HAL open science

Blowup of H1 solutions for a class of the focusing inhomogeneous nonlinear Schrodinger equation

van Duong Dinh

To cite this version:

van Duong Dinh. Blowup of H1 solutions for a class of the focusing inhomogeneous nonlinear Schrodinger equation. Nonlinear Analysis: Theory, Methods and Applications, 2018, 174, pp.169-188. hal-01649943

HAL Id: hal-01649943

https://hal.science/hal-01649943v1

Submitted on 28 Nov 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

BLOWUP OF H^1 SOLUTIONS FOR A CLASS OF THE FOCUSING INHOMOGENEOUS NONLINEAR SCHRÖDINGER EQUATION

VAN DUONG DINH

Abstract. In this paper, we consider a class of the focusing inhomogeneous nonlinear Schrödinger equation

$$i\partial_t u + \Delta u + |x|^{-b}|u|^\alpha u = 0, \quad u(0) = u_0 \in H^1(\mathbb{R}^d),$$

with $0 < b < \min\{2, d\}$ and $\alpha_\star \leq \alpha < \alpha^\star$ where $\alpha_\star = \frac{4-2b}{d}$ and $\alpha^\star = \frac{4-2b}{d-2}$ if $d \geq 3$ and $\alpha^\star = \infty$ if $d = 1, 2$. In the mass-critical case $\alpha = \alpha_\star$, we prove that if u_0 has negative energy and satisfies either $xu_0 \in L^2$ with $d \geq 1$ or u_0 is radial with $d \geq 2$, then the corresponding solution blows up in finite time. Moreover, when $d = 1$, we prove that if the initial data (not necessarily radial) has negative energy, then the corresponding solution blows up in finite time. In the mass and energy intercritical case $\alpha_\star < \alpha < \alpha^\star$, we prove the blowup below ground state for radial initial data with $d \geq 2$. This result extends the one of Farah in [9] where the author proved blowup below ground state for data in the virial space $H^1 \cap L^2(|x|^2 dx)$ with $d \geq 1$.

1. Introduction

Consider the Cauchy problem for the inhomogeneous nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u + \mu|x|^{-b}|u|^\alpha u &= 0, \\ u(0) &= u_0, \end{cases} \tag{INLS}$$

where $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}, u_0 : \mathbb{R}^d \to \mathbb{C}$, $\mu = \pm 1$ and $\alpha, b > 0$. The parameters $\mu = 1$ and $\mu = -1$ correspond to the focusing and defocusing cases respectively. The case $b = 0$ is the well-known nonlinear Schrödinger equation which has been studied extensively over the last three decades. The inhomogeneous nonlinear Schrödinger equation arises naturally in nonlinear optics for the propagation of laser beams, and it is of a form

$$i\partial_t u + \Delta u + K(x)|u|^\alpha u = 0. \tag{1.1}$$

The (INLS) is a particular case of (1.1) with $K(x) = |x|^{-b}$. The equation (1.1) has been attracted a lot of interest in a past several years. Bergé in [1] studied formally the stability condition for soliton solutions of (1.1). Towers-Malomed in [29] observed by means of variational approximation and direct simulations that a certain type of time-dependent nonlinear medium gives rise to completely stabe beams. Merle in [19] and Raphaël-Szeftel in [22] studied (1.1) for $k_1 < K(x) < k_2$ with $k_1, k_2 > 0$. Fibich-Wang in [12] investigated (1.1) with $K(x) := K(\epsilon|x|)$ where $\epsilon > 0$ is small and $K \in C^4(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. The case $K(x) = |x|^b$ with $b > 0$ is studied by many authors (see e.g. [4, 17, 32] and references therein).

In order to recall known results for the (INLS), let us give some facts for this equation. We firstly note that the (INLS) is invariant under the scaling,

$$u_\lambda(t, x) := \lambda^{\frac{2-b}{\alpha}} u(\lambda^2 t, \lambda x), \quad \lambda > 0.$$

2010 *Mathematics Subject Classification.* 35B44, 35Q55.

Key words and phrases. Inhomogeneous nonlinear Schrödinger equation; Blowup; Virial identity; Gagliardo-Nirenberg inequality.

An easy computation shows

$$\|u_\lambda(0)\|_{\dot H^\gamma} = \lambda^{\gamma+\frac{2-b}{\alpha}-\frac{d}{2}}\|u_0\|_{\dot H^\gamma}.$$

Thus, the critical Sobolev exponent is given by

$$\gamma_c := \frac{d}{2} - \frac{2-b}{\alpha}. \tag{1.2}$$

Moreover, the (INLS) has the following conserved quantities:

$$M(u(t)) := \int |u(t,x)|^2 dx = M(u_0), \tag{1.3}$$

$$E(u(t)) := \int \frac{1}{2}|\nabla u(t,x)|^2 - \frac{\mu}{\alpha+2}|x|^{-b}|u(t,x)|^{\alpha+2}dx = E(u_0). \tag{1.4}$$

The well-posedness for the (INLS) was firstly studied by Genoud-Stuart in [13, Appendix] by using the argument of Cazenave [3] which does not use Strichartz estimates. More precisely, the authors showed that the focusing (INLS) with $0 < b < \min\{2, d\}$ is well posed in H^1:

- locally if $0 < \alpha < \alpha^\star$,
- globally for any initial data if $0 < \alpha < \alpha_\star$,
- globally for small initial data if $\alpha_\star \le \alpha < \alpha^\star$,

where $\alpha_\star$ and $\alpha^\star$ are defined by

$$\alpha_\star := \frac{4-2b}{d}, \quad \alpha^\star := \begin{cases} \frac{4-2b}{d-2} & \text{if } d \ge 3, \\ \infty & \text{if } d = 1, 2. \end{cases} \tag{1.5}$$

In the case $\alpha = \alpha_\star$ (L^2-critical), Genoud in [14] showed that the focusing (INLS) with $0 < b < \min\{2, d\}$ is globally well-posed in H^1 assuming $u_0 \in H^1$ and

$$\|u_0\|_{L^2} < \|Q\|_{L^2},$$

where Q is the unique nonnegative, radially symmetric, decreasing solution of the ground state equation

$$\Delta Q - Q + |x|^{-b}|Q|^{\frac{4-2b}{d}}Q = 0.$$

Also, Combet-Genoud in [6] established the classification of minimal mass blow-up solutions for the focusing L^2-critical (INLS).

In the case $\alpha_\star < \alpha < \alpha^\star$, Farah in [9] showed that the focusing (INLS) with $0 < b < \min\{2, d\}$ is globally well-posedness in $H^1, d \ge 1$ assuming $u_0 \in H^1$ and

$$E(u_0)^{\gamma_c}M(u_0)^{1-\gamma_c} < E(Q)^{\gamma_c}M(Q)^{1-\gamma_c}, \tag{1.6}$$

$$\|\nabla u_0\|_{L^2}^{\gamma_c}\|u_0\|_{L^2}^{1-\gamma_c} < \|\nabla Q\|_{L^2}^{\gamma_c}\|Q\|_{L^2}^{1-\gamma_c},$$

where Q is the unique nonnegative, radially symmetric, decreasing solution of the ground state equation

$$\Delta Q - Q + |x|^{-b}|Q|^\alpha Q = 0. \tag{1.7}$$

He also proved that if $u_0 \in H^1 \cap L^2(|x|^2dx) =: \Sigma$ satisfies (1.6) and

$$\|\nabla u_0\|_{L^2}^{\gamma_c}\|u_0\|_{L^2}^{1-\gamma_c} > \|\nabla Q\|_{L^2}^{\gamma_c}\|Q\|_{L^2}^{1-\gamma_c}, \tag{1.8}$$

then the blow-up in H^1 must occur. Afterwards, Farah-Guzman in [10, 11] proved that the above global solution is scattering under the radial condition of the initial data.

Guzman in [16] used Strichartz estimates and the contraction mapping argument to establish the local well-posedness as well as the small data global well-posedness for the (INLS) in Sobolev space. Recently, the author in [7] improved the local well-posedness in H^1 of Guzman by extending

the validity of b in the two and three dimensional spatial spaces. Note that the results of Guzman [16] and Dinh [7] about the local well-posedness of (INLS) in H^1 are a bit weaker than the one of Genoud-Stuart [13]. More precisely, they do not treat the case $d = 1$, and there is a restriction on the validity of b when $d = 2$ or 3. However, the local well-posedness proved in [16, 7] provides more information on the solutions, for instance, one knows that the global solutions to the defocusing (INLS) satisfy $u \in L^p_{loc}(\mathbb{R}, W^{1,q})$ for any Schrödinger admissible pair (p, q). This property plays an important role in proving the scattering for the (INLS). Note also that the author in [7] pointed out that one cannot expect a similar local well-posedness result for (INLS) in H^1 as in [16, 7] holds in the one dimensional case by using Strichartz estimates.

In [7], the author used the so-called pseudo-conformal conservation law to show the decaying property of global solutions to the defocusing (INLS) by assuming the initial data in Σ (see before (1.8)). In particular, he showed that in the case $\alpha \in [\alpha_\star, \alpha^\star)$, global solutions have the same decay as the solutions of the linear Schrödinger equation, that is for $2 \le q \le \frac{2d}{d-2}$ when $d \ge 3$ or $2 \le q < \infty$ when $d = 2$ or $2 \le q \le \infty$ when $d = 1$,

$$\|u(t)\|_{L^q(\mathbb{R}^d)} \lesssim |t|^{-d\left(\frac{1}{2}-\frac{1}{q}\right)}, \quad \forall t \neq 0.$$

This allows the author proved the scattering in Σ for a certain class of the defocusing (INLS). Later, the author in [8] made use of the classical Morawetz inequality and an argument of [30] to derive the decay of global solutions to the defocusing (INLS) with the initial data in H^1. Using the decaying property, he was able to show the energy scattering for a class of the defocusing (INLS). We refer the reader to [7, 8] for more details.

