\mathrm{U}(\mathfrak{n})})=0$$
since $\mathfrak{h}$ acts on $\mathrm{U}(\mathfrak{n})$.

□

3.3.3. *The induced representation.* The aim of this section is to prove the following theorem:

Theorem 3.15. *Given a tame triplet* $(\mathfrak{g}, \mathfrak{h}, \mathfrak{n})$*, the induced* $\mathfrak{g}$*-module* $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}} \mathbf{1}_C$ *of the trivial* $\mathfrak{h}$*-module* $\mathbf{1}_C$ *exists, and is naturally isomorphic to the* $\mathfrak{g}$*-module* $\mathrm{U}(\mathfrak{n})$[9].

Proof. The proof goes in several steps. We want to prove that $\mathrm{U}(\mathfrak{n})$ satisfies the universal property of the induced representation, that is that for any $\mathfrak{g}$-module V, $\mathrm{Hom}_{\mathfrak{h}}(\mathbf{1}_C, V) \simeq \mathrm{Hom}_{\mathfrak{g}}(\mathrm{U}(\mathfrak{g}), V)$.

First, we claim that the induced representation $\mathrm{Ind}_0^{\mathfrak{g}} \mathbf{1}_C$ exists and is isomorphic to $\mathrm{U}(\mathfrak{g})$. This means that $\mathrm{Hom}_C(\mathbf{1}_C, V) \simeq \mathrm{Hom}_{\mathfrak{g}}(\mathrm{U}(\mathfrak{g}), V)$. The morphism is obtained by attaching to each φ in $\mathrm{Hom}_C(\mathbf{1}_C, V)$ the map

$$\mathrm{U}(\mathfrak{g}) \simeq \mathrm{U}(\mathfrak{g}) \otimes \mathbf{1}_C \xrightarrow{\mathrm{id} \otimes \varphi} \mathrm{U}(\mathfrak{g}) \otimes V \to V.$$

Its inverse if simply the precomposition with the map $\mathbf{1}_C \to \mathrm{U}(\mathfrak{g})$. Let us consider the following diagram:

$$\begin{tikzcd}
\mathbf{1}_C \arrow[rr] \arrow[dr] \arrow[ddr, dashed] & & V \\
& \mathrm{U}(\mathfrak{n}) & \\
& \mathrm{U}(\mathfrak{g}) \arrow[u, Rightarrow] \arrow[uur, Rightarrow] &
\end{tikzcd}$$

The vertical arrow is simply given by the left action of $\mathrm{U}(\mathfrak{g})$ on the unit element element $\mathbb{1}$ of $\mathrm{U}(\mathfrak{n})$. Here the dashed arrow is a morphism in C, the plain arrows are morphism of $\mathfrak{h}$-modules and the double arrows are morphisms of $\mathfrak{g}$-modules. We have a diagram of morphism spaces

$$\begin{tikzcd}
\mathrm{Hom}_C(\mathbf{1}_C, V) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathfrak{g}}(\mathrm{U}(\mathfrak{g}), V) \\
\mathrm{Hom}_{\mathfrak{h}}(\mathbf{1}_C, V) \arrow[u, hook] & \mathrm{Hom}_{\mathfrak{g}}(\mathrm{U}(\mathfrak{n}), V) \arrow[l] \arrow[u]
\end{tikzcd}$$

We now claim the following:

- The map $\mathrm{U}(\mathfrak{g}) \to \mathrm{U}(n)$ admits a section, in particular it has a kernel $\mathfrak{N}$ in the category C.
- The map $\mathrm{U}(\mathfrak{g}) \otimes \mathfrak{h} \to \mathrm{U}(\mathfrak{g})$ factors through the kernel $\mathfrak{N}$, and the induced map $\mathrm{U}(\mathfrak{g}) \otimes \mathfrak{h} \to \mathfrak{N}$ is an isomorphism.

To prove the two claims, we make heavy use of the categorical PBW theorem (Proposition 2.11). If we endow $\mathrm{U}(\mathfrak{n})$ and $\mathrm{U}(\mathfrak{g})$ with their natural filtrations, then the action of $\mathfrak{g}$ on $\mathrm{U}(\mathfrak{n})$ is of degree 1 with respect to this filtration. The induced map $\mathfrak{g} \otimes \mathrm{Gr}^p\mathrm{U}(\mathfrak{n}) \to \mathrm{Gr}^{p+1}\mathrm{U}(\mathfrak{n})$ is simply given by the projection $\mathfrak{g} \to \mathfrak{n}$ followed by the multiplication map $\mathfrak{n} \otimes \mathrm{S}^p\mathfrak{n} \to \mathrm{S}^{p+1}\mathfrak{n}$. Therefore, we see that the natural map $\mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{n})$ is a filtered map of degree zero, and that the induced graded map $\mathrm{S}(\mathfrak{g}) \to \mathrm{S}(\mathfrak{n})$ is induced by the projection from $\mathfrak{g}$ to $\mathfrak{n}$. We can now consider the map

$$\delta : \mathrm{U}(\mathfrak{n}) \simeq \mathrm{S}(\mathfrak{n}) \hookrightarrow \mathrm{T}(\mathfrak{n}) \to \mathrm{T}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g})$$

Then the composite map $\mathrm{U}(\mathfrak{n}) \xrightarrow{\delta} \mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{n})$ is an isomorphism, since the associated graded map is the identity, and the filtration splits (as objects of C).

Let us prove the second claim. First we remark that the composite map

$$\mathfrak{h} \hookrightarrow \mathfrak{g} \xrightarrow{.\mathbb{1}} \mathrm{U}(\mathfrak{n})$$

vanishes. This gives the factorization of the map $\mathrm{U}(\mathfrak{g}) \otimes \mathfrak{h} \to \mathrm{U}(\mathfrak{g})$ by $\mathfrak{N}$. We can endow $\mathfrak{N}$ with the filtration induced by $\mathrm{U}(\mathfrak{g})$. Since for each non-negative integer p the morphism $\mathrm{F}^p\mathrm{U}(\mathfrak{g}) \to \mathrm{F}^p\mathrm{U}(\mathfrak{n})$

[9] The $\mathfrak{g}$-module structure on $\mathrm{U}(\mathfrak{n})$ has been introduced in §3.3.2.

admits a section, the natural map from $\mathrm{Gr}^p\mathfrak{N}$ to the kernel of $\mathrm{Gr}^p\mathrm{U}(\mathfrak{g}) \to \mathrm{Gr}^p\mathrm{U}(\mathfrak{n})$ is an isomorphism. Thus, using the PBW isomorphism, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Gr}^{p-1}\mathrm{U}(\mathfrak{g}) \otimes \mathfrak{h} \arrow[r] \arrow[dd, "\sim"] & \mathrm{Gr}^p\mathfrak{N} \arrow[d, "\sim"] \\
& \mathrm{Ker}\,(\mathrm{Gr}^p\mathrm{U}(\mathfrak{g}) \to \mathrm{Gr}^p\mathrm{U}(\mathfrak{n})) \arrow[d, "\sim"] \\
\mathrm{S}^{p-1}\mathfrak{g} \otimes \mathfrak{h} \arrow[r, equal] & \mathrm{S}^{p-1}\mathfrak{g} \otimes \mathfrak{h}
\end{tikzcd}$$

Hence the map $\mathrm{U}(\mathfrak{g}) \otimes \mathfrak{h} \to \mathfrak{N}$ is a degree one map whose associated graded map is an isomorphism. Hence it is an isomorphism.

We now come back to the main proof. The only thing that remains to prove is that the composite morphism

$$\mathrm{Hom}_{\mathfrak{h}}(\mathbf{1}_C, V) \to \mathrm{Hom}_C(\mathbf{1}_C, V) \simeq \mathrm{Hom}_{\mathfrak{g}}(\mathrm{U}(\mathfrak{g}), V)$$

factors through $\mathrm{Hom}_{\mathfrak{g}}(\mathrm{U}(\mathfrak{n}), V)$. This is equivalent to prove that the composition

$$\mathrm{Hom}_{\mathfrak{h}}(\mathbf{1}_C, V) \to \mathrm{Hom}_C(\mathbf{1}_C, V) \simeq \mathrm{Hom}_{\mathfrak{g}}(\mathrm{U}(\mathfrak{g}), V) \to \mathrm{Hom}_{\mathfrak{g}}(\mathrm{U}(\mathfrak{g}) \otimes \mathfrak{h}, V)$$

vanishes. The image of a map φ is the morphism

$$\mathrm{U}(\mathfrak{g}) \otimes \mathfrak{h} \to \mathrm{U}(\mathfrak{g}) \simeq \mathrm{U}(\mathfrak{g}) \otimes \mathbf{1}_C \xrightarrow{\mathrm{id}_{\mathrm{U}(\mathfrak{g})} \otimes \varphi} \mathrm{U}(\mathfrak{g}) \otimes V \to V$$

which can also be written as the composition

$$\mathrm{U}(\mathfrak{g}) \otimes \mathfrak{h} \simeq \mathrm{U}(\mathfrak{g}) \otimes \mathfrak{h} \otimes \mathbf{1}_C \xrightarrow{\mathrm{id}_{\mathrm{U}(\mathfrak{g}) \otimes \mathfrak{h}} \otimes \varphi} \mathrm{U}(\mathfrak{g}) \otimes V \to V$$

Now we have a commutative diagram

$$\begin{tikzcd}
\mathfrak{h} \otimes \mathbf{1}_C \arrow[r, "\mathrm{id}_{\mathfrak{h}} \otimes \varphi"] \arrow[d, "\text{zero map}"'] & \mathfrak{h} \otimes V \arrow[d] \\
\mathbf{1}_C \arrow[r, "\varphi"] & V
\end{tikzcd}$$

which allows to finish the proof. □

Remark now that we have a priori two algebra structures on the algebra $\mathrm{U}(\mathfrak{n})^{\mathfrak{h}}$: the first one is the natural one induced by the algebra structure on $\mathrm{U}(\mathfrak{n})$, and the second one is obtained using Frobenius duality, and Theorem 3.15, and the natural composition on $\mathrm{Hom}_{\mathfrak{g}}(\mathrm{U}(\mathfrak{n}), \mathrm{U}(\mathfrak{n}))$:

$$\mathrm{U}(\mathfrak{n})^{\mathfrak{h}} \simeq \mathrm{Hom}_{\mathfrak{h}}(\mathbf{1}_C, \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{h}}\mathrm{U}(\mathfrak{n})) \simeq \mathrm{Hom}_{\mathfrak{g}}(\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{h}}\mathbf{1}_C, \mathrm{U}(\mathfrak{n})) \simeq \mathrm{Hom}_{\mathfrak{g}}(\mathrm{U}(\mathfrak{n}), \mathrm{U}(\mathfrak{n})).$$

These two structure are in fact compatible, this is the content of the following:

Theorem 3.16. *Given a tame triple* $(\mathfrak{g}, \mathfrak{h}, \mathfrak{n})$*, the natural isomorphism* $\mathrm{U}(\mathfrak{n})^{\mathfrak{h}} \simeq \mathrm{Hom}_{\mathfrak{g}}(\mathrm{U}(\mathfrak{n}), \mathrm{U}(\mathfrak{n}))$ *is a morphism of algebra objects.*

Proof. DAMIEN □

4. Generalities on formal neighbourhoods

4.1. Sheaves on split square zero extensions.

4.1.1. *Theta morphisms.* In this section, we study some properties related to sheaves on a trivial first order thickening of a smooth scheme. Let us fix the setting: X is a smooth $\mathbf{k}$-scheme, $\mathcal{V}$ is a locally free sheaf on X, and S is the (split) first order thickening of X by V; that is $\mathcal{O}_S = \mathcal{V} \oplus \mathcal{O}_X$ and $\mathcal{V}$ is a square zero ideal in $\mathcal{O}_S$. We denote by $j\colon X \to S$ and $\sigma\colon S \to X$ the natural morphisms. Let $\mathcal{F}$ be an element of $\mathrm{D}^-(X)$. Then the exact sequence

$$0 \longrightarrow \mathcal{V} \otimes \mathcal{F} \longrightarrow \sigma^*\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow 0 \tag{8}$$

defines a morphism $\Theta_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathcal{V} \otimes \mathcal{F}[1]$ in $\mathrm{D}^-(S)$.

