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FORMATION OF FACETS FOR AN EFFECTIVE MODEL OF CRYSTAL GROWTH

DMITRY IOFFE AND SENYA SHLOSMAN

Abstract. We study an effective model of microscopic facet formation for low temperature three dimensional microscopic Wulff crystals above the droplet condensation threshold. The model we consider is a 2+1 solid on solid surface coupled with high and low density bulk Bernoulli fields. At equilibrium the surface stays flat. Imposing a canonical constraint on excess number of particles forces the surface to "grow" through the sequence of spontaneous creations of macroscopic size monolayers. We prove that at all sufficiently low temperatures, as the excess particle constraint is tuned, the model undergoes an infinite sequence of first order transitions, which traces an infinite sequence of first order transitions in the underlying variational problem. Away from transition values of canonical constraint we prove sharp concentration results for the rescaled level lines around solutions of the limiting variational problem.

1. Introduction

Low temperature three dimensional equilibrium crystal shapes exhibit flat facets, see e.g. [8], [7], [31]. It is known that lattice oriented low temperature microscopic interfaces stay flat under Dobrushin boundary conditions [17]. But the $\mathbb{L}_1$-theory, which is the base of the microscopic justification of Wulff construction [2, 15, 3] in three and higher dimensions, does not directly address fluctuations of microscopic shapes on scales smaller than the linear size of the system. In particular, the existence and the microscopic structure of facets remains an open question even at very low temperatures.

In a sense, this issue is complementary to a large body of works, see for instance [16, 14, 22, 30] and references to later papers, and also a recent review [32], which focus on study of the corners of zero or low temperature microscopic crystals.

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For low temperature $2+1$ SOS (solid on solid) interfaces under canonical constraints on the volume below the microscopic surface, existence of flat microscopic facets was established in [4]. Here we consider facet formation for a SOS model coupled to high and low density bulk Bernoulli fields which are supposed to mimic coexisting phases of the three dimensional model.

The phenomenon of droplet condensation in the framework of the Ising model was first described in the papers [20], [19]. There it was considered the Ising model at low temperature β^{-1}, occupying a d-dimensional box T_N^d of the linear size $2N$ with periodic boundary conditions. The ensemble was the canonical one: the total magnetization,

$$M_N = \sum \sigma_t,$$

was fixed. In case $M_N = m^*(\beta)\left|T_N^d\right|$, where $m^*(\beta) > 0$ is the spontaneous magnetization, the typical configuration looks as a configuration of the $(+)$-phase: the spins are taking mainly the values $+1$, while the values -1 are seen rarely, and the droplets of minuses in the box T_N^d are at most of the size of $K(d)\ln N$. In order to observe the condensation of small $(-)$-droplets into a big one it is necessary to increase their amount, which can be achieved by considering a different canonical constraint:

$$M_N = m^*(\beta)\left|T_N^d\right| - b_N,$$

$b_N > 0$. It turns out that if $b_N/\left|T_N^d\right|^{\frac{d}{d+1}} \to 0$ as $N \to \infty$, then in the corresponding canonical ensemble all the droplets are still microscopic, not exceeding $K(d)\ln N$ in linear size. On the other hand, once $\liminf_{N\to\infty} b_N/\left|T_N^d\right|^{\frac{d}{d+1}} > 0$, the situation becomes different: among many $(-)$-droplets there is one, $\mathcal{D}$, of the linear size of the order of $(b_N)^{1/d} \geq N^{\frac{d}{d+1}}$, while all the rest of the droplets are still at most logarithmic. Therefore $b_N \sim \left|T_N^d\right|^{\frac{d}{d+1}}$ can be called the *condensation threshold, or dew-point.*

When b_N grows beyond the condensation threshold, the big droplet $\mathcal{D}$ grows as well. To study this growth process, or specifically to try to get an insight of the process of formation of new atomic-scale layers on microscopic facets, we have suggested in our paper "Ising model fog drip: the first two droplets" [24] a simplified growth model, where one puts the observer on the surface $\mathcal{S}$ of $\mathcal{D}$ and studies the evolution of this surface $\mathcal{S}$ as the volume of $\mathcal{D}$ grows.

It was argued in [24] that the evolution of $\mathcal{S}$ proceeds through the spontaneous creations of extra monolayers. Each new monolayer has one-particle thickness, while the breadth of the k-th monolayer is $\sim c_k N$, with $c_k \geq c_{crit} = c_{crit}(\beta) > 0$. It then grows in breadth for some time, until a new monolayer is spontaneously created at its top, of the size of $c_{k+1}N$.

In [24] we were able to analyze this process only for the first two monolayers. Our technique at this time was not sufficient, and we were unable to control the

effect of the interaction between the two monolayers when their boundaries come into contact and start to influence each other. This technique was later developed in our paper [25], so we are able now to conclude our studies. The present paper thus contains the material of what we have promised in [24] to publish under the provisional title "Ising model fog drip, II: the puddle".

In the present work we can handle any finite number of monolayers. What we find quite interesting is that the qualitative picture of the process of growth of monolayers changes, as k increases. Namely, for the few initial values of $k = 1, 2, ..., k_c$ the size $c_k N$ of the k-th monolayer at the moment of its creation is strictly smaller than the size of the underlying $(k-1)$-th layer. Thus, the picture is qualitatively the same as that of the lead crystal - there is an extensive physical literature on the latter subject, for instance see Figure 2 in [5] and discussions in [21, 6] However, for $k > k_c$ this is not the case any more, and the size $c_k N$ is exactly the same as the size of the underlying $(k-1)$-th layer at the creation moment.

Still, creation of each new layer k bears a signature of first order transition - at the moment of creation all the underlying layers shrink. This transition resembles spontaneous creation of mesoscopic size droplets in two-dimensional Ising model [1], and as in the latter work it is related to first order transitions in the underlying variational problem.

Our picture has to be compared with a similar one, describing the layer formation in the SOS model above the wall, studied in a series of works [10, 12, 11]. Unlike our model, all the layers of the SOS surface above the wall have different asymptotic shapes. The reason is that the repulsion from the wall results in different area tilts for different layers there, and, accordingly, gives rise to different solutions of the corresponding variational problem. Another important difference is that in the SOS model [10] one never sees the top monolayer detached from the rest of them, as in the model we consider. Nevertheless, we believe that in our model with k monolayers the fluctuations of their boundaries in the vicinity of the (vertical) wall are, as in the case of entropic repulsion [11], of the order of $N^{1/3}$, and their behavior is given, after appropriate scaling, by k non-intersecting Ferrari-Spohn diffusions [23], as in [26, 29]. See [28] for a review.

2. The Model and the Results

2.1. **The Model.** We will study the following simple model of facets formation on interfaces between two coexisting phases which was introduced in [24]: The system is confined to the 3D box

$$\Lambda_N = B_N \times \left\{ -\frac{N+1}{2}, -\frac{N-1}{2}, ..., \frac{N-1}{2}, N + \frac{N+1}{2} \right\},$$

where $N \in 2\mathbb{N}$ is even, B_N is a two-dimensional $N \times N$ box;

$$B_N = \{-N, \ldots, N\}^2 = N\mathbb{B}_1 \cap \mathbb{Z}^2,$$

and $\mathbb{B}_1 = [-1,1]^2$. The interface Γ between two phases is supposed to be an SOS-type surface; it is uniquely defined by a function

$$h_\Gamma : \mathring{B}_N \to \left\{ -\frac{N}{2}, -\frac{N}{2}+1, ..., \frac{N}{2} \right\}, \tag{2.1}$$

where $\mathring{B}_N$ is the interior of B_N. We assume that the interface Γ is pinned at zero height on the boundary ∂B_N, that is $h_\Gamma \equiv 0$ on ∂B_N. Such a surface Γ splits Λ_N into two parts; let us denote by $V_N(\Gamma)$ and $S_N(\Gamma)$ the upper and the lower halves. We suppose that Γ separates the low density phase (vapor) in the upper half of the box from the high density phase (solid) in the lower half. This is modeled in the following fashion: First of all, the marginal distribution of Γ obeys the SOS statistics at an inverse temperature β. That is we associate with Γ a weight,

$$w_\beta(\Gamma) = \exp\left\{ -\beta \sum_{x\sim y} |h_\Gamma(x) - h_\Gamma(y)| \right\}, \tag{2.2}$$

where we extended $h_\Gamma \equiv 0$ outside B_N, and the sum is over all unordered pairs of nearest neighbors of $\mathbb{Z}^d$.

Next, Γ is coupled to high and low density Bernoulli bulk fields: Let $0 < p_v = p_v(\beta) < p_s = p_s(\beta) < 1$. A relevant choice of p_v, p_s with a simplification of the three dimensional Ising model in mind would be $p_v(\beta) = \mathrm{e}^{-6\beta} = 1 - p_s(\beta)$. In the sequel we shall assume[1]

$$\liminf_{\beta\to\infty} \frac{1}{\beta} \log\left(\min\left\{p_v(\beta), 1-p_s(\beta)\right\}\right) > -\infty. \tag{2.3}$$

At each site $i \in V_N$ we place a particle with probability p_v, while at each site $i \in S_N$ we place a particle with probability p_s. Alternatively, let $\{\xi_i^v\}$ and $\{\xi_i^s\}$ be two independent collection of Bernoulli random variables with parameters p_v and p_s. Then the empirical field of particles *given* interface Γ is

$$\sum_{i\in V_N(\Gamma)} \xi_i^v \delta_i + \sum_{j\in S_N(\Gamma)} \xi_j^s \delta_j.$$

All together, the joint distribution of the triple (Γ, ξ^v, ξ^s) is given by

$$\mathbb{P}_{N,\beta}(\Gamma, \xi^v, \xi^s) \propto w_\beta(\Gamma) \prod_{i\in V_N} p_v^{\xi_i^v} (1-p_v)^{1-\xi_i^v} \prod_{j\in S_N} p_s^{\xi_j^s} (1-p_s)^{1-\xi_j^s}. \tag{2.4}$$

We denote the total number of particles in vapor and solid phases, and total number of particles in the system as

$$\Xi_v = \sum_{i\in V_N(\Gamma)} \xi_i^v, \ \Xi_s = \sum_{j\in S_N(\Gamma)} \xi_j^s \quad \text{and} \quad \Xi_N = \Xi_v + \Xi_s \tag{2.5}$$

[1]Actually, main results hold even with faster decay than (2.3).

respectively. The conditional distributions of Ξ_v and Ξ_s given Γ are binomial $\mathrm{Bin}\left(|V_N|, p_v\right)$ and $\mathrm{Bin}\left(|S_N|, p_s\right)$.

By the definition of the model, the expected total number of particles

$$\mathbb{E}_{N,\beta}\left(\Xi_N\right) = \frac{p^s + p^v}{2}\left|\Lambda_N\right| \equiv \rho_\beta N^3. \tag{2.6}$$

Formation of facets is modeled in the following way: Consider

$$\mathbb{P}^A_{N,\beta}\left(\,\cdot\,\right) = \mathbb{P}_{N,\beta}\left(\,\cdot\,\middle|\Xi_N \geq \rho_\beta N^3 + AN^2\right). \tag{2.7}$$

We claim that the model exhibits a sequence of first order transitions as A in (2.7) varies. The geometric manifestation of these transitions is the spontaneous creation of macroscopic size monolayers. In [24] we have investigated the creation of the first two monolayers. The task of the current paper is to provide an asymptotic (as $N \to \infty$) description of typical surfaces Γ under $\mathbb{P}_{N,\beta}\left(\,\cdot\,\middle|\,\Xi_N \geq \rho_\beta N^3 + AN^2\right)$ for *any* A fixed.

To study this conditional distribution we rely on Bayes rule,

$$\mathbb{P}^A_{N,\beta}\left(\Gamma\right) \;=\; \frac{\mathbb{P}_{N,\beta}\left(\Xi_N \;\geq\; \rho_\beta N^3 + AN^2 \,\middle|\, \Gamma\right)\mathbb{P}_{N,\beta}\left(\Gamma\right)}{\sum_{\Gamma'}\mathbb{P}_{N,\beta}\left(\Xi_N \;\geq\; \rho_\beta N^3 + AN^2 \,\middle|\, \Gamma'\right)\mathbb{P}_{N,\beta}\left(\Gamma'\right)}. \tag{2.8}$$

The control over the conditional probabilities $\mathbb{P}_{N,\beta}\left(\cdot\,\middle|\,\Gamma\right)$ comes from volume order local limit theorems for independent Bernoulli variables, whereas a-priori probabilities $\mathbb{P}_{N,\beta}\left(\Gamma\right)$ are derived from representation of Γ in terms of a gas of non-interacting contours. Models with bulk fields give an alternative approximation of interfaces in low temperature 3D Ising model, and they enjoy certain technical advantages over the usual SOS model with weights $w_\beta(\Gamma)$ (see (4.3)). In particular volume order limit results enable a simple control over the phase of intermediate contours.

2.2. **Heuristics and informal statement of the main result.** Let us describe the heuristics behind the claimed sequence of first order transitions: To each surface Γ corresponds a signed volume $\alpha(\Gamma)$. In terms of the height function h_Γ which was defined in (2.1),

$$\alpha(\Gamma) \;=\; \int\int h_\Gamma(x,y)dxdy, \tag{2.9}$$

Main contribution to $\alpha(\Gamma)$ comes from large microscopic facets, which are encoded by large microscopic level lines $\Gamma_1, \ldots \Gamma_\ell$;

$$\alpha(\Gamma) \approx \sum_1^\ell \mathsf{a}\left(\Gamma_i\right) = N^2 \sum_1^\ell \mathsf{a}\left(\frac{1}{N}\Gamma_i\right). \tag{2.10}$$

The notions of level lines are defined and discussed in Section 4. Locally large level lines $\Gamma_1, \ldots, \Gamma_\ell$ have structure of low temperature Ising polymers, and they give rise to a *two-dimensional* surface tension τ_β. As we shall explain below the a-priori

probability of creating surface with a prescribed volume aN^2 is asymptotically given by

$$\log \mathbb{P}_{N,\beta}\left(\alpha(\Gamma) = aN^2\right) \approx -N\tau_\beta(a), \tag{2.11}$$

where $\tau_\beta(a)$ is the minimal surface tension of a compatible collection of simple curves $\gamma_1, \dots, \gamma_\ell \subset \mathbb{B}_1$ with total area $\sum_i \mathsf{a}(\gamma_i) = a$. The notion of compatibility is explained in the beginning of Section 3, which is devoted to a careful analysis of the minimization problem we consider here. In fact the only relevant compatible collections happen to be ordered stacks

$$\mathring{\gamma}_\ell \subseteq \mathring{\gamma}_2 \subseteq \dots \subseteq \mathring{\gamma}_1 \subseteq \mathbb{B}_1. \tag{2.12}$$

Informally, the asymptotic relation (2.11) is achieved when rescaled $\frac{1}{N}\Gamma_i$ microscopic level lines stay close to optimal γ_i-s. On the other hand, the presence of Γ-interface shifts the expected number of particles in the bulk by the quantity $\Delta_\beta\alpha(\Gamma)$, where $\Delta_\beta = (p^s - p^v)$. That is,

$$\mathbb{E}_{N,\beta}\left(\Xi_N \middle| \Gamma\right) = \rho_\beta N^3 + \Delta_\beta \alpha(\Gamma). \tag{2.13}$$

Therefore, in view of local limit asymptotics for bulk Bernoulli fields,

$$\log \mathbb{P}_{N,\beta}\left(\Xi_N \ \geq \ \rho_\beta N^3 + AN^2 \middle| \alpha(\Gamma) = aN^2\right) \approx -\frac{(AN^2 - \Delta_\beta a N^2)^2}{2N^3 R_\beta} = -N\frac{(\delta_\beta - a)^2}{2D_\beta}, \tag{2.14}$$

where

$$R_\beta := 2\left(p^s(1-p^s) + p^v(1-p^v)\right), \quad D_\beta := \frac{R_\beta}{\Delta_\beta^2} \quad \text{and} \quad \delta_\beta := A/\Delta_\beta. \tag{2.15}$$

Consequently, the following asymptotic relation should hold:

$$\frac{1}{N}\log \mathbb{P}_{N,\beta}\left(\Xi \ \geq \ \rho_\beta N^3 + AN^2\right) \approx -\min_a \left\{\frac{(\delta_\beta - a)^2}{2D_\beta} + \tau_\beta(a)\right\}, \tag{2.16}$$

and, moreover, if a^* is the unique minimizer for the right hand side of (2.16), then the conditional distribution $\log \mathbb{P}_{N,\beta}\left(\cdot \middle| \Xi \geq \rho_\beta N^3 + AN^2\right)$ should concentrate on surfaces Γ which have an optimal volume close to a^*N^2 or, in view of (2.10) and (2.11) whose rescaled large level lines $\frac{1}{N}\Gamma_1, \dots, \frac{1}{N}\Gamma_\ell$ are macroscopically close to the optimal stack $\mathring{\gamma}_\ell^* \subseteq \dots \subseteq \mathring{\gamma}_1^*$ which produces $\tau_\beta(a^*)$.

Theorem A. *Assume that bulk occupation probabilities p_v and p_s satisfy* (2.3)*. Then there exists $\beta_0 < \infty$ such that for any $\beta > \beta_0$ there exists a sequence of numbers $0 < A_1(\beta) < A_2(\beta) < A_3(\beta) < \dots$ and a sequence of areas $0 < a_1^-(\beta) < a_1^+(\beta) < a_2^-(\beta) < a_2^+(\beta) < \dots$ which satisfies properties* **A1** *and* **A2** *below:*

A1. *For any $A \in [0, A_1)$ the minimizer in the right hand side* (2.16) *(with $\delta_\beta = A/\Delta_\beta$ as in* (2.15)*) is $a^* = 0$ which, in terms of contours in* (2.12) *corresponds to the empty stack. For any $\ell = 1, 2, \dots$ and for any $A \in (A_\ell, A_{\ell+1})$ the unique minimizer a^* in the right hand side of* (2.16) *satisfies $a^* \in (a_\ell^-, a_\ell^+)$, and it corresponds to a unique, up to compatible shifts within $\mathbb{B}_1$, stack of exactly ℓ-contours $\mathring{\gamma}_\ell^* \subseteq \dots \subseteq$*

$\mathring{\gamma}_1^* \subseteq \mathbb{B}_1$.

A2. *For any $A \in [0, A_1)$ there are no large level lines (no microscopic facets, that is Γ stays predominately flat on zero-height level) with $\mathbb{P}^A_{N,\beta}$-probability tending to one. On the other hand, for any $\ell = 1, 2, \dots$ and for any $A \in (A_\ell, A_{\ell+1})$, there are exactly ℓ microscopic facets of Γ with $\mathbb{P}^A_{N,\beta}$-probability tending to one and, moreover, the rescaled large level lines $\frac{1}{N}\Gamma_1, \dots, \frac{1}{N}\Gamma_\ell$ concentrate in Hausdorff distance $\mathsf{d_H}$ near the optimal stack $\{\gamma_1^*, \dots, \gamma_\ell^*\}$, as described in part* **A1** *of the Theorem, in the following sense: For any $\epsilon > 0$,*

$$\lim_{N\to\infty} \mathbb{P}^A_{N,\beta}\left(\sum_{i=1}^{\ell-1} \mathsf{d_H}\left(\frac{1}{N}\Gamma_i, \gamma_i^*\right) + \min_{x: x+\gamma_\ell^* \subset \mathbb{B}_1} \mathsf{d_H}\left(\frac{1}{N}\Gamma_\ell, x + \gamma_\ell^*\right) \geq \epsilon\right) = 0. \tag{2.17}$$

2.3. **Structure of the paper.** Section 3 is devoted to a careful analysis of the multi-layer minimization problem in the right hand side (2.16). Our results actually go beyond **A1** in Theorem A, namely we give a rather complete description of optimal stacks $\{\gamma_1^*, \dots, \gamma_\ell^*\}$ in terms of Wulff shapes and Wulff plaquettes associated to surface tension τ_β, and in particular we explain geometry behind the claimed infinite sequence of first order transitions.
The notions of microscopic contours and level lines are defined in Section 4.
The surface tension τ_β is defined in the very beginning of Section 5.
The proof part **A2** of Theorem A, or more precisely the proof of the corresponding statement for the reduced model of large contours, Theorem C in the end of Section 4, is relegated to Section 5.

Throughout the paper we rely on techniques and ideas introduced and developed in [18] and [25]. Whenever possible we only sketch proofs which follow closely the relevant parts of these papers.

2.4. **Some notation.** Let us introduce the following convenient notation: Given two indexed families of numbers $\{a_\alpha\}$ and $\{b_\alpha\}$ we shall say that $a_\alpha \lesssim b_\alpha$ *uniformly* in α if there exists C such that $a_\alpha \leq C b_\alpha$ for all indices α. Furthermore, we shall say that $a_\alpha \cong b_\alpha$ if both $a_\alpha \lesssim b_\alpha$ and $b_\alpha \lesssim a_\alpha$ hold. Finally, the family $\{a_{\alpha,N}\}$ is $o_N(1)$ if $\sup_\alpha |a_{\alpha,N}|$ tends to zero as N tends to infinity.

3. The variational problem

The variational problems we shall deal with are constrained isoperimetric type problems for curves lying inside the box $\mathbb{B}_1 = [-1, 1]^2 \subset \mathbb{R}^2$.