The main purpose of this paper is to show the finite time blowup for the focusing (INLS). Thanks to the well-posedness of Genoud-Stuart [13], we only expect blowup in H^1 when $\alpha_\star \le \alpha < \alpha_\star$ which correspond to the mass-critical and the mass and energy intercritical cases. Note that the local well-posedness for the energy-critical (INLS), i.e. $\alpha = \alpha^\star$ is still an open problem.

Our first result is the following finite time blowup for the (INLS) in the mass-critical case $\alpha = \alpha_\star$.

Theorem 1.1. *Let $0 < b < \min\{2, d\}$ and $u_0 \in H^1$. Then the corresponding solution to the focusing mass-critical* (INLS) *blows up in finite time if one of the following conditions holds true:*

1. *$d \ge 1, E(u_0) < 0$ and $xu_0 \in L^2$,*
2. *$d \ge 2, E(u_0) < 0$ and u_0 is radial,*
3. *$d = 1$ and $E(u_0) < 0$.*

Remark 1.2.

1. This theorem extends the well-known finite time blowup for the focusing mass-critical nonlinear Schrödinger equation (i.e. $b = 0$ in (INLS)) [20, 21] to the focusing mass-critical (INLS).
2. The condition $E(u_0) < 0$ is a sufficient condition for the finite time blowup but it is not necessary. In fact, one can show (see Remark 4.2) that for any $E > 0$, there exists $u_0 \in H^1$ satisfying $E(u_0) = E$ and the corresponding solution blows up in finite time.

We now consider the intercritical (i.e. mass-supercritical and energy-subcritical) case $\alpha_\star < \alpha < \alpha^\star$. Our next result is the following blowup for the intercritical (INLS).

Theorem 1.3. *Let*

$$d \ge 3, \quad 0 < b < 2, \quad \alpha_\star < \alpha < \alpha^\star,$$

or

$$d = 2, \quad 0 < b < 2, \quad \alpha_\star < \alpha < 4.$$

Let $u_0 \in H^1$ be radial and satisfy

$$E(u_0)M(u_0)^\sigma < E(Q)M(Q)^\sigma, \tag{1.9}$$

and

$$\|\nabla u_0\|_{L^2}\|u_0\|_{L^2}^{\sigma} > \|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma}, \tag{1.10}$$

where

$$\sigma := \frac{1-\gamma_c}{\gamma_c} = \frac{2(2-b)-(d-2)\alpha}{d\alpha - 2(2-b)}, \tag{1.11}$$

and Q is the unique solution the ground state equation (1.7). Then the corresponding solution to the focusing intercritical (INLS) *blows up in finite time. Moreover,*

$$\|\nabla u(t)\|_{L^2}\|u(t)\|_{L^2}^{\sigma} > \|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma}, \tag{1.12}$$

for any t in the existence time.

Remark 1.4.
1. The restriction $\alpha < 4$ when $d = 2$ is technical due to the Young inequality (see Lemma 3.4).
1. In [9], the author proved the finite time blowup for the focusing intercritical (INLS) for initial data $u_0 \in H^1 \cap L^2(|x|^2dx), d \geq 1$ satisfying (1.9) and (1.10).
2. It was proved in [9] that if the initial data u_0 satisfies (1.9) and $\|\nabla u_0\|_{L^2}\|u_0\|_{L^2}^{\sigma} < \|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma}$, then the corresponding solution exists globally in time.

This paper is organized as follows. In Section 2, we recall the sharp Gagliardo-Nirenberg inequality related to the focusing (INLS) due to Farah [9]. In Section 3, we derive the standard virial identity and localized virial estimates for the focusing (INLS). We will give the proof of Theorem 1.1 in Section 4. Finally, the proof of Theorem 1.3 will be given in Section 5.

2. Sharp Gagliardo-Nirenberg inequality

In this section, we recall the sharp Gagliardo-Nirenberg inequality related to the focusing (INLS) due to Farah [9].

Theorem 2.1 (Sharp Gagliardo-Nirenberg inequality [9])**.** *Let $d \geq 1, 0 < b < \min\{2, d\}$ and $0 < \alpha < \alpha^\star$. Then the Gagliardo-Nirenberg inequality*

$$\int |x|^{-b}|u(x)|^{\alpha+2}dx \leq C_{\mathrm{GN}}\|u\|_{L^2}^{\frac{4-2b-(d-2)\alpha}{2}}\|\nabla u\|_{L^2}^{\frac{d\alpha+2b}{2}}, \tag{2.1}$$

holds true, and the sharp constant C_{GN} is attended by a function Q, i.e.

$$C_{\mathrm{GN}} = \int |x|^{-b}|Q(x)|^{\alpha+2}dx \div \Big[\|Q\|_{L^2}^{\frac{4-2b-(d-2)\alpha}{2}}\|\nabla Q\|_{L^2}^{\frac{d\alpha+2b}{2}}\Big], \tag{2.2}$$

where Q is the unique non-negative, radially symmetric, decreasing solution to the elliptic equation

$$\Delta Q - Q + |x|^{-b}|Q|^{\alpha}Q = 0. \tag{2.3}$$

Remark 2.2.
1. In [9], the author proved this result for $\alpha_\star < \alpha < \alpha^\star$. However, the proof and so the result are still valid for $0 < \alpha \leq \alpha_\star$.
2. The existence and uniqueness of the solution Q to the elliptic equation (2.3) was proved by Toland [28] and Yanagida [31] (see also Genoud-Stuart [13]).
3. We also have the following Pohozaev identities:
$$\|Q\|_{L^2}^2 = \frac{4-2b-(d-2)\alpha}{d\alpha+2b}\|\nabla Q\|_{L^2}^2 = \frac{4-2b-(d-2)\alpha}{2(\alpha+2)}\int |x|^{-b}|Q(x)|^{\alpha+2}dx. \tag{2.4}$$
In particular,
$$C_{\mathrm{GN}} = \frac{2(\alpha+2)}{4-2b-(d-2)\alpha}\Big[\frac{4-2b-(d-2)\alpha}{d\alpha+2b}\Big]^{\frac{d\alpha+2b}{4}}\frac{1}{\|Q\|_{L^2}^{\alpha}}. \tag{2.5}$$

3. Virial identities

In this section, we derive virial identities and virial estimates related to the focusing (INLS). Given a smooth real valued function a, we define the virial potential by

$$V_a(t) := \int a(x)|u(t,x)|^2 dx. \tag{3.1}$$

By a direct computation, we have the following result (see e.g. [27, Lemma 5.3] for the proof.)

Lemma 3.1 ([27])**.** *If u is a smooth-in-time and Schwartz-in-space solution to*

$$i\partial_t u + \Delta u = N(u),$$

with $N(u)$ satisfying $\mathrm{Im}\ (N(u)\overline{u}) = 0$*, then we have*

$$\frac{d}{dt}V_a(t) = 2\int_{\mathbb{R}^d} \nabla a(x) \cdot \mathrm{Im}\ (\overline{u}(t,x)\nabla u(t,x))dx, \tag{3.2}$$

and

$$\begin{aligned}\frac{d^2}{dt^2}V_a(t) = -\int \Delta^2 a(x)|u(t,x)|^2 dx + 4\sum_{j,k=1}^{d}\int \partial_{jk}^2 a(x)\mathrm{Re}\ (\partial_k u(t,x)\partial_j \overline{u}(t,x))dx \\ + 2\int \nabla a(x)\cdot\{N(u),u\}_p(t,x)dx,\end{aligned} \tag{3.3}$$

where $\{f,g\}_p := \mathrm{Re}\ (f\nabla\overline{g} - g\nabla\overline{f})$ is the momentum bracket.

We note that if $N(u) = -|x|^{-b}|u^\alpha u$, then

$$\{N(u),u\}_p = \frac{\alpha}{\alpha+2}\nabla(|x|^{-b}|u|^{\alpha+2}) + \frac{2}{\alpha+2}\nabla(|x|^{-b})|u|^{\alpha+2}.$$

Using this fact, we immediately have the following result.

Corollary 3.2. *If u is a smooth-in-time and Schwartz-in-space solution to the focusing* (INLS)*, then we have*

$$\begin{aligned}\frac{d^2}{dt^2}V_a(t) = -\int \Delta^2 a(x)|u(t,x)|^2 dx + 4\sum_{j,k=1}^{d}\int \partial_{jk}^2 a(x)\mathrm{Re}\ (\partial_k u(t,x)\partial_j \overline{u}(t,x))dx \\ -\frac{2\alpha}{\alpha+2}\int \Delta a(x)|x|^{-b}|u(t,x)|^{\alpha+2}dx + \frac{4}{\alpha+2}\int \nabla a(x)\cdot\nabla(|x|^{-b})|u(t,x)|^{\alpha+2}dx.\end{aligned} \tag{3.4}$$

A direct consequence of Corollary 3.2 is the following standard virial identity for the (INLS).