Proposition 4.1. *The following properties are valid:*

(i) *For any $\varphi\colon \mathcal{F} \to \mathcal{G}$ in $\mathrm{D}^-(X)$, we have*

$$\Theta_{\mathcal{G}} \circ \varphi = (\mathrm{id}_{\mathcal{V}} \otimes \varphi) \circ \Theta_{\mathcal{F}}.$$

(ii) *For any $\mathcal{F}$ in $\mathrm{D}^-(X)$, we have*

$$\Theta_{\mathcal{F}} = \Theta_{\mathcal{O}_X} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_S} \mathrm{id}_{\sigma^*\mathcal{F}}.$$

(iii) *For any $\mathcal{F}$ in $\mathrm{D}^-(X)$, the composition*

$$\mathcal{F} \xrightarrow{\mathrm{at}_S(\mathcal{F})} \Omega^1_S \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_S} \mathcal{F}[1] \to \Omega^1_S \otimes_{\mathcal{O}_S} \mathcal{F}[1] \simeq \mathcal{V} \otimes \mathcal{F}[1] \oplus \Omega^1_X \otimes \mathcal{F}[1]$$

is the couple $(\Theta_{\mathcal{F}}, \mathrm{at}_X(\mathcal{F}))$.

(iv) *The morphism $\sigma_*\Theta_{\mathcal{F}}$ vanishes.*

Proof. (i), (ii) and (iv) are straightforward. For (iii), $\Theta_{\mathcal{F}}$ is a special occurrence of a residual Atiyah morphism, and we apply [9, Prop. 4.9]. □

4.1.2. *Infinitesimal HKR isomorphism.* In this section, we describe the infinitesimal cohomological HKR isomorphism attached to a split square-zero extension of a smooth scheme.

For any non-negative integer p, we define a morphism $\Delta_p\colon \mathcal{O}_X \to \mathrm{T}^p(\mathcal{V}[1])$ in the derived category $\mathrm{D}^{\mathrm{b}}(S)$ as follows: $\Delta_0 = \mathrm{id}$ and $\Delta_p = -\Theta_{\mathrm{T}^{p-1}(\mathcal{V}[1])} \circ \Delta_{p-1}$ for $p \geq 1$.

Proposition 4.2. *For any vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ on X, there is a canonical isomorphism*

$$\begin{aligned}\mathrm{T}(\mathcal{V}[-1]) \overset{\mathbb{L}}{\otimes} \mathcal{RH}om(\mathcal{E}_1, \mathcal{E}_2) &\simeq \bigoplus_{p\in\mathbb{N}} \mathcal{RH}om(\mathcal{V}^{\otimes p} \otimes \mathcal{E}_1, \mathcal{E}_2)[-p] \\ &\to \bigoplus_{p\in\mathbb{N}} \sigma_*\mathcal{RH}om_{\mathcal{O}_S}(\mathcal{V}^{\otimes p}[p] \otimes \mathcal{E}_1, \mathcal{E}_2) \\ &\to \bigoplus_{p\in\mathbb{N}} \sigma_*\mathcal{RH}om_{\mathcal{O}_S}(\mathcal{E}_1, \mathcal{E}_2)\end{aligned}$$

obtained by precomposing by $\Delta_p \otimes \mathrm{id}_{\sigma^*\mathcal{E}_1}$. Besides, this isomorphism is compatible with the Yoneda product for the pair $(\mathcal{E}_1, \mathcal{E}_2)$.

Proof. The first part of the proof is well known and follows from the existence of a canonical locally $\mathcal{O}_S$-free resolution of $\mathcal{V}$ on S (*see* [1]). The compatibility with the Yoneda product follows from routine calculations using Proposition 4.1. □

It is also possible to derive the internal Hom with respect to the *second* variable instead of the first one: this gives the infinitesimal counterpart of Kashiwara's *dual* HKR isomorphism (*see* [8]). To do so, we replace the morphism Δ_1 by the *dual Atiyah morphism* $\Delta'_1\colon \mathcal{V}^*[-1] \to \mathcal{O}_X$, which is obtained by the composition

$$\mathcal{V}^*[-1] \xrightarrow{\mathrm{id}_{\mathcal{V}^*[-1]} \otimes \Delta_1} \mathcal{V}^*[-1] \otimes \mathcal{V}[1] \xrightarrow{\mathrm{ev}} \mathcal{O}_X.$$

Then for any integer p we construct the morphism $\Delta'_p\colon \mathrm{T}^p(\mathcal{V}^*[-1]) \to \mathcal{O}_X$, as well as the symmetric components $\Delta'^{+}_p\colon \mathrm{S}^p(\mathcal{V}^*[-1]) \to \mathcal{O}_X$ and $\Delta'^{-}_p\colon \widetilde{\Lambda}^p(\mathcal{V}^*[-1]) \to \mathcal{O}_X$ as we did before. Then the dual infinitesimal HKR isomorphism takes the following form:

Proposition 4.3. *For any vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ on X, there is a canonical isomorphism*

$$\begin{aligned}\mathrm{T}(\mathcal{V}[-1]) \overset{\mathbb{L}}{\otimes} \mathcal{RH}om(\mathcal{E}_1, \mathcal{E}_2) &\simeq \bigoplus_{p\in\mathbb{N}} \mathcal{RH}om(\mathcal{E}_1, (\mathcal{V}^*)^{\otimes p} \otimes \mathcal{E}_2)[-p] \\ &\to \bigoplus_{p\in\mathbb{N}} \sigma_* \mathcal{RH}om_{\mathcal{O}_S}(\mathcal{E}_1, (\mathcal{V}^*)^{\otimes p}[-p] \otimes \mathcal{E}_2) \\ &\to \bigoplus_{p\in\mathbb{N}} \sigma_* \mathcal{RH}om_{\mathcal{O}_S}(\mathcal{E}_1, \mathcal{E}_2)\end{aligned}$$

obtained by postcomposing by $\Delta_p^{'} \otimes \mathrm{id}_{\sigma^*\mathcal{E}_2}$. Besides, this isomorphism is compatible with the Yoneda product for the pair $(\mathcal{E}_1, \mathcal{E}_2)$.

Remark 4.4. The object $\sigma_*\mathcal{RH}om_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X)$ is a ring object in $\mathrm{D}^+(X)$. Propositions 4.2 and 4.3 give two a priori different isomorphisms between this ring object and $\mathrm{T}\mathcal{V}^*[-1]$. In the next Lemma, we will provide an identity relating Δ_p and $\Delta_p^{'}$ that shows that these two isomorphisms are in fact the same.

Lemma 4.5. *Let $\mathcal{W}$ be any element in $\mathrm{D}^{\mathrm{b}}(X)$, let $\mathcal{W}^* = \mathcal{RH}om_{\mathcal{O}_X}(\mathcal{W}, \mathcal{O}_X)$ be the (naïve) derived dual of $\mathcal{W}$, and let φ be in $\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(\mathcal{W}, \mathrm{T}^k\mathcal{V}^*[-k])$. Then the following diagram*

$$\begin{tikzcd} \mathcal{W} \arrow[rr, "\varphi"] \arrow[d, "\mathrm{id}_{\mathcal{W}}\otimes\Delta_k"] & & \mathrm{T}^k\mathcal{V}^*[-k] \arrow[d, "\Delta_k^{'}"] \\ \mathcal{W} \otimes \mathrm{T}^k\mathcal{V}[k] \arrow[r, "\varphi\otimes\mathrm{id}_{\mathrm{T}^k\mathcal{V}[k]}"] & \mathrm{T}^k\mathcal{V}^*[-k] \otimes \mathrm{T}^k\mathcal{V}[k] \arrow[r] & \mathcal{O}_X \end{tikzcd}$$

commutes in $\mathrm{D}^{\mathrm{b}}(X)$. In particular, the composition

$$\mathcal{O}_X \xrightarrow{\text{co-ev}} \mathcal{W}^* \otimes \mathcal{W} \xrightarrow{\mathrm{id}_{\mathcal{W}^*}\otimes(\Delta_k^{'}\circ\varphi)} \mathcal{W}^*$$

is $\varphi \circ \Delta_k$, where we implicitly use the isomorphism

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(\mathcal{W}, \mathrm{T}^k\mathcal{V}^*[-k]) \simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(\mathrm{T}^k\mathcal{V}[k], \mathcal{W}^*).$$

Proof. The first point follows directly from Proposition 4.1 (i). The second point follows from the fact that for any $\mathcal{A}$, $\mathcal{B}$ in $\mathrm{D}^{\mathrm{b}}(X)$, the isomorphism

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(\mathcal{A}, \mathcal{B}) \simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(\mathcal{B}^*, \mathcal{A}^*).$$

makes the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\text{co-ev}"] \arrow[dd] & \mathcal{B} \otimes \mathcal{B}^* \otimes \mathcal{A} \arrow[d] \\ & \mathcal{B} \otimes \mathcal{A}^* \otimes \mathcal{A} \arrow[d, "\mathrm{ev}"] \\ \mathcal{B} \arrow[r] & \mathcal{B} \end{tikzcd}$$

commute. Details are left to the reader. □

4.1.3. *Torsion and Atiyah class.* For any locally free sheaf $\mathcal{E}$ on S, let $\overline{\mathcal{E}} = j^*\mathcal{E}$. The exact sequence

$$0 \longrightarrow \mathcal{V} \otimes \overline{\mathcal{E}} \longrightarrow \sigma_* \mathcal{E} \longrightarrow \overline{\mathcal{E}} \longrightarrow 0$$

defines a morphism $\tau_{\mathcal{E}} \colon \overline{\mathcal{E}} \longrightarrow \mathcal{V} \otimes \overline{\mathcal{E}}[1]$ in the derived category $\mathrm{D}^{\mathrm{b}}(X)$, we call it the torsion of $\mathcal{E}$. It is easy to see that the vector bundle $\mathcal{E}$ is entirely determined by the couple $(\overline{\mathcal{E}}, \tau_{\mathcal{E}})$. In particular, $\tau_{\mathcal{E}}$ vanishes if and only if $\mathcal{E}$ is isomorphic to $\sigma^*\overline{\mathcal{E}}$

Proposition 4.6. *For any locally free sheaf $\mathcal{E}$ on S, the components of the morphism*

$$\overline{\mathcal{E}} \xrightarrow{\mathbb{L}j^* \operatorname{at}_S(\mathcal{E})} \mathbb{L}j^*(\Omega^1_S \otimes \mathcal{E}[1]) \longrightarrow j^*(\Omega^1_S \otimes \mathcal{E}[1]) \simeq \mathcal{V} \otimes \overline{\mathcal{E}}[1] \oplus \Omega^1_X \otimes \overline{\mathcal{E}}[1]$$

are $\tau_{\mathcal{E}}$ and $\operatorname{at}_X(\overline{\mathcal{E}})$.

Proof. There is a natural morphism $j^*\mathrm{P}^1_S(\mathcal{E}) \to \mathrm{P}^1_X(\overline{\mathcal{E}})$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & j^*\Omega^1_S \otimes \overline{\mathcal{E}} & \longrightarrow & j^*\mathrm{P}^1_S(\mathcal{E}) & \longrightarrow & \overline{\mathcal{E}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^1_X \otimes \overline{\mathcal{E}} & \longrightarrow & \mathrm{P}^1_X(\overline{\mathcal{E}}) & \longrightarrow & \overline{\mathcal{E}} & \longrightarrow & 0 \end{array}$$

commutative. On the other hand, $j^*\mathrm{P}^1_S(\mathcal{E})/(\Omega^1_X \otimes \overline{\mathcal{E}})$ is the cokernel of the map

$$\mathcal{V} \otimes \overline{\mathcal{E}} \longrightarrow (\mathcal{V} \otimes \overline{\mathcal{E}}) \oplus \mathcal{E}$$

which is the identity of the first factor and the natural inclusion on the second one. Hence it is isomorphic to $\sigma_*\mathcal{E}$ (by taking the difference of the two factors). This gives a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & j^*\Omega^1_S \otimes \overline{\mathcal{E}} & \longrightarrow & j^*\mathrm{P}^1_S(\mathcal{E}) & \longrightarrow & \overline{\mathcal{E}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{V} \otimes \overline{\mathcal{E}} & \longrightarrow & \sigma_*\mathcal{E} & \longrightarrow & \overline{\mathcal{E}} & \longrightarrow & 0 \end{array}$$

and the result follows. □

4.2. **Sheaves on a second order thickening.**

4.2.1. *Setting and cohomological invariants.* As before, let us fix a pair $(X, \mathcal{V})$ where $\mathcal{V}$ is a locally free sheaf on X, and let S be the split first order thickening of X by V. We are interested by ring spaces W which underlying topological space X satisfying the following conditions:

$$\begin{cases} \mathcal{O}_W \text{ is locally isomorphic to } \mathcal{O}_X \oplus \mathcal{V} \oplus \mathrm{S}^2\mathcal{V}. \\ \text{There exists a map } S \to W \text{ which is locally the quotient by } \mathrm{S}^2\mathcal{V}. \end{cases} \tag{9}$$

Let $k\colon X \to W$ be the composite map, and let us denote by $\langle \mathcal{V} \rangle$ the ideal sheaf of X in W, which is a sheaf of $\mathcal{O}_S$-modules. We can attach to W two cohomology classes:

- The class α in $\operatorname{Ext}^1(\mathcal{V}, \mathrm{S}^2\mathcal{V})$ is the extension class of the exact sequence
$$0 \to \mathrm{S}^2\mathcal{V} \to \sigma_*\langle \mathcal{V} \rangle \to \mathcal{V} \to 0$$
- The class β in $\operatorname{Ext}^1(\Omega^1_X, \mathrm{S}^2\mathcal{V})$ is the obstruction of lifting σ to W. To see how this class is defined, it suffices to remark that the sheaf of retractions of k inducing σ on S is an affine bundle directed by the vector bundle $\mathcal{D}er(\mathcal{O}_X, \mathrm{S}^2\mathcal{V})$, which is $\mathcal{H}om(\Omega^1_X, \mathrm{S}^2\mathcal{V})$.

