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Temporal Here and There

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Abstract. Temporal Here and There (THT) constitutes the logical foundations of Temporal Equilibrium Logic. Nevertheless, it has never been studied in detail since results about axiomatisation and interdefinability of modal operators remained unknown. In this paper we provide a sound and complete axiomatic system for THT together with several results on interdefinability of modal operators.

1 Introduction

In [10], Michael Gelfond and Vladimir Lifschitz introduced the so-called *0* semantics that subsumed many of the existing Logic Programming alternatives but without the syntactic restrictions made by previous approaches. The model-based orientation of this semantics led to a paradigm suitable for constraint-satisfaction problems that is known nowadays as *Answer Set Programming* (ASP) [17,18] and that became one of the most prominent and successful approaches for Knowledge Representation. During the evolution of ASP, many hints have pointed out its relevance inside the theoretical foundations of Non-Monotonic Reasoning. One result had a particular success in the study of foundations of ASP: *Equilibrium Logic* (EQL). Introduced by David Pearce [19], this characterisation has shown interesting features such as the theorem of *Strong equivalence* [15] as well as extensions to first-order and modal logics [4,8,20] without imposing any syntactic restriction on the formulas.

Among this modal extensions, we remark Temporal Equilibrium Logic (TEL) [4], which extends the language of EQL with temporal operators from Linear Time Temporal Logic (LTL) [21]. Following the same spirit as EQL, TEL strongly relies on Logic of Temporal Here and There (THT), an extension of the logic of Here and There (HT) [12]. However, contrary to HT, THT has not been studied in detail. Only its role in the theorem of *Temporal Strong Equivalence* [2]

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and a pair of connections with other logics based on HT [5] are known. In this paper we deal with two problems that remained open in THT. The first problem consists in determining whether modal operators are interdefinable or not while the second problem corresponds to the definition of a sound an complete axiomatic system for THT.

The temporal constructs of THT will be $\Box$, $\Diamond$, $\boxed{\star}$ and $\diamondsuit\!\!\!\!\!\star$, the constructs $\Box$ and $\Diamond$ being interpreted by the successor relation between integers whereas the constructs $\boxed{\star}$ and $\diamondsuit\!\!\!\!\!\star$ being interpreted by the precedence relation between integers. As usual when one has to axiomatise modal logics where some modal constructs are interpreted by the reflexive transitive closure of the accessibility relation used to interpret other modal constructs, our axiomatisation will use inference rules for induction. In this setting, traditional proofs of completeness (see [11, Chap. 9]) are based on canonical model and filtration. In our HT setting, however, the usual filtration method does not allow to transform, as it is the case in ordinary temporal logic, the canonical model into a model where $\boxed{\star}$ and $\diamondsuit\!\!\!\!\!\star$ are interpreted by the precedence relation between integers. For this reason, we had to redefine the filtration method in an appropriate way (see Sect. 6 for details). Moreover, the determinisation of the filtrated model requires, in the case of ordinary temporal logic, the use of a characteristic formula that cannot be expressed in our language. As a result, we had to redefined the determinisation of the filtrated model.

This paper is organised as follows: in Sect. 2 we introduce syntax and two equivalent semantics for THT. In Sect. 3 we go through the problem of interdefinability by defining the notion of bisimulation in the HT setting. The proof of completeness of the axiomatic system described in Sect. 4 is described along Sects. 5–7 and we finish the paper with conclusions and future work.

2 Syntax and Semantics

Let At be a finite or countable set of atomic formulas (with typical members denoted by p, q, etc.). We inductively define the set of all formulas (with typical members denoted by ϕ, ψ, etc.) as follows:

$$\phi ::= p \mid \bot \mid (\phi \vee \psi) \mid (\phi \wedge \psi) \mid (\phi \rightarrow \psi) \mid \Box\phi \mid \Diamond\phi \mid \boxed{\star}\phi \mid \diamondsuit\!\!\!\!\!\star\phi.$$

Note that, following the tradition in Intuitionistic Modal Logic, we have added the new temporal constructs $\Box$, $\Diamond$, $\boxed{\star}$ and $\diamondsuit\!\!\!\!\!\star$ to the ordinary language of IPL. As it will soon become clear, the constructs $\Box$ and $\Diamond$ are equivalent in THT while $\boxed{\star}$ and $\diamondsuit\!\!\!\!\!\star$ are independent. We define $\neg\phi$ as the abbreviation $\neg\phi ::= \phi \rightarrow \bot$. For all sets of formulas x, let $\Box x = \{\varphi \mid \Box\varphi \in x\}$ and $\Diamond x = \{\Diamond\varphi \mid \varphi \in x\}$. The sets $\boxed{\star}x$ and $\diamondsuit\!\!\!\!\!\star x$ are similarly defined. We shall say that a set Σ of formulas is closed if (1) Σ is closed under subformulas; (2) if $\boxed{\star}\varphi \in \Sigma$ then $\Box\boxed{\star}\varphi \in \Sigma$; (3) if $\diamondsuit\!\!\!\!\!\star\varphi \in \Sigma$ then $\Diamond\diamondsuit\!\!\!\!\!\star\varphi \in \Sigma$; (4) if $\varphi \in \Sigma$ then $\neg\varphi \in \Sigma$. Remark that the least closed set of formulas containing a given formula is infinite. Nevertheless, its quotient by the relation of logical equivalence will be finite in the context of THT. We define the *degree* of a formula ϕ (in symbols $deg(\phi)$) by induction as follows: (i) $deg(p) =$

$deg(\bot) = 0$; (ii) $deg(\phi \vee \psi) = deg(\phi \wedge \psi) = deg(\phi \to \psi) = max\{deg(\phi), deg(\psi)\}$; (iii) $deg(\Box\phi) = deg(\Diamond\phi) = 1 + deg(\phi)$; (iv) $deg(\boxstar\phi) = deg(\diamondstar\phi) = deg(\phi)$. We define a temporal model as a structure $\mathcal{M} = \langle H, T\rangle$ where $H : \mathbb{N} \to 2^{At}$ and $T : \mathbb{N} \to 2^{At}$ are such that $H(i) \geqslant T(i)$ for all $i \geqslant 0$. If an atomic formula belongs to $H(i)$, $i \in \mathbb{N}$, then it means that p holds here in $\mathcal{M}$ at time i whereas if p belongs to $T(i)$ then it means that p holds there at time i. The satisfaction relation in a temporal model $\mathcal{M} = \langle H, T\rangle$ of a formula φ at the pair $(i, \alpha) \in \mathbb{N} \times \{h, t\}$, denoted by $\mathcal{M}, (i, \alpha) \models \varphi$, is inductively defined as follows:

- $\mathcal{M}, (i, h) \models p$ iff $p \in H(i)$;
- $\mathcal{M}, (i, t) \models p$ iff $p \in T(i)$;
- $\mathcal{M}, (i, \alpha) \models \varphi \wedge \psi$ iff $\mathcal{M}, (i, \alpha) \models \varphi$ and $\mathcal{M}, (i, \alpha) \models \psi$;
- $\mathcal{M}, (i, \alpha) \models \varphi \vee \psi$ iff $\mathcal{M}, (i, \alpha) \models \varphi$ or $\mathcal{M}, (i, \alpha) \models \psi$;
- $\mathcal{M}, (i, \alpha) \models \varphi \to \psi$ iff for all $\beta \in \{\alpha, t\}$ $\mathcal{M}, (i, \beta) \not\models \varphi$ or $\mathcal{M}, (i, \beta) \models \psi$;
- $\mathcal{M}, (i, \alpha) \models \Box\varphi$ iff $\mathcal{M}, (i + 1, \alpha) \models \varphi$;
- $\mathcal{M}, (i, \alpha) \models \Diamond\varphi$ iff $\mathcal{M}, (i + 1, \alpha) \models \varphi$;
- $\mathcal{M}, (i, \alpha) \models \boxstar\varphi$ iff for all $j \geqslant i$, $\mathcal{M}, (j, \alpha) \models \varphi$;
- $\mathcal{M}, (i, \alpha) \models \diamondstar\varphi$ iff there exists $j \geqslant i$ s.t. $\mathcal{M}, (j, \alpha) \models \varphi$;

We will say that a formula φ is THT-valid (denoted by $THT \models \varphi$) iff $\mathcal{M}, (0, h) \models \varphi$ for all THT models $\mathcal{M}$.

