Dynamics of a Geostationary Satellite

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Technical Report

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1 Introduction

The study of the dynamics of a body around the Earth began with the study of the gravitation equation attributed to Newton and the three Kepler laws for the motion around a spherical body. This motion is called keplerian motion. For space applications as the station keeping of a geostationary satellite, this keplerian motion is not enough to describe the spacecraft trajectory. Orbital disturbances have to be added in the description of the motion, namely the fact that Earth is not a spherical body, that the Sun and the Moon create a gravitational attraction on the spacecraft and that the Sun radiation pressure change the spacecraft trajectory.

From the study of the keplerian dynamics and the orbital perturbations, a dynamical model of the spacecraft is set up. The state vector can consist either in the cartesian positions and velocities or the orbital elements. As for station keeping, the satellite stay in the vicinity of its station keeping point, it is possible to linearize the dynamics with respect to the gap between the actual position of the spacecraft and the station keeping one.

This document is organized as follows. The first section describes the physics of the keplerian motion and the orbital perturbations disturbing this keplerian motion. The second section focuses on the state representation of the spacecraft flying around the Earth on the quasi-geostationary orbit. The following section derives the non linear and the linearized equation of motion and the last section computed the linearized dynamics matrices for each orbital perturbation.

2 Orbital Mechanics

2.1 Keplerian Motion

In this section, the equations for the keplerian motion are recalled (see for instance the references [Battin, 1999], [Sidi, 1997] ou [Vallado, 1997]) and some notations used through this document are presented.

2.1.1 Dynamics Equations for the Keplerian Motion

As a first approximation, the Earth can be considered as a spherical body. A spacecraft flying in the gravitational field of the Earth undergoes a force whose expression is given by the Newton gravitational law:

$$\vec{F}_g = -\frac{\mathcal{G} m_\oplus m_{\text{sat}}}{r^2}\frac{\vec{r}}{r}, \tag{1}$$

with:

- $\mathcal{G}$ the gravitational constant,
- $m_\oplus$ the Earth mass,
- m_{sat} the satellite mass,
- $\vec{r}$ the Earth-satellite radius vector,
- r the norm of this vector.

The standard geocentric gravitational parameter is defined by $\mu_\oplus = \mathcal{G} m_\oplus$.

According to the Newton second law, the variation of the spacecraft momentum is equal to the sum of the external forces. As the spacecraft mass is supposed to remain constant, the equations of motion in the inertial geocentric frame supposed to be galilean (see the Figure 8 page 47) are given by:

$$\left.\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2}\right|_{\mathcal{B}_G} = -\frac{\mu_\oplus}{r^2}\frac{\vec{r}}{r} \tag{2}$$

This differential equation describes the motion of the spacecraft and of the Earth around the center of mass of the system {Earth-spacecraft}. As the spacecraft mass is very small with respect to the Earth mass, the Earth-spacecraft center of mass center of mass is supposed to be the center of the Earth. Therefore, the Equation (2) describes the motion of the spacecraft around the Earth.

2.1.2 Keplerian Trajectory

In order to solve the Equation (2), integrals of the motion are first derived. The specific angular momentum is defined by:

$$\vec{h} = \vec{r} \times \vec{v}, \tag{3}$$

and is constant. As the angular momentum is perpendicular to the position and velocity vectors, the plane defined by the initial position and the initial velocity is constant. Therefore, the spacecraft trajectory lies in a plane.

The solution of the Equation (2) is a conic. With a polar parametrisation, the equation of this conic is given by:

$$r = \frac{a(1-e^2)}{1+e\cos(\nu)}, \tag{4}$$

where a is the semi-major axis of the conic, e its eccentricity and ν the angular parameter. a and e describe the shape of this conic. In particular, e defines the conic type:

- if $0 \leqslant e < 1$, the trajectory is an ellipse (or a circle in the case $e = 0$),
- if $e = 1$, the conic is a parabola,
- if $e > 1$, the conic is an hyperbola.

The trajectory is bounded and the motion is periodic in the first case only. Therefore, in the sequel, only trajectories with eccentricities strictly smaller than 1 will be studied.

The semi-minor axis b is also used and is defined by:

$$b = a\sqrt{1-e^2}. \tag{5}$$

In the case $e < 1$, the trajectory is closed and the orbital period is defined as:

$$T = 2\pi\sqrt{\frac{a^3}{\mu_\oplus}}. \tag{6}$$

The mean motion is defined as:

$$n = \frac{2\pi}{T} = \sqrt{\frac{\mu_\oplus}{a^3}}. \tag{7}$$

Conversion formulas between the position and the velocities on one hand and the parameters defining the spacecraft trajectory on the other hand are given in the Appendix B page 55.

2.1.3 The Geostationary Orbit

For spacecraft orbiting the Earth, the orbits are sorted in the following categories :

- Low Earth Orbits (LEO), for an altitude smaller than 800 km ($a < 7178$ km),
- Mid Earth Orbits (MEO), for an altitude between 800 km and 30 000 km (7178 km $<$ $a < 36378$ km),
- Geosynchronous Orbits, for which the orbital period of the spacecraft is equal to the rotation period of the Earth,
- Geostationary Earth Orbits (GEO) for which the spacecraft stays over the same point of the Earth. The GEO orbit is a particular geosynchronous orbit with a zero inclination and a zero eccentricity.

The GEO semi-major axis can be computed thanks to the Equation (6):

$$a = \left(\frac{\mu_\oplus T_\oplus^2}{4\pi^2}\right)^{\frac{1}{3}}. \tag{8}$$

With the sidereal Earth rotation period $T_\oplus = 86164$ s and the geocentric standard gravitational parameter $\mu_\oplus = 3.986.10^{14}$ m^3/s^2, it follows $a = 42158$ km.

2.2 Orbital Perturbations

The orbitals perturbations are all the forces, except the Earth gravitational one, that a spacecraft undergoes. Among this forces, the references [Shrivastava, 1978], [Valorge, 1995] and [Vallado, 1997] introduce the following ones relevant for a spacecraft on a GEO orbit:

- the part of the Earth gravitational potential produced by the non spherical distribution of the mass,
- the Sun and and the Moon gravitational attractions,
- the solar radiation pressure: photons emitted by the Sun are absorbed and re-emitted by the spacecraft, what induces a variation of its momentum.

The following subsections describe the modeling of the disturbing accelerations and potentials for a spacecraft orbiting the Earth on a GEO orbit.

2.2.1 Gravitational Attraction of a non Spherical Earth

If the Earth is supposed to be a perfect sphere, the gravitational potential is expressed as:

$$\mathcal{E}(r, \varphi, \lambda) = -\frac{\mu_\oplus}{r}. \tag{9}$$

However, the Earth is not a spherical body. Therefore, adopting the expression given in the reference [Vallado, 1997], the gravitational potential is given by:

$$\mathcal{E}_{p,\mathrm{pot}}(r, \varphi, \lambda) = \mathcal{G} \int_{\Omega(\mathrm{Terre})} \frac{dm(r_P, \varphi_P, \lambda_P)}{\|\vec{r} - \vec{r}_P\|} = \mathcal{G} \int_{\Omega(\mathrm{Terre})} \frac{dm(r_P, \varphi_P, \lambda_P)}{r\sqrt{1 - \frac{r_P}{r}\cos\Lambda + \frac{r_P^2}{r^2}}}, \tag{10}$$

with $\vec{r}$ the vector from the Earth center to the potential computation point, $\vec{r}_P$ the vector from the Earth center to a point of the Earth whose mass is dm and Λ the angle between these two vectors (see Figure 1).

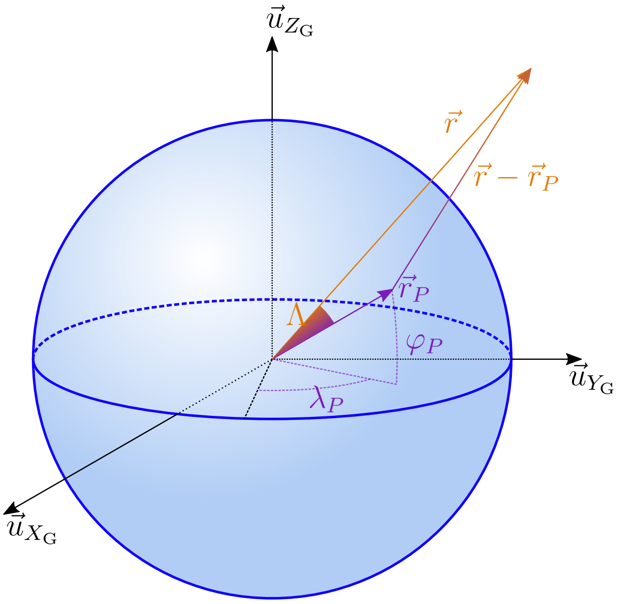

Figure 1 – Angle Λ between the direction of the point in which the gravitational potential is computed and the direction of the current integration point (scheme from [Vallado, 1997]).

Assuming that $\frac{r_P^2}{r^2} \ll 1$, the integrand can be expanded in series of Legendre polynomials:

$$\frac{1}{\sqrt{1 - \frac{r_P}{r}\cos\Lambda + \frac{r_P^2}{r^2}}} = \sum_{l=0}^{\infty} \left(\frac{r}{r_P}\right)^l P_l(\cos\Lambda), \tag{11}$$

where P_l is the Legendre polynomial of degree l given by:

$$P_l(x) = \frac{1}{2^l l!}\frac{\mathrm{d}^l(x^2-1)^l}{\mathrm{d}x^l} = \frac{1}{2^l}\sum_{j=0}^{l}\frac{(-1)^j(2l-2j)!}{j!(l-j)!(l-2j)!}x^{l-2j}. \tag{12}$$

The gravitational potential of a non spherical Earth is thus:

$$\mathcal{E}_{p,\mathrm{pot}}(r,\varphi,\lambda) = \frac{\mu_\oplus}{r}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\left(\frac{r_\oplus}{r}\right)^l P_{l,m}(\sin\varphi)\Big[C_{lm}\cos(m\lambda) + S_{lm}\sin(m\lambda)\Big], \tag{13}$$

where the $P_{l,m}$ are the associated Legendre functions:

$$P_{l,m}(x) = \frac{1}{2^l l!}(1-x^2)^{\frac{m}{2}}\frac{\mathrm{d}^{l+m}(x^2-1)^l}{\mathrm{d}x^{l+m}}, \tag{14}$$

and the C_{lm} and S_{lm} are the coefficients of the decomposition of the potential in spherical harmonics.

The gravitational potential can be rewritten as the sum of the keplerian gravitational potential and a disturbing gravitational potential (see for instance the references [Soop, 1994, Montenbruck and Gill, 2000, Sidi, 1997, Losa, 2007]). This potential is a function of the radius, the latitude and the longitude expanded on the spherical basis of the associated Legendre functions:

$$\begin{aligned}\mathcal{E}_{p,\mathrm{pot}}(r,\varphi,\lambda) &= \frac{\mu_\oplus}{r} + \mathcal{U}_P(r,\varphi,\lambda)\\ &= \frac{\mu_\oplus}{r} + \frac{\mu_\oplus}{r}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\left(\frac{r_\oplus}{r}\right)^l P_{l,m}(\sin\varphi)\Big[C_{lm}\cos(m\lambda) + S_{lm}\sin(m\lambda)\Big].\end{aligned} \tag{15}$$

In some textbooks, the J_l coefficients are used instead of the C_{l0} ones. These coefficients are linked by $J_l = -C_{l0}$.

2.2.2 Sun and Moon Disturbing Gravitational Potential

The Sun and the Moon create a gravitational force acting on a spacecraft orbiting the Earth. These attraction forces tend to change the spacecraft orbital plane due to the fact that the Sun and the Moon orbital planes are inclined with respect to the Earth equator. Thus these gravitational forces act on the orbital plane of the spacecraft.

Denoting:

- $\mu_\odot$ et $\mu_☾$ the Sun and Moon gravitational parameters respectively,
- $\vec{r}_\odot$ et $\vec{r}_☾$ the position vectors of the Sun and the Moon respectively in the geocentric inertial reference frame, with $r_\odot$ and $r_☾$ their norms (see Figure 2),

the acceleration acting on the spacecraft due to the gravitational attraction of the Sun and the Moon is given by:

$$\vec{a}_{p,\odot☾} = -\frac{\mu_\odot}{r^3}\vec{r} + \mu_\odot\left[\frac{\vec{r}_\odot - \vec{r}}{\|\vec{r}_\odot - \vec{r}\|^3} - \frac{\vec{r}_\odot}{r_\odot^3}\right] + \mu_☾\left[\frac{\vec{r}_☾ - \vec{r}}{\|\vec{r}_☾ - \vec{r}\|^3} - \frac{\vec{r}_☾}{r_☾^3}\right], \tag{16}$$

and is usually expanded with the two first Legendre polynomials:

$$\vec{a}_{p,\odot☾} \approx -\frac{\mu_\odot}{r^3}\vec{r} + \frac{\mu_\odot}{r_\odot^3}\left[\frac{3}{r_\odot^2}(\vec{r}_\odot \cdot \vec{r})\vec{r}_\odot - \vec{r}\right] + \frac{\mu_☾}{r_☾^3}\left[\frac{3}{r_☾^2}(\vec{r}_☾ \cdot \vec{r})\vec{r}_☾ - \vec{r}\right]. \tag{17}$$

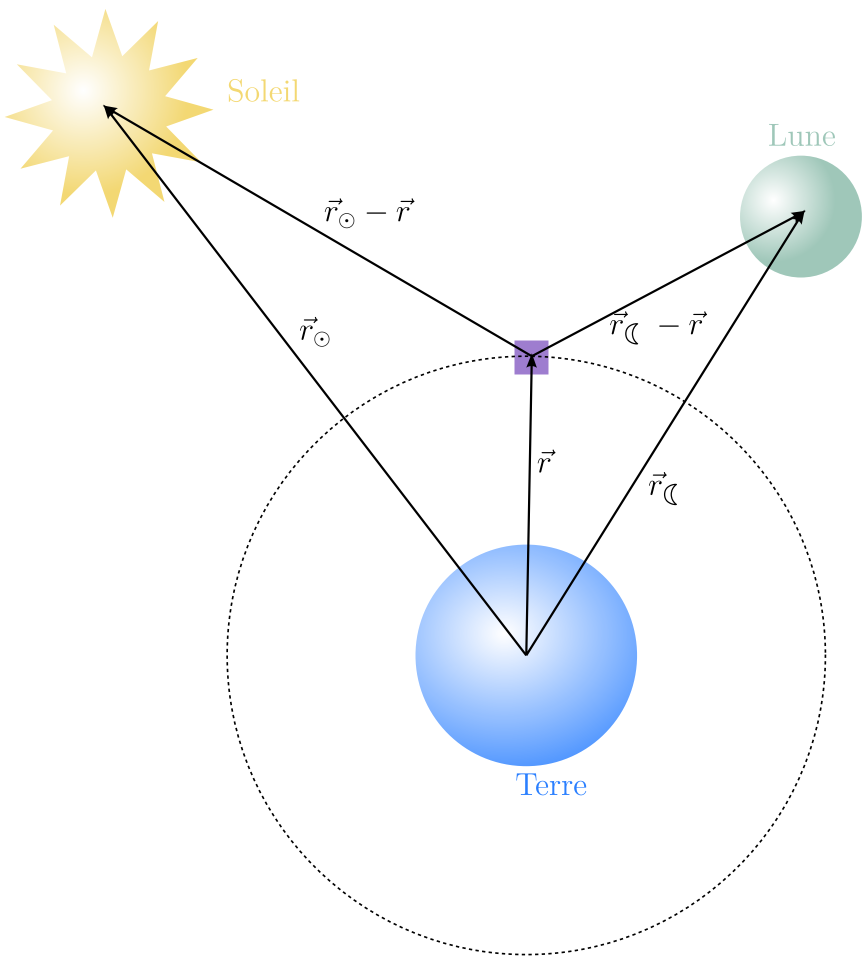

Figure 2 – Illustration des vecteurs $\vec{r}_\odot$ et $\vec{r}_☾$.

The Sun and Moon gravitational attraction potential is derived from the Equation (16) (see the reference [Cot, 1984] and [Campan et al., 1995]):

$$
\begin{aligned}
\mathcal{E}_{p,\odot☾} &= \mu_\odot \left(\frac{1}{\|\vec{r}_\odot - \vec{r}\|} - \frac{\vec{r}\cdot\vec{r}_\odot}{\|\vec{r}_\odot\|^3} \right) - \mu_☾ \left(\frac{1}{\|\vec{r}_☾ - \vec{r}\|} - \frac{\vec{r}\cdot\vec{r}_☾}{\|\vec{r}_☾\|^3} \right), \\
&\approx \frac{\mu_\odot}{r_\odot} \left[1 + \left(\frac{r}{r_\odot}\right)^2 \frac{3(\vec{r}\cdot\vec{r}_\odot)^2 - 1}{2} \right] + \frac{\mu_☾}{r_☾} \left[1 + \left(\frac{r}{r_☾}\right)^2 \frac{3(\vec{r}\cdot\vec{r}_☾)^2 - 1}{2} \right].
\end{aligned}
\tag{18}
$$

2.2.3 Sun Radiation Pressure (SRP)

The radiations emitted by the Sun create a pressure on the spacecraft because of the absorption and emission of photons. According to [Soop, 1994] and [Montenbruck and Gill, 2000], the mean solar power received per surface unit in the vicinity of the Earth is $1.4\ 10^{-6}\ \mathrm{kW/m^2}$. This solar power creates a mean pressure on a surface in the Earth-Sun direction $P_\odot = 4.56\ 10^{-6}\mathrm{N/m^2}$.

In order to compute the value of the force on the satellite from the pressure, it is mandatory to know the area projected on the spacecraft-Sun direction. Several models exist for the computation of this projected area. The most common one is the cannonball model that model the spacecraft as a sphere (see the reference [Lucchesi, 2001]).

If S is the projected area in the Earth-Sun direction, c the radiation pressure coefficient, $\vec{u}_{sS}$ the unit vector in the spacecraft-Sun direction and ϱ a parameter being 0 if the satellite is in the shadow of the Earth and 1 otherwise, the acceleration created by the radiation pressure is written as:

$$
\vec{a}_{\mathrm{SRP}} = \varrho \frac{S P_\odot c}{m} \vec{u}_{sS} \tag{19}
$$

Neglecting the eclipses and assuming that the spacecraft-Sun and Earth-Sun directions are almost the same, it is possible to define the pseudo-potential function for the Sun radiation pressure:

$$
\mathcal{E}_{p,\mathrm{SRP}}(r,\varphi,\lambda) = \varrho \frac{S P_\odot c}{m} \|\vec{r}_\odot - \vec{r}\|, \tag{20}
$$

More complex models exist (see [McMahon and Scheeres, 2010] and the references therein) and take into account the attitude of the spacecraft for instance for a more precise computation of the effect of the radiation pressure.

2.2.4 Summary

In the reference [Valorge, 1995], the relative effects of the disturbing forces with respect to the central gravitational force are compared for a LEO satellite and a GEO satellite. These results are summarized in the Table 1. The J_2 effect is in both cases 100 times as strong as the other disturbing effects. For GEO satellite, the atmospheric drag is completely negligible.

Disturbing force	LEO	GEO
Central gravitational force	1	1
J2 effect	10^{-2}	10^{-5}
Non spherical potential (except J2)	10^{-4}	10^{-7}
Sun and Moon attraction	$10^{-7} \sim 10^{-5}$	$10^{-7} \sim 10^{-4}$
Sun radiation pressure	$10^{-13} \sim 10^{-7}$	$10^{-13} \sim 10^{-7}$
Atmospheric drag	$10^{-9} \sim 10^{-7}$	0

Table 1 – Comparison between orbital disturbing with respect to the central gravitational force for a LEO and a GEO spacecraft.

3 State Representation for the Spacecraft Motion

Through this document, the system is a GEO spacecraft supposed to be a point mass that evolves in the Earth gravitational field with the orbital disturbing forces introduced in the previous section. Six parameters or generalized coordinates are mandatory for the description of the satellite motion. These six parameters form the state vector of the spacecraft. Several state representations can be used. In the sequel, the state vector is written with the cartesian positions and velocities (cartesian parameters), then with the classical or equinoctial orbital elements. The reference [Hintz, 2008] introduce other orbital elements sets that can be used to describe the spacecraft motion.

3.1 Cartesian State Representation

The positions and the velocities of the spacecraft in the geocentric inertial reference frame as depicted in the Figure 8 page 47 are:

$$\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathcal{R}_G} = x\vec{u}_{X_G} + y\vec{u}_{Y_G} + z\vec{u}_{Z_G}, \tag{21}$$

$$\vec{v} = \left.\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right|_{\mathcal{R}_G} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}_{\mathcal{R}_G} = \dot{x}\vec{u}_{X_G} + \dot{y}\vec{u}_{Y_G} + \dot{z}\vec{u}_{Z_G}. \tag{22}$$

The cartesian state vector reads thus as:

$$x_{\text{cart}}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \\ \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{bmatrix}. \tag{23}$$

3.2 Orbital Elements State Representation

3.2.1 Classical Orbital Elements

In the case where the spacecraft flies in a central body gravitation field, the trajectory is a conic. If the trajectory is closed, it is an ellipse or a circle. In that case, the trajectory can be represented by the following parameters:

- two parameters for the shape of the ellipse:
  - ◇ the semi-major axis a,
  - ◇ the eccentricity e,
- three parameters for the orientation of the ellipse in space:
  - ◇ the inclination i of the orbit with respect to the ecliptic plane,
  - ◇ the right ascension of the ascending node Ω : angle in the equatorial plane between the vernal equinox and the intersection direction between the orbital and the equatorial planes in the ascending direction,
  - ◇ the perigee argument ω : angle between the ascending node and the perigee,
- a parameter called anomaly describing the position of the spacecraft on its orbit (time-varying parameter).

These six parameters are called classical orbital elements or keplerian orbital elements (see for instance the reference [Battin, 1999]). The physical meaning of these orbital elements is depicted on the Figure 3. The mean motion of the spacecraft on its orbit n is usually associated to these parameters:

$$n = \sqrt{\frac{\mu_\oplus}{a^3}}, \tag{24}$$

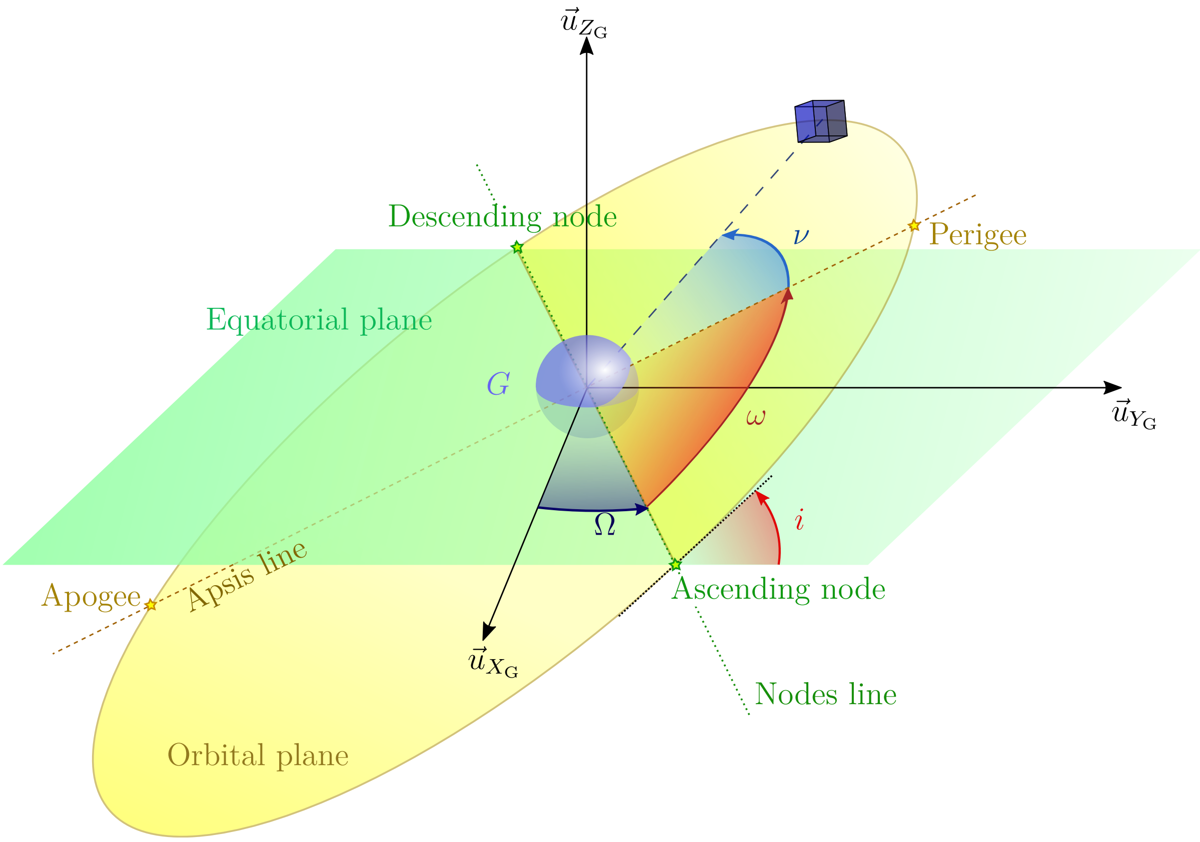

Figure 3 – Classical orbital elements

Three anomalies are classically handled in order to describe the spacecraft position on its orbit:

- the true anomaly ν,
- the eccentric anomaly E,
- the mean anomaly M.

The true anomaly is the angle between the perigee direction and the direction of the radius $\vec{r}$.

The eccentric anomaly is an angle defined as illustrated by the Figure 4. The sine, cosine and tangent of the eccentric anomaly are related to the sine, consine and tangent of the true anomaly thanks to the following relations:

$$\sin E = \frac{\sin \nu \sqrt{1-e^2}}{1 + e\cos\nu}, \tag{25a}$$

$$\cos E = \frac{e + \cos\nu}{1 + e\cos\nu}, \tag{25b}$$

$$\tan\left(\frac{E}{2}\right) = \sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\nu}{2}\right). \tag{25c}$$

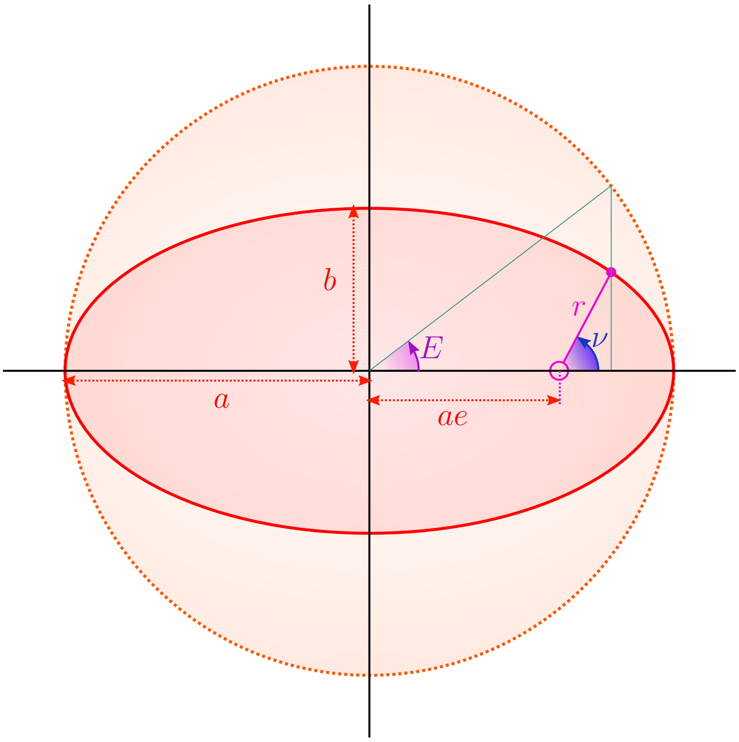

Figure 4 – True and eccentric anomalies

The mean anomaly corresponds to the position of a fictitious spacecraft with a constant velocity on a circular orbit with a period equal to the one of the real spacecraft. The mean anomaly satisfies thus the following equality:

$$M(t) = M(t_0) + n(t - t_0). \tag{26}$$

The eccentric anomaly and the mean one are related through the Kepler equation:

$$E - e \sin E = M. \tag{27}$$

This transcendental equation has no simple solutions. An approximated solution can be computed with an iterative process (Newton method). In the case where the eccentricity is small, the reference [Hull, 2003] gives a method for computing an approximate solution. This method is described in the Appendix D page 69.

A state vector is created with the classical orbital elements:

$$x_{\text{COE}}(t) = \begin{bmatrix} a \\ e \\ i \\ \Omega \\ \omega \\ \nu(t) \text{ ou } E(t) \text{ ou } M(t) \end{bmatrix}, \tag{28}$$

and in the sequel, we will use the mean anomaly M as the time-varying parameter. The state vector is thus:

$$x_{\text{COE}}(t) = \begin{bmatrix} a \\ e \\ i \\ \Omega \\ \omega \\ M(t) \end{bmatrix}. \tag{29}$$

Conversion formulas between the cartesian parameters and the classical orbital elements can be found in the Appendix B page 55.

3.2.2 Éléments orbitaux équinoxiaux

Two singular cases may arise with the classical orbital elements:

- when the orbit is circular, the eccentricity is zero and the perigee argument ω is not defined (because the perigee is not defined),
- when the orbit lies in the equatorial plane, the inclination is zero and the right ascension of the ascending node is not defined (because the ascending node is not defined).

In order to overcome these difficulties, the equinoctial orbital elements are introduced:

$$\begin{cases} a \\ e_x = e\cos(\omega + \Omega) \\ e_y = e\sin(\omega + \Omega) \\ i_x = \tan(i/2)\cos(\Omega) \\ i_y = \tan(i/2)\sin(\Omega) \\ \text{equinoctial anomaly} \end{cases} \tag{30}$$

(e_x, e_y) is called the eccentricity vector and (i_x, i_y) the inclination vector.

In the references [Battin, 1999] and [Cefola, 1972], the equinoctial orbital elements are defined with switched e_x and e_y as well as switched i_x and i_y. Moreover, the inclination vector is sometimes defined using alternative forms:

$$\begin{cases} i_x = \sin(i/2)\cos(\Omega), \\ i_y = \sin(i/2)\sin(\Omega), \end{cases} \quad \text{or} \quad \begin{cases} i_x = \sin(i)\cos(\Omega), \\ i_y = \sin(i)\sin(\Omega). \end{cases} \tag{31}$$

The anomalies for the equinoctial elements are usually called longitudes (see [Battin, 1999] for instance): true longitude L, mean longitude l and eccentric longitude K. For clarity reasons and in order not to mix these anomalies with the geographical longitude, the time-varying parameters are renamed and written as:

- true longitude : $L \rightarrow$ true equinoctial anomaly : $\nu_Q = \Omega + \omega + \nu$,
- mean longitude : $l \rightarrow$ mean equinoctial anomaly : $M_Q = \Omega + \omega + M$,
- eccentric longitude : $K \rightarrow$ eccentric equinoctial anomaly : $E_Q = \Omega + \omega + E$.

Three other angular time-varying parameter called true longitude $\ell_{\nu\Theta}$, mean longitude $\ell_{M\Theta}$ and eccentric longitude $\ell_{E\Theta}$ are defined by:

$$\begin{aligned} \ell_{\nu\Theta} &= \Omega + \omega + \nu - \Theta = \nu_Q - \Theta, \\ \ell_{M\Theta} &= \Omega + \omega + M - \Theta = M_Q - \Theta, \\ \ell_{E\Theta} &= \Omega + \omega + E - \Theta = E_Q - \Theta. \end{aligned} \tag{32}$$

$\Theta(t)$ is the right ascension of the Greenwich meridian and is defined by the Equation (129), page 49.

We choose to replace the sixth parameter of the equinoctial orbital elements state vector by the mean longitude $l_{M\Theta}$ in order to form the following state vector:

$$x_{\mathrm{EOE}}(t) = \begin{bmatrix} a \\ e_x \\ e_y \\ i_x \\ i_y \\ \ell_{M\Theta}(t) \end{bmatrix}. \tag{33}$$

The anomalies defined in this section are summarized in the Table 2.

True equinoctial anomaly	$\nu_Q = \Omega + \omega + \nu$
Mean equinoctial anomaly	$M_Q = \Omega + \omega + M$
Eccentric equinoctial anomaly	$E_Q = \Omega + \omega + E$
Geographical longitude	λ
True longitude	$\ell_{\nu\Theta} = \nu_Q - \Theta$
Mean longitude	$\ell_{M\Theta} = M_Q - \Theta$
Eccentric longitude	$\ell_{E\Theta} = E_Q - \Theta$

Table 2 – Equinoctial anomalies and longitudes.

4 Free Evolution Equations

The previous sections described the environmental disturbing forces acting on the spacecraft and the state vector that will be used for the description of the motion. In this section, the dynamics equations will be derived for a satellite undergoing the Earth central gravitational attraction as well as the disturbing forces: the non spherical Earth gravitational field, the Sun and the Moon gravitational forces and the force from the sun radiation pressure. The acceleration created by the spacecraft thrusters is not taken into account in this section.