Let τ_β be a norm on $\mathbb{R}^2$ which possesses all the lattice symmetries of $\mathbb{Z}^2$. For a closed piecewise smooth Jordan curves (called loops below) $\gamma \subset \mathbb{B}_1$ we define $\mathsf{a}(\gamma)$ to be the area of its interior $\mathring{\gamma}$ and $\tau_\beta(\gamma)$ to be its surface tension,

$$\tau_\beta(\gamma) = \int_\gamma \tau_\beta(\mathfrak{n}_s)\mathrm{d}s.$$

A finite family $\mathcal{L} = \{\gamma_1, \gamma_2, \dots, \gamma_n\} \subset \mathbb{B}_1$ of loops is said to be compatible if

$$\forall\, i \neq j \text{ either } \mathring{\gamma}_i \cap \mathring{\gamma}_j = \emptyset \text{ or } \mathring{\gamma}_i \subseteq \mathring{\gamma}_j \text{ or } \mathring{\gamma}_j \subseteq \mathring{\gamma}_i.$$

Thus compatible families have a structure of finite number of disjoint stacks of loops, with loops within each stack being ordered by inclusion. Given $\mathsf{D}_\beta > 0$ consider the following family, indexed by $\delta > 0$, of minimization problems:

$$\min_{\mathcal{L}} \mathcal{E}_\beta\left(\mathcal{L}\,|\delta\right) := \min_{\mathcal{L}} \left\{ \frac{\left(\delta - \mathsf{a}(\mathcal{L})\right)^2}{2\mathsf{D}_\beta} + \tau_\beta(\mathcal{L}) \right\}, \tag{VP$_\delta$}$$

where

$$\mathsf{a}\left(\mathcal{L}\right) = \sum_{\gamma \in \mathcal{L}} \mathsf{a}\left(\gamma\right) \quad \text{and} \quad \tau_\beta(\mathcal{L}) = \sum_{\gamma \in \mathcal{L}} \tau_\beta(\gamma)$$

3.1. **Rescaling of** (VP$_\delta$)**.** Let e be a lattice direction. Set

$$v = \frac{\delta}{\tau_\beta(\mathsf{e})\mathsf{D}_\beta},\ \sigma_\beta = \mathsf{D}_\beta \tau_\beta(\mathsf{e}) \text{ and } \tau(\cdot) = \frac{\tau_\beta(\cdot)}{\tau_\beta(\mathsf{e})}. \tag{3.1}$$

Since

$$\mathcal{E}_\beta\left(\mathcal{L}\,|\delta\right) = \frac{\left(\delta - \mathsf{a}(\mathcal{L})\right)^2}{2\mathsf{D}_\beta} + \tau_\beta(\mathcal{L}) = \frac{\delta^2}{2\mathsf{D}_\beta} + \tau_\beta(\mathsf{e}) \left\{ -v\mathsf{a}(\mathcal{L}) + \tau(\mathcal{L}) + \frac{\mathsf{a}(\mathcal{L})^2}{2\sigma_\beta} \right\},$$

we can reformulate the family of variational problems (VP$_\delta$) as follows: for $a \geq 0$ define

$$\tau(a) = \min\left\{\tau(\mathcal{L}) \,:\, \mathsf{a}(\mathcal{L}) = a\right\}. \tag{3.2}$$

Then study

$$\min_{a \geq 0} \left\{ -va + \tau(a) + \frac{a^2}{2\sigma_\beta} \right\}. \tag{VP$_v$}$$

The problem (VP$_v$) has a clear geometric interpretation: we want to find $a = a(v) \geq 0$ such that the straight line with slope v is the support line at $a(v)$ to the graph $a \mapsto \tau(a) + \frac{a^2}{2\sigma_\beta}$ on $[0, \infty)$.

3.2. **Wulff shapes, Wulff plaquettes and optimal stacks.** By construction τ inherits lattice symmetries of $\mathbb{Z}^d$ and $\tau\left(\mathsf{e}\right) = 1$. In this case, the Wulff shape

$$\mathsf{W} \triangleq \partial\left\{x : x \cdot \mathfrak{n} \leq \tau(\mathfrak{n})\ \forall\ \mathfrak{n} \in \mathbb{S}^1\right\}$$

has the following properties: Its radius (half of its projection on the x or y axis) equals to 1. Its area $\mathsf{a}(\mathsf{W})$ satisfies

$$\mathsf{a}(\mathsf{W}) \equiv w = \frac{1}{2}\tau(\mathsf{W}),$$

For any $r > 0$ the radius, the area and the surface tension of the scaled curve $r\mathsf{W}$ equal to r, $r^2 w$ and $2rw$ respectively. The curves $r\mathsf{W}$ are (modulo shifts) unique minimizers of $\tau(\gamma)$ restricted to the loops γ with area $\mathsf{a}(\gamma) = r^2 w$:

$$2rw = \tau(r\mathsf{W}) = \min_{\gamma:\mathsf{a}(\gamma)=r^2 w} \tau(\gamma). \tag{3.3}$$

Since $\tau(\mathsf{e}) = 1$, the maximal radius and, accordingly, the maximal area of the rescaled Wulff shape which could be inscribed into the box $\mathbb{B}_1$ are precisely 1 and w. For $b \in [0, w]$ let W_b be the Wulff shape of area b. By convention, W_b is centered at the origin. Its radius r_b and its surface tension $\tau(\mathsf{W}_b)$ are given by

$$r_b = \sqrt{\frac{b}{w}} \quad \text{and} \quad \tau(\mathsf{W}_b) = 2r_b w = 2\sqrt{bw}. \tag{3.4}$$

For $b \in (w, 4]$ the optimal (minimal $\tau(\gamma)$) loop $\gamma \subseteq \mathbb{B}_1$ with $\mathsf{a}(\gamma) = b$ is what we call the Wulff plaquette P_b. It is defined as follows. One takes four Wulff shapes of radius $r \leq 1$ and puts them in four corners of $\mathbb{B}_1$, so that each touches two corresponding sides of $\mathbb{B}_1$; then, one takes the convex envelope of the union of these four Wulff shapes. We will call such Wulff plaquette as having the radius r. It contains four segments of length $2(1-r)$, and so its surface tension is $8\,(1-r) + 2wr$. Its area is $4 - (4-w)\,r^2$. In this way, the Wulff plaquette P_b of area $b \in [w, 4]$ has the radius r_b and surface tension $\tau(\mathsf{P}_b)$ given by

$$r_b = \sqrt{\frac{4-b}{4-w}} \quad \text{and} \quad \tau(\mathsf{P}_b) = 8 - 2r_b(4-w) = 8 - 2\sqrt{(4-w)(4-b)}. \tag{3.5}$$

Remark 1. *By convention* $\mathsf{W}_w = \mathsf{P}_w$. *Also note that both in the case of Wulff shapes and Wulff plaquettes,*

$$\frac{\mathrm{d}}{\mathrm{d}b}\tau\left(\mathsf{W}_b\right) = \frac{1}{r_b} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}b}\tau\left(\mathsf{P}_b\right) = \frac{1}{r_b}, \tag{3.6}$$

for any $b \in (0, w)$ *and, respectively, for any* $b \in (w, 4)$.

Definition. For $b \in [0, 4]$ define the optimal shape

$$\mathsf{S}_b = \mathsf{W}_b \mathbb{1}_{b\in[0,w)} + \mathsf{P}_b \mathbb{1}_{b\in[w,4]}. \tag{3.7}$$

Let now $\mathcal{L}$ be a family of compatible loops. For any $x \in \mathbb{B}_1$ let $n_{\mathcal{L}}(x)$ be the number of loops $\gamma \in \mathcal{L}$ such that $x \in \mathring{\gamma}$. The areas $b_\ell = |x \in \mathbb{B}_1 : n_{\mathcal{L}}(x) \geq \ell|$ form a non-increasing sequence. Therefore $\mathcal{L}^* = \{\mathsf{S}_{b_1}, \mathsf{S}_{b_2}, \ldots\}$ is also a compatible family. Obviously $\mathsf{a}(\mathcal{L}^*) = \mathsf{a}(\mathcal{L})$, but $\tau(\mathcal{L}^*) \leq \tau(\mathcal{L})$. Consequently, we can restrict attention only to compatible families which contain exactly one stack of optimal shapes S_{b_ℓ}.

Furthermore, (3.6) implies that for each $\ell \in \mathbb{N}$ optimal ℓ-stacks could be only of two types: Let $a \in \mathbb{R}^+$.

Definition (Stacks $\mathcal{L}^1_\ell(a)$ of type 1). These contain $\ell - 1$ identical Wulff plaquettes and a Wulff shape, all of the same radius $r = r^{1,\ell}(a)$, which should be recovered

from

$$a = (\ell-1)(4-(4-w)r^2) + wr^2 = 4(\ell-1) - r^2\left(\ell(4-w)-4\right). \tag{3.8}$$

That is, if $\ell(4-w) \neq 4$, then

$$r^{1,\ell}(a) = \sqrt{\frac{4(\ell-1)-a}{4(\ell-1)-w}}. \tag{3.9}$$

Remark 2. *Of course, if* $\frac{4}{4-w} \in \mathbb{N}$, *then* (3.8) *does not determine* r *for* $\ell^* = \frac{4}{4-w}$. *In other words in this case for any* $r \in [0,1]$ *the stack of* (ℓ^*-1) *Wulff plaquettes of radius* r *and the Wulff shape of the very same radius* r *on the top of them has area* $4(\ell^*-1)$ *and surface tension* $8(\ell^*-1)$. *This introduces a certain degeneracy in the problem, but, upon inspection, the inconvenience happens to be of a purely notational nature, and in the sequel we shall ignore this case, and assume that the surface tension* τ *satisfies*

$$\ell^* := \frac{4}{4-w} \notin \mathbb{N}. \tag{3.10}$$

We proceed to work under assumption (3.10). The range $Range(\mathcal{L}^1_\ell)$ of areas a, for which ℓ-stacks of type 1 are defined is:

$$Range(\mathcal{L}^1_\ell) = \begin{cases} [4(\ell-1), \ell w], & \text{if } \ell < \ell^*, \\ [\ell w, 4(\ell-1)], & \text{if } \ell > \ell^*. \end{cases} \tag{3.11}$$

In either of the two cases above the surface tension

$$\tau\left(\mathcal{L}^1_\ell(a)\right) = 8(\ell-1) + 2\text{sign}(\ell^*-\ell)\sqrt{(4(\ell-1)-a)(4(\ell-1)-w)}. \tag{3.12}$$

In view of (3.9) and (3.10)

$$\frac{\mathrm{d}}{\mathrm{d}a}\tau\left(\mathcal{L}^1_\ell(a)\right) = \sqrt{\frac{4(\ell-1)-w}{4(\ell-1)-a}} = \frac{1}{r^{1,\ell}(a)}. \tag{3.13}$$

Definition (Stacks $\mathcal{L}^2_\ell(a)$ of type 2). These contain ℓ identical Wulff plaquettes of radius

$$r^{2,\ell}(a) = \sqrt{\frac{4\ell-a}{(4-w)\ell}}, \tag{3.14}$$

as it follows from $a = \ell(4-(4-w)r^2)$. The range $Range(\mathcal{L}^2_\ell)$ of areas a, for which stacks of type 2 are defined (for a given value of $\ell \in \mathbb{N}$), is:

$$Range(\mathcal{L}^2_\ell) = [\ell w, 4\ell] \tag{3.15}$$

Substituting the value of the radius (3.14) into (3.5) we infer that the surface tension of a stack of type 2 equals to:

$$\tau\left(\mathcal{L}^2_\ell(a)\right) = 8\ell - 2\sqrt{(4\ell-a)(4-w)\ell} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}a}\tau\left(\mathcal{L}^2_\ell(a)\right) = \frac{1}{r^{2,\ell}(a)}. \tag{3.16}$$

Note that by definition

$$\mathcal{L}^1_\ell(4(\ell-1)) = \mathcal{L}^2_{\ell-1}(4(\ell-1)) \text{ and } \mathcal{L}^1_\ell(\ell w) = \mathcal{L}^2_\ell(\ell w). \tag{3.17}$$

Also for notational convenience we set $\mathcal{L}^2_0(0) = \emptyset$.

Set $\tau_\ell(a) = \min\{\tau(\mathcal{L}^1_\ell(a)), \tau(\mathcal{L}^2_\ell(a))\}$, where we define

$$\tau\left(\mathcal{L}^i_\ell(a)\right) = \infty \quad \text{if } a \notin Range(\mathcal{L}^i_\ell) \tag{3.18}$$

We can rewrite (VP$_v$) as

$$\min_{a\geq 0, \ell\in\mathbb{N}} \left\{-va + \tau_\ell(a) + \frac{a^2}{2\sigma_\beta}\right\} = \min_{a\geq 0, \ell\in\mathbb{N}, i=1,2} \left\{-va + \mathcal{G}^i_\ell(a)\right\}, \tag{3.19}$$

where we put

$$\mathcal{G}^i_\ell(a) = \tau\left(\mathcal{L}^i_\ell(a)\right) + \frac{a^2}{2\sigma_\beta}, \quad i = 1, 2. \tag{3.20}$$

3.3. **Irrelevance of $\mathcal{L}^1_\ell$-stacks for $\ell > \ell^*$.** For $\ell > \ell^*$,

$$Range(\mathcal{L}^1_\ell) = [\ell w, 4(\ell-1)] \subset Range(\mathcal{L}^2_\ell) = [\ell w, 4\ell].$$

By the second of (3.17) the values of $\mathcal{L}^1_\ell$ and $\mathcal{L}^2_\ell$ coincide at the left end point $a = \ell w$. On the other hand, $r^1_\ell(a) \leq r^2_\ell(a)$ for any $a \in [\ell w, 4(\ell-1)]$. Hence, (3.13) and (3.16) imply that $\tau(\mathcal{L}^1_\ell(a)) \geq \tau(\mathcal{L}^2_\ell(a))$ whenever $\ell > \ell^*$.

3.4. **Variational problem for stacks of type 2.** We start solving the problem (3.19), by considering the simpler one:

$$\min_{a\in\cup_{\ell\geq 0}[w\ell, 4\ell], \ell\in\mathbb{N}_0} \left\{-va + \mathcal{G}^2_\ell(a)\right\} \stackrel{\Delta}{=} \min_{a\in\cup_{\ell\geq 0}[w\ell, 4\ell], \ell\in\mathbb{N}_0} F_v(\ell, a), \tag{3.21}$$

where we have defined (see (3.16)).

$$F_v(\ell, a) = -va + \frac{a^2}{2\sigma_\beta} + 8\ell - 2\sqrt{(4\ell - a)(4 - w)\ell}. \tag{3.22}$$

Recall (3.18) that we set $\mathcal{G}^2_\ell(a) = \infty$ whenever $a \notin Range(\mathcal{L}^2_\ell)$, as described in (3.15). In this way the functions $\mathcal{G}^2_\ell$ are defined on $\mathbb{R}$; each one has a support line at any slope v. In the variational problem (3.21) we are looking for the lowest such support line, which is precisely the support line to the graph of $\mathcal{G}^2 = \min_\ell \mathcal{G}^2_\ell$. For a generic slope v there is exactly one value $\ell(v)$ for which the minimum is realized. However, for certain critical values v^* of the slope v it might happen that the minimal support line touches the graphs of $\mathcal{G}^2_\ell$ for several different ℓ-s. As the following theorem shows, at every such critical value v^*, the index $\ell = \ell(v)$ of the optimal stack $\mathcal{G}^2_\ell$ jumps exactly by one unit up, that is $\ell(v^* + \varepsilon) = \ell(v^* - \varepsilon) + 1$ for $\varepsilon > 0$ small. Furthermore, these transition are of the first order, both in terms of the radii and the areas of optimal stacks.

Theorem 3. *There exists an increasing sequence of critical slopes* $0 = v_0^* < v_1^* < v_2^* < \dots$ *and an increasing sequence of the area values*

$$0 = a_0^+ < a_1^- < a_1^+ < a_2^- < a_1^+ < a_2^- < \dots$$

such that $a_\ell^\pm \in \mathit{Range}\,(\mathcal{L}_\ell^2) = [\ell w, 4\ell]$ *for every* $\ell \in \mathbb{N}$*, and:*
1. For $v \in [0, v_1^*)$ *the empty stack* $\mathcal{L}_0^2(0)$ *is the unique solution to* (3.21)*.*
2. For each $\ell \in \mathbb{N}$ *such that* $v_\ell^* < 1 + \frac{\ell w}{\sigma_\beta}$*, the minimum in* (3.21) *is attained, for all* $v \in (v_\ell^*, 1 + \frac{\ell w}{\sigma_\beta})$*, at* $a = a_\ell^- = \ell w$*. The corresponding stack is* $\mathcal{L}_\ell^2(\ell w)$*.*
3. For the remaining values of $v \in (v_\ell^* \vee (1 + \frac{\ell w}{\sigma_\beta}), v_{\ell+1}^*)$*, the picture is the following: for each* $\ell \in \mathbb{N}$ *there exists a continuous increasing bijection*

$$a_\ell : [v_\ell^* \vee (1 + \frac{\ell w}{\sigma_\beta}), v_{\ell+1}^*] \mapsto [a_\ell^-, a_\ell^+],$$

such that for each $v \in (v_\ell^* \vee (1 + \frac{\ell w}{\sigma_\beta}), v_{\ell+1}^*)$ *the stack* $\mathcal{L}_\ell^2\left(a_\ell(v)\right)$ *corresponds to the unique solution to* (3.21)*.*
4. At critical slopes $v_1^*, v_2^*, \dots$ *the transitions happen. There is a coexistence:*

$$\frac{(v - a_{\ell-1}^+)^2}{2\sigma_\beta} + \tau\left(\mathcal{L}_{\ell-1}^2(a_{\ell-1}^+)\right) = \frac{(v - a_\ell^-)^2}{2\sigma_\beta} + \tau\left(\mathcal{L}_\ell^2(a_\ell^-)\right). \tag{3.23}$$

Also, the radii of plaquettes of optimal stacks at coexistence points are increasing: Set $b_\ell^\pm = a_\ell^\pm/\ell$*. Then,* $b_{\ell-1}^+ > b_\ell^-$*, and hence*

$$r_{b_{\ell-1}^+} < r_{b_\ell^-} \tag{3.24}$$

for every $\ell \in \mathbb{N}$*.*

We shall prove Theorem 3 under additional non-degeneracy assumption (3.10). However, the proof could be easily modified in order to accommodate the degenerate case as well.

The fact that the problem should exhibit first order transitions could be easily understood from (3.16). The crux of the proof below is to show that when v increases, the number $\ell = \ell(v)$ of layers of the corresponding optimal stack $\mathcal{L}_\ell^2\left(a_\ell(v)\right)$ either stays the same or increases by one, and, above all, to deduce all the results without resorting to explicit and painful computations.

Proof of Theorem 3. Let us start with the following two facts:
Fact 1. For every $\ell \geq 1$, the function $\{\mathcal{G}_\ell^2\left(a\right)\}$ is strictly convex. Let $a_\ell = a_\ell(v)$ be the point where a line with the slope v supports its graph. If

$$v \leq \frac{\mathrm{d}^+}{\mathrm{d}a}\Big|_{a=\ell w} \left\{\mathcal{G}_\ell^2\left(a\right)\right\} \overset{(3.16)}{=} \frac{1}{r^{2,\ell}(\ell w)} + \frac{\ell w}{\sigma_\beta} = 1 + \frac{\ell w}{\sigma_\beta}, \tag{3.25}$$

then $a_\ell(v) = \ell w$. In the remaining region $v > 1 + \frac{\ell w}{\sigma_\beta}$ the value $a_\ell(v)$ is recovered from:

$$v = \frac{\mathrm{d}}{\mathrm{d}a}\Big|_{a=a_\ell} \tau\left(\mathcal{L}_\ell^2(a)\right) + \frac{a_\ell}{\sigma_\beta} \overset{(3.16),(3.14)}{=} \sqrt{\frac{4-w}{4-a_\ell/\ell}} + \frac{a_\ell}{\sigma_\beta} \triangleq \sqrt{\frac{4-w}{4-b_\ell}} + \frac{\ell b_\ell}{\sigma_\beta}. \quad (3.26)$$

Thus, both $a_\ell(v)$ and $b_\ell(v) \triangleq a_\ell(v)/\ell$ are well defined for any $v \in \mathbb{R}$. Of course we consider only $v \in [0, \infty)$. If $m > \ell$, then, by definition, we have for all $v > 1 + \frac{mw}{\sigma_\beta}$ (i.e. when both curves $\mathcal{G}_\ell^2$ and $\mathcal{G}_m^2$ have tangent lines with slope v) that

$$v = \sqrt{\frac{4-w}{4-b_\ell(v)}} + \frac{\ell b_\ell(v)}{\sigma_\beta} = \sqrt{\frac{4-w}{4-b_m(v)}} + \frac{m b_m(v)}{\sigma_\beta}. \quad (3.27)$$

If $v \in (1 + \frac{\ell w}{\sigma_\beta}, 1 + \frac{mw}{\sigma_\beta}]$, then $b_m(v) = w$ and

$$v = \sqrt{\frac{4-w}{4-b_\ell(v)}} + \frac{\ell b_\ell(v)}{\sigma_\beta} < \sqrt{\frac{4-w}{4-b_m(v)}} + \frac{m b_m(v)}{\sigma_\beta}, \quad (3.28)$$

Finally, if $v \le 1 + \frac{\ell w}{\sigma_\beta}$ then $b_\ell(v) = b_m(v) = w$, and the second inequality in (3.28) trivially holds. Together (3.27) and (3.28) imply that for any $v \in [0, \infty)$,

$$a_m(v) > a_\ell(v) \quad \text{and} \quad b_m(v) \le b_\ell(v). \quad (3.29)$$

Fact 2. It is useful to think about F_v as a function of continuous variables $\ell, a \in \mathbb{R}_+$. By direct inspection $-\sqrt{\ell(4\ell - a)}$ is strictly convex on $\mathbb{R}_+^2$ and thus also on the convex sector

$$\mathbb{D}_w \triangleq \{(\ell, a) : 0 \le \ell w \le a \le 4\ell\} \subset \mathbb{R}_+^2.$$

Hence, F_v is strictly convex on $\mathbb{D}_w$ as well. This has the following implication: If $(\ell_1, a_1) \neq (\ell_2, a_2)$ are such that $F_v(\ell_1, a_1) = F_v(\ell_2, a_2)$, then

$$F_v(\ell_1, a_1) > F_v(\lambda\ell_1 + (1-\lambda)\ell_2, \lambda a_1 + (1-\lambda)a_2) \quad (3.30)$$

for any $\lambda \in (0, 1)$.