Lemma 3.3. *Let $u_0 \in H^1$ be such that $|x|u_0 \in L^2$ and $u : I \times \mathbb{R}^d \to \mathbb{C}$ the corresponding solution to the focusing* (INLS)*. Then, $|x|u \in C(I, L^2)$. Moreover, for any $t \in I$,*

$$\frac{d^2}{dt^2}\|xu(t)\|_{L^2_x}^2 = 8\|\nabla u(t)\|_{L^2}^2 - \frac{4(d\alpha+2b)}{\alpha+2}\int |x|^{-b}|u(t,x)|^{\alpha+2}dx. \tag{3.5}$$

Proof. The first claim follows from the standard approximation argument, we omit the proof and refer the reader to [3, Proposition 6.5.1] for more details. The identity (3.5) follows from Corollary 3.2 by taking $a(x) = |x|^2$. □

In order to prove the blowup for the focusing (INLS) with radial data, we need localized virial estimates. To do so, we introduce the smooth, non-negative function $\theta : [0,\infty) \to [0,\infty)$ satisfying

$$\theta(r) = \begin{cases} r^2 & \text{if } 0 \le r \le 1, \\ \text{const.} & \text{if } r \ge 2, \end{cases} \quad \text{and} \quad \theta''(r) \le 2 \quad \text{for } r \ge 0. \tag{3.6}$$

Note that the precise constant here is not important. For $R > 1$, we define the radial function

$$\varphi_R(x) = \varphi_R(r) := R^2\theta(r/R), \quad r = |x|. \tag{3.7}$$

It is easy to see that

$$2 - \varphi_R''(r) \geq 0, \quad 2 - \frac{\varphi_R'(r)}{r} \geq 0, \quad 2d - \Delta\varphi_R(x) \geq 0. \tag{3.8}$$

Lemma 3.4. *Let $d \geq 2, 0 < b < 2, 0 < \alpha < 4, R > 1$ and φ_R be as in (3.7). Let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a radial solution to the focusing* (INLS)*. Then for any $\epsilon > 0$ and any $t \in I$,*

$$\begin{aligned}\frac{d^2}{dt^2}V_{\varphi_R}(t) \leq 8\|\nabla u(t)\|_{L^2}^2 - \frac{4(d\alpha+2b)}{\alpha+2}\int |x|^{-b}|u(t,x)|^{\alpha+2}dx \\ +O\Big(R^{-2} + \epsilon^{-\frac{\alpha}{4-\alpha}}R^{-\frac{2[(d-1)\alpha+2b]}{4-\alpha}} + \epsilon\|\nabla u(t)\|_{L^2}^2\Big).\end{aligned} \tag{3.9}$$

Remark 3.5.
1. The condition $d \geq 2$ comes from the radial Sobolev embedding. This is due to the fact that radial functions in dimension 1 do not have any decaying property. The restriction $0 < \alpha < 4$ comes from the Young inequality below.
2. If we consider $\alpha_\star \leq \alpha \leq \alpha^\star$, then there is a restriction on the validity of α in 2D. More precisely, we need $\alpha_\star \leq \alpha < 4$ when $d = 2$.

Proof of Lemma 3.4. We apply (3.4) for $a(x) = \varphi_R(x)$ to get

$$\begin{aligned}\frac{d^2}{dt^2}V_{\varphi_R}(t) = -\int \Delta^2\varphi_R|u(t)|^2dx + 4\sum_{j,k=1}^d\int \partial_{jk}^2\varphi_R\mathrm{Re}\,(\partial_k u(t)\partial_j\overline{u}(t))dx \\ -\frac{2\alpha}{\alpha+2}\int \Delta\varphi_R|x|^{-b}|u(t)|^{\alpha+2}dx + \frac{4}{\alpha+2}\int \nabla\varphi_R\cdot\nabla(|x|^{-b})|u(t)|^{\alpha+2}dx.\end{aligned}$$

Since $\varphi_R(x) = |x|^2$ for $|x| \leq R$, we use (3.5) to have

$$\begin{aligned}\frac{d^2}{dt^2}V_{\varphi_R}(t) = 8\|\nabla u(t)\|_{L^2}^2 - \frac{4(d\alpha+2b)}{\alpha+2}\int |x|^{-b}|u(t)|^{\alpha+2}dx \\ -8\|\nabla u(t)\|_{L^2(|x|>R)}^2 + \frac{4(d\alpha+2b)}{\alpha+2}\int_{|x|>R}|x|^{-b}|u(t)|^{\alpha+2}dx \\ -\int_{|x|>R}\Delta^2\varphi_R|u(t)|^2dx + 4\sum_{j,k=1}^d\int_{|x|>R}\partial_{jk}^2\varphi_R\mathrm{Re}\,(\partial_k u(t)\partial_j\overline{u}(t))dx \\ -\frac{2\alpha}{\alpha+2}\int_{|x|>R}\Delta\varphi_R|x|^{-b}|u(t)|^{\alpha+2}dx + \frac{4}{\alpha+2}\int_{|x|>R}\nabla\varphi_R\cdot\nabla(|x|^{-b})|u(t)|^{\alpha+2}dx.\end{aligned} \tag{3.10}$$

Since $|\Delta\varphi_R| \lesssim 1, |\Delta^2\varphi_R| \lesssim R^{-2}$ and $|\nabla\varphi_R\cdot\nabla(|x|^{-b})| \lesssim |x|^{-b}$, we have

$$\begin{aligned}\frac{d^2}{dt^2}V_{\varphi_R}(t) = 8\|\nabla u(t)\|_{L^2}^2 - \frac{4(d\alpha+2b)}{\alpha+2}\int |x|^{-b}|u(t)|^{\alpha+2}dx \\ +4\sum_{j,k=1}^d\int_{|x|>R}\partial_{jk}^2\varphi_R\mathrm{Re}\,(\partial_k u(t)\partial_j\overline{u}(t))dx - 8\|\nabla u(t)\|_{L^2(|x|>R)}^2 \\ +O\Big(\int_{|x|>R}R^{-2}|u(t)|^2 + |x|^{-b}|u(t)|^{\alpha+2}dx\Big).\end{aligned}$$

Using (3.8) and the fact that

$$\partial_{jk}^2 = \Big(\frac{\delta_{jk}}{r} - \frac{x_jx_k}{r^3}\Big)\partial_r + \frac{x_jx_k}{r^2}\partial_r^2,$$

we see that

$$\sum_{j,k=1}^{d} \partial_{jk}^2 \varphi_R \partial_k u \partial_j \overline{u} = \varphi_R''(r)|\partial_r u|^2 \leq 2|\partial_r u|^2 = 2|\nabla u|^2.$$

Therefore

$$4\sum_{j,k=1}^{d} \int_{|x|>R} \partial_{jk}^2 \varphi_R \mathrm{Re}\,(\partial_k u(t)\partial_j \overline{u}(t))dx - 8\|\nabla u(t)\|_{L^2(|x|>R)}^2 \leq 0.$$

The conservation of mass then implies

$$\frac{d^2}{dt^2}V_{\varphi_R}(t) \leq 8\|\nabla u(t)\|_{L^2}^2 - \frac{4(d\alpha+2b)}{\alpha+2}\int |x|^{-b}|u(t)|^{\alpha+2}dx + O\Big(R^{-2} + \int_{|x|>R}|x|^{-b}|u(t)|^{\alpha+2}dx\Big).$$

It remains to bound $\int_{|x|>R}|x|^{-b}|u(t)|^{\alpha+2}dx$. To do this, we recall the following radial Sobolev embedding ([23, 5]).

Lemma 3.6 (Radial Sobolev embedding [23, 5])**.** *Let $d \geq 2$ and $\frac{1}{2} \leq s < 1$. Then for any radial function f,*

$$\sup_{x\neq 0}|x|^{\frac{d-2s}{2}}|f(x)| \leq C(d,s)\|f\|_{L^2}^{1-s}\|f\|_{\dot{H}^1}^{s}. \tag{3.11}$$

Moreover, the above inequality also holds for $d \geq 3$ and $s = 1$.

Using (3.11) with $s = \frac{1}{2}$ and the conservation of mass, we estimate

$$\begin{aligned}\int_{|x|>R}|x|^{-b}|u(t)|^{\alpha+2}dx &\leq \Big(\sup_{|x|>R}|x|^{-b}|u(t,x)|\Big)^{\alpha}\|u(t)\|_{L^2}^2\\ &\lesssim R^{-\left[\frac{(d-1)\alpha}{2}+b\right]}\Big(\sup_{|x|>R}|x|^{\frac{d-1}{2}}|u(t,x)|\Big)^{\alpha}\|u(t)\|_{L^2}^2\\ &\lesssim R^{-\left[\frac{(d-1)\alpha}{2}+b\right]}\|\nabla u(t)\|_{L^2}^{\frac{\alpha}{2}}\|u(t)\|_{L^2}^{\frac{\alpha}{2}+2}\\ &\lesssim R^{-\left[\frac{(d-1)\alpha}{2}+b\right]}\|\nabla u(t)\|_{L^2}^{\frac{\alpha}{2}}.\end{aligned}$$

We next recall the Young inequality: for a, b non-negative real numbers and p, q positive real numbers satisfying $\frac{1}{p}+\frac{1}{q}=1$, then for any $\epsilon > 0$, $ab \lesssim \epsilon a^p + \epsilon^{-\frac{q}{p}}b^q$. Applying the Young inequality for $a = \|\nabla u(t)\|_{L^2}^{\frac{\alpha}{2}}, b = R^{-\left[\frac{(d-1)\alpha}{2}+b\right]}$ and $p = \frac{4}{\alpha}, q = \frac{4}{4-\alpha}$, we get for any $\epsilon > 0$,

$$R^{-\left[\frac{(d-1)\alpha}{2}+b\right]}\|\nabla u(t)\|_{L^2}^{\frac{\alpha}{2}} \lesssim \epsilon\|\nabla u(t)\|_{L^2}^2 + \epsilon^{-\frac{\alpha}{4-\alpha}}R^{-\frac{2[(d-1)\alpha]+2b}{4-\alpha}}.$$

Note that the condition $0 < \alpha < 4$ ensures $1 < p, q < \infty$. The proof is complete. □

In the mass-critical case $\alpha = \alpha_\star$, we have the following refined version of Lemma 3.4. The proof of this result is based on an argument of [20] (see also [2]).