4.1 Keplerian Motion

When the Earth central gravitational attraction force is acting on the spacecraft, the non linear dynamics equation reads:

$$\begin{cases} \left.\dfrac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right|_{\mathcal{R}_G} = \vec{v}, \\ \left.\dfrac{\mathrm{d}\vec{v}}{\mathrm{d}t}\right|_{\mathcal{R}_G} = -\mu_{\oplus}\dfrac{\vec{r}}{||\vec{r}||^3}. \end{cases} \tag{34}$$

It can be rewritten as:

$$\frac{\mathrm{d}x_{\mathrm{cart}}}{\mathrm{d}t} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ -\mu_{\oplus}\dfrac{x}{\sqrt{x^2+y^2+z^2}^3} \\ -\mu_{\oplus}\dfrac{y}{\sqrt{x^2+y^2+z^2}^3} \\ -\mu_{\oplus}\dfrac{z}{\sqrt{x^2+y^2+z^2}^3} \end{bmatrix}. \tag{35}$$

With the classical orbital elements, the anomaly is the only time-varying parameter. The evolution of the systems is thus given by:

$$\frac{\mathrm{d}x_{\mathrm{COE}}}{\mathrm{d}t} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ n = \sqrt{\dfrac{\mu}{a^3}} \end{bmatrix} = K(x_{\mathrm{COE}}). \tag{36}$$

The keplerian dynamics can be also expressed using the equinoctial orbital elements. Using the transformation between the classical orbital elements and the equinoctial ones $x_{\mathrm{EOE}} = x_{\mathrm{EOE}}(x_{\mathrm{COE}}, t)$, the keplerian dynamics reads:

$$\begin{aligned} \frac{\mathrm{d}x_{\mathrm{EOE}}}{\mathrm{d}t} &= \frac{\partial x_{\mathrm{EOE}}}{\partial x_{\mathrm{COE}}}\frac{\mathrm{d}x_{\mathrm{COE}}}{\mathrm{d}t} - \frac{\partial x_{\mathrm{COE}}}{\partial t}, \\ &= \frac{\partial x_{\mathrm{EOE}}}{\partial x_{\mathrm{COE}}}\frac{\mathrm{d}x_{COE}}{\mathrm{d}t} - \omega_T, \\ &= K(x_{\mathrm{EOE}}) - \omega_T, \end{aligned} \tag{37}$$

with

$$\omega_T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \omega_{\oplus} \end{bmatrix}, \tag{38}$$

and $\dfrac{\partial x_{\mathrm{EOE}}}{\partial x_{\mathrm{COE}}}$ the jacobian transformation matrix between the classical and the equinoctial orbital elements:

$$\frac{\partial x_{\mathrm{EOE}}}{\partial x_{\mathrm{COE}}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{e_x}{\sqrt{e_x^2+e_y^2}} & 0 & -e_y & -e_y & 0 \\ 0 & \dfrac{e_y}{\sqrt{e_x^2+e_y^2}} & 0 & e_x & e_x & 0 \\ 0 & 0 & \dfrac{i_x(1+i_x^2+i_y^2)}{2\sqrt{i_x^2+i_y^2}} & -i_y & 0 & 0 \\ 0 & 0 & \dfrac{i_y(1+i_x^2+i_y^2)}{2\sqrt{i_x^2+i_y^2}} & i_x & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}. \tag{39}$$

The dynamics equation for the equinoctial state vector reads thus:

$$\frac{\mathrm{d}x_{\mathrm{EOE}}}{\mathrm{d}t} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \sqrt{\dfrac{\mu_\oplus}{a^3}} - \omega_\oplus \end{bmatrix} = K(x_{\mathrm{EOE}}) - \omega_T. \tag{40}$$

4.2 Non Keplerian Motion

The non keplerian motion refers to the motion of the spacecraft taking into account all the disturbing forces described in the section 2.2.

4.2.1 Osculating Orbits

When disturbing forces act on the spacecraft, the transformation between the cartesian positions and velocities and the classical orbital elements is time-dependent. At each time, a set of orbital elements can be defined. These elements are thus time-varying and are called osculating orbital elements, defining an osculating trajectory. This trajectory is the one the satellite would follow if the orbital disturbances were instantly removed. The osculating ellipse is thus tangent to the spacecraft trajectory, but their curvature are different.

The state vectors in terms of the classical or equinoctial orbital elements is thus:

$$x_{\mathrm{COE}}(t) = \begin{bmatrix} a(t) \\ e(t) \\ i(t) \\ \Omega(t) \\ \omega(t) \\ M(t) \end{bmatrix} \text{ et } x_{\mathrm{EOE}}(t) = \begin{bmatrix} a(t) \\ e_x(t) \\ e_y(t) \\ i_x(t) \\ i_y(t) \\ \ell_{M\Theta}(t) \end{bmatrix}. \tag{41}$$

The aim of this section is to derive the dynamics equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, \vec{a}_p, t), \tag{42}$$

with $\vec{a}_p$ the disturbing accelerations, i.e. the accelerations produced by the forces except the Earth central gravitational one. All these disturbing forces are deriving from a potential function. Therefore:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, \mathcal{E}_p, t), \tag{43}$$

where $\mathcal{E}_p$ is the disturbing potential.

With the Lagrange perturbation technique (see the reference [Zarrouati, 1987]), it is possible to derive the dynamics equation for the state vector expressed in terms of the orbital elements with respect to the perturbing potential, assuming that the perturbations are small with respect to the central acceleration term.

4.2.2 Lagrange Equations

The Lagrange equation expresses the evolution of the state vector written with orbital elements for a spacecraft undergoing small perturbations deriving from a potential. Expressing this gradient with respect to the orbital elements, it comes:

$$\vec{a}_p = \left(\frac{\partial \mathcal{E}_p}{\partial x_{\mathrm{COE}}}\right)^T. \tag{44}$$

The motion equation is thus rewritten as:

$$\frac{\mathrm{d}x_{\mathrm{COE}}}{\mathrm{d}t} = K(x_{\mathrm{COE}}) + L_{\mathrm{COE}}(x_{\mathrm{COE}}, t)\left(\frac{\partial \mathcal{E}_p}{\partial x_{\mathrm{COE}}}\right)^T, \tag{45}$$

where the first term is the keplerian part and the second term is the effect of the disturbing forces.

$L_{\mathrm{COE}}(x_{\mathrm{COE}}, t)$ is the Lagrange matrix defined in the reference [Zarrouati, 1987] by:

$$L_{\mathrm{COE}}(x_{\mathrm{COE}}, t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{2}{na} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{1-e^2}}{na^2 e} & \frac{e^2-1}{na^2 e} \\ 0 & 0 & 0 & \frac{1}{na^2\sqrt{1-e^2}\sin i} & -\frac{\cos i}{na^2\sqrt{1-e^2}\sin i} & 0 \\ 0 & 0 & -\frac{1}{na^2\sqrt{1-e^2}\sin i} & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{1-e^2}}{na^2 e} & \frac{\cos i}{na^2\sqrt{1-e^2}\sin i} & 0 & 0 & 0 \\ \frac{2}{na} & \frac{1-e^2}{na^2 e} & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{46}$$

The equation of motion (45) is then transformed in order to express the evolution of the state vector in terms of the equinoctial orbital elements:

$$
\begin{aligned}
\frac{\mathrm{d}x_{\mathrm{EOE}}}{\mathrm{d}t} &= \left(\frac{\partial x_{\mathrm{EOE}}}{\partial x_{\mathrm{COE}}}\right) \frac{\mathrm{d}x_{\mathrm{COE}}}{\mathrm{d}t} - \frac{\partial x_{\mathrm{COE}}}{\partial t}, \\
&= K(x_{\mathrm{EOE}}) - \omega_T + \left(\frac{\partial x_{\mathrm{EOE}}}{\partial x_{\mathrm{COE}}}\right) L_{\mathrm{COE}}(x_{\mathrm{COE}}, t) \left(\frac{\partial \mathcal{E}_p}{\partial x_{\mathrm{COE}}}\right)^T, \\
&= K(x_{\mathrm{EOE}}) - \omega_T + \left(\frac{\partial x_{\mathrm{EOE}}}{\partial x_{\mathrm{COE}}}\right) L_{\mathrm{COE}}(x_{\mathrm{COE}}, t) \left(\frac{\partial \mathcal{E}_p}{\partial x_{\mathrm{EOE}}} \frac{\partial x_{\mathrm{EOE}}}{\partial x_{\mathrm{COE}}}\right)^T, \\
&= K(x_{\mathrm{EOE}}) - \omega_T + \left(\frac{\partial x_{\mathrm{EOE}}}{\partial x_{\mathrm{COE}}}\right) L_{\mathrm{COE}}(x_{\mathrm{COE}}, t) \left(\frac{\partial x_{\mathrm{EOE}}}{\partial x_{\mathrm{COE}}}\right)^T \left(\frac{\partial \mathcal{E}_p}{\partial x_{\mathrm{EOE}}}\right)^T, \\
&= K(x_{\mathrm{EOE}}) - \omega_T + L_{\mathrm{EOE}}(x_{\mathrm{EOE}}, t) \left(\frac{\partial \mathcal{E}_p}{\partial x_{\mathrm{EOE}}}\right)^T.
\end{aligned}
\tag{47}
$$

The Lagrange matrix with the equinoctial orbital elements is:

$$
L_{\mathrm{EOE}}(x_{\mathrm{EOE}}, t) = \left(\frac{\partial x_{\mathrm{EOE}}}{\partial x_{\mathrm{COE}}}\right) L_{\mathrm{COE}}(x_{\mathrm{COE}}, t) \left(\frac{\partial x_{\mathrm{EOE}}}{\partial x_{\mathrm{COE}}}\right)^T. \tag{48}
$$

The Lagrange matrix transformed with the equinoctial orbital elements is computed with the jacobian matrix of the transformation between the classical and the equinoctial orbital elements (39) and the transformations of the section 3.2.2. Denoting $p = 1 + i_x^2 + i_y^2$ and recalling:

$$
\begin{aligned}
n &= \sqrt{\frac{\mu}{a}}, \\
b &= a\sqrt{1 - e^2}, \\
e^2 &= e_x^2 + e_y^2,
\end{aligned}
\tag{49}
$$

the Lagrange matrix reads:

$$
L_{\mathrm{EOE}}(x_{\mathrm{EOE}}, t) =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & -\dfrac{2}{na} \\
0 & 0 & \dfrac{b}{na^3} & \dfrac{e_y i_x p}{2nab} & \dfrac{e_y i_y p}{2nab} & \dfrac{e_x b}{na^2(a+b)} \\
0 & -\dfrac{b}{na^3} & 0 & -\dfrac{e_x i_x p}{2nab} & -\dfrac{e_x i_y p}{2nab} & \dfrac{e_y b}{na^2(a+b)} \\
0 & -\dfrac{e_y i_x p}{2nab} & \dfrac{e_x i_x p}{2nab} & 0 & \dfrac{p^2}{4nab} & \dfrac{i_x p}{2nab} \\
0 & -\dfrac{e_y i_y p}{2nab} & \dfrac{e_x i_y p}{2nab} & -\dfrac{p^2}{4nab} & 0 & \dfrac{i_y p}{2nab} \\
\dfrac{2}{na} & -\dfrac{e_x b}{na^2(a+b)} & -\dfrac{e_y b}{na^2(a+b)} & -\dfrac{i_x p}{2nab} & -\dfrac{i_y p}{2nab} & 0
\end{bmatrix}. \tag{50}
$$

In the sequel, the equinoctial orbital elements will be used. Therefore, the index EOE will be omitted and the dynamics equation reads:

$$
\frac{\mathrm{d}^{\mathrm{real}} x}{\mathrm{d}t} = K(x) - \omega_T + \sum_{\text{perturbation } i} L(x, t) \left(\frac{\partial \mathcal{E}_{p_i}}{\partial x}\right)^T = f_L(x, t). \tag{51}
$$

The f_L function encompass the Earth central acceleration term as well as the orbital disturbances.

4.3 Linearized Evolution Equation

In the case of station keeping, the spacecraft has to stay in the vicinity of the station keeping position. Hence, the distance between the spacecraft and the station keeping point is very small with respect to the distance to the center of the Earth. It is thus possible to linearized the spacecraft dynamics with respect to the ideal trajectory of the station keeping point evolving on the geostationary orbit.

4.3.1 Linearization Point

The nominal station keeping position is a point defined by the state vector:

$$x_{sk} = \begin{bmatrix} a_{sk} \\ 0 \\ 0 \\ 0 \\ 0 \\ \ell_{M\Theta,sk} \end{bmatrix}, \tag{52}$$

with $a_{sk} = 42165.765$ km and $\ell_{M\Theta,mp}$ the station keeping mean longitude. This point is a fictitious point on a geostationary orbit and evolves following an unperturbed keplerian orbit:

$$\begin{aligned} \dot{x}_{sk} &= \left.\frac{\mathrm{d}^{\mathrm{kep}}x}{\mathrm{d}t}\right|_{x=x_{sk}} \\ &= 0 \text{ by hypothesis.} \end{aligned} \tag{53}$$

The gap between the station keeping position and the spacecraft position is written:

$$\Delta x = x - x_{sk}. \tag{54}$$

$\Delta x(t)$ is at each time the gap between the fictitious position of a point that would evolve on a geostationary unperturbed keplerian orbit and the real position of the spacecraft following the equation of motion:

$$\dot{x} = \frac{\mathrm{d}^{\mathrm{real}}x}{\mathrm{d}t}. \tag{55}$$

We assume that this gap is small enough, so that it is possible to develop the dynamics equation at the order 1 in Δx.

4.3.2 Derivative of the State Vector at Order 1

The dynamics of the gap between the real orbit and the fictive one linearized with respect to Δx is computed as:

$$\begin{aligned} \frac{\mathrm{d}\Delta x}{\mathrm{d}t} &= \frac{\mathrm{d}^{\mathrm{real}}x}{\mathrm{d}t} - \left.\frac{\mathrm{d}^{\mathrm{kep}}x}{\mathrm{d}t}\right|_{x=x_{sk}} \\ &= \frac{\mathrm{d}^{\mathrm{real}}x}{\mathrm{d}t} \\ &= \left.\frac{\mathrm{d}^{\mathrm{real}}x}{\mathrm{d}t}\right|_{x=x_{sk}} + \left.\frac{\partial}{\partial x}\left(\frac{\mathrm{d}^{\mathrm{real}}x}{\mathrm{d}t}\right)\right|_{x=x_{sk}} \Delta x \end{aligned}$$

$$= f_L(x_{sk}, t) + \frac{\partial}{\partial x}\left(f_L(x(t), t)\right)\bigg|_{x=x_{sk}} \Delta x \tag{56}$$

Identifying:

$$A(t) = \frac{\partial}{\partial x}\left(f_L(x(t), t)\right)\bigg|_{x=x_{mp}}, \tag{57a}$$

$$D(t) = f_L(x_{mp}, t), \tag{57b}$$

it is possible to recover the time-varying linear system:

$$\frac{\mathrm{d}\Delta x}{\mathrm{d}t} = [A(t)]\Delta x + D(t). \tag{58}$$

5 Computation of the Matrices for the Linearized Dynamics

This section aims at computing the $A(t)$ and $D(t)$ matrices for each of the disturbing forces described in Section 2.2 according to the formulas given by the Equation (57). As the motion has been linearized, the matrix $A(t)$ and the vector $D(t)$ can be decomposed in a keplerian and a disturbing parts:

$$A(t) = A_K(t) + \tilde{A}(t), \tag{59a}$$

$$D(t) = D_K(t) + \tilde{D}(t). \tag{59b}$$

Moreover, the term corresponding to the orbital disturbances is further decomposed in one term for each perturbation:

$$\tilde{A}(t) = \tilde{A}_{\oplus}(t) + \tilde{A}_{\odot}(t) + \tilde{A}_{☾}(t) + \tilde{A}_{\mathrm{SRP}}(t), \tag{60a}$$

$$\tilde{D}(t) = \tilde{D}_{\oplus}(t) + \tilde{D}_{\odot}(t) + \tilde{D}_{☾}(t) + \tilde{D}_{\mathrm{SRP}}(t), \tag{60b}$$

where $\tilde{A}_{\oplus}$ and $\tilde{D}_{\oplus}$ denote the effect of the non spherical Earth gravitational potential, $\tilde{A}_{\odot}$ and $\tilde{D}_{\odot}$ the effect of the gravitational attraction of the Sun, $\tilde{A}_{☾}$ and $\tilde{D}_{☾}$ the gravitational attraction of the Moon, and $\tilde{A}_{\mathrm{SRP}}$ and $\tilde{D}_{\mathrm{SRP}}$ the effect of the Sun radiation pressure.

The physical parameters needed for the numerical computation of dynamics are summarized in the Appendix E page 74.

5.1 Keplerian Part

The non linear keplerian equation of motion is given by the Equation (40). Linearizing this equation according to the formulas given by the Equation (57) leads to:

$$A_K(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{2}\sqrt{\frac{\mu_{\oplus}}{a_{sk}^5}} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{61a}$$

$$D_K(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \sqrt{\frac{\mu_\oplus}{a_{sk}^3}} - \omega_\oplus \end{bmatrix}. \tag{61b}$$

5.2 Non Keplerian Earth Gravitational Part

5.2.1 Introduction

As stated in the section 2.2.1, the gravitational potential of a non spherical Earth can be expanded in spherical harmonics. The potential is first expressed in terms of the geographical position (radius, latitude and longitude). This geographical position is then converted in the cartesian position in the geocentric inertial reference frame, and finally in the equinoctial orbital elements.

Following the process written in the reference [Losa, 2007], the Earth gravitational potential $\mathcal{E}_\oplus$ is expressed in terms of the radius, latitude and longitude of the spacecraft up to the degree and order 3 as:

$$\begin{aligned}
\mathcal{E}_\oplus = & \frac{\mu_\oplus R_\oplus^2}{2r^3} C_{20}\left(3\sin^2\varphi - 1\right) \\
& + \frac{\mu_\oplus R_\oplus^2}{r^3} 3\cos\varphi\sin\varphi\left(C_{21}\cos\lambda + S_{21}\sin\lambda\right) \\
& + \frac{\mu_\oplus R_\oplus^2}{r^3} 3\cos^2\varphi\left(C_{22}\cos(2\lambda) + S_{22}\sin 2\lambda\right) \\
& + \frac{\mu_\oplus R_\oplus^3}{2r^4} C_{30}\sin\varphi\left(5\sin^2\varphi - 3\right) \\
& + \frac{\mu_\oplus R_\oplus^3}{2r^4} \cos\varphi\left(15\sin^2\varphi - 3\right)\left(C_{31}\cos\lambda + S_{31\sin\lambda}\right) \\
& + \frac{\mu_\oplus R_\oplus^3}{r^4} 15\sin\varphi\cos^2\varphi\left(C_{32}\cos(2\lambda) + S_{32}\sin 2\lambda\right) \\
& + \frac{\mu_\oplus R_\oplus^3}{r^4} 15\cos^3\varphi\left(C_{33}\cos(3\lambda) + S_{33}\sin 3\lambda\right),
\end{aligned} \tag{62}$$

with $R_\oplus$ the radius of the Earth and C_{lm} and S_{lm} the coefficient of the Earth gravitational spherical harmonics. Values for these coefficients can be found for instance in the table D-1 of the reference [Vallado, 1997]. Using the conversion formulas (130) between the geographical position and the position in the geocentric inertial reference frame, the gravitational potential of the Earth is written as:

$$\begin{aligned}
\mathcal{E}_\oplus = & \mathcal{E}_{\oplus,C20} + \mathcal{E}_{\oplus,C21} + \mathcal{E}_{\oplus,S21} + \mathcal{E}_{\oplus,C22} + \mathcal{E}_{\oplus,S22} \\
& + \mathcal{E}_{\oplus,C30} + \mathcal{E}_{\oplus,C31} + \mathcal{E}_{\oplus,S31} + \mathcal{E}_{\oplus,C32} + \mathcal{E}_{\oplus,S32} + \mathcal{E}_{\oplus,C33} + \mathcal{E}_{\oplus,S33},
\end{aligned} \tag{63}$$

with:

$$\mathcal{E}_{\oplus,C20} = \frac{\mu_\oplus R_\oplus^2 C_{20}}{2} \frac{2z^2 - x^2 - y^2}{\sqrt{x^2+y^2+z^2}^5}, \tag{64a}$$

$$\mathcal{E}_{\oplus,C21} = 3\mu_\oplus R_\oplus^2 C_{21} \frac{z(x\cos\Theta + y\sin\Theta)}{\sqrt{x^2+y^2+z^2}^5}, \tag{64b}$$

$$\mathcal{E}_{\oplus,S21} = 3\mu_\oplus R_\oplus^2 S_{21} \frac{z(y\cos\Theta - x\sin\Theta)}{\sqrt{x^2+y^2+z^2}^5}, \tag{64c}$$

$$\mathcal{E}_{\oplus,C22} = 3\mu_\oplus R_\oplus^2 C_{22} \frac{(x^2-y^2)\cos(2\Theta) + 2xy\sin 2\Theta}{\sqrt{x^2+y^2+z^2}^5}, \tag{64d}$$

$$\mathcal{E}_{\oplus,S22} = 3\mu_\oplus R_\oplus^2 S_{22} \frac{(y^2-y^2)\sin(2\Theta) + 2xy\cos(2\Theta)}{\sqrt{x^2+y^2+z^2}^5}, \tag{64e}$$

$$\mathcal{E}_{\oplus,C30} = \frac{\mu_\oplus R_\oplus^3 C_{30}}{2} \frac{z(2z^2-3x^2-3y^2)}{\sqrt{x^2+y^2+z^2}^7}, \tag{64f}$$

$$\mathcal{E}_{\oplus,C31} = \frac{\mu_\oplus R_\oplus^3 C_{31}}{2} \frac{(12z^2-3x^2-3y^2)(x\cos\Theta + y\sin\Theta)}{\sqrt{x^2+y^2+z^2}^7}, \tag{64g}$$

$$\mathcal{E}_{\oplus,S31} = \frac{\mu_\oplus R_\oplus^3 S_{31}}{2} \frac{(12z^2-3x^2-3y^2)(y\cos\Theta - x\sin\Theta)}{\sqrt{x^2+y^2+z^2}^7}, \tag{64h}$$

$$\mathcal{E}_{\oplus,C32} = 15\mu_\oplus R_\oplus^3 C_{32} \frac{z[(x^2-y^2)\cos(2\Theta) + 2xy\sin(2\Theta)]}{\sqrt{x^2+y^2+z^2}^7}, \tag{64i}$$

$$\mathcal{E}_{\oplus,S32} = 15\mu_\oplus R_\oplus^3 S_{32} \frac{z[(y^2-x^2)\sin(2\Theta) + 2xy\cos(2\Theta)]}{\sqrt{x^2+y^2+z^2}^7}, \tag{64j}$$

$$\mathcal{E}_{\oplus,C33} = 15\mu_\oplus R_\oplus^3 C_{33} \frac{(x\cos\Theta + y\sin\Theta)[x^2+y^2-4(y\cos\Theta - x\sin\Theta)^2]}{\sqrt{x^2+y^2+z^2}^7}, \tag{64k}$$

$$\mathcal{E}_{\oplus,S33} = 15\mu_\oplus R_\oplus^3 S_{33} \frac{(y\cos\Theta - x\sin\Theta)[x^2+y^2-4(x\cos\Theta + y\sin\Theta)^2]}{\sqrt{x^2+y^2+z^2}^7}. \tag{64l}$$

Using the conversion formulas (216) between the cartesian position in the geocentric inertial reference frame and the equinoctial orbital elements, the Earth gravitational potential is expressed in terms of the equinoctial orbital elements.

In the potential $\mathcal{E}_\oplus$, the contributions of each harmonic are summed together. Therefore, by linearity of the derivation, the matrix $\tilde{A}_\oplus(t)$ and the vector $\tilde{D}_\oplus(t)$ for the overall Earth gravitational potential are the sum of the linearization of each component of the spherical expansion. In the next subsections, the matrix $\tilde{A}_\oplus$ and the vector $\tilde{D}_\oplus$ for each harmonic are derived. In the sequel, the following notations will be used:

$$\begin{aligned}
\kappa(t) &= \ell_{M\Theta} + \Theta(t), & \kappa_{sk}(t) &= \ell_{M\Theta,sk} + \Theta(t),\\
c_\kappa &= \cos\kappa(t), & s_\kappa &= \sin\kappa(t),\\
c_{\kappa,sk} &= \cos\kappa_{sk}(t), & s_{\kappa,sk} &= \sin\kappa_{sk}(t),\\
c_{2\kappa,sk} &= \cos(2\kappa_{sk}(t)), & s_{2\kappa,sk} &= \sin(2\kappa_{sk}(t)),\\
c_{\kappa+\Theta} &= \cos(\kappa(t)+\Theta(t)), & s_{\kappa+\Theta} &= \sin(\kappa(t)+\Theta(t)),\\
c_{2\kappa+2\Theta} &= \cos(2\kappa(t)+2\Theta(t)), & s_{2\kappa+2\Theta} &= \sin(2\kappa(t)+2\Theta(t)),\\
c_{\ell,sk} &= \cos\ell_{M\Theta,sk}, & s_{\ell,sk} &= \sin\ell_{M\Theta,sk},\\
c_{2\ell,sk} &= \cos(2\ell_{M\Theta,sk}), & s_{2\ell,sk} &= \sin(2\ell_{M\Theta,sk}),\\
c_{3\ell,sk} &= \cos(3\ell_{M\Theta,sk}), & s_{3\ell,sk} &= \sin(3\ell_{M\Theta,sk}),\\
c_{2\Theta} &= \cos(2\Theta(t)), & s_{2\Theta} &= \sin(2\Theta(t)),\\
c_{3\Theta} &= \cos(3\Theta(t)), & s_{3\Theta} &= \sin(3\Theta(t)).
\end{aligned} \tag{65}$$

5.2.2 Coefficient C_{20}

The C_{20} term of the Earth potential is:

$$
\begin{aligned}
\mathcal{E}_{\oplus,C20} &= \frac{\alpha_{20}}{r^3}\Big(3\sin^2\varphi - 1\Big),\\
&= \alpha_{20}\frac{2z^2 - x^2 - y^2}{\sqrt{x^2+y^2+z^2}^5},\\
&= \alpha_{20}\frac{(1+e_xc_\kappa+e_ys_\kappa)^3\Big[(i_x^2+i_y^2)^2 + 2i_x^2(6c_\kappa^2-5) + 2i_y^2(1-6c_\kappa^2) + 1 + 24c_\kappa s_\kappa i_x i_y\Big]}{a^3(1-e_x^2-e_y^2)^3(1+i_x^2+i_y^2)^2},
\end{aligned}
\tag{66}
$$

with $\alpha_{20} = \frac{\mu_\oplus R_\oplus^2 C_{20}}{2}$.

The Equation (57) leads to the dynamic matrix:

$$
\tilde{A}_{\oplus,C20} = \frac{\alpha_{20}}{a_{sk}^4\sqrt{\mu_\oplus a_{sk}}}\begin{bmatrix}
0 & A^{12}_{\oplus,C20} & A^{13}_{\oplus,C20} & 0 & 0 & 0\\
A^{21}_{\oplus,C20} & A^{22}_{\oplus,C20} & A^{23}_{\oplus,C20} & 0 & 0 & A^{26}_{\oplus,C20}\\
A^{31}_{\oplus,C20} & A^{32}_{\oplus,C20} & A^{33}_{\oplus,C20} & 0 & 0 & A^{36}_{\oplus,C20}\\
0 & 0 & 0 & A^{44}_{\oplus,C20} & A^{45}_{\oplus,C20} & 0\\
0 & 0 & 0 & A^{54}_{\oplus,C20} & A^{55}_{\oplus,C20} & 0\\
A^{61}_{\oplus,C20} & A^{62}_{\oplus,C20} & A^{63}_{\oplus,C20} & 0 & 0 & 0
\end{bmatrix},
\tag{67}
$$

with:

$$
\begin{aligned}
A^{21}_{\oplus,C20} &= \frac{21s_{\kappa,sk}}{2}, & A^{31}_{\oplus,C20} &= -\frac{21c_{\kappa,sk}}{2}, && \text{(68a)}\\
A^{61}_{\oplus,C20} &= -21, & A^{12}_{\oplus,C20} &= -6a_{sk}^2 s_{\kappa,sk}, && \text{(68b)}\\
A^{22}_{\oplus,C20} &= -6a_{sk}c_{\kappa,sk}s_{\kappa,sk}, & A^{32}_{\oplus,C20} &= 6a_{sk}(1+c_{\kappa,sk}^2), && \text{(68c)}\\
A^{62}_{\oplus,C20} &= \frac{39a_{sk}c_{\kappa,sk}}{2}, & A^{13}_{\oplus,C20} &= 6a_{sk}^2 c_{\kappa,sk}, && \text{(68d)}\\
A^{23}_{\oplus,C20} &= -6a_{sk}(1+s_{\kappa,sk}^2), & A^{33}_{\oplus,C20} &= 6a_{sk}c_{\kappa,sk}s_{\kappa,sk}, && \text{(68e)}\\
A^{63}_{\oplus,C20} &= \frac{39a_{sk}s_{\kappa,sk}}{2}, & A^{44}_{\oplus,C20} &= -6a_{sk}c_{\kappa,sk}s_{\kappa,sk}, && \text{(68f)}\\
A^{54}_{\oplus,C20} &= -6a_{sk}s_{\kappa,sk}^2, & A^{45}_{\oplus,C20} &= 6a_{sk}c_{\kappa,sk}^2, && \text{(68g)}\\
A^{55}_{\oplus,C20} &= 6a_{sk}c_{\kappa,sk}s_{\kappa,sk}, & A^{26}_{\oplus,C20} &= -3a_{sk}c_{\kappa,sk}, && \text{(68h)}\\
A^{36}_{\oplus,C20} &= -3a_{sk}s_{\kappa,sk}, & & && \text{(68i)}
\end{aligned}
$$

and:

$$
\tilde{D}_{\oplus,C20} = \frac{\alpha_{20}}{a_{sk}^3\sqrt{\mu_\oplus a_{sk}}}\begin{bmatrix}
0\\ -3s_{\kappa,sk}\\ 3c_{\kappa,sk}\\ 0\\ 0\\ 6
\end{bmatrix}
\tag{69}
$$

5.2.3 Coefficient C_{21}

The C_{21} term of the Earth potential is:

$$
\begin{aligned}
\mathcal{E}_{\oplus,C21} &= \frac{\alpha_{21}}{r^3}\cos\varphi\sin\varphi\cos\lambda,\\
&= \alpha_{21}\frac{z(x\cos\Theta + y\sin\Theta)}{\sqrt{x^2+y^2+z^2}^5},\\
&= 2\alpha_{21}\frac{(1+e_x c_\kappa + e_y s_\kappa)^3(i_y c_\kappa - i_x s_\kappa)}{a^3(1-e_x^2-e_y^2)^3(1+i_x^2+i_y^2)^2}\\
&\qquad \times\left[c_{\kappa+\Theta}(i_x^2-i_y^2) + 2s_{\kappa+\Theta}i_x i_y + \cos(\ell_{M\Theta})\right],
\end{aligned}
\tag{70}
$$

with $\alpha_{21} = 3\mu_\oplus R_\oplus^2 C_{21}$.

The Equation (57) leads to the dynamic matrix:

$$
\tilde{A}_{\oplus,C21} = \frac{\alpha_{21}}{a_{sk}^4\sqrt{\mu_\oplus a_{sk}}}\begin{bmatrix}
0 & 0 & 0 & A^{14}_{\oplus,C21} & A^{15}_{\oplus,C21} & 0\\
0 & 0 & 0 & A^{24}_{\oplus,C21} & A^{25}_{\oplus,C21} & 0\\
0 & 0 & 0 & A^{34}_{\oplus,C21} & A^{35}_{\oplus,C21} & 0\\
A^{41}_{\oplus,C21} & A^{42}_{\oplus,C21} & A^{43}_{\oplus,C21} & 0 & 0 & A^{46}_{\oplus,C21}\\
A^{51}_{\oplus,C21} & A^{52}_{\oplus,C21} & A^{53}_{\oplus,C21} & 0 & 0 & A^{56}_{\oplus,C21}\\
0 & 0 & 0 & A^{64}_{\oplus,C21} & A^{65}_{\oplus,C21} & 0
\end{bmatrix},
\tag{71}
$$

$$A^{41}_{\oplus,C21} = -\frac{7c_{\kappa,sk}c_{\ell,sk}}{4}, \tag{72a}$$
$$A^{51}_{\oplus,C21} = -\frac{7s_{\kappa,sk}c_{\ell,sk}}{4}, \tag{72b}$$
$$A^{42}_{\oplus,C21} = \frac{3a_{sk}c^2_{\kappa,sk}c_{\ell,sk}}{2}, \tag{72c}$$
$$A^{52}_{\oplus,C21} = \frac{3a_{sk}c_{\kappa,sk}s_{\kappa,sk}c_{\ell,sk}}{2}, \tag{72d}$$
$$A^{43}_{\oplus,C21} = \frac{3a_{sk}c_{\kappa,sk}s_{\kappa,sk}c_{\ell,sk}}{2}, \tag{72e}$$
$$A^{53}_{\oplus,C21} = \frac{3a_{sk}s^2_{\kappa,sk}c_{\ell,sk}}{2}, \tag{72f}$$
$$A^{14}_{\oplus,C21} = 4a_{sk}^2\cos(2\kappa_{sk}-\Theta), \tag{72g}$$
$$A^{24}_{\oplus,C21} = -6a_{sk}s^2_{\kappa,sk}c_{\ell,sk}, \tag{72h}$$
$$A^{34}_{\oplus,C21} = 6a_{sk}c_{\kappa,sk}s_{\kappa,sk}c_{\ell,sk}, \tag{72i}$$
$$A^{64}_{\oplus,C21} = 13a_{sk}s_{\kappa,sk}c_{\ell,sk}, \tag{72j}$$
$$A^{15}_{\oplus,C21} = 4a_{sk}^2\sin(2\kappa_{sk}-\Theta), \tag{72k}$$
$$A^{25}_{\oplus,C21} = 6a_{sk}c_{\kappa,sk}s_{\kappa,sk}c_{\ell,sk}, \tag{72l}$$
$$A^{35}_{\oplus,C21} = -6a_{sk}c^2_{\kappa,sk}c_{\ell,sk}, \tag{72m}$$
$$A^{65}_{\oplus,C21} = -13a_{sk}c_{\kappa,sk}c_{\ell,sk}, \tag{72n}$$
$$A^{46}_{\oplus,C21} = -\frac{a_{sk}\sin(2\kappa_{sk}-\Theta)}{2}, \tag{72o}$$
$$A^{56}_{\oplus,C21} = \frac{a_{sk}\cos(2\kappa_{sk}-\Theta)}{2}, \tag{72p}$$

and:

$$\tilde{D}_{\oplus,C21} = \frac{\alpha_{21} c_{\ell,sk}}{2a_{sk}^3\sqrt{\mu_\oplus a_{sk}}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ c_{\kappa,sk} \\ s_{\kappa,sk} \\ 0 \end{bmatrix} \tag{73}$$

5.2.4 Coefficient S_{21}

The S_{21} term of the Earth potential is:

$$\begin{aligned} \mathcal{E}_{\oplus,S21} &= \frac{\beta_{21}}{r^3}\cos\varphi\sin\varphi\sin\lambda, \\ &= \beta_{21}\frac{z(y\cos\Theta - x\sin\Theta)}{\sqrt{x^2+y^2+z^2}^5}, \\ &= 2\beta_{21}\frac{(1+e_x c_\kappa + e_y s_\kappa)^3(i_y c_\kappa - i_x s_\kappa)}{a^3(1-e_x^2-e_y^2)^3(1+i_x^2+i_y^2)^2} \\ &\quad\times\left[s_{\kappa+\Theta}(i_y^2-i_x^2) + 2c_{\kappa+\Theta} i_x i_y + \sin(\ell_{M\Theta})\right], \end{aligned} \tag{74}$$

with $\beta_{21} = 3\mu_\oplus R_\oplus^2 S_{21}$.

