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A continuum of pure states in the Ising model on a halfplane

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Abstract

We study the homogeneous nearest-neighbor Ising ferromagnet on the right half plane with a Dobrushin type boundary condition — say plus on the top part of the boundary and minus on the bottom. For sufficiently low temperature T, we completely characterize the pure (i.e., extremal) Gibbs states, as follows. There is exactly one for each angle $\theta \in [-\pi/2, +\pi/2]$; here θ specifies the asymptotic angle of the interface separating regions where the spin configuration looks like that of the plus (respectively, minus) full-plane state. Some of these conclusions are extended all the way to $T = T_c$ by developing new Ising exact solution results – in particular, there is at least one pure state for each θ.

1 Introduction

In equilibrium statistical mechanics and particularly in the study of phase transitions, an important role is played by the collection of Gibbs states (or measures or distributions) at temperature T of a fully infinite system, such as the standard (homogeneous nearest-neighbor) Ising ferromagnet on an infinite lattice, such as $\mathbb{Z}^d$. The precise mathematical formulation of such Gibbs states was pioneered by Dobrushin, Lanford and Ruelle [D1, LR].

A nice discussion of the connection between the phenomenon of phase transitions and nonuniqueness of infinite-volume Gibbs distributions may be found in the Introduction section of [G]. The pure phases of a physical system correspond to the extremal (also called pure) Gibbs states — the ones which cannot be decomposed further as convex combinations of other states. Thus for models such as the standard Ising ferromagnet on $\mathbb{Z}^d$, considerable effort has been devoted to characterizing the collection of all pure states.

One of the major accomplishments in the field is a complete characterization for $\mathbb{Z}^2$ [A, H] that there are exactly two pure states for $T < T_c$ (corresponding respectively to plus and to minus boundary conditions). Another major accomplishment was the demonstration by Dobrushin [D2] that for $\mathbb{Z}^d$ with $d \geq 3$ at low enough T, there are other, non-translation invariant, states which display an interface (parallel to a coordinate axis). Together with two translation-invariant states they form a countable set of extremal states. Other results concern the analysis of pure states on infinite graphs such as homogeneous trees [B, BG, GRS, GMRS].

The subject of this paper is one which has not been much investigated previously — namely, the analysis of the pure states when the underlying lattice or graph is infinite, but with a nontrivial boundary. In particular, we consider a half plane — say, the right half-plane. Here one may specify a boundary condition on the (left) boundary of the half-plane. In the absence of boundary conditions (i.e., with free boundary conditions), the pure states would mimic those of the full plane — a single pure state for $T \geq T_c$ and exactly two for $T < T_c$.

If one instead specified a plus (or similarly, minus) left boundary, there would be a unique pure state for all T. Instead we use a Dobrushin type boundary condition — plus on the top and minus on the bottom of the left boundary. Our results prove that in this case, there are uncountably many pure states – exactly one for for each $\theta \in [-\pi/2, +\pi/2]$ where θ is the asymptotic angle of an interface that starts at the origin between plus and minus like regions of the half-plane. We think these are all the pure states of our model, though we prove this only for temperatures low enough. Similar results were formulated earlier by one of us in [DS] and called the 'Meniscus theorem', but without a proof.

We believe that the situation in higher dimensions d is somewhat similar but with possible new phenomena such as needing at least $d - 2 > 1$ continuous variables to parameterize the half-space pure states. For example in the $d = 3$ half-space, $\{(s, t, u) : s > 0\}$ one may take a boundary condition in the $(t, u))$-plane consisting of four alternating plus and minus regions — say, for simplicity, the four quadrants separated by the two interface lines along the t and u axes. Then there could be a family of mutually singular Gibbs states, each with two approximately planar interfaces, emanating from the two boundary interface lines. It should take two continuous angular variables, say θ_1 and θ_2, to parameterize this family, each giving the asympotic angular deviation of one of the two interfaces from the corresponding coordinate plane (the (t, s) or the (u, s) plane).

Other models with uncountably many pure states can be constructed on homogeneous trees that have as much surface as bulk volume (in the thermodynamic limit), see, e.g., [GMRS]. Examples of a different nature are provided by models with quasiperiodic order, as considered, e.g., in [vEM, vEMZ], and stacked models [WF]. We also note that models in half-spaces were studied earlier — see, e.g., [Ba, S].

The remaining sections of the paper are organized as follows. In Section 2, we give a precise definition of our halfplane Ising model and in Section 3 the

main results are stated. Proofs for the simplest case of ground states where $T = 0$ are given in Section 4 while the proofs for temperature $T > 0$ but small enough for cluster expansions to apply are presented in Section 5. Finally, the analysis valid for all $T < T_c$ is given in Sections 6 and 7 of the paper; there we show that there is at least one pure state with an interface at angle θ for each $\theta \in [-\pi/2, +\pi/2]$. We have no proof that these are all the pure states.

2 Definition of the model

We consider the Ising model, defined by the usual Ising Hamiltonian

$$H(\sigma) = -\sum_{x,y \text{ n.n.}} \sigma_x \sigma_y, \tag{1}$$

but on $\mathbb{Z}^{2*}_+ = \left\{x = (s,t) : s,t \in \mathbb{Z}^1 + \frac{1}{2}, s > 0\right\}$, the halfplane (of the dual lattice). We will be interested in a specific boundary condition – which in our case is a configuration $\sigma^\pm$ on $\partial\mathbb{Z}^2_+ \equiv \left\{\left(-\frac{1}{2}, t\right) : t \in \mathbb{Z}^1 + \frac{1}{2}\right\}$ – defined by

$$\sigma^\pm_{\left(-\frac{1}{2},t\right)} = \begin{cases} +1 & \text{if } t > 0 \\ -1 & \text{if } t < 0 \end{cases}.$$

The corresponding relative Hamiltonian $H(\sigma|\sigma^\pm)$ is given, accordingly, by

$$H\left(\sigma|\sigma^\pm\right) = -\sum_{x,y \in \mathbb{Z}^2_+ \text{ n.n.}} \sigma_x \sigma_y - \sum_{t>0} \sigma_{\left(-\frac{1}{2},t\right)} + \sum_{t<0} \sigma_{\left(-\frac{1}{2},t\right)}.$$

We will also consider the strips

$$\mathbb{Z}^2_N = \left\{x = (s,t) : s,t \in \mathbb{Z}^1 + \frac{1}{2}, 0 < s < N\right\},$$

and we will need different boundary conditions on their right boundaries. It is convenient to view the boundary condition as a configuration on all the lattice $\mathbb{Z}^2_+$, and we will use the following family σ^θ:

$$\sigma^\theta_{(s,t)} = \begin{cases} +1 & \text{if } \frac{t}{s} \geq \tan\theta \\ -1 & \text{if } \frac{t}{s} < \tan\theta \end{cases}, \ \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

We denote by $\langle * \rangle^\theta_N$ the Gibbs state in the strip $\mathbb{Z}^2_N$ with the boundary condition $\sigma^\pm$ on its left edge and with the boundary condition σ^θ on its right edge.