Lemma 3.7. *Let $d \geq 2, 0 < b < 2, R > 1$ and φ_R be as in (3.7). Let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a radial solution to the focusing mass-critical* (INLS)*, i.e. $\alpha = \alpha_\star$. Then for any $\epsilon > 0$ and any $t \in I$,*

$$\begin{aligned}\frac{d^2}{dt^2}V_{\varphi_R}(t) \leq 16E(u_0) - 2\int_{|x|>R}\Big(\chi_{1,R} - \frac{\epsilon}{d+2-b}\chi_{2,R}^{\frac{d}{2-b}}\Big)|\nabla u(t)|^2dx\\ + O\Big(R^{-2} + \epsilon R^{-2} + \epsilon^{-\frac{2-b}{d-2+b}}R^{-\frac{2d-2+b}{d-2+b}}\Big),\end{aligned} \tag{3.12}$$

where

$$\chi_{1,R} = 2(2-\varphi_R''), \quad \chi_{2,R} = (2-b)(2d-\Delta\varphi_R) + db\Big(2 - \frac{\varphi_R'}{r}\Big). \tag{3.13}$$

Proof. We firstly notice that

$$\sum_{j,k} \partial^2_{jk}\varphi_R \partial_k u \partial_j \overline{u} = \varphi''_R |\partial_r u|^2, \quad \nabla\varphi_R \cdot \nabla(|x|^{-b}) = -b\frac{\varphi'_R}{r}.$$

Using (3.10) with $\alpha = \alpha_\star = \frac{4-2b}{d}$ and rewriting $\varphi''_R = 2 - (2 - \varphi''_R)$, $\frac{\varphi'_R}{r} = 2 - \Big(2 - \frac{\varphi'_R}{r}\Big)$ and $\Delta\varphi_R = 2d - (2d - \Delta\varphi_R)$, we have

$$\begin{aligned}
\frac{d^2}{dt^2}V_{\varphi_R}(t) &= 16E(u(t)) - \int_{|x|>R} \Delta^2\varphi_R |u(t)|^2 dx - 4\int_{|x|>R} (2-\varphi''_R)|\partial_r u(t)|^2 dx \\
&\quad + \frac{4-2b}{d+2-b}\int_{|x|>R} (2d - \Delta\varphi_R)|x|^{-b}|u(t)|^{\frac{4-2b}{d}+2} dx \\
&\quad + \frac{2db}{d+2-b}\int_{|x|>R} \Big(2 - \frac{\varphi'_R}{r}\Big)|x|^{-b}|u(t)|^{\frac{4-2b}{d}+2} dx \\
&\le 16E(u(t)) + O(R^{-2}) - 2\int_{|x|>R} \chi_{1,R}|\nabla u(t)|^2 dx \\
&\quad + \frac{2}{d+2-b}\int_{|x|>R} \chi_{2,R}|x|^{-b}|u(t)|^{\frac{4-2b}{d}+2} dx,
\end{aligned}$$

where $\chi_{1,R}$ and $\chi_{2,R}$ are as in (3.13). Using the radial Sobolev embedding (3.11) with $s = \frac{1}{2}$ and the conservation of mass, we estimate

$$\begin{aligned}
\int_{|x|>R} \chi_{2,R}|x|^{-b}|u(t)|^{\frac{4-2b}{d}+2} dx &= \int_{|x|>R} |x|^{-b}\Big|\chi_{2,R}^{\frac{d}{4-2b}} u(t)\Big|^{\frac{4-2b}{d}} |u(t)|^2 dx \\
&\le \Big(\sup_{|x|>R} |x|^{-b}\Big|\chi_{2,R}^{\frac{d}{4-2b}}(x) u(t,x)\Big|\Big)^{\frac{4-2b}{d}} \|u(t)\|^2_{L^2} \\
&\lesssim R^{-\left[\frac{(2-b)(d-1)}{d}+b\right]} \Big\|\nabla\Big(\chi_{2,R}^{\frac{d}{4-b}} u(t)\Big)\Big\|_{L^2}^{\frac{2-b}{d}} \|u(t)\|_{L^2}^{2+\frac{2-b}{d}} \\
&\lesssim R^{-\left[\frac{(2-b)(d-1)}{d}+b\right]} \Big\|\nabla\Big(\chi_{2,R}^{\frac{d}{4-2b}} u(t)\Big)\Big\|_{L^2}^{\frac{4-2b}{d}}.
\end{aligned}$$

We next apply the Young inequality with $p = \frac{2d}{2-b}$ and $q = \frac{d}{d-2+b}$ to get for any $\epsilon > 0$

$$R^{-\left[\frac{(2-b)(d-1)}{d}+b\right]} \Big\|\nabla\Big(\chi_{2,R}^{\frac{d}{4-2b}} u(t)\Big)\Big\|_{L^2}^{\frac{4-2b}{d}} \lesssim \epsilon \Big\|\nabla\Big(\chi_{2,R}^{\frac{d}{4-2b}} u(t)\Big)\Big\|_{L^2}^2 + O\Big(\epsilon^{-\frac{2-b}{d-2+b}} R^{-\frac{2d-2+b}{d-2+b}}\Big).$$

Moreover, using (3.6), (3.7) and (3.8), it is easy to check that $|\nabla(\chi_{2,R}^{d/(4-2b)})| \lesssim R^{-1}$ for $|x| > R$. Thus the conservation of mass implies

$$\Big\|\nabla\Big(\chi_{2,R}^{\frac{d}{4-2b}} u(t)\Big)\Big\|_{L^2}^2 \lesssim R^{-2} + \Big\|\chi_{2,R}^{\frac{d}{4-2b}} \nabla u(t)\Big\|_{L^2}^2.$$

Combining the above estimates, we prove (3.12). □

To prove the blowup in the 1D mass-critical case $\alpha = 4-2b$, we need the following version of localized virial estimates due to [21]. Let ϑ be a real-valued function in $W^{3,\infty}$ satisfying

$$\vartheta(x) = \begin{cases} 2x & \text{if} \quad 0 \le |x| \le 1, \\ 2[x-(x-1)^3] & \text{if} \quad 1 < x \le 1+1/\sqrt{3}, \\ 2[x-(x+1)^3] & \text{if} \quad -(1+1/\sqrt{3}) \le x < -1, \\ \vartheta' < 0 & \text{if} \quad 1+1/\sqrt{3} < |x| < 2, \\ 0 & \text{if} \quad |x| \ge 2. \end{cases} \tag{3.14}$$

Set

$$\theta(x) = \int_0^x \vartheta(s)ds. \tag{3.15}$$

Lemma 3.8. *Let $0<b<1$ and θ be as in (3.15). Let $u : I \times \mathbb{R} \to \mathbb{C}$ be a solution to the focusing mass-critical* (INLS)*, i.e. $\alpha = 4-2b$. There exists $a_0 > 0$ such that if*

$$\|u(t)\|_{L^2(|x|>1)} \le a_0, \tag{3.16}$$

for any $t \in I$, then there exists $C>0$ such that

$$\frac{d^2}{dt^2}V_\theta(t) \le 16E(u_0) + C(1+N)^{2-b}\|u(t)\|^{6-2b}_{L^2(|x|>1)} + N\|u(t)\|^2_{L^2(|x|>1)}, \tag{3.17}$$

for any $t \in I$, where $N := \|\partial_x\vartheta\|_{L^\infty} + \|\partial_x^2\vartheta\|_{L^\infty} + \|\partial_x^3\vartheta\|_{L^\infty}$.

Proof. We apply (3.4) with $a(x) = \theta(x)$ to get

$$\begin{aligned}\frac{d^2}{dt^2}V_\theta(t) = &-\int \partial_x^4\theta|u(t)|^2dx + 4\int \partial_x^2\theta|\partial_x u(t)|^2dx - \frac{4-2b}{3-b}\int \partial_x^2\theta|x|^{-b}|u(t)|^{6-2b}dx \\ &+\frac{2}{3-b}\int \partial_x\theta\partial_x(|x|^{-b})|u(t)|^{6-2b}dx.\end{aligned}$$

Since $\theta(x) = x^2$ on $|x| \le 1$, the definition of energy implies

$$\begin{aligned}\frac{d^2}{dt^2}V_\theta(t) &= 16E(u(t)) - \int_{|x|>1}\partial_x^4\theta|u(t)|^2dx - 4\int_{|x|>1}(2-\partial_x^2\theta)|\partial_x u(t)|^2dx \\ &\quad +\frac{4-2b}{3-b}\int_{|x|>1}(2-\partial_x^2\theta)|x|^{-b}|u(t)|^{6-2b}dx + \frac{2b}{3-b}\int_{|x|>1}\Big(2-\frac{\partial_x\theta}{x}\Big)|x|^{-b}|u(t)|^{6-2b}dx \\ &= 16E(u_0) - \int_{|x|>1}\partial_x^4\theta|u(t)|^2dx - 2\int_{|x|>1}\chi_1|\partial_x u(t)|^2dx \\ &\qquad +\frac{2}{3-b}\int_{|x|>1}\chi_2|x|^{-b}|u(t)|^{6-2b}dx,\end{aligned} \tag{3.18}$$

where

$$\chi_1 := 2(2-\partial_x^2\theta), \quad \chi_2 := (2-b)(2-\partial_x^2\theta) + b\Big(2-\frac{\partial_x\theta}{x}\Big).$$

We now estimate

$$\begin{aligned}\int_{|x|>1}\chi_2|x|^{-b}|u(t)|^{6-2b}dx &\le \Big(\sup_{|x|>1}\chi_2|x|^{-b}|u(t)|^{4-2b}\Big)\|u(t)\|^2_{L^2(|x|>1)} \\ &\le \|\rho u(t)\|^{4-2b}_{L^\infty(|x|>1)}\|u(t)\|^2_{L^2(|x|>1)},\end{aligned}$$

where $\rho(x) := \chi_2^{\frac{1}{4-2b}}(x)$. Using a variant of the Gagliardo-Nirenberg inequality (see e.g. [21, Lemma 2.1]), we bound

$$\|\rho u(t)\|_{L^\infty(|x|>1)} \leq \|u(t)\|^{1/2}_{L^2(|x|>1)} \Big[2\|\rho^2\partial_x u(t)\|_{L^2(|x|>1)} + \|u(t)\partial_x(\rho^2)\|_{L^2(|x|>1)}\Big]^{1/2}.$$

