Homogenization of nonconvex unbounded singular integrals

Omar Anza Hafsa, Nicolas Clozeau, Jean-Philippe Mandallena

ANNALES MATHÉMATIQUES

BLAISE PASCAL

Omar Anza Hafsa,
Nicolas Clozeau & Jean-Philippe Mandallena

Homogenization of nonconvex unbounded singular integrals

Volume 24, n° 2 (2017), p. 135-193.

<http://ambp.cedram.org/item?id=AMBP_2017__24_2_135_0>

Publication éditée par le laboratoire de mathématiques Blaise Pascal
de l'université Clermont Auvergne, UMR 6620 du CNRS
Clermont-Ferrand — France

cedram

Article mis en ligne dans le cadre du
Centre de diffusion des revues académiques de mathématiques
http://www.cedram.org/

Annales mathématiques Blaise Pascal **24**, 135-193 (2017)

Homogenization of nonconvex unbounded singular integrals

Omar Anza Hafsa
Nicolas Clozeau
Jean-Philippe Mandallena

Abstract

We study periodic homogenization by Γ-convergence of integral functionals with integrands $W(x,\xi)$ having no polynomial growth and which are both not necessarily continuous with respect to the space variable and not necessarily convex with respect to the matrix variable. This allows to deal with homogenization of composite hyperelastic materials consisting of two or more periodic components whose the energy densities tend to infinity as the volume of matter tends to zero, i.e., $W(x,\xi)=\sum_{j\in J}\mathbb{1}_{V_j}(x)H_j(\xi)$ where $\{V_j\}_{j\in J}$ is a finite family of open disjoint subsets of $\mathbb{R}^N$, with $|\partial V_j|=0$ for all $j\in J$ and $|\mathbb{R}^N\setminus\bigcup_{j\in J}V_j|=0$, and, for each $j\in J$, $H_j(\xi)\to\infty$ as $\det\xi\to 0$. In fact, our results apply to integrands of type $W(x,\xi)=a(x)H(\xi)$ when $H(\xi)\to\infty$ as $\det\xi\to 0$ and $a\in L^\infty(\mathbb{R}^N;[0,\infty[)$ is 1-periodic and is either continuous almost everywhere or not continuous. When a is not continuous, we obtain a density homogenization formula which is a priori different from the classical one by Braides–Müller. Although applications to hyperelasticity are limited due to the fact that our framework is not consistent with the constraint of noninterpenetration of the matter, our results can be of technical interest to analysis of homogenization of integral functionals.

1. Introduction

In this paper we prove homogenization theorems (see Theorems 2.8, 2.19 and 2.33) in the sense of De Giorgi's Γ-convergence (see Definition 2.1) for functionals of type

$$\int_\Omega W\left(\frac{x}{\varepsilon},\nabla\phi(x)\right)\mathrm{d}x\,, \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded open set and $\phi\in W^{1,p}(\Omega;\mathbb{R}^m)$ with $p>1$, when the 1-periodic integrand $W:\mathbb{R}^N\times\mathbb{M}^{m\times N}\to[0,\infty]$ has not p-growth and is both not necessarily continuous with respect to its first

Keywords: Homogenization, Γ-convergence, Unbounded integrand, Singular growth, Determinant constraint type, hyperelasticity.

variable and not necessarily convex with respect to the second variable. Our homogenization results can be summarized as follows (see §1.1 for details on the classes of integrands).

Theorem. *If* $W \in \mathcal{I}^p_{\mathrm{per}} \cup \mathcal{J}^p_{\mathrm{per}} \cup \mathcal{K}^p_{\mathrm{per}}$ *then* (1.1) Γ*-converges as* $\varepsilon \to 0$ *to the homogenized functional*

$$\int_\Omega W_{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x\,.$$

If $W \in \mathcal{I}^p_{\mathrm{per}} \cup \mathcal{J}^p_{\mathrm{per}}$ *then* W_{hom} *is given by the classical density homogenization formula of Braides–Müller, i.e.,*

$$\begin{aligned} W_{\mathrm{hom}}(\xi) &= \mathcal{H}W(\xi) \\ &:= \inf_{k\geq 1}\inf\left\{ ⨍_{kY} W(x,\xi+\nabla\varphi(x))\,\mathrm{d}x : \varphi \in W^{1,p}_0(kY;\mathbb{R}^m)\right\}, \end{aligned}$$

where $Y := \left]-\frac{1}{2},\frac{1}{2}\right[^N$ *and* $W^{1,p}_0(kY;\mathbb{R}^m)$ *denotes the space of* p*-Sobolev functions from* kY *to* $\mathbb{R}^m$ *which are null on the boundary of* kY*.*

If $W \in \mathcal{K}^p_{\mathrm{per}}$ *then* W_{hom} *is given by a priori different formula from the classical one, i.e.,*

$$W_{\mathrm{hom}}(\xi) = \mathcal{H}[\mathcal{G}W](\xi)$$

with

$$\mathcal{G}W(x,\xi) := \varlimsup_{\rho\to 0}\inf\left\{ ⨍_{Q_\rho(x)} W(y,\xi+\nabla\varphi(y))\,\mathrm{d}y : \varphi \in \mathrm{Aff}_0(Q_\rho(x);\mathbb{R}^m)\right\},$$

where $Q_\rho(x) := x + \rho Y$ *and* $\mathrm{Aff}_0(Q_\rho(x);\mathbb{R}^m)$ *denotes the space of continuous piecewise affine functions from* $Q_\rho(x)$ *to* $\mathbb{R}^m$ *which are null on the boundary of* $Q_\rho(x)$*.*

The distinguishing feature of our homogenization results is that they can be applied with integrands $W : \mathbb{R}^N \times \mathbb{M}^{N\times N} \to [0,\infty]$ having a singular behavior of the type

$$\lim_{\det\xi\to 0} W(x,\xi) = \infty\,, \tag{1.2}$$

i.e., when $W(x,\cdot\,)$ is compatible with one of the two basic facts of hyperelasticity, namely the necessity of an infinite amount of energy to compress a finite volume into zero volume (see Corollaries 2.13, 2.17, 2.22 and 2.36). However, our results are not consistent with the noninterpenetration of the matter.