Let us go back to (3.21). Clearly $\min_{a \ge 0, \ell \in \mathbb{N}_0} F_v(\ell, a)$ is attained for all v, and, furthermore $(0,0) = \mathrm{argmin}(F_v)$ for all v sufficiently small. It is also clear that $(0,0) \notin \mathrm{argmin}(F_v)$ whenever v is sufficiently large.

Therefore there exists the unique minimal values $v_1^* > 0$ and $\ell_1^* \ge 1$, and, accordingly the value $a_1^- \in \mathit{Range}\left(\mathcal{L}_{\ell_1^*}^2\right) = [\ell_1^* w, 4\ell_1^*]$ satisfying the condition

$$F_{v_1^*}(0,0) = F_{v_1^*}\left(\ell_1^*, a_1^-\right)$$

Let us show that $\ell_1^* = 1$. Indeed, assume that $\ell_1^* > 1$. But then for the value $\ell = 1$, intermediate between $\ell = 0$ and ℓ_1^*, we have $F_{v_1^*}(1, \frac{a_1^-}{\ell_1^*}) > F_{v_1^*}(0,0)$, which

contradicts the convexity property (3.30). Hence $a_1^- \in [w, 4)$. By the same strict convexity argument,

$$F_{v_1^*}(1, a_1^-) = \min_a F_{v_1^*}(1, a) < \min_{\ell > 1, a} F_{v_1^*}(\ell, a)$$

By continuity the inequality above will persist for $v > v_1^*$ with $v - v_1^*$ sufficiently small. Also, $F_v(1, a_1(v)) < F_v(0, 0)$ for every $v > v_1^*$, since the function $a \mapsto \tau\left(\mathcal{L}_1^2(a)\right)$ is strictly convex. This means that there exists the maximal $v_2^* > v_1^*$, $a_1^+ > a_1^-$ and a continuous bijection $a_1 : [v_1^*, v_2^*] \mapsto [a_1^-, a_1^+]$ such that $(1, a(v)) = \mathrm{argmin} F_v$ on (v_1^*, v_2^*).

Now we can proceed by induction. Let us define the segment $[v_\ell^*, v_{\ell+1}^*]$ as the maximal segment of values of the parameter v, for which there exist the corresponding segment $[a_\ell^-, a_\ell^+]$ and a continuous non-decreasing function $a_\ell : [v_\ell^*, v_{\ell+1}^*] \mapsto [a_\ell^-, a_\ell^+]$ such that $(\ell, a_\ell(v)) = \mathrm{argmin} F_v$ for $v \in (v_\ell^*, v_{\ell+1}^*)$. (If there are several such segments for the same value of ℓ, we take for $[v_\ell^*, v_{\ell+1}^*]$, by definition, the leftmost one. Of course, we will show below that it can not be the case, but we do not suppose it now.) Our induction hypothesis is that the open segment $(v_\ell^*, v_{\ell+1}^*)$ is non-empty, and that

$$\min_{m < \ell} \min_a F_v(m, a) > \min_a F_v(\ell, a), \tag{3.31}$$

for $v > v_\ell^*$. We have already checked it for $\ell = 1$.

Clearly, $(\ell, a) \notin \mathrm{argmin} F_v$ whenever v is sufficiently large. Thus, $v_{\ell+1}^* < \infty$. By induction hypothesis (3.31),

$$F_{v_{\ell+1}^*}(m, a) = F_{v_{\ell+1}^*}(\ell, a_\ell^+) = \min F_{v_{\ell+1}^*} \tag{3.32}$$

implies that $m > \ell$. As before, using convexity and continuity arguments we can check that if (3.32) holds for some $m > \ell$ and a (with $(m, a) \in \mathbb{D}_w$) then necessarily $m = \ell + 1$ and, furthermore,

$$\min_a F_v(\ell + 1, a) < \min_{m > \ell + 1, a} F_v(m, a)$$

for $v > v_{\ell+1}^*$ with $v - v_{\ell+1}^*$ sufficiently small. The first part of the induction step is justified.

Assume finally that $F_v(\ell, a_\ell) = F_v(\ell + 1, a_{\ell+1}) = \min F_v$. Then $a_\ell < a_{\ell+1}$, as it is stated in (3.29). By the same authority, $b_\ell \geq b_{\ell+1}$, and hence $r_{b_\ell} \leq r_{b_{\ell+1}}$. By convexity of both $\tau\left(\mathcal{L}_\ell^2(a)\right)$ and $\tau\left(\mathcal{L}_{\ell+l}^2(a)\right)$ the inequality $a_\ell < a_{\ell+1}$ implies that

$$\min_a F_u(\ell, a) > \min_a F_u(\ell + 1, a) \text{ for any } u > v.$$

Consequently, $\min_a F_u(\ell, a) > \min_a F_u(\ell + 1, a)$ for any $u > v_{\ell+1}^*$, and we are home. $\square$

3.5. **Analysis of** (VP_v)**.** As we already know, ℓ-stacks of type 1 cannot be optimal whenever $\ell > \ell^*$ (see definition (3.10)). Let us explore what happens if $\ell < \ell^*$. In this case

$$Range(\mathcal{L}^1_\ell) = [4(\ell-1), \ell w] \quad \text{and} \quad Range(\mathcal{L}^2_\ell) = [\ell w, 4\ell].$$

Thus, $Range(\mathcal{L}^1_\ell)$ shares endpoints $4(\ell-1)$ and ℓw with $Range(\mathcal{L}^2_{\ell-1})$ and, respectively, $Range(\mathcal{L}^2_\ell)$, and all these ranges have disjoint interiors. So, in principle, ℓ-stack of type 1 may become optimal. Note that by our convention, (3.17) and (3.20),

$$\mathcal{G}^2_{\ell-1}(4(\ell-1)) = \mathcal{G}^1_\ell(4(\ell-1)) \quad \text{and} \quad \mathcal{G}^1_\ell(\ell w) = \mathcal{G}^2_\ell(\ell w), \tag{3.33}$$

so, for $\ell < \ell^*$ the two families $\mathcal{G}^1_\ell$, $\mathcal{G}^2_\ell$ merge together into a single continuous function. In fact, it is even smooth, except that the tangent to its graph becomes vertical at endpoints 4ℓ. Let v^*_ℓ be the critical slope for variational problem (3.21), as described in Theorem 3. By construction, there is a line $\mathfrak{l}(v^*_\ell)$ with slope v^*_ℓ which supports both the graphs of $\mathcal{G}^2_{\ell-1}$ and of $\mathcal{G}^2_\ell$.

Definition. Let us say that the graph of $\mathcal{G}^1_\ell$ *sticks out below* $\mathfrak{l}(v^*_\ell)$ if there exists $a \in Range(\mathcal{L}^1_\ell)$ such that

$$\mathcal{G}^1_\ell(a) < \mathcal{G}^2_\ell(a^-_\ell) - v^*_\ell(a^-_\ell - a). \tag{3.34}$$

Obviously, there are optimal ℓ-stacks of type 1 iff the graph of $\mathcal{G}^1_\ell$ sticks out below $\mathfrak{l}(v^*_\ell)$.

Proposition 4. *For any $\ell < \ell^*$ the graph of $\mathcal{G}^1_\ell$ sticks out below $\mathfrak{l}(v^*_\ell)$ iff*

$$v^*_\ell < 1 + \frac{\ell w}{\sigma_\beta}. \tag{3.35}$$

Equivalently, this happens iff

$$\mathcal{G}^2_{\ell-1}(a) > \mathcal{G}^2_\ell(\ell w) - (\ell w - a)\left(1 + \frac{\ell w}{\sigma_\beta}\right), \tag{3.36}$$

for any $a \in Range(\mathcal{L}^2_{\ell-1}) = [(\ell-1)w, 4(\ell-1)]$.

Proof. Note first of all that in view of (3.25), the condition (3.35) necessarily implies that $a^-_\ell = \ell w$. Consequently the fact that (3.35) and (3.36) are equivalent is straightforward, since by construction $\mathfrak{l}(v^*_\ell)$ supports both graphs.

Note that since $\frac{\mathrm{d}^+}{\mathrm{d}a}\mathcal{G}^2_{\ell-1}(4(\ell-1)) = \infty$ and since $\mathcal{L}^2_{\ell-1}(4(\ell-1)) = \mathcal{L}^1_\ell(4(\ell-1))$ the graph of $\mathcal{G}^1_\ell$ has to stay above $\mathfrak{l}(v^*_\ell)$ for values of $a \in Range(\mathcal{L}^1_\ell)$ which are sufficiently close to $4(\ell-1)$.

Let us compute the second derivative

$$\frac{\mathrm{d}^2}{\mathrm{d}a}\mathcal{G}^1_\ell(a) \overset{(3.12)}{=} \frac{1}{\sigma_\beta} - \frac{1}{2}\sqrt{\frac{\ell w - (4(\ell-1)}{(a-4(\ell-1))^3}}. \tag{3.37}$$

The expression on the right hand side above is increasing (from $-\infty$) on $Range(\mathcal{L}^1_\ell)$. If it is non-positive on the whole interval, then the graph of $\mathcal{G}^1_\ell$ is concave and it

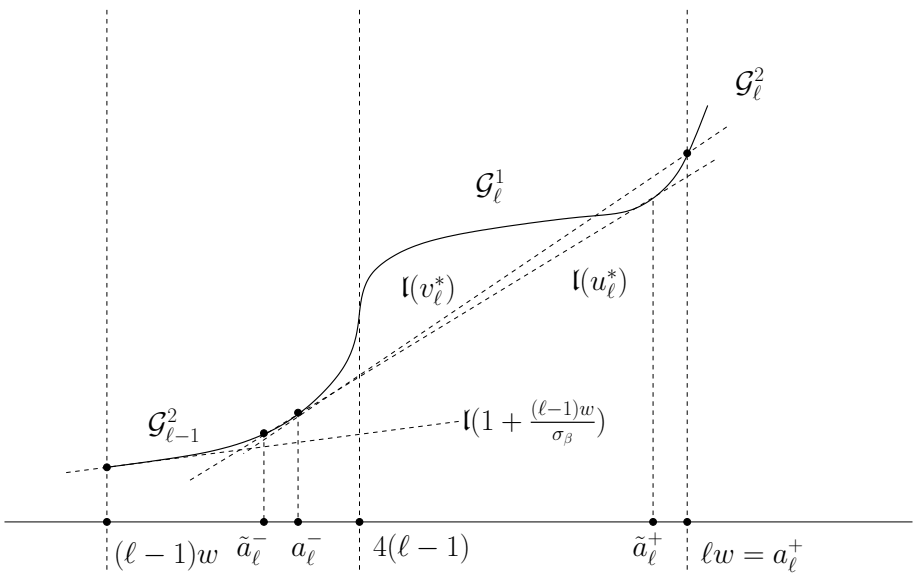

FIGURE 1. The graph of $\mathcal{G}^1_\ell$ sticks out below $\mathfrak{l}(v^*_\ell)$. The transition slope $\tilde{v}^*_\ell$ satisfies $1+\frac{(\ell-1)w}{\sigma_\beta} < \tilde{v}^*_\ell < v^*_\ell < 1+\frac{\ell w}{\sigma_\beta}$.

cannot stick out. Otherwise, the graph of $\mathcal{G}^1_\ell$ is convex near the right end point ℓw and hence its maximal derivative on the convex part is attained at $a=\ell w$ and equals to $1+\frac{\ell w}{\sigma_\beta}$. Therefore, (3.35) is a necessary condition for the graph of $\mathcal{G}^1_\ell$ to stick out.

To see that it is also a sufficient condition recall once again that $v^*_\ell < 1+\frac{\ell w}{\sigma_\beta}$ means that $\mathfrak{l}(v^*_\ell)$ supports $\mathcal{G}^2_\ell$ at the left end point $a=\ell w$. But then, $\mathcal{G}^1_\ell$ goes below $\mathfrak{l}(v^*_\ell)$ for all values of $a \in Range(\mathcal{L}^1_\ell)$ which are sufficiently close to ℓw, because "the union of $\mathcal{G}^1_\ell$ and $\mathcal{G}^2_\ell$" is smooth at $a=\ell w$. In particular, it should have a convex part. $\square$

Remark 5. *The argument above does not imply that for $\ell < \ell^* = \frac{4}{4-w}$ the graph of $\mathcal{G}^1_\ell$ sticks out if it has a convex part near ℓw. The latter is a necessary condition which gives the following upper bound on the maximal number of layers ℓ such that*

$\mathcal{G}^1_\ell$ may stick out: Let us substitute $a = \ell w$ into (3.37)*:*

$$\frac{1}{\sigma_\beta} - \frac{1}{2}\sqrt{\frac{\ell w - 4(\ell-1)}{(\ell w - 4(\ell-1))^3}} > 0 \overset{(3.10)}{\Leftrightarrow} \ell < \ell^* \left(1 - \frac{\sigma_\beta}{8}\right). \tag{3.38}$$

Proposition 6. *If $w \le 2\sigma_\beta$, then the graph of $\mathcal{G}^1_\ell$ does not stick out for any value of $\ell < \ell^* \left(1 - \frac{\sigma_\beta}{8}\right)$ (and hence stacks of type 1 are never optimal).*

If, however, $w > 2\sigma_\beta$, then there exists a number k^*, $1 \le k^* < \ell^* \left(1 - \frac{\sigma_\beta}{8}\right)$ such that the graphs of $\mathcal{G}^1_\ell$ stick out below $\mathfrak{l}(v^*_\ell)$-s for any $\ell = 1, \dots, k^*$, and they do not stick out for $\ell > k^*$.

Proof. The proof comprises two steps.

STEP 1 We claim that the graph of $\mathcal{G}^1_1$ sticks out below $\mathfrak{l}(v^*_1)$ iff $w > 2\sigma_\beta$.

Indeed, recall that $\mathfrak{l}(v^*_1)$ is the line which passes through the origin and which is tangent to the graph of $\mathcal{G}^2_1$. Since the latter is convex and increasing, $v^*_1 < 1 + \frac{w}{\sigma_\beta}$ iff

$$\mathcal{G}^2_1(w) < w\left(1 + \frac{w}{\sigma_\beta}\right) \Leftrightarrow 2w + \frac{w^2}{2\sigma_\beta} < w + \frac{w^2}{\sigma_\beta} \Leftrightarrow w > 2\sigma_\beta,$$

so the claim follows from Proposition 4.

STEP 2 We claim that for any $1 \le m < \ell$, if the graph of $\mathcal{G}^1_\ell$ sticks out below $\mathfrak{l}(v^*_\ell)$, then the graph of $\mathcal{G}^1_m$ sticks out below $\mathfrak{l}(v^*_m)$.

Assume that (3.36) holds. First of all take $a = (\ell-1)w$. Recall that $\mathcal{G}^2_\ell(\ell w) = 2\ell w + (\ell w)^2/2\sigma_\beta$. Therefore,

$$\mathcal{G}^2_{\ell-1}\left((\ell-1)w\right) - \mathcal{G}^2_\ell(\ell w) + w\left(1 + \frac{\ell w}{\sigma_\beta}\right) = \frac{w^2}{2\sigma_\beta} - w > 0. \tag{3.39}$$

Furthermore, if we record the range $a \in Range(\mathcal{L}^2_{(\ell-1)})$ as $a = (\ell-1)w + c$; $c \in [0, (4-w)(\ell-1)]$, then, in view of (3.39), the necessary and sufficient condition (3.36) for the graph of $\mathcal{G}^1_\ell$ to stick out reads as:

$$\begin{aligned}
&\mathcal{G}^2_{\ell-1}\left((\ell-1)w + c\right) - \mathcal{G}^2_\ell(\ell w) + (w-c)\left(1 + \frac{\ell w}{\sigma_\beta}\right) \\
&= \left(\frac{w^2}{2\sigma_\beta} - w\right) + \int_0^c \left(\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{G}^2_{\ell-1}\left((\ell-1)w + \tau\right) - \left(1 + \frac{\ell w}{\sigma_\beta}\right)\right)\mathrm{d}\tau \\
&\overset{(3.16)}{=} \left(\frac{w^2}{2\sigma_\beta} - w\right) + \int_0^c \left(\frac{1}{r^{2,\ell-1}((\ell-1)w + \tau)} - \left(1 + \frac{\ell w}{\sigma_\beta}\right)\right)\mathrm{d}\tau > 0.
\end{aligned} \tag{3.40}$$

for any $c \in [0, (4-w)(\ell-1)]$.

Now for any $k \in \mathbb{N}$ (and $\tau \in [0, (4-w)k]$),

$$r^{2,k}(kw + \tau) \overset{(3.14)}{=} \sqrt{\frac{4k - (kw + \tau)}{(4-w)k}} \overset{(3.10)}{=} \sqrt{1 - \frac{\tau}{(4-w)k}}.$$

That means that for a fixed τ the function $k \mapsto r^{2,k}(kw+\tau)$ is increasing in k. Therefore, (3.40) implies that,

$$\left(\frac{w^2}{2\sigma_\beta} - w\right) + \int_0^c \left(\frac{1}{r^{2,m-1}((m-1)w+\tau)} - \left(1+\frac{mw}{\sigma_\beta}\right)\right) \mathrm{d}\tau > 0, \tag{3.41}$$

for any $m = 1, \dots, \ell$ and, accordingly, any $c \in [0, (m-1)(4-w)]$. Consequently, (3.36) holds for any $m \le \ell$. □

Assume now that $\ell < \ell^*$ and that the graph of $\mathcal{G}^1_\ell$ sticks out below $\mathfrak{l}(v^*_\ell)$. This means that there exits a range of slopes $(\tilde{v}^*_\ell, v^*_\ell)$ such that for any $v \in (\tilde{v}^*_\ell, v^*_\ell)$, one can find $a = a(v)$, such that $\mathcal{L}^1_\ell(a(v))$ is the unique solution to the variational problem (VP_v). By (3.13),

$$v = \frac{1}{r^{1,\ell}(a(v))} + \frac{a(v)}{\sigma_\beta} \ge 1 + \frac{4(\ell-1)}{\sigma_\beta}.$$

Hence, in view of Proposition 4 (and in view of the fact that by Proposition 6 the graph of $\mathcal{G}^1_{\ell-1}$ has to stick out as well and consequently $v^*_{\ell-1} < 1 + \frac{(\ell-1)w}{\sigma_\beta}$),

$$\tilde{v}^*_\ell > 1 + \frac{(\ell-1)w}{\sigma_\beta} > v^*_{\ell-1}.$$

Which means that in the range of slopes $[1 + \frac{(\ell-1)w}{\sigma_\beta}, \tilde{v}^*_\ell)$ the $(\ell-1)$-stacks of type 2 continue to be optimal.

The structure of solutions and first order transitions in terms of optimal layers and optimal areas is summarized in Theorem B and depicted on Figure 2.

Theorem B. *If $w \le 2\sigma_\beta$, then solutions to the variational problem* (VP_v) *are as described in Theorem 3.*

If, however, $w > 2\sigma_\beta$, then there exists a number $1 \le k^* < \ell^*\left(1 - \frac{\sigma_\beta}{8}\right)$ such that the following happens: For every $\ell = 1, \dots, k^*$ there exists a slope $\tilde{v}^*_\ell$;

$$1 + \frac{(\ell-1)w}{2\sigma_\beta} < \tilde{v}^*_\ell < v^*_\ell < 1 + \frac{\ell w}{2\sigma_\beta},$$

such that

1. The empty stack $\mathcal{L}^2_0$ is the unique solution to (VP_v) *for $v \in [0, \tilde{v}^*_1)$.*

2. For every $\ell = 1, \dots, k^*$ there is an area $\tilde{a}^-_\ell \in (4(\ell-1), \ell w)$ and a continuous increasing bijection $\tilde{a}_\ell : [\tilde{v}^*_\ell, 1 + \frac{\ell w}{2\sigma_\beta}] \mapsto [\tilde{a}^-_\ell, \ell w]$ such that the ℓ-stack $\mathcal{L}^1_\ell(\tilde{a}_\ell(v))$ of type 1 is the unique solution to (VP_v) *for every $v \in (\tilde{v}^*_\ell, 1 + \frac{\ell w}{2\sigma_\beta})$.*

3. For every $\ell < k^*$ the ℓ-stack $\mathcal{L}^2_\ell(a_\ell(v))$ of type 2 is the unique solution to (VP_v) *for every $v \in (1 + \frac{\ell w}{\sigma_\beta}, \tilde{v}^*_{\ell+1})$, where a_ℓ is the bijection described in Theorem 3.*

4. The stack $\mathcal{L}^2_{k^*}(a_{k^*}(v))$ is the unique solution to (VP_v) *for every $v \in [1 + \frac{k^* w}{\sigma_\beta}, v_{k^*+1})$.*

5. Finally, for every $\ell > k^*$ the transition slope $v^*_\ell \geq 1 + \frac{\ell w}{\sigma_\beta}$, and the stack $\mathcal{L}^2_\ell(a_\ell(v))$ of type 2 is the unique solution to (VP$_v$) *for every $v \in (v^*_\ell, v^*_{\ell+1})$.*

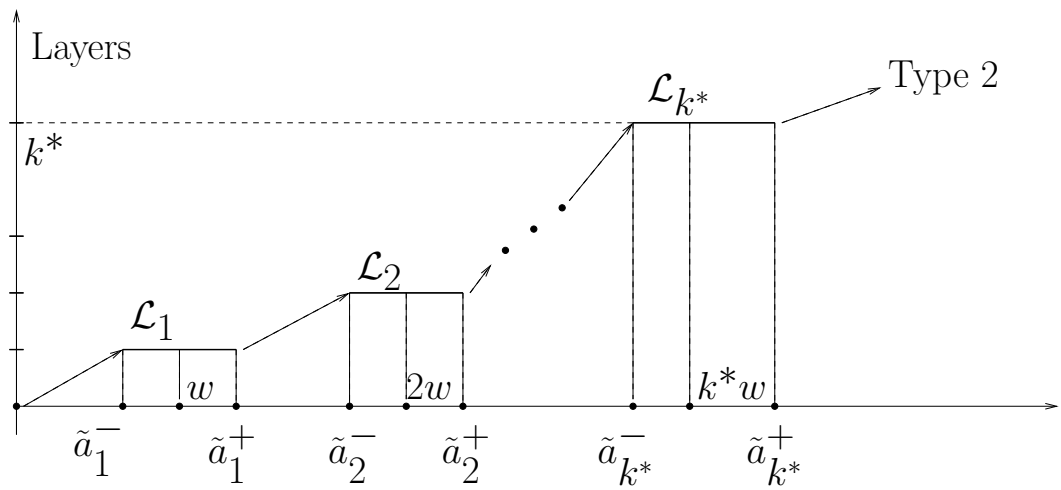

FIGURE 2. For $\ell = 1, \ldots, k^* < \ell^*$ families $\mathcal{L}^1_\ell$ of type one are optimal for $a \in (\tilde{a}^-_\ell, \ell w)$, whereas families $\mathcal{L}^2_\ell$ of type 2 are optimal for $a \in (\ell w, \tilde{a}^+_\ell)$. First order transitions - jumps in terms of number of optimal layers from $\ell - 1$ to ℓ, and in terms of sizes of optimal areas from $\tilde{a}^+_{\ell-1}$ to $\tilde{a}^-_\ell$ (where we set $\tilde{a}^+_0 = 0$) - occur at transition slopes $\tilde{v}^*_\ell$. For $\ell > k^*$ only families of type two are optimal, and first order transitions occur as described in Theorem 3.