The Equation (57) leads to the dynamic matrix:

$$\tilde{A}_{\oplus,S21} = \frac{\beta_{21}}{a_{sk}^4\sqrt{\mu_\oplus a_{sk}}} \begin{bmatrix} 0 & 0 & 0 & A^{14}_{\oplus,S21} & A^{15}_{\oplus,S21} & 0 \\ 0 & 0 & 0 & A^{24}_{\oplus,S21} & A^{25}_{\oplus,S21} & 0 \\ 0 & 0 & 0 & A^{34}_{\oplus,S21} & A^{35}_{\oplus,S21} & 0 \\ A^{41}_{\oplus,S21} & A^{42}_{\oplus,S21} & A^{43}_{\oplus,S21} & 0 & 0 & A^{46}_{\oplus,S21} \\ A^{51}_{\oplus,S21} & A^{52}_{\oplus,S21} & A^{53}_{\oplus,S21} & 0 & 0 & A^{56}_{\oplus,S21} \\ 0 & 0 & 0 & A^{64}_{\oplus,S21} & A^{65}_{\oplus,S21} & 0 \end{bmatrix}, \tag{75}$$

$$A^{41}_{\oplus,S21} = -\frac{7c_{\kappa,sk}s_{\ell,sk}}{4}, \tag{76a}$$

$$A^{51}_{\oplus,S21} = -\frac{7s_{\kappa,sk}s_{\ell,sk}}{4}, \tag{76b}$$

$$A^{42}_{\oplus,S21} = \frac{3a_{sk}c_{\kappa,sk}^2 s_{\ell,sk}}{2}, \tag{76c}$$

$$A^{52}_{\oplus,S21} = \frac{3a_{sk}c_{\kappa,sk}s_{\kappa,sk}s_{\ell,sk}}{2}, \tag{76d}$$

$$A^{43}_{\oplus,S21} = \frac{3a_{sk}c_{\kappa,sk}s_{\kappa,sk}s_{\ell,sk}}{2}, \tag{76e}$$

$$A^{53}_{\oplus,S21} = \frac{3a_{sk}s_{\kappa,sk}^2 s_{\ell,sk}}{2}, \tag{76f}$$

$$A^{14}_{\oplus,S21} = 4a_{sk}^2\sin(2\kappa_{sk} - \Theta), \tag{76g}$$

$$A^{24}_{\oplus,S21} = -6a_{sk}s_{\kappa,sk}^2 s_{\ell,sk}, \tag{76h}$$

$$A^{34}_{\oplus,S21} = 6a_{sk}c_{\kappa,sk}s_{\kappa,sk}s_{\ell,sk}, \tag{76i}$$

$$A^{64}_{\oplus,S21} = 13a_{sk}s_{\kappa,sk}s_{\ell,sk}, \tag{76j}$$

$$A^{15}_{\oplus,S21} = -4a_{sk}^2\cos(2\kappa_{sk} - \Theta), \tag{76k}$$

$$A^{25}_{\oplus,S21} = 6a_{sk}c_{\kappa,sk}s_{\kappa,sk}s_{\ell,sk}, \tag{76l}$$
$$A^{35}_{\oplus,S21} = -6a_{sk}c^2_{\kappa,sk}s_{\ell,sk}, \tag{76m}$$
$$A^{65}_{\oplus,S21} = -13a_{sk}c_{\kappa,sk}s_{\ell,sk}, \tag{76n}$$
$$A^{46}_{\oplus,S21} = \frac{a_{sk}\cos(2\kappa_{sk}-\Theta)}{2}, \tag{76o}$$
$$A^{56}_{\oplus,S21} = \frac{a_{sk}\sin(2\kappa_{sk}-\Theta)}{2}, \tag{76p}$$

and:

$$\tilde{D}_{\oplus,S21} = \frac{\beta_{21}s_{\ell,sk}}{2a^3_{sk}\sqrt{\mu_\oplus a_{sk}}}\begin{bmatrix} 0 \\ 0 \\ 0 \\ c_{\kappa,sk} \\ s_{\kappa,sk} \\ 0 \end{bmatrix} \tag{77}$$

5.2.5 Coefficient C_{22}

The C_{22} term of the Earth potential is:

$$\begin{aligned}
&\mathcal{E}_{\oplus,S22} \\
&= \frac{\alpha_{22}}{r^3}\cos^2\varphi\cos(2\lambda), \\
&= \alpha_{22}\frac{(x^2-y^2)c_{2\Theta}+2xys_{2\Theta}}{\sqrt{x^2+y^2+z^2}^5}, \\
&= \alpha_{22}\frac{(1+e_xc_\kappa+e_ys_\kappa)^3}{a^3(1-e_x^2-e_y^2)^3(1+i_x^2+i_y^2)^2} \\
&\quad \times\left[c_{2\kappa+2\Theta}(-i_x^4-i_y^4+6i_x^2i_y^2)+(4s_{2\kappa+2\Theta}i_xi_y-2c_{2\Theta})(i_y^2-i_x^2)-c_{2\ell,sk}-4s_{2\Theta}i_xi_y\right],
\end{aligned} \tag{78}$$

with $\alpha_{22} = 3\mu_\oplus R^2_\oplus C_{22}$.

The Equation (57) leads to the dynamic matrix:

$$\tilde{A}_{\oplus,C22} = \frac{\alpha_{22}}{a^4_{sk}\sqrt{\mu_\oplus a_{sk}}}\begin{bmatrix} A^{11}_{\oplus,C22} & A^{12}_{\oplus,C22} & A^{13}_{\oplus,C22} & 0 & 0 & A^{16}_{\oplus,C22} \\ A^{21}_{\oplus,C22} & A^{22}_{\oplus,C22} & A^{23}_{\oplus,C22} & 0 & 0 & A^{26}_{\oplus,C22} \\ A^{31}_{\oplus,C22} & A^{32}_{\oplus,C22} & A^{33}_{\oplus,C22} & 0 & 0 & A^{36}_{\oplus,C22} \\ 0 & 0 & 0 & A^{44}_{\oplus,C22} & A^{45}_{\oplus,C22} & 0 \\ 0 & 0 & 0 & A^{54}_{\oplus,C22} & A^{55}_{\oplus,C22} & 0 \\ A^{61}_{\oplus,C22} & A^{62}_{\oplus,C22} & A^{63}_{\oplus,C22} & 0 & 0 & A^{66}_{\oplus,C22} \end{bmatrix}, \tag{79}$$

$$A^{11}_{\oplus,C22} = -10s_{2\ell,sk}a_{sk}, \tag{80a}$$
$$A^{21}_{\oplus,C22} = -\frac{21s_{\kappa,sk}c_{2\ell,sk}}{2}, \tag{80b}$$
$$A^{31}_{\oplus,C22} = \frac{21c_{\kappa,sk}c_{2\ell,sk}}{2}, \tag{80c}$$
$$A^{61}_{\oplus,C22} = -21c_{2\ell,sk}, \tag{80d}$$
$$A^{12}_{\oplus,C22} = -6a^2_{sk}\left[s_{\kappa,sk}c_{2\Theta}(6c^2_{\kappa,sk}-1)+2s_{2\Theta}c_{\kappa,sk}(2-3c^2_{\kappa,sk})\right], \tag{80e}$$

$$A^{22}_{\oplus,C22} = a_{sk}\left[4c_{2\Theta}c_{\kappa,sk}s_{\kappa,sk}(3c^2_{\kappa,sk}-2) + s_{2\Theta}(-12c^4_{\kappa,sk} + 14c^2_{\kappa,sk} - 1)\right], \tag{80f}$$
$$A^{32}_{\oplus,C22} = -6a_{sk}(1 + c^2_{\kappa,sk})c_{2\ell,sk}, \tag{80g}$$
$$A^{62}_{\oplus,C22} = -\frac{39a_{sk}c_{\kappa,sk}c_{2\ell,sk}}{2}, \tag{80h}$$
$$A^{13}_{\oplus,C22} = -6a^2_{sk}\left[2s_{\kappa,sk}s_{2\Theta}(3c^2_{\kappa,sk}-2) + c_{2\Theta}c_{\kappa,sk}(6c^2_{\kappa,sk}-5)\right], \tag{80i}$$
$$A^{23}_{\oplus,C22} = 6a_{sk}(1 + s^2_{\kappa,sk})c_{2\ell,sk}, \tag{80j}$$
$$A^{33}_{\oplus,C22} = -a_{sk}\left[4c_{\kappa,sk}s_{\kappa,sk}c_{2\Theta}(3c^2_{\kappa,sk}-1) + s_{2\Theta}(-12c^4_{\kappa,sk} + 10c^2_{\kappa,sk} + 1)\right], \tag{80k}$$
$$A^{63}_{\oplus,C22} = -\frac{39a_{sk}s_{\kappa,sk}c_{2\ell,sk}}{2}, \tag{80l}$$
$$A^{44}_{\oplus,C22} = -2a_{sk}c_{\kappa,sk}\sin(\kappa_{sk} - 2\Theta), \tag{80m}$$
$$A^{54}_{\oplus,C22} = -2a_{sk}s_{\kappa,sk}\sin(\kappa_{sk} - 2\Theta), \tag{80n}$$
$$A^{45}_{\oplus,C22} = -2a_{sk}c_{\kappa,sk}\cos(\kappa_{sk} - 2\Theta), \tag{80o}$$
$$A^{55}_{\oplus,C22} = -2a_{sk}s_{\kappa,sk}\cos(\kappa_{sk} - 2\Theta), \tag{80p}$$
$$A^{16}_{\oplus,C22} = 8a^2_{sk}c_{2\ell,sk}, \tag{80q}$$
$$A^{26}_{\oplus,C22} = 3a_{sk}\left[2s_{2\Theta}s_{\kappa,sk}(3c^2_{\kappa,sk}-1) + c_{2\Theta}c_{\kappa,sk}(6c^2_{\kappa,sk}-5)\right], \tag{80r}$$
$$A^{36}_{\oplus,C22} = 3a_{sk}\left[c_{2\Theta}s_{\kappa,sk}(6c^2_{\kappa,sk}-1) + 2s_{2\Theta}c_{\kappa,sk}(2 - 3c^2_{\kappa,sk})\right], \tag{80s}$$
$$A^{66}_{\oplus,C22} = 12a_{sk}s_{2\ell,sk}, \tag{80t}$$

and:

$$\tilde{D}_{\oplus,C22} = \frac{\alpha_{22}}{a^3_{sk}\sqrt{\mu_\oplus a_{sk}}}\begin{bmatrix} 4s_{2\ell,sk}a_{sk} \\ 3s_{\kappa,sk}c_{2\ell,sk} \\ -3c_{\kappa,sk}c_{2\ell,sk} \\ 0 \\ 0 \\ -6c_{2\ell,sk} \end{bmatrix} \tag{81}$$

5.2.6 Coefficient S_{22}

The S_{22} term of the Earth potential is:

$$\begin{aligned}
&\mathcal{E}_{\oplus,S22} \\
&= \beta_{22}\cos^2\varphi\cos(2\lambda), \\
&= \beta_{22}\frac{(x^2-y^2)\cos(2\Theta) + 2xy\sin 2\Theta}{\sqrt{x^2+y^2+z^2}^5}, \\
&= \beta_{22}\frac{(1+e_xc_\kappa+e_ys_\kappa)^3}{a^3(1-e_x^2-e_y^2)^3(1+i_x^2+i_y^2)^2} \\
&\qquad \times\left[s_{2\kappa+2\Theta}(-i_x^4 - i_y^4 + 6i_x^2i_y^2) + (4c_{2\kappa+2\Theta}i_xi_y - 2s_{2\Theta})(i_x^2 - i_y^2) - c_{2\ell,sk} + 4c_{2\Theta}i_xi_y\right],
\end{aligned} \tag{82}$$

with $\beta_{22} = 3\mu_\oplus R^2_\oplus S_{22}$.

The Equation (57) leads to the dynamic matrix:

$$\tilde{A}_{\oplus,S22} = \frac{\beta_{22}}{a_{sk}^4\sqrt{\mu_\oplus a_{sk}}}\begin{bmatrix} A^{11}_{\oplus,S22} & A^{12}_{\oplus,S22} & A^{13}_{\oplus,S22} & 0 & 0 & A^{16}_{\oplus,S22} \\ A^{21}_{\oplus,S22} & A^{22}_{\oplus,S22} & A^{23}_{\oplus,S22} & 0 & 0 & A^{26}_{\oplus,S22} \\ A^{31}_{\oplus,S22} & A^{32}_{\oplus,S22} & A^{33}_{\oplus,S22} & 0 & 0 & A^{36}_{\oplus,S22} \\ 0 & 0 & 0 & A^{44}_{\oplus,S22} & A^{45}_{\oplus,S22} & 0 \\ 0 & 0 & 0 & A^{54}_{\oplus,S22} & A^{55}_{\oplus,S22} & 0 \\ A^{61}_{\oplus,S22} & A^{62}_{\oplus,S22} & A^{63}_{\oplus,S22} & 0 & 0 & A^{66}_{\oplus,S22} \end{bmatrix}, \tag{83}$$

$$A^{11}_{\oplus,S22} = 10c_{2\ell,sk}a_{sk}, \tag{84a}$$
$$A^{21}_{\oplus,S22} = -\frac{21s_{\kappa,sk}s_{2\ell,sk}}{2}, \tag{84b}$$
$$A^{31}_{\oplus,S22} = \frac{21c_{\kappa,sk}s_{2\ell,sk}}{2}, \tag{84c}$$
$$A^{61}_{\oplus,S22} = -21s_{2\ell,sk}, \tag{84d}$$
$$A^{12}_{\oplus,S22} = -6a_{sk}^2\left[s_{\kappa,sk}s_{2\Theta}(6c^2_{\kappa,sk}-1)+2c_{2\Theta}c_{\kappa,sk}(3c^2_{\kappa,sk}-2)\right], \tag{84e}$$
$$A^{22}_{\oplus,S22} = -a_{sk}\left[4c_{\kappa,sk}s_{\kappa,sk}s_{2\Theta}(3c^2_{\kappa,sk}-2)+c_{2\Theta}(12c^4_{\kappa,sk}-14c^2_{\kappa,sk}+1)\right], \tag{84f}$$
$$A^{32}_{\oplus,S22} = -6a_{sk}(1+c^2_{\kappa,sk})s_{2\ell,sk}, \tag{84g}$$
$$A^{62}_{\oplus,S22} = -\frac{39a_{sk}c_{\kappa,sk}s_{2\ell,sk}}{2}, \tag{84h}$$
$$A^{13}_{\oplus,S22} = -6a_{sk}^2\left[2s_{\kappa,sk}c_{2\Theta}(3c^2_{\kappa,sk}-1)+s_{2\Theta}c_{\kappa,sk}(5-6c^2_{\kappa,sk})\right], \tag{84i}$$
$$A^{23}_{\oplus,S22} = 6a_{sk}(1+s^2_{\kappa,sk})s_{2\ell,sk}, \tag{84j}$$
$$A^{33}_{\oplus,S22} = a_{sk}\left[4c_{\kappa,sk}s_{\kappa,sk}s_{2\Theta}(3c^2_{\kappa,sk}-1)+c_{2\Theta}(12c^4_{\kappa,sk}-10c^2_{\kappa,sk}-1)\right], \tag{84k}$$
$$A^{63}_{\oplus,S22} = -\frac{39a_{sk}s_{\kappa,sk}s_{2\ell,sk}}{2}, \tag{84l}$$
$$A^{44}_{\oplus,S22} = 2a_{sk}c_{\kappa,sk}\cos(\kappa_{sk}-2\Theta), \tag{84m}$$
$$A^{54}_{\oplus,S22} = 2a_{sk}s_{\kappa,sk}\cos(\kappa_{sk}-2\Theta), \tag{84n}$$
$$A^{45}_{\oplus,S22} = -2a_{sk}c_{\kappa,sk}\sin(\kappa_{sk}-2\Theta), \tag{84o}$$
$$A^{55}_{\oplus,S22} = -2a_{sk}s_{\kappa,sk}\sin(\kappa_{sk}-2\Theta), \tag{84p}$$
$$A^{16}_{\oplus,S22} = 8a^2_{sk}s_{2\ell,sk}, \tag{84q}$$
$$A^{26}_{\oplus,S22} = 3a_{sk}\left[2c_{2\Theta}s_{\kappa,sk}(3c^2_{\kappa,sk}-1)+s_{2\Theta}c_{\kappa,sk}(5-6c^2_{\kappa,sk})\right], \tag{84r}$$
$$A^{36}_{\oplus,S22} = -3a_{sk}\left[s_{2\Theta}c_{\kappa,sk}(6c^2_{\kappa,sk}-1)+2c_{2\Theta}(3c^2_{\kappa,sk}-2)\right], \tag{84s}$$
$$A^{66}_{\oplus,S22} = -12a_{sk}c_{2\ell,sk}, \tag{84t}$$

and:

$$\tilde{D}_{\oplus,S22} = \frac{\beta 22}{a_{sk}^3\sqrt{\mu_\oplus a_{sk}}}\begin{bmatrix} -4c_{2\ell,sk}a_{sk} \\ 3s_{\kappa,sk}s_{2\ell,sk} \\ -3c_{\kappa,sk}s_{2\ell,sk} \\ 0 \\ 0 \\ -6s_{2\ell,sk} \end{bmatrix} \tag{85}$$

5.2.7 COEFFICIENT C_{30}

The C_{30} term of the Earth potential is:

$$
\begin{aligned}
&\mathcal{E}_{\oplus,C30}\\
&= \frac{\alpha_{30}}{r^4} C_{30} \sin\varphi \left(5\sin^2\varphi - 3\right),\\
&= \alpha_{30} \frac{z(2z^2 - 3x^2 - 3y^2)}{\sqrt{x^2+y^2+z^2}^7},\\
&= -2\alpha_{30} \frac{(1+e_x c_\kappa + e_y s_\kappa)^4 (i_y c_\kappa - i_x s_\kappa)}{a^4(1-e_x^2-e_y^2)^4(1+i_x^2+i_y^2)^3}\\
&\qquad \times \left[3(i_x^2+i_y^2)^2 + 2i_y^2(3-10c_\kappa^2) + 2i_x^2(10c_\kappa^2-7) + 3 + 40c_\kappa s_\kappa i_x i_y\right],
\end{aligned}
\tag{86}
$$

with $\alpha_{30} = \dfrac{\mu_\oplus R_\oplus^3 C_{30}}{2}$.

The Equation (57) leads to the dynamic matrix:

$$
\tilde{A}_{\oplus,C30} = \frac{\alpha_{30}}{a_{sk}^5\sqrt{\mu_\oplus a_{sk}}}
\begin{bmatrix}
0 & 0 & 0 & A^{14}_{\oplus,C30} & A^{15}_{\oplus,C30} & 0\\
0 & 0 & 0 & A^{24}_{\oplus,C30} & A^{25}_{\oplus,C30} & 0\\
0 & 0 & 0 & A^{34}_{\oplus,C30} & A^{35}_{\oplus,C30} & 0\\
A^{41}_{\oplus,C30} & A^{42}_{\oplus,C30} & A^{43}_{\oplus,C30} & 0 & 0 & A^{46}_{\oplus,C30}\\
A^{51}_{\oplus,C30} & A^{52}_{\oplus,C30} & A^{53}_{\oplus,C30} & 0 & 0 & A^{56}_{\oplus,C30}\\
0 & 0 & 0 & A^{64}_{\oplus,C30} & A^{65}_{\oplus,C30} & 0
\end{bmatrix},
\tag{87}
$$

$$A^{41}_{\oplus,C30} = \frac{27c_{\kappa,sk}}{4}, \tag{88a}$$

$$A^{51}_{\oplus,C30} = \frac{27s_{\kappa,sk}}{4}, \tag{88b}$$

$$A^{42}_{\oplus,C30} = -6a_{sk}c^2_{\kappa,sk}, \tag{88c}$$

$$A^{52}_{\oplus,C30} = -6a_{sk}c_{\kappa,sk}s_{\kappa,sk}, \tag{88d}$$

$$A^{43}_{\oplus,C30} = -6a_{sk}c_{\kappa,sk}s_{\kappa,sk}, \tag{88e}$$

$$A^{53}_{\oplus,C30} = -6a_{sk}s^2_{\kappa,sk}, \tag{88f}$$

$$A^{14}_{\oplus,C30} = -12a^2_{sk}c_{\kappa,sk}, \tag{88g}$$

$$A^{24}_{\oplus,C30} = 24a_{sk}s^2_{\kappa,sk}, \tag{88h}$$

$$A^{34}_{\oplus,C30} = -24a_{sk}c_{\kappa,sk}s_{\kappa,sk}, \tag{88i}$$

$$A^{64}_{\oplus,C30} = -51a_{sk}s_{\kappa,sk}, \tag{88j}$$

$$A^{15}_{\oplus,C30} = -12a^2_{sk}s_{\kappa,sk}, \tag{88k}$$

$$A^{25}_{\oplus,C30} = -24a_{sk}c_{\kappa,sk}s_{\kappa,sk}, \tag{88l}$$

$$A^{35}_{\oplus,C30} = 24a_{sk}c^2_{\kappa,sk}, \tag{88m}$$

$$A^{65}_{\oplus,C30} = 51a_{sk}c_{\kappa,sk}, \tag{88n}$$

$$A^{46}_{\oplus,C30} = \frac{3a_{sk}s_{\kappa,sk}}{2}, \tag{88o}$$

$$A^{56}_{\oplus,C30} = -\frac{3a_{sk}c_{\kappa,sk}}{2}, \tag{88p}$$

and:

$$\tilde{D}_{\oplus,C30} = -\frac{3\alpha_{30}}{2a_{sk}^4\sqrt{\mu_\oplus a_{sk}}}\begin{bmatrix} 0 \\ 0 \\ 0 \\ c_{\kappa,sk} \\ s_{\kappa,sk} \\ 0 \end{bmatrix} \tag{89}$$

5.2.8 Coefficient C_{31}

The C_{31} term of the Earth potential is:

$$\begin{aligned}
&\mathcal{E}_{\oplus,C31} \\
&= \frac{\alpha_{31}}{r^4}\cos\varphi\Big(15\sin^2\varphi - 3\Big)\cos\lambda, \\
&= \alpha_{31}\frac{(12z^2 - 3x^2 - 3y^2)(x\cos\Theta + y\sin\Theta)}{\sqrt{x^2+y^2+z^2}^7}, \\
&= -3\alpha_{31}\frac{(1+e_xc_\kappa+e_ys_\kappa)^4}{a^4(1-e_x^2-e_y^2)^4(1+i_x^2+i_y^2)^3} \\
&\qquad \times\left[c_{\kappa+\Theta}(i_x^2 - i_y^2) + 2s_{\kappa+\Theta}i_xi_y + \cos(\ell_{M\Theta})\right] \\
&\qquad \times\left[(i_x^2+i_y^2)^2 + 2i_x^2(10c_\kappa^2 - 9) + 2i_y^2(1-10c_\kappa^2) + 1 + 40c_\kappa s_\kappa i_x i_y\right],
\end{aligned} \tag{90}$$

with $\alpha_{31} = 3\dfrac{\mu_\oplus R_\oplus^3 C_{31}}{2}$.

The Equation (57) leads to the dynamic matrix:

$$\tilde{A}_{\oplus,C31} = \frac{\alpha_{31}}{a_{sk}^5\sqrt{\mu_\oplus a_{sk}}}\begin{bmatrix}
A^{11}_{\oplus,C31} & A^{12}_{\oplus,C31} & A^{13}_{\oplus,C31} & 0 & 0 & A^{16}_{\oplus,C31} \\
A^{21}_{\oplus,C31} & A^{22}_{\oplus,C31} & A^{23}_{\oplus,C31} & 0 & 0 & A^{26}_{\oplus,C31} \\
A^{31}_{\oplus,C31} & A^{32}_{\oplus,C31} & A^{33}_{\oplus,C31} & 0 & 0 & A^{36}_{\oplus,C31} \\
0 & 0 & 0 & A^{44}_{\oplus,C31} & A^{45}_{\oplus,C31} & 0 \\
0 & 0 & 0 & A^{54}_{\oplus,C31} & A^{55}_{\oplus,C31} & 0 \\
A^{61}_{\oplus,C31} & A^{62}_{\oplus,C31} & A^{63}_{\oplus,C31} & 0 & 0 & A^{66}_{\oplus,C31}
\end{bmatrix}, \tag{91}$$

$$A^{11}_{\oplus,C31} = 21a_{sk}s_{\ell,sk}, \tag{92a}$$

$$A^{21}_{\oplus,C31} = 54s_{\kappa,sk}c_{\ell,sk}, \tag{92b}$$

$$A^{31}_{\oplus,C31} = -54c_{\kappa,sk}c_{\ell,sk}, \tag{92c}$$

$$A^{61}_{\oplus,C31} = 108c_{\ell,sk}, \tag{92d}$$

$$A^{12}_{\oplus,C31} = -24a_{sk}^2\sin(2\kappa_{sk} - \Theta), \tag{92e}$$

$$A^{22}_{\oplus,C31} = 3a_{sk}\left[s_{\ell,sk}(24c_{\kappa,sk}^2 - 1) + 24c_{\kappa,sk}\sin\Theta\right], \tag{92f}$$

$$A^{32}_{\oplus,C31} = 12a_{sk}(3c_{\kappa,sk}^2 + 2)c_{\ell,sk}, \tag{92g}$$

$$A^{62}_{\oplus,C31} = 102a_{sk}c_{\kappa,sk}c_{\ell,sk}, \tag{92h}$$

$$A^{13}_{\oplus,C31} = 24a_{sk}^2\cos(2\kappa_{sk} - \Theta), \tag{92i}$$

$$A^{23}_{\oplus,C31} = 12a_{sk}(3c_{\kappa,sk}^2 - 5)c_{\ell,sk}, \tag{92j}$$

$$A^{33}_{\oplus,C31} = 3a_{sk}\left[s_{\ell,sk}(24c_{\kappa,sk}^2 + 1) + 24c_{\kappa,sk}\sin\Theta\right], \tag{92k}$$

$$A^{63}_{\oplus,C31} = 102 a_{sk} s_{\kappa,sk} c_{\ell,sk}, \tag{92l}$$
$$A^{44}_{\oplus,C31} = -3 a_{sk} c_{\kappa,sk} \left[10 c_{\kappa,sk} s_{\ell,sk} + 11 \sin\Theta\right], \tag{92m}$$
$$A^{54}_{\oplus,C31} = -3 a_{sk} s_{\kappa,sk} \left[10 s_{\kappa,sk} s_{\ell,sk} + \sin\Theta\right], \tag{92n}$$
$$A^{45}_{\oplus,C31} = 3 a_{sk} c_{\kappa,sk} \left[10 c_{\kappa,sk} c_{\ell,sk} + \cos\Theta\right], \tag{92o}$$
$$A^{55}_{\oplus,C31} = -3 a_{sk} s_{\kappa,sk} \left[10 s_{\kappa,sk} s_{\ell,sk} - 11 \cos\Theta\right], \tag{92p}$$
$$A^{16}_{\oplus,C31} = -6 a^2_{sk} c_{\ell,sk}, \tag{92q}$$
$$A^{26}_{\oplus,C31} = -12 a_{sk} \cos(2\kappa_{sk} - \Theta), \tag{92r}$$
$$A^{36}_{\oplus,C31} = -12 a_{sk} \sin(2\kappa_{sk} - \Theta), \tag{92s}$$
$$A^{66}_{\oplus,C31} = -24 a_{sk} s_{\ell,sk}, \tag{92t}$$

and:

$$\tilde{D}_{\oplus,C31} = \frac{\alpha_{31}}{a^4_{sk}\sqrt{\mu_\oplus a_{sk}}} \begin{bmatrix} -6 s_{\ell,sk} a_{sk} \\ -12 s_{\kappa,sk} c_{\ell,sk} \\ 12 c_{\kappa,sk} c_{\ell,sk} \\ 0 \\ 0 \\ 24 c_{\ell,sk} \end{bmatrix} \tag{93}$$

5.2.9 Coefficient S_{31}

The S_{31} term of the Earth potential is:

$$\begin{aligned}
&\mathcal{E}_{\oplus,S31} \\
&= \frac{\beta_{31}}{r^4} \cos\varphi \left(15 \sin^2\varphi - 3\right) \cos\lambda, \\
&= \beta_{31} \frac{(12z^2 - 3x^2 - 3y^2)(x\cos\Theta + y\sin\Theta)}{\sqrt{x^2+y^2+z^2}^7}, \\
&= -3\beta_{31} \frac{(1 + e_x c_\kappa + e_y s_\kappa)^4}{a^4(1-e_x^2-e_y^2)^4(1+i_x^2+i_y^2)^3} \\
&\quad \times \left[s_{\kappa+\Theta}(i_y^2 - i_x^2) + 2 c_{\kappa+\Theta} i_x i_y - \sin(\ell_{M\Theta})\right] \\
&\quad \times \left[(i_x^2+i_y^2)^2 + 2 i_x^2(10 c_\kappa^2 - 9) + 2 i_y^2(1 - 10 c_\kappa^2) + 1 + 40 c_\kappa s_\kappa i_x i_y\right],
\end{aligned} \tag{94}$$

with $\beta_{31} = 3\dfrac{\mu_\oplus R^3_\oplus S_{31}}{2}$.

The Equation (57) leads to the dynamic matrix:

$$\tilde{A}_{\oplus,S31} = \frac{\beta_{31}}{a^5_{sk}\sqrt{\mu_\oplus a_{sk}}} \begin{bmatrix} A^{11}_{\oplus,S31} & A^{12}_{\oplus,S31} & A^{13}_{\oplus,S31} & 0 & 0 & A^{16}_{\oplus,S31} \\ A^{21}_{\oplus,S31} & A^{22}_{\oplus,S31} & A^{23}_{\oplus,S31} & 0 & 0 & A^{26}_{\oplus,S31} \\ A^{31}_{\oplus,S31} & A^{32}_{\oplus,S31} & A^{33}_{\oplus,S31} & 0 & 0 & A^{36}_{\oplus,S31} \\ 0 & 0 & 0 & A^{44}_{\oplus,S31} & A^{45}_{\oplus,S31} & 0 \\ 0 & 0 & 0 & A^{54}_{\oplus,S31} & A^{55}_{\oplus,S31} & 0 \\ A^{61}_{\oplus,S31} & A^{62}_{\oplus,S31} & A^{63}_{\oplus,S31} & 0 & 0 & A^{66}_{\oplus,S31} \end{bmatrix}, \tag{95}$$

$$A^{11}_{\oplus,S31} = -21 a_{sk} s_{\ell,sk}, \tag{96a}$$
$$A^{21}_{\oplus,S31} = 54 s_{\kappa,sk} s_{\ell,sk}, \tag{96b}$$

$$A^{31}_{\oplus,S31} = -54 c_{\kappa,sk} s_{\ell,sk}, \tag{96c}$$
$$A^{61}_{\oplus,S31} = 108 s_{\ell,sk}, \tag{96d}$$
$$A^{12}_{\oplus,S31} = 24 a_{sk}^2 \cos(2\kappa_{sk} - \Theta), \tag{96e}$$
$$A^{22}_{\oplus,S31} = -3 a_{sk} \left[c_{\ell,sk}(24 c_{\kappa,sk}^2 - 1) - 24 c_{\kappa,sk} \cos\Theta \right], \tag{96f}$$
$$A^{32}_{\oplus,S31} = 12 a_{sk} (3 c_{\kappa,sk}^2 + 2) s_{\ell,sk}, \tag{96g}$$
$$A^{62}_{\oplus,S31} = 102 a_{sk} c_{\kappa,sk} s_{\ell,sk}, \tag{96h}$$
$$A^{13}_{\oplus,S31} = 24 a_{sk}^2 \sin(2\kappa_{sk} - \Theta), \tag{96i}$$
$$A^{23}_{\oplus,S31} = 12 a_{sk} (3 c_{\kappa,sk}^2 - 5) s_{\ell,sk}, \tag{96j}$$
$$A^{33}_{\oplus,S31} = -3 a_{sk} \left[c_{\ell,sk}(24 c_{\kappa,sk}^2 + 1) - 24 c_{\kappa,sk} \cos\Theta \right], \tag{96k}$$
$$A^{63}_{\oplus,S31} = 102 a_{sk} s_{\kappa,sk} s_{\ell,sk}, \tag{96l}$$
$$A^{44}_{\oplus,S31} = 3 a_{sk} c_{\kappa,sk} \left[10 c_{\kappa,sk} c_{\ell,sk} - 11 \cos\Theta \right], \tag{96m}$$
$$A^{54}_{\oplus,S31} = -3 a_{sk} s_{\kappa,sk} \left[10 s_{\kappa,sk} s_{\ell,sk} + \cos\Theta \right], \tag{96n}$$
$$A^{45}_{\oplus,S31} = 3 a_{sk} c_{\kappa,sk} \left[10 c_{\kappa,sk} s_{\ell,sk} - \sin\Theta \right], \tag{96o}$$
$$A^{55}_{\oplus,S31} = 3 a_{sk} s_{\kappa,sk} \left[10 s_{\kappa,sk} c_{\ell,sk} - 11 \sin\Theta \right], \tag{96p}$$
$$A^{16}_{\oplus,S31} = -6 a_{sk}^2 s_{\ell,sk}, \tag{96q}$$
$$A^{26}_{\oplus,S31} = -12 a_{sk} \sin(2\kappa_{sk} - \Theta), \tag{96r}$$
$$A^{36}_{\oplus,S31} = 12 a_{sk} \cos(2\kappa_{sk} - \Theta), \tag{96s}$$
$$A^{66}_{\oplus,S31} = 24 a_{sk} c_{\ell,sk}, \tag{96t}$$

and:

$$\tilde{D}_{\oplus,S31} = \frac{\beta_{31}}{a_{sk}^4 \sqrt{\mu_\oplus a_{sk}}} \begin{bmatrix} 6 c_{\ell,sk} a_{sk} \\ -12 s_{\kappa,sk} s_{\ell,sk} \\ 12 c_{\kappa,sk} s_{\ell,sk} \\ 0 \\ 0 \\ 24 s_{\ell,sk} \end{bmatrix} \tag{97}$$

5.2.10 Coefficient C_{32}

The C_{32} term of the Earth potential is:

$$\begin{aligned} &\mathcal{E}_{\oplus,C32} \\ &= \frac{\alpha_{32}}{r^4} \sin\varphi \cos^2\varphi \cos(2\lambda), \\ &= \alpha_{32} \frac{z[(x^2 - y^2)\cos(2\Theta) + 2xy \sin(2\Theta)]}{\sqrt{x^2 + y^2 + z^2}^7}, \end{aligned} \tag{98}$$

with $\beta_{32} = 15 \mu_\oplus R_\oplus^3 S_{32}$.

