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Decomposition and informational decentralization for the computation of economic equilibrium

P. Mahey*
H.P.L. Luna†

Abstract

We classify different approaches to compute multidivisional sectorial economic equilibrium by decomposing the corresponding structured optimization model. We analyze in particular the different levels of decentralization w.r.t. informational and decisional flows between the central agency and the divisions. We stress the advantages of the optimization approach to solve real-life problems of spatial economic equilibrium and show how mixed procedures based on prices and resource allocations are the most flexible and able to reach complete decentralization. We conclude by a critical analysis of parallel architectures to serve as a simulation tool for the central decisor actions and the decentralized behavior of economic agents.

1 Introduction

The present paper analyzes the qualitative properties of informational decentralization among the so-called decomposition methods used for solving large-scale optimization problems. The main motivation for such a study resides in the search for decentralized iterative schemes to compute economic equilibrium which result from competition between firms (or regions, markets, etc ...) sharing limited resources. Problems of this type arise in a variety of fields like economics, regional science, transportation or production systems. We shall focus on sectorial economic markets (or sectorial price equilibrium) in which commodities are produced and consumed such that they can be represented by some optimization problem with a single criterion. Thus, we shall not include here more general situations where no equivalent optimization problem can be formulated : neither the case of cross-effects among commodities giving rise to non symmetric complementarity problems, nor the case of institutional constraints on prices even if recent advances on the parallel computation of the solution of variational inequalities seem to allow to extend the present discussion on decentralized procedures (see Kim [30], Pang [49] or Nagurney [46]). We shall also focus on the decentralization of sectorial equilibrium w.r.t. divisional processes and not w.r.t.

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regional submarkets, which means that the coupling between the local subsystems is done through the commodities and not through the transportation flows of these commodities from one region to another. Finally, we shall restrict ourselves to static models to avoid multiperiodic formulations of these equilibrium.

What is meant by decentralized information is the capability of the procedure to avoid the storage at a central decision level of the local data concerning each production subsystem. That central level will treat only the coupling constraints or activities between these subsystems, called the divisions hereafter. The divisional decisions are then based on their own local data and objective and on some external parameters updated by the central level, called the coordinator. An immediate consequence is that a decentralized procedure should be able to recover in a decentralized manner the optimal solution when it exists. That point induces a new level of the decentralization which we may call decisional decentralization where each subsystem is able to compute its local optimal decision coinciding with the global one.

On the other hand, decomposition methods may have different motivations : to reduce the dimension by splitting the computation among independent subsystems, to partition the difficulties brought in the model by some coupling variables or constraints, to parallelize the computation among different processors or to decentralize the global optimal decisions. In fact, all these motivations have common features and we shall discuss them along the present paper.

We begin by presenting the general sectorial economic equilibrium as a structured mathematical program. This is followed by a classification of the bibliography w.r.t. the organizational structure of the iterative process, where we analyze the decentralization among the so-called decomposition-coordination schemes. Then, we focus on the completely decentralized schemes where both information and decisions are treated locally until the final 'optimal' decision.

Finally, we comment on the discrepancies between decentralization and parallelization in the context of decomposition methods. Indeed, common features of parallel algorithms like redundancies and shared memory are not easily compatible with decentralization.

2 Decentralized planning

Decentralization is a well-known requirement in economics but the link with the convergence of computational schemes to compute economic equilibrium have been first analyzed in the seminal paper by Arrow and Hurwicz [1]. To define what is meant by decentralization in a multidivisional economy, we may use their own definition first : "For a given set of prices, a process manager need know only the prices and the technology of his own process in order to arrive at the optimal level for his process.[...] the equilibrium on each market may be checked separately; for any given market, the test requires knowing only the net demand, which is an aggregate of many individual decisions, the initial supply, and the price." They observe that it is a severe requirement for applications as convergence of such totally decentralized schemes needs in general strict concavity of the utility function and convexity of the production functions, or in economic terms "diminishing or constant