Proposition 1 (Persistence). *For all formulas φ, for all THT models $\mathcal{M}$ and for all $i \in \mathbb{N}$, if $\mathcal{M}, (i, h) \models \varphi$ then $\mathcal{M}, (i, t) \models \varphi$.*

Our aim, in this paper, is to completely axiomatise the set of all THT-valid formulas. This will be done from Sect. 4 on. In the meantime we study an alternative semantics for THT formulas that will be used in the proof of completeness of our axiomatisation. A *birelational model* is a structure of the form $\mathcal{M} = \langle W, \leqslant, R^{\Box}, R^{\boxstar}, V\rangle$ such that:

- W is a non-empty set of worlds;
- $\leqslant$ is a partial order on W;
- $R^{\Box}$ and $R^{\boxstar}$ are binary relations on W;
- $V : W \to 2^{At}$ is such that for all x, $y \in W$, if $x \leqslant y$ then $V(x) \subseteq V(y)$.

Given a birelational model $\mathcal{M} = \langle W, \leqslant, R^{\Box}, R^{\boxstar}, V\rangle$, a world $x \in W$ and a formula φ, the satisfaction relation is defined as follows:

- $\mathcal{M}, x \models p$ iff $p \in V(x)$;
- $\mathcal{M}, x \models (\varphi \wedge \psi)$ iff $\mathcal{M}, x \models \varphi$ and $\mathcal{M}, x \models \psi$;
- $\mathcal{M}, x \models (\varphi \vee \psi)$ iff $\mathcal{M}, x \models \varphi$ or $\mathcal{M}, x \models \psi$;
- $\mathcal{M}, x \models \varphi \to \psi$ iff for all $x' \in W$, if $x \leqslant x'$ then $\mathcal{M}, x' \not\models \varphi$ or $\mathcal{M}, x' \models \psi$;
- $\mathcal{M}, x \models \Box\varphi$ iff for all $x', y \in W$, if $x \leqslant x' R^{\Box} y$ then $\mathcal{M}, y \models \varphi$;
- $\mathcal{M}, x \models \Diamond\varphi$ iff there exists $y \in W$ s. t. $xR^{\Box}y$ and $\mathcal{M}, y \models \varphi$;
- $\mathcal{M}, x \models \boxstar\varphi$ iff for all $x', y \in W$, if $x \leqslant x' R^{\boxstar} y$ then $\mathcal{M}, y \models \varphi$;
- $\mathcal{M}, x \models \diamondstar\varphi$ iff there exists $y \in W$ s. t. $xR^{\boxstar}y$ and $\mathcal{M}, y \models \varphi$.

Notice that the clauses concerning the temporal constructs $\Box$ and $\boxed{\star}$ imitate the clause for the quantifier $\forall$ in first-order intuitionistic logic whereas the clauses concerning $\Diamond$ and $\diamondsuit\!\!\!\!\star$ imitate the clause for $\exists$. See [7, Lemma 5.3.2] for details. We shall say that $\mathcal{M}$ is *normal* if (1) for all $x,\ y,\ z \in W$, if $x \leqslant y$ and $x \leqslant z$ then either $x = y$ or $x = z$ or $y = z$; (2) for all $x,\ y,\ z \in W$, if $x \leqslant y$ and $xR^{\Box}z$ (respectively $xR^{\boxed{\star}}z$) then $yR^{\Box}t$ (respectively $yR^{\boxed{\star}}t$) and $z \leqslant t$ for some $t \in W$; (3) for all $x,\ y,\ z \in W$, if $xR^{\Box}y$ (respectively $xR^{\boxed{\star}}y$) and $y \leqslant z$ then $x \leqslant t$ and $tR^{\Box}z$ (respectively $tR^{\boxed{\star}}z$) for some $t \in W$. If $\mathcal{M}$ is normal then for all $x \in W$, either x is a maximal element with respect to $\leqslant$, or there exists $y \in W$ such that $x \leqslant y$ and $x \neq y$. In the former case let $\widehat{x}$ be x. In the latter case, there exists exactly one $y \in W$ such that $x \leqslant y$ and $x \neq y$; let $\widehat{x}$ be this y. A normal model $\mathcal{M} = \langle W, \leqslant, R^{\Box}, R^{\boxed{\star}}, V\rangle$ is said to be *standard* if $R^{\Box}$ is serial, $R^{\Box}$ is deterministic and $R^{\boxed{\star}}$ is equal to the reflexive transitive closure of $R^{\Box}$. We say that a formula φ is *standard-valid* iff for all standard birelational models $\mathcal{M} = \langle W, \leqslant, R^{\Box}, R^{\boxed{\star}}, V\rangle$ and for all $x_0 \in W$, $\mathcal{M}, x_0 \models \varphi$. We now relate this alternative semantics to the THT semantics. Let $\mathcal{M} = \langle H, T\rangle$ be a THT model. We define the birelational model $\mathcal{M}' = \langle W, \leqslant, R^{\Box}, R^{\boxed{\star}}, V\rangle$ as follows:

- $W = \mathbb{N} \times \{h, t\}$;
- $(i_1, \alpha_1) \leqslant (i_2, \alpha_2)$ iff $i_1 = i_2$ and either $\alpha_1 = h$ or $\alpha_2 = t$;
- $(i_1, \alpha_1)R^{\Box}(i_2, \alpha_2)$ iff $i_1 + 1 = i_2$ and $\alpha_1 = \alpha_2$;
- $(i_1, \alpha_1)R^{\boxed{\star}}(i_2, \alpha_2)$ iff $i_1 \leqslant i_2$ and $\alpha_1 = \alpha_2$;
- $V((i, \alpha)) = H(i)$ if $\alpha = h$ else $T(i)$.

Obviously, $\mathcal{M}'$ is a standard birelational model. Moreover, as the reader can show by structural induction, for all formulas φ, for all $i \in \mathbb{N}$ and for all $\alpha \in \{h, t\}$, $\mathcal{M}, (i, \alpha) \models \varphi$ iff $\mathcal{M}', (i, \alpha) \models \varphi$. Reciprocally, let $\mathcal{M}' = \langle W, \leqslant, R^{\Box}, R^{\boxed{\star}}, V\rangle$ be a standard birelational model. Hence, $R^{\Box}$ is serial, deterministic and for all $x \in W$ and for all $i \in \mathbb{N}$, there exists exactly one $y \in W$ such that $x \left(R^{\Box}\right)^i y$; let $\left(R^{\Box}\right)^i (x_0)$ be this y. Let $x_0 \in W$. We define the functions $H, T : \mathbb{N} \to 2^{At}$ as $H(i) = V(\left(R^{\Box}\right)^i (x_0))$ and $T(i) = V(\widehat{\left(R^{\Box}\right)^i (x_0)})$. Remark that for all $i \in \mathbb{N}$, $H(i) \subseteq T(i)$. Let $\mathcal{M} = \langle H, T\rangle$. Thus, $\mathcal{M}$ is a THT model. As the reader can show by structural induction, for all formulas φ and for all $i \in \mathbb{N}$, $\mathcal{M}', \left(R^{\Box}\right)^i (x_0) \models \varphi$ iff $\mathcal{M}, (i, h) \models \varphi$ and $\mathcal{M}', \widehat{\left(R^{\Box}\right)^i (x_0)} \models \varphi$ iff $\mathcal{M}, (i, t) \models \varphi$. As a result, THT semantics and the alternative semantics are equivalent.

3 Interdefinability

As it is well-known, disjunction is definable in terms of conjunction and implication within the context of *HT* [16].

Lemma 1. *For all formulas ϕ, ψ, THT $\models \phi \vee \psi \leftrightarrow ((\phi \to \psi) \to \psi) \wedge ((\psi \to \phi) \to \phi)$.*

Below, we show the non-interdefinability of conjunction in THT.

Lemma 2. *Let $\mathcal{M}_1 = \langle H_1, T_1\rangle$, $\mathcal{M}_2 = \langle H_2, T_2\rangle$ and $\mathcal{M}_3 = \langle H_3, T_3\rangle$ be the THT models such that for all $i \in \mathbb{N}$, $H_1(i) = \{p, q\}$, $T_i(i) = \{p, q\}$, $H_2(i) = \{p\}$, $T_2(i) = \{p, q\}$, $H_3(i) = \{q\}$ and $T_3(i) = \{p, q\}$. For all $\wedge$-free formulas φ and for all $i \in \mathbb{N}$, $\mathcal{M}_1, (i, h) \models \varphi$ iff $\mathcal{M}_2, (i, h) \models \varphi$ or $\mathcal{M}_3, (i, h) \models \varphi$.*

Proof. By structural induction on φ.

Lemma 3. *Let $p, q \in At$. There is no $\wedge$-free formula ψ such that THT $\models p \wedge q \leftrightarrow \psi$.*

Proof. Remark that $\mathcal{M}_1, (0, h) \models p\wedge q$, $\mathcal{M}_2, (0, h) \not\models p\wedge q$ and $\mathcal{M}_3, (0, h) \not\models p\wedge q$. Hence, by Lemma 2, $p \wedge q$ is THT-equivalent to no $\wedge$-free formula.