Lemma 4.7. *The map $W \to (\alpha, \beta)$ is a bijection between isomorphism classes of ring spaces W satisfying* (9) *and* $\operatorname{Ext}^1(\mathcal{V}, \mathrm{S}^2\mathcal{V}) \oplus \operatorname{Ext}^1(\Omega^1_X, \mathrm{S}^2\mathcal{V})$.

Proof. The subsheaf of the sheaf of automorphisms of the ringed space $\{X, \mathcal{O}_X \oplus \mathcal{V} \oplus \mathrm{S}^2\mathcal{V}\}$ that induce the indentity morphism after taking the quotient by the square zero ideal $\mathrm{S}^2\mathcal{V}$ is isomorphic to $\mathcal{D}er(\mathcal{O}_X, \mathrm{S}^2\mathcal{V}) \oplus \mathcal{H}om(\mathcal{V}, \mathrm{S}^2\mathcal{V})$, a couple (D, φ) corresponding to the automorphism given by the 3×3 matrix

$$\begin{pmatrix} \mathrm{id} & 0 & 0 \\ 0 & \mathrm{id} & 0 \\ D & \varphi & \mathrm{id} \end{pmatrix}.$$

This gives the required result. □

4.2.2. *The second order HKR class.* Let $\mathcal{E}$ be a locally free sheaf on X. The functor that associates to any open set of U the set of locally O_W-free extensions of $\sigma^*\mathcal{E}$ to W is an abelian gerbe, whose automorphism sheaf is $\mathcal{H}om(\mathcal{E}, \mathrm{S}^2\mathcal{V} \otimes \mathcal{E})$. Hence this gerbe is classified by a cohomology class in $\mathrm{Ext}^2(\mathcal{E}, \mathrm{S}^2\mathcal{V} \otimes \mathcal{E})$.

Definition 4.8. For any locally free sheaf $\mathcal{E}$ on X, the class of the gerbe of locally free extensions of $\sigma^*\mathcal{E}$ on W is called the second order HKR class of $\mathcal{E}$, and is denoted by $\gamma_\sigma(\mathcal{E})$.

Remark 4.9. The terminology is justified as follows: in [1], the authors introduce the HKR class of a vector bundle on X in the case S is not globally split, it mesures the obstruction to lift the bundle from X to S. Here we are defining the same kind of obstruction classes when S is trivial, but W is not.

As in [1], the class $\gamma_\sigma(\mathcal{E})$ can be computed explicitly:

Proposition 4.10. *For any locally free sheaf $\mathcal{E}$ on X, the class $\gamma_\sigma(\mathcal{E})$ is obtained (up to a nonzero scalar) as the composition*

$$\mathcal{E} \xrightarrow{\mathrm{at}_X(\mathcal{E})} \Omega^1_X \otimes \mathcal{E}[1] \xrightarrow{\beta \otimes \mathrm{id}_{\mathcal{E}}} \mathrm{S}^2\mathcal{V} \otimes \mathcal{E}[2]$$

Proof. For the first HKR class, this is [1, Prop. 2.11]. We present an alternative and somehow more down to earth proof. Let us assume that the map $X \to W$ admits two retractions β_1 and β_2, and let $D = \beta_1 - \beta_2$ be the corresponding element in $\mathcal{H}om(\Omega^1_X, \mathcal{V})$. Assume also that $\mathcal{E}$ admits a regular connexion ∇ on X. We claim that the map $\Xi\colon \beta_1^*\mathcal{E} \to \beta_2^*\mathcal{E}$ defined by

$$\Xi(f \otimes s) = f \otimes s + f(D \otimes \mathrm{id}_{\mathcal{E}})(\nabla s)$$

is a well-defined isomorphism. Since $\mathrm{Hom}_{O_W}(\beta_1^*\mathcal{E}, \beta_2^*\mathcal{E}) \simeq \mathrm{Hom}_{O_X}(\mathcal{E}, \beta_{1*}\beta_2^*\mathcal{E})$, it suffices to show that $\Xi(1 \otimes gs) = \beta_1^*(g) \,.\, \Xi(s)$ for any section g of O_X. We compute:

$$\begin{aligned}
\Xi(1 \otimes gs) &= 1 \otimes gs + (D \otimes \mathrm{id}_{\mathcal{E}})(g\nabla s + dg \otimes s) \\
&= \beta_2(g) \otimes s + g(D \otimes \mathrm{id}_{\mathcal{E}})(\nabla s) + D(dg) \otimes s \\
&= \beta_1(g) \otimes s + g(D \otimes \mathrm{id}_{\mathcal{E}})(\nabla s) + D(dg) \otimes s \\
&= \beta_1(g).(s + (D \otimes \mathrm{id}_{\mathcal{E}})(\nabla s)) \\
&= \beta_1^*(g) \,.\, \Xi(s).
\end{aligned}$$

Let us now fix a covering $(U_i)_{i \in I}$ of X, and assume that on each U_i, there is a retraction β_i of the map $X \to W$ and $\mathcal{E}$ admits a regular connexion ∇_i. We put $D_{ij} = \beta_i - \beta_j$. Then we have isomorphisms

$$\begin{cases}
\Xi^i\colon \beta_i^*\mathcal{E}_{|U_{ij}} \xrightarrow{\sim} \beta_j^*\mathcal{E}_{|U_{ij}} & \text{depending on } \nabla_i \\
\Xi^j\colon \beta_j^*\mathcal{E}_{|U_{jk}} \xrightarrow{\sim} \beta_k^*\mathcal{E}_{|U_{jk}} & \text{depending on } \nabla_j \\
\Xi^k\colon \beta_k^*\mathcal{E}_{|U_{ki}} \xrightarrow{\sim} \beta_i^*\mathcal{E}_{|U_{ki}} & \text{depending on } \nabla_k
\end{cases}$$

The composition $\Xi^k_{|U_{ijk}} \circ \Xi^j_{|U_{ijk}} \circ \Xi^i_{|U_{ijk}}$ yields an automorphism of $\beta_i^*\mathcal{E}_{|U_{ijk}}$, which corresponds to the element

$$\lambda = (\mathrm{D}_{ij} \otimes \mathrm{id}_{\mathcal{E}}) \circ \nabla_i + (\mathrm{D}_{jk} \otimes \mathrm{id}_{\mathcal{E}}) \circ \nabla_j + (\mathrm{D}_{ki} \otimes \mathrm{id}_{\mathcal{E}}) \circ \nabla_k$$

of $\Gamma(U_{ijk}, \mathcal{H}om(\mathcal{E}, \mathrm{S}^2\mathcal{V} \otimes \mathcal{E}))$. This element is a Čhech representative of the class of the gerbe of locally free extensions of $\sigma^*\mathcal{E}$ on W. If $c_{ij} = (\mathrm{D}_{ij} \otimes \mathrm{id}_{\mathcal{E}}) \circ (\nabla_i - \nabla_j)$, then (c_{ij}) defines a 1-cochain with values in $\mathcal{H}om(\mathcal{E}, \mathrm{S}^2\mathcal{V} \otimes \mathcal{E})$ and we have

$$\begin{aligned}
\lambda = c_{ij} + c_{jk} + (\mathrm{D}_{ij} \otimes \mathrm{id}_{\mathcal{E}}) \circ (\nabla_j - \nabla_k) &= \frac{2}{3}(c_{ij} + c_{jk} + c_{ki}) \\
&+ \frac{1}{3}((\mathrm{D}_{ij} \otimes \mathrm{id}_{\mathcal{E}}) \circ (\nabla_j - \nabla_k) + (\mathrm{D}_{jk} \otimes \mathrm{id}_{\mathcal{E}}) \circ (\nabla_k - \nabla_i) + (\mathrm{D}_{kj} \otimes \mathrm{id}_{\mathcal{E}}) \circ (\nabla_i - \nabla_j))
\end{aligned}$$

which can be split us to some nonzero constant factors as the sum of the boundary of the cochain $(c_{ij})_{i,j}$ and the Yoneda product of the 1-cocycles $(\nabla_i - \nabla_j)_{i,j}$ and $(D_{ij} \otimes \mathrm{id}_{\mathcal{E}})_{i,j}$ that represent $\mathrm{at}_X(\mathcal{E})$ and $\beta \otimes \mathrm{id}_{\mathcal{E}}$ respectively. This gives the required formula. □

4.3. **Quantized cycles.**

4.3.1. *Setting, and tameness condition.* Let Y be a smooth $\mathbf{k}$-scheme, and let X be a smooth and closed subscheme of Y. We denote by $i\colon X \hookrightarrow Y$ the injection of X in Y.

Let S denote the first formal neighbourhood of X in Y. Let us assume that the closed immersion $j\colon X \hookrightarrow S$ admits a retraction $\sigma\colon S \to X$ (that is S is a globally trivial square zero extension of X by $\mathrm{N}^*_{X/Y}$); this is equivalent to say that the conormal sequence of the pair (X, Y) splits. In this case, we say that (X, σ) is a quantized cycle in Y (*see* [8]).

Assume that (X, σ) is a quantized cycle, and let W be the second formal neighborhood of X in Y, and let $k\colon X \hookrightarrow W$ be the corresponding inclusion. According to Lemma 4.7, the ringed space W is entirely encoded by two classes α and β introduced in the previous section; α is the extension class of the exact sequence

$$0 \to \sigma_* \frac{\mathcal{I}_X^2}{\mathcal{I}_X^3} \to \sigma_* \frac{\mathcal{I}_X}{\mathcal{I}_X^3} \to \sigma_* \frac{\mathcal{I}_X}{\mathcal{I}_X^2} \to 0$$

that lives in $\mathrm{Ext}^2(\mathrm{N}^*_{X/Y}, \mathrm{S}^2\mathrm{N}^*_{X/Y})$, and β in the class in $\mathrm{Ext}^1(\Omega^1_X, \mathrm{S}^2\mathrm{N}^*_{X/Y})$ that measures the obstruction to the existence of a retraction of k that extends σ.

Definition 4.11. A (X, σ) quantized cycle (X, σ) is *tame* if the locally free sheaf $\sigma^*\mathrm{N}^*_{\mathrm{X/Y}}$ on S extends to a locally free sheaf on W.

Remark 4.12. If k admits a retraction $q\colon W \to X$ such that $q_{|S} = \sigma$, that is if β vanishes, then (X, σ) is automatically tame: the locally free sheaf $q^*\mathrm{N}^*_{X/Y}$ on W extends $\sigma^*\mathrm{N}^*_{X/Y}$. In this case, we say that (X, σ) is *2-split*.

4.3.2. *Restriction of the Atiyah class.* The aim of this section is to describe another intrinsic description of the classes α and β.

Proposition 4.13. *The torsion of the locally free sheaf* $(\Omega^1_Y)_{|S}$ *is the morphism*

$$\mathrm{N}^*_{X/Y} \oplus \Omega^1_X \to \mathrm{N}^*_{X/Y} \otimes (\mathrm{N}^*_{X/Y} \oplus \Omega^1_X)\,[1]$$

given by the 2×2 *matrix* $\begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}$.