returns in production and diminishing marginal utility for the consumption of final demands". In the general case, one must distinguish between informational decentralization which guarantees that each process manager only need know its own local data with additional information about prices or quotas, and decisional decentralization where each manager is able to compute and recognize its own local globally optimal decision. Beside that, many authors have focussed on the hierarchical aspects of the effective planning relying on the concept of coordinability. It supposes (see Mesarovic et al [45]) that a global decision problem (P) is decomposed in local decision subproblems $(P_j(\gamma), j = 1, \dots, m)$ depending on a coordination input vector γ and a supremal decision process (MP) yielding a two-level hierarchical procedure. We assume there exists a mapping Π which associates to any bundle of local decisions (x_1, \dots, x_m) a global decision for (P). Then, the subproblems are said to be coordinable if :

- a) There exists at least one γ satisfying some coordinating conditions (optimality conditions for (MP)) such that each local manager computes a local optimal decision $(x_1(\gamma), \dots, x_m(\gamma))$.
- b) For any coordinating input γ satisfying a), and for any optimal local solution $(x_1(\gamma), \dots, x_m(\gamma))$ the corresponding global solution $\Pi(x_1(\gamma), \dots, x_m(\gamma))$ is optimal for (P).

As observed by Jennergren [28], the point is that, in many practical cases, there are local alternative solutions for an optimal γ which do not yield an optimal solution and the mapping Π must rely on some global information to reconstitute global optimality. Then, decentralization is not complete. This is what happens in the linear case with the classical coordination schemes and it has been observed very early (see e.g. [5]). Coordinability is indeed a particular case for the present analysis as some procedures may avoid the use of a coordination level. We resume the conditions for a given process to be decentralized in three parts :

1. Informational Decentralization : The local managers may compute local decisions by solving subproblems depending on local information and on a few number of external parameters, the coordination input γ .
2. Coordinability : There exists some coordination input such that any optimal global decision corresponds to an optimal local solution in the subproblems.
3. Complete Decentralization : There exists some coordination input such that any local optimal solution corresponds (via the mapping Π) to a global optimal solution.

Of course, when unicity of the solutions of the subproblems is guaranteed, coordinability implies complete decentralization. More implications about decentralization may be found in Malinvaud [41], Bensoussan et al [7], Atkins [2], Luna [35], Van de Panne [57] or Dirickx and Jennergren [15].

3 The sectorial economic equilibrium

Following Arrow and Hurwicz [1], we define a general nonlinear model of resource allocation among m different processes (or regions) producing and consuming n commodities for which initial availabilities are supposed known quantities. Let x_j be the m_j -vector of scales (or activity levels) of the j th process and q_i be the final demand of the i th commodity ($x = (x_1, \dots, x_m)$ and $q = (q_1, \dots, q_n)$ are the corresponding vectors in \mathbb{R}^M and \mathbb{R}^n respectively, $M = \sum_j m_j$). Then, the search for an economic equilibrium for that particular sector of the economy should be represented by the following mathematical program :

$$\begin{aligned} & \text{Minimize } c(x) - f(q) \\ & \text{subject to } q - g(x) \leq b \\ & q \geq 0, \quad x_j \in S_j, j = 1, \dots, m \end{aligned} \tag{P}$$

Here, the cost function is the balance between the production cost of the processes $c(x)$ and the utility function of the sector $f(q)$. The production cost is supposed to be separable w.r.t. the m processes, i.e. :

$$c(x) = \sum_{j=1}^m c_j(x_j)$$

The n constraints

$$q_i - g_i(x) \leq b_i \tag{1}$$

represent the feasibility of the productive sector for each commodity. We do not distinguish here between external commodities for which a final demand q_i has to be computed and primary commodities which enter in the production of the external commodities directly or indirectly (for these latter commodities the demand should be forced to zero). We suppose too that the total output for commodity i , is given by a separable function $g_i, i = 1, \dots, n$ w.r.t. the processes, i.e. that :

$$g_i(x) = \sum_{j=1}^m g_{ij}(x_j)$$

and that b_i is the known available quantity of the i th commodity.