In the HT setting, the non-interdefinability of $\wedge$ has also been proved by Aguado et al. [1] by means of a denotational semantics based on sets of models. Our proof is simpler, seeing that it does not require the use of sets of models. Before considering the interdefinability of the modal operators in THT, we must remark that the following equivalences are THT-valid:

- $\Box\bot \leftrightarrow \bot$;
- $\Box(\varphi \vee \psi) \leftrightarrow \Box\varphi \vee \Box\psi$;
- $\Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi$;
- $\Box(\varphi \rightarrow \psi) \leftrightarrow (\Box\varphi \rightarrow \Box\psi)$;
- $\Box\boxed{\star}\varphi \leftrightarrow \boxed{\star}\Box\varphi$;
- $\Box\diamondsuit\varphi \leftrightarrow \diamondsuit\Box\varphi$;
- $\Diamond\bot \leftrightarrow \bot$;
- $\Diamond(\varphi \vee \psi) \leftrightarrow \Diamond\varphi \vee \Diamond\psi$;
- $\Diamond(\varphi \wedge \psi) \leftrightarrow \Diamond\varphi \wedge \Diamond\psi$;
- $\Diamond(\varphi \rightarrow \psi) \leftrightarrow (\Diamond\varphi \rightarrow \Diamond\psi)$;
- $\Diamond\boxed{\star}\varphi \leftrightarrow \boxed{\star}\Diamond\varphi$;
- $\Diamond\diamondsuit\varphi \leftrightarrow \diamondsuit\Diamond\varphi$.

As a result, every formula is equivalent to a formula in which $\bot$, $\vee$, $\wedge$, $\rightarrow$, $\boxed{\star}$ and $\diamondsuit$ do not appear within the scope of $\Box$ or $\Diamond$. In order to prove the non-interdefinability of $\boxed{\star}$ and $\diamondsuit$, we introduce the notions of $\boxed{\star}$-bisimulation and $\diamondsuit$-bisimulation between THT models. Let $\mathcal{D} = \{(i, \alpha) \mid i \in \mathbb{N} \text{ and } \alpha \in \{h, t\}\}$ and let $k \in \mathbb{N}$. A binary relation $\mathcal{Z}$ on $\mathcal{D}$ is said to be a *k-$\boxed{\star}$-bisimulation* between the THT models $\mathcal{M}_1$ and $\mathcal{M}_2$ if the following conditions are satisfied:

1. if $(i_1, \alpha_1)\ \mathcal{Z}\ (i_2, \alpha_2)$ then for all j, $0 \leqslant j \leqslant k$, and for all propositional variables p, $\mathcal{M}_1\,(i_1 + j, \alpha_1) \models p$ iff $\mathcal{M}_2, (i_2 + j, \alpha_2) \models p$;
2. if $(i_1, \alpha_1)\ \mathcal{Z}\ (i_2, \alpha_2)$ then $(i_1, t)\,\mathcal{Z}\,(i_2, t)$ or both $(i_1, \alpha_1)\ \mathcal{Z}\ (i_2, t)$ and $(i_1, t)\ \mathcal{Z}\ (i_2, \alpha_2)$;
3. if $(i_1, \alpha_1)\ \mathcal{Z}\ (i_2, \alpha_2)$ and $i_1 \leqslant j_1$ then there exists $j_2 \in \mathbb{N}$ s.t. $i_2 \leqslant j_2$ and either $(j_1, \alpha_1)\mathcal{Z}\ (j_2, \alpha_2)$ or $(j_1, \alpha_1)\ \mathcal{Z}\ (j_2, t)$;
4. if $(i_1, \alpha_1)\ \mathcal{Z}\ (i_2, \alpha_2)$ and $i_2 \leqslant j_2$ then there exists $j_1 \in \mathbb{N}$ s.t. $i_1 \leqslant j_1$ and $(j_1, \alpha_1)\ \mathcal{Z}\ (j_2, \alpha_2)$ or $(j_1, t)\ \mathcal{Z}\ (j_2, \alpha_2)$.

A binary relation $\mathcal{Z}$ on $\mathcal{D}$ is said to be a *k-$\diamondsuit$-bisimulation* between the THT models $\mathcal{M}_1$ and $\mathcal{M}_2$ if the following conditions are satisfied:

1. if $(i_1, \alpha_1)\ \mathcal{Z}\ (i_2, \alpha_2)$ then for all j, $0 \leqslant j \leqslant k$, and for all propositional variable p, $\mathcal{M}_1, (i_1 + j, \alpha_1) \models p$ iff $\mathcal{M}_2, (i_2 + j, \alpha_2) \models p$;

2. if $(i_1, \alpha_1)\ \mathcal{Z}\ (i_2, \alpha_2)$ then $(i_1, t)\ \mathcal{Z}\ (i_2, t)$ or both $(i_1, \alpha_1)\ \mathcal{Z}\ (i_2, t)$ and $(i_1, t)\ \mathcal{Z}\ (i_2, \alpha_2)$;
3. if $(i_1, \alpha_1)\ \mathcal{Z}\ (i_2, \alpha_2)$ and $i_1 \leqslant j_1$ then there exists $j_2 \in \mathbb{N}$ s.t. $i_2 \leqslant j_2$ and either $(j_1, \alpha_1)\ \mathcal{Z}\ (j_2, \alpha_2)$ or $(j_1, t)\ \mathcal{Z}\ (j_2, \alpha_2)$;
4. if $(i_1, \alpha_1)\ \mathcal{Z}\ (i_2, \alpha_2)$ and $i_2 \leqslant j_2$ then there exists $j_1 \in \mathbb{N}$ s.t. $i_1 \leqslant j_1$ and $(j_1, \alpha_1)\ \mathcal{Z}\ (j_2, \alpha_2)$ or $(j_1, \alpha_1)\ \mathcal{Z}\ (j_2, t)$;

The proof of the following lemmas can be done by induction on ϕ.

Lemma 4 (Bisimulation Lemma 1). *Given THT models $\mathcal{M}_1$ and $\mathcal{M}_2$ and a k-$\boxstar$-bisimulation $\mathcal{Z}$ between them, for all $\diamondminus$-free formulas ϕ, $deg(\phi) \leqslant k$, and for all (i_1, α_1) and $(i_2, \alpha_2) \in \mathcal{D}$, if $(i_1, \alpha_1)\, \mathcal{Z}\, (i_2, \alpha_2)$ then $\mathcal{M}_1, (i_1, \alpha_1) \models \phi$ iff $\mathcal{M}_2, (i_2, \alpha_2) \models \phi$.*

Lemma 5 (Bisimulation Lemma 2). *Given THT models $\mathcal{M}_1$ and $\mathcal{M}_2$ and a k-$\diamondminus$-bisimulation $\mathcal{Z}$ between them, for all $\boxstar$-free formulas ϕ, $deg(\phi) \leqslant k$, and for all (i_1, α_1) and $(i_2, \alpha_2) \in \mathcal{D}$. if $(i_1, \alpha_1)\, \mathcal{Z}\, (i_2, \alpha_2)$ then $\mathcal{M}_1, (i_1, \alpha_1) \models \phi$ iff $\mathcal{M}_2, (i_2, \alpha_2) \models \phi$.*

Proposition 2. *Let $p \in At$. There is no $\diamondminus$-free formula ψ such that $THT \models \diamondminus p \leftrightarrow \psi$.*

Proof. Suppose that ψ is a $\diamondminus$-free formula such that $THT \models \diamondminus p \leftrightarrow \psi$. Let $k \geqslant 0$ be the degree of ψ. Without loss of generality we can assume that $\bot$, $\vee$, $\wedge$, $\rightarrow$ and $\boxstar$ do not appear in ψ within the scope of the connectives $\Box$ and $\Diamond$. Let $\mathcal{M}_1 = \langle H_1, T_1 \rangle$ and $\mathcal{M}_2 = \langle H_2, T_2 \rangle$ be the THT models such that for all $i \in \mathbb{N}$, $H_1(i) = \varnothing$, $T_1(i) = \{p\}$ if $i \bmod k+2 = k+1$ and $\varnothing$ otherwise, $H_2(i) = \{p\}$ if $i = k+1$ and $\varnothing$ otherwise, $T_2(i) = \{p\}$ if $i \bmod k+2 = k+1$ and $\varnothing$ otherwise. Let $\mathcal{Z}$ be the binary relation on $\mathcal{D}$ such that $(i_1, \alpha_1)\, \mathcal{Z}\, (i_2, \alpha_2)$ iff one of the following condition holds: (1) either $\alpha_1 = \alpha_2 = h$ and $i_1 = i_2 = 0$, or $\alpha_1 = \alpha_2 = t$ and $i_1 = i_2$; (2) $\alpha_1 = \alpha_2 = h$ and $i_2 = i_1 + k + 2$; (3) $\alpha_1 = \alpha_2 = t$ and $i_2 = i_1 + k + 2$; (4) $\alpha_1 = t$, $\alpha_2 = h$ and $i_1 = i_2 < k + 2$. The reader may easily verify that $\mathcal{Z}$ is a k-$\boxstar$-bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$. Since $\mathcal{M}_1, (0, h) \not\models \diamondminus p$, therefore $\mathcal{M}_1, (0, h) \not\models \psi$. Hence, by Lemma 4, $\mathcal{M}_2, (0, h) \not\models \psi$. Thus $\mathcal{M}_2, (0, h) \not\models \diamondminus p$: a contradiction.