The plan of the paper is as follows. In the next section we state our main results, see Theorems 2.8, 2.19 and 2.33 and Corollaries 2.13, 2.17, 2.22 and 2.36 (see also Remark 2.37) establishing new homogenization results for functionals with 1-periodic integrands W which are consistent with (1.2). Theorems 2.8, 2.19 and 2.33 are proved in Section 4. The proofs of Theorems 2.8 and 2.19 use both Braides–Müller's homogenization theorem (see Theorem 2.2) and new relaxation theorems (see Theorems 3.8 and 3.15) whose statements and proofs are given in Section 3. In the appendix, we recall some standard and less standard results on relaxation of singular integrands (see §A.1), approximation of the relaxation formula (see §A.2), approximation of the homogenization formula (see §A.3) and integral representation of the Vitali envelope of a set function (see §A.4). In fact, Corollaries 2.13 and 2.17 are based upon Theorem 2.12 which is proved in §A.1. On the other hand, Theorem A.4 is used in the proof of Corollary 2.22, and Propositions A.5 and A.8 and Theorem A.16 are used in the proof of Theorem 2.33.

1.1. **Notation, hypotheses and classes of integrands**

Troughout the paper, the symbol $\fint$ stands for the mean value integral, i.e.,

$$\fint_B f(x)\,\mathrm{d}x = \frac{1}{|B|}\int_B f(x)\,\mathrm{d}x\,.$$

Several general hypotheses are stated troughout the paper. For the convenience of the reader we summarize it below.

(A_0) W is p-coercive, i.e., $W(x,\xi)\geq C|\xi|^p$ for all $(x,\xi)\in\mathbb{R}^N\times\mathbb{M}^{m\times N}$ and some $C>0$.

(A_1) W is 1-periodic, i.e., for every $\xi\in\mathbb{M}^{m\times N}$ and every $i\in\{1,\dots,N\}$, $W(x+e_i,\xi)=W(x,\xi)$ for a.a. $x\in\mathbb{R}^N$, where $(e_1,\dots,e_N)$ is the standard basis of $\mathbb{R}^N$.

(A_2) there exists a function $\omega:[0,\infty[\to[0,\infty[$ continuous at the origin with $\omega(0)=0$ such that for every $x_1,x_2\in\mathbb{R}^N$ and every $\xi\in\mathbb{M}^{m\times N}$,

$$W(x_1,\xi)\leq\omega(|x_1-x_2|)(1+W(x_2,\xi))+W(x_2,\xi)\,.$$

(A_3) $\mathcal{Z}W : \mathbb{R}^N \times \mathbb{M}^{m\times N} \to [0,\infty]$ defined by

$$\mathcal{Z}W(x,\xi) := \inf\left\{\int_Y W(x,\xi+\nabla\varphi(y))\,\mathrm{d}y : \varphi \in W_0^{1,\infty}(Y;\mathbb{R}^m)\right\}$$

has p-growth, i.e., $\mathcal{Z}W(x,\xi) \leq c(1+|\xi|^p)$ for all $(x,\xi) \in \mathbb{R}^N \times \mathbb{M}^{m\times N}$ and some $c > 0$.

(A_4) there exists $\lambda \in \mathcal{L}$ such that for every $x_1, x_2 \in \mathbb{R}^N$ and every $\xi \in \mathbb{M}^{m\times N}$,

$$W(x_1,\xi) \leq |\lambda(x_1) - \lambda(x_2)|(1 + W(x_2,\xi)) + W(x_2,\xi)\,.$$

(A_5) $\mathcal{S}_\xi^W(U) \leq c|U|(1+|\xi|^p)$ for all $\xi \in \mathbb{M}^{m\times N}$, all bounded open sets $U \subset \mathbb{R}^N$ and some $c > 0$, where

$$\mathcal{S}_\xi^W(U) := \inf\left\{\int_U W(y,\xi+\nabla\varphi(y))\,\mathrm{d}y : \varphi \in \mathrm{Aff}_0(U;\mathbb{R}^m)\right\}.$$

(A_6) $\mathcal{G}W(x,\cdot) \leq W(x,\cdot)$ for a.a. $x \in \mathbb{R}^N$, where

$$\mathcal{G}W(x,\xi) := \varlimsup_{\rho\to 0} \frac{\mathcal{S}_\xi^W(Q_\rho(x))}{|Q_\rho(x)|}\,.$$

(A_7) $\widehat{\mathcal{Z}}[\widehat{\mathcal{H}}[\mathcal{G}W]] \leq \mathcal{H}[\mathcal{G}W]$, where, for $L : \mathbb{M}^{m\times N} \to [0,\infty]$,

$$\widehat{\mathcal{Z}}L(\xi) := \inf\left\{\int_Y L(\xi+\nabla\varphi(y))\,\mathrm{d}y : \varphi \in \mathrm{Aff}_0(Y;\mathbb{R}^m)\right\}$$

and

$$\widehat{\mathcal{H}}[\mathcal{G}W](\xi) := \inf_{k\geq 1}\inf\left\{\fint_{kY} \mathcal{G}W(y,\xi+\nabla\varphi(y))\,\mathrm{d}y : \varphi \in \mathrm{Aff}_0(kY;\mathbb{R}^m)\right\}.$$

Other more particular hypotheses are stated troughout the paper. For the convenience of the reader, we list the main ones below.

(P) for every bounded open subset U of $\mathbb{R}^N$ with $|\partial U| = 0$ and every $\delta \in]0,\delta_0]$ with $\delta_0 > 0$ small enough, there exists a compact $K_\delta \subset \overline{U}$ such that

$$\begin{cases} |\partial K_\delta| = 0 \\ |\overline{U} \setminus K_\delta| < \delta \\ \lambda|_{K_\delta} \text{ is continuous} \end{cases}$$

with $\overline{U}$ denoting the closure of U.

(H) there exist $\alpha, \beta > 0$ such that for every $\xi \in \mathbb{M}^{N\times N}$,

$$\text{if } |\det \xi| \geq \alpha \text{ then } H(\xi) \leq \beta(1 + |\xi|^p)\,.$$

(S) there exist a finite family $\{V_j\}_{j\in J}$ of open disjoint subsets of $\mathbb{R}^N$, with $|\partial V_j| = 0$ for all $j \in J$ and $|\mathbb{R}^N \setminus \bigcup_{j\in J} V_j| = 0$, and a finite family $\Big\{H_j : \mathbb{M}^{m\times N} \to [0,\infty]\Big\}_{j\in J}$ of Borel measurable functions such that W is defined by

$$W(x,\xi) = \sum_{j\in J} \mathbb{1}_{V_j}(x) H_j(\xi)\,.$$

Several different classes of integrands are defined troughout the paper (see Definitions 2.6, 2.9, 2.10, 2.14, 2.18, 2.20, 2.25, 2.34, 2.40). For the convenience of the reader, we summarize it below.