3 Main theorems for low temperature

Theorem 1 *Let the temperature $T = \beta^{-1}$ be low enough. Then for every $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ there exists a Gibbs state $\langle * \rangle^{\theta}$ of the model* (1) *at the temperature T, such that*

$$\left\langle \sigma_{(s,t)} \right\rangle^{\theta} \to \left\{ \begin{array}{c} +m^{*}(\beta) \\ -m^{*}(\beta) \end{array} \right\} \text{ when } s,t \to \infty \text{ s.t. } \left\{ \begin{array}{c} \liminf \\ \limsup \end{array} \right\} \left(\frac{t}{s}\right) \left\{ \begin{array}{c} > \\ < \end{array} \right\} \tan\theta. \tag{2}$$

It follows that the states $\langle * \rangle^{\theta}$ with different θ's are mutually singular.

The cases $\theta = \pm\frac{\pi}{2}$ require obvious modifications, as given in the next theorem.

Theorem 2 *Let the temperature $T = \beta^{-1}$ be low enough. Then for $\theta = \pm\frac{\pi}{2}$ there exist two Gibbs states $\langle * \rangle^{\pm\frac{\pi}{2}}$ of the model* (1) *at temperature T, such that for any $C \in (-\infty, +\infty)$*

$$\left\langle \sigma_{(s,t)} \right\rangle^{\pm\frac{\pi}{2}} \to \mp m^{*}(\beta) \text{ for } s,t \to \infty \text{ s.t. } \frac{t}{s} \to C. \tag{3}$$

At the same time, there exists a function $t_0(s) \to +\infty$ as $s \to \infty$, s.t. for any function $t(s) \geq t_0(s)$

$$\left\langle \sigma_{(s,\pm t(s))} \right\rangle^{\pm\frac{\pi}{2}} \to \pm m^{*}(\beta) \text{ as } s \to \infty. \tag{4}$$

In fact, $t_0(s)$ can be any function such that $\frac{t_0(s)}{s^2} \to \infty$ as $s \to \infty$.

Theorem 3 *Let the temperature $T = \beta^{-1}$ be low enough. For $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ the state $\langle * \rangle^{\theta}$ can be obtained as a limit,*

$$\langle * \rangle^{\theta} = \lim_{N\to\infty} \langle * \rangle_N^{\theta_N},$$

where θ_N is any sequence of angles satisfying the condition: $\lim_{N\to\infty} \theta_N = \theta$.

Theorem 4 *Let the temperature $T = \beta^{-1}$ be low enough, and μ be any half-plane Gibbs state with boundary condition $\sigma^{\pm}$. Then there is a (unique) probability measure d_μ on the segment $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that*

$$\mu = \int \langle * \rangle^{\theta} d_\mu(\theta).$$

This implies that the family $\left\{\langle * \rangle^{\theta}, \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right\}$ of states coincides with the family of all extremal Gibbs states of the half-plane Ising model with boundary condition $\sigma^{\pm}$.

4 Zero temperature case

We start by considering the simplest case of zero temperature. Here, some of the above theorems have to be modified slightly. Namely, the states $\langle * \rangle^{+\frac{\pi}{2}}$ and $\langle * \rangle^{-\frac{\pi}{2}}$ of Theorem 2 become trivial; they are each concentrated on a single ground state configuration, $\sigma \equiv +1$ or $\sigma \equiv -1$. Another trivial state is $\langle * \rangle^{0}$ – it is also concentrated on a single configuration, $\sigma^{\theta=0}$ which is $+1$ (resp., -1) for $t > 0$ (resp., $t < 0$); all other states $\langle * \rangle^{\theta}$ are supported on infinitely many ground state configurations.

The zero-temperature case is simpler because the relevant configurations have only open contours (i.e., there are no loop contours), which do not interact, except for a non-intersection condition — see below.

Let $V_N \subset \mathbb{Z}^2$ be the box

$$V_N = \{(x, y) : 0 \leq x \leq N, -N \leq y \leq N\}$$

and let

$$V_N^* = \left\{(s, t) \in \mathbb{Z}^{2*} : 0 < s < N, -N < t < N\right\}$$

with boundary ∂V_N^*. A spin boundary condition $\sigma_{\partial V_N^*}$ on ∂V_N^* is specified by a collection of an even number of distinct points $z_1, ..., z_{2k} \in \partial V_N$, where the configuration $\sigma_{\partial V_N^*}$ changes its value from ± 1 to ∓ 1. In our case we can put z_1 to be the point $(0, 0)$, and we have to suppose that all the other points z_i are in

$$\partial^- V_N \; = \; \partial V_N \setminus \{(x, y) : x = 0, -N < y < N\}.$$

Every spin configuration σ on V_N^* with this boundary condition defines a partition p of the set $\{z_1, ..., z_{2k}\}$ into pairs $\left\{z_i, z_{p(i)}\right\}$, in the following way: among the Peierls contours of σ there are precisely k open contours γ_i, $1 \leq i \leq k$, with $\cup_i \partial(\gamma_i) = \{z_1, ..., z_{2k}\}$; the rest of the contours of σ are closed contours, i.e., loops. Then the partition into pairs is defined by

$$\left\{z_i, z_{p(i)}\right\} = \partial(\gamma_i).$$

If the configuration σ is a ground state configuration, then it has no loops, while the corresponding partition has minimal length: for any other partition q we have

$$\sum_i \left|z_i - z_{p(i)}\right|_1 \leq \sum_i \left|z_i - z_{q(i)}\right|_1,$$

where $|*|_1$ denotes L_1-distance. Moreover,

$$\sum_i |\gamma_i| = \sum_i \left| z_i - z_{p(i)} \right|_1 .$$

Without loss of generality we can suppose that the indices are chosen such that the partition $\{\{z_1, z_2\}, ..., \{z_{2k-1}, z_{2k}\}\}$ is a ground state partition.