Finally, S_j is a subset of the j th activity space which represent the technological constraints of the j th process (for instance $S_j = \{x_j \in \mathbb{R}^{m_j} | h_j(x_j) \leq 0, x_j \geq 0\}$).

The economic equilibrium of the sector, when it exists, should be written using the Kuhn-Tucker optimality conditions for (P), assuming that all functions are smooth and that the constraints satisfy some appropriate constraint qualification. They are, for the optimal vector of multipliers $p \in \mathbb{R}^n$:

$$\nabla f(q) = p \tag{2}$$

$$\nabla c_j(x_j) - \sum_{i=1}^n p_i \nabla g_{ij}(x_j) + \nabla h_j(x_j)^T \mu_j = 0, j = 1, \dots, m \tag{3}$$

$$p^T (q - g(x) - b) = 0 \tag{4}$$

$$\begin{aligned} \mu_j^T h_j(x_j) &= 0, j = 1, \dots, m \\ p &\geq 0 \quad \mu \geq 0 \end{aligned}$$

$$\begin{aligned}
q - g(x) &\leq b \\
x_j &\in S_j \quad j = 1, \dots, m \\
q &\geq 0
\end{aligned}$$

Observations :

- (2) expresses the multipliers p_i as the marginal prices for each external commodity.
- (3) expresses the marginal equilibrium for each process. ∇h_j stands for the Jacobian matrix of the multiapplication h_j .
- (4) is the complementarity slackness condition which says that offer in excess yields null prices.

Another more restrictive way to traduce that equilibrium is to associate to it an eventual saddle-point of the Lagrangian function $L(q, x, p)$, defined for $q \geq 0, x_j \in S_j, j = 1, \dots, m, p \geq 0$:

$$L(q, x, p) = c(x) - f(q) + p^T(q - g(x) - b)$$

Of course, it is well known that strong additional conditions on problem (P) should be added to ensure the existence of saddle-points. These are for example :

- Convex production costs and concave utility function
- Concave production rates for each process
- Convex subsystems feasible sets S_j
- Slater constraint qualification for (1)

In the following sections, we shall analyze some decomposition schemes to solve (P). These schemes treat the coupling constraints (1) in order to build the divisional subproblems w.r.t. the partition of the variables, i.e. the divisional scales $x_j, j = 1, \dots, m$ and the final demands variables q . As the decentralization is mainly concerned with the splitting among the divisions, the demand subproblem will be omitted hereafter, the corresponding computations being performed in general by the central level itself.

4 Classification of the literature

In order to classify the literature on decomposition methods which are candidate to solve the general equilibrium problem (P), we have focussed on the kind of information that flow between the coordination level and the divisional processes. Typically, we distinguish between price-directive schemes where the upper level fix the prices of the commodities, resource-directive schemes where the upper level fixes resource quotas for each process and mixed schemes. Inside these three classes, we comment on the following features concerning decentralization and convergence of the iterative schemes :

- a) Conditions for convergence and applications of the decomposition strategy

- b) Computational complexity and rate of convergence of the coordination task
- c) Computational complexity and exigence of accurate solutions in the sub-problems
- d) Feasibility of the resource constraints at each cycle
- e) Rate of convergence
- f) Complete decentralization

In order to organize the answers in tables, the algorithms have been grouped in families which depend on the solution strategy used to solve the coordination problem. To the classical classification of the different manipulation/strategy pairs proposed early by Geoffrion [20], we have preferred the equivalent but simpler partition based on the mathematical programming strategy to solve the corresponding coordination problem. This is addressed in the following sections.

5 Price Coordination

The classical price-directive scheme has been formalized early by Koopmans [31] and Arrow-Hurwicz [1]. It is based on an iterative search for the saddle-point of the Lagrangian where the prices of the commodities are adjusted by a coordination level (the Central Agency of the Walrasian tatonnement procedure). These prices are then sent to the subsystems which compute their activity levels according to the following price-directive subproblems :

$$\begin{aligned} & \text{Minimize } c_j(x_j) - p^T g_j(x_j) \\ & x_j \in S_j \end{aligned} \quad (\mathbf{PP}_j)$$

Many algorithms use these subproblems and they differ only on the coordination strategy where the possibly nonsmooth dual function $h(p)$ defined below has to be maximized over all non negative price vectors p .

$$h(p) = \sum_{j=1}^m h_j(p) + p^T(q - b)$$

where h_j are the optimal value functions for \mathbf{PP}_j .