Proof. The locally free sheaf $(\Omega^1_Y)_{|S}$ depends only on the second formal neighbourhood W of X in Y. Let us consider an automorphism of the trivial ringed space $\mathcal{O}_X \oplus \mathrm{N}^*_{X/Y} \oplus \mathrm{S}^2\mathrm{N}^*_{X/Y}$ given by a couple (d, φ), where d is a derivation from $\mathcal{O}_X$ to $\mathrm{S}^2\mathrm{N}^*_{X/Y}$ and φ is a linear morphism from $\mathrm{N}^*_{X/Y}$ to $\mathrm{S}^2\mathrm{N}^*_{X/Y}$. Assume that we are in the local situation, so that we can take coordinates: $X = U \subset \mathbf{k}^n$ and $Y = U \times V$ where $V \subset \mathbf{k}^r$. Then we can represent the morphisms d and φ by sequences of regular maps $(Z^k_{i,j}(\mathbf{x}))_{1\le k\le n, 1\le i,j\le r}$ and $(\Lambda^\ell_{i,j}(\mathbf{x}))_{1\le i,j,\ell\le r}$ that are symmetric in the indices (i, j). The automorphism of W can be lifted to an automorphism of Y given by the formula

$$(\mathbf{x}, \mathbf{t}) \to \left(\{\mathbf{x}^k + \sum_{i,j} Z^k_{i,j}(\mathbf{x}) t^i t^j)\}_{1\le k\le n}, \{\mathbf{t}^\ell + \sum_{i,j} \Lambda^\ell_{i,j}(\mathbf{x}) t^i t^j)\}_{1\le \ell\le r} \right)$$

The result follows by computing the pullback of the forms $dx^k, t^i dx^k$, dt^j, $t^i dt^j$ restricted to S by the above automorphism. □

Corollary 4.14. *The composition*

$$\mathrm{N}^*_{X/Y} \oplus \Omega^1_X \simeq \Omega^1_{Y|X} \xrightarrow{\mathbb{L}i^*\mathrm{at}_Y(\Omega^1_Y)} \mathrm{S}^2\Omega^1_{Y|X}[1] \simeq \mathrm{S}^2\mathrm{N}^*_{X/Y}[1] \oplus \Omega^1_X \otimes \mathrm{N}^*_{X/Y}[1] \oplus \mathrm{S}^2\Omega^1_X[1]$$

is given by the matrix $\begin{pmatrix} \alpha & \beta \\ \mathrm{at}_X(\mathrm{N}^*_{X/Y}) & 0 \\ 0 & \mathrm{at}_X(\Omega^1_X) \end{pmatrix}$.

Proof. This is obtained by putting together Proposition 4.6 and Proposition 4.13, together with the functoriality of the Atiyah class. □

4.3.3. *Quantized HKR isomorphism.* As noticed in [1], the composition

$$\mathrm{S}(\mathrm{N}_{X/Y}[-1]) \longrightarrow \mathrm{T}(\mathrm{N}_{X/Y}[-1]) \simeq \mathcal{RH}om_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X) \longrightarrow \mathcal{RH}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \tag{10}$$

is an isomorphism in $\mathrm{D}^+(Y)$, where the first map is the antisymetrization map. For quantized cycles, it is also possible to produce a left resolution of the sheaf $\mathcal{O}_X$ (the Atiyah-Kashiwara resolution) that computes directly $\mathcal{RH}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$, this construction is done in [8]. Both constructions are in fact compatible (*see* [8, Thm. 4.13]). Let us now give the corresponding HKR isomorphism. For any non-negative integer p, we decompose the morphism Δ_p as the sum $\Delta_p^- + \Delta_p^+$ according to the decomposition

$$\mathrm{T}^p(\mathrm{N}^*_{X/Y}[1]) \simeq \widetilde{\Lambda}^p(\mathrm{N}^*_{X/Y}[1]) \oplus \mathrm{S}^p(\mathrm{N}^*_{X/Y}[1]).$$

For any sheaf $\mathcal{F}$ on X, the functor $\mathcal{H}om_{\mathcal{O}_Y}(\star, \mathcal{F})\colon \mathrm{Mod}(\mathcal{O}_Y) \to \mathrm{Mod}(\mathcal{O}_Y)$ factors through a functor

$$\mathcal{H}om^{\ell}_{\mathcal{O}_Y}(\star, \mathcal{F})\colon \mathrm{Mod}(\mathcal{O}_Y) \to \mathrm{Mod}(\mathcal{O}_X).$$

Proposition 4.15. *For any locally free sheaves $\mathcal{E}_1$, $\mathcal{E}_2$ on X, there is a canonical isomorphism*

$$\mathcal{RH}om^{\ell}_{\mathcal{O}_Y}(\mathcal{E}_1, \mathcal{E}_2) \simeq \bigoplus_{p\in\mathbb{N}} \mathcal{RH}om(\Lambda^p \mathrm{N}^*_{X/Y} \otimes \mathcal{E}_1, \mathcal{E}_2)[-p] \simeq \mathrm{S}(\mathrm{N}_{X/Y}[-1]) \overset{\mathbb{L}}{\otimes} \mathcal{RH}om(\mathcal{E}_1, \mathcal{E}_2)$$

in $\mathrm{D}^{\mathrm{b}}(X)$ *obtained by precomposing with* $\Delta_p^+ \otimes \mathrm{id}_{\sigma^*\mathcal{E}_1}$.

We also give Kashiwara's *dual* version:

Proposition 4.16. *For any locally free sheaves $\mathcal{E}_1$, $\mathcal{E}_2$ on X, there is a canonical isomorphism*

$$\mathcal{RH}om^{r}_{\mathcal{O}_Y}(\mathcal{E}_1, \mathcal{E}_2) \simeq \bigoplus_{p\in\mathbb{N}} \mathcal{RH}om(\mathcal{E}_1, \Lambda^p \mathrm{N}_{X/Y} \otimes \mathcal{E}_2)[-p] \simeq \mathrm{S}(\mathrm{N}_{X/Y}[-1]) \overset{\mathbb{L}}{\otimes} \mathcal{RH}om(\mathcal{E}_1, \mathcal{E}_2)$$

in $\mathrm{D}^{\mathrm{b}}(X)$ *obtained by postcomposing by* $\Delta_p'^+ \otimes \mathrm{id}_{\sigma^*\mathcal{E}_2}$.

5. Tame quantized cycles

5.1. **Structure constants.**

5.1.1. *Definition.* We fix a quantized analytic cycle (X, σ) in Y. By Proposition 4.15, there exist unique coefficients $(c_p'^k)_{0\le k\le p}$ such that:

- Each c_p^k belongs to $\mathrm{Hom}\,(\mathrm{S}^k(\mathrm{N}^*_{X/Y}[1]), \mathrm{T}^p(\mathrm{N}^*_{X/Y}[1]))$, that is $\mathrm{Ext}^{p-k}(\Lambda^k \mathrm{N}^*_{X/Y}, \mathrm{T}^p \mathrm{N}^*_{X/Y})$.
- The relation $\Delta_p = \sum_{k=0}^p c_p^k \circ \Delta_k^+$ holds in $\mathrm{D}^{\mathrm{b}}(Y)$.

Similarly, using Proposition 4.16, we can define *dual* coefficients: there exist unique coefficients $(c_p'^k)_{0\le k\le p}$ such that:

- Each $c_p'^k$ belongs to $\mathrm{Hom}\,(\mathrm{T}^p(\mathrm{N}_{X/Y}[-1]), \mathrm{S}^k(\mathrm{N}_{X/Y}[-1]))$, that is $\mathrm{Ext}^{p-k}(\mathrm{T}^p \mathrm{N}_{X/Y}, \Lambda^k \mathrm{N}_{X/Y})$.
- The relation $\Delta_p' = \sum_{k=0}^p \Delta_k'^+ \circ c_p'^k$ holds in $\mathrm{D}^{\mathrm{b}}(Y)$.

Lemma 5.1. *Via the isomorphism* $\mathrm{Ext}^{p-k}(\mathrm{T}^p \mathrm{N}_{X/Y}, \Lambda^k \mathrm{N}_{X/Y}) \simeq \mathrm{Ext}^{p-k}(\Lambda^k \mathrm{N}^*_{X/Y}, \mathrm{T}^p \mathrm{N}^*_{X/Y})$*, the coefficients* $c_p'^k$ *and* c_p^k *are equal.*

Proof. We take the relation $\Delta_p' = \sum_{k=0}^p \Delta_k'^+ \circ c_p'^k$, take the tensor product by $\mathrm{id}_{\mathrm{T}^p \mathrm{N}_{X/Y}[-p]}$, and precompose with the co-evaluation map $\mathcal{O}_X \to \mathrm{id}_{\mathrm{T}^p \mathrm{N}_{X/Y}[-p]} \otimes \mathrm{id}_{\mathrm{T}^p \mathrm{N}^*_{X/Y}[p]}$. Applying Lemma 4.5 with $\mathcal{W} = \mathrm{T}^p \mathrm{N}_{X/Y}[-p]$, we get $\Delta_p = \sum_{k=0}^p c_p'^k \circ \Delta_k^+$, so $c_p'^k = c_k^p$. □

Lemma 5.2. *Let* $p \ge 1$.

(i) *The coefficient* c_p^0 *vanish.*
(ii) *If* $1 \le k \le p-1$*, the* $\mathrm{S}^p(\mathrm{N}^*_{X/Y}[1])$ *component of* c_p^k *vanishes, so* c_p^k *factors through* $\widetilde{\Lambda}^p(\mathrm{N}^*_{X/Y}[1])$.
(iii) *The coefficient* c_p^p *is the canonical inclusion of* $\mathrm{S}^p(\mathrm{N}^*_{X/Y}[1])$ *in* $\mathrm{T}^p(\mathrm{N}^*_{X/Y}[1])$.

Proof. For (i), we apply σ_* to the relation defining the c_p^k and use Proposition 4.1 (iv). To prove (ii), let π_p be the projection from $T^p(\mathrm{N}^*_{X/Y}[1])$ to $S^p(\mathrm{N}^*_{X/Y}[1])$. Then

$$\Delta_p^+ = \sum_{k=0}^{p} \pi_p \circ c_p^k \circ \Delta_k^+$$

so thanks to Proposition 4.15, $\pi_p \circ c_p^k = 0$ if $0 \leq k \leq p-1$. For (iii) this is a purely local question, and we can use the Koszul complex as in [8]. □

5.1.2. *The decomposition lemma.* Let us first compute the first nontrivial structure constant:

Proposition 5.3. *The morphism* $\Delta_2^- - \alpha \circ \Delta_1$ *vanishes in* $\mathrm{D}^{\mathrm{b}}(W)$. *In particular,* $c_2^1 = \alpha$.

Proof. Let us consider the exact sequence

$$0 \to \frac{\mathcal{I}_X^2}{\mathcal{I}_X^3} \to \frac{\mathcal{I}_X}{\mathcal{I}_X^3} \to \frac{\mathcal{I}_X}{\mathcal{I}_X^2} \to 0. \tag{11}$$

It defines a morphism f from $\mathrm{N}^*_{X/Y}$ to $\mathrm{S}^2\mathrm{N}^*_{X/Y}[1]$ in $\mathrm{D}^{\mathrm{b}}(S)$. Applying Proposition 4.2, f can be written as $u + v \circ \Theta_{\mathrm{N}^*_{X/Y}}$ where u is in $\mathrm{Ext}^1(\mathrm{N}^*_{X/Y}, \mathrm{S}^2\mathrm{N}^*_{X/Y})$ and v is in $\mathrm{Hom}(\mathrm{T}^2\mathrm{N}^*_{X/Y}, \mathrm{S}^2\mathrm{N}^*_{X/Y})$. The morphism u is the image under σ_* of (11), so it is α. The morphism v can be computed locally, it is the the natural symetrization map from $\mathrm{T}^2\mathrm{N}^*_{X/Y}$ to $\mathrm{S}^2\mathrm{N}^*_{X/Y}$. Let us now consider the diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & \frac{\mathcal{I}_X^2}{\mathcal{I}_X^3} \arrow[r] \arrow[d] & \frac{\mathcal{I}_X}{\mathcal{I}_X^3} \arrow[r] \arrow[d] & \frac{\mathcal{I}_X}{\mathcal{I}_X^2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \frac{\mathcal{I}_X^2}{\mathcal{I}_X^3} \arrow[r] & \frac{\mathcal{O}_Y}{\mathcal{I}_X^3} \arrow[r] & \frac{\mathcal{O}_Y}{\mathcal{I}_X^2} \arrow[r] \arrow[d] & 0 \\
 & & & \frac{\mathcal{O}_Y}{\mathcal{I}_X} \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

It gives a commutative diagram

$$\begin{tikzcd}
\mathcal{O}_X \arrow[r, "\Theta_{\mathcal{O}_X}"] & \mathrm{N}^*_{X/Y}[1] \arrow[r, "f"] \arrow[d] & \mathrm{S}^2\mathrm{N}^*_{X/Y}[2] \arrow[d, equal] \\
 & \mathcal{O}_S[1] \arrow[r] & \mathrm{S}^2\,\mathrm{N}^*_{X/Y}[2]
\end{tikzcd}$$

in $\mathrm{D}^{\mathrm{b}}(W)$. Since the composite arrow from $\mathcal{O}_X$ to $\mathcal{O}_S[1]$ vanishes, the morphism $f \circ \Theta_{\mathcal{O}_X}$ also vanishes. But $f \circ \Theta_{\mathcal{O}_X} = (u + v \circ \Theta_{\mathrm{N}^*_{X/Y}}) \circ \Theta_{\mathcal{O}_X} = \alpha \circ \Delta_1 - \Delta_2^-$. This gives the result. □

As a corollary, we get our key technical result:

Proposition 5.4 (Decomposition lemma)**.** *Assume that the quantized cycle (X,σ) is tame. For any non-negative integer i, we can write*

$$\Theta_{\mathrm{T}^{i+1}\mathrm{N}^*_{X/Y}} \circ \Theta_{\mathrm{T}^i\mathrm{N}^*_{X/Y}} = \mathfrak{A}_i - (\alpha \otimes \mathrm{id}_{\mathrm{T}^i\mathrm{N}^*_{X/Y}}) \circ \Theta_{\mathrm{T}^i\mathrm{N}^*_{X/Y}} + \mathfrak{R}_i$$

in the derived category $\mathrm{D}^{\mathrm{b}}(S)$*, where* $\mathfrak{A}_i$ *factors through* $\Lambda^2\mathrm{N}^*_{X/Y} \otimes \mathrm{T}^i\mathrm{N}^*_{X/Y}[2]$*, and* $\mathfrak{R}_i$ *vanishes in the derived category* $\mathrm{D}^{\mathrm{b}}(W)$*.*

Proof. Denote by μ the natural map from S to W, and let $\mathcal{S}$ be a locally free extension of $\sigma^*\mathrm{N}^*_{X/Y}$ on W. Then

$$\begin{aligned}\Theta_{\mathrm{T}^{i+1}\mathrm{N}^*_{X/Y}} \circ \Theta_{\mathrm{T}^i\mathrm{N}^*_{X/Y}} &= \Delta_2 \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_S} \mathrm{id}_{\sigma^*\mathrm{T}^i\mathrm{N}^*_{X/Y}} \\ &= \Delta_2^+ \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_S} \mathrm{id}_{\sigma^*\mathrm{T}^i\mathrm{N}^*_{X/Y}} + (\alpha \circ \Delta_1) \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_S} \mathrm{id}_{\sigma^*\mathrm{T}^i\mathrm{N}^*_{X/Y}} + (\Delta_2^- - \alpha \circ \Delta_1) \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_S} \mathrm{id}_{\sigma^*\mathrm{T}^i\mathrm{N}^*_{X/Y}}.\end{aligned}$$

We claim that $(\Delta_2^- - \alpha \circ \Delta_1) \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_S} \mathrm{id}_{\sigma^*\mathrm{T}^i\mathrm{N}^*_{X/Y}}$ vanishes in $\mathrm{D}^{\mathrm{b}}(W)$. Indeed,

$$\begin{aligned}\mu_*\left((\Delta_2^- - \alpha \circ \Delta_1) \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_S} \mathrm{id}_{\sigma^*\mathrm{T}^i\mathrm{N}^*_{X/Y}}\right) &= \mu_*\left((\Delta_2^- - \alpha \circ \Delta_1) \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_S} \mathbb{L}\mu^*\mathrm{id}_{\mathrm{T}^i\mathcal{S}}\right) \\ &= \mu_*\left(\Delta_2^- - \alpha \circ \Delta_1\right) \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_W} \mathrm{id}_{\mathrm{T}^i\mathcal{S}}.\end{aligned}$$

□

Remark 5.5. ⚠ Proposition 5.4 is false in general when the quantized cycle (X,σ) is not tame: the morphism $(\Delta_2^- - \alpha \circ \Delta_1) \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_S} \mathrm{id}_{\sigma^*\mathrm{T}^i\mathrm{N}^*_{X/Y}}$ can be nonzero in $\mathrm{D}^{\mathrm{b}}(W)$.

For $1 \leq p \leq p-1$, let

$$\Psi_p \colon \mathrm{T}^p(\mathrm{N}^*_{X/Y}[1]) \to \bigoplus_{i=1}^{p-1} \mathrm{T}^{i-1}(\mathrm{N}^*_{X/Y}[1]) \otimes \Lambda^2(\mathrm{N}^*_{X/Y}[1]) \otimes \mathrm{T}^{p-i-1}(\mathrm{N}^*_{X/Y}[1])$$

be the map obtained by (graded) antisymetrization of two consecutive factors. We also denote by $\chi_{p,i}$ the components of Ψ_p.

Corollary 5.6. *Assume to be given a tame cycle (X,σ) in Y. Then*

$$\Psi_p \circ c_p^k = \{(\mathrm{id}_{\mathrm{T}^{i-1}\mathrm{N}^*_{X/Y}} \otimes \alpha \otimes \mathrm{id}_{\mathrm{T}^{p-i-1}\mathrm{N}^*_{X/Y}}) \circ c_{p-1}^k\}_{1\leq i\leq p-1} \ \ \textit{if} \ \ 1 \leq k \leq p-1.$$

Proof. Using Proposition 5.4, we have in $\mathrm{D}^{\mathrm{b}}(S)$

$$\begin{aligned}\Delta_p &= (-1)^p\, \Theta_{\mathrm{T}^{p-1}\mathrm{N}^*_{X/Y}} \circ \cdots \circ (\Theta_{\mathrm{T}^{i+1}\mathrm{N}^*_{X/Y}} \circ \Theta_{\mathrm{T}^i\mathrm{N}^*_{X/Y}}) \circ \cdots \circ \Theta_{\mathcal{O}_X} \\ &= (\mathrm{id}_{\mathrm{T}^{p-i-1}\mathrm{N}^*_{X/Y}} \otimes \alpha \otimes \mathrm{id}_{\mathrm{T}^i\mathrm{N}^*_{X/Y}}) \circ \Delta_{p-1} + (-1)^p\, \Theta_{\mathrm{T}^{p-1}\mathrm{N}^*_{X/Y}} \circ \cdots \circ (\mathfrak{U}_i + \mathfrak{R}_i) \circ \cdots \circ \Theta_{\mathcal{O}_X}\end{aligned}$$

It implies that in $\mathrm{D}^{\mathrm{b}}(W)$,

$$\begin{aligned}\chi_{p,i} \circ \Delta_p &= (\mathrm{id}_{\mathrm{T}^{p-i-1}\mathrm{N}^*_{X/Y}} \otimes \alpha \otimes \mathrm{id}_{\mathrm{T}^i\mathrm{N}^*_{X/Y}}) \circ \Delta_{p-1} \\ &= \sum_{k=1}^{p-1} (\mathrm{id}_{\mathrm{T}^{p-i-1}\mathrm{N}^*_{X/Y}} \otimes \alpha \otimes \mathrm{id}_{\mathrm{T}^i\mathrm{N}^*_{X/Y}}) \circ c_{p-1}^k \circ \Delta_k^+.\end{aligned}$$

On the other hand

$$\chi_{p,i} \circ \Delta_p = \sum_{k=1}^{p-1} \chi_{p,i} \circ c_p^k \circ \Delta_k^+.$$

We conclude using Proposition 4.15. □

Corollary 5.7. *If (X,σ) is tame, $\Delta_p = \sum_{k=1}^p c_p^k \circ \Delta_k^+$ in $\mathrm{D}^{\mathrm{b}}(W)$.*

Proof. We argue by induction on p. For $p = 2$, this is Proposition 5.3. Assume that the property holds at the order $p-1$. The morphism $\Delta_p - \sum_{k=1}^{p} c_p^k \circ \Delta_k^+$ takes values in $\widetilde{\Lambda}^p(\mathrm{N}^*_{X/Y}[1])$. Hence it suffices to prove that $\Psi_p \circ (\Delta_p - \sum_{k=1}^{p} c_p^k \circ \Delta_k^+)$ vanishes in $\mathrm{D}^{\mathrm{b}}(W)$. Using the notation of the preceding proof,

$$\chi_{p,i} \circ (\Delta_p - \sum_{k=1}^{p} c_p^k \circ \Delta_k^+) = (\mathrm{id}_{\mathrm{T}^{p-i-1}\mathrm{N}^*_{X/Y}} \otimes \alpha \otimes \mathrm{id}_{\mathrm{T}^i\mathrm{N}^*_{X/Y}}) \circ \left(\Delta_{p-1} - \sum_{k=1}^{p-1} c_{p-1}^k \circ \Delta_k^+\right)$$

in $\mathrm{D}^{\mathrm{b}}(W)$. Then we use the induction hypothesis. □

5.1.3. *Main result.* We can come to our main result.

Theorem 5.8. *Let (X, σ) be a tame quantized cycle in Y. The class α defines a Lie coalgebra structure on $\mathrm{N}^*_{X/Y}[1]$, hence a Lie algebra structure on $\mathrm{N}_{X/Y}[-1]$. Besides, the objects $\mathcal{RH}om^{\ell}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ and $\mathcal{RH}om^{r}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ are naturally algebra objects in the derived category $\mathrm{D}^{\mathrm{b}}(X)$, and there are commutative diagrams*

$$\begin{tikzcd}
\sigma_*\mathcal{RH}om_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X) \arrow[r] \arrow[dd, "\mathrm{HKR}"'] & \mathcal{RH}om^{\ell}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \arrow[d, "\mathrm{HKR}"] \\
& \mathrm{S}(\mathrm{N}_{X/Y}[-1]) \arrow[d, "\mathrm{PBW}"] \\
\mathrm{T}(\mathrm{N}_{X/Y}[-1]) \arrow[r] & \mathrm{U}(\mathrm{N}_{X/Y}[-1])
\end{tikzcd}$$

and

$$\begin{tikzcd}
\sigma_*\mathcal{RH}om_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X) \arrow[r] \arrow[dd, "\text{dual HKR}"'] & \mathcal{RH}om^{r}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \arrow[d, "\text{dual HKR}"] \\
& \mathrm{S}(\mathrm{N}_{X/Y}[-1]) \arrow[d, "\mathrm{PBW}"] \\
\mathrm{T}(\mathrm{N}_{X/Y}[-1]) \arrow[r] & \mathrm{U}(\mathrm{N}_{X/Y}[-1])
\end{tikzcd}$$

where all horizontal arrows are algebra morphisms.

Strategy of proof. Let us first discuss the statement, as well as the main points involved in the proof. The object $\mathcal{RH}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ is always an algebra object in the category $\mathrm{D}^{\mathrm{b}}(Y)$, but this structure doesn't always come in full generality from an algebra object structure on $\mathcal{RH}om^{r}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ or on $\mathcal{RH}om^{\ell}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ in $\mathrm{D}^+(X)$. Indeed, if $\mathcal{RH}om^{\ell}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ has such an algebra structure, the natural morphism

$$\mathrm{N}_{X/Y}[-1] \to \mathcal{RH}om^{\ell}_{\mathcal{O}_Y}(\mathrm{N}^*_{X/Y}[1], \mathcal{O}_X) \to \mathcal{RH}om^{\ell}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$$

obtained by precomposition with Δ_1 yields a morphism $\mathrm{T}(\mathrm{N}_{X/Y}[-1]) \to \mathcal{RH}om^{\ell}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$, and the composite morphism

$$\mathrm{S}(\mathrm{N}_{X/Y}[-1]) \hookrightarrow \mathrm{T}(\mathrm{N}_{X/Y}[-1]) \to \mathcal{RH}om^{\ell}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$$

is an isomorphism in $\mathrm{D}^+(X)$, which is not always the case.

Assuming that we have an algebra structure on $\mathcal{RH}om^{r}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ and on $\mathcal{RH}om^{\ell}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ making the top row of each diagram if Theorem 5.8 multiplicative, the statement follows directly from Proposition 5.3 the reverse PBW Theorem (that is Theorem 2.7), without using the tameness condition at all: indeed, condition (A1) is (10), and condition (A2) is Lemma 5.2. Therefore, the main difficulty lies in the construction of this algebra structure on $\mathcal{RH}om^{r}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ and on $\mathcal{RH}om^{\ell}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$. We will provide three different proofs corresponding to different geometric contexts:

Case A: ∞-split Assume that X admits a global retract in Y that lifts σ. This is the easiest case, but it covers the case of the diagonal injection and is therefore sufficient to prove the results of Kapranov, Markarian and Ramadoss that are presented in the next section. The reader only interested in this can skip the two other cases.

Case B: 2-split Assume that X admits a retract at order two that lifts σ (that is $\beta = 0$). In this case, we can adapt the former proof to the second formal neighborhood, using the second Corollary of the decomposition Lemma (Corollary 5.7).