Proposition 3. *Let $p \in At$. There is no $\boxstar$-free formulas ψ such that $THT \models \boxstar p \leftrightarrow \psi$.*

Proof. Similarly to the proof of Proposition 2, by using the THT models $\mathcal{M}_1 = \langle H_1, T_1 \rangle$ and $\mathcal{M}_2 = \langle H_2, T_2 \rangle$ such that for all $i \in \mathbb{N}$, $H_1(i) = \{p\}$, $T_1(i) = \{p\}$, $H_2(i) = \varnothing$ if $i = k+1$ and $\{p\}$ otherwise, $T_2(i) = \{p\}$ and the binary relation $\mathcal{Z}$ on $\mathcal{D}$ such that $(i_1, \alpha_1)\, \mathcal{Z}\, (i_2, \alpha_2)$ iff one of the following condition holds: (1) either $\alpha_1 = \alpha_2 = h$ and $i_1 = i_2 = 0$; (2) $\alpha_1 = \alpha_2 = t$ and $i_1 = i_2$; (3) $\alpha_1 = \alpha_2 = h$ and $i_2 = i_1 + k + 2$; (4) $\alpha_1 = \alpha_1 = h$ and $i_1 = i_2 > k + 1$; (5) $\alpha_1 = h$, $\alpha_2 = t$ and $i_1 = i_2$.

4 Axiomatisation

The axiomatic system of THT consists of the axioms of Intuitionistic Propositional Logic [6, Chap. 5] plus the following axioms and inference rules:

Hosoi axiom: (1) $p \vee (p \to q) \vee \neg q$;

Axioms for $\Box$ and $\Diamond$:

(2) $\Box p \leftrightarrow \Diamond p$;

(3) $\Box(p \to q) \leftrightarrow (\Box p \to \Box q)$;

(4) $\Box(p \vee q) \leftrightarrow \Box p \vee \Box q$;

(5) $\Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q$;

(6) $\Box\bot \leftrightarrow \bot$

Fisher Servi axioms for $\boxed{\star}$ and $\Diamond\!\!\!\!\star$:

(7) $\boxed{\star}\bot \leftrightarrow \bot$

(8) $\boxed{\star}(p \to q) \to (\boxed{\star}p \to \boxed{\star}q)$;

(9) $\boxed{\star}(p \to q) \to (\Diamond\!\!\!\!\star p \to \Diamond\!\!\!\!\star q)$;

(10) $\Diamond\!\!\!\!\star(p \vee q) \to \Diamond\!\!\!\!\star p \vee \Diamond\!\!\!\!\star q$;

(11) $(\Diamond\!\!\!\!\star p \to \boxed{\star}q) \to \boxed{\star}(p \to q)$;

Axioms combining $\Box$, $\Diamond$, $\boxed{\star}$ and $\Diamond\!\!\!\!\star$: (12) $\boxed{\star}p \to p \wedge \Box\boxed{\star}p$; (13) $p \vee \Diamond\Diamond\!\!\!\!\star p \to \Diamond\!\!\!\!\star p$;

Induction: (14) $\dfrac{p \to \Box p}{p \to \boxed{\star}p}$; (15) $\dfrac{\Diamond p \to p}{\Diamond\!\!\!\!\star p \to p}$;

Modus ponens: (16) $\dfrac{p \to q,\ p}{q}$. **Necessitation**: (17) $\dfrac{p}{\boxed{\star}p}$; (18) $\dfrac{p}{\Box p}$.

Proposition 4 (Soundness). *The axiomatic system presented in this section is sound.*

Proof. Left to the reader. It is sufficient to check that all axioms are valid and the inference rules preserve validity.

The Hosoi axiom corresponds to the fact that, in a normal model, $\mathcal{M}=\langle W, \leqslant, R^{\Box}, R^{\boxed{\star}}, V\rangle$, if $x \leqslant y$ and $x \leqslant z$ then either $x = y$ or $x = z$ or $y = z$. Axioms (2)–(4) correspond to the fact that in a standard model $\mathcal{M} = \langle W, \leqslant, R^{\Box}, R^{\boxed{\star}}, V\rangle$, $R^{\Box}$ is serial and deterministic. The Fisher Servi axioms for $\boxed{\star}$ and $\Diamond\!\!\!\!\star$ are similar to the axioms considered in [9,22]. Remark that the corresponding Fisher Servi axioms for $R^{\Box}$ are easily derivable. Axioms combining $\Box$, $\Diamond$, $\boxed{\star}$ and $\Diamond\!\!\!\!\star$ correspond to the fact that, in a standard model $\mathcal{M} = \langle W, \leqslant, R^{\Box}, R^{\boxed{\star}}, V\rangle$, $R^{\boxed{\star}}$ is reflexive and $R^{\Box} \circ R^{\boxed{\star}} \subseteq R^{\boxed{\star}}$. As for the rules of inference (14) and (15), they will be used in the proof of Lemma 15, where the canonical model $\mathcal{M}_c = \langle W_c, \leqslant_c, R_c^{\Box}, R_c^{\boxed{\star}}, V_c\rangle$ of THT is filtrated into a model $\mathcal{M}_{\Sigma} = \langle W_{\Sigma}, \leqslant_{\Sigma}, R_{\Sigma}^{\Box}, R_{\Sigma}^{\boxed{\star}}, V_{\Sigma}\rangle$ such that $R_{\Sigma}^{\boxed{\star}}$ is the reflexive transitive closure of $R_{\Sigma}^{\Box}$.

Lemma 6. *For all $m, n \in \mathbb{N}$, the following rules are derivable:*

1. $\dfrac{\psi_1 \wedge \ldots \wedge \psi_m \to \phi \vee \chi_1 \vee \ldots \vee \chi_n}{\boxed{\star}\psi_1 \wedge \ldots \wedge \boxed{\star}\psi_m \to \boxed{\star}\phi \vee \Diamond\!\!\!\!\star\chi_1 \vee \ldots \vee \Diamond\!\!\!\!\star\chi_n}$;

2. $$\frac{\phi \wedge \psi_1 \wedge \ldots \wedge \psi_m \rightarrow \chi_1 \vee \ldots \vee \chi_n}{\diamondplus\phi \wedge \boxed{\star}\psi_1 \wedge \ldots \wedge \boxed{\star}\psi_m \rightarrow \diamondplus\chi_1 \vee \ldots \vee \diamondplus\chi_n}\ .$$

Proof. These rules are derivable by means of Fisher Servi axioms. See [9,22].

Lemma 7. *The following formulas are derivable:* $\varphi \wedge \Box\boxed{\star}\varphi \rightarrow \boxed{\star}\varphi$, $\diamondplus\varphi \rightarrow \varphi \vee \Diamond\diamondplus\varphi$

5 Canonical Model Construction

As usual, we will base our proof of completeness on the canonical model construction.

5.1 Prime Sets

Given two sets of formulas x and y, we say that y is a consequence of x (denoted by $x \vdash y$) iff there exists $\phi_1, \ldots, \phi_m \in x$ and $\psi_1, \ldots, \psi_n \in y$ such that $\phi_1 \wedge \ldots \wedge \phi_m \rightarrow \psi_1 \vee \ldots \vee \psi_n \in THT$. We shall say that a set x of formulas is *prime* if it satisfies the following conditions: (1) $\bot \notin x$; (2) for all formulas ϕ, ψ, if $\phi \vee \psi \in x$ then either $\phi \in x$, or $\psi \in x$; (3) for all formulas ϕ, if $x \vdash \phi$ then $\phi \in x$.

Lemma 8 (Lindenbaum Lemma). *Let x and y be sets of formulas. If $x \nvdash y$ then there exists a prime set z of formulas such that $x \subseteq z$ and $z \nvdash y$.*

The next Lemma shows the connection between Hosoi axiom and the relation of inclusion between prime sets of formulas.