4. Low temperature level lines

The main contribution to the event $\{\Xi_N \geq \rho_\beta N^3 + AN^2\}$ comes from bulk fluctuations and creations of macroscopic size facets (large contours - see below) of the interface Γ. In order to formulate the eventual reduced model let us first of all collect the corresponding notions and facts from [24].

4.1. **Bulk Fluctuations.** For each β fixed bulk fluctuations are governed by local limit results for sums of Bernoulli random variables, as the linear size of the system $N \to \infty$. Let us record a more quantitative version of (2.14): For every K fixed

$$\log \mathbb{P}_{N,\beta}\left(\Xi_N = \rho_\beta N^3 + AN^2 \middle| \Gamma\right) = -N \frac{\left(\delta_\beta - \frac{\alpha(\Gamma)}{N^2}\right)^2}{2D_\beta} + O\left(\log N\right), \tag{4.1}$$

uniformly in $A \le K$ and $|\alpha(\Gamma)| \le KN^2$.

4.2. **Contours and their Weights.** There is a natural contour parametrization of surfaces Γ. Namely, given an interface Γ and, accordingly, the height function h_Γ which, by definition, is identically zero outside Λ_N°, define the following semi-infinite subset $\widehat{\Gamma}$ of $\mathbb{R}^3$,

$$\widehat{\Gamma} \,=\, \bigcup_{\substack{(x,y,k)\\ k<h_\Gamma(x,y)}} \Big((x,y,k)+\widehat{C}\Big)\,,$$

where $\widehat{C} = [-1/2, 1/2]^3$ is the unit cube. The above union is over all $(x,y) \in \mathbb{Z}^2$ and $k \in 1/2 + \mathbb{Z}$.

Consider now the level sets of Γ, i.e. the sets

$$H_k \,=\, H_k\left(\widehat{\Gamma}\right) \,=\, \Big\{(x,y)\in\mathbb{R}^2 : (x,y,k)\in\widehat{\Gamma}\Big\}\,,\ k=-N,\,-N+1,\dots,\,N.$$

We define *contours* as the connected components of sets ∂H_k, subject to south-west splitting rules, see Section 2.1 in [25] or [18]. The length $|\gamma|$ of a contour is defined in an obvious way. Since, by construction all contours are closed self-avoiding polygons composed of the nearest neighbor bonds of Λ_N^*, the notions of interior $\mathrm{int}(\gamma)$ and exterior $\mathrm{ext}(\gamma)$ of a contour γ are well defined. A contour γ is called a $\oplus$-contour ($\ominus$-contour), if the values of the function h_Γ at the immediate exterior of γ are smaller (bigger) than those at the immediate interior of γ.

Alternatively, let us orient the bonds of each contours $\gamma \subseteq \partial H_k$ in such a way that when we traverse γ the set H_k remains to the right. Then $\oplus$-contours are those which are clockwise oriented with respect to their interior, whereas $\ominus$-contours are counter-clockwise oriented with respect to their interior.

Let us say that two oriented contours γ and γ' are compatible, $\gamma \sim \gamma'$, if

(1) Either $\mathrm{int}(\gamma)\cap\mathrm{int}(\gamma')=\emptyset$ or $\mathrm{int}(\gamma)\subseteq\mathrm{int}(\gamma')$ or $\mathrm{int}(\gamma')\subseteq\mathrm{int}(\gamma)$.
(2) Whenever γ and γ' share a bond b, b has the same orientation in both γ and γ'.

A family $\Gamma = \{\gamma_i\}$ of oriented contours is called consistent, if contours of Γ are pair-wise compatible. It is clear that the interfaces Γ are in one-to-one correspondence with consistent families of oriented contours. The height function h_Γ could be reconstructed from a consistent family $\Gamma = \{\gamma\}$ in the following way: For every contour γ the sign of γ, which we denote as $\mathrm{sign}(\gamma)$, could be read from it orientation. Then,

$$h_\gamma(x,y) \,=\, \mathrm{sign}(\gamma)\chi_{\mathrm{int}(\gamma)}(x,y) \quad\text{and}\quad h_\Gamma \,=\, \sum_{\gamma\in\Gamma} h_\gamma, \tag{4.2}$$

where χ_A is the indicator function of A.

In this way the weights $w_\beta(\Gamma)$ which appear in (2.2) could be recorded as follows: Let $\Gamma = \{\gamma\}$ be a consistent family of oriented (signed) contours, Then,

$$w_\beta(\Gamma) \,=\, \exp\left\{-\beta \sum_{\gamma\in\Gamma} |\gamma|\right\}. \tag{4.3}$$

In the sequel we shall assume that β is sufficiently large. By definition the weight of the flat interface **is** $w_\beta(\Gamma_0) = 1$.

4.3. **Absence of Intermediate and Negative Contours.** In the sequel a claim that a certain property holds *for all β sufficiently large* means that one can find β_0, such that it holds for all $\beta \geq \beta_0$.

For all β sufficiently large the following rough apriori bound holds (see (6.2) in [24]): There exist positive combinatorial (that is independent of β) constant ν such for every $b_0 > 0$ fixed,

$$\log \mathbb{P}_{N,\beta}\left(|\alpha(\Gamma)| \,>\, bN^2\right) \,\lesssim\, -\nu\beta N\sqrt{b}, \tag{4.4}$$

uniformly in $b \geq b_0$ and in N large enough. A comparison with (4.1) reveals that we may fix $K_\beta = K_\beta(A)$ and ignore Γ with $\alpha(\Gamma) \geq K_\beta N^2$. Now let the interface Γ with $\alpha(\Gamma) \leq K_\beta N^2$ be given by a consistent collection of contours, and assume that $\gamma \sim \Gamma$. Of course $\alpha(\Gamma\cup\gamma) = \alpha(\Gamma) + \alpha(\gamma)$. Then [24] there exists a constant $c_\beta = c_\beta(A)$ such that

$$-\log \mathbb{P}_{N,\beta}\left(\Gamma\cup\gamma \,\middle|\, \Xi_N \geq \rho_\beta N^3 + AN^2\right) \lesssim c_\beta \frac{|\gamma|^2}{N} \mathbb{1}_{\mathrm{sign}(\gamma)=1} - \beta|\gamma|. \tag{4.5}$$

Consequently, there exists $\epsilon_\beta = \epsilon_\beta(A) > 0$, such that we can rule out all contours γ with $\epsilon_\beta^{-1}\log N \leq |\gamma| \leq \epsilon_\beta N$, and all negative contours γ with $|\gamma| > \epsilon_\beta N$.

It is not difficult to see that (2.3) implies that

$$\liminf_{\beta\to\infty} \frac{1}{\beta}\log \epsilon_\beta(A) > -\infty \tag{4.6}$$

for any A fixed.

Definition. The notion of big and small contours depends on the running value of excess area A, on the linear size of the system N and on the inverse temperature β. Namely, a contour γ is said to be large, respectively small, if

$$|\gamma| \geq \epsilon_\beta(A)N \quad \text{and, respectively, if} \quad |\gamma| \leq \frac{1}{\epsilon_\beta(A)}\log N. \tag{4.7}$$

Since we already know that intermediate contours and large $\ominus$-type contours could be ignored, let us use $\hat{\mathbb{P}}_{N,\beta}$ for the restricted ensemble which contains only $\oplus$-type large or small contours.

4.4. **Irrelevance of Small Contours.** Let $\Gamma = \Gamma^l \cup \Gamma^s$ is a compatible collection of contours with Γ^l being the corresponding set of large contours of Γ and, accordingly, Γ^s being the collection of small contours of Γ. Clearly,

$$\log \mathbb{P}_{N,\beta}\left(\left|\Gamma^l\right| \geq cN\right) \leq -\beta c N \left(1 - \mathrm{o}_N(1)\right), \tag{4.8}$$

uniformly in c and all β sufficiently large. Hence, again by comparison with (4.1) we can ignore all collections of large contours with total length $\left|\Gamma^l\right| \geq K_\beta N$.

On the other hand, elementary low density percolation arguments and the $\pm$-symmetry of height function imply that

$$\log \mathbb{P}_{N,\beta}\left(|\alpha(\Gamma^s)| \geq b \ \middle|\ \Gamma^l\right) \lesssim -\frac{\mathrm{e}^{4\beta} b^2}{N^2} \wedge \frac{\epsilon_\beta b}{\log N}, \tag{4.9}$$

uniformly in Γ^l and in $b \lesssim K_\beta N^2$. Again, a comparison with (4.1) implies that we may restrict attention to the case of $|\alpha(\Gamma^s)| \lesssim N^{3/2}$. Such corrections to the total value of Ξ are invisible on the scales (2.7) we work with and, consequently, the area shift induced by small contours may be ignored.

4.5. **The Reduced Model of Big Contours.** In the sequel we shall employ the following notation: $\mathcal{C}$ for clusters of non-compatible *small* contours and $\Phi_\beta(\mathcal{C})$ for the corresponding cluster weights which shows up in the cluster expansion representation of partition functions. Note that although the family of clusters $\mathcal{C}$ does depend on the running microscopic scale N, the weights $\Phi_\beta(\mathcal{C})$ remains the same. By usual cluster expansion estimates, for all β sufficiently large

$$\sup_{\mathcal{C} \neq \emptyset} \mathrm{e}^{2\beta(\mathrm{diam}_\infty(\mathcal{C})+1)} \left|\Phi_\beta(\mathcal{C})\right| \lesssim 1. \tag{4.10}$$

We are ready now to describe the reduced model which takes into account only large contours: The probability of a compatible collection $\Gamma = \{\Gamma_1, \ldots, \Gamma_k\}$ of *large* contours is given by

$$\mathbb{Q}_{N,\beta}(\Gamma) \cong \exp\left\{-\beta \sum |\Gamma_i| - \sum_{\mathcal{C} \subset \Lambda_N} \mathbb{1}_{\{\mathcal{C} \not\sim \Gamma\}} \Phi_\beta(\mathcal{C})\right\}, \tag{4.11}$$

The conditional distributions of Ξ_N^v and Ξ_N^s given such Γ^l are still $\mathrm{Bin}\left(|V_N(\Gamma)|, p^v\right)$ and $\mathrm{Bin}\left(|S_N(\Gamma)|, p^s\right)$, and we shall use $\mathbb{Q}_{N,\beta}$ for the corresponding joint distribution.

For future references let us reformulate the bulk fluctuation bound (4.1) in terms of the reduced measure $\mathbb{Q}_{N,\beta}$: For every K fixed

$$\log \mathbb{Q}_{N,\beta}\left(\Xi_N \geq \rho_\beta N^3 + A N^2 \middle| \Gamma\right) = -N \frac{\left(\delta_\beta - \frac{\alpha(\Gamma)}{N^2}\right)^2}{2D_\beta} + O\left(\log N\right), \tag{4.12}$$

uniformly in $A \leq K$ and $|\alpha(\Gamma)| \leq K N^2$.

The notation for conditional reduced measures is

$$\mathbb{Q}^{A}_{N,\beta} = \mathbb{Q}_{N,\beta}\left(\cdot \,\middle|\, \Xi_N \geq \rho_\beta N^3 + AN^2\right). \tag{4.13}$$

From now on we shall concentrate on proving **A2** of Theorem A for $\mathbb{Q}^{A}_{N,\beta}$-measures instead of $\mathbb{P}^{A}_{N,\beta}$-measures. Specifically, we shall prove:

Theorem C. *Assume that bulk occupation probabilities p_v and p_s satisfy* (2.3)*. Then there exists $\beta_0 < \infty$ such that for any $\beta > \beta_0$ the following holds: Let $0 < A_1(\beta) < A_2(\beta) < A_3(\beta) < \dots$ and, respectively $0 < a_1^-(\beta) < a_1^+(\beta) < a_2^-(\beta) < a_2^+(\beta) < \dots$ be the sequences of as described in part* **A1** *of Theorem A.*

Then, for any $A \in [0, A_1)$ the set Γ of large contours is empty with $\mathbb{Q}^{A}_{N,\beta}$-probability tending to one. On the other hand, for any $\ell = 1, 2, \dots$ and for any $A \in (A_\ell, A_{\ell+1})$, the set Γ contains, with $\mathbb{Q}^{A}_{N,\beta}$-probability tending to one, exactly ℓ large contours $\Gamma = \{\Gamma_1, \dots, \Gamma_\ell\}$. Moreover, the rescaled contours from Γ concentrate in Hausdorff distance $\mathsf{d_H}$ near the optimal stack $\{\gamma_1^*, \dots, \gamma_\ell^*\}$, as described in part **A1** *of Theorem A, in the following sense: For any $\epsilon > 0$,*

$$\lim_{N\to\infty} \mathbb{Q}^{A}_{N,\beta}\left(\sum_{i=1}^{\ell-1} \mathsf{d_H}\left(\frac{1}{N}\Gamma_i, \gamma_i^*\right) + \min_{x\,:\,x+\gamma_\ell^{`}\subset\mathbb{B}_1} \mathsf{d_H}\left(\frac{1}{N}\Gamma_\ell, x+\gamma_\ell^*\right) \geq \epsilon\right) = 0. \tag{4.14}$$

5. Proofs

5.1. **Surface tension.** Let us say that a nearest-neighbor path γ on $\mathbb{Z}^2$ is admissible if it can be a realized as a portion of a level line of the height function h_γ in (4.2). Following (4.11) we associate with γ its *free* weight

$$w^{\mathsf{f}}_\beta(\gamma) = \mathrm{e}^{-\beta|\gamma| - \sum_{\mathcal{C} \not\sim \Gamma} \Phi_\beta(\mathcal{C})}. \tag{5.1}$$

We say that an admissible $\gamma = (\gamma_0, \dots, \gamma_n)$ is $\gamma : 0 \to x$ if its end-points satisfy $\gamma_0 = 0$ and $\gamma_n = x$. The corresponding two-point function and the surface tension are

$$G_\beta(x) \triangleq \sum_{\gamma:0\to x} w^{\mathsf{f}}_\beta(\gamma) \quad \text{and} \quad \tau_\beta(x) = -\lim_{n\to\infty} \frac{1}{n} \log G_\beta(\lfloor nx \rfloor). \tag{5.2}$$

Recall that we are considering only sufficiently large values of β. In particular, (4.10) applies, and the surface tension τ_β in (5.2) is well defined.

In the notation of (5.1) given a large level line $\Gamma \subset B_N$ we define its *free* weight

$$w^{\mathsf{f}}_\beta(\Gamma) = \mathrm{e}^{-\beta|\Gamma| - \sum_{\mathcal{C} \not\sim \Gamma} \Phi_\beta(\mathcal{C})}. \tag{5.3}$$

In this way the measure $\mathbb{Q}_N$ in (4.11) describes a gas of large level lines which interact between each other and with the boundary ∂B_N. The statement below is well understood (see e.g. [DKS]), and it holds for all sufficiently low temperatures:

Lemma 7. *Let $\eta \subset \mathbb{B}_1$ be a rectifiable Jordan curve. Given any sequence of positive numbers ϵ_N such that $\lim_{N\to\infty}\epsilon_N = 0$ and $\lim_{N\to\infty} N\epsilon_N = \infty$, the following holds:*

$$\lim_{N\to\infty}\frac{1}{N}\log\left(\sum_{\Gamma\subset B_N} w^{\mathsf{f}}_\beta(\Gamma)\mathbb{1}_{\left\{\mathrm{d}_{\mathsf{H}}\left(\frac{1}{N}\Gamma,\eta\right)\le\epsilon_N\right\}}\right) = -\tau_\beta(\eta). \tag{5.4}$$

5.2. **Lower bounds on $\mathbb{Q}_{N,\beta}\left(\Xi_N \ge \rho_\beta N^3 + AN^2\right)$.** Let us apply part **A.1** of Theorem A for the surface tension τ_β defined in (5.2) and bulk occupation probabilities $p_v(\beta), p_s(\beta)$ which satisfy (2.3). Assume that $A \in (A_\ell(\beta), A_{\ell+1}(\beta))$ for some $\ell = 0, 1, \dots$. Let $a^* = 0$ if $\ell = 0$ and $a^* \in (a^-_\ell, a^+_\ell)$ be the optimal rescaled area as described in Theorem A. Then,

Proposition 8. *In the notation of* (2.15) *and* (VP$_\delta$) *the following lower bound holds:*

$$\begin{aligned}\liminf_{N\to\infty}\frac{1}{N}\log\mathbb{Q}_{N,\beta}\left(\Xi_N \ge \rho_\beta N^3 + AN^2\right) \ge -\min_{\mathcal{L}}\mathcal{E}_\beta\left(\mathcal{L}\ |\delta_\beta\right) = \\ = -\left(\frac{(\delta_\beta - a^*)^2}{2D_\beta} + \tau_\beta(a^*)\right).\end{aligned} \tag{5.5}$$

Proof. If $a^* = 0$, then the claim follows from (4.12).
Assume that $a^* \in (a^-_\ell, a^+_\ell)$ for some $\ell \ge 1$. Let $\gamma_1 \subseteq \gamma_2 \subseteq \dots \subseteq \gamma_\ell \subset \mathbb{B}_1$ be the optimal stack as described in Theorem B. Pick a sequence ϵ_N which satisfies conditions of Lemma 7 and, for $J = 1, \dots, \ell$ define tubes

$$A^j_N = (1 - (1 + 3(j-1))\epsilon_N)N\mathring{\gamma}_j \setminus (1 - (2 + 3(j-1))\epsilon_N)N\mathring{\gamma}_j.$$

Lemma 7 implies that for any $j = 1, \dots, \ell$

$$\lim_{N\to\infty}\frac{1}{N}\log\left(\sum_{\Gamma_j\subset A^j_N} w^{\mathsf{f}}_\beta(\Gamma_j)\right) = -\tau_\beta(\gamma_j). \tag{5.6}$$

By construction A^j_N-s are disjoint and there exists $c_1 = c_1(\beta, a^*) > 0$ such that

$$\min_{1\le j\ell}\mathrm{d}_{\mathsf{H}}(A^{j-1}_N, A^j_N) \ge c_1 N\epsilon_N \tag{5.7}$$

where we put $A^0_N = \partial B_N$. Hence, in view of (4.10),

$$\exp\left\{-\beta\sum_{j=1}^{\ell}|\Gamma_j| - \sum_{\substack{\mathcal{C}\not\sim\Gamma\\ \mathcal{C}\subset\Lambda_N}}\Phi_\beta(\mathcal{C})\right\} \gtrsim \mathrm{e}^{-c_2\ell N\mathrm{e}^{-2c_1N\epsilon_N}}\prod_1^\ell w^{\mathsf{f}}_\beta(\Gamma_j) \tag{5.8}$$

for any collection $\Gamma = (\Gamma_1, \dots, \Gamma_\ell)$ of level lines satisfying $\Gamma_j \subset A^j_N$ for $j = 1, \dots, \ell$.

Note also that, for any $j = 1, \dots \ell$, if a large level line $\Gamma_j \subset A^j_N$, then

$$N^2\mathsf{a}(\gamma_j)(1 - 3\ell\epsilon_N)^2 \le \mathsf{a}(\Gamma_j) \le N^2\mathsf{a}(\gamma_j). \tag{5.9}$$

Hence, by (4.12),

$$\frac{1}{N}\log \mathbb{Q}_{N,\beta}\left(\Xi_N \geq \rho_\beta N^3 + AN^2 \middle| \Gamma\right) \geq -\frac{(\delta_\beta - a^*)^2}{2D_\beta} - \frac{6\delta_\beta^2 \ell \epsilon_N}{D_\beta}, \tag{5.10}$$

for any collection $\Gamma = (\Gamma_1, \dots, \Gamma_\ell)$ of such level lines.

Putting (5.6), (5.8) and (5.10) together (and recalling that ϵ_N was chosen to satisfy conditions of Lemma 7) we deduce (8). □

5.3. **Strategy for proving upper bounds.** Below we explain the strategy which we employ for proving (4.14). For the rest of the section let us assume that a sufficiently large β and an excess area value $A \in (A_\ell(\beta), A_{\ell+1}(\beta))$ are fixed.