Case C: tame The general case: (X, σ) is tame. This requires the first Corollary of the decomposition Lemma (Corollary 5.6) as well as the full strength of the categorical PBW Theorem (Theorem 2.11). □

Proof of Theorem 5.8. We follow the aforementioned plan of proof, and discuss successively the three cases A, B and C. We recall the following notation:

$$\begin{tikzcd} X \arrow[r, shift left, "j"'] \arrow[d, "i"'] \arrow[dr, "k"] & S \arrow[l, bend right, "\sigma"'] \arrow[d] \\ Y & W \arrow[l] \end{tikzcd}$$

Case A Let $f\colon Y \to X$ a retraction of X in Y. There is a natural morphism of algebra objects

$$j_{S/Y*}\mathcal{RH}om_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X) \to \mathcal{RH}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$$

in $\mathrm{D}^+(Y)$, which gives a morphism of algebra objects

$$f_* j_{S/Y*}\mathcal{RH}om_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X) \to f_*\mathcal{RH}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \tag{12}$$

since f_* is monoidal. Now it suffices to remark that

$$f_*\mathcal{RH}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \simeq f_* i_*\mathcal{RH}om^{\ell}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \simeq \mathcal{RH}om^{\ell}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X),$$

so that $\mathcal{RH}om^{\ell}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ inherits naturally from an algebra structure and the morphism (12) becomes an algebra morphism

$$\sigma_*\mathcal{RH}om_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X) \to \mathcal{RH}om^{\ell}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X).$$

The same trick works for the functor $\mathcal{RH}om^r$. This settles Case A.

Case B Let us consider the map $\mathfrak{p}\colon \sigma_*\mathcal{RH}om_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X) \to \mathcal{RH}om^{\ell}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)$. It admits a section, given by the composition

$$\mathcal{RH}om^{\ell}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \simeq \mathrm{S}(\mathrm{N}^*_{X/Y}[1]) \hookrightarrow \mathrm{T}(\mathrm{N}_{X/Y}[-1]) \simeq \sigma_*\mathcal{RH}om_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X).$$

Hence $\mathfrak{p}$ admits a kernel $\mathcal{K}$, which can be explicitly described as follows: $\mathcal{K}$ is isomorphic to $\widetilde{\Lambda}(\mathrm{N}_{X/Y}[-1])$, and the (split) embedding of $\mathcal{K}$ in $\sigma_*\mathcal{RH}om_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X)$ is obtained by applying σ_* to the composition

$$\bigoplus_{p\geq 0} j_*\widetilde{\Lambda}^p(\mathrm{N}_{X/Y}[-1]) \to \bigoplus_{p\geq 0} \mathcal{RH}om_{\mathcal{O}_S}(\widetilde{\Lambda}^p(\mathrm{N}^*_{X/Y}[1]), \mathcal{O}_X) \to \mathcal{RH}om_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X)$$

where the last map is obtained componentwise by precomposing with $\Delta_p - \sum_{k=1}^{p} c_p^k \Delta_k^+$, which is a morphism in $\mathrm{Hom}_{\mathrm{D}^b(S)}(\mathcal{O}_X, \widetilde{\Lambda}^p(\mathrm{N}^*_{X/Y}[1])$. Assume now that there exists a retraction $q\colon W \to X$ that extends the first order retraction σ. Then the composition

$$\mathcal{K} \to \sigma_*\mathcal{RH}om_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X) \to q_*\mathcal{RH}om_{\mathcal{O}_W}(\mathcal{O}_X, \mathcal{O}_X)$$

is obtained by applying q_* to the map

$$\bigoplus_{p\geq 0} j_{X/W*}\,\widetilde{\Lambda}^p(\mathrm{N}_{X/Y}[-1]) \to \bigoplus_{p\geq 0} \mathcal{RH}om_{\mathcal{O}_W}(\widetilde{\Lambda}^p(\mathrm{N}^*_{X/Y}[1]), \mathcal{O}_X) \to \mathcal{RH}om_{\mathcal{O}_W}(\mathcal{O}_X, \mathcal{O}_X).$$

But this map is identically zero, since according to proposition 5.7, $\Delta_p - \sum_{k=1}^{p} c_p^k \Delta_k^+$ vanishes in the derived category $\mathrm{D}^{\mathrm{b}}(W)$. We can now conclude: the map

$$\sigma_* \mathcal{RH}om_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X) \to q_* \mathcal{RH}om_{\mathcal{O}_W}(\mathcal{O}_X, \mathcal{O}_X)$$

is a morphism of algebra objects and $\mathcal{K}$ is a split sub-object that maps to zero, so that the composition

$$\mathcal{K} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \sigma_* \mathcal{RH}om_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X) \to \sigma_* \mathcal{RH}om_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X) \to q_* \mathcal{RH}om_{\mathcal{O}_W}(\mathcal{O}_X, \mathcal{O}_X)$$

is zero. It follows that the composition

$$\mathcal{K} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \sigma_* \mathcal{RH}om_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X) \to \sigma_* \mathcal{RH}om_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X) \to \mathcal{RH}om^{\ell}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$$

is zero. Therefore, in the decomposition

$$\sigma_* \mathcal{RH}om_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X) \simeq \mathcal{K} \oplus \mathcal{RH}om^{\ell}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X),$$

the object $\mathcal{K}$ is an ideal object, so that $\mathcal{RH}om^{\ell}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ inherits of a natural algebra structure, for which $\mathfrak{p}$ is a multiplicative morphism. This finishes the proof. Again, the whole proof works in the same way for the functor $\mathcal{RH}om^{r}_{\mathcal{O}_Y}$.

Case C If we consider α as a morphism from $\Lambda^2(\mathrm{N}^*_{X/Y}[1])$ to $\mathrm{N}^*_{X/Y}[1]$ in the opposite category of $\mathrm{D}^{\mathrm{b}}(X)$, we notice that the induction relations provided by Corollary 5.6 are exactly the same as the ones proved in Proposition 2.6 (this is why we took the same notation c_p^k). Hence $(\mathrm{N}^*_{X/Y}[1], \alpha)$ is a Lie algebra object in the opposite derived category of X. Now according to the second part of the categorical PBW Theorem (Theorem 2.11), we can define an algebra structure on $\mathrm{S}(\mathrm{N}^*_{X/Y}[1])$ using the coefficients c_p^k and there is a natural multiplicative morphism from $\mathrm{T}(\mathrm{N}^*_{X/Y}[1])$ to $\mathrm{S}(\mathrm{N}^*_{X/Y}[1])$ endowed with this structure. To conclude, it suffices to notice that the following diagram

$$\begin{array}{ccc} \sigma_* \mathcal{RH}om_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X) & \longrightarrow & \mathcal{RH}om^{\ell}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \\ \downarrow \sim & & \downarrow \sim \\ \mathrm{T}(\mathrm{N}^*_{X/Y}[1]) & \longrightarrow & \mathrm{S}(\mathrm{N}^*_{X/Y}[1]) \end{array}$$

is commutative, which is nothing but the fact that the "geometric" coefficients c_p^k are the same as the "algebraic" coefficients c_p^k, that is Corollary 5.6. □

5.2. **The results of Kapranov, Markarian, Ramadoss and Yu.** In this section, we provide a new light on the foundational result in this theory: the construction of the Lie algebra structure on $\mathrm{T}_X[-1]$, due to Kapranov [11] and Markarian [14], and the computation of its universal enveloping algebra, due to Markarian [14], and Ramadoss [16]. Then we prove Ramadoss formula [15] that computes the big Chern classes of a vector bundle introduced by Kapranov in [11].

5.2.1. *The Lie algebra* $\mathrm{T}_X[-1]$. Given a smooth scheme/manifold X, we consider the special case of the quantized cycle $(\Delta_X, \mathrm{pr}_1)$ in the product $X \times X$.

Theorem 5.9 ([11], [14], [16])**.**

- *The object* $(\Omega^1_X[1], \mathrm{at}_{\Omega^1_X[1]})$ *is a Lie coalgebra in* $\mathrm{D}^{\mathrm{b}}(X)$.
- *The ring object* $\mathrm{pr}_{1*}\mathcal{RH}om_{\mathcal{O}_{X\times X}}(\mathcal{O}_X, \mathcal{O}_X)$ *is isomorphic to* $\mathrm{U}(\mathrm{T}_X[-1])$.
- *The map* $\mathrm{pr}_{1*}\mathcal{RH}om_{\mathcal{O}_{\overline{X}}}(\mathcal{O}_X, \mathcal{O}_X) \to \mathrm{pr}_{1*}\mathcal{RH}om_{\mathcal{O}_{X\times X}}(\mathcal{O}_X, \mathcal{O}_X)$ *identifies with the natural projection map* $\mathrm{T}(\mathrm{T}_X[-1]) \to \mathrm{U}(\mathrm{T}_X[-1])$.

Proof. The only thing we must prove is that α identifies with the Atiyah class of Ω^1_X, which is well-known: this follows from looking at the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{pr}_{1*}\dfrac{\mathcal{I}^2_{\Delta_X}}{\mathcal{I}^3_{\Delta_X}} \arrow[r] \arrow[d] & \mathrm{pr}_{1*}\dfrac{\mathcal{I}_{\Delta_X}}{\mathcal{I}^3_{\Delta_X}} \arrow[r] \arrow[d] & \mathrm{pr}_{1*}\dfrac{\mathcal{I}_{\Delta_X}}{\mathcal{I}^2_{\Delta_X}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{pr}_{1*}\left(\dfrac{\mathcal{I}_{\Delta_X}}{\mathcal{I}^2_{\Delta_X}} \otimes \mathrm{pr}_2^*\Omega^1_X\right) \arrow[r] & \mathrm{pr}_{1*}\left(\dfrac{\mathcal{O}_{X\times X}}{\mathcal{I}^2_{\Delta_X}} \otimes \mathrm{pr}_2^*\Omega^1_X\right) \arrow[r] & \mathrm{pr}_{1*}\left(\dfrac{\mathcal{O}_{X\times X}}{\mathcal{I}_{\Delta_X}} \otimes \mathrm{pr}_2^*\Omega^1_X\right) \arrow[r] & 0
\end{tikzcd}$$

where the vertical maps are given by differentation with respect to the second variable (so that they are all linear). The right vertical map is the isomorphism given by the quantization pr_1, and the left vertical map is the symetrization morphism. □

Remark 5.10. Any object $\mathcal{F}$ in $\mathrm{D}^{\mathrm{b}}(X)$ defines a representation of the Lie algebra $\mathrm{T}_X[-1]$, which is obtained by the chain of morphisms

$$\mathrm{T}_X[-1] \otimes \mathcal{F} \xrightarrow{\mathrm{id}\otimes \mathrm{at}_{\mathcal{F}}} \mathrm{T}_X[-1] \otimes \Omega^1_X[1] \otimes \mathcal{F} \xrightarrow{\mathrm{ev}\otimes \mathrm{id}} \mathcal{F}$$

For any $\mathcal{F}, \mathcal{G}$ in $\mathrm{D}^{\mathrm{b}}(X)$ viewed as representations of $\mathrm{T}_X[-1]$, the naturality of Atiyah classes implies that

$$\mathrm{Hom}_{\mathrm{Rep}(\mathrm{T}_X[-1])}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(\mathcal{F}, \mathcal{G}).$$

In other words, $\mathrm{D}^{\mathrm{b}}(X)$ embeds as a full subcategory of $\mathrm{Rep}\,(\mathrm{T}_X[-1])$.

Theorem 5.9 allows to give a Lie-theoretic interpretation of the tameness condition (Definition 4.11) for quantized cycles. To a quantized cycle (X, σ) in Y, we have an exact sequence

$$0 \to \mathrm{T}X[-1] \to \mathrm{T}Y[-1]_{|X} \to \mathrm{N}_{X/Y}[-1] \to 0$$

and σ provides a splitting of this sequence.

Theorem 5.11. *The triplet* $(\mathrm{T}X[-1], \mathrm{T}Y[-1]_{|X}, \mathrm{N}_{X/Y}[-1])$ *is a reductive pair of Lie objects, as defined in §3.3. Besides this pair is tame in the sense of Definition 3.12 if and only if* (X, σ) *is tame in the sense of Definition 4.11. In this case, the dual of* α *defines a Lie structure on* $\mathrm{N}_{X/Y}[-1]$.