The most popular, designed originally for the linear case, is the Dantzig-Wolfe decomposition method [14]. The principle is the following : the subproblems \mathbf{PP}_j send the scales $x_j(p), j = 1, \dots, m$, to the coordinator who computes the global balance :

$$\gamma(p) = q - g(x(p)) - b$$

which is a subgradient of the dual function h . As that function is in general nonsmooth (smoothness relies on the strict convexity of the objective function in (P)), the coordinator needs to store the old subgradients to compute a feasible solution and test optimality. As

Baumol and Fabian did observe early [5], decentralization is not complete at least in the linear case, because the subproblems are unable to compute their own part of the optimal global solution.

Within this general coordination framework, what makes the differences between the various approaches proposed in the literature is the solution strategy to solve the dual problem and we may classify them (we propose a representative rather than exhaustive list of algorithms) in the following way :

1. *Gradient methods*: in the strictly convex case, Arrow and Hurwicz [1] proved the convergence of a fixed-step gradient method to maximize the dual function, as the latter is indeed concave and smooth. The divisional subproblems need not be solved to optimality, but the scales are updated w.r.t. the directional derivative of the Lagrangian function. In the general case, they propose two distinct modifications : first, a nonlinear price-adjustment where the prices p_i are substituted by $p'_i = (1 + \eta_i)p_i(1 - g_i(x))^{\eta_i}$. The second modification is a price speculation mechanism where prices are substituted by $p''_i = p_i + 2\lambda(q_i - g_i(x) - b_i)^+$. The latter case may be seen as a linearization procedure for an Augmented Lagrangian method.
2. *Cutting planes methods* : here, all past subgradients are kept to build a piecewise linear approximation of the dual function. Solving that approximate problem for the new prices is equivalent to substitute to the divisional constraints some inner linearization of their constraint set (forming the so-called Master Program). This is the Dantzig-Wolfe decomposition [14] in the linear case. The main difficulties reported with that method are :
 - i) Accumulation of the cutting planes (column generation in the master program);
 - ii) Non unique solutions in the subproblems turning the decentralization incomplete;
 - iii) Large jumps in the price adjustment turning the process numerically unstable.

To cope with these difficulties, some variants have been proposed : In the Boxstep method, Marsten et al [42] add some 'move limits' in the divisional subproblems to regularize the erratic convergence of the cutting plane approach. Mahey [39] combines a subgradient approach with Dantzig-Wolfe algorithm to accelerate the convergence.

3. *Simplex-like procedures* : in the linear case, the basic structure of the subproblems may be used to build some variants of the Dantzig-Wolfe algorithm : in Balas [4], a parametric representation of the alternative solutions in the subproblems is substituted to the convex combination of extreme points used in the Dantzig-Wolfe master. Thus, the divisions must transfer more information to the upper level, indeed, all simplex tableau columns with a zero reduced cost. In compensation, no effort is made in the restricted master to get feasibility and no accumulation of columns (i.e. cutting planes) occur turning the coordination task lighter. In the same spirit, a primal-dual steepest-ascent decomposition has been proposed by Grinold [24]. It results in a procedure where part of the divisions have to be solved twice to get all extreme-point solutions, and it is indeed a high price to pay for decentralization. Abadie and

<i>Properties</i>	<i>Classes</i>		
	<i>Arrow-Hurwicz</i>	<i>Dantzig-Wolfe</i>	<i>Others</i>
a. Conditions of convergence	Strict convexity	Linearity	Convexity
b. Easy coordination task	yes	no	[4],[24]
c. Easy divisional task	yes	yes	[17], [39], [43], [56]
d. Feasibility	no	yes	[4]
e. Fast convergence	yes	yes	[22], [39], [43], [42]
f. Decentralization	yes	no	[52], [4], [17], [24]

Table 1: Price-directive schemes

Williams proposed to solve the master by the dual simplex method and Bell used the primal-dual simplex, but both need some specific fractional programming routine at the upper level (see [34]). Very few attempts to adapt the Dantzig-Wolfe method to non convex objective have been made. We can cite Phan [50] who has considered a specific class of min concave cost flow problems with a hierarchical structure.