STEP 1 (Hausdorff distance and Isoperimetric rigidity). We shall employ the same notation d_{H} for two and three dimensional Hausdorff distances.

Given a family $\mathcal{L}$ of compatible loops define the height function $h[\mathcal{L}] : \mathbb{B}_1 \mapsto \mathbb{N}$,

$$h[\mathcal{L}](y) = \sum_{\gamma \in \mathcal{L}} \mathbb{1}_{\{y \in \mathring{\gamma}\}}. \tag{5.11}$$

In the case of the optimal stack (recall that we fixed β and A, so the latter notion is well defined) $\mathcal{L}^* = (\gamma_1^*, \dots, \gamma_\ell^*)$ as described in Theorem B, we say that x is admissible if $x + \gamma_\ell^* \subset \mathbb{B}_1$. If x is admissible, then we use $h_x^* = h_x^*(A, \beta)$ for the height function of $(\gamma_1^*, \dots, x + \gamma_\ell^*)$. Of course if $\mathcal{L}^*$ is of type-2, the only admissible x is $x = 0$.

We can think about $h[\mathcal{L}]$ in terms of its epigraph, which is a three dimensional subset of $\mathbb{B}_1 \times \mathbb{R}_+$. In this way the notion of Hausdorff distance $\mathrm{d}_{\mathsf{H}}\left(h[\mathcal{L}], h_x^*\right)$ is well defined.

We will need the qualitative stability properties of the minima of the functional

$$\mathcal{E}_\beta\left(\mathcal{L} \mid \delta_\beta\right) = \sum_{\gamma \in \mathcal{L}} \tau_\beta(\gamma) + \frac{(\delta_\beta - \mathsf{a}(\mathcal{L}))^2}{2D_\beta}.$$

Namely, we claim that for every $\nu > 0$ there exists $\rho = \rho_\beta(\nu, A) > 0$ such that

$$\mathcal{E}_\beta\left(\mathcal{L} \mid \delta_\beta\right) > \tau_\beta(a^*) + \frac{(\delta_\beta - a^*)^2}{2D_\beta} + \rho, \tag{5.12}$$

whenever $\min_x \mathrm{d}_{\mathsf{H}}\left(h[\mathcal{L}], h_x^*\right) > \nu$. Such stability properties are known for the Wulff and constrained Wulff isoperimetric problems. That is, if W is the Wulff loop with area $a\left(\mathsf{W}\right)$ inside, see Section 3.2, and γ is a loop with the same area, $a\left(\gamma\right) = a\left(\mathsf{W}\right)$, which satisfies $\min_x \mathrm{d}_{\mathsf{H}}\left(\gamma, \mathsf{W} + \mathsf{x}\right) > \nu$, then for some $\rho = \rho\left(\nu\right) > 0$ we have

$$\tau\left(\gamma\right) \geq \tau\left(\mathsf{W}\right) + \rho.$$

That is the content of the *generalized Bonnesen inequality*, proven in [18], see relation 2.4.1 there. The same stability property holds for the constrained case, that is when we impose the additional constrain that the loop γ has to fit inside a square, and where the role of the Wulff shape W is replaced by the Wulff plaquette P, see [33] for

more details. We will show now that the above quantitative stability of the surface tension functional implies the quantitative stability of the functional $\mathcal{E}_\beta(*|\delta_\beta)$.

Let us prove (5.12). Suppose d_{H} is bigger than ν. There can be several reasons for that. The simplest is that the number of levels in the epigraph of $h[\mathcal{L}]$ is different from that for h^*. (In which case the Hausdorff distance in question is at least 1.) In order to estimate the discrepancy ρ in that case we have to look for the minimizer of the functional $\mathcal{E}_\beta(\mathcal{L}|\delta_\beta)$ under additional constraint on $\mathcal{L}$ to have a different number of levels than the optimal stack h^*. Let $\mathcal{E}^w_\beta(\delta_\beta)$ be the *minimal value* of the functional $\mathcal{E}_\beta(\mathcal{L}|\delta_\beta)$ over these 'wrong' loop configurations. The function $\mathcal{E}^w_\beta(\delta_\beta)$, as a function of δ_β, is piecewise continuous, as is the true optimal value $\mathcal{E}_\beta(\delta_\beta) = \tau_\beta(a^*) + \frac{(\delta_\beta - a^*)^2}{2D_\beta}$ (where $a^* = a^*(\delta_\beta)$ is the optimal area corresponding to the excess value δ_β). The difference $\mathcal{E}^w_\beta(\delta_\beta) - \mathcal{E}_\beta(\delta_\beta)$ is continuous and non-negative, and it vanishes precisely at the values δ_β corresponding to critical values $A_\ell(\beta)$ of the parameter A. Therefore the difference $\mathcal{E}^w_\beta(\delta_\beta) - \mathcal{E}_\beta(\delta_\beta) \equiv \rho_0(A,\beta)$ is positive for our fixed value $A \in (A_\ell(\beta), A_{\ell+1}(\beta))$.

Let now the number of levels l in the epigraph of $h[\mathcal{L}]$ is $l(h^*)$ – i.e. the same as that for h^*. Let $\mathcal{L}_a$ be the minimizer of $\mathcal{E}_\beta(\mathcal{L}|\delta_\beta)$ over all the families with $l(h^*)$ levels and with $a(\mathcal{L}) = a$. If our loop collection $\mathcal{L}$ is far from the optimal stack $\mathcal{L}_{a^*}$ – i.e. if $\min_x \mathrm{d}_{\mathsf{H}}(h[\mathcal{L}], h^*_x) > \nu$ – as well as from all other stacks: $\min_x \mathrm{d}_{\mathsf{H}}(h[\mathcal{L}], h[\mathcal{L}_a]_x) > \frac{\nu}{10}$, then our claim (5.12) follows just from the stability properties of the surface tension functional $\tau(*)$. If, on the other hand, $\mathcal{L}$ is far from the optimal stack $\mathcal{L}_{a^*}$, i.e. $\min_x \mathrm{d}_{\mathsf{H}}(h[\mathcal{L}], h^*_x) > \nu$, but it is close to some other stack $\mathcal{L}_{\bar{a}}$ from 'wrong area stacks': $\min_x \mathrm{d}_{\mathsf{H}}(h[\mathcal{L}], h[\mathcal{L}_{\bar{a}}]_x) < \frac{\nu}{10}$, then we are done, since in that case $\mathrm{d}_{\mathsf{H}}(h[\mathcal{L}_{a^*}], h[\mathcal{L}_{\bar{a}}]) > \frac{9\nu}{10}$, and we already know the minimizer of $\mathcal{E}_\beta(*|\delta_\beta)$ over the set $\left\{\mathcal{L} : \min_x \mathrm{d}_{\mathsf{H}}(h[\mathcal{L}], h[\mathcal{L}_{\bar{a}}]_x) < \frac{\nu}{10}\right\}$ to be far from $\mathcal{L}_{a^*}$.

STEP 2 (Upper bound on the number of large contours, compactness considerations and skeleton calculus). In principle there should be a nice way to formulate and prove large deviation results using compactness of the space of closed connected subsets of $\mathbb{B}_1$ endowed with Hausdorff distance, see [13]. However, as in the latter work, we shall eventually resort to skeleton calculus developed in [18].

By (4.11) and (4.10),

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{Q}_{N,\beta}\left(\sum |\Gamma_i| \geq KN\right) \leq -K\beta\left(1 - \mathrm{O}\left(\mathrm{e}^{-4\beta}\right)\right). \tag{5.13}$$

In view of Proposition 8 we can reduce attention to families Γ of large contours which satisfy $\sum |\Gamma_i| \leq K_\beta(A)N$ for some $K_\beta(A)$ large enough. By (4.7) this means that we can restrict attention to collections Γ of large contours whose cardinality is at most

$$\#\{\Gamma_i \in \Gamma\} \leq n_\beta(A) = K_\beta(A)/\epsilon_\beta(A) \tag{5.14}$$

Given a compatible collection Γ of large contours we can talk about the rescaled surface $h_N[\Gamma] := h\left[\frac{1}{N}\Gamma\right]$ and, accordingly, consider events

$$\min_x \mathrm{d}_{\mathsf{H}}\left(h_N[\Gamma], h^*_x(A,\beta)\right) > \nu. \tag{5.15}$$

Fix a sequence ϵ_N satisfying conditions of Lemma 7 and set $\ell_N = N\epsilon_N$. Consider a large contour $\Gamma_i \in \Gamma$. By (5.14) there are at most $n_\beta(A)$ such contours, and each of them has a bounded length $|\Gamma_i| \le KN$. In view of south-west splitting rules we can view Γ_i as a parameterized nearest neighbor loop $\Gamma_i = \{u_0, \dots, u_n = u_0\}$ with $n \le KN$.

Definition. The ℓ_N-skeleton γ_i of Γ_i is defined as follows: Set $N\mathsf{u}_0 = u_0$ and $\tau_0 = 0$. Then, given $k = 0, 1, \dots$ with u_k and τ_k already defined set

$$\tau_{k+1} = \min\left\{m > \tau_k \ : \ |u_m - N\mathsf{u}_k|_1 > \ell_N\right\} \quad \text{and} \quad N\mathsf{u}_{k+1} = u_{\tau_{k+1}}, \tag{5.16}$$

provided $\{m > \tau_k \ : \ |u_m - \mathsf{u}_k| > \ell_N\} \neq \emptyset$. Otherwise stop and set $\gamma_i \subset \mathbb{B}_1$ to be the polygonal approximation through the vertices $\mathsf{u}_0, \dots, \mathsf{u}_k, \mathsf{u}_0$. □

We write $\Gamma_i \overset{\ell_N}{\sim} \gamma_i$ and, accordingly, $\Gamma \overset{\ell_N}{\sim} \mathcal{S}$, if $\mathcal{S} = (\gamma_1, \gamma_2, \dots)$ is a collection of ℓ_N skeletons compatible with family Γ of large contours. Since $|\Gamma_i| \le KN$ any compatible ℓ_N skeleton γ_i has at most $\frac{K}{\epsilon_N}$ vertices. Therefore, there are at most

$$\left(\left(N^2\right)^{\frac{K}{\epsilon_N}}\right)^{n_\beta(A)} = \exp\left\{\frac{2KK_\beta(A)\log N}{\epsilon_N \epsilon_\beta(A)}\right\} = \mathrm{e}^{\mathrm{o}_N(1)N} \tag{5.17}$$

different collections of ℓ_N-skeletons to consider. Thus the entropy of the skeletons does not present an issue, and it would suffice to give uniform upper bounds on

$$\mathbb{Q}^A_{N,\beta}\left(\min_x \mathrm{d}_{\mathsf{H}}\left(h_N[\Gamma], h^*_x(A,\beta)\right) > \nu; \Gamma \overset{\ell_N}{\sim} \mathcal{S}\right) \tag{5.18}$$

for fixed ℓ_N-skeleton collections $\mathcal{S}$.

If $\gamma \subset \mathbb{B}_1$ is a closed polygon – for instance an ℓ_N-skeleton of some large contour – then its surface tension $\tau_\beta(\gamma)$ and its signed area $\mathsf{a}(\gamma)$ are well defined. Accordingly one defines $\tau_\beta(\mathcal{S}) = \sum_i \tau_\beta(\gamma_i)$ and $\mathsf{a}(\mathcal{S}) = \sum_i \mathsf{a}(\gamma_i)$ for finite collections $\mathcal{S}$ of polygonal lines. We apply now the isoperimetric rigidity bound (5.12): For every $\nu > 0$ there exists $\rho = \rho_\beta(\nu, A) > 0$ such that for all N sufficiently large the following holds:

$$\mathcal{E}_\beta\left(\mathcal{S} \mid \delta_\beta\right) = \sum_{\gamma \in \mathcal{S}} \tau_\beta(\gamma) + \frac{(\delta_\beta - \mathsf{a}(\mathcal{S}))^2}{2D_\beta} > \tau_\beta(a^*) + \frac{(\delta_\beta - a^*)^2}{2D_\beta} + \rho, \tag{5.19}$$

whenever $\mathcal{S}$ is an ℓ_N-skeleton; $\mathcal{S} \overset{\ell_N}{\sim} \Gamma$ of a collection Γ of large contours which satisfies (5.15).

For $n \le n_\beta(A)$ (see (5.14)) and $\rho > 0$ consider the collection $\mathfrak{S}_N(\rho)$ of families of ℓ_N-skeletons $\mathcal{S} = (\gamma_1, \dots, \gamma_n)$ which satisfy (5.19).

Then (4.14) would be a consequence of the following statement:

Theorem 9. *There exists a positive function α on $(0,\infty)$ such that the following happens: Fix β sufficiently large and let A be as in the conditions of Theorem C. Then for any $\rho>0$ fixed,*

$$\max_{\mathcal{S}\in\mathfrak{S}_N(\rho)}\frac{1}{N}\log\mathbb{Q}^A_{\beta,N}\left(\Gamma\overset{\ell_N}{\sim}\mathcal{S}\right)<-\alpha(\rho), \tag{5.20}$$

as soon as N is sufficiently large.

Similar upper bounds were derived in [18] for collections consisting of one large and several small skeletons. Here we have somewhat more delicate situation, since we need to control the weights of stacks of almost optimal contours, which are interacting. This requires additional tools and efforts.

STEP 3 (Refined estimates in terms of graph structure of Γ).

Let us elaborate on the upper bounds derived in [18]. Consider the ensemble of (single) large microscopic loops Γ with weights $w^\beta_{\mathsf{f}}(\Gamma)$ as in (5.3). Given a (polygonal) skeleton $\gamma\subset\mathbb{B}_1$ define

$$w^\beta_{\mathsf{f}}\left(\Gamma\overset{\ell_N}{\sim}\gamma\right):=\sum_{\Gamma\overset{\ell_N}{\sim}\gamma}w^\beta_{\mathsf{f}}(\Gamma).$$

More generally, given a function $F(\Gamma_1,\Gamma_2,...)$ we put

$$\bigotimes_i w^{\mathsf{f}}_\beta\left(F(\Gamma_1,\Gamma_2,...)\right):=\int\bigotimes_i w^{\mathsf{f}}_\beta\left(d\Gamma_i\right)F(\Gamma_1,\Gamma_2,...).$$

Upper bounds derived in [18] imply that here exists a positive non-decreasing function α_0 on $(0,\infty)$ such that the following happens: Fix $a_0>0$. Given a closed polygon γ, define its excess surface tension

$$\Omega_\beta(\gamma)=\tau_\beta(\gamma)-\tau_\beta\left(\mathsf{a}(\gamma)\right). \tag{5.21}$$

Then, for all N and β sufficiently large,

$$\frac{1}{N}\log w^\beta_{\mathsf{f}}\left(\Gamma\overset{\ell_N}{\sim}\gamma\right)<-\left(\tau_\beta\left(\mathsf{a}(\gamma)\right)+\alpha_0\left(\Omega_\beta(\gamma)\right)\right)\left(1-\mathrm{o}_N(1)\right) \tag{5.22}$$

uniformly in $\mathsf{a}(\gamma)>a_0$. This estimate, is explained in the beginning of Subsection 6.1.

Should we be able to bound $\mathbb{Q}_{\beta,N}\left(\Gamma\overset{\ell_N}{\sim}\mathcal{S}\right)$ by product weights

$$\bigotimes w^{\mathsf{f}}_\beta\left(\Gamma\overset{\ell_N}{\sim}\mathcal{S}\right):=\prod_{\gamma_i\in\mathcal{S}}w^\beta_{\mathsf{f}}\left(\Gamma_i\overset{\ell_N}{\sim}\gamma_i\right),$$

then (5.22) and Proposition 8 would readily imply (5.20). However, due to cluster sharing in (4.11) and due to confined geometry of clusters; $\mathcal{C}\subset B_N$, contours in Γ do interact both between each other and with ∂B_N, which a priori may lead to a modification of surface tension. Therefore, one should proceed with care.

A compatible collection $\Gamma = \{\Gamma_v\}_{v\in\mathcal{V}}$ of large level lines has a natural graph structure: Namely let us say that $\Gamma_u \sim \Gamma_v$ if there is a continuous path in $\mathbb{R}^2$ which connects between Γ_u and Γ_v without hitting any other element of Γ. This notion of hitting is ambiguous because by construction different Γ_v-s may share bounds or even coincide. We resolve this ambiguity as follows: If there is a strict inclusion $\mathring{\Gamma}_u \subset \mathring{\Gamma}_v$, then any path from the infinite component of $\mathbb{R}^2 \setminus \Gamma_v$ to Γ_u by definition hits Γ_v. If $\Gamma_{v_1} = \cdots = \Gamma_{v_k}$, then we fix an ordering and declare that any path from the infinite component of $\mathbb{R}^2 \setminus \Gamma_{v_1}$ to Γ_{v_j}; $j > 1$, hits Γ_{v_i} for any $i < j$.

In this way we label collections Γ of large level lines by finite graphs $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. If $\mathcal{S} = \{\gamma_u\}$ is a family of ℓ_N-skeletons of Γ (meaning that $\Gamma_u \overset{\ell_N}{\sim} \gamma_u$ for every $u \in \mathcal{V}$, then by definition $\mathcal{S}$ has the same graph structure.

We write $\Gamma \in \mathcal{G}$ if $\mathcal{G}$ is the above graph of Γ. The chromatic number of this graph plays a role. In Subsection 5.4 we show how, once the chromatic number is under control, to reduce complex many-body interactions in $\mathbb{Q}_{\beta,N}\left(\Gamma \overset{\ell_N}{\sim} \mathcal{S}\right)$ to upper bounds on pairs of interacting contours.

STEP 4 (Entropic repulsion versus interaction). In Subsection 5.5 we formulate decoupling upper bounds for two interacting contours. In view of these bounds (4.10) implies that at sufficiently low temperatures entropic repulsion always beats our weak interactions. The proof is relegated to Section 6.

STEP 5 (Bounds on chromatic numbers). In Subsection 5.6 we derive exponential upper bounds on chromatic numbers, which enable reduction to the decoupling estimates for pairs of contours.

5.4. **A chromatic number upper bound on a collection of large contours.** Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a finite graph, for instance associated to a collection $\Gamma = \{\Gamma_v\}_{v\in\mathcal{V}}$ of large level lines. Let $\mathcal{S} = \{\gamma_v\}_{v\in\mathcal{V}}$ be a collections of polygonal lines. We wish to derive an upper bound on $\mathbb{Q}_{N,\beta}$-probabilities (see (4.11))

$$\mathbb{Q}_{N,\beta}\left(\mathbb{1}_{\{\Gamma\in\mathcal{G}\}}\prod_{v\in\mathcal{V}}\mathbb{1}_{\left\{\Gamma_v \overset{\ell_N}{\sim} \gamma_v\right\}}\right) \simeq \sum_{\Gamma} \mathrm{e}^{-\beta\sum_v|\Gamma_v| - \sum^*_{\mathcal{C}\not\sim\Gamma}\Phi_\beta(\mathcal{C})}\mathbb{1}_{\{\Gamma\in\mathcal{G}\}}\prod_{v\in\mathcal{V}}\mathbb{1}_{\left\{\Gamma_v \overset{\ell_N}{\sim} \gamma_v\right\}}. \tag{5.23}$$

Above $\sum^*$ restricts summation to connected clusters $\mathcal{C} \subset B_N$. Since we are trying to derive upper bounds in terms of surface tension τ_β which was introduced in (5.4) in terms of infinite volume weights, it happens to be convenient to augment $\mathcal{V}$ with an additional root vertex 0 which corresponds to $\Gamma_0 = \partial B_N$, and connect it to other vertices of $\mathcal{V}$ using exactly the same rules as specified above (under the convention that if Γ contains other copies of ∂B_N, then Γ_0 is the external one in the ordering of these copies). Let $\mathcal{G}_0 = (\mathcal{V}_0, \mathcal{E}_0)$ to be the augmented graph, and let $\hat{\mathcal{G}}_0$ to be the line graph of $\mathcal{G}_0$. That is vertices of $\hat{\mathcal{G}}_0$ are undirected edges $e = (u, v)$ of $\mathcal{G}_0$, and we say that e and g are neighbors in $\hat{\mathcal{G}}_0$ if they are adjacent in $\mathcal{G}_0$. Let $\kappa_{\mathcal{G}}$ be the

chromatic number of $\hat{\mathcal{G}}_0$, and consider a disjoint decomposition $\hat{\mathcal{G}}_0 = \cup_{i=1}^{\kappa_{\mathcal{G}}} \hat{\mathcal{G}}_i$. By definition each class $\hat{\mathcal{G}}_i$ contains pair-wise non-adjacent edges.

Now, if $\Gamma \in \mathcal{G}$, then,

$$\sum_{\substack{\mathcal{C} \not\sim \Gamma \\ \mathcal{C} \subset B_N}} \Phi_\beta(\mathcal{C}) \geq \sum_{v \in \mathcal{V}} \Phi_\beta(\mathcal{C}) \mathbb{1}_{\{\mathcal{C} \not\sim \Gamma_v\}} - \sum_{e \in \mathcal{E}_0} |\Phi_\beta(\mathcal{C})| \, \mathbb{1}_{\{\mathcal{C} \not\sim e\}}, \tag{5.24}$$

where we write $\mathbb{1}_{\{\mathcal{C} \not\sim e\}} = \mathbb{1}_{\{\mathcal{C} \not\sim \Gamma_u\}} \mathbb{1}_{\{\mathcal{C} \not\sim \Gamma_v\}}$ for an undirected edge $e = (u, v) \in \mathcal{E}_0$.