The Equation (57) leads to the dynamic matrix:

$$\tilde{A}_{\oplus,C32} = \frac{\alpha_{32}}{a_{sk}^5 \sqrt{\mu_\oplus a_{sk}}} \begin{bmatrix} 0 & 0 & 0 & A^{14}_{\oplus,C32} & A^{15}_{\oplus,C32} & 0 \\ 0 & 0 & 0 & A^{24}_{\oplus,C32} & A^{25}_{\oplus,C32} & 0 \\ 0 & 0 & 0 & A^{34}_{\oplus,C32} & A^{35}_{\oplus,C32} & 0 \\ A^{41}_{\oplus,C32} & A^{42}_{\oplus,C32} & A^{43}_{\oplus,C32} & 0 & 0 & A^{46}_{\oplus,C32} \\ A^{51}_{\oplus,C32} & A^{52}_{\oplus,C32} & A^{53}_{\oplus,C32} & 0 & 0 & A^{56}_{\oplus,C32} \\ 0 & 0 & 0 & A^{64}_{\oplus,C32} & A^{65}_{\oplus,C32} & 0 \end{bmatrix}, \tag{99}$$

$$A^{41}_{\oplus,C32} = -\frac{9c_{\kappa,sk}s_{2\ell,sk}}{4}, \tag{100a}$$
$$A^{51}_{\oplus,C32} = -\frac{9s_{\kappa,sk}s_{2\ell,sk}}{4}, \tag{100b}$$
$$A^{42}_{\oplus,C32} = 2a_{sk}c^2_{\kappa,sk}s_{2\ell,sk}, \tag{100c}$$
$$A^{52}_{\oplus,C32} = 2a_{sk}c_{\kappa,sk}s_{\kappa,sk}c_{2\ell,sk}, \tag{100d}$$
$$A^{43}_{\oplus,C32} = 2a_{sk}c_{\kappa,sk}s_{\kappa,sk}s_{2\ell,sk}, \tag{100e}$$
$$A^{53}_{\oplus,C32} = 2a_{sk}s^2_{\kappa,sk}s_{2\ell,sk}, \tag{100f}$$
$$A^{14}_{\oplus,C32} = 4a^2_{sk}\left[2c_{2\Theta}s_{\kappa,sk}(3c^2_{\kappa,sk}-1) + s_{2\Theta}c_{\kappa,sk}(5-6c^2_{\kappa,sk})\right], \tag{100g}$$
$$A^{24}_{\oplus,C32} = -8a_{sk}s^2_{\kappa,sk}s_{2\ell,sk}, \tag{100h}$$
$$A^{34}_{\oplus,C32} = 8a_{sk}c_{\kappa,sk}s_{\kappa,sk}s_{2\ell,sk}, \tag{100i}$$
$$A^{64}_{\oplus,C32} = 17a_{sk}s_{\kappa,sk}s_{2\ell,sk}, \tag{100j}$$
$$A^{15}_{\oplus,C32} = -4a^2_{sk}\left[s_{2\Theta}s_{\kappa,sk}(6c^2_{\kappa,sk}-1) + 2c_{2\Theta}c_{\kappa,sk}(3c^2_{\kappa,sk}-2)\right], \tag{100k}$$
$$A^{25}_{\oplus,C32} = 8a_{sk}c_{\kappa,sk}s_{\kappa,sk}s_{2\ell,sk}, \tag{100l}$$
$$A^{35}_{\oplus,C32} = -8a_{sk}c^2_{\kappa,sk}s_{2\ell,sk}, \tag{100m}$$
$$A^{65}_{\oplus,C32} = -17a_{sk}c_{\kappa,sk}s_{2\ell,sk}, \tag{100n}$$
$$A^{46}_{\oplus,C32} = \frac{a_{sk}}{2}\left[s_{2\Theta}s_{\kappa,sk}(6c^2_{\kappa,sk}-1) + 2c_{2\Theta}c_{\kappa,sk}(3c^2_{\kappa,sk}-2)\right], \tag{100o}$$
$$A^{56}_{\oplus,C32} = \frac{a_{sk}}{2}\left[2c_{2\Theta}s_{\kappa,sk}(3c^2_{\kappa,sk}-1) + s_{2\Theta}c_{\kappa,sk}(5-6c^2_{\kappa,sk})\right], \tag{100p}$$

and:

$$\tilde{D}_{\oplus,C32} = \frac{\alpha_{32}s_{2\ell,sk}}{2a^4_{sk}\sqrt{\mu_\oplus a_{sk}}}\begin{bmatrix} 0 \\ 0 \\ 0 \\ c_{\kappa,sk} \\ s_{\kappa,sk} \\ 0 \end{bmatrix} \tag{101}$$

5.2.11 COEFFICIENT S_{32}

The S_{32} term of the Earth potential is:

$$\begin{aligned} &\mathcal{E}_{\oplus,S32} \\ &= \frac{\beta_{32}}{r^4}\sin\varphi\cos^2\varphi\sin(2\lambda), \\ &= \beta_{32}\frac{z[(y^2-x^2)\sin(2\Theta)+2xy\cos(2\Theta)]}{\sqrt{x^2+y^2+z^2}^7}, \end{aligned} \tag{102}$$

with $\beta_{32} = 15\mu_\oplus R^3_\oplus C_{32}$.

The Equation (57) leads to the dynamic matrix:

$$\tilde{A}_{\oplus,S32} = \frac{\beta_{32}}{a_{sk}^5\sqrt{\mu_\oplus a_{sk}}}\begin{bmatrix} 0 & 0 & 0 & A^{14}_{\oplus,S32} & A^{15}_{\oplus,S32} & 0 \\ 0 & 0 & 0 & A^{24}_{\oplus,S32} & A^{25}_{\oplus,S32} & 0 \\ 0 & 0 & 0 & A^{34}_{\oplus,S32} & A^{35}_{\oplus,S32} & 0 \\ A^{41}_{\oplus,S32} & A^{42}_{\oplus,S32} & A^{43}_{\oplus,S32} & 0 & 0 & A^{46}_{\oplus,S32} \\ A^{51}_{\oplus,S32} & A^{52}_{\oplus,S32} & A^{53}_{\oplus,S32} & 0 & 0 & A^{56}_{\oplus,S32} \\ 0 & 0 & 0 & A^{64}_{\oplus,S32} & A^{65}_{\oplus,S32} & 0 \end{bmatrix}, \tag{103}$$

$$\begin{aligned}
A^{41}_{\oplus,S32} &= -\frac{9c_{\kappa,sk}c_{2\ell,sk}}{4}, && \text{(104a)}\\
A^{51}_{\oplus,S32} &= -\frac{9s_{\kappa,sk}c_{2\ell,sk}}{4}, && \text{(104b)}\\
A^{42}_{\oplus,S32} &= 2a_{sk}c^2_{\kappa,sk}c_{2\ell,sk}, && \text{(104c)}\\
A^{52}_{\oplus,S32} &= 2a_{sk}c_{\kappa,sk}s_{\kappa,sk}c_{2\ell,sk}, && \text{(104d)}\\
A^{43}_{\oplus,S32} &= 2a_{sk}c_{\kappa,sk}s_{\kappa,sk}c_{2\ell,sk}, && \text{(104e)}\\
A^{53}_{\oplus,S32} &= 2a_{sk}s^2_{\kappa,sk}c_{2\ell,sk}, && \text{(104f)}\\
A^{14}_{\oplus,S32} &= 4a^2_{sk}\left[2s_{2\Theta}s_{\kappa,sk}(3c^2_{\kappa,sk}-1)+c_{2\Theta}c_{\kappa,sk}(6c^2_{\kappa,sk}-5)\right], && \text{(104g)}\\
A^{24}_{\oplus,S32} &= -8a_{sk}s^2_{\kappa,sk}c_{2\ell,sk}, && \text{(104h)}\\
A^{34}_{\oplus,S32} &= 8a_{sk}c_{\kappa,sk}s_{\kappa,sk}c_{2\ell,sk}, && \text{(104i)}\\
A^{64}_{\oplus,S32} &= 17a_{sk}s_{\kappa,sk}c_{2\ell,sk}, && \text{(104j)}\\
A^{15}_{\oplus,S32} &= 4a^2_{sk}\left[c_{2\Theta}s_{\kappa,sk}(6c^2_{\kappa,sk}-1)+2s_{2\Theta}c_{\kappa,sk}(2-3c^2_{\kappa,sk})\right], && \text{(104k)}\\
A^{25}_{\oplus,S32} &= 8a_{sk}c_{\kappa,sk}s_{\kappa,sk}c_{2\ell,sk}, && \text{(104l)}\\
A^{35}_{\oplus,S32} &= -8a_{sk}c^2_{\kappa,sk}c_{2\ell,sk}, && \text{(104m)}\\
A^{65}_{\oplus,S32} &= -17a_{sk}c_{\kappa,sk}c_{2\ell,sk}, && \text{(104n)}\\
A^{46}_{\oplus,S32} &= \frac{a_{sk}}{2}\left[c_{2\Theta}s_{\kappa,sk}(6c^2_{\kappa,sk}-1)+2s_{2\Theta}c_{\kappa,sk}(2-3c^2_{\kappa,sk})\right], && \text{(104o)}\\
A^{56}_{\oplus,S32} &= \frac{a_{sk}}{2}\left[2s_{2\Theta}s_{\kappa,sk}(3c^2_{\kappa,sk}-1)+c_{2\Theta}c_{\kappa,sk}(6c^2_{\kappa,sk}-5)\right], && \text{(104p)}
\end{aligned}$$

and:

$$\tilde{D}_{\oplus,S32} = \frac{\beta_{32}c_{2\ell,sk}}{2a^4_{sk}\sqrt{\mu_\oplus a_{sk}}}\begin{bmatrix} 0 \\ 0 \\ 0 \\ c_{\kappa,sk} \\ s_{\kappa,sk} \\ 0 \end{bmatrix} \tag{105}$$

5.2.12 Coefficient C_{33}

The C_{33} term of the Earth potential is:

$$\begin{aligned}
&\mathcal{E}_{\oplus,C33} \\
&= \frac{\alpha_{33}}{r^4}\cos^3\varphi\cos(3\lambda), \\
&= \alpha_{33}\frac{(x\cos\Theta+y\sin\Theta)[x^2+y^2-4(y\cos\Theta-x\sin\Theta)^2]}{\sqrt{x^2+y^2+z^2}^7},
\end{aligned} \tag{106}$$

with $\alpha_{33} = 15\mu_\oplus R_\oplus^2 C_{33}$.

The Equation (57) leads to the dynamic matrix:

$$\tilde{A}_{\oplus,C33} = \frac{\alpha_{33}}{a_{sk}^4\sqrt{\mu_\oplus a_{sk}}}\begin{bmatrix} A^{11}_{\oplus,C33} & A^{12}_{\oplus,C33} & A^{13}_{\oplus,C33} & 0 & 0 & A^{16}_{\oplus,C33} \\ A^{21}_{\oplus,C33} & A^{22}_{\oplus,C33} & A^{23}_{\oplus,C33} & 0 & 0 & A^{26}_{\oplus,C33} \\ A^{31}_{\oplus,C33} & A^{32}_{\oplus,C33} & A^{33}_{\oplus,C33} & 0 & 0 & A^{36}_{\oplus,C33} \\ 0 & 0 & 0 & A^{44}_{\oplus,C33} & A^{45}_{\oplus,C33} & 0 \\ 0 & 0 & 0 & A^{54}_{\oplus,C33} & A^{55}_{\oplus,C33} & 0 \\ A^{61}_{\oplus,C33} & A^{62}_{\oplus,C33} & A^{63}_{\oplus,C33} & 0 & 0 & A^{66}_{\oplus,C33} \end{bmatrix}, \tag{107}$$

$$A^{11}_{\oplus,C33} = -21 s_{3\ell,sk} a_{sk}, \tag{108a}$$
$$A^{21}_{\oplus,C33} = -18 s_{\kappa,sk} c_{3\ell,sk}, \tag{108b}$$
$$A^{31}_{\oplus,C33} = 18 c_{\kappa,sk} c_{3\ell,sk}, \tag{108c}$$
$$A^{61}_{\oplus,C33} = -36 c_{3\ell,sk}, \tag{108d}$$
$$A^{12}_{\oplus,C33} = 8a_{sk}^2\left[2c_{3\Theta}c_{\kappa,sk}s_{\kappa,sk}(8c_{\kappa,sk}^2-3)+s_{3\Theta}(-16c_{\kappa,sk}^4+14c_{\kappa,sk}^2-1)\right], \tag{108e}$$
$$A^{22}_{\oplus,C33} = \frac{3a_{sk}}{2}\left[c_{3\Theta}s_{\kappa,sk}(32c_{\kappa,sk}^4-28c_{\kappa,sk}^2+1)+c_{\kappa,sk}s_{3\Theta}(-32c_{\kappa,sk}^4+44c_{\kappa,sk}^2-1))\right], \tag{108f}$$
$$A^{32}_{\oplus,C33} = 4a_{sk}\left[s_{3\Theta}s_{\kappa,sk}(12c_{\kappa,sk}^4+5c_{\kappa,sk}^2-2)+c_{3\Theta}c_{\kappa,sk}(12c_{\kappa,sk}^2-c_{\kappa,sk}^2-6)\right], \tag{108g}$$
$$A^{62}_{\oplus,C33} = 34 a_{sk} c_{\kappa,sk} c_{3\ell,sk}, \tag{108h}$$
$$A^{13}_{\oplus,C33} = -8a_{sk}^2\left[2s_{3\Theta}c_{\kappa,sk}s_{\kappa,sk}(8c_{\kappa,sk}^2-5)+c_{3\Theta}(16c_{\kappa,sk}^4-18c_{\kappa,sk}^2+3)\right], \tag{108i}$$
$$A^{23}_{\oplus,C33} = 4a_{sk}\left[s_{3\Theta}s_{\kappa,sk}(12c_{\kappa,sk}^4-23c_{\kappa,sk}^2+5)+c_{3\Theta}c_{\kappa,sk}(12c_{\kappa,sk}^2-29c_{\kappa,sk}^2+15)\right], \tag{108j}$$
$$A^{33}_{\oplus,C33} = 3a_{sk}\left[c_{3\Theta}s_{\kappa,sk}(-32c_{\kappa,sk}^4+20c_{\kappa,sk}^2+1)+c_{\kappa,sk}s_{3\Theta}(32c_{\kappa,sk}^4-36c_{\kappa,sk}^2+5))\right], \tag{108k}$$
$$A^{63}_{\oplus,C33} = -34 a_{sk} s_{\kappa,sk} c_{3\ell,sk}, \tag{108l}$$
$$A^{44}_{\oplus,C33} = -3a_{sk}c_{\kappa,sk}\sin(2\kappa_{sk}-3\Theta), \tag{108m}$$
$$A^{54}_{\oplus,C33} = -3a_{sk}s_{\kappa,sk}\sin(2\kappa_{sk}-3\Theta), \tag{108n}$$
$$A^{45}_{\oplus,C33} = -3a_{sk}c_{\kappa,sk}\cos(2\kappa_{sk}-3\Theta), \tag{108o}$$
$$A^{55}_{\oplus,C33} = -3a_{sk}s_{\kappa,sk}\cos(2\kappa_{sk}-3\Theta), \tag{108p}$$
$$A^{16}_{\oplus,C33} = -18a_{sk}^2 c_{3\ell,sk}, \tag{108q}$$
$$A^{26}_{\oplus,C33} = 4a_{sk}\left[2s_{3\Theta}c_{\kappa,sk}s_{\kappa,sk}(8c_{\kappa,sk}^2-5)+c_{3\Theta}(16c_{\kappa,sk}^4-18c_{\kappa,sk}^2+3)\right], \tag{108r}$$
$$A^{36}_{\oplus,C33} = 4a_{sk}\left[2c_{3\Theta}c_{\kappa,sk}s_{\kappa,sk}(8c_{\kappa,sk}^2-3)+s_{3\Theta}(-16c_{\kappa,sk}^4+14c_{\kappa,sk}^2-1)\right], \tag{108s}$$
$$A^{66}_{\oplus,C33} = 24 a_{sk} s_{3\ell,sk} \tag{108t}$$

and:

$$\tilde{D}_{\oplus,C33} = \frac{\alpha_{33}}{a_{sk}^3\sqrt{\mu_\oplus a_{sk}}}\begin{bmatrix} 6s_{3\ell,sk}a_{sk} \\ 3s_{\kappa,sk}c_{3\ell,sk} \\ -3c_{\kappa,sk}c_{3\ell,sk} \\ 0 \\ 0 \\ -8c_{3\ell,sk} \end{bmatrix} \tag{109}$$

5.2.13 Coefficient S_{33}

The S_{33} term of the Earth potential is:

$$\begin{aligned}
&\mathcal{E}_{\oplus,S33}\\
&= \frac{\beta_{33}}{r^4}\cos^3\varphi\cos(3\lambda),\\
&= \beta_{33}\frac{(x\cos\Theta + y\sin\Theta)[x^2+y^2-4(y\cos\Theta - x\sin\Theta)^2]}{\sqrt{x^2+y^2+z^2}^7},
\end{aligned} \tag{110}$$

with $\beta_{33} = 15\mu_\oplus R_\oplus^2 S_{33}$.

The Equation (57) leads to the dynamic matrix:

$$\tilde{A}_{\oplus,S33} = \frac{\beta_{33}}{a_{sk}^4\sqrt{\mu_\oplus a_{sk}}}\begin{bmatrix}
A^{11}_{\oplus,S33} & A^{12}_{\oplus,S33} & A^{13}_{\oplus,S33} & 0 & 0 & A^{16}_{\oplus,S33}\\
A^{21}_{\oplus,S33} & A^{22}_{\oplus,S33} & A^{23}_{\oplus,S33} & 0 & 0 & A^{26}_{\oplus,S33}\\
A^{31}_{\oplus,S33} & A^{32}_{\oplus,S33} & A^{33}_{\oplus,S33} & 0 & 0 & A^{36}_{\oplus,S33}\\
0 & 0 & 0 & A^{44}_{\oplus,S33} & A^{45}_{\oplus,S33} & 0\\
0 & 0 & 0 & A^{54}_{\oplus,S33} & A^{55}_{\oplus,S33} & 0\\
A^{61}_{\oplus,S33} & A^{62}_{\oplus,S33} & A^{63}_{\oplus,S33} & 0 & 0 & A^{66}_{\oplus,S33}
\end{bmatrix}, \tag{111}$$

$$A^{11}_{\oplus,S33} = 21 c_{3\ell,sk} a_{sk}, \tag{112a}$$
$$A^{21}_{\oplus,S33} = 18 s_{\kappa,sk} s_{3\ell,sk}, \tag{112b}$$
$$A^{31}_{\oplus,S33} = 18 c_{\kappa,sk} c_{3\ell,sk}, \tag{112c}$$
$$A^{61}_{\oplus,S33} = -36 s_{3\ell,sk}, \tag{112d}$$
$$A^{12}_{\oplus,S33} = 8a_{sk}^2\left[2s_{3\Theta}c_{\kappa,sk}s_{\kappa,sk}(8c_{\kappa,sk}^2-3) + c_{3\Theta}(16c_{\kappa,sk}^4 - 14c_{\kappa,sk}^2+8)\right], \tag{112e}$$
$$A^{22}_{\oplus,S33} = 3a_{sk}\left[s_{3\Theta}s_{\kappa,sk}(32c_{\kappa,sk}^4 - 28c_{\kappa,sk}^2+1) + c_{\kappa,sk}c_{3\Theta}(32c_{\kappa,sk}^4 - 44c_{\kappa,sk}^2+11))\right], \tag{112f}$$
$$A^{32}_{\oplus,S33} = 4a_{sk}\left[c_{3\Theta}s_{\kappa,sk}(12c_{\kappa,sk}^4 + 5c_{\kappa,sk}^2-2) + s_{3\Theta}c_{\kappa,sk}(-12c_{\kappa,sk}^2 + c_{\kappa,sk}^2+6)\right], \tag{112g}$$
$$A^{62}_{\oplus,S33} = 34 a_{sk} c_{\kappa,sk} s_{3\ell,sk}, \tag{112h}$$
$$A^{13}_{\oplus,S33} = 8a_{sk}^2\left[2c_{3\Theta}c_{\kappa,sk}s_{\kappa,sk}(8c_{\kappa,sk}^2-5) + s_{3\Theta}(-16c_{\kappa,sk}^4 + 18c_{\kappa,sk}^2-3)\right], \tag{112i}$$
$$A^{23}_{\oplus,S33} = 4a_{sk}\left[c_{3\Theta}s_{\kappa,sk}(12c_{\kappa,sk}^4 - 23c_{\kappa,sk}^2+5) + c_{3\Theta}s_{\kappa,sk}(-12c_{\kappa,sk}^2 + 29c_{\kappa,sk}^2-15)\right], \tag{112j}$$
$$A^{33}_{\oplus,S33} = \frac{3a_{sk}}{2}\left[s_{3\Theta}s_{\kappa,sk}(-32c_{\kappa,sk}^4 + 20c_{\kappa,sk}^2+1) + c_{\kappa,sk}c_{3\Theta}(-32c_{\kappa,sk}^4 + 36c_{\kappa,sk}^2-5))\right], \tag{112k}$$
$$A^{63}_{\oplus,S33} = -34 a_{sk} s_{\kappa,sk} s_{3\ell,sk}, \tag{112l}$$
$$A^{44}_{\oplus,S33} = -3a_{sk}c_{\kappa,sk}\cos(2\kappa_{sk} - 3\Theta), \tag{112m}$$
$$A^{54}_{\oplus,S33} = -3a_{sk}s_{\kappa,sk}\cos(2\kappa_{sk} - 3\Theta), \tag{112n}$$
$$A^{45}_{\oplus,S33} = 3a_{sk}c_{\kappa,sk}\sin(2\kappa_{sk} - 3\Theta), \tag{112o}$$
$$A^{55}_{\oplus,S33} = 3a_{sk}s_{\kappa,sk}\sin(2\kappa_{sk} - 3\Theta), \tag{112p}$$
$$A^{16}_{\oplus,S33} = -18a_{sk}^2 s_{3\ell,sk}, \tag{112q}$$
$$A^{26}_{\oplus,S33} = -4a_{sk}\left[2c_{3\Theta}c_{\kappa,sk}s_{\kappa,sk}(8c_{\kappa,sk}^2-5) + s_{3\Theta}(-16c_{\kappa,sk}^4 + 18c_{\kappa,sk}^2-3)\right], \tag{112r}$$
$$A^{36}_{\oplus,S33} = 4a_{sk}\left[2s_{3\Theta}c_{\kappa,sk}s_{\kappa,sk}(8c_{\kappa,sk}^2-3) + c_{3\Theta}(16c_{\kappa,sk}^4 - 14c_{\kappa,sk}^2+1)\right], \tag{112s}$$
$$A^{66}_{\oplus,S33} = 24 a_{sk} c_{3\ell,sk}, \tag{112t}$$

and:

$$\tilde{D}_{\oplus,S33} = \frac{\beta_{33}}{a_{sk}^5\sqrt{\mu_\oplus a_{sk}}} \begin{bmatrix} 6s_{3\ell,sk} \\ -4a_{sk}s_{\kappa,sk}s_{3\ell,sk} \\ 4a_{sk}c_{\kappa,sk}s_{3\ell,sk} \\ 0 \\ 0 \\ 8a_{sk}s_{3\ell,sk} \end{bmatrix} \tag{113}$$

5.3 Sun and Moon Gravitational Attractions

As presented the Section 2.2.2, the gravitational potential of the Sun and the Moon atraction is often expanded in Legendre polynomials (see for instance the thesis [Losa, 2007]). For the derivation of the linearized matrix, we choose not to make this approximation and to use the exact expression of the potential. The expression of the relative dynamics matrices derived in this subsection are valid for any disturbing body whose gravitational potential acts on the spacecraft orbiting the Earth on the GEO orbit. For this study, these disturbing bodies are only the Sun and the Moon.

5.3.1 Sun and Moon Positions

The positions of the Sun and the Moon are involved in their gravitational potential. The ephemeris have been retrieved from the NASA JPL's Horizons System (see [Giorgini and JPL Solar System Dynamics Group, 2005]). The Figures 5 and 6 display the Sun and Moon cartesian position in the geocentric inertial reference frame from January 1st, 2034 to January 1st, 2039.

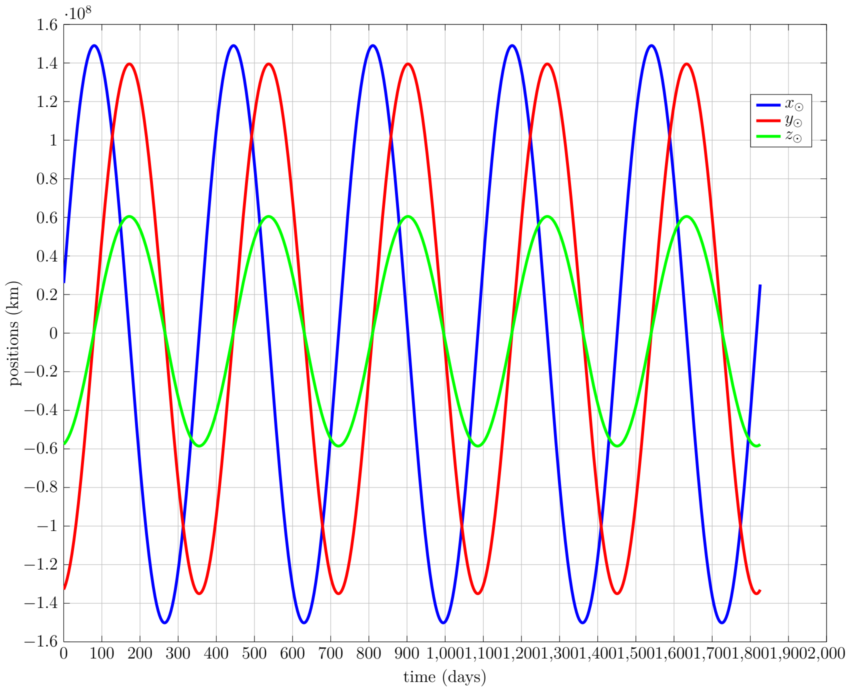

Figure 5 – Cartesian position of the Sun in the geocentric inertial reference frame from January 1st, 2034 to January 1st, 2039.

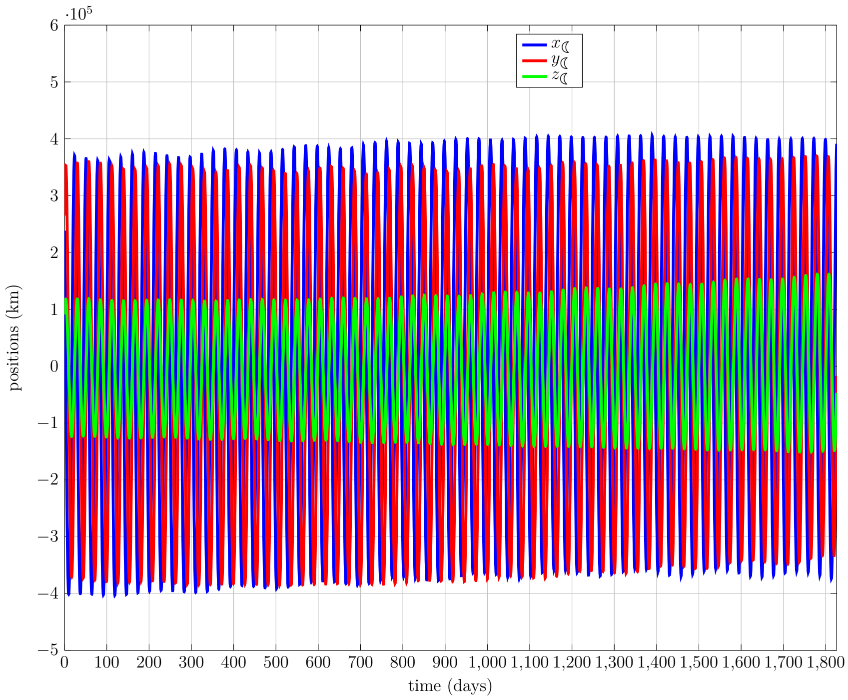

Figure 6 – Cartesian position of the Moon in the geocentric inertial reference frame from January 1st, 2034 to January 1st, 2039.

The radial position of these bodies are written:

$$r_P = \|\vec{r}_P\| = \sqrt{x_P^2 + y_P^2 + z_P^2} \qquad \text{for } P = \odot \text{ or } P = ☾\,. \tag{114}$$

The notations presented in the Equation (65) are still valid here. The distance between the station keeping point (in the equatorial plane) and the disturbing bodies is written:

$$r_{\oplus P} = \sqrt{x_{\oplus P}^2 + y_{\oplus P}^2 + z_P^2}, \tag{115}$$

with:

$$\begin{aligned} x_{\oplus P}(t) &= x_P(t) - a_{sk}\cos\kappa_{sk}, \\ y_{\oplus P}(t) &= y_P(t) - a_{sk}\sin\kappa_{sk}, \end{aligned} \qquad \text{with } P = \odot \text{ or } P = ☾\,, \tag{116}$$

and illustrated on the Figure 7.

5.3.2 Relative Dynamics Matrices

The gravitational potential of the Sun and the Moon are expressed in terms of the cartesian position of the sapcecraft and of the disturbing body in the geocentric inertial reference frame. As the chosen state vector is made of the equinoctial orbital elements, the cartesian position is transformed into the equinoctial orbital elements thanks to the formulas given by the Equation (216). The positions of the Sun and the Moon are interpolated with the values from the Horizon ephemeris generator.

The matrices $\tilde{A}_\odot$ and $\tilde{A}_☾$ are written:

$$\tilde{A}_P = \mu_P \sqrt{\frac{a_{sk}}{\mu_\oplus}} \left[\tilde{A}_P^{ij}\right]_{i=1,\ldots,6,\ j=1,\ldots,6}, \qquad \text{with } P = \odot \text{ or } P = ☾\,. \tag{117}$$

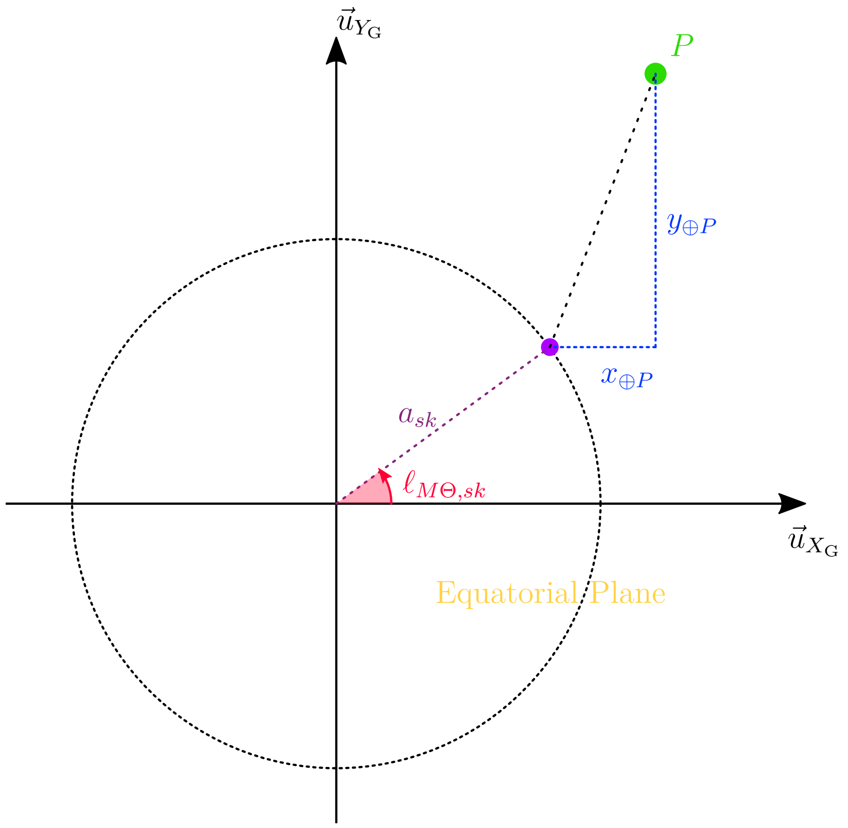

Figure 7 – Illustration of $x_{\oplus P}$ and $y_{\oplus P}$ for $P = \odot$ or $P = ☾$.