Thus,

$$\begin{aligned}\int_{|x|>1} \chi_2|x|^{-b}|u(t)|^{6-2b}dx &\leq \|u(t)\|^{4-b}_{L^2(|x|>1)} \Big[2\|\rho^2\partial_x u(t)\|_{L^2(|x|>1)} + \|u(t)\partial_x(\rho^2)\|_{L^2(|x|>1)}\Big]^{2-b} \\ &\leq \|u(t)\|^{4-b}_{L^2(|x|>1)} 2^{1-b}\Big[2^{2-b}\|\rho^2\partial_x u(t)\|^{2-b}_{L^2(|x|>1)} + \|u(t)\partial_x(\rho^2)\|^{2-b}_{L^2(|x|>1)}\Big] \\ &\leq 2^{3-2b}\|u(t)\|^{4-b}_{L^2(|x|>1)}\|\rho^2\partial_x u(t)\|^{2-b}_{L^2(|x|>1)} \\ &\qquad + 2^{1-b}\|u(t)\|^{6-2b}_{L^2(|x|>1)}\|\partial_x(\rho^2)\|^{2-b}_{L^\infty(|x|>1)}. \end{aligned} \tag{3.19}$$

We next estimate $\|\partial_x(\rho^2)\|_{L^\infty(|x|>1)}$. By the definition of ρ, we write

$$\partial_x(\rho^2) = \frac{1}{2-b}\frac{\partial_x\chi_2}{\chi_2^{\frac{1-b}{2-b}}}.$$

On $1 < |x| \leq 1 + 1/\sqrt{3}$, a direct computation shows

$$2 - \frac{\partial_x\theta}{x} = 2\frac{(|x|-1)^3}{|x|}, \quad 2 - \partial^2\theta = 6(|x|-1)^2.$$

Thus,

$$\chi_2 = 6(2-b)(|x|-1)^2 + 2b\frac{(|x|-1)^3}{|x|},$$

and

$$\partial_x\chi_2 = \begin{cases} (x-1)\left[12(2-b) + 2b\frac{(x-1)(3x-1)}{x^2}\right] & \text{if} \quad 1 < x \leq 1+1/\sqrt{3}, \\ (x+1)\left[12(2-b) + 2b\frac{(x+1)(3x-1)}{x^2}\right] & \text{if} \quad -(1+1/\sqrt{3}) \leq x < -1. \end{cases}$$

Thus

$$\frac{\partial_x\chi_2}{\chi_2^{\frac{1-b}{2-b}}} = \begin{cases} (x-1)^{\frac{b}{2-b}}\frac{\left[12(2-b)+2b\frac{(x-1)(3x-1)}{x^2}\right]}{\left[6(2-b)+2b\frac{x-1}{x}\right]^{\frac{1-b}{2-b}}} & \text{if} \quad 1 < x \leq 1+1/\sqrt{3}, \\ (x+1)^{\frac{b}{2-b}}\frac{\left[12(2-b)+2b\frac{(x+1)(3x-1)}{x^2}\right]}{\left[6(2-b)+2b\frac{x+1}{x}\right]^{\frac{1-b}{2-b}}} & \text{if} \quad -(1+1/\sqrt{3}) \leq x < -1. \end{cases}$$

This implies that $\partial_x\chi_2/\chi_2^{\frac{1-b}{2-b}}$ is uniformly bounded on $1 < |x| \leq 1 + 1/\sqrt{3}$.

On $|x| > 1 + 1/\sqrt{3}$, we note that $\chi_2 \geq 4$ since $\partial_x^2\theta$ and $\partial_x\theta/x$ are both non-positive there by the choice of ϑ. We thus simply bound

$$\left|\partial_x\chi_2/\chi_2^{\frac{1-b}{2-b}}\right| \lesssim \|\partial_x\vartheta\|_{L^\infty} + \|\partial_x^2\vartheta\|_{L^\infty} \lesssim N.$$

Therefore,

$$\|\partial_x(\rho^2)\|_{L^\infty(|x|>1)} \lesssim 1 + N.$$

□

Combining this with (3.19), we obtain

$$\int_{|x|>1} \chi_2|x|^{-b}|u(t)|^{6-2b}dx \leq 2^{3-2b}\|u(t)\|^{4-b}_{L^2(|x|>1)}\|\rho^2\partial_x u(t)\|^{2-b}_{L^2(|x|>1)} + C(1+N)^{2-b}\|u(t)\|^{6-2b}_{L^2(|x|>1)}, \tag{3.20}$$

for some constant $C>0$. We thus get from (3.18) and (3.20) that

$$\begin{aligned}\frac{d^2}{dt^2}V_\theta(t) &\le 16E(u_0)+N\|u(t)\|^2_{L^2(|x|>1)}-2\int_{|x|>1}\chi_1|\partial_x u(t)|^2dx\\ &\qquad+\frac{2^{4-2b}}{3-b}\|u(t)\|^{4-b}_{L^2(|x|>1)}\|\rho^2\partial_x u(t)\|^{2-b}_{L^2(|x|>1)}+C(1+N)^{2-b}\|u(t)\|^{6-2b}_{L^2(|x|>1)}\\ &\le 16E(u_0)-2\int_{|x|>1}\Big(\chi_1-\frac{2^{3-2b}}{3-b}\chi_2\|u(t)\|^{4-b}_{L^2(|x|>1)}\Big)|\partial_x u(t)|^2dx\\ &\qquad+C(1+N)^{2-b}\|u(t)\|^{6-2b}_{L^2(|x|>1)}+N\|u(t)\|^2_{L^2(|x|>1)}.\end{aligned}$$

We will show that if $\|u(t)\|_{L^2(|x|>1)}\le a_0$ for some $a_0>0$ small enough, then

$$\chi_1-\frac{2^{3-2b}}{3-b}\chi_2\|u(t)\|^{4-b}_{L^2(|x|>1)}\ge 0, \tag{3.21}$$

for any $|x|>1$. It immediately yields (3.17). It remains to prove (3.21). To do so, it is enough to show for some $a_0>0$ small enough,

$$\chi_1-a_0\frac{2^{3-2b}}{3-b}\chi_2\ge 0, \tag{3.22}$$

for any $|x|>1$.

On $1<|x|\le 1+1/\sqrt{3}$, we have

$$\chi_1=2(2-\partial_x^2\theta)=12(|x|-1)^2,$$

and

$$\begin{aligned}\chi_2=(2-b)(2-\partial_x^2\theta)+b\Big(2-\frac{\partial_x\theta}{x}\Big)&=6(2-b)(|x|-1)^2+2b\frac{(|x|-1)^3}{|x|}\\ &=6(|x|-1)^2\Big[2-b+b\frac{|x|-1}{3|x|}\Big]<6(|x|-1)^2\Big[2-b+\frac{b}{3\sqrt{3}}\Big].\end{aligned}$$

Thus, by taking $a_0>0$ small enough, we have (3.22).

On $|x|>1+1/\sqrt{3}$, since $\partial_x^2\theta=\partial_x\vartheta\le 0$, we have $\chi_1\ge 4$. Moreover, $\chi_2\le C$ for some constant $C>0$. We thus get (3.22) by taking $a_0>0$ small enough. The proof is complete.

4. Mass-critical case $\alpha=\alpha_\star$

In this section, we will give the proof of Theorem 1.1.

4.1. The case $d\ge 1, E(u_0)<0$ and $xu_0\in L^2$.

Applying (3.5) with $\alpha=\alpha_\star$, we see that

$$\frac{d^2}{dt^2}\|xu(t)\|^2_{L^2}=8\|\nabla u(t)\|^2_{L^2}-\frac{16}{\alpha_\star+2}\int|x|^{-b}|u(t,x)|^{\alpha_\star+2}dx=16E(u_0)<0.$$

By the classical argument of Glassey [15], the solution must blow up in finite time.

4.2. The case $d\ge 2, E(u_0)<0$ and u_0 is radial.

We use the localized virial estimate (3.12) to have

$$\begin{aligned}\frac{d^2}{dt^2}V_{\varphi_R}(t)\le 16E_c(u_0)-2\int_{|x|>R}\Big(\chi_{1,R}-\frac{\epsilon}{d+2-b}\chi_{2,R}^{\frac{d}{2-b}}\Big)|\nabla u(t)|^2dx\\ +O\Big(R^{-2}+\epsilon R^{-2}+\epsilon^{-\frac{2-b}{d-2+b}}R^{-\frac{2d-2+b}{d-2+b}}\Big),\end{aligned}$$

where

$$\chi_{1,R} = 2(2-\varphi_R''), \quad \chi_{2,R} = (2-b)(2d-\Delta\varphi_R) + db\Big(2-\frac{\varphi_R'}{r}\Big).$$

If we choose a suitable radial function φ_R defined by (3.7) so that

$$\chi_{1,R} - \frac{\epsilon}{d+2-b}\chi_{2,R}^{\frac{d}{2-b}} \geq 0, \quad \forall r > R, \tag{4.1}$$

for a sufficiently small $\epsilon > 0$, then by choosing $R > 1$ sufficiently large depending on ϵ, we see that

$$\frac{d^2}{dt^2}V_{\varphi_R}(t) \leq 8E(u_0) < 0,$$

for any t in the existence time. This shows that the solution u blows up in finite time. It remains to find φ_R so that (4.1) holds true. To do so, we follow the argument of [20]. Let us define the smooth function

$$\vartheta(r) := \begin{cases} 2r & \text{if} \quad 0 \leq r \leq 1, \\ 2[r-(r-1)^3] & \text{if} \quad 1 < r \leq 1+1/\sqrt{3}, \\ \vartheta' < 0 & \text{if} \quad 1+1/\sqrt{3} < r < 2, \\ 0 & \text{if} \quad r \geq 2, \end{cases}$$

and

$$\theta(r) := \int_0^r \vartheta(s)ds.$$

It is easy to see that θ satisfies (3.6). We thus define φ_R as in (3.7). We show that (4.1) holds true for this choice of φ_R. Using the fact

$$\Delta\varphi_R(x) = \varphi_R''(r) + \frac{d-1}{r}\varphi_R'(r),$$

we have

$$\chi_{2,R} = (2-b)(2-\varphi_R'') + (2d-2+b)\Big(2-\frac{\varphi_R'}{r}\Big).$$

By the definition of φ_R,

$$\varphi_R'(r) = R\theta'(r/R) = R\vartheta(r/R), \quad \varphi_R''(r) = \theta''(r/R) = \vartheta'(r/R).$$

When $R < r \leq (1+1/\sqrt{3})R$, we have

$$\chi_{1,R}(r) = 12\Big(\frac{r}{R}-1\Big)^2,$$

and

$$\chi_{2,R}(r) = 6\Big(\frac{r}{R}-1\Big)^2\Big[2-b+\frac{(2d-2+b)(r/R-1)}{3r/R}\Big] < 6\Big(\frac{r}{R}-1\Big)^2\Big(2-b+\frac{2d-2+b}{3\sqrt{3}}\Big).$$

Since $0 < r/R - 1 < 1/\sqrt{3}$, we can choose $\epsilon > 0$ small enough so that (4.1) is satisfied.