We will argue now that all the straight-line segments $[z_3, z_4], ..., [z_{2k-1}, z_{2k}] \subset \mathbb{R}^2$ are far away from the segment $[z_1, z_2]$ 'in the bulk'. It is enough to consider the case $k = 2$. Suppose the point $w \in [z_1, z_2]$ at (L_2) distance at least $c_1 N$ from z_2, is at distance $c_2 N$ from the segment $[z_3, z_4]$, see Fig. 1. If $\arcsin \frac{c_2}{c_1} < \alpha$, then α has to exceed $\arctan \frac{1}{2}$ – otherwise the ground state condition is violated. Hence, $c_2 > c_1 \sin\left(\arctan \frac{1}{3}\right)$.

By the same token, all the segments $[z_{2l-1}, z_{2l}]$ with $l > 1$ cannot pass too close to the origin:

$$[z_{2l-1}, z_{2l}] \cap V_{N/4} = \varnothing.$$

Let now $\langle * \rangle_{\{z_1,z_2\},...,\{z_{2k-1},z_{2k}\}}$ be the ground state in V_N, corresponding to the ground state partition $\{\{z_1, z_2\}, ..., \{z_{2k-1}, z_{2k}\}\}$, and σ be a ground state spin configuration from that state $\langle * \rangle_{\{z_1,z_2\},...,\{z_{2k-1},z_{2k}\}}$. Let $\gamma_1, ..., \gamma_k$ be its collection of open contours. Then the probability in $\langle * \rangle_{\{z_1,z_2\},...,\{z_{2k-1},z_{2k}\}}$

$$\Pr\left(\{\gamma_2, ..., \gamma_k\} \cap V_{N/4} \neq \varnothing\right) \to 0$$

as $N \to \infty$, because for each $\varepsilon > 0$ and each l the probability

$$\Pr\left(\operatorname{dist}\left(\gamma_l, (z_{2l-1}, z_{2l})\right) > N^{1/2+\varepsilon}\right) \to 0 \text{ as } N \to \infty. \tag{5}$$

Summarizing, we can say that the distance between the projected states,

$$\operatorname{dist}\left(\left\langle *|_{V_{N/4}} \right\rangle_{\{z_1,z_2\},...,\{z_{2k-1},z_{2k}\}}, \left\langle *|_{V_{N/4}} \right\rangle_{\{z_1,z_2\}}\right) \to 0$$

as $N \to \infty$, where $z_i = z_i(N), i = 2, ..., 2k \in \partial^- V_N$. Therefore, all the ground states of our model are among the limit points of the states $\langle * \rangle_{\{z_1,z_2(N)\}}$.

In the case when the sequence $z_2(N)$ is in the ray $r_\theta = \{(s,t) : t = s \tan\theta\}$ (or, rather, to its neighborhood of radius 1/2, due to the rounding off to the closest integer point) the existence of the limit ground state $\langle * \rangle^\theta = \lim_{N\to\infty} \langle * \rangle_{\{z_1,z_2(N)\}}$ is evident. It is supported by the set of infinite staircase contours γ starting from $(0,0)$ and having asymptotic slope θ. The probability in the

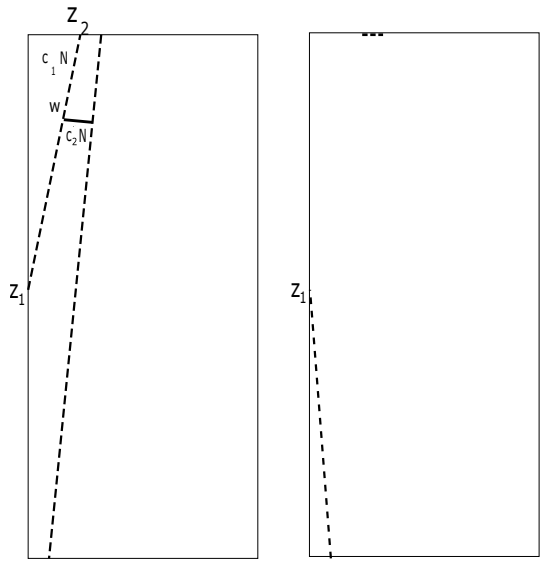

Figure 1: The surgery on long paths.

state $\langle * \rangle^{\theta}$ that the initial piece $\gamma_N = \gamma \cap Z_N^2$ of the contour γ has n vertical steps at n prescribed locations (not necessarily different) on the Ox axis is given by $p^n (1-p)^N$, with $p = p(\theta) = \tan\theta / (1 + \tan\theta)$.

In case the sequence $\frac{z_2(N)}{N}$ has at least two different subsequence limit points, the sequence of ground states $\langle * \rangle_{(z_1, z_2(N))}$ also has at least two different limit points, due to (5) ,and so does not have a unique limit.

Finally, suppose the sequence $\frac{z_2(N)}{N}$ has a limit, and so defines the limiting ray r_θ, $\theta = \theta[(z_2(*))]$. Let us show that $\lim_{N\to\infty} \langle * \rangle_{(z_1, z_2(N))} = \langle * \rangle^{\theta}$. We suppose additionally that $\theta \in \left(0, \frac{\pi}{4}\right)$, since the case where $\theta \in [\pi/4, \pi/2)$ can be handled similarly. Let $\bar{z}_2(N) \equiv \left(\bar{a}(N), \bar{b}(N)\right) = \lfloor r_\theta \cap \partial V_N \rfloor$ be the 'integer part' of $r_\theta \cap \partial V_N$, and let $z_2(N) \equiv (a(N), b(N)) = \bar{z}_2(N) + o(N)$ as $N \to \infty$. Let us fix a point $(m, n) \in \mathbb{R}^2$ with $m, n > 0$; for all N large enough we have $(m, n) \in V_N$. To simplify our exposition, we consider the case when $a(N) = \bar{a}(N)$, while $b(N) \geq \bar{b}(N)$, so $b(N) = \bar{b}(N) + o(N)$. The relevant 'ratio of partition functions' in our case is just the ratio of binomial coefficients,

$$\frac{\binom{a(N) + b(N) - m - n}{b(N) - n}}{\binom{a(N) + \bar{b}(N) - m - n}{\bar{b}(N) - n}},$$

which goes to 1 as $N \to \infty$, for any m, n (though not uniformly in m, n) in case $b(N) = \bar{b}(N) + o(N)$. That follows from Stirling's formula.