Proof. We switch from Lie algebras objects to Lie coalgebras objects, so that we see $\Omega^1_{Y|X}[1]$ as a Lie coalgebra in $\mathrm{D}^{\mathrm{b}}(X)$. This Lie coalgebra is described by Corollary 4.14: first the diagram

$$\begin{tikzcd}
\Omega^1_{Y|X}[1] \arrow[r] \arrow[d] & \Omega^1_{Y|X}[1] \otimes \Omega^1_{Y|X}[1] \arrow[r] & \Omega^1_X[1] \otimes \Omega^1_X[1] \arrow[d, equal] \\
\Omega^1_X[1] \arrow[rr] & & \Omega^1_X[1] \otimes \Omega^1_X[1]
\end{tikzcd}$$

commutes, so the morphism $\Omega^1_{Y|X} \to \Omega^1_X$ is a morphism of Lie coalgebras objects. Next the tameness of the pair (in the sens of Definition 3.12) can be explicited as follows: if consider the morphism

$$\Omega^1_X[1] \to \Omega^1_{Y|X}[1] \to \Omega^1_{Y|X}[1] \otimes \Omega^1_{Y|X}[1] \to \mathrm{N}^*_{X/Y}[1] \otimes \mathrm{N}^*_{X/Y}[1],$$

tameness means the vanishing of the composite morphism

$$\mathrm{N}^*_{X/Y}[1] \to \Omega^1_X[1] \otimes \mathrm{N}^*_{X/Y}[1] \to (\mathrm{N}^*_{X/Y}[1] \otimes \mathrm{N}^*_{X/Y}[1]) \otimes \mathrm{N}^*_{X/Y}[1].$$

The first morphism is the class β, and the second one is $(\beta \otimes \mathrm{id}_{\mathrm{N}^*_{X/Y}[1]}) \circ \mathrm{at}_{\mathrm{N}^*_{X/Y}[1]}$. Thanks to Proposition 4.10, this is $\gamma_\sigma(\mathrm{N}^*_{X/Y})$, and we are done. For the last point, we remark that the composition

$$\mathrm{N}^*_{X/Y}[1] \to \Omega^1_{Y|X}[1] \to \Omega^1_{Y|X}[1] \otimes \Omega^1_{Y|X}[1] \to \mathrm{N}^*_{X/Y}[1] \otimes \mathrm{N}^*_{X/Y}[1]$$

is exactly α. □

5.2.2. *Big Chern classes.* Let us recall that for any vector bundle E on X, the big Chern classes $\hat{c}_p(E)$ of E live in $\mathrm{H}^p(X, \mathrm{T}^p\Omega^1_X)$, they are obtained by composing the morphisms $\mathrm{at}_E, \mathrm{id}_{\Omega^1_X} \otimes \mathrm{at}_E, \dots, \mathrm{id}_{\Omega^{p-1}_X} \otimes \mathrm{at}_E$ and then taking the trace on E (without antisymmetrizing on the factor $\mathrm{T}^p\Omega^1_X$).

Theorem 5.12. *For any vector bundle E on X, we have $\hat{c}_p(E) = \sum_{p=1}^k c_p^k(X) \circ c_k(E)$ where c_p^k are the universal elements in $\mathrm{Ext}^{p-k}(\Lambda^k X, \mathrm{T}^p X)$ associated to the Lie algebra $\mathrm{T}_X[-1]$.*

Proof. On $X \times X$, we have $\Delta_p = \sum_{k=1}^p c_p^k \circ \Delta_k^+$. This gives

$$\mathrm{pr}_{2*}(\Delta_p \otimes \mathrm{id}_{\mathrm{pr}_1^*\mathrm{E}}) = \sum_{k=1}^p (c_p^k \otimes \mathrm{id}_E) \circ \mathrm{pr}_{2*}(\Delta_k^+ \otimes \mathrm{id}_{\mathrm{pr}_1^*\mathrm{E}}).$$

The result follows by taking the trace on E. □

Remark 5.13. As explained in [15], the total Chern class of a vector bundle E can be interpreted in terms of the representation of the Lie algebra $\mathrm{T}_X[-1]$. Indeed E defines a representation of $\mathrm{T}_X[-1]$, whith is (the dual of) the Atiyah class of E, hence a map from $\mathrm{U}(\mathrm{T}_X[-1])$ to $\mathrm{End}\,(E)$. Its trace defines a map from $\mathrm{U}(\mathrm{T}_X[-1])$ to $\mathcal{O}_X$, which is exactly $\sum_p c_p(E)$ via the isomorphism

$$\mathrm{Hom}_{\mathrm{D^b}(X)}(\mathrm{U}(\mathrm{T}_X[-1]), \mathcal{O}_X) \simeq \bigoplus_p \mathrm{H}^p(X, \Omega^p_X).$$

5.2.3. *The quantized cycle class.* Let us recall the definition of the quantized cycle class introduced in [8]. For any quantized analytic cycle (X, σ) in Y, we consider the composition

$$\omega_{X/Y} \simeq \mathcal{RH}om^{\mathrm{r}}(\mathcal{O}_X, \mathcal{O}_Y) \to \mathcal{RH}om^{\mathrm{r}}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \simeq \mathrm{S}(\mathrm{N}_{X/Y}[-1]) \tag{13}$$

where the last isomorphism is the dual HKR isomorphism. Let d be the codimension of X in Y.

Proposition 5.14. *Assume that (X, σ) is tame. Then the morphism (13) is a d-torsion morphism for the Lie algebra $\mathrm{N}_{X/Y}[-1]$.*

Proof. We must prove that the composition

$$\mathrm{N}_{X/Y}[-1] \otimes \omega_{X/Y} \to \mathrm{N}_{X/Y}[-1] \otimes \mathrm{S}(\mathrm{N}_{X/Y}[-1]) \simeq \mathrm{N}_{X/Y}[-1] \otimes \mathrm{U}(\mathrm{N}_{X/Y}[-1]) \to \mathrm{U}(\mathrm{N}_{X/Y}[-1])$$

vanishes, where the last map is given the multiplication in the algebra $\mathrm{U}(\mathrm{N}_{X/Y}[-1])$. We can rewrite this map (using duality) as a morphism from $\mathrm{U}(\mathrm{N}_{X/Y}[-1])$ to $\mathrm{N}^*_{X/Y}[1] \otimes \mathrm{U}(\mathrm{N}_{X/Y}[-1])$, and the question reduces to the vanishing of the map

$$\omega_{X/Y} \to \mathrm{S}(\mathrm{N}_{X/Y}[-1]) \simeq \mathrm{U}(\mathrm{N}_{X/Y}[-1]) \to \mathrm{N}^*_{X/Y}[1] \otimes \mathrm{U}(\mathrm{N}_{X/Y}[-1])$$

Using Theorem 5.8, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{RH}om^r_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) & \xrightarrow{\Delta_1 \circ} & \mathcal{RH}om^r_{\mathcal{O}_Y}(\mathcal{O}_X, \mathrm{N}^*_{X/Y}[1]) \\ \sim \downarrow & & \downarrow \sim \\ \mathrm{U}(\mathrm{N}_{X/Y}[-1]) & \longrightarrow & \mathrm{N}^*_{X/Y}[1] \otimes \mathrm{U}(\mathrm{N}_{X/Y}[-1]) \end{array}$$

Hence the morphism we want to look at is (modulo isomorphism on the target) the composition

$$\omega_{X/Y} \simeq \mathcal{RH}om^{\mathrm{r}}(\mathcal{O}_X, \mathcal{O}_Y) \to \mathcal{RH}om^{\mathrm{r}}(\mathcal{O}_X, \mathcal{O}_X) \to \mathcal{RH}om^{\mathrm{r}}(\mathcal{O}_X, \mathrm{N}^*_{X/Y}[1]).$$

We can factor the first arrow through $\mathcal{RH}om^{\mathrm{r}}(\mathcal{O}_X, \mathcal{O}_S)$, and the composition

$$\mathcal{RH}om^{\mathrm{r}}(\mathcal{O}_X, \mathcal{O}_S) \to \mathcal{RH}om^{\mathrm{r}}(\mathcal{O}_X, \mathcal{O}_X) \to \mathcal{RH}om^{\mathrm{r}}(\mathcal{O}_X, \mathrm{N}^*_{X/Y}[1])$$

vanishes, since it is obtained by composing two successive arrows of a distinguished triangle. □

Theorem 5.15. [18] *Let (X, σ) be a tame quantized cycle in Y. Via the isomorphism*

$$\mathrm{RHom}_{\mathrm{D^b}(Y)}(\omega_{X/Y}, \mathrm{S}(\mathrm{N}_{X/Y}[1])) \simeq \mathbb{H}^0(X, \mathrm{S}(\mathrm{N}^*_{X/Y}[1])) \simeq \mathrm{S}(\mathrm{N}^*_{X/Y}[1])^{\mathrm{N}_{X/Y}[-1]},$$

the image of the morphism (13) is the Duflo element of $\mathrm{N}_{X/Y}[-1]$.

Proof. Thanks to Proposition 5.14 and Theorem 3.11, the morphism (13) is obtained by contracting the morphism

$$\omega_{X/Y} \simeq \mathrm{S}^d(\mathrm{N}_{X/Y}[-1]) \to \mathrm{S}(\mathrm{N}_{X/Y}[-1])$$

by the Duflo element of $\mathrm{N}_{X/Y}[-1]$. This gives the result. □

6. The Ext algebras

6.1. **Definitions.** Let (X, Y, σ) be a fixed quantized cycle. For any k, we denote by $X_Y^{(k)}$ is the kth formal neighborhood of X in Y.

Definition 6.1. $\mathcal{A}_{X/Y}^{(k)}$ is the algebra $\bigoplus_{n=0}^{\infty} \mathrm{Ext}^n_{X_Y^{(k)}}(\mathcal{O}_X, \mathcal{O}_X)$, the algebra structure being given by the Yoneda product.

For any k, there are natural algebra morphisms

$$\mathcal{A}_{X/Y}^{(0)} \hookrightarrow \mathcal{A}_{X/Y}^{(1)} \longrightarrow \mathcal{A}_{X/Y}^{(2)} \longrightarrow \cdots \longrightarrow \mathcal{A}_{X/Y}^{(k)} \longrightarrow \cdots \longrightarrow \mathcal{A}_{X/Y}^{(\infty)}$$

Note that all the algebras $\mathcal{A}_{X/Y}^{(k)}$ are naturally graded by the integer n, we call this grading the degree grading. Thanks to the Proposition (4.15), the algebra $\mathcal{A}_{X/Y}^{(1)}$ is canonically isomorphic (as an algebra) to the algebra

$$\bigoplus_{n=0}^{\infty} \bigoplus_{p=0}^{n} \mathrm{Ext}^{n-p}(\mathrm{T}^p \mathrm{N}^*_{X/Y}, \mathcal{O}_X)$$

via the map that attaches to any φ in $\mathrm{Ext}^{n-p}(\mathrm{T}^p \mathrm{N}^*_{X/Y}, \mathcal{O}_X)$ the element $\varphi \circ \Delta_p$. As a corollary, the algebra $\mathcal{A}_{X/Y}^{(1)}$ carries another natural grading given by the integer p, which is completely different from the degree grading; we call this grading the Lie grading. Elements of depth zero correspond to the sub-algebra $\mathcal{A}_{X/Y}^{(0)}$. Using the sequence (10), the composition

$$\bigoplus_{n=0}^{\infty} \bigoplus_{p=0}^{n} \mathrm{Ext}^{n-p}(\Lambda^p \mathrm{N}^*_{X/Y}, \mathcal{O}_X) \longrightarrow \bigoplus_{n=0}^{\infty} \bigoplus_{p=0}^{n} \mathrm{Ext}^{n-p}(\mathrm{T}^p \mathrm{N}^*_{X/Y}, \mathcal{O}_X) \simeq \mathcal{A}_{X/Y}^{(1)} \longrightarrow \mathcal{A}_{X/Y}^{(\infty)}$$

is an isomorphism of $\mathbf{k}$-vector spaces. Hence the map $\mathcal{A}_{X/Y}^{(1)} \longrightarrow \mathcal{A}_{X/Y}^{(\infty)}$ is surjective, and there is an isomorphism

$$\mathcal{A}_{X/Y}^{(\infty)} \simeq \bigoplus_{n=0}^{\infty} \bigoplus_{p=0}^{n} \mathrm{Ext}^{n-p}(\Lambda^p \mathrm{N}^*_{X/Y}, \mathcal{O}_X).$$

of $\mathbf{k}$-vector spaces obtained by attaching to any φ in $\mathrm{Ext}^{n-p}(\Lambda^p \mathrm{N}^*_{X/Y}, \mathcal{O}_X)$ the push forward of the element $\varphi \circ \Delta_p^-$ from S to Y. The integer p defines a grading on $\mathcal{A}_{X/Y}^{(\infty)}$ but it no longer respects the algebra structure, however the corresponding ascending filtration does. We call this filtration the Lie filtration.

6.2. **Enrichment.** For any scheme (or complex manifold) Z, the category $\mathrm{D}^{\mathrm{b}}(Z)$ is enriched over $\mathrm{D}^{\mathrm{b}}(\mathbf{k})$ in the following way:

$$\mathrm{D}^{\mathrm{b}}(Z)(\mathcal{F}, \mathcal{G}) := \mathrm{R}\Gamma(X, \mathcal{RH}om_{\mathrm{D}^{\mathrm{b}}(Z)}(\mathcal{F}, \mathcal{G}))\,.$$

Remark 6.2.