Lemma 9. *Let x, y, z be prime sets of formulas. If $x \subseteq y$ and $x \subseteq z$ then either $x = y$, or $x = z$, or $y = z$.*

Proof. By Hosoi axiom.

Proposition 5. *Let x be a prime set of formulas. There exists at most one prime set of formulas strictly containing x.*

Hence, for all prime sets x, either x is maximal, for inclusion, among all prime sets, or there exists a prime set y such that $x \subseteq y$ and $x \neq y$. In the former case, let $\widehat{x} = x$. In the latter case, there exists exactly one prime set y such that $x \subseteq y$ and $x \neq y$; let $\widehat{x}$ be this y. In our HT setting, one can easily show that for all formulas φ, $\varphi \in \widehat{x}$ iff $\neg\neg\varphi \in x$.

5.2 Canonical Model

The canonical model $\mathcal{M}_c$ is defined as the structure $\mathcal{M}_c = \langle W_c, \leqslant_c, R_c^{\Box}, R_c^{\boxed{\star}}, V_c \rangle$ where:

- W_c is the set of all prime sets;
- $\leqslant_c$ is the partial order on W_c defined by: $x \leqslant_c y$ iff $x \subseteq y$;
- $R_c^{\Box}$ is the binary relation on W_c defined by: $xR_c^{\Box}y$ iff $\Box x \subseteq y$ and $\Diamond y \subseteq x$;

– $R_c^{\boxed{\star}}$ is the binary relation on W_c defined by: $xR_c^{\boxed{\star}}y$ iff $\boxed{\star}x \subseteq y$ and $\diamondsuit y \subseteq x$;
– $V_c : W_c \to 2^{At}$ is the valuation function defined by: $p \in V_c(x)$ iff $p \in x$;

Proposition 6. *$\mathcal{M}_c$ is normal.*

Proof. The condition (1) of normality follows from Lemma 9. In order to prove the conditions (2) and (3) it suffices to prove that for all $x, y \in W_c$, if $xR_c^{\Box}y$ then $\widehat{x}R_c^{\Box}\widehat{y}$. Firstly remark that $\neg\neg\Box p \to \Box\neg\neg p$ and $\Diamond\neg\neg p \to \neg\neg\Diamond p$ are in THT seeing that these formulas are derivable in the axiom systems considered in [9, 22]. Secondly, let x and y be prime sets such that $xR_c^{\Box}y$ and suppose that $\widehat{x}\not{R}_c^{\Box}\widehat{y}$. Hence, either $\Box\widehat{x} \not\subseteq \widehat{y}$ or $\Diamond\widehat{y} \not\subseteq \widehat{x}$. Let φ be a formula such that either $\Box\varphi \in \widehat{x}$ but $\varphi \notin \widehat{y}$ or $\varphi \in \widehat{y}$ but $\Diamond\varphi \notin \widehat{x}$. In the former case, $\neg\neg\Box\varphi \in x$, $\Box\neg\neg\varphi \in x$, $\neg\neg\varphi \in y$ and $\varphi \in \widehat{y}$: a contradiction. In the latter case, $\neg\neg\varphi \in y$, $\Diamond\neg\neg\varphi \in x$, $\neg\neg\Diamond\varphi \in x$ and $\Diamond\varphi \in \widehat{x}$: a contradiction.

Proposition 7. *$R_c^{\Box}$ is serial and deterministic.*

Proof. **Seriality:** Let $x \in W_c$. We define $y = \Box x$. By means of Axiom (2) the reader can easily show that y is a prime set such that $\Box x \subseteq y$ and $\Diamond y \subseteq x$, thus $xR_c^{\Box}y$. **Determinism:** Suppose that there exists $x, y, z \in W_c$ such that $xR_c^{\Box}y$ and $xR_c^{\Box}z$ but $y \neq z$. Without loss of generality, let $\varphi \in y$ be such that $\varphi \notin z$. As a consequence $\Diamond\varphi \in x$ but $\Box\varphi \notin x$, which contradicts Axiom (2).

Proposition 8. *$R_c^{\boxed{\star}}$ is reflexive and transitive.*

Proof. **Reflexivity:** Use the first parts of Axioms (12) and (13). **Transitivity:** Use the second parts of Axioms (12) and (13) together with the induction rules.

Remark also that Axioms (12) and (13) guarantee that $\left(R_c^{\Box}\right)^{\star} \subseteq R_c^{\boxed{\star}}$. Nevertheless, as it is usually the case when one axiomatises a modal logic in which one connective is interpreted by the reflexive transitive closure of the relation interpreting another connective, it might be the case that $\left(R_c^{\Box}\right)^{\star} \neq R_c^{\boxed{\star}}$.

Lemma 10 (Truth Lemma). *For all formulas φ and for all $x \in W_c$, (1) If $\varphi \in x$ then $\mathcal{M}_c, x \models \varphi$; (2) if $\varphi \notin x$ then $\mathcal{M}_c, x \not\models \varphi$.*

Proof. By induction on φ. We only present the proof for the case of $\boxed{\star}$. Assume that $\boxed{\star}\psi \in x$ but $\mathcal{M}_c, x \not\models \boxed{\star}\psi$. From the latter assumption it follows that there exists $x', y \in W_c$ such that $x \leqslant_c x'$, $x'R_c^{\boxed{\star}}y$ and $\mathcal{M}_c, y \not\models \psi$. Since $x \leqslant_c x'$ then $\boxed{\star}\psi \in x'$. On the other hand, from $x'R_c^{\boxed{\star}}y$, $\mathcal{M}_c, y \not\models \psi$ and the induction hypothesis we conclude that $\boxed{\star}\psi \notin x'$, which is a contradiction.

Reciprocally, assume that $\mathcal{M}_c, x \models \boxed{\star}\psi$ but $\boxed{\star}\psi \notin x$. Let $u = \boxed{\star}x$. Remark that $u \not\vdash \{\psi\} \cup \{\chi \mid \diamondsuit\chi \notin x\}$. By Lindenbaum Lemma, let $y \in W_c$ be such that $u \subseteq y$ and $y \not\vdash \{\psi\} \cup \{\chi \mid \diamondsuit\chi \notin x\}$. Note that $\boxed{\star}x \subseteq y$ and $\diamondsuit y \subseteq x$. Hence, $xR_c^{\boxed{\star}}y$. Since $y \not\vdash \psi$, therefore $\psi \notin y$ and, by induction hypothesis, $\mathcal{M}_c, y \not\models \psi$, which contradicts $\mathcal{M}_c, x \models \boxed{\star}\psi$ and $xR_c^{\boxed{\star}}y$.

6 Filtration

In order to repair the main defect of $\mathcal{M}_c$, namely $\left(R_c^{\Box}\right)^{\star} \neq R_c^{\boxed{\star}}$, the traditional tool, filtration, consists in identifying prime sets in W_c that contain the same formulas from the least closed set of formulas containing a given formula. We had to change the definition of filtration, seeing that, within the context of THT, the ordinary definition of filtration as the one presented in [11, Chap. 9] is not appropriate. Given a normal THT model $\mathcal{M} = \langle W, \leqslant, R^{\Box}, R^{\boxed{\star}}, V\rangle$ and a closed set Σ of formulas, we define the equivalence relation $\equiv_\Sigma$ on W as: $x \equiv_\Sigma y$ iff for all $\varphi \in \Sigma$, $\mathcal{M}, x \models \varphi$ iff $\mathcal{M}, y \models \varphi$.