We arrive to the following upper bound in terms of product free weights defined in (5.3):

$$\mathbb{Q}_{N,\beta} \left(\mathbb{1}_{\{\Gamma \in \mathcal{G}\}} \prod_{v \in \mathcal{V}} \mathbb{1}_{\left\{ \Gamma_v \overset{\ell_N}{\sim} \gamma_v \right\}} \right) \lesssim \bigotimes_{v \in \mathcal{V}} w_\beta^{\mathsf{f}} \left(1_{\{\Gamma \in \mathcal{G}\}} \prod_{v \in \mathcal{V}} \mathbb{1}_{\left\{ \Gamma_v \overset{\ell_N}{\sim} \gamma_v \right\}} \exp \left\{ \sum_{i=1}^{\kappa_{\mathcal{G}}} \sum_{\substack{e \in \hat{\mathcal{G}}_i \\ \mathcal{C} \not\sim e}} |\Phi_\beta(\mathcal{C})| \right\} \right). \tag{5.25}$$

By the (generalized) Hölder inequality,

$$\begin{aligned} &\log \mathbb{E}_{N,\beta} \left(\mathbb{1}_{\{\Gamma \in \mathcal{G}\}} \prod_{v \in \mathcal{V}} \mathbb{1}_{\left\{ \Gamma_v \overset{\ell_N}{\sim} \gamma_v \right\}} \right) \\ &\quad \lesssim \frac{1}{\kappa_{\mathcal{G}}} \sum_{i=1}^{\kappa_{\mathcal{G}}} \log \left(\bigotimes_{v \in \mathcal{V}} w_\beta^{\mathsf{f}} \left(1_{\{\Gamma \in \mathcal{G}\}} \prod_{v \in \mathcal{V}} \mathbb{1}_{\left\{ \Gamma_v \overset{\ell_N}{\sim} \gamma_v \right\}} \exp \left\{ \kappa_{\mathcal{G}} \sum_{\substack{e \in \hat{\mathcal{G}}_i \\ \mathcal{C} \not\sim e}} |\Phi_\beta(\mathcal{C})| \right\} \right) \right). \end{aligned} \tag{5.26}$$

For each $i = 1, \ldots, \kappa_{\mathcal{G}}$ we can relax constraints and write

$$1_{\{\Gamma \in \mathcal{G}\}} \leq \prod_{e = (u,v) \in \hat{\mathcal{G}}_i} \mathbb{1}_{\{\Gamma_u \sim \Gamma_v\}}. \tag{5.27}$$

Above $\Gamma_u \sim \Gamma_v$ just means that Γ_u and Γ_v are two compatible large level lines.

Let us say that $u \notin \hat{\mathcal{G}}_i$ if no edge of $\hat{\mathcal{G}}_i$ contains u as a vertex. Each of the $i = 1, \ldots, \kappa_{\mathcal{G}}$ summands on the right hand side of (5.26) is bounded above (under notation convention $w_\beta^{\mathsf{f}}(\Gamma_0) = 1$ for the auxiliary vertex $\Gamma_0 = \partial B_N$) by

$$\sum_{u \notin \hat{\mathcal{G}}_i} \log w_\beta^{\mathsf{f}}(\Gamma_u \overset{\ell_N}{\sim} \gamma_u) + \sum_{(u,v) \in \hat{\mathcal{G}}_i} \log w_\beta^{\mathsf{f}} \bigotimes w_\beta^{\mathsf{f}} \left(\mathbb{1}_{\{\Gamma_u \sim \Gamma_v\}} \mathbb{1}_{\left\{ \Gamma_v \overset{\ell_N}{\sim} \gamma_v \right\}} \mathbb{1}_{\left\{ \Gamma_u \overset{\ell_N}{\sim} \gamma_u \right\}} \mathrm{e}^{\kappa_{\mathcal{G}} \sum_{\mathcal{C} \not\sim e} |\Phi_\beta(\mathcal{C})|} \right). \tag{5.28}$$

In order to apply (5.28) we need, first of all, to control the chromatic number $\kappa_{\mathcal{G}}$. After that we shall be left with studying only the case of two compatible contours, and we shall need to show that in the latter case at all sufficiently low temperatures entropic repulsion which is triggered by compatibility constraint $\Gamma_u \sim \Gamma_v$ wins over the attractive potential $\kappa_{\mathcal{G}} \sum_{\mathcal{C} \not\sim e} |\Phi_\beta(\mathcal{C})|$.

5.5. **Entropic repulsion versus interaction.** In this subsection we formulate upper bounds on probabilities related to two compatible interacting large contours.

Theorem 10. *Assume that a number $\chi > 1/2$ and a sequence $\{\kappa_\beta\}$ are such that*

$$\limsup_{\beta\to\infty} \sup_{\mathcal{C}\neq\emptyset} \kappa_\beta e^{\chi\beta(\mathrm{diam}_\infty(\mathcal{C})+1)} \left|\Phi_\beta(\mathcal{C})\right| < 1. \tag{5.29}$$

Fix $a_0 > 0$. Recall the definition of excess surface tension Ω_β in (5.21)*. Then,*

$$\begin{aligned} &\frac{1}{N}\log\left(w^{\mathsf{f}}_\beta \bigotimes w^{\mathsf{f}}_\beta \left(\mathbb{1}_{\{\Gamma_u\sim\Gamma_v\}}\mathbb{1}_{\left\{\Gamma_v \overset{\ell_N}{\sim} \gamma_v\right\}}\mathbb{1}_{\left\{\Gamma_u \overset{\ell_N}{\sim} \gamma_u\right\}} \mathrm{e}^{\kappa_\beta\sum_{\mathcal{C}\not\sim e}\left|\Phi_\beta(\mathcal{C})\right|}\right)\right) \\ &\qquad \leq -\left[\tau_\beta(\mathsf{a}(\gamma_u)) + \tau_\beta(\mathsf{a}(\gamma_v)) + \alpha_0\left(\Omega_\beta(\gamma_u)\right) + \alpha_0\left(\Omega_\beta(\gamma_v)\right)\right](1-\mathrm{o}_N(1)), \end{aligned} \tag{5.30}$$

uniformly in β and N sufficiently large and uniformly in closed polygonal lines γ_v, γ_u satisfying $\mathsf{a}(\gamma_v), \mathsf{a}(\gamma_u) \geq a_0$. The function α_0 is the function appearing in (5.22)*.*

We sketch the proof of Theorem 10 in the concluding Section 6.

5.6. **Upper bound on $\kappa_{\mathcal{G}}$.** In this Subsection we shall show that for all β sufficiently large one can restrict attention to graphs $\mathcal{G}$ satisfying $\kappa_{\mathcal{G}} \leq \kappa_\beta$ where the sequence $\{\kappa_\beta\}$ complies with (5.29).

We start with a simple general combinatorial observation.

Let G_N be a graph with no loops and double edges, having N vertices. Its edge coloring is called proper, if at every vertex all the bonds entering it have different colors. The minimal number of colors needed for creating a proper edge coloring will be denoted by $\varkappa(G_N)$. It is called the *edge chromatic number.* We need the upper bound on $\varkappa(G_N)$. Evidently, the complete graph with N vertices has the highest edge chromatic number, so it is sufficient to consider only the case of G_N being a complete graph.

Theorem 11. *Let G_N be a complete graph with N vertices. Then*

$$\varkappa(G_N) = \begin{cases} N & \text{if } N \text{ is odd,} \\ N-1 & \text{if } N \text{ is even.} \end{cases}$$

Proof. Let us produce a proper edge coloring of G_N by N colors, for N odd. To do this, let us draw G_N on the plane as a regular N-gon P_N, with all its diagonals. Let us color all the N sides s of P_N using all the N colors. What remains now is to color all diagonals. Note, that every diagonal d of P_N is parallel to precisely one side s_d of P_N, because N is odd. Let us color the diagonal d by the color of the side e_d. Evidently, the resulting coloring $\mathcal{C}_N$ is proper.

Every vertex of G_N has $N-1$ incoming edges, while we have used N colors to color all the edges. Therefore at each vertex v exactly one color c_v is missing – it is the color of the edge opposite v. By construction, all N missing colors c_v are different. Let us use this property to construct a proper edge coloring of G_{N+1} by N colors. First, we color $G_N \subset G_{N+1}$ by the coloring $\mathcal{C}_N$. Let w be the extra vertex,

$w = G_{N+1} \setminus G_N$. Let us color the bond (v, w) by the color c_v. Evidently, the resulting coloring $\mathcal{C}_{N+1}$ is again proper.

Finally, for N odd the constructed coloring $\mathcal{C}_N$ is best possible: there is no coloring using $N-1$ colors. Indeed, suppose such a coloring does exist. That means that at each vertex of G_N all $N-1$ colors are present. Therefore, the bonds of the first color, say, define a partition of the set of all vertices into pairs. That, however, is impossible, since N is odd. This nice last argument is due to O. Ogievetsky. □

Next let us record a (straightforward) consequence of (4.12) as follows: Fix A. Then,

$$\lim_{N\to\infty}\left(\frac{1}{N}\log\left(\mathbb{Q}_{N,\beta}\left(\Xi_N \geq \rho_\beta N^3 + AN^2 \,\middle|\, \Gamma \overset{\ell_N}{\sim} \mathcal{S}\right)\right) + \frac{(\delta_\beta - \mathsf{a}(\mathcal{S}))^2}{2D_\beta}\right) = 0. \quad (5.31)$$

uniformly in collections $\mathcal{S}$ of closed polygons with $0 \leq \mathsf{a}(\mathcal{S}) \leq \delta_\beta$ (recall (2.15) for the definition of δ_β).

We proceed with the following two observations concerning the variational problem (VP_δ):

Lemma 12. *Fix β sufficiently large and consider (VP_δ). Given $\delta > 0$ let $a^*_\beta(\delta)$ be the optimal area. Then,*

$$\limsup_{\delta\to\infty}\left(\delta - a^*_\beta(\delta)\right) := \xi_\beta < \infty. \quad (5.32)$$

Furthermore,

Lemma 13. *In the notation of Lemma 12,*

$$\min_\ell\left(\tau_\beta\left(\mathcal{L}^2_\ell(a)\right) + \frac{(\delta-a)^2}{2D_\beta}\right) - \left(\tau_\beta\left(a^*_\beta\right) + \frac{(\delta - a^*_\beta)^2}{2D_\beta}\right) \geq \frac{(a-a^*_\beta)^2}{2D_\beta}, \quad (5.33)$$

for all δ and all $a \in [0, \delta]$

Proof of Lemma 12. We may consider only sufficiently large values of δ, such that solutions to (VP_δ) are given by optimal stacks of type 2. By the second of (3.16),

$$\frac{\delta - a^*_\beta(\delta)}{D_\beta} = \frac{\tau_\beta(\mathsf{e})}{r^{2,*}(a^*_\beta(\delta))}, \quad (5.34)$$

where $r^{2,*}(a)$ is the radius of the optimal stack of type-2 at given area a. Hence, it would be enough to check that

$$\liminf_{a\to\infty} r^{2,*}(a) > 0. \quad (5.35)$$

Clearly, if $\ell < m$ and $a \in [mw, 4\ell]$ (that is if area a can be realized by both ℓ and m stacks of type-2), then $r^{2,\ell}(a) < r^{2,m}(a)$. Which means that the map $a \mapsto r^{2,*}(a)$ has the following structure: There is a sequence $w = \hat{a}_1 < \hat{a}_2 < \dots$ of (transition) areas such that:

(a) On each of the intervals $(\hat{a}_\ell, \hat{a}_{\ell+1})$ the optimal radius $r^{2,*}(a) = r^{2,\ell}(a)$ and it is decreasing.
(b) At transition points $r^{2,\ell}(\hat{a}_{\ell+1}) < r^{2,\ell+1}(\hat{a}_{\ell+1})$. Hence we need to show that

$$\liminf_{\ell\to\infty} r^{2,\ell}(\hat{a}_{\ell+1}) > 0. \tag{5.36}$$

Fix ℓ and define $r_\ell = r^{2,\ell}(\hat{a}_{\ell+1})$ and $\rho_\ell = r^{2,\ell+1}(\hat{a}_{\ell+1})$. Then,

$$\hat{a}_{\ell+1} = \ell\left(4-(4-w)r_\ell^2\right) = (\ell+1)\left(4-(4-w)\rho_\ell^2\right). \tag{5.37}$$

By definition, $\tau\left(\mathcal{L}_\ell^2(\hat{a}_{\ell+1})\right) = \tau\left(\mathcal{L}_{\ell+1}^2(\hat{a}_{\ell+1})\right)$. Which reads (recall (3.14) and the first of (3.16)) as

$$\ell\left(8-2r_\ell(4-w)\right) = (\ell+1)\left(8-2\rho_\ell(4-w)\right) \tag{5.38}$$

Solving (5.37) and (5.38) we recover (5.35). □

Remark 14. *A slightly more careful analysis implies that under* (2.3) *there exists* $\nu < \infty$ *such that for all sufficiently large values of* β,

$$\sup_\delta \left(\delta - a_\beta^*(\delta)\right) \le \mathrm{e}^{\beta\nu}. \tag{5.39}$$

Proof of Lemma 13. By the second of (3.16) and then by (3.14) the function $a \mapsto \tau_\beta\left(\mathcal{L}_\ell^2(a)\right)$ is convex for any $\ell \in \mathbb{N}$. Hence,

$$\frac{\mathrm{d}^2}{\mathrm{d}a^2}\left(\tau_\beta\left(\mathcal{L}_\ell^2(a)\right) + \frac{(\delta-a)^2}{2D_\beta}\right) \ge \frac{1}{D_\beta}$$

uniformly in $a \in (\ell w, \delta \wedge 4\ell)$. The same applies at generic values $a \in (w, \delta)$ for the function

$$\min_\ell \left(\tau_\beta\left(\mathcal{L}_\ell^2(a)\right) + \frac{(\delta-a)^2}{2D_\beta}\right).$$

But the latter attains its minimum at $a_\beta^*(\delta)$. Hence the conclusion. □

Consider now a collection $\mathcal{S} = \{\gamma_v\}_{v\in\mathcal{V}}$ of closed polygonal lines. Given a number $\zeta < 1$, such that $2\zeta w > \mathsf{a}\left(\mathbb{B}_1\right) = 4$, let us split $\mathcal{S}$ into a disjoint union,

$$\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1 \cup \ldots, \tag{5.40}$$

where $\mathcal{S}_0$ contains all the polygons γ of $\mathcal{S}$ with area $\mathsf{a}(\gamma) \ge \zeta w$, whereas, for $i = 1, 2, \ldots$

$$\mathcal{S}_i = \left\{\gamma \in \mathcal{S} \,:\, \mathsf{a}(\gamma) \in \left[\zeta^{i+1}w, \zeta^i w\right)\right\}. \tag{5.41}$$

By construction, any compatible collection Γ of large level lines, such that $\Gamma \overset{\ell_N}{\sim} \mathcal{S}_0$ is always an ordered stack. Given numbers $d > 0, m \in \mathbb{N}$ and a value β of the inverse temperature, let us say $\mathcal{S}$ is bad; $\mathcal{S} \in \mathfrak{B}_{d,m}(\beta)$, if either there exists $i > d\beta$ such that $\mathcal{S}_i$ is not empty, or there exists $1 \le i \le d\beta$ such that the cardinality

$$|\mathcal{S}_i| = \#\left\{\gamma \,:\, \gamma \in \mathcal{S}_i\right\} \ge m. \tag{5.42}$$

Alternatively, we may think in terms of graphs $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ associated to bad collections $\mathcal{S} = \{\gamma_v\}_{v\in\mathcal{V}}$.

Proposition 15. *There exist $d < \infty$ and $m \in \mathbb{N}$ such that for all sufficiently large values of β the following holds:*

$$\limsup_{N\to\infty} \frac{1}{N} \max_{\mathcal{S}\in\mathfrak{B}_{d,m}(\beta)} \log \mathbb{Q}^{A}_{N,\beta}\left(\mathbb{I}_{\{\Gamma\in\mathcal{G}\}} \prod_{v\in\mathcal{V}} \mathbb{I}_{\left\{\Gamma_v \overset{\ell_N}{\sim} \mathcal{S}\right\}} \right) < 0, \tag{5.43}$$

for any excess area $A \geq 0$.

In particular for all sufficiently large β we may restrict attention to graphs $\mathcal{G}$ with chromatic number $\kappa_{\mathcal{G}} \leq \beta dm + 2$, independently of the value of excess area A in $\mathbb{Q}^{A}_{N,\beta}$.

Proof of Proposition 15. Note that the estimate $\kappa_{\mathcal{G}} \leq \beta dm + 2$ on the chromatic number is obtained from (5.43) as follows. For good skeleton collections the union $\mathcal{S}_1 \cup \mathcal{S}_2 \cup ...$ contains at most βdm loops, while the collection of large level lines $\Gamma^{(0)} \overset{\ell_N}{\sim} \mathcal{S}_0$ is an ordered stack. Therefore we have to estimate the edge chromatic number of a graph with at most $\beta dm + 2$ vertices, which we do by using our combinatorial theorem above.

In view of the lower bound (5.5) it would be enough to prove the following: There exists $c = c_\beta > 0$ such that If $\mathcal{S} \in \mathfrak{B}_{d,m}(\beta)$

$$\limsup_{N\to\infty} \max_{\mathcal{S}\in\mathfrak{B}_{d,m}(\beta)} \left(\frac{1}{N} \log \left(\mathbb{Q}_{N,\beta}\left(\Gamma \overset{\ell_N}{\sim} \mathcal{S} \right) \right) + \tau_\beta\left(\mathsf{a}\left(\mathcal{S}\right)\right) \right) < -c_\beta \tag{5.44}$$

Let $\mathcal{S} = \{\gamma_u\} \in \mathfrak{B}_{d,m}(\beta)$ be a bad collection of skeletons. Consider its decomposition (5.40). For any $k \geq 0$ set $\mathcal{S}^{(k)}_l = \mathcal{S}_0 \cup \cdots \cup \mathcal{S}_k$ and $\mathcal{S}^{(k)}_s = \mathcal{S}_{k+1} \cup \ldots$. We shall prove (5.44) by a gradual reduction procedure using the following identity: Given $k \geq 0$, the decomposition $\mathcal{S} = \mathcal{S}^{(k)}_l \cup \mathcal{S}^{(k)}_s$ induces the decomposition $\Gamma = \Gamma^{(k)}_l \cup \Gamma^{(k)}_s$ of any collection $\Gamma \overset{\ell_N}{\sim} \mathcal{S}$ of large level lines. Then,

$$\begin{aligned} &\mathbb{Q}_{N,\beta}\left(\Xi \geq \rho_\beta N^3 + AN^2 \,;\, \Gamma \overset{\ell_N}{\sim} \mathcal{S} \right) \\ &= \sum_{\Gamma \overset{\ell_N}{\sim} \mathcal{S}} \mathbb{Q}_{N,\beta}\left(\Xi \geq \rho_\beta N^3 + AN^2 \;\middle|\; \Gamma^{(k)}_l \cup \Gamma^{(k)}_s \right) \mathbb{Q}_{\beta,N}\left(\Gamma^{(k)}_s \middle| \Gamma^{(k)}_l \right) \mathbb{Q}_{\beta,N}\left(\Gamma^{(k)}_l \right). \end{aligned} \tag{5.45}$$

The conditional probability $\mathbb{Q}_{N,\beta}\left(\cdot \,\middle|\, \Gamma \right)$ is a straightforward modification of (4.11): Given a splitting $\Gamma \cup \Gamma'$ of a compatible family of large contours, or, alternatively a splitting $\mathcal{V} \cup \mathcal{V}'$ of the vertices of the associated graph,

$$\mathbb{Q}_{N,\beta}\left(\Gamma' \,\middle|\, \Gamma \right) \overset{\sim}{=} \exp\left\{ -\beta \sum_{v\in\mathcal{V}'} |\Gamma_v| - \sum_{\substack{\mathcal{C}\subset\Lambda_N \\ \mathcal{C}\sim\Gamma}} \mathbb{I}_{\{\mathcal{C}\not\sim\Gamma'\}} \Phi_\beta(\mathcal{C}) \right\}. \tag{5.46}$$

We shall rely on the following upper bounds on conditional weight which holds for all β and N sufficiently large: By (4.10) and (5.2),

$$\frac{1}{N}\log\left(\sum_{\Gamma'\overset{\ell_N}{\sim}\mathcal{S}'}\mathbb{Q}_{N,\beta}\left(\Gamma'\middle|\Gamma\right)\right)\leq-\left(1-\mathrm{O}\left(\mathrm{e}^{-4\beta}\right)\right)\tau_\beta\left(\mathcal{S}'\right). \tag{5.47}$$

for any Γ and any collection $\mathcal{S}'$ of closed polygonal lines.

Let us now turn to proving (5.44) (and consequently (5.43)) proper.

STEP 1. Let us explain how we choose m in $\mathfrak{B}_{d,m}$. We can fix (independently of β two numbers $r>1$ and $c<\infty$ such that

$$\tau_\beta\left(\mathcal{S}_s^{(k)}\right)\geq r\tau_\beta\left(\mathsf{a}\left(\mathcal{S}_s^{(k)}\right)\right), \tag{5.48}$$

whenever $\mathcal{S}$ and a number $k\geq 0$ are such that

$$\mathsf{a}\left(\mathcal{S}_s^{(k)}\right)\geq c\zeta^k. \tag{5.49}$$

Indeed, by construction, the areas of loops from $\mathcal{S}_s^{(k)}$ are bounded above by ζ^{k+1}. Hence,

$$\tau_\beta\left(\mathcal{S}_s^{(k)}\right)\geq 2\tau_\beta(\mathsf{e})\left\lfloor\frac{a}{\zeta^{k+1}}\right\rfloor\sqrt{w_\beta\zeta^{k+1}}$$

whenever $\mathsf{a}\left(\mathcal{S}_s^{(k)}\right)=a$. This should be compared with $\tau_\beta(a)$ which equals to $2\tau_\beta(\mathsf{e})\sqrt{aw_\beta}$ if $a\leq w_\beta$ and, otherwise, bounded above by $2\tau_\beta(\mathsf{e})\left(\left\lfloor\frac{a}{w_\beta}\right\rfloor+1\right)w_\beta$.