4. *Bundle methods* : the way to limit the number of subgradients in the approximation of the dual function is to generate ϵ -subgradients until a descent direction is found. This leads to bundle methods for nonsmooth optimization and the idea has been applied to block-angular linear programs by Medhi [43]. The dual steps correspond to a proximal regularization of the cutting plane method (see Ruszczyński [52]). Furthermore, it is possible to add directly some coercion functions in the divisional subproblems to force unicity and yield complete decentralization (see Feinberg [17]).
5. *Interior-point methods* : Goffin et al [22] have proposed an interior-point based decomposition where the prices are obtained by computing the analytic center of the relaxed dual master. No decentralized version of that computationally efficient technique has been proposed to our knowledge. Todd [56] has given an interesting economic interpretation of the price adjustment of a special version of Karmarkar’s method where the local constraints are also affected by the prices.

The evaluation of the relevant properties of these decomposition methods are resumed in Table 1. The two first columns correspond to the basic algorithms of Arrow-Hurwicz and Dantzig-Wolfe, and the third one is dedicated to their variants and other above discussed schemes.

6 Resource-directive decomposition schemes

In the resource-directive scheme, quantities rather than prices are transferred to the divisions. The latter may propose marginal prices or feasible solutions, depending on whether they take in account explicitly the variable resource constraints in their constraint set or not. In many aspects, this scheme is dual to the former price-directive coordination and the solution strategies to solve the coordination problem are similar to the ones presented

before (see Lasdon [34] for a complete description of that relationship). As a specific difference between both schemes, we may stress the point that the divisional resource allocations $y_j, j = 1, \dots, m$ (which represent the a priori participation of the division in the production of the i th commodity (1)) must be forced to feasibility at the upper level, i.e. they must satisfy :

$$\sum_{j=1}^m y_j = b - q$$

Let $(y_j^0, j = 1, \dots, m)$ be an initial feasible allocation of the resource among the divisions and set $y_j = y_j^0 + \delta y_j, j = 1, \dots, m$. Then, the resource variations $(\delta y_1, \dots, \delta y_m)$ must lie in the subspace $\{(z_1, \dots, z_m) | \sum_{j=1}^m z_j = 0\}$ To understand the duality relation between price- and resource- directive schemes, we can look at the orthogonal subspace :

$$\{(u_1, \dots, u_m) | u_1 = \dots = u_m\}$$

Each u_j is a copy of the price vector p used before. It is the price imputation by the j -th division associated to the resource allocation y_j it has been allocated to. The equilibrium occurs when all divisions propose the same prices. These prices and the corresponding feasible activity levels are given by the solution of the resource-directive subproblems (RP _{j}) :

$$\begin{aligned} & \text{Minimize } c_j(x_j) \\ & g_j(x_j) \geq y_j \\ & x_j \in S_j \end{aligned} \tag{RP}_j$$

Observe that the resource allocation y_j must be chosen such that (RP _{j}) is non empty. Let denote its optimal value, when finite, by $v_j(y_j)$. An equivalent formulation for (P) is then :

$$\begin{aligned} & \text{Minimize } V(y_1, \dots, y_m) = \sum_{j=1}^m v_j(y_j) \\ & y_j = y_j^0 + \delta y_j, j = 1, \dots, m \\ & \sum_{j=1}^m \delta y_j = 0 \\ & y_j \in Y_j, j = 1, \dots, m \end{aligned} \tag{V}$$

where Y_j is the set of the locally acceptable allocations for the j -th division. Again, the objective function V is convex and possibly nonsmooth, but collecting the dual imputed prices $(-u_1, \dots, -u_m)$ we obtain a subgradient which should be used to update the resource allocations.