Lemma 11. *For all $x, y \in W_c$, if $x \equiv_\Sigma y$ then $\widehat{x} \equiv_\Sigma \widehat{y}$.*

The equivalence class of $x \in W$ with respect to $\equiv_\Sigma$ is denoted by $[x]$. We say that a THT model $\mathcal{M}_\Sigma = \langle W_\Sigma, \leqslant_\Sigma, R_\Sigma^{\Box}, R_\Sigma^{\boxed{\star}}, V_\Sigma\rangle$ is a filtration of $\mathcal{M}$, with respect to Σ, iff $W_\Sigma = W_{|\equiv_\Sigma}$ and for all $x, y \in W$:

1. if $x \leqslant y$ then $[x] \leqslant_\Sigma [y]$;
2. for all $\varphi \to \psi \in \Sigma$, if $[x] \leqslant_\Sigma [y]$, $\mathcal{M}, x \models \varphi \to \psi$ and $\mathcal{M}, y \models \varphi$ then $\mathcal{M}, y \models \psi$;
3. if $xR^{\Box}y$ then there exists $z \in W$ s. t. $[x]R_\Sigma^{\Box}[z]$ and $[y] \leqslant_\Sigma [z]$;
4. if $xR^{\Box}y$ then there exists $t \in W$ s. t. $[t]R_\Sigma^{\Box}[y]$ and $[x] \leqslant_\Sigma [t]$;
5. for all $\Box\varphi \in \Sigma$, if $[x]R_\Sigma^{\Box}[y]$ and $\mathcal{M}, x \models \Box\varphi$ then $\mathcal{M}, y \models \varphi$;
6. for all $\Diamond\varphi \in \Sigma$, if $[x]R_\Sigma^{\Box}[y]$ and $\mathcal{M}, y \models \varphi$ then $\mathcal{M}, x \models \Diamond\varphi$;
7. if $xR^{\boxed{\star}}y$ then there exists $z \in W$ s. t. $[x]R_\Sigma^{\boxed{\star}}[z]$ and $[y] \leqslant_\Sigma [z]$;
8. if $xR^{\boxed{\star}}y$ then there exists $t \in W$ s. t. $[t]R_\Sigma^{\boxed{\star}}[y]$ and $[x] \leqslant_\Sigma [t]$;
9. for all $\boxed{\star}\varphi \in \Sigma$, if $[x]R_\Sigma^{\boxed{\star}}[y]$ and $\mathcal{M}, x \models \boxed{\star}\varphi$ then $\mathcal{M}, y \models \varphi$;
10. for all $\diamondsuit\!\!\!\!\star\,\varphi \in \Sigma$, if $[x]R_\Sigma^{\boxed{\star}}[y]$ and $\mathcal{M}, y \models \varphi$ then $\mathcal{M}, x \models \diamondsuit\!\!\!\!\star\,\varphi$;
11. for all $p \in At \cap \Sigma$, $p \in V_\Sigma([x])$ iff $p \in V(x)$.

Lemma 12 (Filtration Lemma). *Let $\mathcal{M} = \langle W, \leqslant, R^{\Box}, R^{\boxed{\star}}, V\rangle$ be a normal THT model, Σ be a closed set of formulas and $\mathcal{M}_\Sigma = \langle W_\Sigma, \leqslant_\Sigma, R_\Sigma^{\Box}, R_\Sigma^{\boxed{\star}}, V_\Sigma\rangle$ be a filtration of $\mathcal{M}$ with respect to Σ. For all $\varphi \in \Sigma$ and for all $x \in W$, $\mathcal{M}, x \models \varphi$ iff $\mathcal{M}_\Sigma, [x] \models \varphi$.*

Proof. By induction on φ.

We will be interested in the filtration $\mathcal{M}_\Sigma$ of $\mathcal{M}_c$ with respect to the least closed set Σ containing a given formula φ_0. Remind that the quotient of W_c by $\equiv_\Sigma$ is finite. The relational structure $\mathcal{M}_\Sigma = \langle W_\Sigma, \leqslant_\Sigma, R_\Sigma^{\Box}, R_\Sigma^{\boxed{\star}}, V_\Sigma\rangle$ is defined as follows:

1. $W_\Sigma = W_{c|\equiv_\Sigma}$;
2. $[x] \leqslant_\Sigma [y]$ iff $x \equiv_\Sigma \circ \leqslant_c \circ \equiv_\Sigma y$;
3. $[x]R_\Sigma^{\Box}[y]$ iff $x \equiv_\Sigma \circ R_c^{\Box} \circ \equiv_\Sigma y$;
4. $[x]R_\Sigma^{\boxed{\star}}[y]$ iff $[x]\left(R_\Sigma^{\Box}\right)^{*}[y]$;
5. $V_\Sigma([x]) = V_c(x) \cap \Sigma$.

Lemma 13. *For all $x, y \in W_c$, $[x] \leqslant_\Sigma [y]$ iff $x \equiv_\Sigma y$ or $\widehat{x} \equiv_\Sigma y$.*

In the sequel, φ and ψ will be Σ-formulas. For each $t \in W_c$, let

$$\Phi_t = \bigwedge_{\varphi \in t} \varphi \wedge \bigwedge_{\varphi \notin t} \neg\varphi \wedge \bigwedge_{\varphi \in\ t\ \setminus\ t} \neg\neg\varphi \wedge \bigwedge_{\varphi,\ \psi \in\ t\ \setminus\ t} (\varphi \to \psi).$$

By using the results proved in [3] we can deduce that for all $s \in W_c$, $\mathcal{M}_c, s \models \Phi_t$ iff $[s] = [t]$ or $[s] = [\widehat{t}]$. Now W_Σ is finite, as Σ is, so for all $D \subseteq W_\Sigma$ let $\Psi_D = \bigvee_{[t] \in D} \Phi_t$.

Lemma 14. *For any set $D \subseteq W_\Sigma$ and for all $x \in W_c$, $\mathcal{M}_c, x \models \Psi_D$ iff $\exists [z] \in D$ s. t. $[x] = [z]$ or $[x] = [\widehat{z}]$.*

Lemma 15 (Filtrated Model). *The aforementioned filtrated model, $\mathcal{M}_\Sigma$, is a filtration of the canonical model $\mathcal{M}_c$.*

Proof. We only study the Conditions 7 and 9.

- **Condition** 7: Suppose $xR_c^{\boxed{\star}}y$ and let $D = \{[z] \in W_\Sigma \mid$ there exists $t \in W_c$ s.t. $[x]R_\Sigma^{\boxed{\star}}[t]$ and $[z] \leqslant_\Sigma [t]\}$. Let us prove that $[y] \in D$. Remark that for all $[z] \in W_\Sigma$, if $[\widehat{z}] \in D$ then $[z] \in D$. Suppose by contradiction that $[y] \in (W_\Sigma \backslash D)$. Remark that $[x] \in D$. Let $\Psi_{W_\Sigma \backslash D}$ be the characteristic formula of $W_\Sigma \backslash D$. Since $[y] \in (W_\Sigma \backslash D)$ then, by Lemma 14, $\mathcal{M}_c, y \models \Psi_{W_\Sigma \backslash D}$. Since $xR_c^{\boxed{\star}}y$, it holds that $\mathcal{M}_c, x \models \Diamond\!\!\!\!\Diamond \Psi_{W_\Sigma \backslash D}$. Since $[x] \in D$, then $\mathcal{M}_c, x \not\models \Psi_{W_\Sigma \backslash D}$ and, therefore, $\mathcal{M}_c, x \not\models \Diamond\!\!\!\!\Diamond \Psi_{W_\Sigma \backslash D} \to \Psi_{W_\Sigma \backslash D}$. Consequently $\Diamond\!\!\!\!\Diamond \Psi_{W_\Sigma \backslash D} \to \Psi_{W_\Sigma \backslash D} \notin THT$. From the Induction rule (15) we conclude that $\Diamond \Psi_{W_\Sigma \backslash D} \to \Psi_{W_\Sigma \backslash D} \notin THT$. This means that there exists $u \in W_c$ such that $\mathcal{M}_c, u \models \Diamond \Psi_{W_\Sigma \backslash D}$ and $\mathcal{M}_c, u \not\models \Psi_{W_\Sigma \backslash D}$ (therefore $[u] \in D$). From the latter it follows that there exists $t \in W_c$ s.t. $[x]R_\Sigma^{\boxed{\star}}[t]$ and $[u] \leqslant [t]$, while from the former we get that there exists $v \in W_c$ such that $uR_c^{\Box}v$ and $\mathcal{M}_c, v \models \Psi_{W_\Sigma \backslash D}$ (thus $[v] \notin D$). Since $[u] \leqslant_\Sigma [t]$, therefore by Lemma 13 either $u \equiv_\Sigma t$ or $\widehat{u} \equiv_\Sigma t$. In the former case $[u] = [t]$ and we have: $[u]R_\Sigma^{\Box}[v]$, $[t]R_\Sigma^{\Box}[v]$ and $[x]R_\Sigma^{\boxed{\star}}[v]$. Thus $[v] \in D$ and $\mathcal{M}_c, v \not\models \Psi_{W_\Sigma \backslash D}$: a contradiction. In the latter case, $[t] = [\widehat{u}]$ and we have: $[\widehat{u}]R_\Sigma^{\Box}[\widehat{v}]$, $[t]R_\Sigma^{\Box}[\widehat{v}]$ and $[x]R_\Sigma^{\boxed{\star}}[\widehat{v}]$. Hence, $[v] \in D$ and $\mathcal{M}_c, v \not\models \Psi_{W_\Sigma \backslash D}$: a contradiction.
- **Condition** 9: Suppose $[x]R_\Sigma^{\boxed{\star}}[y]$ and let $\boxed{\star}\varphi \in \Sigma$. Suppose $\mathcal{M}_c, x \models \boxed{\star}\varphi$ and let $k \in \mathbb{N}$ be such that $[x]\left(R_\Sigma^{\Box}\right)^k[y]$. Such k exists by definition of $R_\Sigma^{\boxed{\star}}$. By induction on k, we demonstrate $\mathcal{M}_c, y \models \varphi$. Firstly, assume $k = 0$, therefore $[x] = [y]$, which means that $x \equiv_\Sigma y$. From $\mathcal{M}_c, x \models \boxed{\star}\varphi$ and Axiom (12) we conclude that $\mathcal{M}_c, y \models \varphi$. For the inductive step, assume $k \geqslant 1$ and let $[z]$ be such that $[x]R_\Sigma^{\Box}[z]$ and $[z]\left(R_\Sigma^{\Box}\right)^{k-1}[y]$. From $\mathcal{M}_c, x \models \boxed{\star}\varphi$ and Axiom (12) we conclude that $\mathcal{M}_c, x \models \Box\boxed{\star}\varphi$. Since Σ is closed and $\boxed{\star}\varphi \in \Sigma$, therefore $\Box\boxed{\star}\varphi \in \Sigma$. From $[x]R_\Sigma^{\Box}[z]$ and Condition 5 of Filtration we conclude that $\mathcal{M}_c, z \models \boxed{\star}\varphi$. Finally from $[z]\left(R_\Sigma^{\Box}\right)^{k-1}[y]$ and the induction hypothesis it follows that $\mathcal{M}_c, y \models \varphi$.