The expressions for each coefficient are:

$$A_p^{11} = 3\left[\left(x_{\oplus P}s_{\kappa,sk} - y_{\oplus P}c_{\kappa,sk}\right)\left(\frac{1}{r_P^3} + \frac{2a_{sk}}{r_{\oplus P}^5}\right) - \frac{x_P s_{\kappa,sk} - y_P c_{\kappa,sk}}{r_P^3}\right], \tag{118a}$$

$$\begin{aligned} A_p^{21} = \frac{s_{\kappa,sk}}{a_{sk}} \Bigg[& -\frac{1}{2}\frac{x_{\oplus P}c_{\kappa,sk} + y_{\oplus P}s_{\kappa,sk} - 2a_{sk}}{r_{\oplus P}^3} + \frac{1}{2}\frac{x_P c_{\kappa,sk} + y_P s_{\kappa,sk}}{r_P^3} \\ & -3a_{sk}\frac{(x_{\oplus P}c_{\kappa,sk} + y_{\oplus P}s_{\kappa,sk})^2}{r_{\oplus P}^5}\Bigg], \end{aligned} \tag{118b}$$

$$\begin{aligned} A_p^{31} = \frac{c_{\kappa,sk}}{a_{sk}} \Bigg[& \frac{1}{2}\frac{x_{\oplus P}c_{\kappa,sk} + y_{\oplus P}s_{\kappa,sk} - 2a_{sk}}{r_{\oplus P}^3} - \frac{1}{2}\frac{x_P c_{\kappa,sk} + y_P s_{\kappa,sk}}{r_P^3} \\ & +3a_{sk}\frac{(x_{\oplus P}c_{\kappa,sk} + y_{\oplus P}s_{\kappa,sk})^2}{r_{\oplus P}^5}\Bigg], \end{aligned} \tag{118c}$$

$$A_p^{41} = \frac{c_{\kappa,sk}z_P}{4a_{sk}}\left[\frac{1}{r_{\oplus P}^3} - \frac{1}{r_P^3} + 6a_{sk}\frac{x_{\oplus P}c_{\kappa,sk} + y_{\oplus P}s_{\kappa,sk}}{r_{\oplus P}^5}\right], \tag{118d}$$

$$A_p^{51} = \frac{s_{\kappa,sk}z_P}{4a_{sk}}\left[\frac{1}{r_{\oplus P}^3} - \frac{1}{r_P^3} + 6a_{sk}\frac{x_{\oplus P}c_{\kappa,sk} + y_{\oplus P}s_{\kappa,sk}}{r_{\oplus P}^5}\right], \tag{118e}$$

$$\begin{aligned} A_p^{61} = & \frac{x_{\oplus P}c_{\kappa,sk} + y_{\oplus P}s_{\kappa,sk} - 2a_{sk}}{r_{\oplus P}^3} - \frac{x_P c_{\kappa,sk} + y_P s_{\kappa,sk}}{r_P^3} \\ & + 2a_{sk}\frac{(x_{\oplus P}c_{\kappa,sk} + y_{\oplus P}s_{\kappa,sk})^2}{r_{\oplus P}^5}, \end{aligned} \tag{118f}$$

$$A_p^{12} = 2a_{sk}\left[-\frac{x_{\oplus P}s_{2\kappa,sk} - y_{\oplus P}c_{2\kappa,sk}}{r_{\oplus P}^3} + \frac{x_P s_{2\kappa,sk} - y_P c_{2\kappa,sk}}{r_P^3}\right.$$
$$\left.-3a_{sk}c_{\kappa,sk}\frac{(x_{\oplus P}s_{\kappa,sk} - y_{\oplus P}c_{\kappa,sk})(x_{\oplus P}c_{\kappa,sk} + y_{\oplus P}s_{\kappa,sk})}{r_{\oplus P}^5}\right], \tag{118g}$$

$$A_p^{22} = c_{\kappa,sk}s_{\kappa,sk}\frac{2(x_{\oplus P}c_{\kappa,sk} + y_{\oplus P}s_{\kappa,sk}) - a_{sk}}{r_{\oplus P}^3} - \frac{x_{\oplus P}s_{\kappa,sk} - y_{\oplus P}c_{\kappa,sk}}{2r_{\oplus P}^3}$$
$$+ \frac{-4c_{\kappa,sk}s_{\kappa,sk}(x_P c_{\kappa,sk} + y_P s_{\kappa,sk}) + x_P s_{\kappa,sk} - y_P c_{\kappa,sk}}{r_P^3}$$
$$+ 3c_{\kappa,sk}s_{\kappa,sk}\frac{(x_{\oplus P}c_{\kappa,sk} + y_{\oplus P}s_{\kappa,sk})^2}{r_{\oplus P}^5}, \tag{118h}$$

$$A_p^{32} = \frac{2s_{\kappa,sk}^2(x_{\oplus P}c_{\kappa,sk} + y_{\oplus P}s_{\kappa,sk}) + a_{sk}c_{\kappa,sk}^2}{r_{\oplus P}^3} - 3c_{\kappa,sk}^2 a_{sk}\frac{(x_P c_{\kappa,sk} + y_P s_{\kappa,sk})^2}{r_{\oplus P}^5}$$
$$- 2s_{\kappa,sk}^2\frac{x_{\oplus P}c_{\kappa,sk} + y_{\oplus P}s_{\kappa,sk}}{r_P^3}, \tag{118i}$$

$$A_p^{42} = \frac{c_{\kappa,sk}^2 z_P}{2}\left[-\frac{1}{r_{\oplus P}^3} + \frac{1}{r_P^3} - 3a_{sk}\frac{x_{\oplus P}c_{\kappa,sk} + y_{\oplus P}s_{\kappa,sk}}{r_{\oplus P}^5}\right], \tag{118j}$$

$$A_p^{52} = \frac{c_{\kappa,sk}s_{\kappa,sk}z_P}{2}\left[-\frac{1}{r_{\oplus P}^3} + \frac{1}{r_P^3} - 3a_{sk}\frac{x_{\oplus P}c_{\kappa,sk} + y_{\oplus P}s_{\kappa,sk}}{r_{\oplus P}^5}\right], \tag{118k}$$

$$A_p^{62} = c_{\kappa,sk}\left[\frac{-3(x_{\oplus P}c_{\kappa,sk} + y_{\oplus P}s_{\kappa,sk}) + 4a_{sk}}{r_{\oplus P}^3} + \frac{3}{2}\frac{x_P c_{\kappa,sk} + y_P s_{\kappa,sk}}{r_P^3}\right.$$
$$\left.-6a_{sk}\frac{(x_{\oplus P}c_{\kappa,sk} + y_{\oplus P}s_{\kappa,sk})^2}{r_{\oplus P}^5}\right], \tag{118l}$$

$$A_p^{13} = -2a_{sk}\left[-\frac{x_{\oplus P}c_{2\kappa,sk} + y_{\oplus P}s_{2\kappa,sk}}{r_{\oplus P}^3} + \frac{x_P s_{2\kappa,sk} - y_P c_{2\kappa,sk}}{r_P^3}\right.$$
$$\left.+3a_{sk}c_{\kappa,sk}\frac{(x_{\oplus P}s_{\kappa,sk} - y_{\oplus P}c_{\kappa,sk})(x_{\oplus P}c_{\kappa,sk} + y_{\oplus P}s_{\kappa,sk})}{r_{\oplus P}^5}\right], \tag{118m}$$

$$A_p^{23} = -\frac{2c_{\kappa,sk}^2(x_{\oplus P}c_{\kappa,sk} + y_{\oplus P}s_{\kappa,sk}) + a_{sk}s_{\kappa,sk}^2}{r_{\oplus P}^3} + 3s_{\kappa,sk}^2 a_{sk}\frac{(x_P c_{\kappa,sk} + y_P s_{\kappa,sk})^2}{r_{\oplus P}^5}$$
$$+ 2c_{\kappa,sk}^2\frac{x_{\oplus P}c_{\kappa,sk} + y_{\oplus P}s_{\kappa,sk}}{r_P^3}, \tag{118n}$$

$$A_p^{33} = -c_{\kappa,sk}s_{\kappa,sk}\frac{2(x_{\oplus P}c_{\kappa,sk} + y_{\oplus P}s_{\kappa,sk}) - a_{sk}}{r_{\oplus P}^3} - \frac{x_{\oplus P}s_{\kappa,sk} - y_{\oplus P}c_{\kappa,sk}}{2r_{\oplus P}^3}$$
$$+ \frac{4c_{\kappa,sk}s_{\kappa,sk}(x_P c_{\kappa,sk} + y_P s_{\kappa,sk}) + x_P s_{\kappa,sk} - y_P c_{\kappa,sk}}{2r_P^3}$$
$$+ 3c_{\kappa,sk}s_{\kappa,sk}\frac{(x_{\oplus P}c_{\kappa,sk} + y_{\oplus P}s_{\kappa,sk})^2}{r_{\oplus P}^5}, \tag{118o}$$

$$A_p^{43} = \frac{c_{\kappa,sk}s_{\kappa,sk}z_P}{2}\left[-\frac{1}{r_{\oplus P}^3} + \frac{1}{r_P^3} - 3a_{sk}\frac{x_{\oplus P}c_{\kappa,sk} + y_{\oplus P}s_{\kappa,sk}}{r_{\oplus P}^5}\right], \tag{118p}$$

$$A_p^{53} = \frac{s_{\kappa,sk}^2 z_P}{2}\left[-\frac{1}{r_{\oplus P}^3} + \frac{1}{r_P^3} - 3a_{sk}\frac{x_{\oplus P}c_{\kappa,sk} + y_{\oplus P}s_{\kappa,sk}}{r_{\oplus P}^5}\right], \tag{118q}$$

$$A_p^{63} = s_{\kappa,sk}\left[\frac{-3(x_{\oplus P}c_{\kappa,sk} + y_{\oplus P}s_{\kappa,sk}) + 4a_{sk}}{r_{\oplus P}^3} + \frac{3}{2}\frac{x_P c_{\kappa,sk} + y_P s_{\kappa,sk}}{r_P^3}\right.$$

$$-6a_{sk}\frac{(x_{\oplus P}c_{\kappa,sk}+y_{\oplus P}s_{\kappa,sk})^2}{r_{\oplus P}^5}\Bigg], \tag{118r}$$

$$A_p^{14}=-4a_{sk}z_P\left[-\frac{c_{\kappa,sk}}{r_{\oplus P}^3}+\frac{c_{\kappa,sk}}{r_P^3}+3a_{sk}s_{\kappa,sk}\frac{x_{\oplus P}s_{\kappa,sk}-y_{\oplus P}c_{\kappa,sk}}{r_{\oplus P}^5}\right], \tag{118s}$$

$$A_p^{24}=2z_Ps_{\kappa,sk}^2\left[\frac{1}{r_{\oplus P}^3}-\frac{1}{r_P^3}+3a_{sk}\frac{x_{\oplus P}c_{\kappa,sk}+y_{\oplus P}s_{\kappa,sk}}{r_{\oplus P}^5}\right], \tag{118t}$$

$$A_p^{34}=-2z_Ps_{\kappa,sk}c_{\kappa,sk}\left[\frac{1}{r_{\oplus P}^3}-\frac{1}{r_P^3}+3a_{sk}\frac{x_{\oplus P}c_{\kappa,sk}+y_{\oplus P}s_{\kappa,sk}}{r_{\oplus P}^5}\right], \tag{118u}$$

$$A_p^{44}=c_{\kappa,sk}\left[\frac{y_\odot}{r_{\oplus P}^3}-\frac{y_P}{r_P^3}-\frac{3a_{sk}s_{\kappa,sk}z_P^2}{r_{\oplus P}^5}\right], \tag{118v}$$

$$A_p^{54}=s_{\kappa,sk}\left[\frac{y_P}{r_{\oplus P}^3}-\frac{y_P}{r_P^3}-\frac{3a_{sk}s_{\kappa,sk}z_P^2}{r_{\oplus P}^5}\right], \tag{118w}$$

$$A_p^{64}=3s_{\kappa,sk}z_P\left[-\frac{1}{r_{\oplus P}^3}+\frac{1}{r_P^3}-4a_{sk}\frac{x_{\oplus P}c_{\kappa,sk}+y_{\oplus P}s_{\kappa,sk}}{r_{\oplus P}^5}\right], \tag{118x}$$

$$A_p^{15}=-4a_{sk}z_P\left[-\frac{s_{\kappa,sk}}{r_{\oplus P}^3}+\frac{s_{\kappa,sk}}{r_P^3}-3a_{sk}c_{\kappa,sk}\frac{x_{\oplus P}s_{\kappa,sk}-y_{\oplus P}c_{\kappa,sk}}{r_{\oplus P}^5}\right], \tag{118y}$$

$$A_p^{25}=-2z_Ps_{\kappa,sk}c_{\kappa,sk}\left[\frac{1}{r_{\oplus P}^3}-\frac{1}{r_P^3}+3a_{sk}\frac{x_{\oplus P}c_{\kappa,sk}+y_{\oplus P}s_{\kappa,sk}}{r_{\oplus P}^5}\right], \tag{118z}$$

$$A_p^{35}=2z_Pc_{\kappa,sk}^2\left[\frac{1}{r_{\oplus P}^3}-\frac{1}{r_P^3}+3a_{sk}\frac{x_{\oplus P}c_{\kappa,sk}+y_{\oplus P}s_{\kappa,sk}}{r_{\oplus P}^5}\right], \tag{118aa}$$

$$A_p^{45}=c_{\kappa,sk}\left[-\frac{x_P}{r_{\oplus P}^3}+\frac{x_P}{r_P^3}+\frac{3a_{sk}c_{\kappa,sk}z_P^2}{r_{\oplus P}^5}\right], \tag{118ab}$$

$$A_p^{55}=s_{\kappa,sk}\left[-\frac{x_\odot}{r_{\oplus P}^3}+\frac{x_P}{r_P^3}+\frac{3a_{sk}c_{\kappa,sk}z_P^2}{r_{\oplus P}^5}\right], \tag{118ac}$$

$$A_p^{65}=3c_{\kappa,sk}z_P\left[\frac{1}{r_{\oplus P}^3}-\frac{1}{r_P^3}+4a_{sk}\frac{x_{\oplus P}c_{\kappa,sk}+y_{\oplus P}s_{\kappa,sk}}{r_{\oplus P}^5}\right], \tag{118ad}$$

$$A_p^{16}=2a_{sk}\Bigg[-\frac{x_{\oplus P}c_{\kappa,sk}+y_{\oplus P}s_{\kappa,sk}+a_{sk}}{r_{\oplus P}^3}+\frac{x_Pc_{\kappa,sk}+y_Ps_{\kappa,sk}}{r_P^3}$$

$$+3a_{sk}\frac{(x_{\oplus P}s_{\kappa,sk}-y_{\oplus P}c_{\kappa,sk})^2}{r_{\oplus P}^5}\Bigg], \tag{118ae}$$

$$A_p^{26}=-\frac{x_{\oplus P}c_{2\kappa,sk}+y_{\oplus P}s_{2\kappa,sk}}{r_{\oplus P}^3}+\frac{x_Pc_{2\kappa,sk}+y_Ps_{2\kappa,sk}}{r_P^3}$$

$$+3a_{sk}s_{\kappa,sk}\frac{(x_{\oplus P}s_{\kappa,sk}-y_{\oplus P}c_{\kappa,sk})(x_{\oplus P}c_{\kappa,sk}+y_{\oplus P}s_{\kappa,sk})}{r_{\oplus P}^5}, \tag{118af}$$

$$A_p^{36}=-\frac{x_{\oplus P}s_{2\kappa,sk}-y_{\oplus P}c_{2\kappa,sk}}{r_{\oplus P}^3}+\frac{x_Ps_{2\kappa,sk}-y_Pc_{2\kappa,sk}}{r_P^3}$$

$$+3a_{sk}c_{\kappa,sk}\frac{(x_{\oplus P}s_{\kappa,sk}-y_{\oplus P}c_{\kappa,sk})(x_{\oplus P}c_{\kappa,sk}+y_{\oplus P}s_{\kappa,sk})}{r_{\oplus P}^5}, \tag{118ag}$$

$$A_p^{46}=\frac{z_P}{2}\left[-\frac{s_{\kappa,sk}}{r_{\oplus P}^3}+\frac{s_{\kappa,sk}}{r_P^3}-3a_{sk}c_{\kappa,sk}\frac{x_{\oplus P}s_{\kappa,sk}-y_{\oplus P}c_{\kappa,sk}}{r_{\oplus P}^5}\right], \tag{118ah}$$

$$A_p^{56}=-\frac{z_P}{2}\left[-\frac{c_{\kappa,sk}}{r_{\oplus P}^3}+\frac{c_{\kappa,sk}}{r_P^3}+3a_{sk}s_{\kappa,sk}\frac{x_{\oplus P}s_{\kappa,sk}-y_{\oplus P}c_{\kappa,sk}}{r_{\oplus P}^5}\right], \tag{118ai}$$

$$A_p^{66} = 2\left[-\frac{x_{\oplus P}s_{\kappa,sk} - y_{\oplus P}c_{\kappa,sk}}{r_{\oplus P}^3} + \frac{x_P c_{\kappa,sk} + y_P s_{\kappa,sk}}{r_P^3}\right.$$
$$\left. -3a_{sk}\frac{(x_{\oplus P}s_{\kappa,sk} - y_{\oplus P}c_{\kappa,sk})(x_{\oplus P}c_{\kappa,sk} + y_{\oplus P}s_{\kappa,sk})}{r_{\oplus P}^5}\right], \tag{118aj}$$

for $P = \odot$ or $P = ☾$.

The vectors $\tilde{D}_\odot$ and $\tilde{D}_☾$ are written:

$$\tilde{D}_P = \mu_P\sqrt{\frac{a_{sk}}{\mu_\oplus}}\left[\tilde{D}_P^i\right]_{i=1,\dots,6}, \qquad \text{with } P = \odot \text{ or } P = ☾ . \tag{119}$$

The expressions for each coefficient are:

$$\tilde{D}_P^1 = 2a_{sk}\left[\frac{x_{\oplus P}s_{\kappa,sk} - y_{\oplus P}c_{\kappa,sk}}{r_{\oplus P}^3} - \frac{x_P s_{\kappa,sk} - y_P c_{\kappa,sk}}{rP^3}\right], \tag{120a}$$

$$\tilde{D}_P^2 = -s_{\kappa,sk}\left[\frac{x_{\oplus P}c_{\kappa,sk} + y_{\oplus P}s_{\kappa,sk}}{r_{\oplus P}^3} - \frac{x_P c_{\kappa,sk} + y_P s_{\kappa,sk}}{r_P^3}\right], \tag{120b}$$

$$\tilde{D}_P^3 = c_{\kappa,sk}\left[\frac{x_{\oplus P}c_{\kappa,sk} + y_{\oplus P}s_{\kappa,sk}}{r_{\oplus P}^3} - \frac{x_P c_{\kappa,sk} + y_P s_{\kappa,sk}}{r_P^3}\right], \tag{120c}$$

$$\tilde{D}_P^4 = \frac{c_{\kappa,sk}z_P}{2}\left[\frac{1}{r_{\oplus P}^3} - \frac{1}{r_P^3}\right], \tag{120d}$$

$$\tilde{D}_P^5 = \frac{s_{\kappa,sk}z_P}{2}\left[\frac{1}{r_{\oplus P}^3} - \frac{1}{r_P^3}\right], \tag{120e}$$

$$\tilde{D}_P^6 = 2\left[\frac{x_{\oplus P}c_{\kappa,sk} + y_{\oplus P}s_{\kappa,sk}}{r_{\oplus P}^3} - \frac{x_P c_{\kappa,sk} + y_P s_{\kappa,sk}}{r_P^3}\right]. \tag{120f}$$

5.4 Solar Radiation Pressure

With the notations used for the previous computations of the linearized dynamics, the SRP pseudo-potential function defined in the Section 2.2.3 is written as:

$$\mathcal{E}_{\mathrm{SRP}} = \alpha\sqrt{(x_\odot - x)^2 + (y_\odot - y)^2 + (z \odot - z)^2}, \tag{121}$$

with $\alpha = \varrho P_\odot \frac{Sc}{m}$ (see the definition of these quantities in the Section 2.2.3). The Sun cartesian position in the geocentric inertial reference frame is interpolated with the data retrieved from the Horizon ephemeris system and the cartesian position pf the spacecraft is transformed into the osculating equinoctial orbital elements with the formulas given by the Equation 216.

The matrix $\tilde{A}_{\mathrm{SRP}}$ is written:

$$\tilde{A}_{\mathrm{SRP}} = \alpha\sqrt{\frac{a_{sk}}{\mu_\oplus}}\left[\tilde{A}_{\mathrm{PRS}}^{ij}\right]_{i=1,\dots,6,\ j=1,\dots,6}. \tag{122}$$

The expressions for each coefficient are:

$$\tilde{A}_{\mathrm{PRS}}^{11} = -\frac{(x_{\oplus\odot}s_{\kappa,sk} - y_{\oplus\odot}c_{\kappa,sk})\left[3z_\odot^2 + x_{\oplus\odot}(3x_\odot a_{sk}c_{\kappa,sk}) + y_{\oplus\odot}(3y_\odot - a_{sk}s_{\kappa,sk})\right]}{r_{\oplus\odot}^3}, \tag{123a}$$

$$\tilde{A}^{21}_{\rm PRS} = \frac{s_{\kappa,sk}}{r_{\oplus\odot}}\left[\frac{x_{\oplus\odot}c_{\kappa,sk}+y_{\oplus\odot}s_{\kappa,sk}-2a_{sk}}{2a_{sk}}+\frac{(x_{\oplus\odot}c_{\kappa,sk}+y_{\oplus\odot}s_{\kappa,sk})^2}{r^2_{\oplus\odot}}\right], \tag{123b}$$

$$\tilde{A}^{31}_{\rm PRS} = -\frac{c_{\kappa,sk}}{r_{\oplus\odot}}\left[\frac{x_{\oplus\odot}c_{\kappa,sk}+y_{\oplus\odot}s_{\kappa,sk}-2a_{sk}}{2a_{sk}}+\frac{(x_{\oplus\odot}c_{\kappa,sk}+y_{\oplus\odot}s_{\kappa,sk})^2}{r^2_{\oplus\odot}}\right], \tag{123c}$$

$$\tilde{A}^{41}_{\rm PRS} = -\frac{c_{\kappa,sk}z_{\odot}}{2r_{\oplus\odot}}\left[\frac{1}{2a_{sk}}+\frac{x_{\oplus\odot}c_{\kappa,sk}+y_{\oplus\odot}s_{\kappa,sk}}{r^2_{\oplus\odot}}\right], \tag{123d}$$

$$\tilde{A}^{51}_{\rm PRS} = -\frac{s_{\kappa,sk}z_{\odot}}{2r_{\oplus\odot}}\left[\frac{1}{2a_{sk}}+\frac{x_{\oplus\odot}c_{\kappa,sk}+y_{\oplus\odot}s_{\kappa,sk}}{r^2_{\oplus\odot}}\right], \tag{123e}$$

$$\tilde{A}^{61}_{\rm PRS} = -\frac{x_{\oplus\odot}c_{\kappa,sk}+y_{\oplus\odot}s_{\kappa,sk}-2a_{sk}}{a_{sk}r_{\oplus\odot}}-2\frac{(x_{\oplus\odot}c_{\kappa,sk}+y_{\oplus\odot}s_{\kappa,sk})^2}{r^2_{\oplus\odot}}, \tag{123f}$$

$$\begin{aligned}\tilde{A}^{12}_{\rm PRS} = \frac{2a_{sk}}{r_{\oplus\odot}}\Big[&x_{\oplus\odot}s_{2\kappa,sk}-y_{\oplus\odot}c_{2\kappa,sk}\\ &+c_{\kappa,sk}a_{sk}\frac{(x_{\oplus\odot}s_{\kappa,sk}-y_{\oplus\odot}c_{\kappa,sk})(x_{\oplus\odot}c_{\kappa,sk}+y_{\oplus\odot}s_{\kappa,sk})}{r^2_{\oplus\odot}}\Bigg],\end{aligned} \tag{123g}$$

$$\begin{aligned}\tilde{A}^{22}_{\rm PRS} = \frac{1}{r_{\oplus\odot}}\Bigg[&c_{\kappa,sk}s_{\kappa,sk}\Big(a_{sk}-2(x_{\oplus\odot}c_{\kappa,sk}+y_{\oplus\odot}s_{\kappa,sk})\Big)\\ &+\frac{1}{2}(x_{\oplus\odot}s_{\kappa,sk}-y_{\oplus\odot}c_{\kappa,sk})-a_{sk}c_{\kappa,sk}s_{\kappa,sk}\frac{(x_{\oplus\odot}c_{\kappa,sk}+y_{\oplus\odot}s_{\kappa,sk})^2}{r^2_{\oplus\odot}}\Bigg],\end{aligned} \tag{123h}$$

$$\tilde{A}^{32}_{\rm PRS} = -\frac{2s^2_{\kappa,sk}(x_{\oplus\odot}c_{\kappa,sk}+y_{\oplus\odot}s_{\kappa,sk})+c^2_{\kappa,sk}a_{sk}}{r_{\oplus\odot}}+a_{sk}c^2_{\kappa,sk}\frac{(x_{\oplus\odot}c_{\kappa,sk}+y_{\oplus\odot}s_{\kappa,sk})^2}{r^3_{\oplus\odot}}, \tag{123i}$$

$$\tilde{A}^{42}_{\rm PRS} = \frac{c^2_{\kappa,sk}z_{\oplus\odot}}{2r_{\oplus\odot}}\left[1+a_{sk}\frac{x_{\oplus\odot}c_{\kappa,sk}+y_{\oplus\odot}s_{\kappa,sk}}{r^2_{\oplus\odot}}\right], \tag{123j}$$

$$\tilde{A}^{52}_{\rm PRS} = \frac{c_{\kappa,sk}s_{\kappa,sk}z_{\oplus\odot}}{2r_{\oplus\odot}}\left[1+a_{sk}\frac{x_{\oplus\odot}c_{\kappa,sk}+y_{\oplus\odot}s_{\kappa,sk}}{r^2_{\oplus\odot}}\right], \tag{123k}$$

$$\tilde{A}^{62}_{\rm PRS} = \frac{c_{\kappa,sk}}{r_{\oplus\odot}}\left[\frac{3}{2}(x_{\oplus\odot}c_{\kappa,sk}+y_{\oplus\odot}s_{\kappa,sk})-2a_{sk}+2a_{sk}\frac{(x_{\oplus\odot}c_{\kappa,sk}+y_{\oplus\odot}s_{\kappa,sk})^2}{r^2_{\oplus\odot}}\right], \tag{123l}$$

$$\begin{aligned}\tilde{A}^{13}_{\rm PRS} = \frac{2a_{sk}}{r_{\oplus\odot}}\Big[&-(x_{\oplus\odot}c_{2\kappa,sk}+y_{\oplus\odot}s_{2\kappa,sk})\\ &+s_{\kappa,sk}a_{sk}\frac{(x_{\oplus\odot}s_{\kappa,sk}-y_{\oplus\odot}c_{\kappa,sk})(x_{\oplus\odot}c_{\kappa,sk}+y_{\oplus\odot}s_{\kappa,sk})}{r^2_{\oplus\odot}}\Bigg],\end{aligned} \tag{123m}$$

$$\tilde{A}^{23}_{\rm PRS} = \frac{2c^2_{\kappa,sk}(x_{\oplus\odot}c_{\kappa,sk}+y_{\oplus\odot}s_{\kappa,sk})+s^2_{\kappa,sk}a_{sk}}{r_{\oplus\odot}}-a_{sk}s^2_{\kappa,sk}\frac{(x_{\oplus\odot}c_{\kappa,sk}+y_{\oplus\odot}s_{\kappa,sk})^2}{r^3_{\oplus\odot}}, \tag{123n}$$

$$\begin{aligned}\tilde{A}^{33}_{\rm PRS} = \frac{1}{r_{\oplus\odot}}\Bigg[&-c_{\kappa,sk}s_{\kappa,sk}\Big(a_{sk}-2(x_{\oplus\odot}c_{\kappa,sk}+y_{\oplus\odot}s_{\kappa,sk})\Big)\\ &+\frac{1}{2}(x_{\oplus\odot}s_{\kappa,sk}-y_{\oplus\odot}c_{\kappa,sk})+a_{sk}c_{\kappa,sk}s_{\kappa,sk}\frac{(x_{\oplus\odot}c_{\kappa,sk}+y_{\oplus\odot}s_{\kappa,sk})^2}{r^2_{\oplus\odot}}\Bigg],\end{aligned} \tag{123o}$$

$$\tilde{A}^{43}_{\rm PRS} = \frac{c_{\kappa,sk}s_{\kappa,sk}z_{\oplus\odot}}{2r_{\oplus\odot}}\left[1+a_{sk}\frac{x_{\oplus\odot}c_{\kappa,sk}+y_{\oplus\odot}s_{\kappa,sk}}{r^2_{\oplus\odot}}\right], \tag{123p}$$

$$\tilde{A}^{53}_{\rm PRS} = \frac{s^2_{\kappa,sk}z_{\oplus\odot}}{2r_{\oplus\odot}}\left[1+a_{sk}\frac{x_{\oplus\odot}c_{\kappa,sk}+y_{\oplus\odot}s_{\kappa,sk}}{r^2_{\oplus\odot}}\right], \tag{123q}$$

$$\tilde{A}^{63}_{\rm PRS} = \frac{s_{\kappa,sk}}{r_{\oplus\odot}}\left[\frac{3}{2}(x_{\oplus\odot}c_{\kappa,sk}+y_{\oplus\odot}s_{\kappa,sk})-2a_{sk}+2a_{sk}\frac{(x_{\oplus\odot}c_{\kappa,sk}+y_{\oplus\odot}s_{\kappa,sk})^2}{r^2_{\oplus\odot}}\right], \tag{123r}$$

$$\tilde{A}^{14}_{\mathrm{PRS}} = \frac{4a_{sk}z_\odot}{r_{\oplus\odot}}\left[-c_{\kappa,sk} + s_{\kappa,sk}a_{sk}\frac{x_{\oplus\odot}s_{\kappa,sk} - y_{\oplus\odot}c_{\kappa,sk}}{r^2_{\oplus\odot}}\right], \tag{123s}$$

$$\tilde{A}^{24}_{\mathrm{PRS}} = -\frac{2s^2_{\kappa,sk}z_\odot}{r_{\oplus\odot}}\left[1 + a_{sk}\frac{x_{\oplus\odot}c_{\kappa,sk} + y_{\oplus\odot}s_{\kappa,sk}}{r^2_{\oplus\odot}}\right], \tag{123t}$$

$$\tilde{A}^{34}_{\mathrm{PRS}} = \frac{2s_{\kappa,sk}c_{\kappa,sk}z_\odot}{r_{\oplus\odot}}\left[1 + a_{sk}\frac{x_{\oplus\odot}c_{\kappa,sk} + y_{\oplus\odot}s_{\kappa,sk}}{r^2_{\oplus\odot}}\right], \tag{123u}$$

$$\tilde{A}^{44}_{\mathrm{PRS}} = \frac{c_{\kappa,sk}}{r_{\oplus\odot}}\left[-\frac{y_\odot}{r_{\oplus\odot}} + \frac{a_{sk}s_{\kappa,sk}z^2_\odot}{r^2_{\oplus\odot}}\right], \tag{123v}$$

$$\tilde{A}^{54}_{\mathrm{PRS}} = \frac{s_{\kappa,sk}}{r_{\oplus\odot}}\left[-\frac{y_\odot}{r_{\oplus\odot}} + \frac{a_{sk}s_{\kappa,sk}z^2_\odot}{r^2_{\oplus\odot}}\right], \tag{123w}$$

$$\tilde{A}^{64}_{\mathrm{PRS}} = \frac{s_{\kappa,sk}z_\odot}{r_{\oplus\odot}}\left[3 + 4a_{sk}\frac{x_{\oplus\odot}c_{\kappa,sk} + y_{\oplus\odot}s_{\kappa,sk}}{r^2_{\oplus\odot}}\right], \tag{123x}$$

$$\tilde{A}^{15}_{\mathrm{PRS}} = -\frac{4a_{sk}z_\odot}{r_{\oplus\odot}}\left[s_{\kappa,sk} + c_{\kappa,sk}a_{sk}\frac{x_{\oplus\odot}s_{\kappa,sk} - y_{\oplus\odot}c_{\kappa,sk}}{r^2_{\oplus\odot}}\right], \tag{123y}$$

$$\tilde{A}^{25}_{\mathrm{PRS}} = \frac{2s_{\kappa,sk}c_{\kappa,sk}z_\odot}{r_{\oplus\odot}}\left[1 + a_{sk}\frac{x_{\oplus\odot}c_{\kappa,sk} + y_{\oplus\odot}s_{\kappa,sk}}{r^2_{\oplus\odot}}\right], \tag{123z}$$

$$\tilde{A}^{35}_{\mathrm{PRS}} = -\frac{2s^2_{\kappa,sk}z_\odot}{r_{\oplus\odot}}\left[1 + a_{sk}\frac{x_{\oplus\odot}c_{\kappa,sk} + y_{\oplus\odot}s_{\kappa,sk}}{r^2_{\oplus\odot}}\right], \tag{123aa}$$

$$\tilde{A}^{45}_{\mathrm{PRS}} = \frac{c_{\kappa,sk}}{r_{\oplus\odot}}\left[\frac{x_\odot}{r_{\oplus\odot}} - \frac{a_{sk}c_{\kappa,sk}z^2_\odot}{r^2_{\oplus\odot}}\right], \tag{123ab}$$

$$\tilde{A}^{55}_{\mathrm{PRS}} = \frac{s_{\kappa,sk}}{r_{\oplus\odot}}\left[\frac{x_\odot}{r_{\oplus\odot}} + \frac{a_{sk}c_{\kappa,sk}z^2_\odot}{r^2_{\oplus\odot}}\right], \tag{123ac}$$

$$\tilde{A}^{65}_{\mathrm{PRS}} = -\frac{c_{\kappa,sk}z_\odot}{r_{\oplus\odot}}\left[3 + 4a_{sk}\frac{x_{\oplus\odot}c_{\kappa,sk} + y_{\oplus\odot}s_{\kappa,sk}}{r^2_{\oplus\odot}}\right], \tag{123ad}$$

$$\tilde{A}^{16}_{\mathrm{PRS}} = \frac{2a_{sk}}{r_{\oplus\odot}}\left[-(x_{\oplus\odot}c_{\kappa,sk} + y_{\oplus\odot}s_{\kappa,sk}) + a_{sk}\frac{(x_{\oplus\odot}s_{\kappa,sk} - y_{\oplus\odot}c_{\kappa,sk})^2}{r^2_{\oplus\odot}}\right], \tag{123ae}$$

$$\begin{aligned}\tilde{A}^{26}_{\mathrm{PRS}} = {} & \frac{x_{\oplus\odot}c_{2\kappa,sk} + y_{\oplus\odot}s_{2\kappa,sk}}{r_{\oplus\odot}} \\ & - a_{sk}s_{\kappa,sk}\frac{(x_{\oplus\odot}s_{\kappa,sk} - y_{\oplus\odot}c_{\kappa,sk})(x_{\oplus\odot}c_{\kappa,sk} + y_{\oplus\odot}s_{\kappa,sk})}{r^3_{\oplus\odot}},\end{aligned} \tag{123af}$$

$$\begin{aligned}\tilde{A}^{36}_{\mathrm{PRS}} = {} & \frac{x_{\oplus\odot}s_{2\kappa,sk} - y_{\oplus\odot}c_{2\kappa,sk}}{r_{\oplus\odot}} \\ & + a_{sk}c_{\kappa,sk}\frac{(x_{\oplus\odot}s_{\kappa,sk} - y_{\oplus\odot}c_{\kappa,sk})(x_{\oplus\odot}c_{\kappa,sk} + y_{\oplus\odot}s_{\kappa,sk})}{r^3_{\oplus\odot}},\end{aligned} \tag{123ag}$$

$$\tilde{A}^{46}_{\mathrm{PRS}} = \frac{z_\odot}{2r_{\oplus\odot}}\left[s_{\kappa,sk} + a_{sk}c_{\kappa,sk}\frac{x_{\oplus\odot}s_{\kappa,sk} - y_{\oplus\odot}c_{\kappa,sk}}{r^2_{\oplus\odot}}\right], \tag{123ah}$$

$$\tilde{A}^{56}_{\mathrm{PRS}} = \frac{z_\odot}{2r_{\oplus\odot}}\left[-c_{\kappa,sk} + a_{sk}s_{\kappa,sk}\frac{x_{\oplus\odot}s_{\kappa,sk} - y_{\oplus\odot}c_{\kappa,sk}}{r^2_{\oplus\odot}}\right], \tag{123ai}$$

$$\begin{aligned}\tilde{A}^{66}_{\mathrm{PRS}} = {} & \frac{2}{r_{\oplus\odot}}\left[x_{\oplus\odot}s_{\kappa,sk} - y_{\oplus\odot}c_{\kappa,sk}\right. \\ & \left. -a_{sk}\frac{(x_{\oplus\odot}s_{\kappa,sk} - y_{\oplus\odot}c_{\kappa,sk})(x_{\oplus\odot}c_{\kappa,sk} + y_{\oplus\odot}s_{\kappa,sk})}{r^3_{\oplus\odot}}\right].\end{aligned} \tag{123aj}$$

The vector $\tilde{D}_{\mathrm{SRP}}$ is written:

$$\tilde{D}_{\mathrm{SRP}} = \alpha\sqrt{\frac{a_{sk}}{\mu_\oplus}}\left[\tilde{D}^i_{\mathrm{PRS}}\right]_{i=1,\ldots,6}. \tag{124}$$

The expressions for each coefficient are:

$$\tilde{D}^1_{\mathrm{PRS}} = -\frac{2a_{sk}(x_{\oplus\odot}s_{\kappa,sk} - y_{\oplus\odot}c_{\kappa,sk})}{r_{\oplus\odot}}, \tag{125a}$$

$$\tilde{D}^2_{\mathrm{PRS}} = s_{\kappa,sk}\frac{x_{\oplus\odot}c_{\kappa,sk} + y_{\oplus\odot}s_{\kappa,sk}}{r_{\oplus\odot}}, \tag{125b}$$

$$\tilde{D}^3_{\mathrm{PRS}} = -c_{\kappa,sk}\frac{x_{\oplus\odot}c_{\kappa,sk} + y_{\oplus\odot}s_{\kappa,sk}}{r_{\oplus\odot}}, \tag{125c}$$

$$\tilde{D}^4_{\mathrm{PRS}} = -\frac{c_{\kappa,sk}z_\odot}{2r_{\oplus\odot}}, \tag{125d}$$

$$\tilde{D}^5_{\mathrm{PRS}} = -\frac{s_{\kappa,sk}z_\odot}{2r_{\oplus\odot}}, \tag{125e}$$

$$\tilde{D}^6_{\mathrm{PRS}} = -2\frac{x_{\oplus\odot}c_{\kappa,sk} + y_{\oplus\odot}s_{\kappa,sk} - a_{sk}}{r_{\oplus\odot}}. \tag{125f}$$

Appendix

A Reference Frames

A.1 Geocentric Inertial Reference Frame

The reference frame for the cartesian positions and velocities is the Geocentric Inertial Reference Frame. A definition of this reference frame can be found in [Vallado, 1997] named *Geocentric Equatorial Coordinate System* or *Earth Center Inertial*. The origin is located at the center of the Earth and the axis are defined by:

- the axis $(G, \vec{u}_{Z_\mathrm{G}})$ in the direction of the Earth North pole,
- the axis $(G, \vec{u}_{X_\mathrm{G}})$ in the direction of the intersection between the equatorial plane and the ecliptic plane (vernal equinox),
- the axis $(G, \vec{u}_{Y_\mathrm{G}})$ completes the basis,

and are illustrated on the Figure 8. The reference frame $(G, \vec{u}_{X_\mathrm{G}}, \vec{u}_{Y_\mathrm{G}}, \vec{u}_{Z_\mathrm{G}})$ is denoted $\mathcal{R}_\mathrm{G}$ and the associated basis $\mathcal{B}_\mathrm{G}$.