In words, the reason for the identity $\lim_{N\to\infty} \langle * \rangle_{(z_1, z_2(N))} = \langle * \rangle^{\theta}$ is that the probability of having the 'extra' $o(N)$ (uniformly distributed) vertical segments (which one has to add to $\bar{b}(N)$ in order to get $b(N)$ of them) to be located at any of the first m positions of the segment $[1, a(N)]$, goes to zero as $N \to \infty$.

5 Low temperature case

For $T > 0$ the problem becomes more involved, since the contours $\gamma_1, ..., \gamma_k$ are interacting. We first describe their joint distribution. The following formula follows from the cluster expansion technique; it can be found in

[DKS]. The weight of our family $\Gamma = \{\gamma_1, ..., \gamma_k\}$ is given by

$$w(\Gamma) = \exp\left\{-\beta|\Gamma| + \sum_{\substack{C\subset V_N:\\ C\cap\Delta_\Gamma\neq\varnothing}} \Phi(C)\right\}, \tag{6}$$

where $\beta = \frac{1}{T}$ is the inverse temperature, $|\Gamma| = |\gamma_1|+...+|\gamma_k|$ is the total length of our contours, while the term $\sum_{C:C\cap\Delta_\Gamma\neq\varnothing}\Phi(C)$, which we explain now, contains the interaction between γ's. The sum is taken over all connected subsets ($\equiv$clusters) $C \subset V_N$. The notation $C \cap \Delta_\gamma \neq \varnothing$ essentially means that C intersects the union Γ of the contours γ_i, while the function $\Phi(C)$ (which, of course, depends also on β) has the following properties:

- Decay: for all β sufficiently large,

$$|\Phi(C)| \leq \exp\{-2\beta(\mathrm{diam}_\infty(C)+1)\}. \tag{7}$$

- Symmetry: the function Φ is translation invariant.

In what follows we will rely on the results of [DKS]. However, there are some important differences between our situation and the one treated there. In [DKS] the analysis is restricted to the case of periodic boundary conditions, while here our contours γ are in the box V_N and so interact with its boundary ∂V_N: – inspecting the relation (6), we see that the part of the contour in the bulk lives in a different potential landscape than the part near the boundary. Indeed, let λ be some fragment of γ, and suppose the cluster $C \subset V_N$ intersects λ and thus contributes to (6) an amount $\Phi(C)$. Let us shift λ by a vector $s \in \mathbb{Z}^2$, in such a way that $\lambda + s$ is still in V_N, but $C + s$ is not. Then the corresponding contribution $\Phi(C+s)\,(=\Phi(C))$ is missing near the boundary. In case $\Phi(C) < 0$ that would mean that γ is effectively attracted to the wall ∂V_N. The study of this issue turns out to be quite complicated technically; it is done in [IST]. A main result of [IST] is that even if such an attraction is present, it is beaten by the entropic repulsion of γ from the wall ∂V_N, provided (7) holds. (In fact, the result of [IST] is more precise: the entropic repulsion beats an attraction of strength $-\exp\{-\chi\beta(\mathrm{diam}_\infty(C)+1)\}$ provided $\chi > \frac{1}{2}$, but can fail against it for $\chi < \frac{1}{2}$.)

Let us show that the states $\langle * \rangle^{\theta}$ are distinct for different $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. For that we will use the following result of [DKS], contained in Theorem 4.16 there (see also relation (2.5) and the Theorem 29 of [IST], where an error in [DKS] is corrected). To lighten the notation, we consider only the case when $\theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

Let us introduce the following cigar-shaped subset $U_{N,d,\varkappa} \subset V_N$:

$$U_{N,d,\varkappa} = \left\{ (x,y) \in V_N : |y - x \tan\theta| \leq d \left(\frac{x(N-x)}{N} \right)^{\frac{1}{2}+\varkappa} \right\},$$

where $d, \varkappa > 0$. Let

$$U_{N,d,\varkappa,R} = U_{N,d,\varkappa} \cup D_R(0,0) \cup D_R(N, N\tan\theta)$$

be the union of $U_{N,d,\varkappa}$ and two disks of radius R centered at the endpoints of the line segment $((0,0),(N,N\tan\theta))$.

Every configuration from the state $\langle * \rangle_N^{\theta}$ possesses exactly one open contour, which we denote by γ. Then for any $\varkappa > 0$ the probability of the event

$$\gamma \subset U_{N,d,\varkappa,R},$$

computed in this state $\langle * \rangle_N^{\theta}$, goes to 1, as R increases, uniformly in N. That shows that $\langle * \rangle^{\theta_1} \neq \langle * \rangle^{\theta_2}$ for different θ's.

Now we will show that $\lim_{N\to\infty} \langle * \rangle_N^{\theta'(N)} = \langle * \rangle^{\theta}$ if $\lim_{N\to\infty} \theta'(N) = \theta$. To see that we will check that the states $\langle * \rangle^{\theta'}$ and $\langle * \rangle^{\theta}$ are absolutely continuous with respect to each other. So we introduce the partition function

$$\begin{aligned} Z(\theta, N) &= \sum_{\gamma:(0,0)\rightsquigarrow(N,N\tan\theta)} w(\gamma) \\ &\equiv \sum_{\gamma:(0,0)\rightsquigarrow(N,N\tan\theta)} \exp\left\{ -\beta|\gamma| + \sum_{\substack{C\subset V_N: \\ C\cap\Delta_\gamma\neq\varnothing}} \Phi(C) \right\}, \end{aligned}$$

where the sum is taken over all paths γ in V_N, connecting $(0,0)$ to $(N, N\tan\theta)$. We denote by $Z(\theta', N)$ the partition function $Z(\theta'(N), N)$. Let us fix an integer n, and let η be a path in V_n, connecting $(0,0)$ to a point $(k,l) \in \partial V_n$. We will need the partition functions

$$Z(\theta, N, \eta) = \sum_{\substack{\gamma:(k,l)\rightsquigarrow(N,N\tan\theta) \\ \eta\cap\gamma=\varnothing}} w(\eta\cup\gamma),$$

where $\eta \cap \gamma = \varnothing$ means that the contours γ and η are compatible, so that their concatenation $\eta \cup \gamma$ is a legitimate contour, $\eta \cup \gamma : (0,0) \rightsquigarrow (N, N \tan\theta)$. The partition function $Z(\theta', N, \eta)$ is defined in the obvious way. Our claim boils down to showing that the cross-ratio

$$\frac{Z(\theta', N, \eta)}{Z(\theta', N)} \frac{Z(\theta, N)}{Z(\theta, N, \eta)} \tag{8}$$

is bounded from above and below, uniformly in N.