Lemma 16. *For all* $[x], [y], [z] \in W_\Sigma$*, if* $[x] \leqslant_\Sigma [y]$ *and* $[x] \leqslant_\Sigma [z]$ *then* $[x] = [y]$ *or* $[x] = [z]$ *or* $[y] = [z]$*.*

Proof. Suppose that $[x] \leqslant_\Sigma [y]$ and $[x] \leqslant_\Sigma [z]$. Let x', x'', y' and z'' in W_c be such that $x \equiv_\Sigma x' \leqslant_c y' \equiv_\Sigma y$ and $x \equiv_\Sigma x'' \leqslant_c z'' \equiv_\Sigma z$. Moreover, suppose that $[x][y]$, $[x] \neq [z]$ and $[y] \neq [z]$. Without loss of generality, let ϕ, ψ and χ in Σ be such that ($\mathcal{M}_c, x \nvDash \phi$ and $\mathcal{M}_c, y \vDash \phi$), ($\mathcal{M}_c, x \nvDash \chi$ and $\mathcal{M}_c, z \vDash \chi$) and ($\mathcal{M}_c, y \vDash \psi$ and $\mathcal{M}_c, z \nvDash \psi$). Since $\mathcal{M}_c, y \vDash \psi$ then $\mathcal{M}_c, y \nvDash \neg\psi$ and, together with the definition of $[x] \leqslant_\Sigma [y]$, $\mathcal{M}_c, x \nvDash \neg\psi$. Moreover, from $x \equiv_\Sigma x''$ and $\mathcal{M}_c, x \nvDash \phi$ we conclude that $\mathcal{M}_c, x'' \nvDash \phi \vee \neg\psi$ and, by means of Hosoi axiom, it follows that $\mathcal{M}_c, x'' \vDash \phi \to \psi$. Since $x'' \leqslant_c z''$ then $M_c, z'' \vDash \phi \to \psi$. Apart from this, since $z'' \equiv_\Sigma z$, $\mathcal{M}_c, z \vDash \phi$ and $\mathcal{M}_c, z \nvDash \psi$ then $\mathcal{M}_c, z'' \nvDash \psi$ and $\mathcal{M}_c, z'' \vDash \phi$. Finally, from $\mathcal{M}_c, z'' \vDash \phi \to \psi$ we reach a contradiction.

Proposition 9. $\mathcal{M}_\Sigma$ *is normal.*

Proof. Condition (1) of normality follows from Lemma 16. To prove Conditions (2) and (3) it is sufficient to prove that if $[x]R^{\Box}_\Sigma[y]$ (respectively $[x]R^{\Box}_\Sigma[y]$) then $[\widehat{x}]R^{\Box}_\Sigma[\widehat{y}]$ (respectively $[\widehat{x}]R^{\boxstar}_\Sigma[\widehat{y}]$). The proof for $R^{\Box}_\Sigma$ follows from Lemma 11 and Proposition 6 while the proof for $R^{\boxstar}_\Sigma$ follows a similar argument.

Lemma 17. *For any formula* φ *and* $x \in W_c$*.*

(1) If $\boxstar\varphi \in \Sigma$ *and* $\boxstar\varphi \notin x$ *then there exists* $y \in W_c$ *such that* $[x]R^{\boxstar}_\Sigma[y]$ *and* $\varphi \notin y$*;*
(2) If $\Diamond\!\!\!\star\varphi \in \Sigma$ *and* $\Diamond\!\!\!\star\varphi \in x$ *then there exists* $y \in W_c$ *such that* $[x]R^{\boxstar}_\Sigma[y]$ *and* $\varphi \in y$*;*

Proof. (1) From $\boxstar\varphi \notin x$ and Lemma 10 we conclude that $\mathcal{M}_c, x \nvDash \boxstar\varphi$, so there exists $z \in W_c$ such that $xR^{\boxstar}_c z$ and $\mathcal{M}_c, z \nvDash \varphi$. From Condition 8 of filtration we conclude that either $[x]R^{\boxstar}_\Sigma[z]$ or $[\widehat{x}]R^{\boxstar}_\Sigma[z]$. In the first case, take $y = z$. In the second case, we follow the argument as follows: from $\mathcal{M}_c, z \nvDash \varphi$ and Condition 9 of filtration we conclude that $\mathcal{M}_c, \widehat{x} \nvDash \boxstar\varphi$. By following an argument as in Proposition 6 $\mathcal{M}_c, x \nvDash \boxstar\neg\neg\varphi$, thus there exists $t \in W_c$ such that $xR^{\boxstar}_c t$ and $\mathcal{M}_c, t \nvDash \neg\neg\varphi$ (as a consequence, $\mathcal{M}_c, t \nvDash \varphi$ and $\mathcal{M}_c, \widehat{t} \nvDash \varphi$). Finally by applying the Condition 7 of filtration, we conclude that $[x]R^{\boxstar}_\Sigma[t]$ or $[x]R^{\boxstar}_\Sigma[\widehat{t}]$. In the first case take $y = t$ while, in the second one take $y = \widehat{t}$. (2) From $\Diamond\!\!\!\star\varphi \in \Sigma$, $\Diamond\!\!\!\star\varphi \in x$ and Lemma 10 we conclude that $\mathcal{M}_c, x \vDash \Diamond\!\!\!\star\varphi$ and, therefore, there exists $y \in W_c$ such that $xR^{\boxstar}_c y$ and $\mathcal{M}_c, y \vDash \varphi$ (and $\mathcal{M}_c, \widehat{y} \vDash \varphi$). Then, due to Condition 7 of filtration it follows that either $[x]R^{\boxstar}_\Sigma[y]$ or $[x]R^{\boxstar}_\Sigma[\widehat{y}]$. We conclude the proof by saying that it is sufficient to take y in the first case and $\widehat{y}$ in the second one to reach the condition.

Lemma 18. *Let* $\Box\varphi \in \Sigma$ *be a temporal formula and* $x \in W_c$*. The following conditions are equivalent: (1)* $\mathcal{M}_c, x \vDash \Box\varphi$*. (2)* $\forall y \in W_c$*, if* $\big([x]R^{\Box}_\Sigma[y]$ *then* $\mathcal{M}_c, y \vDash \varphi\big)$ *(3)* $\exists y \in W_c \big([x]R^{\Box}_\Sigma[y]$ *and* $\mathcal{M}_c, y \vDash \varphi\big)$*.*

Proof. **(1)⇒ (2):** Assume there exists $y \in W_c$ such that $[x]R^{\Box}_{\Sigma}[y]$ and $\mathcal{M}_c, y \not\models \varphi$. Thanks to the Condition 5 of filtration we get $\mathcal{M}_c, y \models \varphi$: a contradiction. **(2)⇒ (3):** Take $[x] \in W_c$. Since $R^{\Box}_{\Sigma}$ is serial, there exists $[y] \in W_{\Sigma}$ such that $[x]R^{\Box}_{\Sigma}[y]$. From 18 and $[x]R^{\Box}_{\Sigma}[y]$ we obtain (3). **(3) ⇒ (1):** By definition of $[x]R^{\Box}_{\Sigma}[y]$, there exist $x', y' \in W_c$ such that $x \equiv_{\Sigma} x' R^{\Box}_c y' \equiv_{\Sigma} y$. From $\mathcal{M}_c, y \models \varphi$ and Axiom (2) it follows that $\mathcal{M}_c, x \models \Box\varphi$.