When $r > (1+1/\sqrt{3})R$, we see that $\vartheta'(r/R) \leq 0$, so $\chi_{1,R}(r) = 2(2-\varphi_R''(r)) \geq 4$. We also have $\chi_{2,R}(r) \leq C$ for some constant $C > 0$. Thus by choosing $\epsilon > 0$ small enough, we have (4.1).

4.3. **The case $d = 1$ and $E(u_0) < 0$.** We follow the argument of [21]. We only consider the positive time, the negative one is treated similarly. We argue by contradiction and assume that the solution exists for all $t \geq 0$. We divide the proof in two steps.

Step 1. We assume that the initial data satisfies

$$\delta := -16E(u_0) - C(1+N)^2\|u_0\|_{L^2}^{6-2b} - N\|u_0\|_{L^2}^2 > 0, \tag{4.2}$$

$$\Big(\int \theta|u_0|^2dx\Big)^{1/2}\Big(\frac{2}{\delta}\|\partial_x u_0\|_{L^2}^2 + 1\Big)^{1/2} \le \frac{1}{2}a_0, \tag{4.3}$$

where C, N, θ and a_0 are defined as in Lemma 3.8. We will show that if u_0 satisfies (4.2) and (4.3), then the corresponding solution satisfies (3.16) for all $t \ge 0$. Since $\theta(x) \ge 1$ for $|x| > 1$ and $\delta > 0$, we have from (4.3) that

$$\|u_0\|_{L^2(|x|>1)} \le \frac{1}{2}a_0. \tag{4.4}$$

Let us define

$$T_0 := \sup\{t > 0 : \|u(s)\|_{L^2(|x|>1)} \le a_0, \quad 0 \le s < t\}.$$

Since $s \mapsto \|u(s)\|_{L^2(|x|>1)}$ is continuous, (4.4) implies $T_0 > 0$. If $T_0 = \infty$, we are done. Suppose that $T_0 < \infty$. The continuity in L^2 of $u(t)$ gives

$$\|u(T_0)\|_{L^2(|x|>1)} = a_0. \tag{4.5}$$

On the other hand, $u(t)$ satisfies the assumption of Lemma 3.8 on $[0, T_0)$. We thus get from Lemma 3.8 and (4.2) that

$$\begin{aligned}
\int \theta|u(t)|^2dx &\le \int \theta|u_0|^2dx - 2t\mathrm{Im}\int \partial_x\theta\overline{u}_0\partial_x u_0dx - \frac{\delta}{2}t^2 \\
&= -\frac{\delta}{2}\Big(t + \frac{1}{\delta}\mathrm{Im}\int \partial_x\theta\overline{u}_0\partial_x u_0dx\Big)^2 + \frac{1}{2\delta}\Big(\mathrm{Im}\int \partial_x\theta\overline{u}_0\partial_x u_0dx\Big)^2 + \int \theta|u_0|^2dx \\
&\le \frac{1}{2\delta}\Big(\mathrm{Im}\int \partial_x\theta\overline{u}_0\partial_x u_0dx\Big)^2 + \int \theta|u_0|^2dx \\
&\le \frac{1}{2\delta}\|\partial_x\theta u_0\|_{L^2}^2\|\partial_x u_0\|_{L^2}^2 + \int \theta|u_0|^2dx,
\end{aligned} \tag{4.6}$$

for all $0 \le t < T_0$. By the definition of θ, it is easy to see that $\theta \ge \vartheta^2/4 = (\partial_x\theta)^2/4$ for any $x \in \mathbb{R}$. Thus, (4.6) yields

$$\int \theta|u(t)|^2dx \le \Big(\frac{2}{\delta}\|\partial_x u_0\|_{L^2}^2 + 1\Big)\int \theta|u_0|^2dx,$$

for all $0 \le t < T_0$. By (4.3) and the fact that $\theta \ge 1$ on $|x| > 1$, we obtain

$$\|u(t)\|_{L^2(|x|>1)} \le \Big(\int \theta|u(t)|^2dx\Big)^{1/2} \le \frac{1}{2}a_0,$$

for all $0 \le t < T_0$. By the continuity of $u(t)$ in L^2, we get

$$\|u(T_0)\|_{L^2(|x|>1)} \le \frac{1}{2}a_0.$$

This contradicts with (4.5). Therefore, the assumptions of Lemma 3.8 are satisfied with $I = [0, \infty)$ and we get

$$\frac{d^2}{dt^2}V_\theta(t) \le -\delta < 0,$$

for all $t \ge 0$. This is impossible. Hence, if the initial data u_0 satisfies (4.2) and (4.3), then the corresponding solution must blow up in finite time.

Step 2. In this step, we will use the scaling

$$u_\lambda(t,x) = \lambda^{-\frac{1}{2}} u(\lambda^{-2}t, \lambda^{-1}x), \quad \lambda > 0 \tag{4.7}$$

to transform all initial data with negative energy into initial data satisfying (4.2) and (4.3). Note that the 1D mass-critical (INLS) is invariant under (4.7), that is, if $u(t)$ is a solution to the 1D mass-critical (INLS) with initial data u_0, then $u_\lambda(t)$ is also a solution to the 1D mass-critical (INLS) with initial data $u_\lambda(0)$. Moreover, we have

$$\|u_\lambda(t)\|_{L^2} = \|u_\lambda(0)\|_{L^2} = \|u_0\|_{L^2}, \tag{4.8}$$

$$E(u_\lambda(t)) = E(u_\lambda(0)) = \lambda^{-2}E(u_0), \tag{4.9}$$

for any t as long as the solution exists.

We will show that there exists $\lambda > 0$ such that

$$\delta_\lambda = -16E(u_\lambda(0)) - C(1+N)^2\|u_\lambda(0)\|_{L^2}^{6-2b} - N\|u_\lambda(0)\|_{L^2}^2 > 0, \tag{4.10}$$

$$\Big(\int \theta|u_\lambda(0)|^2dx\Big)^{1/2}\Big(\frac{2}{\delta_\lambda}\|\partial_x u_\lambda(0)\|_{L^2}^2 + 1\Big)^{1/2} \le \frac{1}{2}a_0. \tag{4.11}$$

By (4.8) and (4.9),

$$\delta_\lambda = -16\lambda^{-2}E(u_0) - C(1+N)^2\|u_0\|_{L^2}^{6-2b} - N\|u_0\|_{L^2}^2. \tag{4.12}$$

Thus, if we choose $\lambda > 0$ so that

$$\lambda < \Big[-16E(u_0)\Big[C(1+N)^2\|u_0\|_{L^2}^{6-2b} + N\|u_0\|_{L^2}^2\Big]^{-1}\Big]^{1/2} =: \lambda_0, \tag{4.13}$$

then (4.10) holds true. Moreover, since $\|\partial_x u_\lambda(0)\|_{L^2}^2 = \lambda^{-2}\|\partial_x u_0\|_{L^2}^2$, we have from (4.12) that

$$\frac{2}{\delta_\lambda}\|\partial_x u_\lambda(0)\|_{L^2}^2 = \frac{2\|\partial_x u_0\|_{L^2}^2}{\lambda^2\delta_\lambda} \le C_0, \quad 0 < \lambda < \lambda_1, \tag{4.14}$$

for some $\lambda_1 > 0$, where C_0 depends on λ_1 but does not depend on λ. We next recall the following fact (see e.g. [21, Lemma 2.3]).

Lemma 4.1. *Let $v \in L^2$ and*

$$H(x) := \begin{cases} |x| & \text{if} \quad |x| \le 1, \\ 1 & \text{if} \quad |x| > 1. \end{cases}$$

Set $v_\lambda(x) = \lambda^{-1/2}v(\lambda^{-1}x)$ for $\lambda > 0$. Then for any $\epsilon > 0$, there exists $\lambda_0 > 0$ such that

$$\|Hv_\lambda\|_{L^2} \le \epsilon, \quad 0 < \lambda < \lambda_0.$$

Applying Lemma 4.1, there exists $\lambda_2 > 0$ such that $\lambda_2 < \lambda_1$ and

$$\int \theta|u_\lambda(0)|^2dx \le 4\|Hu_\lambda(0)\|_{L^2}^2 \le \frac{1}{4}(C_0+1)^{-1}a_0^2, \quad 0 < \lambda < \lambda_2.$$

Combining this and (4.14), the condition (4.11) holds for $0 < \lambda < \lambda_2$. Therefore, if we choose $0 < \lambda < \min\{\lambda_0, \lambda_2\}$, then $u_\lambda(0)$ satisfies (4.10) and (4.11). The proof is complete.