Note that (5.49) will hold if $|\mathcal{S}_{k+1}|\geq\frac{c}{\zeta w}$. We set $m=\frac{c}{\zeta w}$.

STEP 2. The first consequence of (5.48) is that we can rule out collections $\mathcal{S}$ with $\mathsf{a}\left(\mathcal{S}_s^{(0)}\right)\geq c$. Indeed, let $\mathcal{S}$ be such a collection of polygonal lines. By construction $\mathcal{S}_0$ is compatible with an ordered stack of large contours, hence its graph is just a line segment. Hence, the decoupling bound (5.28) and Theorem 10 imply:

$$\frac{1}{N}\log\left(\mathbb{Q}_{N,\beta}\left(\Gamma\overset{\ell_N}{\sim}\mathcal{S}_0\right)\right)\leq-\left(1-\mathrm{o}_N(1)\right)\sum_{\gamma\in\mathcal{S}_0}\left(\tau_\beta\left(\mathsf{a}(\gamma)\right)+\alpha_0\left(\Omega_\beta(\gamma)\right)\right) \tag{5.50}$$

On the other hand, (5.47) and (5.48) imply that

$$\frac{1}{N}\log\left(\mathbb{Q}_{N,\beta}\left(\Gamma_s^{(0)}\overset{\ell_N}{\sim}\mathcal{S}_s^{(0)}\,\middle|\,\Gamma\right)\right)\leq-r\left(1-\mathrm{O}\left(\mathrm{e}^{-4\beta}\right)\right)\tau_\beta\left(\mathsf{a}\left(\mathcal{S}_s^{(0)}\right)\right). \tag{5.51}$$

for all compatible collections of large contours $\Gamma\overset{\ell_N}{\sim}\mathcal{S}_0$. The last expression is strictly smaller than $-\tau_\beta\left(\mathsf{a}\left(\mathcal{S}_s^{(0)}\right)\right)$ for all β satisfying

$$r\left(1-\mathrm{O}\left(\mathrm{e}^{-4\beta}\right)\right)>1.$$

Hence (5.44) .

There are two implications of the above computation which hold for all β sufficiently large. First of all pick $\nu_1 > \nu$ (see (5.39)). Then for any A we can restrict attention to skeleton collections $\mathcal{S}$ satisfying

$$\mathsf{a}\left(\mathcal{S}_0\right) \geq \delta_\beta - \mathrm{e}^{\nu_1 \beta}. \tag{5.52}$$

Indeed, recall β-independent constant c which was defined via (5.48). In view of Proposition 15 we may restrict attention to $\mathsf{a}\left(\mathcal{S}_s^{(0)}\right) \leq c$. Which means that if (5.52)is violated, then

$$\mathsf{a}\left(\mathcal{S}\right) \leq \mathsf{a}_\beta^*(\delta_\beta) - \mathrm{e}^{\nu\beta} + c$$

for all β sufficiently large. On the other hand, as we have already mentioned, $\mathcal{S}_0$ has graph structure of an ordered stack or, in other words, one-dimensional segment, and (5.50) holds. Therefore, (5.52) is secured by Lemma 13 and (5.39) (and, of course, lower bound (5.5)).

Next, assume (5.52). Then, in view of the upper bound on conditional weights (5.47), and proceeding as in derivation of (4.5), one can fix $d < \infty$ and rule out skeletons γ with $\mathsf{a}(\gamma) < \xi^{\beta d} w$.

As a result we need to consider only bad collections $\mathcal{S}$ which satisfy $\mathcal{S}_i = \emptyset$ for any $i > \beta d$, but which still violate (5.42).

STEP 3. We proceed by induction. Assume that $\mathcal{S}$ is such that the cardinalities $|\mathcal{S}_1|, \ldots, |\mathcal{S}_k| \leq m$. Then $\mathcal{S}$ may be ignored if $|\mathcal{S}_{k+1}| > m$. Indeed, the latter would imply that $\mathsf{a}\left(\mathcal{S}_s^{(k)}\right) \geq c\zeta^k$, and hence (5.48) holds. Consequently, as in the case of (5.51)

$$\limsup_{N\to\infty} \frac{1}{N} \log\left(\mathbb{Q}_{N,\beta}\left(\Gamma_s^{(k)} \overset{\ell_N}{\sim} \mathcal{S}_s^{(k)} \,\middle|\, \Gamma\right)\right) \leq -r\left(1 - \mathrm{O}\left(\mathrm{e}^{-4\beta}\right)\right) \tau_\beta\left(\mathsf{a}\left(\mathcal{S}_s^{(k)}\right)\right). \tag{5.53}$$

for all collection $\Gamma \overset{\ell_N}{\sim} \mathcal{S}_l^{(k)}$.

On the other hand, by induction assumption, the chromatic number of $\mathcal{L}_l^{(k)}$ is bounded above by $km + 2 \leq \beta dm + 2 := \kappa_\beta$. Hence, Theorem 10 applies, and in view of the decoupling bound (5.28), we infer:

$$\frac{1}{N} \log\left(\mathbb{Q}_{N,\beta}\left(\Gamma \overset{\ell_N}{\sim} \mathcal{S}_l^{(k)}\right)\right) \leq -\left(1 - \mathrm{o}_N(1)\right) \sum_{\gamma \in \mathcal{S}_l^{(k)}} \left(\tau_\beta\left(\mathsf{a}(\gamma)\right) + \alpha_0\left(\Omega_\beta(\gamma)\right)\right) \tag{5.54}$$

Together with (5.53) this implies (5.44). □

5.7. **Proof of Theorem 9.** Let us complete the proof of Theorem 9 and hence of Theorem C. As we have seen in the previous subsection for each β sufficiently large we may ignore skeleton collections with chromatic numbers exceeding $\kappa_\beta = \beta dm + 2$. If, however, the skeleton $\mathcal{S}$ is *good*, that is if the chromatic number of $\mathcal{S}$ is less or

equal to κ_β, then (5.29) is satisfied, and, in view of (5.26), (5.28) and (5.30) we conclude that

$$\frac{1}{N}\log\left(\mathbb{Q}_{N,\beta}\left(\Gamma \overset{\ell_N}{\sim} \mathcal{S}\right)\right) \leq -\left(\sum_{\gamma_u\in\mathcal{S}}\left(\tau_\beta\left(\mathsf{a}(\gamma_u)\right)+\alpha_0\left(\Omega_\beta(\gamma_u)\right)\right)\right)(1-\mathrm{o}_N(1)). \quad (5.55)$$

In (5.55) there is the same correction term $\mathrm{o}_N(1)\to 0$ for *all* good skeletons. Since,

$$\mathcal{E}_\beta\left(\mathcal{S}\mid\delta_\beta\right)=\sum_{\gamma_u\in\mathcal{S}}\Omega_\beta(\gamma_u)+\left(\sum_{\gamma_u\in\mathcal{L}}\tau_\beta(\mathsf{a}(\gamma_u))+\frac{(\delta_\beta-\mathsf{a}(\mathcal{L}))^2}{2D_\beta}\right),$$

it remains to apply the quantitative isoperimetric bound (5.19).

6. Two interacting contours

In this concluding section we sketch the proof of Theorem 10. The proof relies on the skeleton calculus developed in [18], Ornstein-Zernike theory and random walk representation of polymer models [27], which, in the particular case of Ising polymers, was refined and adjusted in [25]. We shall repeatedly refer to these papers for missing details.

Throughout this section we shall assume that the constants $\chi>1/2$ and κ_β are fixed so that (5.29) holds.

6.1. **Low temperature skeleton calculus and modified surface tension.** We need to recall some ideas and techniques introduced in [18].

Consider an ℓ_N-skeleton $\gamma=(\mathsf{u}_0,\mathsf{u}_1,\dots,\mathsf{u}_n)$. It has $n+1$ edges

$$e_0=(\mathsf{u}_0,\mathsf{u}_1),\dots,e_n=(\mathsf{u}_n,\mathsf{u}_0).$$

The last edge e_n might have a shorter length than ℓ_N, but for the sake of the exposition we shall ignore the corresponding negligible corrections. The edges of γ are classified into being *good* or *bad* as follows: Fix once and for all some small angle $\theta>0$ (note that the value of θ and hence the classification of edges we are going to explain does not depend on β). With each edge $e=(\mathsf{u},\mathsf{v})$ we associate a diamond shape $\mathsf{D}_\theta(e)$,

$$\mathsf{D}(e)=\mathsf{D}_\theta(\mathsf{u},\mathsf{v})=\left(\mathsf{u}+\mathcal{Y}_{\mathsf{v}-\mathsf{u}}\right)\cap\left(\mathsf{v}+\mathcal{Y}_{\mathsf{u}-\mathsf{v}}\right),$$

where for any $\mathsf{x}\in\mathbb{R}^2\setminus 0$ we use $\mathcal{Y}_{\mathsf{x}}$ to denote the symmetric cone of opening θ along the ray passing through x.

An edge e_i of γ is called good if

$$\mathsf{D}(e_i)\cap\mathsf{D}(e_j)=\emptyset \quad \text{for any } j\neq i. \quad (6.1)$$

Otherwise the edge is called *bad*. We use $\mathfrak{g}(\gamma)$ and $\mathfrak{b}(\gamma)$ for, respectively, the sets of good and bad edges of γ.

Let Γ be a large contour, and γ be its skeleton, having the set $\mathfrak{b}(\gamma)$ of bad bonds. We denote by $\Gamma_{\mathfrak{b}}$ the portion of Γ, corresponding to bonds in $\mathfrak{b}(\gamma)$.

The modified surface tension $\hat{\tau}_\beta(\gamma)$ is defined as follows:

$$\hat{\tau}_\beta(\gamma) = \tau_\beta(\gamma) - \sum_{e\in\mathfrak{b}(\gamma)} \psi_\beta(e). \tag{6.2}$$

In its turn the function $\psi_\beta \geq 0$ is defined via functions (compare with (5.2))

$$G^*_\beta(x) := \sum_{\gamma:0\to x} w^{\mathsf{f}}_\beta(\eta) \mathrm{e}^{\kappa_\beta \sum_{\mathcal{C}\not\sim\eta}|\Phi_\beta(\mathcal{C})|} \quad \text{and} \quad \tau^*_\beta(x) = -\lim_{n\to\infty}\frac{1}{n}\log G^*_\beta(\lfloor nx\rfloor), \tag{6.3}$$

as

$$\psi_\beta(x) = \tau_\beta(x) - \tau^*_\beta(x). \tag{6.4}$$

The skeleton calculus developed in [18] implies the following two crucial bounds, which hold uniformly in $\mathsf{a}(\gamma) > a_0$:

$$\hat{\tau}_\beta(\gamma) \geq \tau_\beta(\mathsf{a}(\gamma)) + \alpha_0\left(\Omega_\beta(\gamma)\right). \tag{6.5}$$

and

$$\left|\ln \frac{w^{\mathsf{f}}_\beta\left(\mathbb{1}_{\left\{\Gamma \overset{\ell_N}{\sim} \gamma\right\}} \mathrm{e}^{\kappa_\beta \sum_{\mathcal{C}\not\sim\Gamma_{\mathfrak{b}}}|\Phi_\beta(\mathcal{C})|}\right)}{\prod_{(\mathsf{u},\mathsf{v})\in\mathfrak{g}(\gamma)} G_\beta(\mathsf{v}-\mathsf{u}) \prod_{(\mathsf{u},\mathsf{v})\in\mathfrak{b}(\gamma)} G^*_\beta(\mathsf{v}-\mathsf{u})}\right| \leq \mathrm{o}_N(1)\tau_\beta(\gamma)N. \tag{6.6}$$

The estimate (6.5) with the function $\alpha_0(x) = \frac{1}{2}x$ is nothing else but the estimate (2.16.1) from the Lemma 2.16 of [18]. (Our Ω is what is called Δ there, and the function n_δ there vanishes in the case of our interest.) The estimate (6.6) is a very special case of the Theorem 4.16 of [18], which establishes the asymptotic independence of the surface tension on the shape of the volume.

In view of the Ornstein-Zernike asymptotics (for instance (3.4) in [25]) of low temperature two-point functions G_β in (5.2) and G^*_β in (6.3) the upper bound (5.22) readily follows from (6.5) and (6.6).

6.2. **Decoupling upper bound for two interacting skeletons.** Consider two ℓ_N-skeletons γ_1 and γ_2 as in the formulation of Theorem 10. Upper bound (6.6) implies:

$$\begin{aligned} & w^{\mathsf{f}}_\beta \otimes w^{\mathsf{f}}_\beta \left(\mathbb{1}_{\{\Gamma_1\sim\Gamma_2\}} \mathbb{1}_{\left\{\Gamma_1 \overset{\ell_N}{\sim} \gamma_1\right\}} \mathbb{1}_{\left\{\Gamma_2 \overset{\ell_N}{\sim} \gamma_2\right\}} \mathrm{e}^{\kappa_\beta \sum_{\mathcal{C}\not\sim(\Gamma_1,\Gamma_2)}|\Phi_\beta(\mathcal{C})|}\right) \\ & \leq \mathrm{e}^{\mathrm{o}_N(1)N(\tau_\beta(\gamma_1)+\tau_\beta(\gamma_2))} \prod_{(\mathsf{u},\mathsf{v})\in\mathfrak{b}(\gamma_1)\cup\mathfrak{b}(\gamma_2)} \hat{G}_\beta(\mathsf{v}-\mathsf{u}) \times \\ & \bigotimes_{e\in\mathfrak{g}(\gamma_1), f\in\mathfrak{g}(\gamma_2)} w^{\mathsf{f}}_\beta \left(\prod_{e,f} \mathbb{1}_{\left\{\eta_e \overset{\ell_N}{\sim} e\right\}} \mathbb{1}_{\left\{\eta_f \overset{\ell_N}{\sim} f\right\}} \mathbb{1}_{\{\eta_e\sim\eta_f\}} \mathrm{e}^{\kappa_\beta \sum_{e,f}\sum_{\mathcal{C}\not\sim(\eta_e,\eta_f)}|\Phi_\beta(\mathcal{C})|}.\right) \end{aligned} \tag{6.7}$$

Above $\eta_e \overset{\ell_N}{\sim} e = (\mathsf{u}, \mathsf{v})$ means that η_e is an admissible path from u to v which is, in addition, compatible with the ℓ_N-skeleton construction. As before, $\mathcal{C} \not\sim (\eta_e, \eta_f)$ means that $\mathcal{C}$ is not compatible with both η_e and η_f.

As in the derivation of (6.6) it is possible to check that the contribution coming from $\sum_{\mathcal{C}\not\sim(\eta_e,\eta_f)} |\Phi_\beta(\mathcal{C})|$ could be ignored whenever $\mathsf{D}_\theta(e) \cap \mathsf{D}_\theta(f) = \emptyset$. In the latter case let us say that the edges e and f *are not associated*. Otherwise we say that $e \in \mathfrak{g}(\gamma_1)$ and $f \in \mathfrak{g}(\gamma_2)$ are *associated*. Since by construction $\mathsf{D}_\theta(f) \cap \mathsf{D}_\theta(f') = \emptyset$ for any two good edges $f, f' \in \mathfrak{g}(\gamma_2)$, any good edge $e \in \mathfrak{g}(\gamma_1)$ can be associated to at most m_θ different good edges of $\mathfrak{g}(\gamma_2)$. Proceeding as in the derivation of (5.28) we conclude that we need to derive the upper bound on

$$w_\beta^{\mathsf{f}} \otimes w_\beta^{\mathsf{f}} \left(\mathbb{1}_{\left\{\eta_e \overset{\ell_N}{\sim} e\right\}} \mathbb{1}_{\left\{\eta_f \overset{\ell_N}{\sim} f\right\}} \mathbb{1}_{\{\eta_e \sim \eta_f\}} \mathrm{e}^{m_\theta \kappa_\beta \sum_{\mathcal{C}\not\sim(\eta_e,\eta_f)} \left|\Phi_\beta(\mathcal{C})\right|}, \right) \tag{6.8}$$

uniformly in $\mathsf{D}_\theta(e) \cap \mathsf{D}_\theta(f) \neq \emptyset$.

Since m_θ does not depend on β, $\kappa'_\beta = m_\theta \kappa_\beta$ satisfies (5.29) with any $\chi' < \chi$. Therefore, Theorem 10 would be a consequence of the following claim:

Proposition 16. *Under the conditions of Theorem 10*

$$w_\beta^{\mathsf{f}} \otimes w_\beta^{\mathsf{f}} \left(\mathbb{1}_{\left\{\eta_e \overset{\ell_N}{\sim} e\right\}} \mathbb{1}_{\left\{\eta_f \overset{\ell_N}{\sim} f\right\}} \mathbb{1}_{\{\eta_e \sim \eta_f\}} \mathrm{e}^{\kappa_\beta \sum_{\mathcal{C}\not\sim(\eta_e,\eta_f)} \left|\Phi_\beta(\mathcal{C})\right|} \right) \leq \mathrm{e}^{-\ell_N \left(\tau_\beta(\mathsf{v}-\mathsf{u}) + \tau_\beta(\mathsf{w}-\mathsf{z})\right)(1-\mathrm{o}_N(1))}, \tag{6.9}$$

uniformly in β large and uniformly in all pairs of ℓ_N-edges $e = (\mathsf{u}, \mathsf{v})$ and $f = (\mathsf{v}, \mathsf{z})$ such that $\mathsf{D}_\theta(e) \cap \mathsf{D}_\theta(f) \neq \emptyset$.

The estimate (6.9) is a manifestation of the fact that under (5.29) entropic repulsion between paths η_e and η_f beats the attractive potential $\mathrm{e}^{\kappa_\beta \sum_{\mathcal{C}\not\sim(\eta_e,\eta_f)} \left|\Phi_\beta(\mathcal{C})\right|}$. As a result, typical paths stay far apart and their contributions to the surface tension just add up.

In the concluding Subsection 6.3 we shall prove (6.9) in the most difficult case when e and f stay close to the horizontal axis. This case is the most difficult since it corresponds to the minimal strength of entropic repulsion between η_e and η_f.

6.3. **Effective random walk representation.** Consider edges $f_1 = (\mathsf{z}, \mathsf{w})$ and $f_2 = (\mathsf{u}, \mathsf{v})$ with $\mathsf{u} = (0,0) := \mathsf{0}$, $\mathsf{v} = (\ell_N, 0)$, $\mathsf{z} = (0, z)$ and $\mathsf{w} = (\ell_N, z)$. Define the event (collection of paths (γ_1, γ_2))

$$\mathcal{T}_+^2 = \mathcal{T}_+^2(\ell_N \,|\, z) = \left\{ (\gamma_1, \gamma_2) \ : \ \gamma_1 \overset{\ell_N}{\sim} f_1 \,;\, \gamma_2 \overset{\ell_N}{\sim} f_2 \,;\, \gamma_1 \sim \gamma_2 \right\}. \tag{6.10}$$

In particular, if $z \geq 0$, $\{\gamma_1, \gamma_2\} \in \mathcal{T}_+^2(f_1, f_2)$ implies that γ_1 "stays above" γ_2. Note, however that they can share edges, and that they might have overhangs.

We claim that the following holds:

Proposition 17. *Assume* (5.29). *There exist a finite constant c_+ such that for all β sufficiently large,*

$$\sup_{z\geq 0} w^{\mathsf{f}}_\beta \otimes w^{\mathsf{f}}_\beta \left(\mathcal{T}^2_+(\ell_N \mid z); \mathrm{e}^{\kappa_\beta \sum_{\mathcal{C}} \mathbf{1}_{\mathcal{C}\not\sim\eta_1}\mathbf{1}_{\mathcal{C}\not\sim\eta_2}|\Phi_\beta(\mathcal{C})|}\right) \leq c_+ \mathrm{e}^{-2\tau_\beta(\mathsf{e}_1)\ell_N} \tag{6.11}$$

as soon as N is sufficiently large.

Proving that c_+ does not depend on β is the crux of the matter, and it is based on a careful analysis of non-intersection probabilities for effective random walks in a weak attractive potential.

Let $\mathcal{T}^1 = \mathcal{T}^1(\ell_N)$ be the set of paths $\gamma : \mathsf{0} \mapsto \mathsf{v}$ with $\mathsf{v}\cdot\mathsf{e}_1 = \ell_N$. Note that by definition $(\gamma_1, \gamma_2) \in \mathcal{T}^2_+$ implies that $\gamma_2 \in \mathcal{T}^1$ and $\gamma_1 \in (0, z) + \mathcal{T}^1$.

Let $\mathcal{K}$ be a positive (two-dimensional) symmetric cone around e_1 with an opening strictly between $\pi/2$ and π. A high temperature expansions of polymer weights $\mathrm{e}^{\Phi(\mathcal{C})}$ leads (see (4.9) in [25]) to the following irreducible decomposition of decorated (open) contours $[\gamma, \underline{\mathcal{C}}]$, where $\gamma \in \mathcal{T}^1(\ell_N)$ and $\underline{\mathcal{C}}$ is a collection of γ-incompatible clusters:

$$[\gamma, \underline{\mathcal{C}}] = \mathfrak{a}^\ell \circ \mathfrak{a}^1 \circ \ldots \mathfrak{a}^m \circ \mathfrak{a}^r. \tag{6.12}$$

The irreducible animals $\mathfrak{a}^i = \left[\eta^i, \underline{\mathcal{D}}^i\right]$ belong to the family $\mathsf{A} = \{\mathsf{a} = [\eta, \mathcal{D}]\}$, which could be characterized by the following two properties:

a. If η is a path with endpoints at x, y, then

$$\eta \cup \underline{\mathcal{D}} \subseteq \mathsf{D}(\mathsf{x}, \mathsf{y}) := (\mathsf{x} + \mathcal{K}) \cap (\mathsf{y} - \mathcal{K}). \tag{6.13}$$

b. $\mathfrak{a}$ could not be split into concatenation of two non-trivial animals satisfying **a** above.