Lemma 19. *Let $\Diamond\varphi \in \Sigma$ be a temporal formula and $x \in W_c$. The following conditions are equivalent: (1) $\mathcal{M}_c, x \not\models \Diamond\varphi$. (2) $\exists y \in W_c \left([x]R^{\Box}_{\Sigma}[y] \text{ and } \mathcal{M}_c, y \not\models \varphi\right)$ (3) $\forall y \in W_c$, if $\left([x]R^{\Box}_{\Sigma}[y] \text{ then } \mathcal{M}_c, y \not\models \varphi\right)$.*

Proof. Similar to the proof of Lemma 18.

7 Determinisation

The filtrated model defined in Sect. 6 possesses the normality conditions (1) and (2). Since $R^{\Box}_{\Sigma}$ is serial and $R^{\boxstar}_{\Sigma}$ is equal to the reflexive transitive closure of the $R^{\Box}_{\Sigma}$, $\mathcal{M}_{\Sigma}$ would be standard if $R^{\Box}_{\Sigma}$ were deterministic. The property of determinism is not preserved by filtration. In this section we show how to extract a deterministic model from $\mathcal{M}_{\Sigma}$. Before that, we must introduce the concepts of chain and defect. Let $S = \mathbb{N} \times \{\boxstar, \diamondstar\} \times \Sigma$. Remark that S is countable. Let $(k_0, \sigma_0, \psi_0), (k_1, \sigma_1, \psi_1), \cdots$ be an enumeration on S where each triple is repeated infinitely many times. A *chain* consists of a finite sequence $([x_0], \cdots, [x_n])$ of elements of W_{Σ} such that for all $i < n$, $[x_i]R^{\Box}_{\Sigma}[x_{i+1}]$. A triple $(k, \boxstar, \psi) \in S$ is a *defect* of the chain $([x_0], \cdots [x_n])$ if (1) $k \leqslant n$; (2) $\boxstar\psi \notin x_k$; (3) for all i, $k \leqslant i \leqslant n, \psi \in x_i$. Similarly, a triple $(k, \diamondstar, \psi) \in S$ is a defect of the sequence $([x_0], \cdots [x_n])$ if (1) $k \leqslant n$; (2) $\diamondstar\psi \in x_k$; (3) for all i, $k \leqslant i \leqslant n, \psi \notin x_i$. Let φ_0 be a formula such that $\varphi_0 \notin$ THT. Let $x_0 \in W_c$ be such that $\varphi_0 \notin x_0$. We define an infinite sequence $([x_0], [x_1], \cdots)$ of elements of W_{Σ} such that $[x_0]R^{\Box}_{\Sigma}[x_1]R^{\Box}_{\Sigma}\ [x_3] \cdots$ as follows: let $S_0 = ([x_0])$. Let $a \geqslant 0$ and $S_a = ([x_0], \cdots, [x_m])$ be a sequence of elements of W_{Σ} such that $[x_0]R^{\Box}_{\Sigma} \cdots R^{\Box}_{\Sigma}[x_m]$. We consider the following cases:

- Case "(k_a, σ_a, ψ_a) is not a defect of S_a": In this case let $[y] \in W_{\Sigma}$ be such that $[x_m]R^{\Box}_{\Sigma}[y]$ and define $S_{a+1} = ([x_0], \cdots, [x_m], [y])$.
- Case "(k_a, σ_a, ψ_a) is a defect of S_a and $\sigma_a = \boxstar$": Hence, $k_a \leqslant x_m$, $\boxstar\psi_a \notin x_m$ and for all i, $k_a \leqslant i \leqslant m$, $\psi_a \in x_i$. By Lemma 7, $\boxstar\psi_a \notin x_m$. By Lemma 17, let $[y] \in W_{\Sigma}$ be such that $[x_m]R^{\boxstar}_{\Sigma}[y]$ and $\psi_a \notin y$. Let $[y_0], \cdots, [y_n] \in W_{\Sigma}$ be such that $[y_0] = [x_m]$, $[y_n] = [y]$ and $[y_0]R^{\Box}_{\Sigma}[y_1] \cdots R^{\Box}_{\Sigma}[y_n]$. We define $S_{a+1} = ([x_0], \cdots, [x_m], [y_1], \cdots, [y_n])$.
- Case "(k_a, σ_k, ψ_k) is a defect of S_a and $\sigma_a = \diamondstar$": This case is similar to the previous one.

Now, let $\mathcal{M}_d = \langle W_d, \leqslant_d, R^{\Box}_d, R^{\boxstar}_d, V_d \rangle$ be the model defined as follows:

- $W_d = \mathbb{N} \times \{h, t\}$;

– $(i_1, \alpha_1) \leqslant_d (i_2, \alpha_2)$ iff $i_1 = i_2$ and either $\alpha_1 = h$ or $\alpha_2 = t$;
– $(i_1, \alpha_1) R_d^{\Box} (i_2, \alpha_2)$ iff $i_1 + 1 = i_2$ and $\alpha_1 = \alpha_2$;
– $(i_1, \alpha_1) R_d^{\boxstar} (i_2, \alpha_2)$ iff $i_1 \leqslant i_2$ and $\alpha_1 = \alpha_2$;
– $V_d((i, \alpha)) = \{p \in At \mid p \in x_i \cap \Sigma\}$ if $\alpha = h$ and $\{p \in At \mid p \in \widehat{x_i} \cap \Sigma\}$ otherwise.

Lemma 20 (Truth Lemma). *Let $\varphi \in \Sigma$. For all $i \in \mathbb{N}$ and for all $\alpha \in \{h, t\}$, the following conditions are equivalent: (1) $\mathcal{M}_d, (i, \alpha) \models \varphi$; (2) $\mathcal{M}_\Sigma, [x_i] \models \varphi$.*

Proof. By induction on φ. The case for atomic formulas follows from the definition of V_d. The cases for $\bot$, $\wedge$, $\vee$ and $\rightarrow$ are left to the reader. The cases for $\Box$ and $\Diamond$ follow from Lemmas 18 and 19. The cases for $\boxstar$ and $\diamondstar$ follow from the definition of $\mathcal{M}_d$.

And now, the grand finale:

Proposition 10. *Let φ be a formula. The following conditions are equivalent: (1) $\varphi \in THT$; (2) $THT \models \varphi$.*

Proof. (1) $\Rightarrow$ (2): By proposition 4. (2) $\Rightarrow$ (1): Suppose $\varphi \notin THT$. Let $x_0 \in W_c$ be such that $\varphi \notin x_0$. By Lemma 10, $\mathcal{M}_c, x_0 \not\models \varphi$. Let Σ be the least closed set of formulas containing φ. By Lemmas 12 and 15, $\mathcal{M}_\Sigma, [x_0] \not\models \varphi$. By Lemma 20, $\mathcal{M}_d, (0, \alpha) \not\models \varphi$. Since $\mathcal{M}_d$ is standard, therefore $THT \not\models \varphi$.

8 Conclusion

Much remains to be done. For example, suppose the language is extended by the temporal constructs $\mathcal{U}$ (until) and $\mathcal{R}$ (release). In that case, within the context of THT-models, can we demonstrate that these temporal constructs are not interdefinable? And how to axiomatise the set of all THT-valid formulas? One may also consider, for this extended language, a van Benthem characterization theorem. Its proof will probably necessitates the definition of an appropriate notion of bisimulation similar to the one considered by de Rijke and Kurtonina [14]. Now, what do these problems become when the language, restricted to the temporal constructs $\mathcal{U}$ and $\mathcal{R}$, is interpreted over the nonnegative rationals or the nonnegative reals? In that case, THT-models will be of the form $\mathcal{M} = \langle H, T \rangle$ where $H : \mathbb{Q}^+$ (or $\mathbb{R}^+$) $\rightarrow 2^{At}$ and $T : \mathbb{Q}^+$ (or $\mathbb{R}^+$) $\rightarrow 2^{At}$ are such that $H(i) \subseteq T(i)$ for each $i \geqslant 0$. In other respect, for the language extended by the temporal constructs $\mathcal{U}$ (until), $\mathcal{R}$ (release), $\mathcal{S}$ (since) and $\mathcal{T}$ (trigger), when interpreted over the set of all integers, can we demonstrate that these temporal constructs are not interdefinable? When interpreted over Dedekind-complete linear orders, can one obtain for this language a THT version of Kamp's Theorem [13]? Finally, if one prefers partial orders to linear orders then one may want to axiomatise the HT version of branching time logics like CTL.

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