- $\mathcal{I}^p$ denotes the class of Borel measurable functions $W : \mathbb{R}^N \times \mathbb{M}^{m\times N} \to [0,\infty]$ satisfying (A_0), (A_3) and (A_4), see Definition 2.6, and

$$\mathcal{I}^p_{\text{per}} := \Big\{W \in \mathcal{I}^p : W \text{ satisfies } (A_1)\Big\}.$$

- $\mathcal{J}^p$ denotes the class of Borel measurable functions $W : \mathbb{R}^N \times \mathbb{M}^{m\times N} \to [0,\infty]$ satisfying (A_0) and (A_3) and (S), see Definition 2.18, and

$$\mathcal{J}^p_{\text{per}} := \Big\{W \in \mathcal{J}^p : W \text{ satisfies } (A_1)\Big\}.$$

The Borel measurable functions W belonging to $\mathcal{I}^p$ or $\mathcal{J}^p$ (and so to $\mathcal{I}^p_{\text{per}}$ or $\mathcal{I}^p_{\text{per}}$) are continuous almost everywhere with respect to the space variable (see Lemma 2.5). We consider subclasses $\mathcal{S}_1$ (see Definition 2.10) and $\mathcal{S}_2$ (see Definition 2.14) of $\mathcal{I}^p_{\text{per}}$ and $\mathcal{S}_3$ (see Definition 2.20) of $\mathcal{J}^p_{\text{per}}$ of W having separated space and matrix variables. These classes are defined through the class $\mathcal{H}$ (see Definition 2.9) of functions $H : \mathbb{M}^{N\times N} \to [0,\infty]$ satisfying (H) which are consistent with the singular behavior $H(\xi) \to \infty$ as $\det \xi \to 0$.

- $\mathcal{K}^p_{\text{per}}$ denotes the class of Borel measurable functions $W : \mathbb{R}^N \times \mathbb{M}^{m\times N} \to [0,\infty]$ satisfying (A_0), (A_1), (A_5), (A_6) and (A_7), see Definition 2.25.

The Borel measurable functions W belonging to $\mathcal{K}^p_{\mathrm{per}}$ are not necessarily continuous with respect to the space variable. We consider the subclass $\mathcal{S}_4$ of $\mathcal{K}^p_{\mathrm{per}}$ of W having separated space and matrix variables which is defined through the class $\mathcal{H}_{\mathrm{usc}} := \{H \in \mathcal{H} : H \text{ is upper semicontinuous}\}$ (see Definition 2.34). Finally, to make clear the link between the almost continuous case to the non-continuous one, we consider the classes $\widehat{\mathcal{I}}^p$ and $\widehat{\mathcal{I}}^p_{\mathrm{per}} := \{W \in \widehat{\mathcal{I}}^p : W \text{ satisifes } (\mathrm{A}_0)\}$, see Definition 2.40 and Remark 2.37.

2. **Main results**

Consider the family of integral functionals $\{I_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,\infty]\}_{\varepsilon>0}$ defined by

$$I_\varepsilon(\phi) := \int_\Omega W\left(\frac{x}{\varepsilon}, \nabla\phi(x)\right) \mathrm{d}x\,, \tag{2.1}$$

where $p > 1$, $\varepsilon > 0$ is a (small) parameter, $\Omega \subset \mathbb{R}^N$ is a bounded open set with $|\partial\Omega| = 0$, where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^N$ and $W : \mathbb{R}^N \times \mathbb{M}^{m\times N} \to [0,\infty]$ is a Borel measurable function, where $\mathbb{M}^{m\times N}$ denotes the space of real $m \times N$ matrices with $m, N \geq 1$ two integers, which satisfies the following two assumptions:

- (A_0) W is p-coercive, i.e., $W(x,\xi) \geq C|\xi|^p$ for all $(x,\xi) \in \mathbb{R}^N \times \mathbb{M}^{m\times N}$ and some $C > 0$;

- (A_1) W is 1-periodic, i.e., for every $\xi \in \mathbb{M}^{m\times N}$ and every $i \in \{1,\dots,N\}$, $W(x + e_i, \xi) = W(x,\xi)$ for a.a. $x \in \mathbb{R}^N$, where $(e_1,\dots,e_N)$ is the standard basis of $\mathbb{R}^N$.

In [14] (see also [16, Theorem 14.5]) Braides (and independently Müller in [24]) proved the following homogenization theorem (see Theorem 2.2) in the sense of De Giorgi's Γ-convergence whose definition is given below.

Definition 2.1. We say that I_ε Γ-converges to $I_{\mathrm{hom}} : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,\infty]$ with respect to the $L^p(\Omega;\mathbb{R}^m)$-convergence as $\varepsilon \to 0$, and we write $\Gamma\text{-}\lim_{\varepsilon\to 0} I_\varepsilon = I_{\mathrm{hom}}$, if

$$\left(\Gamma\text{-}\varliminf_{\varepsilon\to 0} I_\varepsilon\right)(\phi) = \left(\Gamma\text{-}\varlimsup_{\varepsilon\to 0} I_\varepsilon\right)(\phi) = I_{\mathrm{hom}}(\phi)$$

for all $\phi \in W^{1,p}(\Omega;\mathbb{R}^m)$ with:

$$\left(\Gamma\text{-}\varliminf_{\varepsilon\to 0} I_\varepsilon\right)(\phi) := \inf\left\{\varliminf_{\varepsilon\to 0} I_\varepsilon(\phi_\varepsilon) : \phi_\varepsilon \to \phi \text{ in } L^p(\Omega;\mathbb{R}^m)\right\};$$

$$\left(\Gamma\text{-}\varlimsup_{\varepsilon\to 0} I_\varepsilon\right)(\phi) := \inf\left\{\varlimsup_{\varepsilon\to 0} I_\varepsilon(\phi_\varepsilon) : \phi_\varepsilon \to \phi \text{ in } L^p(\Omega;\mathbb{R}^m)\right\}.$$

(For more details on the concept of Γ-convergence, we refer the reader to [20, 16, 15].)