Remark 4.2. We now show that the condition $E(u_0) < 0$ is sufficient for the blowup but it is not necessary. Let $E > 0$. We find data $u_0 \in H^1$ so that $E(u_0) = E$ and the corresponding solution u blows up in finite time. We follow the standard argument (see e.g. [3, Remark 6.5.8]). Using the standard virial identity (3.5) with $\alpha = \alpha_\star$, we have

$$\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 = 16E(u_0),$$

hence

$$\|xu(t)\|_{L^2}^2 = 8t^2E(u_0) + 4t\Big(\mathrm{Im}\int \overline{u}_0 x\cdot\nabla u_0 dx\Big) + \|xu_0\|_{L^2}^2 =: f(t).$$

We see that if $f(t)$ takes negative values, then the solution must blow up in finite time. In order to make $f(t)$ takes negative values, we need

$$\Big(\mathrm{Im}\int \overline{u}_0 x\cdot\nabla u_0 dx\Big)^2 > 2E(u_0)\|xu_0\|_{L^2}^2. \tag{4.15}$$

Now fix $\theta\in C_0^\infty(\mathbb{R}^d)$ a real-valued function and set $\psi(x) = e^{-i|x|^2}\theta(x)$. We see that $\psi\in C_0^\infty(\mathbb{R}^d)$ and

$$\mathrm{Im}\int \overline{\psi}x\cdot\nabla\psi dx = -2\int |x|^2\theta^2(x)dx < 0.$$

We now set

$$A = \frac{1}{2}\|\nabla\psi\|_{L^2}^2,\quad B = \frac{1}{\alpha_\star+2}\int |x|^{-b}|\psi(x)|^{\alpha_\star+2}dx,$$
$$C = \|x\psi\|_{L^2}^2,\qquad D = -\mathrm{Im}\int \overline{\psi}x\cdot\nabla\psi dx.$$

Let $\lambda,\mu>0$ be chosen later and set $u_0(x) = \lambda\psi(\mu x)$. We will choose $\lambda,\mu>0$ so that $E(u_0)=E$ and (4.15) holds true. A direct computation shows

$$E(u_0) = \lambda^2\mu^2\mu^{-d}\frac{1}{2}\|\nabla\psi\|_{L^2}^2 - \lambda^{\alpha_\star+2}\mu^b\mu^{-d}\frac{1}{\alpha_\star+2}\int |x|^{-b}|\psi(x)|^{\alpha_\star+2}dx = \lambda^2\mu^{2-d}\Big(A - \frac{\lambda^{\alpha_\star}}{\mu^{2-b}}B\Big),$$

and

$$\mathrm{Im}\int \overline{u}_0 x\cdot\nabla u_0 dx = \lambda^2\mu^{-d}\mathrm{Im}\int \overline{\psi}x\cdot\nabla\psi dx = -\lambda^2\mu^{-d}D,$$

and

$$\|xu_0\|_{L^2}^2 = \lambda^2\mu^{-d-2}\|x\psi\|_{L^2}^2 = \lambda^2\mu^{-d-2}C.$$

Thus, the conditions $E(u_0)=E$ and (4.15) yield

$$\lambda^2\mu^{2-d}\Big(A - \frac{\lambda^{\alpha_\star}}{\mu^{2-b}}B\Big) = E, \tag{4.16}$$

$$\frac{D^2}{C} > 2\Big(A - \frac{\lambda^{\alpha_\star}}{\mu^{2-b}}B\Big). \tag{4.17}$$

Fix $0<\epsilon<\min\Big\{A, \frac{D^2}{2C}\Big\}$ and choose

$$\frac{\lambda^{\alpha_\star}}{\mu^{2-b}}B = A-\epsilon.$$

It is obvious that (4.17) is satisfied. Condition (4.16) implies

$$\epsilon\lambda^2\mu^{2-d} = E \quad\text{or}\quad \epsilon\Big(\frac{B}{A-\epsilon}\Big)^{\frac{2-d}{2-b}}\lambda^{2+\frac{(2-d)\alpha_\star}{2-b}} = E.$$

This holds true by choosing a suitable value of λ.

5. Intercritical case $\alpha_\star < \alpha < \alpha^\star$

In this section, we will give the proof of Theorem 1.3. We firstly show (1.12). By the definition of energy and multiplying both sides of $E(u(t))$ by $M(u(t))^\sigma$, the sharp Gagliardo-Nirenberg inequality (2.1) yields

$$\begin{aligned} E(u(t))M(u(t))^\sigma &= \frac{1}{2}\Big(\|\nabla u(t)\|_{L^2}\|u(t)\|_{L^2}^\sigma\Big)^2 - \frac{1}{\alpha+2}\Big(\int |x|^{-b}|u(t,x)|^{\alpha+2}dx\Big)\|u(t)\|_{L^2}^{2\sigma} \\ &\geq \frac{1}{2}\Big(\|\nabla u(t)\|_{L^2}\|u(t)\|_{L^2}^\sigma\Big)^2 - \frac{C_{\mathrm{GN}}}{\alpha+2}\|u(t)\|^{\frac{4-2b-(d-2)\alpha}{2}+2\sigma}\|\nabla u(t)\|^{\frac{d\alpha+2b}{2}} \\ &= f(\|\nabla u(t)\|_{L^2}\|u(t)\|_{L^2}^\sigma), \end{aligned} \tag{5.1}$$

where

$$f(x) = \frac{1}{2}x^2 - \frac{C_{\mathrm{GN}}}{\alpha+2}x^{\frac{d\alpha+2b}{2}}.$$

Moreover, using (2.4) and (2.5), it is easy to see that

$$f(\|\nabla Q\|_{L^2}\|Q\|_{L^2}^\sigma) = E(Q)M(Q)^\sigma. \tag{5.2}$$

We also have f is increasing on $(0, x_0)$ and decreasing on (x_0, ∞), where

$$x_0 = \Big[\frac{2(\alpha+2)}{(d\alpha+2b)C_{\mathrm{GN}}}\Big]^{\frac{2}{d\alpha-(4-2b)}}.$$

Using again (2.4) and (2.5), we see that x_0 is exactly $\|\nabla Q\|_{L^2}\|Q\|_{L^2}^\sigma$. By (5.1), the conservations of mass and energy together with the assumption (1.9) imply

$$f(\|\nabla u(t)\|_{L^2}\|u(t)\|_{L^2}^\sigma) \leq E(u_0)M(u_0)^\sigma < E(Q)M(Q)^\sigma.$$

Using this, (5.2) and the assumption (1.10), the continuity argument shows

$$\|\nabla u(t)\|_{L^2}\|u(t)\|_{L^2}^\sigma > \|\nabla Q\|_{L^2}\|Q\|_{L^2}^\sigma,$$

for any t as long as the solution exists. This proves (1.12).

We next pick $\delta > 0$ small enough so that

$$E(u_0)M(u_0)^\sigma \leq (1-\delta)E(Q)M(Q)^\sigma. \tag{5.3}$$

This implies

$$f(\|\nabla u(t)\|_{L^2}\|u(t)\|_{L^2}^\sigma) \leq (1-\delta)E(Q)M(Q)^\sigma. \tag{5.4}$$

By Pohozaev identities (2.4), we learn that

$$E(Q)M(Q)^\sigma = \frac{d\alpha-(4-2b)}{2(d\alpha+2b)}(\|\nabla Q\|_{L^2}\|Q\|_{L^2}^\sigma)^2. \tag{5.5}$$

Moreover, we have from the fact $x_0 = \|\nabla Q\|_{L^2}\|Q\|_{L^2}^\sigma$ that

$$C_{\mathrm{GN}} = \frac{2(\alpha+2)}{d\alpha+2b}\frac{1}{(\|\nabla Q\|_{L^2}\|Q\|_{L^2}^\sigma)^{\frac{d\alpha-(4-2b)}{2}}}. \tag{5.6}$$

By dividing both sides of (5.4) by $E(Q)M(Q)^\sigma$ and using (5.5) and (5.6), we obtain

$$\frac{d\alpha+2b}{d\alpha-(4-2b)}\Big(\frac{\|\nabla u(t)\|_{L^2}\|u(t)\|_{L^2}^\sigma}{\|\nabla Q\|_{L^2}\|Q\|_{L^2}^\sigma}\Big)^2 - \frac{4}{d\alpha-(4-2b)}\Big(\frac{\|\nabla u(t)\|_{L^2}\|u(t)\|_{L^2}^\sigma}{\|\nabla Q\|_{L^2}\|Q\|_{L^2}^\sigma}\Big)^{\frac{d\alpha+2b}{2}} \leq 1-\delta.$$

The continuity argument then implies that there exists $\delta' > 0$ depending on δ so that

$$\frac{\|\nabla u(t)\|_{L^2}\|u(t)\|_{L^2}^\sigma}{\|\nabla Q\|_{L^2}\|Q\|_{L^2}^\sigma} \geq 1+\delta' \quad \text{or} \quad \|\nabla u(t)\|_{L^2}\|u(t)\|_{L^2}^\sigma \geq (1+\delta')\|\nabla Q\|_{L^2}\|Q\|_{L^2}^\sigma. \tag{5.7}$$

We also have for $\epsilon > 0$ small enough,

$$8\|\nabla u(t)\|_{L^2}^2 - \frac{4(d\alpha+2b)}{\alpha+2}\int |x|^{-b}|u(t,x)|^{\alpha+2}dx + \epsilon\|\nabla u(t)\|_{L^2}^2 \le -c < 0, \tag{5.8}$$

for any t in the existence time. Indeed, multiplying the left hand side of (5.8) with a conserved quantity $M(u(t))^\sigma$, we get

$$\text{LHS}(5.8) \times M(u(t))^\sigma = 4(d\alpha+2b)E(u(t))M(u(t))^\sigma + (8+\epsilon-2d\alpha-4b)\|\nabla u(t)\|_{L^2}^2 M(u(t))^\sigma.$$

The conservations of mass and energy, (5.3), (5.5) and (5.7) then yield

$$\begin{aligned}\text{LHS}(5.8) \times M(u_0)^\sigma &\le 4(d\alpha+2b)(1-\delta)E(Q)M(Q)^\sigma + (8+\epsilon-2d\alpha-4b)(1+\delta')^2(\|\nabla Q\|_{L^2}\|Q\|_{L^2}^\sigma)^2\\ &= (\|\nabla Q\|_{L^2}\|Q\|_{L^2}^\sigma)^2\Big[2(d\alpha-4+2b)[1-\delta-(1+\delta')^2] + \epsilon(1+\delta')^2\Big].\end{aligned}$$

By taking $\epsilon > 0$ small enough, we prove (5.8).