The additional requirement of the feasibility of the resource allocations (y_1, \dots, y_m) which complicates somewise the coordination task turns the resource-directive approach less attractive than the price-directive one. Anyway, the literature may be classified in a similar way to the former classification. No equivalent decentralized procedure to the one designed by Arrow-Hurwicz has been proposed in the literature, but feasible directions methods proposed by Geoffrion are in the same spirit; on the other side, the method corresponding to Dantzig-Wolfe decomposition is the well-known Benders decomposition :

1. *Projected gradient methods* : in the differentiable case (again, this is guaranteed by unicity of the divisional solutions for any feasible allocations), projected gradient methods can be applied to solve (V) as proposed by Geoffrion [21] and Silverman [54]. Note that the property of differentiability has been justified in the context of economic planning (see Heal [25]). Under the name of parametric decomposition, Ermoliev and Ermolieva [16] have proposed a subgradient approach in the nondifferentiable case. Although the coordination task is relatively easy and the volume of transferred information is low, two major problems must be faced : the difficulty to control the convergence and the slow rate of convergence. Moreover, the divisions cannot be decentralized as they present non unique dual solutions and they need the optimal global dual solution to recognize optimality.
2. *Cutting planes* : the generation of affine cuts based on each computed subgradient to build an outer linearization of $V(y)$ leads to Benders decomposition algorithm [6], indeed a dual version of Dantzig-Wolfe decomposition in the linear case. In fact, the main interest for Benders decomposition lies in the fact that the coupling variables in the dual problem may appear with non convex or even discrete functions or sets. Thus, it permits the treatment of discontinuities and indivisibilities. This is the case of some location models that, even with indivisibility can be solved with the interesting property of feasibility with a reduced number of cycles; within that context, one can consider the upper level as the operational or tactical level, while the lower level is the structural or strategical level (see Luna [36]). Other procedures of the same class require convexity of the global problem (see the application to economic planning by Weitzman [59]).
3. *Simplex-based methods* : In the linear case, Rosen has first proposed a partitioning algorithm [51], guaranteeing feasibility and improvement of the cost function at each cycle. Indeed, this is the right way to decentralize the information, even if complete decentralization is not possible and the transmission of information is rather costly.

To complete that list, we must add the singular approach of Cremer [13] who has analyzed a planning procedure without price adjustment where both levels transfer quantities. As a rare decentralized version of these resource-directive schemes, we can cite Grinold [24], already cited for the price-directive schemes.

Comments on the properties of the cited methods are resumed in Table 2.

7 Mixed coordination schemes

In this section, we consider coordination schemes where both prices and quantities are transferred to the divisions to modify their subproblems. The motivations for such an increase in the volume of information transferred from one level to the other allow to classify the algorithms in three classes :

1. *Kornai-Liptak primal-dual game* : we put apart Kornai and Liptak's method [32], not for its numerical efficiency but for its originality as a perfect balance between price-

<i>Properties</i>	<i>Classes</i>		
	<i>Geoffrion</i>	<i>Benders</i>	<i>Others</i>
a. Conditions of convergence	Strict convexity	Convex subproblems	Linearity
b. Easy coordination task	yes	no	[16], [32], [54]
c. Easy divisional task	yes	yes	[16], [32], [51]
d. Feasibility	yes	yes	[51], [54]
e. Fast convergence	yes	no	[51]
f. Decentralization	yes	no	[24]

Table 2: Resource-directive schemes

and resource-directive schemes. There, prices and resource allocations are updated by averaging the dual solutions of all past resource-directive subproblems and the primal solutions of all past price-directive subproblems. It may then be seen as a balanced mixture of Dantzig-Wolfe and Benders with no master programs. As it is applicable to block-angular linear programs, it may be interpreted too (see Holmberg [27]) as a particular case of Van Roy's cross-decomposition [58] originally designed for problems with both coupling variables and constraints. Convergence of Kornai-Liptak algorithm is related to Brown-Robinson's finite matrix game but it seems very slow in practice.