Theorem 2.2. *Under* (A_0) *and* (A_1) *if in addition* W *has* p*-growth, i.e., there exists* $c > 0$ *such that* $W(x,\xi) \le c(1+|\xi|^p)$ *for all* $(x,\xi) \in \mathbb{R}^N \times \mathbb{M}^{m\times N}$*, then*

$$\left(\Gamma\text{-}\lim_{\varepsilon\to 0} I_\varepsilon\right)(\phi) = \int_\Omega W_{\mathrm{hom}}(\nabla\phi(x))\,\mathrm{d}x \quad \textit{for all } \phi \in W^{1,p}(\Omega;\mathbb{R}^m) \qquad (2.2)$$

with $W_{\mathrm{hom}} : \mathbb{M}^{m\times N} \to [0,\infty]$ *given by*

$$W_{\mathrm{hom}}(\xi) = \inf_{k\ge 1}\inf\left\{ ⨍_{kY} W(x,\xi+\nabla\varphi(x))\,\mathrm{d}x : \varphi \in W_0^{1,p}(kY;\mathbb{R}^m)\right\}, \qquad (2.3)$$

where $Y := \left]-\frac{1}{2},\frac{1}{2}\right[^N$ *and* $W_0^{1,p}(kY;\mathbb{R}^m) := \{\varphi \in W^{1,p}(kY;\mathbb{R}^m) : \varphi = 0$ *on* $\partial(kY)\}$*.*

The interest of Theorem 2.2 is to establish a suitable variational framework to deal with nonconvex homogenization problems in the vectorial case: it is the point of departure of many works on the subject related to hyperelasticity. However, because of the p-growth assumption on the integrand W, Theorem 2.2 is not consistent with (1.2).

In the present paper we establish new homogenization results (see §2.1, §2.2 and §2.3) which are consistent with (1.2). (For other works on homogenization related to hyperelasticity we refer the reader to [21, 6, 7, 10] and the references therein.)

2.1. Homogenization with singular integrands which are continuous almost everywhere with respect to the space variable

In [2] it was proved the following homogenization theorem whose distinguishing feature is to be consistent with (1.2) even though it is not consistent with the noninterpenetration of the matter, see [2, §4] for more details.

Theorem 2.3. *Under* (A_0) *and* (A_1) *if in addition* W *satisfies the following two conditions:*

(A_2) *there exists a function* $\omega : [0,\infty[\to [0,\infty[$ *continuous at the origin with* $\omega(0) = 0$ *such that for every* $x_1, x_2 \in \mathbb{R}^N$ *and every* $\xi \in \mathbb{M}^{m\times N}$,

$$W(x_1,\xi) \le \omega(|x_1 - x_2|)(1 + W(x_2,\xi)) + W(x_2,\xi);$$

(A_3) $\mathcal{Z}W : \mathbb{R}^N \times \mathbb{M}^{m\times N} \to [0,\infty]$ *defined by*

$$\mathcal{Z}W(x,\xi) := \inf\left\{\int_Y W(x,\xi + \nabla\varphi(y))\,\mathrm{d}y : \varphi \in W_0^{1,\infty}(Y;\mathbb{R}^m)\right\} \tag{2.4}$$

has p-growth, i.e., $\mathcal{Z}W(x,\xi) \le c(1 + |\xi|^p)$ *for all* $(x,\xi) \in \mathbb{R}^N \times \mathbb{M}^{m\times N}$ *and some* $c > 0$,

then (2.2) *holds with* W_{hom} *given by* (2.3).

However, since the condition (A_2) implies the continuity of W with respect to its first variable, Theorem 2.3 cannot be applied with W of the form

$$W(x,\xi) = a(x)H(\xi)$$
$$\text{with } a(x) = \begin{cases} \gamma_1 & \text{if } x \in E_1 \\ \gamma_2 & \text{if } x \in E_2 \end{cases} \text{ and } \lim_{\det\xi\to 0} H(\xi) = \infty, \tag{2.5}$$

where $\gamma_1, \gamma_2 > 0$ and E_1 is a 1-periodic open subset of $\mathbb{R}^N$ such that $|\partial E_1| = 0$ and $E_2 := \mathbb{R}^N \setminus E_1$.

Theorem 2.8 below improves Theorem 2.3 by allowing to the integrand W not to be necessarily continuous with respect to its first variable. Theorem 2.8 can be applied with 1-periodic integrands W as in (2.5), see Corollary 2.17.

To state Theorem 2.8 we need to introduce a new class of 1-periodic integrands. Let $\mathcal{L}$ be the class of $\lambda \in L^\infty(\mathbb{R}^N;[0,\infty[)$ satisfying the following property:

(P) for every bounded open subset U of $\mathbb{R}^N$ with $|\partial U| = 0$ and every $\delta \in]0,\delta_0]$ with $\delta_0 > 0$ small enough, there exists a compact $K_\delta \subset \overline{U}$

such that

$$\begin{cases} |\partial K_\delta| = 0 \\ |\overline{U} \setminus K_\delta| < \delta \\ \lambda|_{K_\delta} \text{ is continuous} \end{cases}$$

with $\overline{U}$ denoting the closure of U.

Remark 2.4. If λ is continuous then the property (P) is trivially verified with $K_\delta = \overline{U}$. On the other hand, if λ is continuous almost everywhere, i.e.,

$$|N| = 0 \text{ where } N := \Big\{x \in \mathbb{R}^N : \lambda \text{ is not continuous at } x\Big\},$$

and if

$$\lim_{\delta \to 0} |V_\delta| = 0 \text{ and } |\partial V_\delta| = 0 \text{ where } V_\delta := \Big\{x \in U : \operatorname{dist}(x, N) < \delta\Big\},$$

then (P) is verified with $K_\delta = \overline{U} \setminus V_\delta$.

On the other hand, we have

Lemma 2.5. *If* (P) *is verified then* λ *is continuous almost everywhere.*

Proof. Arguing by induction, it is easily seen that from (P) we can deduce that there exists a disjointed sequence $\{K_n\}_{n \geq 1}$ of compact subsets of $\mathbb{R}^N$ such that

$$\begin{cases} K_1 \subset \overline{B_1} \\ K_n \subset \overline{B_n \setminus \bigcup_{i=1}^{n-1} K_i} \text{ for all } n \geq 2\,, \end{cases}$$

and for every $n \geq 1$,

$$\begin{cases} |B_n \setminus \bigcup_{i=1}^{n} K_i| < \frac{\delta_0}{n} \\ |\partial K_n| = 0 \\ \lambda|_{K_n} \text{ is continuous}\,, \end{cases} \tag{2.6}$$

where $\delta_0 > 0$ is given by (P) and B_n denotes the open ball in $\mathbb{R}^N$ centered at the origin and of radius n. It is sufficient to prove that

$$\left|\mathbb{R}^N \setminus \bigcup_{i=1}^{\infty} \operatorname{int}(K_i)\right| = 0\,. \tag{2.7}$$