The left and right irreducible animals satisfy one-sided versions of diamond-confinement condition (6.13). For instance if $\mathfrak{a}^\ell = \left[\eta^\ell, \underline{\mathcal{D}}^\ell\right]$ and y is the right end-points of η^ℓ, then $\eta^\ell \cup \underline{\mathcal{D}}^\ell \subseteq (\mathsf{y} - \mathcal{K})$.

Given an animal $\mathsf{a} = [\eta, \underline{\mathcal{D}}]$ we use $\mathsf{X}(\mathfrak{a}) = \mathsf{X}(\eta)$ to denote the vector which connects the left and right end-points of η. The horizontal and vertical coordinates of X are denoted by $\mathsf{H} = \mathsf{X}\cdot\mathsf{e}_1$ and $\mathsf{V} = \mathsf{X}\cdot\mathsf{e}_2$. By construction, $\mathsf{H}(\mathfrak{a}) \in \mathbb{N}$ for any irreducible animal a.

Returning to the decomposition (6.12), let us consider for each $k \in \mathbb{N}$ the subset $\mathcal{T}^1_k$ of decorated paths $[\gamma, \underline{\mathcal{C}}]$ from $\mathcal{T}^1$ for which $\mathsf{H}(\mathfrak{a}^\ell) + \mathsf{H}(\mathfrak{a}^r) = k$. Then, by (4.11) in [25] there exists $c_g < \infty$ and $\nu_g > 0$, such that

$$w^{\mathsf{f}}_\beta\left(\mathcal{T}^1_k\left(\ell_N\right)\right) \leq c_g \mathrm{e}^{-\beta\nu_g k} w^{\mathsf{f}}_\beta\left(\mathcal{T}^1\left(\ell_N\right)\right) \tag{6.14}$$

for all β and N sufficiently large. So in the sequel we shall restrict attention to decorated paths $[\gamma, \underline{\mathcal{C}}] \in \mathcal{T}^1_0$ with empty right and left irreducible animals, that is with $\mathfrak{a}^\ell, \mathfrak{a}^r = \emptyset$ in (6.12). In particular, we shall restrict attention to $\mathcal{T}^2_{0,+}(\ell_N|z) := \mathcal{T}^2_+ \cap ((\mathsf{z} + \mathcal{T}^1_0) \times \mathcal{T}^1_0)$, and, instead of (6.11), shall derive an upper bound on

$$\sup_{z\geq 0} w^{\mathsf{f}}_\beta \otimes w^{\mathsf{f}}_\beta \left(\mathcal{T}^2_{0,+}(\ell_N|z); \mathrm{e}^{\kappa_\beta \sum_{\mathcal{C}} \mathbf{1}_{\mathcal{C}\not\sim\gamma_1}\mathbf{1}_{\mathcal{C}\not\sim\gamma_2}|\Phi_\beta(\mathcal{C})|}\right) \tag{6.15}$$

A general case could be done by a straightforward adaptation based on the mass-gap property (6.14).

Decorated paths $[\gamma, \underline{\mathcal{C}}] \in \mathcal{T}_0^1$ have an immediate probabilistic interpretation: Set $\tau_\beta = \tau(\mathsf{e}_1)$. Then (see Theorem 5 in [25])

$$\mathbb{P}_\beta(\mathfrak{a}) := \mathrm{e}^{\tau_\beta \mathsf{H}(\mathfrak{a})} w_\beta^{\mathsf{f}}(\mathfrak{a}) \tag{6.16}$$

is a probability distribution on A with exponentially decaying tail:

$$\sum_{\mathfrak{a}\in\mathsf{A}} \mathbb{I}_{\mathsf{X}(\mathfrak{a})=(h,v)} \mathbb{P}_\beta(\mathfrak{a}) \le c_g \mathrm{e}^{-\nu_g \beta(h+|v|-1)}, \tag{6.17}$$

– i.e. no normalization in (6.16) is needed! Given $\mathsf{x} = (0, x)$ consider random walk $\mathsf{S}_n = \mathsf{x} + \sum_1^n \mathsf{X}^i$, where X^u-s are independent $\mathbb{N}\times\mathbb{Z}$-valued steps distributed according to

$$\mathbb{P}_\beta(\mathsf{X} = \mathsf{y}) = \sum_{\mathfrak{a}\in\mathsf{A}} \mathbb{I}_{\mathsf{X}(\mathfrak{a})=\mathsf{y}} \mathbb{P}_\beta(\mathfrak{a}).$$

Let $\mathbb{P}_{\beta,x}$ be the corresponding measure on random walk paths. In this way $w_\beta^{\mathsf{f}}(\mathcal{T}_0^1)$ equals to

$$w_\beta^{\mathsf{f}}\left(\mathcal{T}_0^1\right) = \mathrm{e}^{-\ell_N \tau_\beta} \mathbb{P}_{\beta,0}\left(\ell_N \in \mathrm{Range}\left(\mathsf{S}_n \cdot \mathsf{e}_1\right)\right). \tag{6.18}$$

Let us adapt the above random walk representation of a single decorated path to the case of a pair of decorated paths, from $(\mathsf{z} + \mathcal{T}_0^1) \times \mathcal{T}_0^1$. These have irreducible decompositions

$$\underline{\mathfrak{a}} = \mathsf{z} + \mathfrak{a}^1 \circ \cdots \circ \mathfrak{a}^n \quad \text{and} \quad \underline{\mathfrak{b}} = \mathfrak{b}^1 \circ \cdots \circ \mathfrak{b}^m. \tag{6.19}$$

Following [9] we shall align horizontal projections of underlying random walks. Given (6.19) consider sets

$$\mathcal{H}_a = \{0, \mathsf{H}(\mathfrak{a}_1), \mathsf{H}(\mathfrak{a}_1) + \mathsf{H}(\mathfrak{a}_2), \ldots, \ell_N\}, \ \mathcal{H}_b = \{0, \mathsf{H}(\mathfrak{b}_1), \mathsf{H}(\mathfrak{b}_1) + \mathsf{H}(\mathfrak{b}_2), \ldots, \ell_N\}. \tag{6.20}$$

Set $\mathcal{H}(\underline{\mathfrak{a}}, \underline{\mathfrak{b}}) = \mathcal{H}_a \cap \mathcal{H}_b$. This intersection $\mathcal{H}$ is the set of horizontal projections of end-points of jointly irreducible pairs of strings of animals. The alphabet A^2 of such pairs could be described as follows: $(\underline{\mathfrak{a}}, \underline{\mathfrak{b}}) \in \mathsf{A}^2$ if $\sum \mathsf{H}(\mathfrak{a}^i) = \sum \mathsf{H}(\mathfrak{b}^j) := \mathsf{H}(\underline{\mathfrak{a}}, \underline{\mathfrak{b}})$, and

$$\mathcal{H}(\underline{\mathfrak{a}}, \underline{\mathfrak{b}}) = \{0, \mathsf{H}(\underline{\mathfrak{a}}, \underline{\mathfrak{b}})\}. \tag{6.21}$$

In the sequel we shall refer to elements $\mathfrak{c} \in \mathsf{A}^2$ as irreducible pairs.

By elementary renewal theory, (6.16) induces a probability distribution on A^2 which inherits exponential tails from (6.17). We continue to call this distribution $\mathbb{P}_\beta$. The i.i.d. $\mathbb{N}\times\mathbb{Z}\times\mathbb{Z}$-valued steps of the induced random walk have distribution

$$\mathbb{P}_\beta\left(\mathsf{X} = (\mathsf{H}, \mathsf{V}_1, \mathsf{V}_2) = (h, v_1, v_2)\right) = \sum_{\mathfrak{c}\in\mathsf{A}^2} \mathbb{I}_{\mathsf{X}(\mathfrak{c})=(h,v_1,v_2)} \mathbb{P}_\beta(\mathfrak{c}).$$

Decorated paths from $(\mathsf{z}+\mathcal{T}_0^1)\times\mathcal{T}_0^1$ give rise to random walks

$$(0,z,0)+\sum_1^n \mathsf{X}_i.$$

Note that in this notation

$$\mathcal{T}_{0,+}^2\left(\ell_N\,|\,z\right)\subset\bigcup_n\left\{\sum_1^n \mathsf{H}_i=n;\mathcal{R}_+^n(z)\right\}, \tag{6.22}$$

where

$$\mathcal{R}_+^n(z)=\{\mathsf{Z}_k\geq 0 \text{ for } k=0,1,\dots,n\} \text{ and } \mathsf{Z}_k=z+\sum_1^k\left(\mathsf{V}_i^1-\mathsf{V}_i^2\right). \tag{6.23}$$

With a slight abuse of notation we shall use the very same symbol $\mathbb{P}_{\beta,z}$ also for the law of the random walk Z_k in (6.23).

6.4. **Recursion and random walk analysis.** Define

$$\mu_\beta(\ell_N|z)=\mathbb{P}_{\beta,z}\left(\mathcal{T}_{0,+}^2(\ell_N\;|\;z);\mathrm{e}^{\kappa_\beta\sum_{\mathcal{C}}\mathbf{1}_{\mathcal{C}\not\sim\gamma_1}\mathbf{1}_{\mathcal{C}\not\sim\gamma_2}|\Phi_\beta(\mathcal{C})|}\right). \tag{6.24}$$

Assume (5.29). Then, following Subsection 6.1 of [25], one can develop the following recursion relation for $\mu_\beta=\sup_N\sup_z\mu_\beta(\ell_N|z)$: There exist β-independent constants c_1 and ν such that

$$\mu_\beta\leq 1+\mu_\beta c_1\mathrm{e}^{-2\chi\beta}\max_z\sum_n\mathbb{E}_{\beta,z}\left(\mathrm{e}^{-\nu\beta\mathsf{Z}_n};\mathcal{R}_+^n\right). \tag{6.25}$$

The importance of (6.25) is that

$$c_1\mathrm{e}^{-2\chi\beta}\sup_{z\geq 0}\sum_n\mathbb{E}_{\beta,z}\left(\mathrm{e}^{-\nu\beta\mathsf{Z}_n};\mathcal{R}_+^n\right)<1 \tag{6.26}$$

implies that μ_β is bounded, and, since by (6.16) and (6.18),

$$\begin{aligned}&w_\beta^{\mathsf{f}}\otimes w_\beta^{\mathsf{f}}\left(\mathcal{T}_{0,+}^2(\ell_N\;|\;z);\mathrm{e}^{\kappa_\beta\sum_{\mathcal{C}}\mathbf{1}_{\mathcal{C}\not\sim\gamma_1}\mathbf{1}_{\mathcal{C}\not\sim\gamma_2}|\Phi_\beta(\mathcal{C})|}\right)\\&\quad=\mathrm{e}^{-2\ell_N\tau_\beta(\mathsf{e}_1)}\mathbb{P}_{\beta,z}\left(\mathcal{T}_{0,+}^2(\ell_N\;|\;z);\mathrm{e}^{\kappa_\beta\sum_{\mathcal{C}}\mathbf{1}_{\mathcal{C}\not\sim\gamma_1}\mathbf{1}_{\mathcal{C}\not\sim\gamma_2}|\Phi_\beta(\mathcal{C})|}\right)\leq\mathrm{e}^{-2\ell_N\tau_\beta(\mathsf{e}_1)}\mu_\beta,\end{aligned} \tag{6.27}$$

one deduces (6.11) as an immediate consequence. It remains to prove:

Lemma 18. *If $\chi>1/2$, then*

$$\lim_{\beta\to\infty}\mathrm{e}^{-2\chi\beta}\sup_{z\geq 0}\sum_n\mathbb{E}_{\beta,z}\left(\mathrm{e}^{-\nu\beta\mathsf{Z}_n};\mathcal{R}_+^n\right)=0. \tag{6.28}$$

In particular, the inequality (6.26) *holds for all β sufficiently large.*

Proof of Lemma 18 . Let us fix $z \geq 0$. Consider decomposition of paths from $\mathcal{R}_+^n(z)$ with respect to the left-most minimum u, $0 \leq u \leq z$. Define the strict version $\hat{\mathcal{R}}_+^n$ as

$$\hat{\mathcal{R}}_+^n = \{\mathsf{Z}_k > 0 \text{ for } k = 0, 1, \ldots, n\}.$$

Since $\mathsf{W} = \mathsf{V}^1 - \mathsf{V}^2$ has symmetric distribution under $\mathbb{P}_\beta$, we can rewrite

$$\sum_n \mathbb{E}_{\beta,z}\left(\mathrm{e}^{-\nu\beta\mathsf{Z}_n}; \mathcal{R}_+^n\right) = \sum_{u=0}^{z} \mathrm{e}^{-\nu\beta u} \sum_n \mathbb{P}_{\beta,0}\left(\hat{\mathcal{R}}_+^n; \mathsf{Z}_n = z - u\right) \sum_m \mathbb{E}_{\beta,0}\left(\mathrm{e}^{-\nu\beta\mathsf{Z}_m}; \mathcal{R}_+^m\right). \tag{6.29}$$

Below we shall use a shorter notation $\mathbb{P}_\beta$ for $\mathbb{P}_{\beta,0}$.

Following Subsection 7.1 in [25] let us describe in more detail the distribution of steps W under $\mathbb{P}_\beta$. The fact that here we take the cone $\mathcal{K}$ to be symmetric with respect to the e_1-axis simplifies the exposition. In particular, W has a symmetric distribution. The analysis of [25] could be summarized as follows:

$$\begin{aligned} 1 - p_\beta := \mathbb{P}_\beta(\mathsf{W} = 0) = 1 - \mathrm{O}\left(\mathrm{e}^{-\beta}\right),\ \mathbb{P}_\beta(\mathsf{W} = \pm 1) = \mathrm{O}\left(\mathrm{e}^{-\beta}\right) \\ \text{and, for } z \neq 0, \pm 1,\ \mathbb{P}_\beta(\mathsf{W} = z) \leq p_\beta \mathrm{e}^{-\nu\beta|z|} \end{aligned} \tag{6.30}$$

There is a natural Wald-type decomposition of random walk Z_k with i.i.d. steps distributed according to (6.30): Let $\xi_1, \xi_2, \ldots$ be i.i.d. Bernoulli random variables with probability of success p_β, and let U_k be another independent i.i.d. sequence with

$$\mathbb{P}_\beta(\mathsf{U} = z) = p_\beta^{-1}\mathbb{P}_\beta(\mathsf{W} = z) \text{ for } z \neq 0. \tag{6.31}$$

Set $\mathsf{M}_n = \sum_1^n \xi_i$. Then Z_n could be represented as

$$\mathsf{Z}_n = \sum_{k=1}^{\mathsf{M}_n} \mathsf{U}_k := \mathsf{Y}_{\mathsf{M}_n}, \tag{6.32}$$

where Y_k is a random walk with i.i.d. steps $\mathsf{U}_1, \mathsf{U}_2, \ldots$. With a slight abuse of notation we continue to use $\mathcal{R}_+^n$ for the corresponding event for Y_k-walk.

Let $y \in \mathbb{N}$ and consider $\sum_n \mathbb{P}_\beta\left(\hat{\mathcal{R}}_+^n; \mathsf{Z}_n = y\right)$. Let $\hat{\mathcal{L}}_y$ be the event that y is a strict ladder height of Z, or equivalently, of Y. Let $\hat{N}(y)$ be the number of strict ladder heights $v \leq y$. Then (see Subsection 7.3 of [25] for more detail),

$$\mathbb{P}_\beta\left(\hat{\mathcal{R}}_+^n; \mathsf{Z}_n = y\right) = \mathbb{P}_\beta\left(\hat{\mathcal{L}}_y; \mathsf{Z}_n = y\right) = \frac{1}{n}\mathbb{E}_\beta\left(\hat{N}(y); \mathsf{Z}_n = y\right) \leq \frac{y}{n}\mathbb{P}(\mathsf{Z}_n = y). \tag{6.33}$$

Hence,

$$\sum_n \mathbb{P}_\beta\left(\hat{\mathcal{R}}_+^n; \mathsf{Z}_n = y\right) \leq y \sum_{\ell=1}^{\infty} \mathbb{P}_\beta(\mathsf{Y}_\ell = y) \sum_{n=1}^{\infty} \frac{1}{n}\mathbb{P}_\beta(\mathsf{M}_n = \ell) \tag{6.34}$$

Now, since M is a process with Bernoulli steps, there is a combinatorial identity:

$$\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{P}_\beta(\mathsf{M}_n=\ell)=\frac{1}{\ell}\sum_{n=\ell}^{\infty}\binom{n-1}{\ell-1}p_\beta^\ell\left(1-p_\beta\right)^{n-\ell}=\frac{1}{\ell}.$$

We claim that there exists $c_1<\infty$ such that

$$\sum_{\ell=1}^{\infty}\frac{1}{\ell}\mathbb{P}_\beta(\mathsf{Y}_\ell=y)\le\frac{c_1}{y} \tag{6.35}$$

holds for every $y\in\mathbb{N}$ and all β sufficiently large. Note that (6.35) would imply that

$$\sum_n\mathbb{P}_\beta\left(\hat{\mathcal{R}}_+^n;\mathsf{Z}_n=y\right)\le c_1 \tag{6.36}$$

for all β sufficiently large.

In its turn (6.35) follows from routine estimates on characteristic functions. Let ϕ_β be the characteristic function of U in (6.31). By direct computation,

$$\sum_{\ell=1}^{\infty}\frac{1}{\ell}\mathbb{P}_\beta(\mathsf{Y}_\ell=y)=\frac{1}{y}\cdot\frac{1}{2\pi}\int_0^{2\pi}F_\beta(\theta)\left(\sum_1^{y-1}\mathrm{e}^{-\mathrm{i}\mathsf{k}\theta}\right)\mathrm{d}\theta=\frac{1}{y}\sum_1^{y-1}\hat{F}_\beta(k), \tag{6.37}$$

where,

$$F_\beta(\theta)=F_\beta\left(\mathrm{e}^{i\theta}\right)=\frac{1-\mathrm{e}^{i\theta}}{1-\phi_\beta(\theta)}\phi_\beta'(\theta), \tag{6.38}$$

and $\hat{F}_\beta$ is its Fourier transform. In view of the exponential decay of probabilities in (6.30), there exists $r>0$, such that F_β have uniformly bounded (in large β) analytic extension to the complex annulus $\{\mathsf{z}\in\mathbb{C}\ :\ |\mathsf{z}|\in(1,1+r)\}$. Hence, $\hat{F}_\beta(k)$ tend to zero uniformly exponentially fast, and in particular $\sum_1^\infty\left|\hat{F}_\beta(k)\right|$ is uniformly bounded in β large. (6.35) follows.

Let us turn to the second term

$$\sum_m\mathbb{E}_\beta\left(\mathrm{e}^{-\nu\beta\mathsf{Z}_m};\mathcal{R}_+^m\right)=\sum_{y\ge0}\mathrm{e}^{-\nu\beta y}\sum_m\mathbb{P}_\beta\left(\mathcal{R}_+^m;\mathsf{Z}_m=y\right) \tag{6.39}$$

in (6.29). We claim that there exists a finite constant c_2 such that

$$\sum_m\mathbb{P}_\beta\left(\mathcal{R}_+^m;\mathsf{Z}_m=y\right)\le\frac{c_2(y+1)}{p_\beta}. \tag{6.40}$$

for all $y\ge0$ and β large.

Note that (6.40) would imply that

$$\sum_m\mathbb{E}_\beta\left(\mathrm{e}^{-\nu\beta\mathsf{Z}_m};\mathcal{R}_+^m\right)\le\frac{c_3}{p_\beta}.$$

By (6.29) and (6.36) this would mean that

$$\sup_{z\geq 0}\sum_n \mathbb{E}_{\beta,z}\left(\mathrm{e}^{-\nu\beta \mathsf{Z}_n};\mathcal{R}_+^n\right)\leq \frac{c_1c_3}{\left(1-\mathrm{e}^{-\nu\beta}\right)p_\beta}$$

Since by (6.30) the order of $p_\beta = \mathrm{O}\left(\mathrm{e}^{-\beta}\right)$, the limit $\lim_{\beta\to\infty}\frac{\mathrm{e}^{-2\chi\beta}}{p_\beta}=0$ whenever $\chi>1/2$. Hence (6.28).

It, therefore, remains to verify (6.40). Let N_y be the number of non-strict ladder times for heights between 0 and y. Let M_y be the same variable for the random walk Y. Clearly,

$$N_y=\sum_{i=1}^{M_y}\zeta_i,$$

where ζ_i-s are i.i.d. Geo(p_β)-random variables. Proceeding as in the proof of (6.36), and in particular relying on Subsection 7.1 in [25], we estimate:

$$\begin{aligned}\sum_m \mathbb{P}_\beta\left(\mathcal{R}_+^m;\mathsf{Z}_m=y\right)&\leq \frac{1}{p_\beta}\sum_n\frac{1}{n}\mathbb{E}_\beta\left(M_y;\mathsf{Y}_n=y\right)\\&\leq \frac{\left(\mathbb{E}_\beta M_y^k\right)^{1/k}}{p_\beta}\sum_n\frac{1}{n}\left(\mathbb{P}_\beta\left(\mathsf{Y}_n=y\right)\right)^{(k-1)/k}.\end{aligned}\tag{6.41}$$

Applying Lemma 22 in [25] it is straightforward to see that under (6.30) for any $k\in\mathbb{N}$ there is a finite constant c_k such that

$$\left(\mathbb{E}_\beta M_y^k\right)^{1/k}\leq c_k(y+1)$$

for all $y\geq 0$ and β sufficiently large. On the other hand, again under (6.30), it is straightforward to check that

$$\max_y \mathbb{P}_\beta\left(\mathsf{Y}_n=y\right)\lesssim n^{-1/2}.$$

uniformly in n and in β large. Hence (6.40), and we are done. □

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