2. *Variable decomposition methods* : the motivation here is to give more flexibility to the coordination scheme when dealing with heterogenous economies. As a general idea, some divisions are treated by price-coordination while the others are treated by resource-coordination. Obel [48] used this idea in the context of Dantzig-Wolfe method and the economic implications are analyzed by Burton and Obel [9]. Heterogenous markets or decisional issues in a real company are the main motivation for these approaches and some real cases have been simulated to compare the impact of the various possibilities of mixed organizations on human decision-making ([11]). Finally, in Van de Panne local decomposition method for linear models [57], the choice of the divisions which are allocated prices (respectively quantities) is dynamic and depends on the basis structure of these divisions. Typically, when a division has alternative proposals, it receives resource allocations and send the imputed prices to the remaining decentralized divisions. The procedure may be interpreted as a one-level coordination but decentralization is unfortunately only limited to the independent divisions (which number could as well be zero). On the other hand, primal-dual schemes where two kinds of subproblems have to be solved for each division are meant to reduce the number of iterations by strengthening the estimate of the optimal value between two bounds. That idea could be of interest when implementing a subgradient method (see [37] and [27]).
3. *Decomposition with mixed subproblems* : The motivation is now to compensate the lack of decentralization of the pure price- or resource- directive schemes. In that third type of mixed methods, prices and resource allocations are sent to the divisions

to form mixed subproblems. Decentralization is often the motivation for such a complication of the divisional problem. The control of the production level of the division can first be made by adding bounds or neighbourhood constraints in the subproblem as in the above cited Boxstep method ([42]). This is the early approach in Younes [60] and Charreton [10]. Now, we consider algorithms where some resource allocations are forced in the price-directive subproblems : Maier and Vanderweide [40] proposed to solve two different subproblems for each division : the first one is the pure price-directive subproblem and the second one adds a resource allocation constraint to the former. In Sengupta and Gruver [53], each division receives prices and resource allocations and a local price imputation is obtained by adding the central price to the local dual variable associated to the resource constraints. These price imputations are needed to update the resource allocations maximizing the net difference between central prices and local price imputations.

As a first subclass, we focus on some mixed algorithms that aim at the complete decentralization of the divisions : first, Kydland [33] introduced a hierarchical allocation of the coupling constraints in the price-directive subproblems. The allocated resource are not charged by their price in the subproblem cost function. This has been generalized by Mahey [38] for any structure in the coupling constraint yielding the following mixed subproblems :

Let (I_1, \dots, I_m) be a partition of the resource index set $\{1, \dots, n\}$. Then the mixed subproblems (MP_j) are :

$$\begin{aligned} \text{Minimize } & c_j(x_j) + \sum_{i \notin I_j} p_i g_{ij}(x_j) \\ & g_{ij}(x_j) \geq y_{ij}, \quad \forall i \in I_j \\ & x_j \in S_j \end{aligned} \tag{MP}_j$$

It is shown in [38] how to choose the partition of the resource in the linear case to yield decentralization and how to derive a one-level procedure where the divisions exchange prices and quantities without the help of a coordinator (see too Cohen [12] for a theoretical study of the linear-quadratic case, Meijboom [44] and Nurminski [47] for related methods).

Finally, decentralization may be obtained forcing unicity in the divisional subproblems by adding a quadratic term in the objective function. Note that Arrow and Hurwicz, as cited above, have mentioned that modification. In the linear case, that idea appears in Jennergren [29] as a decentralized extension of Dantzig-Wolfe decomposition method. When the quadratic term is updated by forcing feasibility of the resource allocations, we obtain a procedure kindred to a separable augmented Lagrangian method. This is what Spingarn has obtained [55] adapting the Proximal Point method to design a decomposition method for block-angular convex programs based on what he called the Partial Inverse operator. The subproblems are of the following form :