- The enrichment is symmetric monoidal. In particular, the tensor product functor is an enriched functor.
- As $\mathbf{k}$ is a field, we can replace $\mathrm{D}^{\mathrm{b}}(\mathbf{k})$ by the equivalent category $\mathbf{k}^{\mathbb{Z}}$. Through this equivalence we have
$$\mathrm{D}^{\mathrm{b}}(Z)(\mathcal{F}, \mathcal{G}) \simeq \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(Z)}(\mathcal{F}, \mathcal{G}[n]).$$
- The Ext algebras admit a very simple description using this formalism, they are given by the formula
$$\mathcal{A}_{X/Y}^{(k)} = \mathrm{D}^{\mathrm{b}}\left(X_Y^{(k)}\right)(\mathcal{O}_X, \mathcal{O}_X)\,.$$

Let E an object in $\mathrm{D}^{\mathrm{b}}(Z)$, which we view as a module over the Lie algebra object $\mathrm{T}_X[-1]$ using the Atiyah class. There are two possible definitions of the V-invariants E^V of E:

(1) the standard one: $E^V := \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(Z)}(\mathcal{O}_Z, E)$, where $\mathcal{O}_Z$ is equipped with the trivial $\mathrm{T}_X[-1]$-module structure.
(2) the enriched one: $E^V := \mathrm{D}^{\mathrm{b}}(Z)(\mathcal{O}_Z, E)$.

Below we always use the enriched version.

6.3. **Structure theorem.** If (X, σ) is tame, we can explicitly describe the Ext algebra $\mathcal{A}^{(\infty)}_{X/Y}$. We have an exact sequence

$$0 \to \mathrm{T}_X[-1] \to (\mathrm{T}_Y[-1])_{|X} \to \mathrm{N}_{X/Y}[-1] \to 0.$$

Recall that $\mathrm{N}_{X/Y}[-1]$ is naturally endowed with a Lie structure (given by α).

Theorem 6.3. *Assume that (X, σ) is a tame quantized cycle in Y. Then using the corresponding Lie structure on $\mathrm{N}_{X/Y}[-1]$, the algebra $\mathcal{A}^{(\infty)}_{X/Y}$ is naturally isomorphic to $\mathrm{U}(\mathrm{N}_{X/Y}[-1])^{\mathrm{T}_X[-1]}$. Besides, there is a commutative diagram*

$$\begin{array}{ccc} \mathcal{A}^{(1)}_{X/Y} & \longrightarrow & \mathcal{A}^{(\infty)}_{X/Y} \\ \downarrow & & \downarrow \\ (\mathrm{T}(\mathrm{N}_{X/Y}[-1]))^{\mathrm{T}_X[-1]} & \longrightarrow & (\mathrm{U}(\mathrm{N}_{X/Y}[-1]))^{\mathrm{T}_X[-1]} \end{array}$$

Proof. We have already proved that there is an isomorphism of algebra objects between $\mathcal{RH}om^{\ell}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ and $\mathrm{U}(\mathrm{N}_{X/Y}[-1])$. Applying the derived global section functor we get an isomorphism of graded algebras between $\mathcal{A}^{(\infty)}_{X/Y}$ and $\mathrm{R}\Gamma(X, \mathrm{U}(\mathrm{N}_{X/Y}[-1]))$. Finally, observe that the $\mathrm{T}_X[-1]$-module structure of $\mathrm{U}(\mathrm{N}_{X/Y}[-1]) \cong \mathrm{S}(\mathrm{N}_{X/Y}[-1])$ is given by the Atiyah class of $\mathrm{S}(\mathrm{N}_{X/Y}[-1])$. Therefore the algebra $\mathrm{R}\Gamma(X, \mathrm{U}(\mathrm{N}_{X/Y}[-1]))$ is indeed $\mathrm{U}(\mathrm{N}_{X/Y}[-1])^{\mathrm{T}_X[-1]}$. □

6.4. **The algebra $\mathcal{A}^{(2)}_{X/Y}$.** In this section, we describe completely the image of $\mathcal{A}^{(1)}_{X/Y}$ in $\mathcal{A}^{(2)}_{X/Y}$ for tame quantized cycles. The result, which seems quite surprising at first sight, runs as follows:

Theorem 6.4. *Assume that (X, σ) is tame in Y. The surjective morphism from $\mathcal{A}^{(2)}_{X/Y}$ to $\mathcal{A}^{(\infty)}_{X/Y}$ admits a canonical section.*

Proof. Let $\mathcal{R}_{X/Y}$ be the kernel of the map $\mathcal{A}^{(1)}_{X/Y} \to \mathcal{A}^{(\infty)}_{X/Y}$. The kernel of the map $\mathcal{A}^{(1)}_{X/Y} \to \mathcal{A}^{(2)}_{X/Y}$ is a subalgebra of $\mathcal{R}_{X/Y}$, and we must prove that any element of $\mathcal{R}_{X/Y}$ maps to zero in $\mathcal{A}^{(2)}_{X/Y}$. We prove this for all the subspaces $\mathrm{F}_p\mathcal{R}_{X/Y}$ (where F_p is the Lie filtration), and argue by induction on p. If $p = 0$, this is trivial since $\mathrm{F}_0\mathcal{R}_{X/Y}$ vanishes. For any i with $1 \leq i \leq p$, we can write in $\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(S)}(\mathcal{O}_X, \mathrm{S}^p\mathrm{N}^*_{X/Y}[p])$ using Lemma 5.4

$$\begin{aligned} \Delta_p &= (-1)^p \Theta_{\mathrm{T}^{p-1}\mathrm{N}^*_{X/Y}} \circ \Theta_{\mathrm{T}^{p-2}\mathrm{N}^*_{X/Y}} \circ \cdots \circ (\Theta_{\mathrm{T}^{i+1}\mathrm{N}^*_{X/Y}} \circ \Theta_{\mathrm{T}^{i}\mathrm{N}^*_{X/Y}}) \circ \cdots \circ \Theta_{\mathcal{O}_X} \\ &= \mathfrak{A}_{p,i} + \Theta_{p,i} + \mathfrak{R}_{p,i} \end{aligned}$$

where:

- $\mathfrak{A}_{p,i} \in \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(S)}(\mathcal{O}_X, \mathrm{T}^{p-i-1}\mathrm{N}^*_{X/Y} \otimes \Lambda^2\mathrm{N}^*_{X/Y} \otimes \mathrm{T}^{i-1}\mathrm{N}^*_{X/Y}[p])$.
- $\Theta_{p,i}$ is in $\mathrm{F}_{p-1}\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(S)}(\mathcal{O}_X, \mathrm{T}^{p-i-1}\mathrm{N}^*_{X/Y} \otimes \mathrm{sym}^2\mathrm{N}^*_{X/Y} \otimes \mathrm{T}^{i-1}\mathrm{N}^*_{X/Y}[p])$.
- $\mathfrak{R}_{p,i}$ vanishes in $\mathcal{A}^{(2)}_{X/Y}$.

For $1 \leq i \leq p-1$, let $\chi_{p,i} \colon \mathrm{T}^p\mathrm{N}^*_{X/Y} \to \mathrm{T}^{i-1}\mathrm{N}^*_{X/Y} \otimes \mathrm{S}^2\mathrm{N}^*_{X/Y} \otimes \mathrm{T}^{p-i-1}\mathrm{N}^*_{X/Y}$ be the partial symmetrization map. Then we have

$$\chi_{p,i} \circ \Delta_p = \chi_{p,i} \circ \mathfrak{R}_{p,i} + \Theta_{p,i}.$$

It follows from the dual version of Lemma 2.1 that there exist an element $\mathfrak{R}_p$ in the kernel of the natural map $\mathcal{A}^{(1)}_{X/Y} \to \mathcal{A}^{(2)}_{X/Y}$ such that $\Delta_p^- + \mathfrak{R}_p$ lies in $\mathrm{F}_{p-1}\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(S)}(\mathcal{O}_X, \mathrm{T}^p\mathrm{N}^*_{X/Y}[p])$. Now any element in $\mathrm{F}_p\mathcal{R}_{X/Y}$

can be written as $\varphi \circ \Delta_p^- + \psi$ where ψ belongs to $\mathrm{F}_{p-1}\mathcal{R}_{X/Y}$, and we conclude by induction since elements in $\mathrm{F}_{p-1}\mathcal{R}_{X/Y}$ map to zero in $\mathcal{A}^{(2)}_{X/Y}$. □

7. Conclusion and perspectives

7.1. **State of the art in the tame case.** In this paper we introduced a tameness condition for a quantized cycle (X, σ) in Y. Under this assumption we proved that the shifted normal sheaf $\mathrm{N}_{X/Y}[-1]$ is endowed with a Lie bracket and that the sheaf of derived $\mathcal{O}_Y$-linear endomorphisms of $\mathcal{O}_X$ is isomorphic to the universal enveloping algebra of $\mathrm{N}_{X/Y}[-1]$. Using this, we were able to:

- identify the quantized cycle class with the Duflo element of the Lie algebra object $\mathrm{N}_{X/Y}[-1]$, recovering and reinterpreting in Lie algebraic terms a result Yu.
- describe explicitly the Ext algebra $\mathrm{Ext}^\bullet_Y(\mathcal{O}_X, \mathcal{O}_X)$.

7.2. **Beyond the tame case.** At first sight, the diagonal cycle $X \hookrightarrow X \times X$ admits two distinguished quantizations (*i.e.* first order retractions): the two projections pr_1 and pr_2. However, since the space of quantizations of a cycle is an affine space, this gives a whole line of quantizations, namely $t\mathrm{pr}_1 + (1-t)\mathrm{pr}_2$ for t in the base field $\mathbf{k}$.

In general these quantizations are never tame except for $t = 0$ or $t = 1$, that is for the two projections. However, the value $t = \dfrac{1}{2}$ is special since the corresponding quantized cycle fits in a very interesting family of quantized cycles: every fixed-point locus $X = Y^\iota$ of an involution ι of Y defines naturally a quantized cycle. In our former example, $Y = X \times X$ and $\iota(x, x') = (x', x)$. In Lie algebraic terms this corresponds to symmetric pairs. We will study further on this case in future work.

References

[1] Dima Arinkin and Andrei Căldăraru. When is the self-intersection of a subvariety a fibration? *Adv. Math.*, 231(2):815–842, 2012.

[2] D. Calaque and M. Van den Bergh. Hochschild cohomology and Atiyah classes. *Adv. Math.*, 224(5):1839–1889, 2010.

[3] Damien Calaque, Andrei Căldăraru, and Junwu Tu. PBW for an inclusion of Lie algebras. *J. of Algebra*, 378:64–79, 2013.

[4] Damien Calaque, Andrei Căldăraru, and Junwu Tu. On the Lie algebroid of a derived self-intersection. *Adv. Math.*, 262:751–783, 2014.

[5] Pierre Deligne and John W. Morgan. Notes on supersymmetry (following Joseph Bernstein). In *Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997)*, pages 41–97. Amer. Math. Soc., Providence, RI, 1999.

[6] Michel Duflo. Opérateurs différentiels bi-invariants sur un groupe de Lie. *Annales scientifiques de l'École normale supérieure*, 10(2):265–288, 1977.

[7] Julien Grivaux. On a conjecture of Kashiwara relating Chern and Euler classes of o-modules. *J. Differential Geom.*, 90(2):267–275, 2012.

[8] Julien Grivaux. The Hochschild-Kostant-Rosenberg isomorphism for quantized analytic cycles. *Int. Math. Res. Not. IMRN*, (4):865–913, 2014.

[9] Julien Grivaux. Derived geometry of the first formal neighbourhood of a smooth analytic cycle. *Preprint*, 2017.

[10] G. Hochschild, Bertram Kostant, and Alex Rosenberg. Differential forms on regular affine algebras. *Trans. Amer. Math. Soc.*, 102:383–408, 1962.

[11] M. Kapranov. Rozansky-Witten invariants via Atiyah classes. *Compositio Math.*, 115(1):71–113, 1999.

[12] M. Kashiwara and P. Schapira. *Deformation quantization modules*. Astérisque. Société mathématique de France, 2012.

[13] Maxim Kontsevich. Deformation quantization of Poisson manifolds. *Lett. Math. Phys.*, 66(3):157–216, 2003.

[14] Nikita Markarian. The Atiyah class, Hochschild cohomology and the Riemann-Roch theorem. *J. Lond. Math. Soc. (2)*, 79(1):129–143, 2009.

[15] Ajay C. Ramadoss. The big Chern classes and the Chern character. *Internat. J. Math.*, 19(6):699–746, 2008.

[16] Ajay C. Ramadoss. The relative Riemann-Roch theorem from Hochschild homology. *New York J. Math.*, 14:643–717, 2008.

[17] Domingo Toledo and Yue Lin L. Tong. Duality and intersection theory in complex manifolds. I. *Mathematische Annalen*, 237(1):41–77, 1978.

[18] Shilin Yu. Todd class via homotopy perturbation theory. *Preprint*, 2016.

Institut Montpelliérain Alexander Grothendieck, CNRS, Univ. Montpellier & Institut Universitaire de France

CNRS, I2M (Marseille) & IHÉS
E-mail address: damien.calaque@umontpellier.fr
E-mail address: jgrivaux@math.cnrs.fr