(Indeed, since $\lambda|_{K_i}$ is continuous, also is $\lambda|_{\operatorname{int}(K_i)}$ and so $\lambda|_{\bigcup_{i=1}^{\infty} \operatorname{int}(K_i)}$ is continuous because $\{\operatorname{int}(K_i)\}_{i \geq 1}$ is a disjointed sequence of open subsets of $\mathbb{R}^N$.) Using the second equality in (2.6) and the fact that $\mathbb{R}^N = \bigcup_{n=1}^\infty B_n$ we have

$$\begin{aligned}\left|\mathbb{R}^N \setminus \bigcup_{i=1}^\infty \mathrm{int}(K_i)\right| &= \left|\left(\bigcup_{n=1}^\infty B_n\right) \setminus \bigcup_{i=1}^\infty K_i\right| \\ &= \left|\bigcup_{n=1}^\infty \left(B_n \setminus \bigcup_{i=1}^\infty K_i\right)\right| = \lim_{n\to\infty}\left|B_n \setminus \bigcup_{i=1}^\infty K_i\right|. \quad (2.8)\end{aligned}$$

But, for each $n \geq 1$, $B_n \setminus \bigcup_{i=1}^\infty K_i \subset B_n \setminus \bigcup_{i=1}^n K_i$ hence $|B_n \setminus \bigcup_{i=1}^\infty K_i| < \frac{\delta_0}{n}$ by using the first inequality in (2.6). Consequently $\lim_{n\to\infty} |B_n \setminus \bigcup_{i=1}^\infty K_i| = 0$, which gives (2.7). $\square$

Definition 2.6. We denote by $\mathcal{I}^p$ the class of Borel measurable functions $W : \mathbb{R}^N \times \mathbb{M}^{m\times N} \to [0,\infty]$ satisfying (A_0), (A_3) and the following additional condition:

(A_4) there exists $\lambda \in \mathcal{L}$ such that for every $x_1, x_2 \in \mathbb{R}^N$ and every $\xi \in \mathbb{M}^{m\times N}$,

$$W(x_1,\xi) \leq |\lambda(x_1) - \lambda(x_2)|(1 + W(x_2,\xi)) + W(x_2,\xi).$$

Let us set

$$\mathcal{I}^p_{\mathrm{per}} := \Big\{W \in \mathcal{I}^p : W \text{ satisfies } (\mathrm{A}_1)\Big\}.$$

Remark 2.7. The following are elementary properties whose the proofs are left to the reader.

(a) If W satisfies (A_4) then $\mathrm{dom} W(x_1,\cdot\,) = \mathrm{dom}\, W(x_2,\cdot\,)$ for all $x_1, x_2 \in \mathbb{R}^N$, where, for $x \in \mathbb{R}^N$, $\mathrm{dom}\, W(x,\cdot\,) := \{\xi \in \mathbb{M}^{m\times N} : W(x,\xi) < \infty\}$.

(b) If λ is continuous then (A_2) is satisfied with ω given by the modulus of continuity of λ.

(c) If W satisfies (A_4) and if λ is continuous at $x \in \mathbb{R}^N$ then $W(\,\cdot\,,\xi)$ is continuous at x for all $\xi \in \mathbb{M}^{m\times N}$. More generally, if (A_4) holds and if $\lambda|_K$ is continuous for $K \subset \mathbb{R}^N$ then $W(\,\cdot\,,\xi)|_K$ is continuous for all $\xi \in \mathbb{M}^{m\times N}$.

(d) If W satisfies (A_4) then for every $x_1, x_2 \in \mathbb{R}^N$ and every $\xi \in \mathbb{M}^{m\times N}$,

$$\mathcal{Z}W(x_1,\xi) \le |\lambda(x_1) - \lambda(x_2)|(1 + \mathcal{Z}W(x_2,\xi)) + \mathcal{Z}W(x_2,\xi)\,.$$

Hence, if (A_4) holds and if $\lambda|_K$ is continuous for $K \subset \mathbb{R}^N$ then $\mathcal{Z}W(\,\cdot\,,\xi)|_K$ is continuous for all $\xi \in \mathbb{M}^{m\times N}$ and $\operatorname{dom}\mathcal{Z}W(x_1,\cdot\,) = \operatorname{dom}\mathcal{Z}W(x_2,\cdot\,)$ for all $x_1, x_2 \in \mathbb{R}^N$.

(e) As a consequence of (c) and (d), if W satisfies (A_4) then $W(\,\cdot\,,\xi)$ and $\mathcal{Z}W(\,\cdot\,,\xi)$ are continuous almost everywhere for all $\xi \in \mathbb{M}^{m\times N}$ because λ is continuous almost everywhere by Lemma 2.5.

Here is the first homogenization result of the paper.

Theorem 2.8. *If $W \in \mathcal{I}^p_{\mathrm{per}}$ then* (2.2) *holds with W_{hom} given by* (2.3)*.*

Definition 2.9. We denote by $\mathcal{H}$ the class of p-coercive and Borel measurable functions $H : \mathbb{M}^{N\times N} \to [0,\infty]$ with the following property:

(H) there exist $\alpha, \beta > 0$ such that for every $\xi \in \mathbb{M}^{N\times N}$,

$$\text{if } |\det\xi| \ge \alpha \text{ then } H(\xi) \le \beta(1 + |\xi|^p)\,.$$

Note that the property (H) is compatible with the singular behavior

$$\lim_{\det\xi\to 0} H(\xi) = \infty\,.$$

A typical example of a function belonging to the class $\mathcal{H}$ is given by

$$H(\xi) = |\xi|^p + h(\det\xi)$$

where $h : \mathbb{R} \to [0,\infty]$ is a measurable function for which there exist $\delta, \delta' > 0$ such that $h(t) \le \delta'$ for all $|t| \ge \delta$. For example, given $s > 0$ and $T \ge 0$ (possibly very large), this latter condition is satisfied with $\delta = 2T$ and $\delta' = \max\left\{\frac{1}{(2T)^s}, T\right\}$ when h is of type

$$h(t) = \begin{cases} T & \text{if } t < -T \\ \infty & \text{if } t \in [-T,0] \\ \dfrac{1}{t^s} & \text{if } t > 0\,. \end{cases}$$