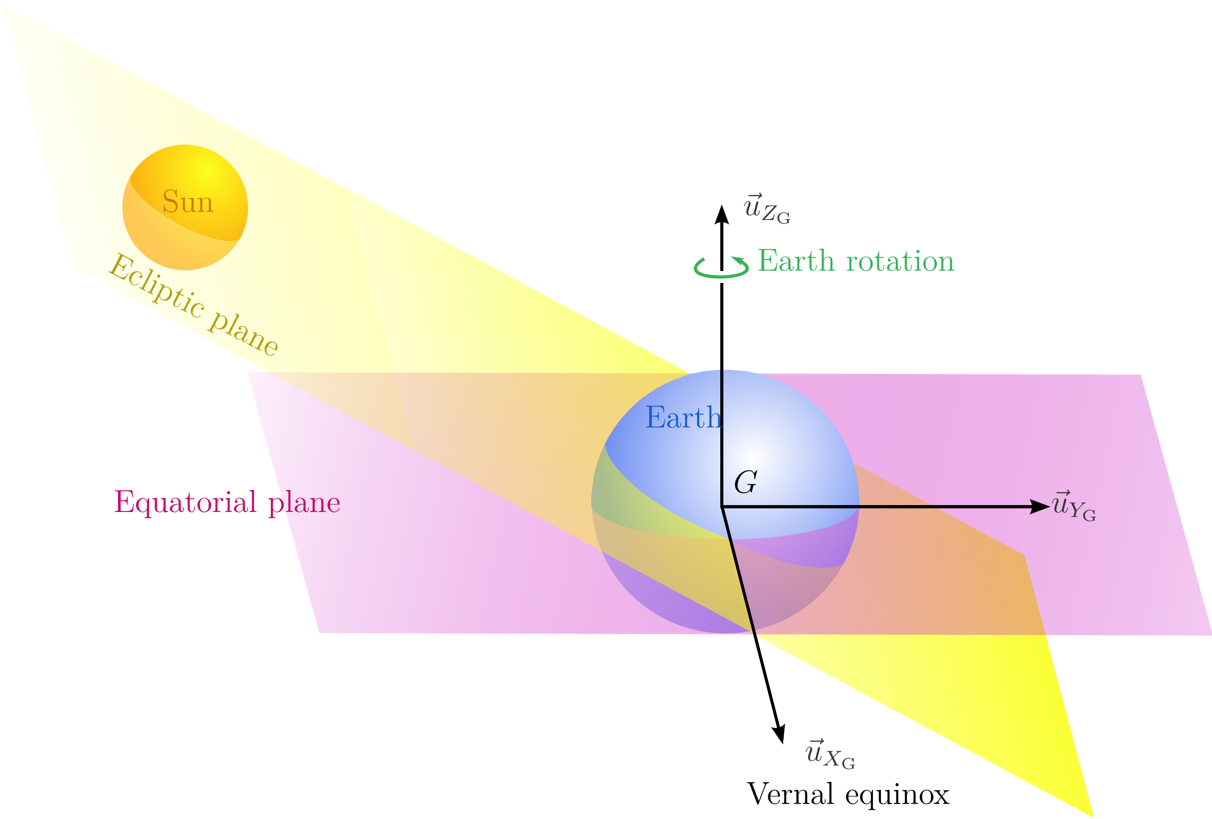

Figure 8 – Geocentric inertial reference frame

A point in space is represented by its three cartesian coordinates x, y and z or its three spherical coordinates r, α (right ascension) and δ (declination). The transformations between the cartesian and spherical coordinates are illustrated on the Figure 9 and given by:

$$x = r\cos\delta\cos\alpha, \tag{126a}$$

$$y = r\cos\delta\sin\alpha, \tag{126b}$$
$$z = r\sin\delta. \tag{126c}$$

The inverse relationships are:

$$r = \sqrt{x^2 + y^2 z^2}, \tag{127a}$$
$$\alpha = \operatorname{atan}\left(\frac{y}{x}\right), \tag{127b}$$
$$\delta = \operatorname{atan}\left(\frac{z}{\sqrt{x^2+y^2}}\right), \tag{127c}$$

where α is chosen such that $\alpha \in [-90°, 90°]$ for $x > 0$ and $\alpha \in [90°, 270°]$ for $x < 0$.

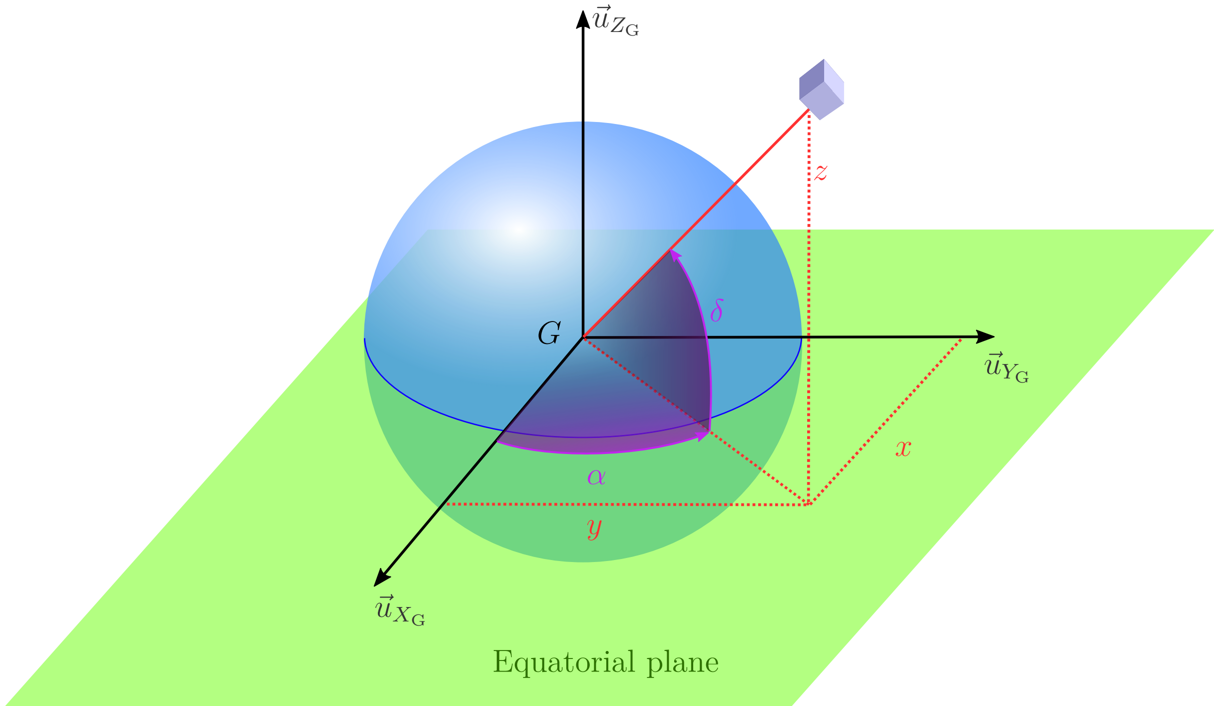

Figure 9 – Right ascension α and declination δ of a point in space

A.2 Rotating Geocentric Reference Frame

The rotating geocentric reference frame is named by the reference [Vallado, 1997] Earth Centered Earth Fixed reference frame. This is a reference frame whose origin is located at the center of the Earth and whose axis are defined by:

- an axis in the direction of the Earth North pole,
- an axis in the direction of the Greenwich meridian,
- an axis that completes the basis.

This reference frame is illustrated on the Figure 10.

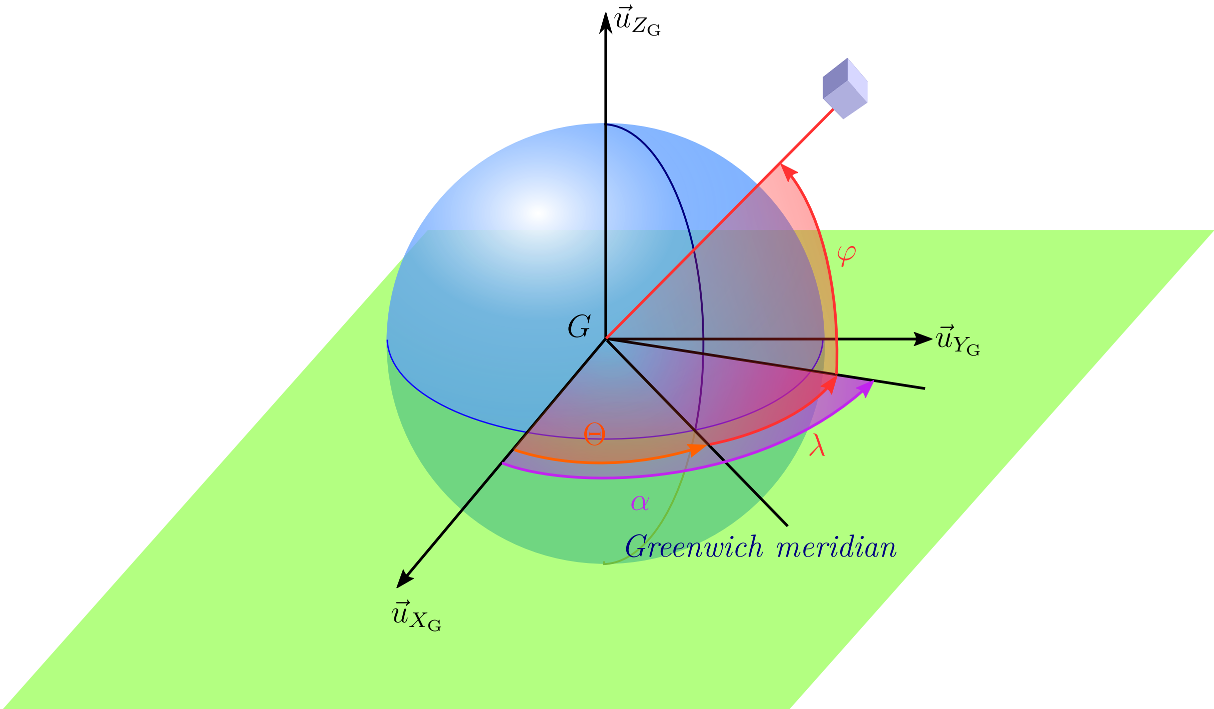

Figure 10 – Rotating Geocentric Reference Frame

In this reference frame, a point is defined by its radius r, its geographical longitude λ and its geographical latitude φ such that:

$$\lambda = \alpha - \Theta(t), \tag{128a}$$
$$\varphi = \delta. \tag{128b}$$

$\Theta(t)$ is the right ascension of the Greenwich meridian (angle between the axis $(G, \vec{u}_{X_G})$ of the Figure 8 and the Greewich meridian). It is thus the Earth rotation angle at time t and is sometimes called sidereal time. Assuming that the Earth rotation rate $\omega_\oplus$ is constant, $\Theta(t)$ verifies:

$$\Theta(t) = \Theta(t_0) + \omega_\oplus (t - t_0). \tag{129}$$

If $[x\ y\ \ z]^T$ denotes the position of a spacecraft in the geocentric inertial reference frame, its radius, latitude and longitude are computed with the following conversion formulas:

$$r = \sqrt{x^2 + y^2 + z^2}, \tag{130a}$$
$$\cos\lambda = \frac{x\cos\Theta(t) + y\sin\Theta(t)}{\sqrt{x^2+y^2}}, \tag{130b}$$
$$\sin\lambda = \frac{y\cos\Theta(t) - x\sin\Theta(t)}{\sqrt{x^2+y^2}}, \tag{130c}$$
$$\cos\varphi = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}, \tag{130d}$$
$$\sin\varphi = \frac{z}{\sqrt{x^2+y^2+z^2}}. \tag{130e}$$

A.3 Local Orbital Frame

It is possible to define a coordinates system linked to the position of the spacecraft on its orbit. This reference frame is labeled RTN or RSW as in [Vallado, 1997]. The axis are defined by:

- the axis $\vec{u}_N$ is in the direction of the angular momentum $\vec{h}$,
- the axis $\vec{u}_R$ is in the direction Earth-spacecraft,
- the axis $\vec{u}_T$ completes the basis,

and are illustrated on the Figure 11. The reference frame $(S, \vec{u}_R, \vec{u}_T, \vec{u}_N)$ is denoted $\mathcal{R}_{\mathrm{OL}}$ and the associated basis $\mathcal{B}_{\mathrm{OL}}$.

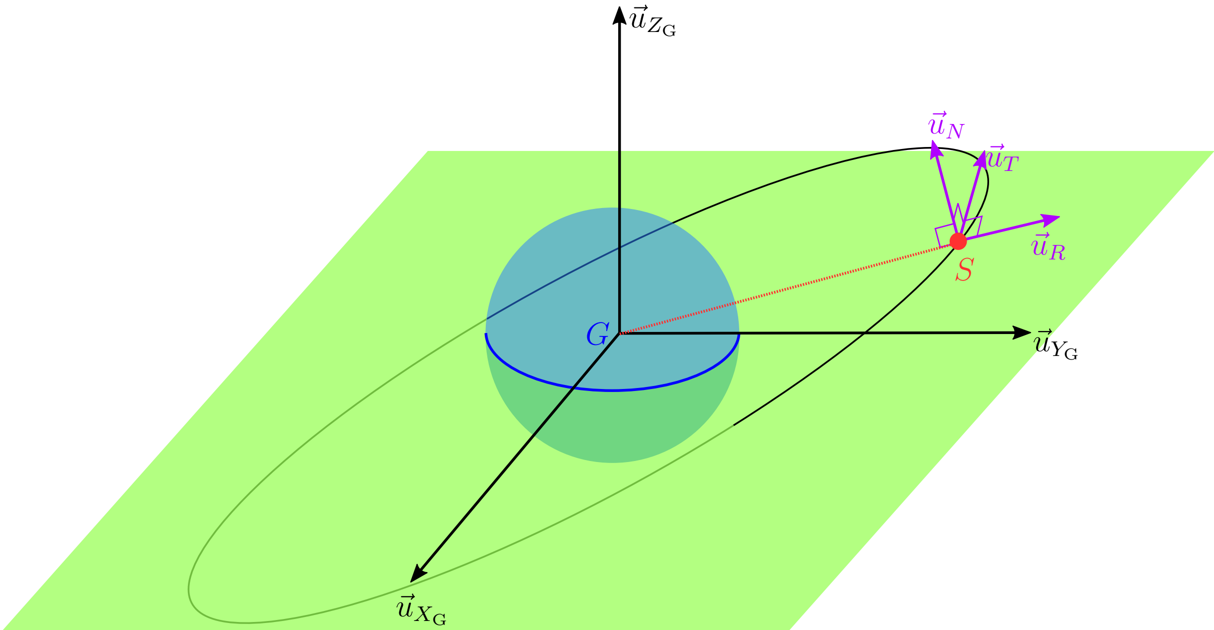

Figure 11 – Local orbital frame

The local orbital frame is defined from the inertial geocentric reference frame thanks to three rotations (see the Figure 12):

- rotation about the axis $(G, \vec{u}_{Z_G})$ with an angle Ω,
- rotation about the axis $(G, \vec{u}_n)$ with an angle i,
- rotation about the axis $(G, \vec{u}_h)$ with an angle $\omega + \nu$,

where $\vec{u}_n$ is a unit vector in the direction of the intersection between the equatorial plan and the orbital plane, and $\vec{u}_h$ is a unit vector perpendicular to the orbit plane. The Figure 13 depicts these rotations in three dimensions and the Figure 14 the associated plane rotations.

If $\vec{\sigma}$ is a vector whose coordinates are:

$$\vec{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{bmatrix}_{\mathcal{B}_{\mathrm{G}}} , \tag{131}$$

in the geocentric inertial frame and:

$$\vec{\sigma} = \begin{bmatrix} \sigma_r \\ \sigma_t \\ \sigma_t \end{bmatrix}_{\mathcal{B}_{\mathrm{OL}}} , \tag{132}$$

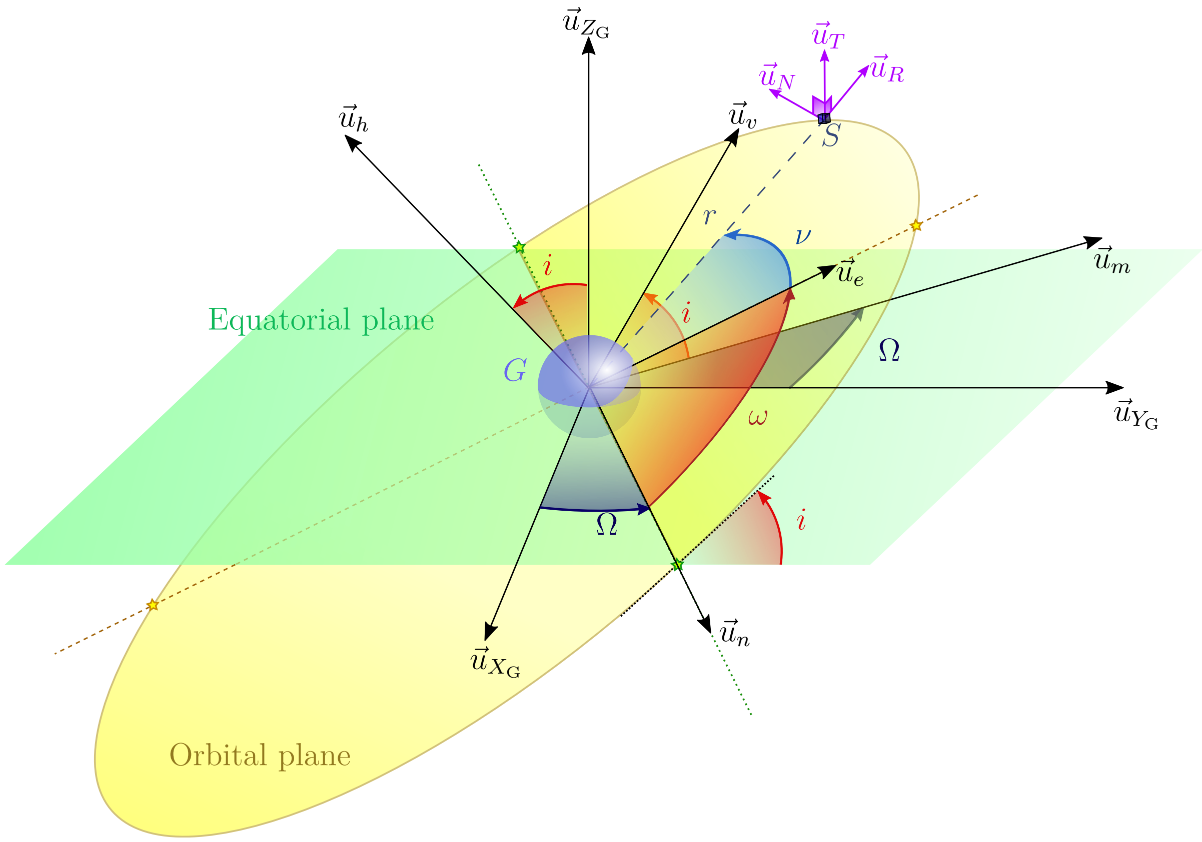

Figure 12 – Local orbital frame.

in the local orbital frame, the coordinates transformation is given by:

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{bmatrix}_{\mathcal{B}_G} = \begin{bmatrix}\cos\Omega & -\sin\Omega & 0\\ \sin\Omega & \cos\Omega & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos i & -\sin i\\ 0 & \sin i & \cos i\end{bmatrix}\begin{bmatrix}\cos(\omega+\nu) & -\sin(\omega+\nu) & 0\\ \sin(\omega+\nu) & \cos(\omega+\nu) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\sigma_r\\ \sigma_t\\ \sigma_t\end{bmatrix}_{\mathcal{B}_{\mathrm{OL}}},$$

$$= \begin{bmatrix}\cos(\omega+\nu)\cos\Omega - \cos i\,\sin(\omega+\nu)\sin\Omega & -\cos i\,\cos(\omega+\nu)\sin\Omega - \sin(\omega+\nu)\cos\Omega & \sin i\,\sin\Omega\\ \cos(\omega+\nu)\sin\Omega + \cos i\,\sin(\omega+\nu)\cos\Omega & \cos i\,\cos(\omega+\nu)\cos\Omega - \sin(\omega+\nu)\sin\Omega & -\sin i\,\cos\Omega\\ \sin i\,\sin(\omega+\nu) & \sin i\,\cos(\omega+\nu) & \cos i\end{bmatrix}\begin{bmatrix}\sigma_r\\ \sigma_t\\ \sigma_t\end{bmatrix}_{\mathcal{B}_{\mathrm{OL}}}. \tag{133}$$

It is also possible to define a local orbital frame from the trajectory tangent vector and the angular momentum vector, denoted NTW. When the orbit is circular, the RTN and NTW are the same.

A.4 Equinoctial Reference Frame

The equinoctial reference frame $\mathcal{B}_{\mathrm{EQX}} = (\vec{u}_h, \vec{u}_p, \vec{u}_q)$ is defined by:

- a rotation of the geocentric inertial reference frame about the axis $(G, \vec{u}_{Z_G})$ with an angle Ω,
- a rotation about the new $(G, \vec{u}_n)$ axis (pointing in the ascending node direction) with an angle i,

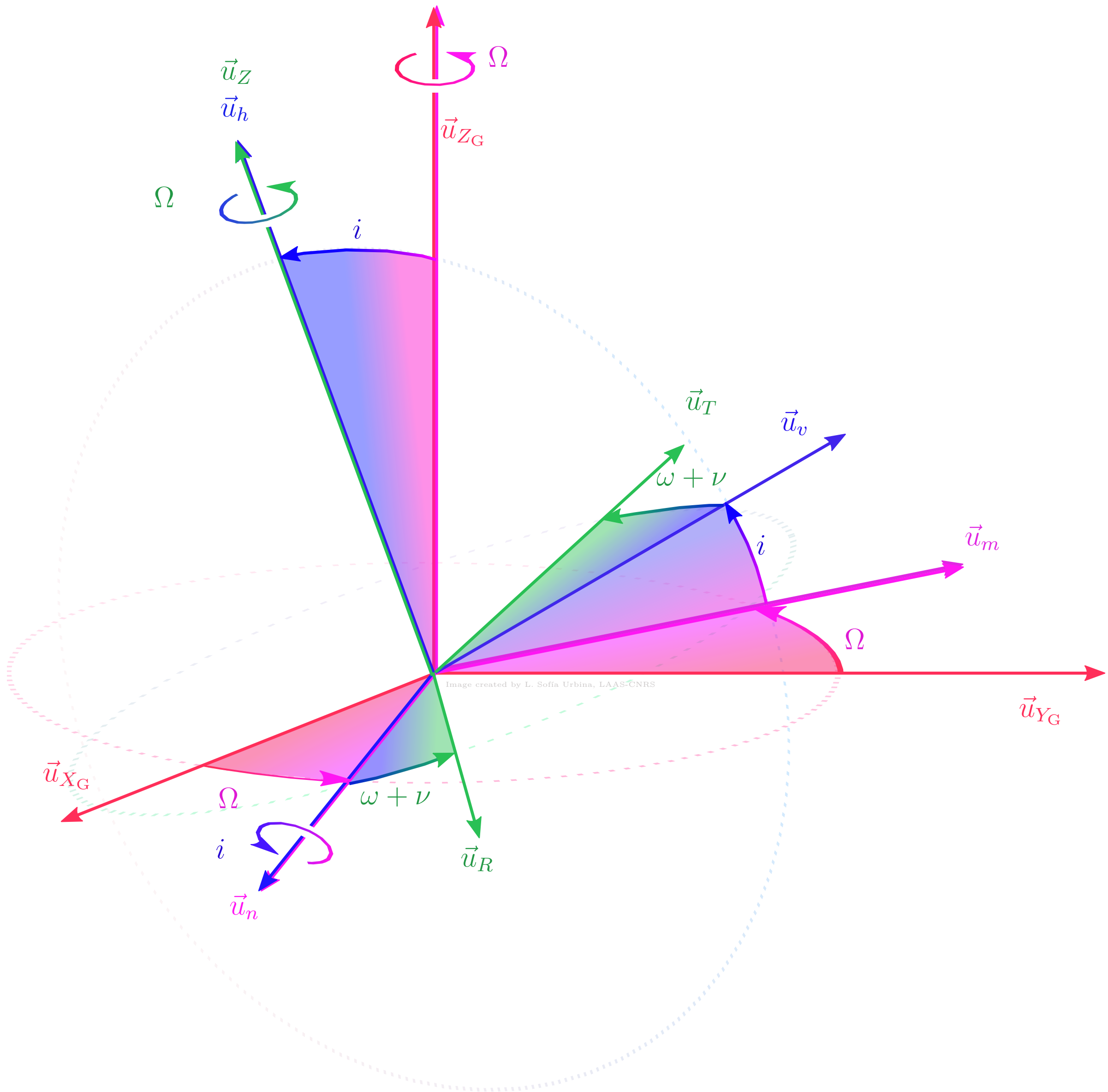

Figure 13 – Three dimensions rotation between the geocentric inertial reference frame and the local orbital frame.

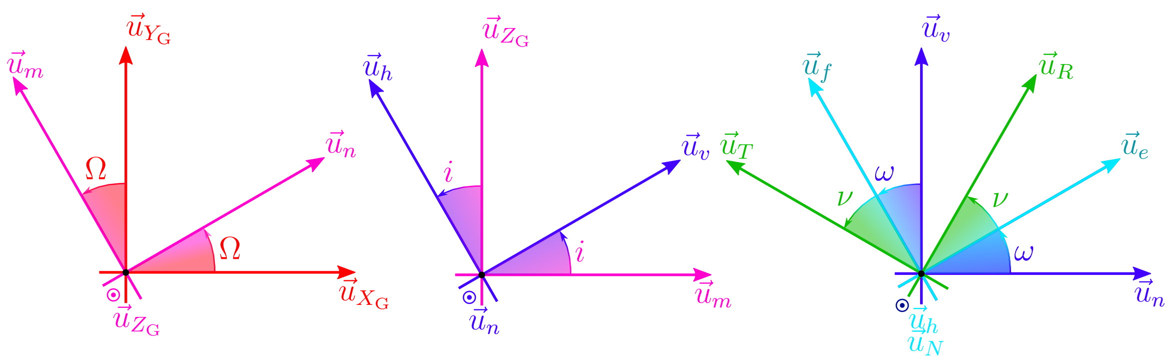

Figure 14 – Planar rotations between the geocentric inertial reference frame and the local orbital frame.

- a rotation about the axis $(G, \vec{u}_h)$ with an angle $-\Omega$,

and is denoted $\mathcal{R}_{\text{EQX}} = (G, \vec{u}_h, \vec{u}_p, \vec{u}_q)$.

The Figure 15 shows the equinoctial reference frame, the Figure 16 the three dimensional rotation and the Figure 17 the planar rotations for the transformations between the geocentric inertial reference frame and the equinoctial reference frame.

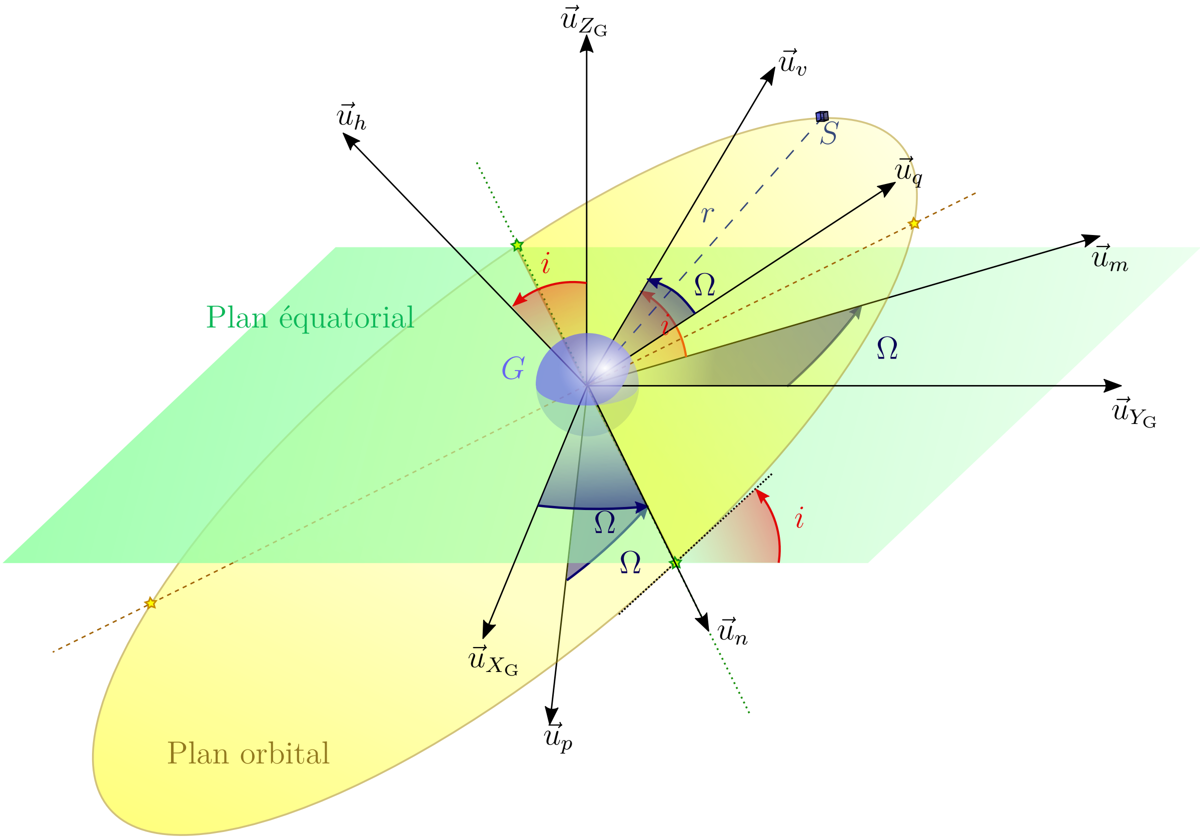

Figure 15 – Equinoctial reference frame.