<i>Properties</i>	<i>Classes</i>		
	<i>Kornai-Liptak</i>	<i>Variable decomposition</i>	<i>Mixed decentralized</i>
a. Conditions of convergence	Linearity	Linearity	Linearity : [29], [33], [38], [47] Convexity : [3], [19], [23], [55] Quadratic : [12]
b. Easy coordination task	yes	no	[23], [55]
c. Easy divisional task	yes	no	[33], [38], [47]
d. Feasibility	yes	yes	[29], [33], [38], [55]
e. Fast convergence	no	yes	[3], [12], [23], [33], [38], [55]
f. Decentralization	no	yes but restricted	yes

Table 3: Mixed schemes

$$\begin{aligned} & \text{Minimize } c_j(x_j) + \sum_i (p_{ij}g_{ij}(x_j) + g_{ij}(x_j)(y_{ij} - g_{ij}(x_{ij}))^+) \\ & x_j \in S_j \end{aligned} \quad (\text{QP}_j)$$

where $(t)^+ = \text{Max}\{0, t\}$. The subproblems receive prices and resource allocations which are then updated and projected on their respective feasibility subspaces by the coordinator. Observe that these subspaces have the following structure :

$$\text{Primal subspace : } A = \{(z_{11}, \dots, z_{nm}) \mid \sum_{j=1}^m z_{ij} = 0, i = 1, \dots, n\}$$

$$\text{Dual subspace : } B = \{(u_{11}, \dots, u_{nm}) \mid u_{i1} = \dots = u_{im}, i = 1, \dots, n\} = A^\perp$$

Related methods based on proximal techniques may be found in Golstein [23] and Fukushima [19]. We may cite too a regularized version of Brown-Robinson matrix game proposed by Auslender [3]. These regularization methods need convexity assumptions and they offer many interesting features for parallelization purpose (see [8] for general comments on the distributed computation of separable convex programs).

Again, some properties of the cited methods are resumes in Table 3. The third column is limited to completely decentralized procedures.

8 Concluding remarks

We have surveyed the different schemes of hierarchical computation in the context of a procedure for management or planning with decentralized information. We have considered both numerical behavior and informational flows, making a critical analysis of the applicability of each class of algorithms.

Our synthesis do not obviously pretend to be exhaustive. In what concerns the application models, our attention has been focussed on papers that are important for economic

planning. Many applications in technology and some particular models of operations research have not been taken in consideration. For instance, stochastic and dynamic aspects have been left in the shade to give more emphasis on the structural aspects of organization. Another important question that has been left apart refers to the stability of the coordination process when getting close to the optimum. Indeed, for a given environment (i.e. type of functions and parameters that represent the preferences, the technologies and resource availabilities), each mechanism must lie on an 'output rule' that specifies the actions that will be taken at the end of the dialogue. The concepts of transition between the dialogue (iterative computation of the equilibrium) and the decision-making (optimal control) is of particular importance for the theory and practice of decentralized planning (see Luna [36]).

As an interesting conclusion of our synthesis, we may say that the optimization approach is a limited but convenient model to discuss global market mechanisms. It suggests a nice balance between the advantages of informational decentralization and incentives of the markets mechanisms on one hand, and the government actions to drive the social 'optimum' in economic sectors with strong economies of scale, which seems easier with state companies for such monopolistic sectors. Recent efforts in Regional Science and Transportation Research are more devoted to the numerical solution of variational inequalities to cope with more general situations than the ones considered here. As it is shown in Nagurney [46], parallel decomposition algorithms are currently being investigated to solve these equilibrium in a decentralized manner. But the mathematical background on which these techniques rely seem curiously poorer than the one which has motivated earlier the research on decomposition methods, mainly with linear programming. Relaxation and block-coordinate methods are the most popular to take full advantage of parallel processing, but we observe that they are quite useless for the purpose of decentralized information and decision-making as they suppose that the interactions between the divisions are distributed on a low-connected grid of processors, turning the coordination very hard to implement. We think that a good compromise between decentralized computation (with a coordination level) and fine-grain distributed computation of equilibrium is given by the proximal-like mixed coordination presented in the last section (subproblems (QP_j)) as they offer nice features for distributed computation (see [8]) and retain most interesting properties of complete decentralization.

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