The bounds on the above partition functions, provided by the analysis of [DKS] – see relations (4.11.3), (4.12.3) there – show that

$$\ln Z(\theta, N) = -\beta \tau_\beta(\theta) \frac{N}{\cos\theta} - \frac{1}{2} \ln \frac{N}{\cos\theta} + O(1), \tag{9}$$

where $\tau_\beta(*)$ is the surface tension function. (Unlike the situation of a 1D Gibbs field with finite spin state space, where the log of the partition function $\ln Z(N) = -\beta f + O(1)$, here we have the additional *universal* 'Ornstein-Zernike' term $-\frac{1}{2}\ln N$, see [CIV].) Plugging (9) into (8) and using analyticity of the function $\tau_\beta(*)$ in θ, we see that the terms which grow in N cancel each other; thus we establish the boundedness of (8) in N. (Of course, it is not uniform in η.)

The same argument shows that the cross-ratio

$$\frac{Z(\theta, N_1, \eta)}{Z(\theta, N_1)} \frac{Z(\theta, N_2)}{Z(\theta, N_2, \eta)}$$

is bounded from above and below uniformly in N_1, N_2 (again, not uniformly in η), which shows the existence of the weak limit $\lim_{N\to\infty} \langle * \rangle_N^\theta = \langle * \rangle^\theta$.

Finally we show that every state of our system is a mixture of various $\langle * \rangle^\theta$'s. This will be obtained as an adaptation of the arguments from the previous section. Namely, we will show that the contour γ_1 is the only contour visible in any finite vicinity of the point $(0,0)$, with overwhelming probability as $N \to \infty$. To that end we will show that the total length of the collection $\gamma_1, ..., \gamma_k$ is sufficiently close to its minimal possible value, with high probability, provided the temperature is low enough. That would imply that among the contours $\gamma_1, ..., \gamma_k$ there is only one – γ_1 – which comes into the vicinity of the point $(0,0)$. To this end we introduce another collection of open Ising contours, $\nu_1, ..., \nu_k$ in V_N, which has the same set of end-points $z_1, ..., z_{2k}$, and which has minimal total length $|\nu_1| + ... + |\nu_k|$ among all such collections. (In other words, the collection $\nu_1, ..., \nu_k$ defines the ground state spin configuration.)

Lemma 5 *Let $s > 0$, and $L > (|\nu_1| + ... + |\nu_k|)(1+s)$. Then there exists $\beta(s)$ such that*

$$\Pr(|\gamma_1| + ... + |\gamma_k| > L) \leq \exp\{-(\beta - \beta(s))[L - (|\nu_1| + ... + |\nu_k|)]\}.$$

Proof. Consider the set of bonds D, which is the symmetric difference $\{\gamma_1 \cup ... \cup \gamma_k\} \triangle \{\nu_1 \cup ... \cup \nu_k\}$. D is a collection of closed contours, and if $|\gamma_1| + ... + |\gamma_k| > L$, then $|D| > L - (|\nu_1| + ... + |\nu_k|)$. Now we define the Peierls transformation, which from every configuration σ containing the contours $\gamma_1, ..., \gamma_k$ produces another configuration $\pi_D(\sigma)$, which satisfies the same boundary condition $\{z_1, ..., z_{2k}\}$ but whose energy is smaller by at least $L - (|\nu_1| + ... + |\nu_k|)$. Moreover, if $\sigma \neq \sigma'$, then $\pi_D(\sigma) \neq \pi_D(\sigma')$. The construction is the following: if σ corresponds to the set $B(\sigma)$ of bonds forming all the contours of σ, then $\pi_D(\sigma)$ corresponds to the bond set $B(\sigma) \triangle D$. That correspondence proves the estimate

$$\Pr(\{\sigma : \gamma_1 \cup ... \cup \gamma_k \subset B(\sigma)\}) \leq \exp\{-\beta(|\gamma_1| + ... + |\gamma_k| - |\nu_1| - ... - |\nu_k|)\}.$$

Assuming $L > (|\nu_1| + ... + |\nu_k|)(1+s)$ we have $L - (|\nu_1| + ... + |\nu_k|) > s'L$ with $s' > 0$. Therefore the entropy factor 3^L is beaten by the energy gain $L - (|\nu_1| + ... + |\nu_k|)$, and the proof follows. ■

From the above lemma we conclude that the contours $\gamma_2, ..., \gamma_k$ stay close to the ground state contours, so the results of the previous section apply.

The only items left to prove are the properties of the states $\langle * \rangle^{\pm\frac{\pi}{2}}$. The relation (3) is proven in the same way as was used in proving that the states $\langle * \rangle^{\theta_1}$ and $\langle * \rangle^{\theta_2}$ are different. What is needed for (4) is that the contour γ_1, distributed according to the state $\langle * \rangle^{+\frac{\pi}{2}}$ fluctuates away from Oy (the positive y-axis). The precise way in which it does this can be analysed by using the *methods* of the paper [IST]. But even the *results* of [IST] show that it does fluctuate away from Oy. In what follows we will prove one version of this phenomenon. First, we introduce some notation.

Let $\gamma \in V_N$ be a contour connecting the points $(0,0)$ and $(0,N)$. For every integer h define the set $Y_{N,h}(\gamma)$ by

$$Y_{N,h}(\gamma) = \{y \in [0,N] : \max\{m : [0,m] \cap \gamma = \varnothing\} > h\}.$$

In words, $Y_{N,h}(\gamma)$ is the set of locations where the contour γ is farther than h from Oy.

Proposition 6 *For all temperatures low enough and for every $h > 0$ we have*

$$\mathbb{E}_N\left(\frac{|Y_{N,h}(\gamma_1)|}{N}\right) \to 1 \text{ as } N \to \infty, \tag{10}$$

where the expectation is computed in the state $\langle * \rangle_N^{+\frac{\pi}{2}}$.