If $\vec{\sigma}$ is a vector whose coordinates are:

$$\vec{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{bmatrix}_{\mathcal{B}_\text{G}}, \tag{134}$$

in the inertial geocentric reference frame and:

$$\vec{\sigma} = \begin{bmatrix} \sigma_p \\ \sigma_q \\ \sigma_h \end{bmatrix}_{\mathcal{B}_\text{EQX}}, \tag{135}$$

in the equinoctial reference frame, the transformations are written:

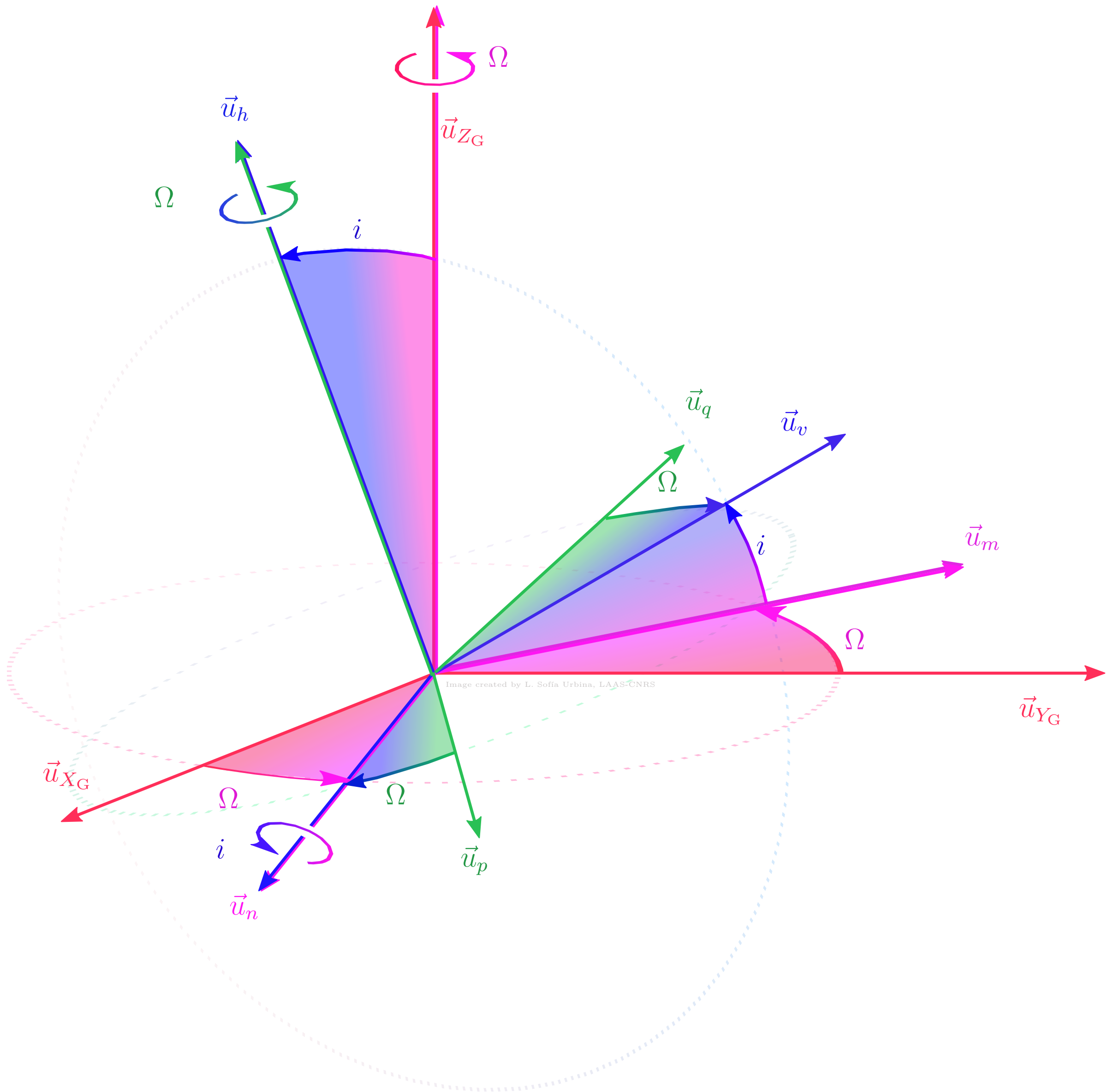

Figure 16 – Three dimensions rotation between the geocentric reference frame and the equinoctial reference frame.

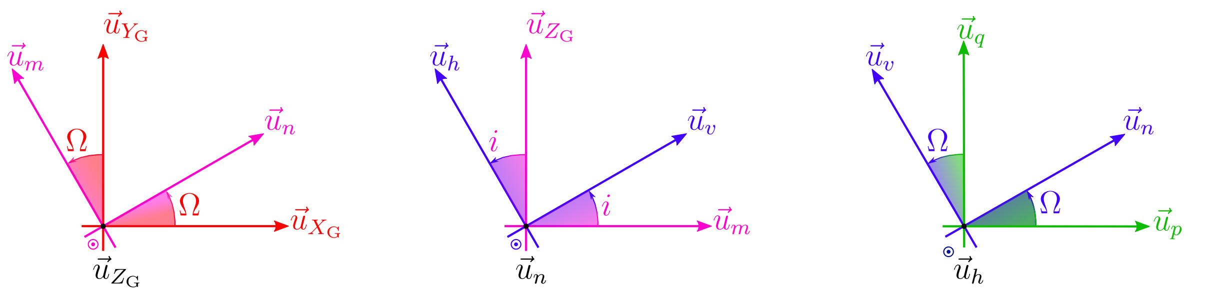

Figure 17 – Planar rotations between the geocentric reference frame and the equinoctial reference frame.

$$
\begin{aligned}
\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{bmatrix}_{\mathcal{B}_G} &= \begin{bmatrix}\cos\Omega & \sin\Omega & 0\\ -\sin\Omega & \cos\Omega & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos i & -\sin i\\ 0 & \sin i & \cos i\end{bmatrix}\begin{bmatrix}\cos\Omega & -\sin\Omega & 0\\ \sin\Omega & \cos\Omega & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\sigma_p\\ \sigma_q\\ \sigma_h\end{bmatrix}_{\mathcal{B}_{EQX}}, \\
&= \begin{bmatrix}\cos i\sin^2\Omega+\cos^2\Omega & \cos\Omega\,\sin\Omega\,(1-\cos i) & -\sin i\,\sin\Omega\\ \cos\Omega\,\sin\Omega\,(1-\cos i) & \sin^2\Omega+\cos i\cos^2\Omega & \sin i\,\cos\Omega\\ \sin i\,\sin\Omega & -\sin i\,\cos\Omega & \cos i\end{bmatrix}\begin{bmatrix}\sigma_p\\ \sigma_q\\ \sigma_h\end{bmatrix}_{\mathcal{B}_{EQX}}.
\end{aligned}
\tag{136}
$$

B Conversion Formulas with the Classical Orbital Elements

This section presents the conversion formulas between the cartesian positions and velocities of a spacecraft given in the inertial geocentric reference frame and the classical orbital elements. This formulas are derived from [Battin, 1999], [Vallado, 1997] and [Lyon, 2004].

The position vector is:

$$
\vec{r} = \begin{bmatrix}x\\ y\\ z\end{bmatrix}_{\mathcal{B}_G}, \tag{137}
$$

and its norm is:

$$
r = \sqrt{x^2+y^2+z^2}. \tag{138}
$$

The velocity vector is:

$$
\vec{v} = \dot{\vec{r}} = \begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix}_{\mathcal{B}_G}, \tag{139}
$$

and its norm is:

$$
v = \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}. \tag{140}
$$

B.1 Keplerian Motion Integrals

From the keplerian motion dynamics equation, some motion integrals can be derived.

The specific angular momentum:

$$
\vec{h} = \vec{r}\times\vec{v} = \begin{bmatrix}y\dot{z}-z\dot{y}\\ z\dot{x}-x\dot{z}\\ x\dot{y}-y\dot{x}\end{bmatrix}_{\mathcal{R}_G} = \begin{bmatrix}h_x\\ h_y\\ h_z\end{bmatrix}_{\mathcal{R}_G}, \tag{141}
$$

is constant and its norm is:

$$
\begin{aligned}
h &= \sqrt{h_x^2+_y^2+h_z^2}, \\
&= \sqrt{(y\dot{z}-z\dot{y})^2+(z\dot{x}-x\dot{z})^2+(x\dot{y}-y\dot{x})^2}.
\end{aligned}
\tag{142}
$$

The specific angular momentum is perpendicular to the position and the velocities of the spacecraft and is a constant vector. Therefore, the spacecraft trajectory lies in a plane called orbital plane. The unit vector perpendicular to this plane is:

$$\vec{u}_h = \frac{1}{\sqrt{h_x^2 +_y^2 + h_z^2}} \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix}_{\mathcal{R}_G}, \tag{143}$$

The total energy conservation gives the following equation:

$$-\frac{\mu}{2a} = \frac{v^2}{2} - \frac{\mu}{r}. \tag{144}$$

The eccentricity vector is defined by:

$$\vec{e} = \frac{\vec{v} \times \vec{h}}{\mu} - \frac{\vec{r}}{r}, \tag{145}$$

and is in the direction Earth-perigee. $\vec{u}_e$ is the unit vector in this direction. the norm e of the eccentricity vector is the eccentricity of the ellipse.

B.2 Anomalies Transformations

Le calcul de l'anomalie excentrique à partir de la l'anomalie vraie est effectué selon : The eccentric anomaly is computed from the true on with the following relationships:

$$\sin E = \frac{\sin \nu \sqrt{1-e^2}}{1 + e\cos \nu}, \tag{146a}$$

$$\cos E = \frac{e + \cos \nu}{1 + e\cos \nu}, \tag{146b}$$

$$\tan\left(\frac{E}{2}\right) = \sqrt{\frac{1-e}{1+e}} \tan\left(\frac{\nu}{2}\right). \tag{146c}$$

The Equations (146) are inverted as:

$$\sin \nu = -\frac{\sin E \sqrt{1-e^2}}{e\cos E - 1}, \tag{147a}$$

$$\cos \nu = \frac{e - \cos E}{e\cos E - 1}, \tag{147b}$$

$$\tan\left(\frac{\nu}{2}\right) = \sqrt{\frac{1+e}{1-e}} \tan\left(\frac{E}{2}\right). \tag{147c}$$

The mean anomaly is computed through the Kepler equation:

$$M = E - e\sin E. \tag{148}$$

B.3 Computation of the Orbital Elements from the Cartesian Positions and Velocities

The equation (144) leads to:

$$a = \frac{1}{\dfrac{2}{\sqrt{x^2+y^2+z^2}} - \dfrac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{\mu}}. \tag{149}$$

The equation (145) leads to:

$$e = \sqrt{\left(\frac{z}{r} + \frac{h_x\,\dot{y} - h_y\,\dot{x}}{\mu}\right)^2 + \left(\frac{y}{r} + \frac{h_z\,\dot{x} - h_x\,\dot{z}}{\mu}\right)^2 + \left(\frac{x}{r} + \frac{h_y\,\dot{z} - h_z\,\dot{y}}{\mu}\right)^2}, \tag{150}$$

and the vector $\vec{u}_e$ is the unit vector in the direction of $\vec{e}$.

L'inclinaison est l'angle entre le plan de l'équateur et le plan de l'orbite, ou encore l'angle entre le vecteur moment cinétique et le vecteur $\vec{u}_{Z_G}$ En notant $\vec{u}_h$ le vecteur unitaire perpendiculaire au plan de l'orbite, c'est-à-dire dans la direction du vecteur moment cinétique $\vec{h}$, il vient :

the inclination os the angle between the equator and the orbital planes, as well as the angle between the angular momentum and the $\vec{u}_{Z_G}$ vectors. Denoting $\vec{u}_h$ the unit vector in the angular momentum direction perpendicular to the orbital plane, the cosine of the inclination is:

$$\cos i = \vec{u}_{Z_G} \cdot \vec{u}_h = \frac{h_z}{h}. \tag{151}$$

As $i \in [0, \pi]$, the Equation (151) can be inverted as:

$$i = \arccos\left(\frac{h_z}{h}\right) \tag{152}$$

The angle Ω is the angle between the direction $(G, \vec{u}_{X_G})$ and the direction of the vector $\vec{n}$, intersection of the orbital and the equatorial planes. $\vec{n}$ verifies thus:

$$\vec{n} = \vec{u}_{Z_G} \times \vec{h} = \begin{bmatrix} -h_y \\ h_x \\ 0 \end{bmatrix}_{\mathcal{R}_G}, \tag{153}$$

with the unit vector:

$$\vec{u}_n = \frac{\vec{n}}{\|\vec{n}\|} = \frac{1}{\sqrt{h_x^2+h_y^2}} \begin{bmatrix} -h_y \\ h_x \\ 0 \end{bmatrix}_{\mathcal{R}_G}. \tag{154}$$

The cosine of Ω is:

$$\cos\Omega = \vec{u}_{X_G} \cdot \vec{u}_n = -\frac{h_y}{\sqrt{h_x^2+h_y^2}}. \tag{155}$$

As $\Omega \in [0, 2\pi]$, $\sin\Omega$ has to be computed in order to recover the value of Ω. Defining:

$$\vec{u}_m = \vec{u}_{Z_G} \times \vec{u}_n = \frac{1}{\sqrt{h_x^2+h_y^2}} \begin{bmatrix} -h_x \\ -h_y \\ 0 \end{bmatrix}_{\mathcal{B}_G}, \tag{156}$$

as depicted on the Figures 13 and 14, the sine of the right ascension of the ascending node is:

$$\sin\Omega = -\vec{u}_{X_G}\cdot\vec{u}_m = \frac{h_x}{\sqrt{h_x^2+h_y^2}} \tag{157}$$

As ω is the angle between $\vec{n}$ et $\vec{e}$, it comes :

$$\cos\omega = \vec{u}_e\cdot\vec{u}_n = \frac{-h_x\left(\frac{y}{r}+\frac{h_z\,\dot{x}-h_x\,\dot{z}}{\mu}\right)+h_y\left(\frac{x}{r}+\frac{h_y\,\dot{z}-h_z\,\dot{y}}{\mu}\right)}{e\sqrt{{h_y}^2+{h_x}^2}} \tag{158}$$

As $\omega\in[0,2\pi]$, $\sin\omega$ has to be computed. With the definition of $\vec{u}_f$:

$$\vec{u}_f = \vec{u}_h\times\vec{u}_e, \tag{159}$$

depicted on the Figures 13 and 14, $\sin\omega$ is computed as:

$$\begin{aligned}\sin\omega &= \vec{u}_e\cdot\vec{u}_v\\ &= \frac{1}{he\sqrt{h_x^2+h_y^2}}\left(h_y\left[-h_y\left(\frac{z}{r}+\frac{h_x\,\dot{y}-h_y\,\dot{x}}{\mu}\right)+h_z\left(\frac{y}{r}+\frac{h_z\,\dot{x}-h_x\,\dot{z}}{\mu}\right)\right]\right.\\ &\quad\left.-h_x\left[-h_z\left(\frac{x}{r}+\frac{h_y\,\dot{z}-h_z\,\dot{y}}{\mu}\right)+h_y\left(\frac{z}{r}+\frac{h_x\,\dot{y}-h_y\,\dot{x}}{\mu}\right)\right]\right)\end{aligned} \tag{160}$$

The local orbital frame $(\vec{u}_R,\vec{u}_T,\vec{u}_N)$ is defined such that:

$$\begin{aligned}\vec{u}_R &= \frac{\vec{r}}{r},\\ \vec{u}_Z &= \vec{u}_h,\\ \vec{u}_T &= \vec{u}_Z\times\vec{u}_R = \vec{u}_h\times\frac{\vec{r}}{r}.\end{aligned} \tag{161}$$

The true anomalie ν is the angle between the vector $\vec{e}$ and the vector $\vec{u}_R$. From this, it is possible to write:

$$\begin{aligned}\cos\nu &= \vec{u}_e\cdot\vec{u}_R = \vec{u}_e\cdot\frac{\vec{r}}{r}\\ &= -\frac{1}{er}\left[x\left(\frac{x}{r}+\frac{h_y\,\dot{z}-h_z\,\dot{y}}{\mu}\right)+y\left(\frac{y}{r}+\frac{h_z\,\dot{x}-h_x\,\dot{z}}{\mu}\right)+z\left(\frac{z}{r}+\frac{h_x\,\dot{y}-h_y\,\dot{x}}{\mu}\right)\right]\end{aligned} \tag{162}$$

As $\nu\in[0,2\pi]$, $\sin\nu$ has to be computed:

$$\begin{aligned}\sin\nu &= -\vec{u}_e\cdot\vec{u}_T = -\vec{u}_e\cdot\left(\vec{u}_Z\times\vec{u}_R\right) = -\vec{u}_e\cdot\left(\vec{u}_h\times\frac{\vec{r}}{r}\right),\\ &= \frac{1}{rhe}\left[\left(h_yz-h_zy\right)\left(\frac{x}{r}+\frac{h_y\,\dot{z}-h_z\,\dot{y}}{\mu}\right)+\left(h_zx-h_xz\right)\left(\frac{y}{r}+\frac{h_z\,\dot{x}-h_x\,\dot{z}}{\mu}\right)\right.\\ &\quad\left.+\left(h_xy-h_yx\right)\left(\frac{z}{r}+\frac{h_x\,\dot{y}-h_y\,\dot{x}}{\mu}\right)\right].\end{aligned} \tag{163}$$

From the value of ν computed through the Equations (162) and (163), the eccentric anomaly is computed with the Equation (146) and the mean anomaly through the Kepler Equation.

B.4 Computation of the Cartesian Positions and Velocities from the Classical Orbital Elements

B.4.1 Computation of the Position

In the local orbital frame, the position vector components read:

$$\vec{r} = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}_{OL}} . \tag{164}$$

Applying the rotation formula (133), it follows:

$$\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathcal{R}_G} = r \begin{bmatrix} \cos(\omega+\nu)\,\cos(\Omega) - \cos(i)\,\sin(\omega+\nu)\,\sin(\Omega) \\ \cos(\omega+\nu)\,\sin(\Omega) + \cos(i)\,\sin(\omega+\nu)\,\cos(\Omega) \\ \sin(i)\,\sin(\omega+\nu) \end{bmatrix}_{\mathcal{R}_G} . \tag{165}$$

Replacing r by its expression in terms of the classical orbital elements, the cartesian position components in the geocentric inertial reference frame are:

$$x = \frac{a(1-e^2)}{1+e\cos\nu}\left(\cos(\omega+\nu)\,\cos(\Omega) - \cos(i)\,\sin(\omega+\nu)\,\sin(\Omega)\right) \tag{166a}$$

$$y = \frac{a(1-e^2)}{1+e\cos\nu}\left(\cos(\omega+\nu)\,\sin(\Omega) + \cos(i)\,\sin(\omega+\nu)\,\cos(\Omega)\right) \tag{166b}$$

$$z = \frac{a(1-e^2)}{1+e\cos\nu}\left(\sin(i)\,\sin(\omega+\nu)\right) \tag{166c}$$

B.4.2 Computation of the Velocity

The velocity in the inertial geocentric reference frame is computed by derivating the position on the inertial geocentric reference frame:

$$\vec{v} = \left.\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right|_{\mathcal{R}_G} \tag{167}$$

The total time derivative is decomposed in the partial derivatives with respect to the orbital elements:

$$\vec{v} = \frac{\partial\vec{r}}{\partial a}\dot{a} + \frac{\partial\vec{r}}{\partial e}\dot{e} + \frac{\partial\vec{r}}{\partial i}\dot{i} + \frac{\partial\vec{r}}{\partial\Omega}\dot{\Omega} + \frac{\partial\vec{r}}{\partial\omega}\dot{\omega} + \frac{\partial\vec{r}}{\partial\nu}\dot{\nu}. \tag{168}$$

Or, les éléments orbitaux classiques définissent une ellipse osculatrice, c'est-à-dire l'ellipse que le satellite suivrait dans le cas où les perturbations disparaissaient instantanément. Ainsi, les éléments orbitaux ci-dessus sont à considérer dans le cas képlérien, et seul ν est à dérivée temporelle non-nulle. On peut alors réécrire :

As the orbital elements define an osculating ellipse that is tangent to the trajectory, the velocity of the spacecraft on its real trajectory is the velocity of the spacecraft if it would fly on the osculating ellipse. Therefor, the classical elements are to be considered in the keplerian case, and all the elements have a zero time derivative except the anomaly:

$$\vec{v} = \frac{\partial\vec{r}}{\partial\nu}\dot{\nu}, \tag{169}$$

with the time derivative of the true anomaly given by the reference [Battin, 1999]:

$$\dot{\nu} = \frac{\sqrt{\mu a\left(1-e^2\right)}\left(1+e\,\cos\left(\nu\right)\right)^2}{a^2\left(1-e^2\right)^2}. \tag{170}$$

Therefore, the cartesian velocity in the inertial geocentric reference frame is given by:

$$\begin{aligned}\dot{x} = \sqrt{\frac{\mu}{a(1-e^2)}}\Bigg[&\Big(e\sin\nu\sin(\omega+\nu)+(1+e\cos\nu)\cos(\omega+\nu)\Big)\sin\Omega\cos i\\ &+\Big((1+e\cos\nu)\sin(\omega+\nu)-e\sin\nu\cos(\omega+\nu)\Big)\Bigg]\end{aligned} \tag{171a}$$

$$\begin{aligned}\dot{x} = \sqrt{\frac{\mu}{a(1-e^2)}}\Bigg[&\Big((1+e\cos\nu)\sin(\omega+\nu)+e\sin\nu\sin(\omega+\nu)\Big)\cos\Omega\\ &-\Big((1+e\cos\nu)\cos(\omega+\nu)+e\sin\nu\sin(\omega+\nu)\Big)\cos\Omega\cos i\Bigg]\end{aligned} \tag{171b}$$

$$\dot{z} = \sqrt{\frac{\mu}{a(1-e^2)}}\Big[e\sin\nu\sin(\omega+\nu)\sin i+(1+e\cos\nu)\cos(\omega+\nu)\sin i\Big] \tag{171c}$$

C Conversion Formulas with the Equinoctial Orbital Elements

C.1 Definition of the Equinoctial Orbital Elements from the Classical Ones

La définition des éléments orbitaux équinoxiaux présentée dans la section 3.2.2 page 14 est rappelée ici : The definition of the equinoctial orbital elements is recalled here:

$$\begin{cases} a \\ e_x = e\cos(\omega+\Omega) \\ e_y = e\sin(\omega+\Omega) \\ i_x = \tan(i/2)\cos(\Omega) \\ i_y = \tan(i/2)\sin(\Omega) \\ \text{equinoctial anomaly} \end{cases} \tag{172}$$

Three anomalies are used

- true equinoctial anomaly: $\nu_Q = \Omega + \omega + \nu$;
- mean equinoctial anomaly: $M_Q = \Omega + \omega + M$;
- eccentric equinoctial anomaly: $E_Q = \Omega + \omega + E$

C.2 Computation of the Classical Orbital Elements from the Equinoctial Ones

The eccentricity is computed as the norm of the eccentricity vector:

$$e = \sqrt{e_x^2 + e_y^2}. \tag{173}$$

The definition of the inclination vector leads to:

$$\tan^2\left(\frac{i}{2}\right) = i_x^2 + i_y^2. \tag{174}$$

Hence:

$$\cos i = \frac{1 - \tan^2\left(\frac{i}{2}\right)}{1 + \tan^2\left(\frac{i}{2}\right)} = \frac{1 - i_x^2 - i_y^2}{1 + i_x^2 + i_y^2}, \tag{175}$$

and:

$$\sin i = \frac{2\tan\left(\frac{i}{2}\right)}{1 + \tan^2\left(\frac{i}{2}\right)} = \frac{2\sqrt{i_x^2 + i_y^2}}{1 + i_x^2 + i_y^2}. \tag{176}$$

As $i \in [0, \pi]$, the inclination reads:

$$i = \arccos\left(\frac{1 - i_x^2 - i_y^2}{1 + i_x^2 + i_y^2}\right). \tag{177}$$

The right ascension of the ascending node is computed from the eccentricity vector:

$$\cos\Omega = \frac{i_x}{\sqrt{i_x^2 + i_y^2}}, \tag{178a}$$

$$\sin\Omega = \frac{i_y}{\sqrt{i_x^2 + i_y^2}}. \tag{178b}$$

If the inclination is zero, i_x and i_y are also zero and Ω is not defined.

It comes from the eccentricity vector:

$$\cos(\Omega + \omega) = \frac{e_x}{\sqrt{e_x^2 + e_y^2}}, \tag{179a}$$

$$\sin(\Omega + \omega) = \frac{e_y}{\sqrt{e_x^2 + e_y^2}}, \tag{179b}$$

then with trigonometric expansions:

$$\cos\Omega\cos\omega - \sin\Omega\sin\omega = \frac{e_x}{\sqrt{e_x^2 + e_y^2}}, \tag{180a}$$

$$\sin\Omega\cos\omega + \cos\Omega\sin\omega = \frac{e_y}{\sqrt{e_x^2 + e_y^2}}. \tag{180b}$$

The system (180) of unknowns $\cos\omega$ and $\sin\omega$ has only one solution because the determinant is $\cos^2\Omega + \sin^2\Omega = 1$. The solution is:

$$\cos\omega = \frac{e_x\cos\Omega + e_y\sin\Omega}{\sqrt{e_x^2 + e_y^2}} = \frac{e_x i_x + e_y i_y}{\sqrt{e_x^2 + e_y^2}\sqrt{i_x^2 + i_y^2}}, \tag{181a}$$

$$\sin\omega = \frac{e_y\cos\Omega - e_x\sin\Omega}{\sqrt{e_x^2+e_y^2}} = \frac{e_y i_x - e_x i_y}{\sqrt{e_x^2+e_y^2}\sqrt{i_x^2+i_y^2}}. \tag{181b}$$

If the eccentricity is zero, e_x and e_y are also zero and ω is not defined.
Expanding the definition of the true equinoctial anomaly leads to:

$$\begin{aligned}
\cos\nu_Q &= \cos(\Omega+\omega)\cos\nu - \sin(\Omega+\omega)\sin\nu,\\
&= \frac{e_x}{\sqrt{e_x^2+e_y^2}}\sin\nu - \frac{e_y}{\sqrt{e_x^2+e_y^2}}\cos\nu, && (182a)\\
\sin\nu_Q &= \sin(\Omega+\omega)\cos\nu + \cos(\Omega+\omega)\sin\nu,\\
&= \frac{e_y}{\sqrt{e_x^2+e_y^2}}\cos\nu + \frac{e_x}{\sqrt{e_x^2+e_y^2}}\sin\nu. && (182b)
\end{aligned}$$

Hence, if the eccentricity is not zero:

$$\cos\nu = \frac{e_x\cos\nu_Q + e_y\sin\nu_Q}{\sqrt{e_x^2+e_y^2}}, \tag{183a}$$

$$\sin\nu = \frac{e_x\sin\nu_Q - e_y\cos\nu_Q}{\sqrt{e_x^2+e_y^2}}. \tag{183b}$$

If the eccentricity is zero but the inclination is not zero, it is possible to define the position of the spacecraft on its orbit by the angle $\omega+\nu$. The definition of the true equinoctial anomaly leads to:

$$\begin{aligned}
\cos\nu_Q &= \cos\Omega\cos(\omega+\nu) - \sin\Omega\sin(\omega+\nu),\\
&= \frac{i_x}{\sqrt{i_x^2+i_y^2}}\cos(\omega+\nu) - \frac{i_y}{\sqrt{i_x^2+i_y^2}}\sin(\omega+\nu), && (184a)\\
\sin\nu_Q &= \sin\Omega\cos(\omega+\nu) + \cos\Omega\sin(\omega+\nu),\\
&= \frac{i_y}{\sqrt{i_x^2+i_y^2}}\cos(\omega+\nu) + \frac{i_x}{\sqrt{i_x^2+i_y^2}}\sin(\omega+\nu). && (184b)\\
& && (184c)
\end{aligned}$$

The solution of this system is:

$$\cos(\omega+\nu) = \frac{i_x\cos\nu_Q + i_y\sin\nu_Q}{\sqrt{i_x^2+i_y^2}}, \tag{185a}$$

$$\sin(\omega+\nu) = \frac{i_x\sin\nu_Q - i_y\cos\nu_Q}{\sqrt{i_x^2+i_y^2}}. \tag{185b}$$

Doing the same calculations with the mean and the eccentric anomalies leads to:

$$\cos E = \frac{e_x\cos E_Q + e_y\sin E_Q}{\sqrt{e_x^2+e_y^2}}, \tag{186a}$$

$$\sin E = \frac{e_x\sin E_Q - e_y\cos E_Q}{\sqrt{e_x^2+e_y^2}}. \tag{186b}$$

$$\cos(\omega + E) = \frac{i_x \cos E_Q + i_y \sin E_Q}{\sqrt{i_x^2 + i_y^2}}, \tag{187a}$$

$$\sin(\omega + E) = \frac{i_x \sin E_Q - i_y \cos E_Q}{\sqrt{i_x^2 + i_y^2}}. \tag{187b}$$

$$\cos M = \frac{e_x \cos M_Q + e_y \sin M_Q}{\sqrt{e_x^2 + e_y^2}}, \tag{188a}$$

$$\sin M = \frac{e_x \sin M_Q - e_y \cos M_Q}{\sqrt{e_x^2 + e_y^2}}. \tag{188b}$$

$$\cos(\omega + M) = \frac{i_x \cos M_Q + i_y \sin M_Q}{\sqrt{i_x^2 + i_y^2}}, \tag{189a}$$

$$\sin(\omega + M) = \frac{i_x \sin M_Q - i_y \cos M_Q}{\sqrt{i_x^2 + i_y^2}}. \tag{189b}$$

Using the Equation (183) in the Equation (146) leads to:

$$\cos E = \frac{e + \dfrac{e_x \cos \nu_Q + e_y \sin \nu_Q}{\sqrt{e_x^2 + e_y^2}}}{1 + e\dfrac{e_x \cos \nu_Q + e_y \sin \nu_Q}{\sqrt{e_x^2 + e_y^2}}} = \frac{e_x^2 + e_y^2 + e_x \cos \nu_Q + e_y \sin \nu_Q}{\sqrt{e_x^2 + e_y^2}(1 + e_x \cos \nu_Q + e_y \sin \nu_Q)}, \tag{190a}$$

$$\sin E = \frac{\dfrac{e_x \sin \nu_Q - e_y \cos \nu_Q}{\sqrt{e_x^2 + e_y^2}}\sqrt{1 - e_x^2 - e_y^2}}{1 + e\dfrac{e_x \cos \nu_Q + e_y \sin \nu_Q}{\sqrt{e_x^2 + e_y^2}}} = \frac{(e_x \sin \nu_Q - e_y \cos \nu_Q)\sqrt{1 - e_x^2 - e_y^2}}{\sqrt{e_x^2 + e_y^2}(1 + e_x \cos \nu_Q + e_y \sin \nu_Q)}. \tag{190b}$$

Identifying $\cos E$ et $\sin E$ with the (186), it follows:

$$e_x \cos E_Q + e_y \sin E_Q = \frac{e_x^2 + e_y^2 + e_x \cos \nu_Q + e_y \sin \nu_Q}{1 + e_x \cos \nu_Q + e_y \sin \nu_Q}, \tag{191a}$$

$$e_x \sin E_Q - e_y \cos E_Q = \frac{(e_x \sin \nu_Q - e_y \cos \nu_Q)\sqrt{1 - e_x^2 - e_y^2}}{1 + e_x \cos \nu_Q + e_y \sin \nu_Q}, \tag{191b}$$

and:

$$\cos E_Q = \frac{e_x\left(e_x^2 + e_y^2 + e_x \cos \nu_Q + e_y \sin \nu_Q\right) - e_y(e_x \sin \nu_Q - e_y \cos \nu_Q)\sqrt{1 - e_x^2 - e_y^2}}{(e_x^2 + e_y^2)(1 + e_x \cos \nu_Q + e_y \sin \nu_Q)}, \tag{192a}$$

$$\sin E_Q = \frac{e_x(e_x \sin \nu_Q - e_y \cos \nu_Q)\sqrt{1 - e_x^2 - e_y^2} + e_y\left(e_x^2 + e_y^2 + e_x \cos \nu_Q + e_y \sin \nu_Q\right)}{(e_x^2 + e_y^2)(1 + e_x \cos \nu_Q + e_y \sin \nu_Q)}. \tag{192b}$$

Using the Equation (186) in the Equation (147) leads to:

$$\cos\nu = \frac{e - \dfrac{e_x\cos E_Q + e_y\sin E_Q}{\sqrt{e_x^2+e_y^2}}}{e\dfrac{e_x\cos E_Q + e_y\sin E_Q}{\sqrt{e_x^2+e_y^2}} - 1} = \frac{e_x^2+e_y^2 - e_x\cos E_Q - e_y\sin E_Q}{\sqrt{e_x^2+e_y^2}(e_x\cos E_Q + e_y\sin E_Q - 1)}, \tag{193a}$$

$$\sin\nu = -\frac{\dfrac{e_x\sin E_Q - e_y\cos E_Q}{\sqrt{e_x^2+e_y^2}}\sqrt{1-e^2}}{e\dfrac{e_x\cos E_Q + e_y\sin E_Q}{\sqrt{e_x^2+e_y^2}}} = -\frac{(e_x\sin E_Q - e_y\cos E_Q)\sqrt{1-e_x^2-e_y^2}}{\sqrt{e_x^2+e_y^2}(e_x\cos E_Q + e_y\sin E_Q - 1)}. \tag{193b}$$

Identifying $\cos\nu$ and $\sin\nu$ with the equation (183), it follows:

$$e_x\cos\nu_Q + e_y\sin\nu_Q = \frac{e_x^2+e_y^2 - e_x\cos E_Q - e_y\sin E_Q}{e_x\cos E_Q + e_y\sin E_Q - 1}, \tag{194a}$$

$$e_x\sin\nu_Q - e_y\cos\nu_Q = -\frac{(e_x\sin E_Q - e_y\cos E_Q)\sqrt{1-e_x^2-e_y^2}}{e_x\cos E_Q + e_y\sin E_Q - 1}, \tag{194b}$$

and:

$$\cos\nu_Q = \frac{e_x\left(e_x^2+e_y^2 - e_x\cos E_Q - e_y\sin E_Q\right) + e_y(e_x\sin E_Q - e_y\cos E_Q)\sqrt{1-e_x^2-e_y^2}}{(e_x^2+e_y^2)(e_x\cos E_Q + e_y\sin E_Q - 1)}, \tag{195a}$$

$$\sin\nu_Q = \frac{-e_x(e_x\sin E_Q - e_y\cos E_Q)\sqrt{1-e_x^2-e_y^2} + e_y\left(e_x^2+e_y^2 - e_x\cos E_Q - e_y\sin E_Q\right)}{(e_x^2+e_y^2)(e_x\cos E_Q + e_y\sin E_Q - 1)}. \tag{195b}$$

The mean equinoctial anomaly is computed from the eccentric one through the Kepler equation expressed by means of the equinoctial orbital elements:

$$E_Q + e_y\cos E_Q - e_x\sin E_Q = M_Q. \tag{196}$$

C.3 Conversion from the Cartesian Position and Velocity to the Equinoctial Orbital Elements

As for the classical orbital elements, the Equation (144) gives the semi-major axis as:

$$a = \frac{1}{\dfrac{2}{\sqrt{x^2+y^2+z^2}} - \dfrac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{\mu}} \tag{197}$$

The Equation (136) gave the transformation between the coordinates in the equinoctial and the inertial geocentric reference frames in terms of the classical orbital elements. Using the definition of the equinoctial orbital elements, the transformation (136) is rewritten as:

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{bmatrix}_{\mathcal{B}_\mathrm{G}} = \frac{1}{1+i_x^2+i_y^2}\begin{bmatrix}1+i_x^2-i_y^2 & 2i_xi_y & -2i_y\\ 2i_xi_y & 1-i_x^2+i_y^2 & 2i_x\\ 2i_y & -2i_x & 1-i_x^2-i_y^2\end{bmatrix}\begin{bmatrix}\sigma_p\\ \sigma_q\\ \sigma_h\end{bmatrix}_{\mathcal{B}_{EQX}} \tag{198}$$

This transformation matrix gives the coordinates of the equinoctial basis in the inertial geocentric one. In particular:

$$\vec{u}_h = \frac{1}{1+i_x^2+i_y^2}\begin{bmatrix}-2i_y\\ 2i_x\\ 1-i_x^2-i_y^2\end{bmatrix}_{\mathcal{B}_\mathrm{G}}, \tag{199}$$

which allows to write the coordinates of the vector $\vec{u}_h$ in the inertial geocentric reference frame:

$$u_{h_x} = \vec{u}_h\cdot\vec{u}_{X_\mathrm{G}} = \frac{-2i_y}{1+i_x^2+i_y^2}, \tag{200a}$$

$$u_{h_y} = \vec{u}_h\cdot\vec{u}_{Y_\mathrm{G}} = \frac{2i_x}{1+i_x^2+i_y^2}, \tag{200b}$$

$$u_{h_z} = \vec{u}_h\cdot\vec{u}_{Z_\mathrm{G}} = \frac{1-i_x^2-i_y^2}{1+i_x^2+i_y^2}. \tag{200c}$$

With the Equation 199, the inclination vector is computed as:

$$i_x = \frac{u_{h_y}}{1+u_{h_z}} = \frac{h_y}{h_z+h}, \tag{201a}$$

$$i_y = -\frac{u_{h_x}}{1+u_{h_z}} = -\frac{h_x}{h_z+h}, \tag{201b}$$

where h is the norm of the vector $\vec{h}$.