5.1. **The case $xu_0 \in L^2$.** The finite time blowup for the intercritical (INLS) with initial data in $H^1 \cap L^2(|x|^2dx)$ satisfying (1.9) and (1.10) was proved in [9]. For the sake of completeness, we recall some details. By the standard virial identity (3.5) and (5.8),

$$\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 = 8\|\nabla u(t)\|_{L^2}^2 - \frac{4(d\alpha+2b)}{\alpha+2}\int |x|^{-b}|u(t,x)|^{\alpha+2}dx \le -c < 0.$$

This shows that the solution blows up in finite time.

5.2. **The case u_0 is radial.** We firstly note that under the assumptions of Theorem 1.3, we can apply Lemma 3.4 to obtain for any $\epsilon > 0$,

$$\begin{aligned}\frac{d^2}{dt^2}V_{\varphi_R}(t) \le 8\|\nabla u(t)\|_{L^2}^2 &- \frac{4(d\alpha+2b)}{\alpha+2}\int |x|^{-b}|u(t,x)|^{\alpha+2}dx\\ &+O\Big(R^{-2} + \epsilon^{-\frac{\alpha}{4-\alpha}}R^{-\frac{2[(d-1)\alpha+2b]}{4-\alpha}} + \epsilon\|\nabla u(t)\|_{L^2}^2\Big),\end{aligned}$$

for any t in the existence time. By taking $\epsilon > 0$ small enough and $R > 1$ large enough depending on ϵ, we learn from (5.8) that

$$\frac{d^2}{dt^2}V_{\varphi_R}(t) \le -c/2 < 0.$$

This shows that the solution must blow up in finite time.

Acknowledgments

The author would like to express his deep thanks to his wife-Uyen Cong for her encouragement and support. He would like to thank his supervisor Prof. Jean-Marc Bouclet for the kind guidance and constant encouragement. He also would like to thank the reviewer for his/her helpful comments and suggestions.

References

[1] **L. Bergé**, *Soliton stability versus collapse*, Phys. Rev. E 62, No. 3 (2000), doi: 10.1103/PhysRevE.62.R3071, R3071-R3074. 1

[2] **T. Boulenger, D. Himmelsbach, E. Lenzmann**, *Blowup for fractional NLS*, J. Funct. Anal. 271 (2016), 2569-2603. 7

[3] **T. Cazenave**, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics 10, Courant Institute of Mathematical Sciences, AMS, 2003. 2, 5, 14

[4] **J. Q. Chen**, *On a class of nonlinear inhomogeneous Schrödinger equation*, J. Appl. Math. Comput. 32 (2010), 237-253. 1

[5] **Y. Cho, T. Ozawa**, *Sobolev inequalities with symmetry*, Commun. Contemp. Math. 11 (2009), No. 3, 355-365. 7

[6] **V. Combet, F. Genoud**, *Classification of minimal mass blow-up solutions for an L^2 critical inhomogeneous NLS*, J. Evol. Equ. 16, No. 2 (2016), doi: 10.1007/s00028-015-0309-z, 483-500. 2

[7] **V. D. Dinh**, *Scattering theory in a weighted L^2 space for a class of the defocusing inhomegeneous nonlinear Schrödinger equation*, preprint arXiv:1710.01392, 2017. 2, 3

[8] **V. D. Dinh**, *Energy scattering for a class of the defocusing inhomogeneous nonlinear Schrödinger equation*, preprint arXiv:1710.05766, 2017. 3

[9] **L. G. Farah**, *Global well-posedness and blow-up on the energy space for the inhomogeneous nonlinear Schrödinger equation*, J. Evol. Equ. 16, No. 1 (2016), doi: 10.1007/s00028-01500298-y, 193-208. 1, 2, 4, 17

[10] **L. G. Farah, C. M. Guzman**, *Scattering for the radial 3D cubic focusing inhomogeneous nonlinear Schrödinger equation*, J. Differential Equations 262, No. 8 (2017), doi: 10.1016/j.jde.2017.01.013, 4175-4231. 2

[11] **L. G. Farah, C. M. Guzman**, *Scattering for the radial focusing INLS equation in higher dimensions*, preprint arXiv:1703.10988, 2017. 2

[12] , **G. Fibich, X. P. Wang**, *Stability of solitary waves for nonlinear Schrödinger equations with inhomogeneous nonlinearities*, Physica D 175 (2003), doi: 10.1016/S0167-2789(02)00626-7, 96-108. 1

[13] **F. Genoud, C. A. Stuart**, *Schrödinger equations with a spatially decaying nonlinearity: existence and stability of standing waves*, Discrete Contin. Dyn. Syst. 21, No. 1 (2008), doi: 10.3934/dcds.2008.21.137, 137-286. 2, 3, 4

[14] **F. Genoud**, *An inhomogeneous, L^2-critical, nonlinear Schrödinger equation*, Z. Anal. Anwend. 31, No. 3 (2012), doi: 10.4171/ZAA/1460, 283-290. 2

[15] **R. T. Glassey**, *On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations*, J. Math. Phys. 18 (1977), 1794-1797. 11

[16] **C. M. Guzmán**, *On well posedness for the inhomogneous nonlinear Schrödinger equation*, Nonlinear Anal. 37 (2017), doi: 10.1016/j.nonrwa.2017.02.018, 249-286. 2, 3

[17] **Y. Liu, X. P. Wang, K. Wang**, *Instability of standing waves of the Schrödinger equations with inhomogeneous nonlinearity*, Trans. Amer. Math. Soc. 358, No. 5 (2006), doi: 10.1090/S0002-9947-05-03763-3, 2105-2122. 1

[18] **F. Merle**, *Nonexistence of minimal blow-up solutions of equations $iu_t = -\Delta u - k(x)|u|^{\frac{4}{d}}u$ in $\mathbb{R}^N$*, Ann. Inst. H. Poincaré Phys. Théor. 64, No. 1 (1996), 35-85. 1

[19] **F. Merle**, *Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equations with critical power*, Duke Math. J. 69 (1993), No. 2, 427-454. 1

[20] **T. Ogawa, Y. Tsutsumi**, *Blow-up of H^1 solutions for the nonlinear Schrödinger equation*, J. Differential Equations 92 (1991), 317-330. 3, 7, 12

[21] **T. Ogawa, Y. Tsutsumi**, *Blow-up of H^1 solutions for the one dimensional nonlinear Schrödinger equation with critical power nonlinearity*, Proc. Amer. Math. Soc. 111 (1991), 487-496. 3, 9, 10, 12, 14

[22] **P. Raphaël, J. Szeftel**, *Existence and uniqueness of minimal blow-up solutions to an inhomogeneous mass critical NLS*, J. Amer. Math. Soc. 24, No. 2 (2011), doi: 10.1090/S0894-0347-2010-00688-1, 471-546. 1

[23] **W. A. Strauss**, *Existence of solitary waves in higher dimensions*, Comm. Math. Phys. 55 (1977), No. 2, 149-162. 7

[24] **T. Suzuki**, *Energy methods for Hartree type equations with inverse-square potentials*, Evol. Equ. Control Theory 2 (2013), 531-542.

[25] **T. Suzuki**, *Critical case of nonlinear Schrödinger equations with inverse-square potentials on bounded domains*, Math. Bohem. 139 (2014), 231-238.

[26] **G. Talenti**, *Best constant in Sobolev inequality*, Ann. Math. Pura. Appl. 110 (1976), 353-372.

[27] **T. Tao, M. Visan, X. Zhang**, *The nonlinear Schrödinger equation with combined power-type nonlinearities*, Comm. Partial Differential Equations 32 (2007), doi: 10.1080/0360530070158880, 1281-1343. 5

[28] **J. F. Toland**, *Uniqueness of positive solutions of some semilinear Sturm-Liouville problems on the half line*, Proc. Roy. Soc. Edinburgh Sect. A 97, 259-263, 1984. 4

[29] **I. Towers, B. A. Malomed**, *Stable (2+1)-dimensional solitons in a layered medium with sign-alternating Kerr nonlinearity*, J. Opt. Soc. Amer. B Opt. Phys. 19, No. 3 (2002), doi: 10.1364/JOSAA.19.000537, 537-543. 1

[30] **N. Visciglia**, *On the decay of solutions to a class of defocusing NLS*, Math. Res. Lett. 16 (2009), No. 5, 919-926. 3

[31] **E. Yanagida**, *Uniqueness of positive radial solutions of $\Delta u + g(r)u + h(r)u^p = 0$ in $\mathbb{R}^N$*, Arch. Ration. Mech. Anal. 115, 257-274, 1991. 4

[32] **S. Zhu**, *Blow-up solutions for the inhomogeneous Schrödinger equation with L^2 supercritical nonlinearity*, J. Math. Anal. Appl. 409 (2014), doi: 10.1016/j.jmaa.2013.07.029, 760-776. 1

(V. D. Dinh) Institut de Mathématiques de Toulouse UMR5219, Université Toulouse CNRS, 31062 Toulouse Cedex 9, France

E-mail address: dinhvan.duong@math.univ-toulouse.fr