Proof. As we know from the main result of [IST], the limit

$$\begin{aligned} &\lim_{N\to\infty} -\frac{\ln\left(\sum\limits_{\substack{\gamma_1\subset V_N:\\ \gamma_1:(0,0)\rightsquigarrow(0,N)}} \exp\left\{-\beta|\gamma_1| + \sum\limits_{\substack{C\subset V_N:\\ C\cap\Delta_{\gamma_1}\neq\varnothing}} \Phi(C)\right\}\right)}{\beta N} \\ &= \tau_\beta\left(\frac{\pi}{2}\right), \end{aligned} \tag{11}$$

where $\tau_\beta\left(\frac{\pi}{2}\right)$ is the surface tension of the Ising model. This is so despite the fact that the contour γ_1 is confined to the right halfplane and despite the expression involving the clusters C entering (11). Moreover, the relation (11) would stay true even if we supplement the set of allowed clusters by some extra clusters – by $\bar{C}$'s, intersecting Δ_{γ_1} but not confined to the right halfplane. These auxiliary clusters $\bar{C}$ can have positive weights $\Phi\left(\bar{C}\right)$, thus introducing an extra attraction to the Oy axis. Still, the relation (11) holds, provided that

$$\left|\Phi\left(\bar{C}\right)\right| < \exp\left\{-\chi\beta(\mathrm{diam}_\infty(\bar{C})+1)\right\} \text{ with } \chi > \frac{1}{2}.$$

In particular, let us add to the set of clusters $\{C \subset V_N : C \cap \Delta_{\gamma_1} \neq \varnothing\}$ extra clusters $\bar{C}$, which are simply horizontal segments of length h, which intersect both the line Oy and the contour γ_1. Define their weight to be $\exp\{-\beta(h+1)\}$. According to [IST], we still have the same limit:

$$\lim_{N\to\infty} -\frac{1}{\beta N}\ln\left(\sum_{\substack{\gamma_1\subset V_N:\\ \gamma_1:(0,0)\rightsquigarrow(0,N)}}\right.$$

$$\left.\exp\left\{-\beta\left|\gamma_1\right|+\sum_{\substack{C\subset V_N:\\ C\cap\Delta_{\gamma_1}\neq\varnothing}}\Phi\left(C\right)+\exp\left\{-\beta(h+1)\right\}\left(N-\left|Y_{N,h}\left(\gamma_1\right)\right|\right)\right\}\right)$$

$$=\tau_\beta\left(\frac{\pi}{2}\right),$$

which proves (10) . ■

6 Main theorems valid for all $T < T_c$

We define for each $T = \beta^{-1} > 0$ and $\theta \in [-\pi/2, +\pi/2]$ a collection G_θ^T of Gibbs states of the model (1) at temperature T as follows. For $\theta \in (-\pi/2, +\pi/2)$ (resp., for $\theta = \pm\pi/2$), $\langle * \rangle$ is in G_θ^T if (2) is valid (resp., (3) is valid for any $C \in (-\infty, +\infty)$).

Theorem 7 *Let $T \in (0, T_c)$; then G_θ^T is nonempty for every $\theta \in [-\pi/2, +\pi/2]$. States from G_θ^T and $G_{\theta'}^T$ with $\theta \neq \theta'$ are mutually singular.*

The next theorem is analogous to Theorem 3 for low T given above, except that now (a) we do not know that all subsequence limits agree and (b) we treat $\theta = \pm\pi/2$ differently. The proofs of both theorems are then given together.

Theorem 8 *Let $T \in (0, T_c)$ and let θ_N and $\langle * \rangle_N^{\theta_N}$ be as in Theorem 3. Then for $\theta \in (-\pi/2, +\pi/2)$, every subsequence limit of of $\langle * \rangle_N^{\theta_N}$ belongs to G_θ^T. Let θ_m be a sequence in $(-\pi/2, +\pi/2)$ converging to $\pm\pi/2$ and let $\langle * \rangle^{\theta_m} \in G_{\theta_m}^T$. Then any subsequence limit as $m \to \infty$ of $\langle * \rangle^{\theta_m}$ is in $G_{\pm\pi/2}^T$.*

Proof. The proof of Theorems 7 and 8 is based on the planar Ising model exact calculations for the profile of an interface at angle θ given in the next section of the paper; earlier exact calculations on interface profiles may be

found in [AR, FFW, AU]. For those calculations, we note that one may obtain the states $\langle * \rangle_N^{\theta_N}$ in the strip that is infinite in the vertical t-coordinate (and of width N in the horizontal s-coordinate) by the following procedure. First take the finite region that is periodic in the t-coordinate with large but finite period K and choose boundary conditions on the left and right boundaries (at $s = 0$ and $s = N$) which require two interfaces as follows: all spins are $+1$ at both the left and right boundaries *except* for -1 spins when t is between a large negative value M and 0 (resp., between M and $N \tan\theta$) on the left (resp., the right) boundary. Then take the limit where first $K \to \infty$ and then $M \to -\infty$ so that the M-interface is eliminated, resulting in the $\langle * \rangle_N^{\theta_N}$ state on the $N \times \infty$ strip.

Now the results of the next section, in particular Equations (33)–(37) give an exact formula for the extension of the limit in (2) when $t - s\tan\theta$ is proportional to $\sqrt{s}$. When $(t - s\tan\theta)/\sqrt{s} \to \pm\infty$, it corresponds to $z = \pm\infty$ in (35)–(36). This proves the first parts of Theorems 7 and 8 for $\theta \in (-\pi/2, +\pi/2)$. The rest of the claims of the theorems then follow from the well-known result about the full plane Ising model for $T < T_c$ that the only pure states are the plus and minus ones [A, H]. ■

7 Exact results for all $T < T_c$

It turns out that Dobrushin boundary conditions in the half-plane Ising ferromagnet has a useful realization in the spinor language of Kaufman [K] and of Schultz, Mattis and Lieb [SML]. To have translational symmetry at one's disposal, which is crucial, it is necessary to wrap the lattice on a cylinder; thus with plus boundary conditions, say, it is necessary to induce two domain walls crossing the cylinder. We take the diameter of the cylinder to infinity and then follow that by imposing infinite separation between the two domain walls. We then focus on one of these domain walls. We select a Dobrushin boundary condition, one that forces the interface to cross at a given mean angle; then we investigate the behavior of the expectation of the magnetization near one face of the cylinder.