Definition 2.10. We denote by $\mathcal{S}_1$ the class of 1-periodic and Borel measurable functions $W : \mathbb{R}^N \times \mathbb{M}^{N\times N} \to [0,\infty]$ for which there exist $H \in \mathcal{H}$ and $a \in \mathcal{L}$ with $a(x) \geq \eta$ for all $x \in \mathbb{R}^N$ and some $\eta > 0$ such that W is defined by

$$W(x,\xi) = a(x)H(\xi).$$

If $W \in \mathcal{S}_1$ then it is compatible with (1.2) and can be as in (2.5). In fact, we have

Lemma 2.11. *Let $m = N$. The class $\mathcal{S}_1$ is a subclass of $\mathcal{I}^p_{\mathrm{per}}$, i.e., $\mathcal{S}_1 \subset \mathcal{I}^p_{\mathrm{per}}$.*

Proof. Let $W \in \mathcal{S}_1$. Firstly, it is clear that (A_0) and (A_1) are verified and (A_4) holds with $\lambda = \frac{1}{\eta}a$. Secondly, The condition (A_3) follows from the following theorem which was proved in [2].

Theorem 2.12. *Let $W : \mathbb{R}^N \times \mathbb{M}^{N\times N} \to [0,\infty]$ be a Borel measurable function satisfying the following property:*

$(\widehat{\mathrm{H}})$ *there exist $\widehat{\alpha}, \widehat{\beta} > 0$ such that for every $(x,\xi) \in \mathbb{R}^N \times \mathbb{M}^{N\times N}$,*

$$\text{if } |\det \xi| \geq \widehat{\alpha} \text{ then } W(x,\xi) \leq \widehat{\beta}(1 + |\xi|^p).$$

Then $\mathcal{Z}W$ has p-growth, i.e., W satisfies the condition (A_3)*.*

(For the convenience of the reader, the proof of Theorem 2.12 will be given in appendix.) Indeed, since $H \in \mathcal{H}$, there exist $\alpha, \beta > 0$ such that for every $\xi \in \mathbb{M}^{N\times N}$, if $|\det \xi| \geq \alpha$ then $H(\xi) \leq \beta(1 + |\xi|^p)$. Setting $\widehat{\alpha} := \alpha$ and $\widehat{\beta} := \beta\|a\|_{L^\infty}$, it is then clear that W satisfies $(\widehat{\mathrm{H}})$. ☐

The following result is a direct consequence of Theorem 2.8 (which will be proved in Section 4) and Lemma 2.11.

Corollary 2.13. *Let $m = N$. If $W \in \mathcal{S}_1$ then* (2.2) *holds with W_{hom} given by* (2.3)*.*

Another application of Theorem 2.8 can be obtained by introducing the following class of integrands.

Definition 2.14. We denote by $\mathcal{S}_2$ the class of 1-periodic and Borel measurable functions $W : \mathbb{R}^N \times \mathbb{M}^{N\times N} \to [0,\infty]$ for which there exist Borel measurable functions $H_1, H_2 : \mathbb{M}^{N\times N} \to [0,\infty]$ with

$$\begin{cases} H_2 \in \mathcal{H} \\ H_2 \leq H_1 \leq \gamma H_2 \text{ for some } \gamma > 1 \end{cases} \tag{2.9}$$

such that W is defined by

$$W(x,\xi) = \mathbb{1}_{E_1}(x)H_1(\xi) + \mathbb{1}_{E_2}(x)H_2(\xi), \tag{2.10}$$

where E_1 is a 1-periodic open subset of $\mathbb{R}^N$ such that $|\partial E_1| = 0$ and $E_2 := \mathbb{R}^N \setminus E_1$, with $\mathbb{1}_{E_1}$ and $\mathbb{1}_{E_2}$ denoting the characteristic functions of E_1 and E_2 respectively.

Remark 2.15. If $H_2 \le H_1 \le \gamma H_2$ for some $\gamma > 1$ then $\operatorname{dom} H_1 = \operatorname{dom} H_2$ and if futhermore $H_2 \in \mathcal{H}$ then $H_1 \in \mathcal{H}$.

The following lemma makes clear the link between $\mathcal{S}_2$ and $\mathcal{I}^p_{\rm per}$.

Lemma 2.16. *Let $m = N$. The class $\mathcal{S}_2$ is a subclass of $\mathcal{I}^p_{\rm per}$, i.e., $\mathcal{S}_2 \subset \mathcal{I}^p_{\rm per}$.*

Proof. Let $W \in \mathcal{S}_2$. Then, by definition, W is given by (2.10) with $H_1, H_2 \in \mathcal{H}$ (see Remark 2.15). As (A_0) and (A_1) are clearly verified, to prove that $W \in \mathcal{I}^p_{\rm per}$ it is sufficient to show that the conditions (A_3) and (A_4) are satisfied.

Proof of (A_3). The condition (A_3) follows from the following Theorem 2.12. Indeed, since $H_1, H_2 \in \mathcal{H}^p$, for $i = 1,2$ there exist $\alpha_i, \beta_i > 0$ such that for every $\xi \in \mathbb{M}^{N\times N}$, if $|\det \xi| \ge \alpha_i$ then $H_i(\xi) \le \beta_i(1 + |\xi|^p)$. Setting $\widehat{\alpha} := \min\{\alpha_1, \alpha_2\}$ and $\widehat{\beta} := \max\{\beta_1, \beta_2\}$, it is then clear that W satisfies ($\widehat{\mathrm{H}}$).

Proof of (A_4). We are going to prove that (A_4) is verified with $\lambda = (\gamma - 1)\mathbb{1}_{E_1}$ where $\gamma > 1$ is given by (2.9). (Clearly $(\gamma-1)\mathbb{1}_{E_1} \in \mathcal{L}$, see Remark 2.4.) Fix $x_1, x_2 \in \mathbb{R}^N$ and $\xi \in \mathbb{M}^{N\times N}$. By definition of $\mathcal{S}^p_{\rm per}$ we have $\operatorname{dom} H_1 = \operatorname{dom} H_2$ (see Remark 2.15) and so $\operatorname{dom} W(x_1, \cdot\,) = \operatorname{dom} W(x_2, \cdot\,)$. Hence, without loss of generality we can assume that $W(x_i, \xi) < \infty$ for $i = 1, 2$. Then, we have