With the Equation (198), the basis vectors are written:

$$\vec{u}_p = \frac{1}{1+i_x^2+i_y^2}\begin{bmatrix}1+i_x^2-i_y^2\\ 2i_xi_y\\ -2i_y\end{bmatrix}_{\mathcal{B}_\mathrm{G}}, \tag{202a}$$

$$\vec{u}_q = \frac{1}{1+i_x^2+i_y^2}\begin{bmatrix}2i_xi_y\\ 1+i_x^2-i_y^2\\ 2i_x\end{bmatrix}_{\mathcal{B}_\mathrm{G}}. \tag{202b}$$

Combining the Figures 14 and 17, $\Omega+\omega$ appears to be the angle between the vectors $\vec{u}_p$ and $\vec{u}_e$ on one hand and $\vec{u}_q$ and $\vec{u}_f$ on the other hand (see the Figure 18).

Hence:

$$\cos(\Omega+\omega) = \vec{u}_e\cdot\vec{u}_p,\ \sin(\Omega+\omega) = \vec{u}_e\cdot\vec{u}_q, \tag{203a}$$

and the components of the eccentricity vector are:

$$e_x = e\vec{u}_e\cdot\vec{u}_p = \vec{e}\cdot\vec{u}_p, \tag{204a}$$

$$e_y = e\vec{u}_e\cdot\vec{u}_q = \vec{e}\cdot\vec{u}_q, \tag{204b}$$

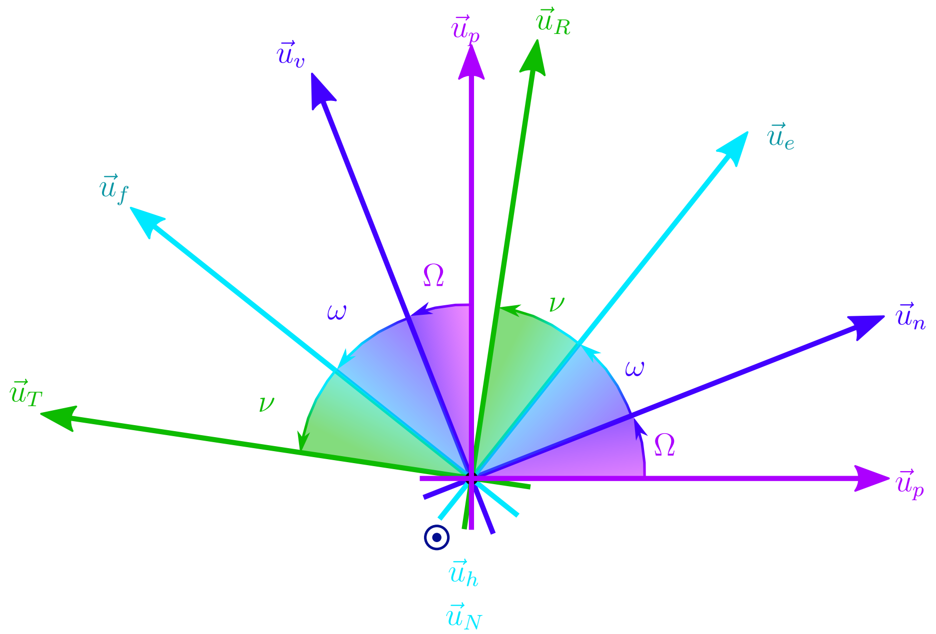

Figure 18 – Rotations planes dans le plan orbital autour du vecteur $\vec{u}_h$.

and:

$$e_x = \frac{1}{1+\frac{h_x^2}{(h_z+h)^2}+\frac{h_y^2}{(h_z+h)^2}}\left[-\left(\frac{x}{r}+\frac{h_y\,\dot z - h_z\,\dot y}{\mu}\right)\left(1+\frac{h_y^2}{(h_z+h)^2}-\frac{h_x^2}{(h_z+h)^2}\right)\right.$$
$$\left.+\frac{2h_x\left(\frac{z}{r}+\frac{h_x\,\dot y - h_y\,\dot x}{\mu}\right)}{h_z+h}-\frac{2h_xh_y\left(\frac{y}{r}+\frac{h_z\,\dot x - h_x\,\dot z}{\mu}\right)}{(h_z+h)^2}\right] \quad (205a)$$

$$e_y = \frac{1}{1+\frac{h_x^2}{(h_z+h)^2}+\frac{h_y^2}{(h_z+h)^2}}\left[-\left(\frac{y}{r}+\frac{h_z\,\dot x - h_x\,\dot z}{\mu}\right)\left(1-\frac{h_y^2}{(h_z+h)^2}-\frac{h_x^2}{(h_z+h)^2}\right)\right.$$
$$\left.+\frac{2h_y\left(\frac{z}{r}+\frac{h_x\,\dot y - h_y\,\dot x}{\mu}\right)}{h_z+h}-\frac{2h_xh_y\left(\frac{x}{r}+\frac{h_y\,\dot z - h_z\,\dot y}{\mu}\right)}{(h_z+h)^2}\right] \quad (205b)$$

From the Figure 18, the true anomaly appears to be $\nu_Q = \Omega + \omega + \nu$ the angle between the vectors $\vec{u}_p$ and $\vec{u}_R$ on one hand, and $\vec{u}_q$ and $\vec{u}_T$ on the other hand. Therefore:

$$\cos\nu_Q = \vec{u}_R \cdot \vec{u}_p, \quad (206a)$$
$$\sin\nu_Q = \vec{u}_R \cdot \vec{u}_q, \quad (206b)$$

and:

$$\cos\nu_Q = \frac{1}{r\left(1+\frac{h_x^2}{(h_z+h)^2}+\frac{h_y^2}{(h_z+h)^2}\right)}\left[x\left(1+\frac{h_y^2}{(h_z+h)^2}-\frac{h_x^2}{(h_z+h)^2}\right)\right.$$
$$\left.-\frac{2h_xz}{h_z+h}+\frac{2h_xh_yy}{(h_z+h)^2}\right], \quad (207a)$$

$$\sin\nu_Q = \frac{1}{r\left(1+\frac{h_x^2}{(h_z+h)^2}+\frac{h_y^2}{(h_z+h)^2}\right)}\left[y\left(1-\frac{h_y^2}{(h_z+h)^2}-\frac{h_x^2}{(h_z+h)^2}\right)\right.$$
$$\left.+\frac{2h_y z}{h_z+h}+\frac{2h_x h_y x}{(h_z+h)^2}\right]. \tag{207b}$$

C.4 Conversion from the Equinoctial Orbital Elements to Cartesian Position and Velocity

The transformation of the equinoctial orbital elements to the cartesian position and velocity in the inertial geocentric reference frame uses the expression of the position vector in the equinoctial frame and the rotation matrix 136.

The radius is expressed in terms of the equinoctial orbital elements as (see the reference [McClain, 1977]):

$$r = \frac{a(1-e_x^2-e_y^2)}{1+e_x\cos\nu_Q+e_y\sin\nu_Q}, \tag{208}$$

This expression of r uses the true equinoctial anomaly. As the sixth parameter of the state vector is the mean longitude one, it is mandatory to transform ν_Q in $\ell_{M\Theta}$. To this end, the Kepler equation expressed in equinoctial orbital elements is solved with the approximated method of the Appendix D:

$$\begin{aligned}\nu_Q &= \Omega+\omega+\nu,\\ &= \Omega+\omega+M+2e\sin(M),\\ &= M_Q+2e\sin(M),\\ &= \ell_{M\Theta}+\Theta+2e\sin(M).\end{aligned} \tag{209}$$

At the order 0 in eccentricity:

$$\cos\nu_Q = \cos M_Q = \cos(\ell_{M\Theta}+\Theta), \tag{210a}$$
$$\sin\nu_Q = \sin M_Q = \sin(\ell_{M\Theta}+\Theta), \tag{210b}$$

and the radius reads then:

$$r = \frac{a(1-e_x^2-e_y^2)}{1+e_x\cos(\ell_{M\Theta}+\Theta)+e_y\sin(\ell_{M\Theta}+\Theta)}. \tag{211}$$

The radius is expressed by means of the eccentric equinoctial anomaly as:

$$r = a(1-e_x\cos E_Q-e_y\sin E_Q), \tag{212}$$

According to the Figure 18:

$$\begin{aligned}\vec{u}_R &= \cos(\Omega+\omega+\nu)\vec{u}_p+\sin(\Omega+\omega+\nu)\vec{u}_q,\\ &= \cos(\ell_{M\Theta}+\Theta)\vec{u}_p+\sin(\ell_{M\Theta}+\Theta)\vec{u}_q.\end{aligned} \tag{213}$$

Therefore :

$$\vec{r} = \frac{a(1-e_x^2-e_y^2)}{1+e_x\cos(\ell_{M\Theta}+\Theta)+e_y\sin(\ell_{M\Theta}+\Theta)}\begin{bmatrix}\cos(\ell_{M\Theta}+\Theta)\\ \sin(\ell_{M\Theta}+\Theta)\\ 0\end{bmatrix}_{\mathcal{B}_{\mathrm{EQX}}}. \tag{214}$$

The coordinates of the position vector in the equinoctial reference frame are transformed in the inertial geocentric one with the rotation matrix (198), what leads to:

$$\begin{aligned}\vec{r} &= \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathcal{B}_{\mathrm{G}}} \\ &= \frac{a(1-e_x^2-e_y^2)}{1+e_x\cos(\ell_{M\Theta}+\Theta)+e_y\sin(\ell_{M\Theta}+\Theta)}\frac{1}{1+i_x^2+i_y^2} \\ &\quad\times\begin{bmatrix} 1+i_x^2-i_y^2 & 2i_xi_y & -2i_y \\ 2i_xi_y & 1-i_x^2+i_y^2 & 2i_x \\ 2i_y & -2i_x & 1-i_x^2-i_y^2 \end{bmatrix}\begin{bmatrix} \cos(\ell_{M\Theta}+\Theta) \\ \sin(\ell_{M\Theta}+\Theta) \\ 0 \end{bmatrix}_{\mathcal{B}_{\mathrm{EQX}}},\end{aligned} \tag{215}$$

and thus:

$$x = \frac{\left[a(1-e_x^2-e_y^2)\right]\left[(1+i_x^2-i_y^2)\cos(\ell_{M\Theta}+\Theta)+2i_xi_y\sin(\ell_{M\Theta}+\Theta)\right]}{\left[1+e_x\cos(\ell_{M\Theta}+\Theta)+e_y\sin(\ell_{M\Theta}+\Theta)\right]\left[1+i_x^2+i_y^2\right]}, \tag{216a}$$

$$y = \frac{\left[a(1-e_x^2-e_y^2)\right]\left[2i_xi_y\cos(\ell_{M\Theta}+\Theta)+(1-i_x^2+i_y^2)\sin(\ell_{M\Theta}+\Theta)\right]}{\left[1+e_x\cos(\ell_{M\Theta}+\Theta)+e_y\sin(\ell_{M\Theta}+\Theta)\right]\left[1+i_x^2+i_y^2\right]}, \tag{216b}$$

$$z = \frac{\left[a(1-e_x^2-e_y^2)\right]\left[2i_y\cos(\ell_{M\Theta}+\Theta)-2i_x\sin(\ell_{M\Theta}+\Theta)\right]}{\left[1+e_x\cos(\ell_{M\Theta}+\Theta)+e_y\sin(\ell_{M\Theta}+\Theta)\right]\left[1+i_x^2+i_y^2\right]}. \tag{216c}$$

The velocity in the inertial geocentric reference frame is computed by derivation of the position in the inertial reference frame:

$$\vec{v} = \left.\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right|_{\mathcal{R}_G} \tag{217}$$

The total time derivative is decomposed in the partial derivatives with respect to the equinoctial orbital elements:

$$\vec{v} = \frac{\partial\vec{r}}{\partial a}\dot{a} + \frac{\partial\vec{r}}{\partial e_x}\dot{e}_x + + \frac{\partial\vec{r}}{\partial e_y}\dot{e}_y + \frac{\partial\vec{r}}{\partial i_x}\dot{i}_x + \frac{\partial\vec{r}}{\partial i_y}\dot{i}_y + \frac{\partial\vec{r}}{\partial(\ell_{M\Theta}+\Theta)}\frac{\mathrm{d}(\ell_{M\Theta}+\Theta)}{\mathrm{d}t}. \tag{218}$$

However $\ell_{M\Theta+\Theta} = M_Q$, the velocity reads:

$$\vec{v} = \frac{\partial\vec{r}}{\partial a}\dot{a} + \frac{\partial\vec{r}}{\partial e_x}\dot{e}_x + + \frac{\partial\vec{r}}{\partial e_y}\dot{e}_y + \frac{\partial\vec{r}}{\partial i_x}\dot{i}_x + \frac{\partial\vec{r}}{\partial i_y}\dot{i}_y + \frac{\partial\vec{r}}{\partial M_Q}\dot{M}_Q. \tag{219}$$

Following the argumentation developed for the computation of the velocity with the classical orbital elements, the velocity is computed as:

$$\vec{v} = \frac{\partial\vec{r}}{\partial M_Q}\dot{M}_Q, \tag{220}$$

with

$$\dot{M}_Q = n = \sqrt{\frac{\mu}{a^3}}. \tag{221}$$

Hence :

$$\begin{aligned}
\dot{x} = \sqrt{\frac{\mu}{a^3}} & \frac{a(1-e_x^2-e_y^2)}{\Big[1+e_x\cos(\ell_{M\Theta}+\Theta)+e_y\sin(\ell_{M\Theta}+\Theta)\Big]^2\Big[1+i_x^2+i_y^2\Big]} \\
\times \Bigg(& \Big[-(1+i_x^2-i_y^2)\sin(\ell_{M\Theta}+\Theta)+2i_xi_y\cos(\ell_{M\Theta}+\Theta)\Big]\Big[1+e_x\cos(\ell_{M\Theta}+\Theta)+e_y\sin(\ell_{M\Theta}+\Theta)\Big] \\
- & \Big[e_y\cos(\ell_{M\Theta}+\Theta)-e_x\sin(\ell_{M\Theta}+\Theta)\Big]\Big[(1+i_x^2-i_y^2)\cos(\ell_{M\Theta}+\Theta)+2i_xi_y\sin(\ell_{M\Theta}+\Theta)\Big]\Bigg),
\end{aligned} \tag{222a}$$

$$\begin{aligned}
\dot{y} = \sqrt{\frac{\mu}{a^3}} & \frac{a(1-e_x^2-e_y^2)}{\Big[1+e_x\cos(\ell_{M\Theta}+\Theta)+e_y\sin(\ell_{M\Theta}+\Theta)\Big]^2\Big[1+i_x^2+i_y^2\Big]} \\
\times \Bigg(& \Big[(1-i_x^2+i_y^2)\cos(\ell_{M\Theta}+\Theta)-2i_xi_y\sin(\ell_{M\Theta}+\Theta)\Big]\Big[1+e_x\cos(\ell_{M\Theta}+\Theta)+e_y\sin(\ell_{M\Theta}+\Theta)\Big] \\
- & \Big[e_y\cos(\ell_{M\Theta}+\Theta)-e_x\sin(\ell_{M\Theta}+\Theta)\Big]\Big[(1-i_x^2+i_y^2)\sin(\ell_{M\Theta}+\Theta)+2i_xi_y\cos(\ell_{M\Theta}+\Theta)\Big]\Bigg),
\end{aligned} \tag{222b}$$

$$\begin{aligned}
\dot{z} = \sqrt{\frac{\mu}{a^3}} & \frac{a(1-e_x^2-e_y^2)}{\Big[1+e_x\cos(\ell_{M\Theta}+\Theta)+e_y\sin(\ell_{M\Theta}+\Theta)\Big]^2\Big[1+i_x^2+i_y^2\Big]} \\
\times \Bigg(& \Big[-2i_x\cos(\ell_{M\Theta}+\Theta)-2i_y\sin(\ell_{M\Theta}+\Theta)\Big]\Big[1+e_x\cos(\ell_{M\Theta}+\Theta)+e_y\sin(\ell_{M\Theta}+\Theta)\Big] \\
- & \Big[e_y\cos(\ell_{M\Theta}+\Theta)-e_x\sin(\ell_{M\Theta}+\Theta)\Big]\Big[-2i_x\sin(\ell_{M\Theta}+\Theta)+2i_y\cos(\ell_{M\Theta}+\Theta)\Big]\Bigg).
\end{aligned} \tag{222c}$$

D Approximation Methods for Solving the Kepler Equation

With the state space representation using the orbital elements, the sixth parameter is one of the three anomalies: the true one ν, the mean one M or the eccentric one E. Through this document, the mean anomaly is used for the simplicity of its time derivative. However, the Lagrange matrix is usually expressed in terms of the true anomaly. It is therefore mandatory to compute the former from the latter. This can be done in two ways:

- solving the Kepler equation:

$$E - e \sin E = M, \tag{223}$$

in order to compute E from M and computing ν with the formula:

$$\nu = 2 \arctan \left(\sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} \right); \tag{224}$$

- solving the Kepler equation directly in true anomaly:

$$2 \arctan \left(\sqrt{\frac{1-e}{1+e}} \tan \frac{\nu}{2} \right) - e \sin \left[2 \arctan \left(\sqrt{\frac{1-e}{1+e}} \tan \frac{\nu}{2} \right) \right] = M. \tag{225}$$

As the Kepler equation can not be solved analytically, approximations techniques have to be used. The following sections present some of these techniques.

D.1 Newton Algorithm

D.1.1 Kepler Equation Expressed in E

The Newton algorithm is an iterative method aimed at finding an approximation of the zero of the equation:

$$f(x) = 0, \tag{226}$$

from a first guess of the solution x_0. A sequence of zeros $(x_n)_{n\in\mathbb{N}}$ converging towards the solution $\tilde{x}$ is built. Each approximate solution x_{n+1} is computed from the previous one x_n according to:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{227}$$

In the case of the Kepler equation expressed in E, the algorithm reads:

$$E_{n+1} = E_n - \frac{E_n - e \sin E_n - M}{1 - e \cos E_m} \tag{228}$$

Once E is found, ν is computed with the Equation (224).

A way to make the algorithm faster is to choose a starting point close to the actual solution. In the reference [Battin, 1999], a choice for the starting point is proposed. If the function f is monotonic on the interval $[x_1, x_2]$ such that $f(x_1)f(x_2) \leqslant 0$, then a good starting point may the the intersection between the abscissa axis and the line passing through $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

For the Kepler equation, the function $f : E \mapsto E - e \sin E - M$ is increasing, and [Battin, 1999] suggests the following evaluation points:

$$f(M) = -e \sin M \leqslant 0, \ \forall M \in [0, \pi],$$
$$f(M + e) = e(1 - \sin(M + e)) \geqslant 0, \ \forall M.$$

The initial point is thus the intersection with the abscissa line (for $M \in [0, \pi]$):

$$E_0 = M + \frac{e \sin M}{1 - \sin(M + E) + \sin M}. \tag{229}$$

D.1.2 Kepler Equation Expressed in ν

The Kepler equation expressed in ν can be solved directly with the Newton algorithm using the functions:

$$f(\nu) = 2\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\nu}{2}\right) - e\sin\left[2\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\nu}{2}\right)\right] - M, \tag{230a}$$

$$f'(\nu) = \left(1 - e\cos\left[2\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\nu}{2}\right)\right]\right)\cdot\frac{\sqrt{\frac{1-e}{1+e}}\left(1+\tan^2\frac{\nu}{2}\right)}{1+\frac{1-e}{1+e}\tan^2\frac{\nu}{2}}, \tag{230b}$$

and :

$$\nu_{n+1} = \nu_n - \frac{f(\nu_n)}{f'(\nu_n)}. \tag{231}$$

D.2 Hull's Method

D.2.1 With the Equations (223) and (224)

The Equation (223) is a transcendental equation with e as a small parameter as the orbit is almost a geostationary one. the reference [Hull, 2003] proposes a technique to solve the Kepler equation.

Considering the funcion:

$$f : (E;e) \mapsto f(E;e) = E - e\sin E - M, \tag{232}$$

the equation to be solved is:

$$f(E;e) = 0, \tag{233}$$

for E as unknown and e close to zero. The solution E_* of this equation is a function of e. As e is close to 0, the solution can be expanded power of e:

$$\begin{aligned} E_* &= E_0 + E_1 e + \frac{1}{2}E_2 e^2 \\ &= E_0 + \Delta E. \end{aligned} \tag{234}$$

The function $f(E;e) = f(E_0 + \Delta E; e)$ is then expanded with respect to $(E_0;0)$ up to the order 2. The following notations will be used in the sequel:

$$\begin{aligned} f(E_0;0) &\longrightarrow f, & \frac{\partial f(E;e)}{\partial e}\Big|_{(E_0;0)} &\longrightarrow f_e, \\ \frac{\partial f(E;e)}{\partial E}\Big|_{(E_0;0)} &\longrightarrow f_E, & \frac{\partial^2 f(E;e)}{\partial e^2}\Big|_{(E_0;0)} &\longrightarrow f_{ee}, \\ \frac{\partial^2 f(E;e)}{\partial E^2}\Big|_{(E_0;0)} &\longrightarrow f_{EE}, & \frac{\partial^2 f(E;e)}{\partial e \partial E}\Big|_{(E_0;0)} &\longrightarrow f_{eE}, \end{aligned}$$

$$
\begin{aligned}
f(E;e) &= f(E_0 + \Delta E; e),\\
&= f + f_E \Delta E + f_e e + \frac{1}{2} f_{EE} \Delta E^2 + \frac{1}{2} f_{ee} e^2 + f_{eE} e \Delta E,\\
&= f + f_E \left(E_1 e + \frac{1}{2} E_2 e^2 \right) + f_e e + \frac{1}{2} f_{EE} E_1^2 e^2 + \frac{1}{2} f_{ee} e^2 + f_{eE} E_1 e^2.
\end{aligned} \tag{235}
$$

Identifying the coefficients for each order of e leads to the following equations:

$$
\begin{cases}
f = 0,\\
f_E E_1 + f_e = 0,\\
\frac{1}{2} f_E E_2 + \frac{1}{2} f_{EE} E_1^2 + \frac{1}{2} f_{ee} + f_{eE} E_1 = 0.
\end{cases} \tag{236}
$$

Solving this previous system leads to:

$$
\begin{cases}
E_0 = M,\\
E_1 = \sin(M),\\
E_2 = \sin(2M),
\end{cases} \tag{237}
$$

and then:

$$
E_{\text{hull}} = M + e \sin M + \frac{1}{2} e^2 \sin(2M). \tag{238}
$$

From this point, the Equation (224) gives ν.

D.2.2 WITH THE EQUATION (225)

The Hull's method is applied to compute an approximate solution of the transcendental Equation (225). The handled function is:

$$
f(\nu; e) = 2 \arctan \left(\sqrt{\frac{1-e}{1+e}} \tan \frac{\nu}{2} \right) - e \sin \left[2 \arctan \left(\sqrt{\frac{1-e}{1+e}} \tan \frac{\nu}{2} \right) \right] - M. \tag{239}
$$

With similar developments as (245), the true anomaly reads:

$$
\nu = M + 2e \sin M + \frac{5}{2} e^2 \sin(2M). \tag{240}
$$

D.2.3 KEPLER EQUATION WITH THE EQUINOCTIAL ORBITAL ELEMENTS

The Kepler equation in terms of the equinoctial orbital elements reads:

$$
E_Q + e_y \cos E_Q - e_x \sin E_Q = \ell_{M\Theta} + \Theta(t). \tag{241}
$$

For a quasi-circular orbit, the eccentricity is very small and this transcendental equation can be solved in E_Q with the Hull's method applied to the function:

$$
f : (E_Q; e_x, e_y) \mapsto f(E_Q; e_x, e_y) = E_Q + e_y \cos E_Q - e_x \sin E_Q - \ell_{M\Theta} + \Theta(t). \tag{242}
$$

The solution E_{Q_*} of the equation:

$$f(E_Q; e_x, e_y) = 0, \tag{243}$$

with E_Q unknown and e_x and e_y small can be expanded in power series of e_x and e_y:

$$\begin{aligned} E_{Q_*} &= E_{Q_0} + E_{Q_x} e_x + E_{Q_y} e_y + \frac{1}{2} E_{Q_{xx}} e_x^2 + \frac{1}{2} E_{Q_{yy}} e_y^2 + E_{Q_{xy}} e_x e_y, \\ &= E_{Q_0} + \Delta E_Q. \end{aligned} \tag{244}$$

$f(E_Q; e_x, e_y) = f(E_{Q_0} + \Delta E_Q; e_x, e_y)$ is then expanded with respect to $(E_{Q_0}; 0, 0)$ up to the order 2. For clarity reasons, the following notations will be used:

$$\begin{aligned}
&f(E_{Q_0}; 0, 0) \longrightarrow f, && \frac{\partial f(E_Q; e_x, e_y)}{\partial e_x}|_{(E_{Q_0};0,0)} \longrightarrow f_{e_x}, \\
&\frac{\partial f(E_Q; e_x, e_y)}{\partial e_y}|_{(E_{Q_0};0,0)} \longrightarrow f_{e_y}, && \frac{\partial f(E_Q; e_x, e_y)}{\partial E_Q}|_{(E_{Q_0};0,0)} \longrightarrow f_{E_Q}, \\
&\frac{\partial^2 f(E_Q; e_x, e_y)}{\partial e_x^2}|_{(E_{Q_0};0,0)} \longrightarrow f_{e_x e_x}, && \frac{\partial^2 f(E_Q; e_x, e_y)}{\partial e_y^2}|_{(E_{Q_0};0,0)} \longrightarrow f_{e_y e_y}, \\
&\frac{\partial^2 f(E_Q; e_x, e_y)}{\partial E_Q^2}|_{(E_{Q_0};0,0)} \longrightarrow f_{E_Q E_Q}, && \frac{\partial^2 f(E_Q; e_x, e_y)}{\partial e_x \partial e_y}|_{(E_{Q_0};0,0)} \longrightarrow f_{e_x e_y}, \\
&\frac{\partial^2 f(E_Q; e_x, e_y)}{\partial e_x \partial E_Q}|_{(E_{Q_0};0,0)} \longrightarrow f_{e_x E_Q}, && \frac{\partial^2 f(E_Q; e_x, e_y)}{\partial e_y \partial E_Q}|_{(E_{Q_0};0,0)} \longrightarrow f_{e_y E_Q}.
\end{aligned}$$

$$\begin{aligned}
f(E_Q; e_x, e_y) &= f(E_{Q_0} + \Delta E_Q; e_x, e_y), \\
&= f + f_{E_Q} \Delta E_Q + f_{e_x} e_x + f_{e_y} e_y + \\
&\quad + \frac{1}{2} f_{E_Q E_Q} \Delta E_Q^2 + \frac{1}{2} f_{e_x e_x} e_x^2 + \frac{1}{2} f_{e_y e_y} e_y^2 \\
&\quad + f_{E_Q e_x} \Delta E_Q e_x + f_{E_Q e_y} \Delta E_Q e_y + f_{e_x e_y} e_x e_y, \\
&= f + f_{E_Q} \left(E_{Q_x} e_x + E_{Q_y} e_y + \frac{1}{2} E_{Q_{xx}} e_x^2 + \frac{1}{2} E_{Q_{yy}} e_y^2 + E_{Q_{xy}} e_x e_y \right) + f_{e_x} e_x + f_{e_y} e_y \\
&\quad + \frac{1}{2} f_{E_Q E_Q} \left({E_{Q_x}}^2 e_x^2 + E_{Q_y} e_y^2 + 2 E_{Q_x} E_{Q_y} e_x e_y \right) + \frac{1}{2} f_{e_x e_x} e_x^2 + \frac{1}{2} f_{e_y e_y} e_y^2 \\
&\quad + f_{E_Q e_x} \left(E_{Q_x} e_x + E_{Q_y} e_y \right) e_x + f_{E_Q e_y} \left(E_{Q_x} e_x + E_{Q_y} e_y \right) e_y + f_{e_x e_y} e_x e_y.
\end{aligned} \tag{245}$$

Identifying the coefficients of the several orders of e_x and e_y so that the Equation (243) holds leads to the following system:

$$\begin{cases}
f = 0, \\
f_{E_Q} E_{Q_x} + f_{e_x} = 0, \\
f_{E_Q} E_{Q_y} + f_{e_y} = 0, \\
\frac{1}{2} f_{E_Q} E_{Q_{xx}} + \frac{1}{2} f_{E_Q E_Q} {E_{Q_x}}^2 + \frac{1}{2} f_{e_x e_x} + f_{E_Q e_x} E_{Q_x} = 0, \\
\frac{1}{2} f_{E_Q} E_{Q_{yy}} + \frac{1}{2} f_{E_Q E_Q} {E_{Q_y}}^2 + \frac{1}{2} f_{e_y e_y} + f_{E_Q e_y} E_{Q_y} = 0, \\
f_{E_Q E_Q} E_{Q_x} E_{Q_y} + f_{E_Q e_x} E_{Q_y} + f_{E_Q e_y} E_{Q_x} + f_{e_x e_y} + f_{E_Q} E_{Q_{xy}} = 0.
\end{cases} \tag{246}$$

Solving this system leads to:

$$\begin{cases} E_{Q_0} = \ell_{M\Theta} + \Theta(t), \\ E_{Q_x} = \sin(\ell_{M\Theta} + \Theta(t)), \\ E_{Q_y} = -\cos(\ell_{M\Theta} + \Theta(t)), \\ E_{Q_{xx}} = \cos(\ell_{M\Theta} + \Theta(t))\sin(\ell_{M\Theta} + \Theta(t)), \\ E_{Q_{yy}} = -\cos(\ell_{M\Theta} + \Theta(t))\sin(\ell_{M\Theta} + \Theta(t)), \\ E_{Q_{xy}} = \cos^2(\ell_{M\Theta} + \Theta(t)) - \sin^2(\ell_{M\Theta} + \Theta(t)). \end{cases} \tag{247}$$

Hence, the eccentric equinoctial anomaly is expressed in terms of the mean longitude as:

$$\begin{aligned} E_Q = \ell_{M\Theta} + \Theta(t) + e_x \sin(\ell_{M\Theta} + \Theta(t)) - e_y \cos(\ell_{M\Theta} + \Theta(t)) \\ + \sin\left(\frac{\ell_{M\Theta} + \Theta(t)}{2}\right)(e_x^2 - e_y^2) + \cos\left(\frac{\ell_{M\Theta} + \Theta(t)}{2}\right) e_x e_y. \end{aligned} \tag{248}$$

At the order 0 in e_x and e_y, we get:

$$\cos E_Q = \cos(\ell_{M\Theta} + \Theta(t)), \tag{249a}$$

$$\sin E_Q = \sin(\ell_{M\Theta} + \Theta(t)). \tag{249b}$$

E Physical Parameters

This appendix gives numerical values for the physical parameters involved for the computation of the orbit of a geostationary satellite. In the sequel, the unit d stands for "day".

The physical and orbital parameters of the Earth are:

- geocentric gravitational parameter: $\mu_\oplus = 3.986\ 10^5\ \text{km}^3/\text{s}^2 = 2.9755\ 10^{15}\ \text{km}^3/\text{d}^2$,
- rotation rate: $\omega_\oplus = 7.2921\ 10^{-5}\ \text{rad/s} = 6.3004\ \text{rad/d}$,
- radius: $r_\oplus = 6378.137$ km
- coefficient of the spherical decomposition of the Earth gravitational field: see the Table 3 (these values have been taken from the reference [Vallado, 1997])
- sidereal angle $\Theta(t)$: using the computation algorithm from [Vallado, 1997], for January 1$^{\text{st}}$, 2034, the value of the sidereal angle is: $\Theta_0 = 1.7579$ rad. If t denotes the elapsed time since January 1$^{\text{st}}$, 2034, the sidereal angle at time t is computed as: $\Theta(t) = \Theta_0 + \omega_\oplus t$.
- geostationary semi-major axis: $a_{sk} = 42165.8$ km.

The gravitational parameter of the Sun is:

$$\mu_\odot = 1.32712\ 10^{11}\ \text{km}^3/\text{s}^2 = 9.9069\ 10^{20}\ \text{km}^3/\text{d}^2. \tag{250}$$

The gravitational parameter of the Moon is:

$$\mu_☾ = 4.9028\ 10^4\ \text{km}^3/\text{s}^2 = 3.6599\ 10^{14}\ \text{km}^3/\text{d}^2. \tag{251}$$

degree l	order m	C_{lm}	S_{lm}
2	0	$-1.083\ 10^{-3}$	$\times$
2	1	$-2.414\ 10^{-10}$	$1.543\ 10^{-9}$
2	2	$1.574\ 10^{-6}$	$-9.038\ 10^{-7}$
3	0	$2.532\ 10^{-6}$	$\times$
3	1	$2.191\ 10^{-6}$	$2.687\ 10^{-7}$
3	2	$3.089\ 10^{-7}$	$-2.115\ 10^{-7}$
3	3	$1.006\ 10^{-}7$	$1.972\ 10^{-7}$

Table 3 – C_{lm} and S_{lm} coefficients for the spherical harmonics decomposition of the Earth gravitational potential (values taken from [Vallado, 1997]). The $\times$ means that for order 0, the S_{l0} coefficient does not exist.

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