The arrangement is shown in Fig. 2, the caption of which explains the spatial coordinates. The partition function for such a domain wall, beginning at $y = 1$ and ending at, $y = s + 1$ is given by

$$Z^{\times} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\omega \, \exp\Big[-N\gamma(\omega) + is\omega\Big] |g(\omega)|^2 \,. \tag{12}$$

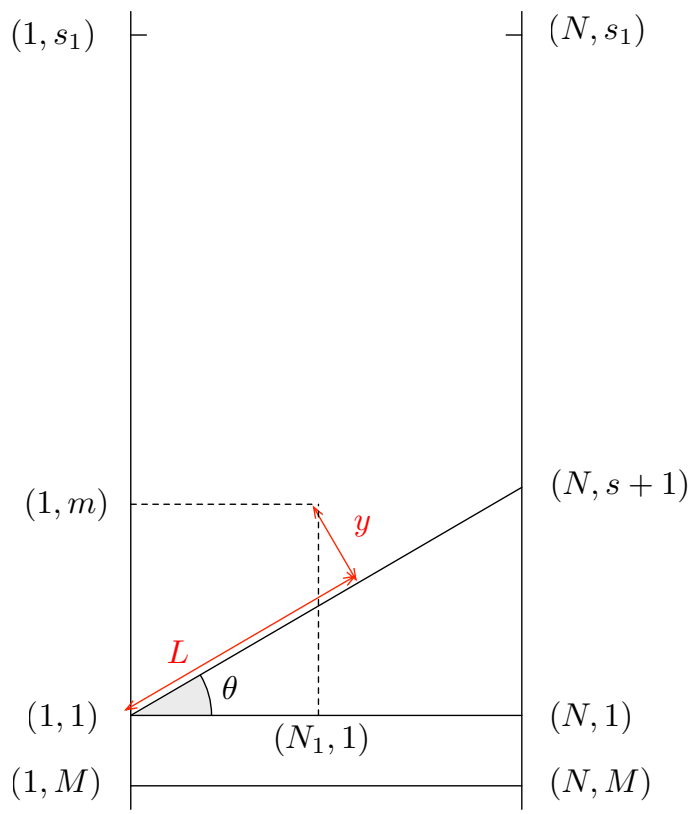

Figure 2: The diagonal interface and the definitions used. Note that $m = [L \sin\theta + y \cos\theta]$, $N_1 = [L \cos\theta - y \sin\theta]$, $[\cdots]$ denotes the integer part.

Here, the function $\gamma(\omega)$ is one of the hyperbolic triangle elements of Onsager [O], given by

$$\cosh\gamma(\omega) = \cosh 2K_1^\star \cosh 2K_2 - \sinh 2K_1^\star \sinh 2K_2 \cos\omega\,,\ \ \gamma(\omega) \geqslant 0\,, \omega \in \mathbb{R}\,. \tag{13}$$

The interactions are in units of $k_B T$ and are taken different in the two directions, something mathematically useful but physically irrelevant. The variables $K_j^\star$, $j = 1, 2$ are dual ones given by

$$\exp\left(2K_j^\star\right) = \coth K_j\,. \tag{14}$$

The function g has the same domain of analyticity as γ, a matter of some significance, which we merely quote for the next step: we are interested in an interface crossing at a given angle, say θ, so we take $N \to \infty$, $s = [N \tan\theta]$. The analytical tool here is saddle point integration. The saddle point in $-\pi \leqslant \mathrm{Re}(\omega) \leqslant \pi$ is at $\omega_s = i\nu(\theta)$ where

$$\gamma^{(1)}\left(i\nu(\theta)\right) = i\tan\theta\,. \tag{15}$$

It is straightforward to see that there is a unique solution for $0 \leqslant \theta < \pi/2$ in the interval $0 \leqslant \nu(\theta) < 2|K_1 - K_2^\star|$. The asymptotics of (12) are given by

$$Z^\times \sim |g\left(i\nu(\theta)\right)|^2 \exp\Big[-L\tau(\theta)\Big] \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}u\ \exp\Big[-u^2 L\cos\theta\gamma^{(2)}\left(i\nu(\theta)\right)/2\Big]\,. \tag{16}$$

The length L is given by $N = [L\cos\theta]$ and the surface tension is

$$\tau(\theta) = \cos\theta\,\gamma\left(i\nu(\theta)\right) + \sin\theta\,\nu(\theta)\,. \tag{17}$$

This is an exact derivation of the angle-dependent surface tension for Dobrushin boundary conditions. The finite-size corrections are easily obtained by integration in (16):

$$Z^\times \sim |g\left(i\nu(\theta)\right)|^2 \exp\Big[-L\tau(\theta)\Big] \Big[2\pi L\cos\theta\,\gamma^{(2)}\left(i\nu(\theta)\right)\Big]^{-1/2}\,. \tag{18}$$

We have just given the leading term, as soon we shall be interested only in limiting behavior: (16) is equivalent to the first term of a Laplace method. Some comments are in order: first, we have to get onto the steepest descent path which actually goes monotone-upwards in the ω plane, intersecting the

line $\nu = \infty$, $\omega = u + iv$ at $-\pi < u < 0$ to the left and symmetrically at $0 < u < \pi$ to the right. We have evaluated the function g and assure the reader that we do not have to cross a singularity of it to get onto the steepest descent path. A good contemporary source on steepest descent methods is Ablowitz and Fokas [AF]. Another point, which must be outlined here, is that the Fermionic realisation of the Dobrushin boundary allows emission of more than one Fermion. These terms can be controlled; they do not report in the limiting behavior given in (12), nor do they in the magnetization profile, which we are about to specify. Consider the magnetization at a position (N_1, m) in the Dobrushin boundary condition used above. It is known that the leading term, when $M \to \infty$ and when the second interface has been clustered away, is

$$\langle \sigma(N_1, m) \rangle = \mathfrak{m}^* \left(Z^\times\right)^{-1} Y(N, N_1, m, s)\,. \tag{19}$$