$$\begin{aligned} &W(x_1,\xi) - W(x_2,\xi) \\ &\qquad= \left(\mathbb{1}_{E_1}(x_1) - \mathbb{1}_{E_1}(x_2)\right) H_1(\xi) + \left(\mathbb{1}_{E_2}(x_1) - \mathbb{1}_{E_2}(x_2)\right) H_2(\xi) \\ &\qquad= \left(\mathbb{1}_{E_1}(x_1) - \mathbb{1}_{E_1}(x_2)\right)\left(H_1(\xi) - H_2(\xi)\right) \\ &\qquad\le \left|\mathbb{1}_{E_1}(x_1) - \mathbb{1}_{E_1}(x_2)\right|\left(H_1(\xi) - H_2(\xi)\right). \end{aligned}$$

because $\mathbb{1}_{E_2} = 1 - \mathbb{1}_{E_1}$ and $H_1 \ge H_2$. Let us set

$$\begin{cases} \widehat{H}_1(\xi) := \frac{\gamma}{\gamma+1} H_1(\xi) - \frac{1}{\gamma+1} H_2(\xi) \\ \widehat{H}_2(\xi) := \frac{\gamma}{\gamma+1} H_2(\xi) - \frac{1}{\gamma+1} H_1(\xi). \end{cases}$$

Then, it is easy to see that

$$\begin{cases} H_1(\xi) - H_2(\xi) = \widehat{H}_1(\xi) - \widehat{H}_2(\xi) \\ \widehat{H}_1(\xi) \geq \widehat{H}_2(\xi) \geq 0 \text{ by using the inequalities in (2.9)} \\ W(x_2,\xi) = \left(\mathbb{1}_{E_1}(x_2) + \frac{1}{\gamma-1}\right)\widehat{H}_1(\xi) + \left(\mathbb{1}_{E_2}(x_2) + \frac{1}{\gamma-1}\right)\widehat{H}_2(\xi). \end{cases}$$

From the above it follows that

$$\begin{aligned} &W(x_1,\xi) - W(x_2,\xi) \\ &\qquad\qquad \leq |\mathbb{1}_{E_1}(x_1) - \mathbb{1}_{E_1}(x_2)| \left(\widehat{H}_1(\xi) - \widehat{H}_2(\xi)\right) \\ &\qquad\qquad \leq |\mathbb{1}_{E_1}(x_1) - \mathbb{1}_{E_1}(x_2)| \,\widehat{H}_1(\xi) \\ &\qquad\qquad \leq (\gamma-1)\,|\mathbb{1}_{E_1}(x_1) - \mathbb{1}_{E_1}(x_2)| \left(\mathbb{1}_{E_1}(x_2) + \frac{1}{\gamma-1}\right)\widehat{H}_1(\xi) \\ &\qquad\qquad \leq (\gamma-1)\,|\mathbb{1}_{E_1}(x_1) - \mathbb{1}_{E_1}(x_2)|\, W(x_2,\xi), \end{aligned}$$

which proves the condition (A_4). □

As a direct consequence of Theorem 2.8 and Lemma 2.16 we have the following result.

Corollary 2.17. *Let $m = N$. If $W \in \mathcal{S}_2$ then* (2.2) *holds with W_{hom} given by* (2.3).

2.2. **Homogenization with a sum of singular integrands**

We introduce the following class of integrands.

Definition 2.18. We denote by $\mathcal{J}^p$ the class of Borel measurable functions $W : \mathbb{R}^N \times \mathbb{M}^{m\times N} \to [0,\infty]$ satisfying (A_0) and (A_3) and the following condition:

(S) there exist a finite family $\{V_j\}_{j\in J}$ of open disjoint subsets of $\mathbb{R}^N$, with $|\partial V_j| = 0$ for all $j \in J$ and $|\mathbb{R}^N \setminus \bigcup_{j\in J} V_j| = 0$, and a finite family $\left\{H_j : \mathbb{M}^{m\times N} \to [0,\infty]\right\}_{j\in J}$ of Borel measurable functions such that W is defined by

$$W(x,\xi) = \sum_{j\in J} \mathbb{1}_{V_j}(x) H_j(\xi). \tag{2.11}$$

Let us set

$$\begin{aligned}\mathcal{J}^p_{\mathrm{per}} &:= \Big\{W \in \mathcal{J}^p : W \text{ satisfies } (\mathrm{A}_1)\Big\}\\ &= \Big\{W \in \mathcal{J}^p : \mathbb{1}_{V_j} \text{ is 1-periodic for all } i \in J\Big\}.\end{aligned}$$

Here is the second homogenization result of the paper.

Theorem 2.19. *If $W \in \mathcal{J}^p_{\mathrm{per}}$ then* (2.2) *holds with W_{hom} given by* (2.3)*.*

Definition 2.20. Let $\mathcal{S}_3$ be the class of 1-periodic and Borel measurable functions $W : \mathbb{R}^N \times \mathbb{M}^{N\times N} \to [0,\infty]$ satisfying (S) with $H_j \in \mathcal{H}$ for all $j \in J$.

The following lemma makes clear the link between $\mathcal{S}_3$ and $\mathcal{J}^p_{\mathrm{per}}$.

Lemma 2.21. *Let $m = N$. The class $\mathcal{S}_3$ is a subclass of $\mathcal{J}^p_{\mathrm{per}}$, i.e., $\mathcal{S}_3 \subset \mathcal{J}^p_{\mathrm{per}}$.*

Proof. Let $W \in \mathcal{S}_3$. Then, by definition, W is given by (2.11) with $H_j \in \mathcal{H}$ for all $j \in J$. Firstly, (A_0) and (A_1) are clearly verified. Secondly, since every H_j belongs to $\mathcal{H}$, for each $j \in J$ there exists $\alpha_j, \beta_j > 0$ such that for every $\xi \in \mathbb{M}^{N\times N}$, if $|\det \xi| \geq \alpha_j$ then $H_j(\xi) \leq \beta_j(1 + |\xi|^p)$. Setting $\widehat{\alpha} := \min\{\alpha_j : j \in J\}$ and $\widehat{\beta} := \max\{\beta_j : j \in J\}$, it is then clear that W satisfies $(\widehat{\mathrm{H}})$ and (A_3) follows from Theorem 2.12. □

As a direct consequence of Theorem 2.19 and Lemma 2.21 we have the following result.

Corollary 2.22. *Let $m = N$. If $W \in \mathcal{S}_3$ then* (2.2) *holds with W_{hom} given by* (2.3)*.*

Remark 2.23. The class $\mathcal{S}_2$ is a subclass of $\mathcal{S}_3$, i.e., $\mathcal{S}_2 \subset \mathcal{S}_3$. (Indeed, if W is given by (2.10) then (2.11) holds with $J = \{1,2\}$, $V_1 = E_1$ and $V_2 = \mathrm{int}(E_2)$, where $\mathrm{int}(E_2)$ denotes the interior of E_2.) Thus, Corollary 2.22 generalizes Corollary 2.17.