Here, $\mathfrak{m}^*$ is the spontaneous magnetization and Y has the integral representation:

$$\begin{aligned} Y = {} & \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\omega_1\, g(\omega_1)\, \mathrm{e}^{-N_1 \gamma(\omega_1)} \\ & \times \frac{\mathcal{P}}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\omega_2\, \mathrm{e}^{-(N-N_1)\gamma(\omega_2)}\, \frac{j(\omega_1, \omega_2)}{\mathrm{e}^{i(\omega_1+\omega_2)} - 1} \mathrm{e}^{-im(\omega_1+\omega_2)+is\omega_2} g^*(-\omega_2)\,, \end{aligned} \tag{20}$$

in which $*$ denotes complex conjugation. This is what is left after multiple domain wall configurations have been eliminated. The function $j(\omega_1, \omega_2)$ is 2π-periodic in both variables and has in each variable the same domain of analyticity as $\sinh \gamma$, that is, square root branch cuts at $\omega = \pm 2i(K_1 \pm K_2^\star)$. Further, we have $j(\omega, -\omega) = 2$. We are interested in the limit as $N \to \infty$, $s = [N \tan \theta]$, $0 \leqslant \theta < \pi/2$; here, $[\cdots]$ denotes nearest integer. We apply the Plemelj theorem, bearing in mind that we want to get onto the steepest descent path for the ω_2 integral; this is in the upper half plane. We find that

$$\left(Z^\times\right)^{-1} Y(N, N_1, m, s) \sim 1 + \left(Z^\times\right)^{-1} X\,, \tag{21}$$

where

$$X = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\omega_1\, g(\omega_1)\, \mathrm{e}^{-N_1\gamma(\omega_1) - im\omega_1} W(N, s, m|\omega_1)\,, \tag{22}$$

finally, we have:

$$W = \frac{1}{2\pi} \int_{\mathcal{C}} \mathrm{d}\omega_2\, \mathrm{e}^{-(N-N_1)\gamma(\omega_2)} \mathrm{e}^{i(s-m)\omega_2} \frac{j(\omega_1, \omega_2)}{\mathrm{e}^{i(\omega_1+\omega_2)} - 1} g^*(-\omega_2)\,, \tag{23}$$

where $\mathcal{C}$ is exactly the steepest descent path used in the partition function investigation above. We now evaluate the limit:

$$\lim \left(Z^{\times}\right)^{-1} W = g^{*}\left(-i\nu(\theta)\right) \mathrm{e}^{N_1 \gamma(i\nu(\theta))} \mathrm{e}^{m\nu(\theta)} \frac{j(\omega_1, i\nu(\theta))}{\mathrm{e}^{i\omega_1}\mathrm{e}^{-\nu(\theta)} - 1}\,, \tag{24}$$

thus we have the limiting result that $\left(Z^{\times}\right)^{-1}$ converges to

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\omega_1\, \mathrm{e}^{-N_1\left[\gamma(\omega_1)-\gamma(i\nu(\theta))\right]} \mathrm{e}^{-i(\omega_1+i\nu(\theta))} \frac{j(\omega_1, i\nu(\theta))}{\mathrm{e}^{i\omega_1}\mathrm{e}^{-\nu(\theta)} - 1} \frac{g(\omega_1)}{g\left(i\nu(\theta)\right)}\,. \tag{25}$$

We now investigate this by saddle-point integration: care is needed. The integrand in (25) has a simple pole at $\omega_1 = -i\nu(\theta)$. The saddle point is given by

$$N_1 \gamma^{(1)}\left(i\nu_s(1)\right) = -im\,. \tag{26}$$

It is natural to express the result in terms of the Euclidean distance along the "flattened" interface pointing in the direction given by θ and the normal coordinate y. Thus we have

$$\begin{aligned} N_1 &= L\cos\theta - y\sin\theta\,, \\ m &= L\sin\theta + y\cos\theta\,, \end{aligned} \tag{27}$$

equation (26) becomes

$$\gamma^{(1)}\left(i\nu_s(1)\right) = -i\frac{L\sin\theta + y\cos\theta}{L\cos\theta - y\sin\theta}\,. \tag{28}$$

Referring back to (15), we see that

$$\gamma^{(1)}\left(i\nu_s(1)\right) - \gamma^{(1)}\left(i\nu(\theta)\right) = -i\frac{y}{L\cos^2\theta}\left(1 - \frac{y}{L}\tan\theta\right)^{-1}\,. \tag{29}$$

We are interested in the case L large and $\alpha \neq 0$ where

$$y = \alpha L^{1/2}\,, \tag{30}$$

this means we can use a Taylor-series approximation on the left hand side of (29)

$$\nu_s(1) = -\nu(\theta) - \frac{\alpha}{L^{1/2}}\frac{1}{\cos^2\theta\,\gamma^{(2)}\left(i\nu(\theta)\right)} + \mathcal{O}\left(\frac{1}{L}\right)\,. \tag{31}$$

Working up (25) with (31) must take careful account of the simple pole and whether it is crossed in getting onto the steepest descents contour. We encounter the following integral representation:

$$F(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}u \, \frac{\mathrm{e}^{-u^2}}{z + iu} \,, \tag{32}$$

it is easy to see that $F(-z) = -F(z)$ and also to derive the identity

$$F(z) = \frac{2}{\sqrt{\pi}} \int_{z}^{+\infty} \mathrm{d}u \, \mathrm{e}^{-u^2} \,, \qquad z > 0 \,. \tag{33}$$

This applies to the development of (25) by steepest descents with

$$z = \frac{\alpha (\sec\theta)^{3/2}}{2\gamma^{(2)}\left(i\nu(\theta)\right)} \,. \tag{34}$$

With the definition

$$G(z) = \frac{2}{\sqrt{\pi}} \int_{0}^{z} \mathrm{d}u \, \mathrm{e}^{-u^2} \,, \qquad z > 0 \,, \tag{35}$$

we have

$$s \lim \langle \sigma(N_1, m) \rangle = -\mathfrak{m}^* \mathrm{sgn}(z) G(|z|) \,, \tag{36}$$

where the rather complex limiting procedure $s \lim$ is specified in the text. As a final remark, note that a full development of the saddle point equations and the surface tension function (17) allow (34) to be expressed in terms of thermodynamic quantities

$$z = \alpha \Big[\sec\theta \left(\tau(\theta) + \tau^{(2)}(\theta) \right) \Big]^{1/2} \,. \tag{37}$$

This combination of surface tension and derivatives is known as the surface stiffness; it is interesting that such a meso-scale quantity, representing a contraction of description, occurs here.

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