2.3. Homogenization with singular integrands which are not continuous with respect to the space variable

Let $\mathcal{O}_{\mathrm{b}}(\mathbb{R}^N)$ be the class of bounded open subsets of $\mathbb{R}^N$. Given any $U \in \mathcal{O}_{\mathrm{b}}(\mathbb{R}^N)$, we denote the space of continuous piecewise affine functions from U to $\mathbb{R}^m$ by $\mathrm{Aff}(U;\mathbb{R}^m)$, i.e., $\varphi \in \mathrm{Aff}(U;\mathbb{R}^m)$ if and only if φ is continuous and there exists a finite family $\{U_i\}_{i\in I}$ of open disjoint subsets of U such that $|\partial U_i| = 0$ for all $i \in I$, $|U \setminus \bigcup_{i\in I} U_i| = 0$ and for every $i \in I$, $\nabla\varphi \equiv \xi_i$ in U_i with $\xi_i \in \mathbb{M}^{m\times N}$, and we set $\mathrm{Aff}_0(U;\mathbb{R}^m) := \{\varphi \in \mathrm{Aff}(U;\mathbb{R}^m) : \varphi = 0 \text{ on } \partial U\}$. Given any Borel measurable function $L : \mathbb{R}^N \times \mathbb{M}^{m\times N} \to [0,\infty]$, we define $\mathcal{S}^L_\xi : \mathcal{O}_\mathrm{b}(\mathbb{R}^N) \to [0,\infty]$ with $\xi \in \mathbb{M}^{m\times N}$, $\mathcal{G}L : \mathbb{R}^N \times \mathbb{M}^{m\times N} \to [0,\infty]$ and $\widehat{\mathcal{H}}L, \mathcal{H}L : \mathbb{M}^{m\times N} \to [0,\infty]$ by:

- $\mathcal{S}^L_\xi(U) := \inf\left\{\int_U L(y,\xi+\nabla\varphi(y))\,\mathrm{d}y : \varphi \in \mathrm{Aff}_0(U;\mathbb{R}^m)\right\};$

- $$\begin{aligned}\mathcal{G}L(x,\xi) &:= \varlimsup_{\rho\to 0} \frac{\mathcal{S}^L_\xi(Q_\rho(x))}{|Q_\rho(x)|}\\ &= \varlimsup_{\rho\to 0} \inf\left\{\fint_{Q_\rho(x)} L(y,\xi+\nabla\varphi(y))\,\mathrm{d}y : \varphi \in \mathrm{Aff}_0(Q_\rho(x);\mathbb{R}^m)\right\};\end{aligned}$$

- $\widehat{\mathcal{H}}L(\xi) := \inf_{k\geq 1} \inf\left\{\fint_{kY} L(y,\xi+\nabla\varphi(y))\,\mathrm{d}y : \varphi \in \mathrm{Aff}_0(kY;\mathbb{R}^m)\right\};$

- $\mathcal{H}L(\xi) := \inf_{k\geq 1} \inf\left\{\fint_{kY} L(y,\xi+\nabla\varphi(y))\,\mathrm{d}y : \varphi \in W^{1,p}_0(kY;\mathbb{R}^m)\right\},$

where $Y := \left]-\frac{1}{2},\frac{1}{2}\right[^N$ and $Q_\rho(x) := x + \rho Y$.

Remark 2.24. If L does not depend on the space variable then $\mathcal{S}^L_\xi(U) = |U|\widehat{\mathcal{Z}}L(\xi)$ for all $U \in \mathcal{O}_\mathrm{b}(\mathbb{R}^N)$ with $|\partial U| = 0$ and all $\xi \in \mathbb{M}^{m\times N}$, where $\widehat{\mathcal{Z}}L : \mathbb{M}^{m\times N} \to [0,\infty]$ is defined by

$$\widehat{\mathcal{Z}}L(\xi) := \inf\left\{\int_Y L(\xi+\nabla\varphi(y))\,\mathrm{d}y : \varphi \in \mathrm{Aff}_0(Y;\mathbb{R}^m)\right\}.$$

Consequently, in such a case we have $\mathcal{G}L = \widehat{\mathcal{Z}}L$ (see also Remark 2.37).

Definition 2.25. We denote by $\mathcal{K}^p_\mathrm{per}$ the class of Borel measurable functions $W : \mathbb{R}^N \times \mathbb{M}^{m\times N} \to [0,\infty]$ satisfying (A_0), (A_1) and the following additional conditions:

(A_5) $\mathcal{S}^W_\xi(U) \leq c|U|(1+|\xi|^p)$ for all $\xi \in \mathbb{M}^{m\times N}$, all $U \in \mathcal{O}_\mathrm{b}(\mathbb{R}^N)$ and some $c > 0$;

(A_6) $\mathcal{G}W(x,\cdot\,) \leq W(x,\cdot\,)$ for a.a. $x \in \mathbb{R}^N$;

(A$_7$) $\widehat{\mathcal{Z}}[\widehat{\mathcal{H}}[\mathcal{G}W]] \le \mathcal{H}[\mathcal{G}W]$.

Remark 2.26. If (A$_5$) holds then $\mathcal{G}W$ has p-growth.

Remark 2.27. If $W(x,\xi) = a(x)H(\xi)$ with $a \in L^\infty(\mathbb{R}^N;[0,\infty[)$ and $H : \mathbb{M}^{m\times N} \to [0,\infty]$ Borel measurable such that $\widehat{\mathcal{Z}}H$ has p-growth, then (A$_5$) holds.

Remark 2.28. It is clear that $\mathcal{G}W(x,\xi) \le W(x,\xi)$ for all $(x,\xi) \in \mathbb{R}^N \times \mathbb{M}^{m\times N}$ such that $W(x,\xi) = \infty$. On the other hand, if for $\xi \in \mathbb{M}^{m\times N}$ one has $W(\,\cdot\,,\xi) \in L^1(\mathbb{R}^N)$ then $\lim_{\rho\to 0} \fint_{Q_\rho(x)} W(y,\xi)\,\mathrm{d}y = W(x,\xi)$ for a.a. $x \in \mathbb{R}^N$, and so $\mathcal{G}W(x,\xi) \le W(x,\xi)$ for a.a. $x \in \mathbb{R}^N$. Thus, if $W(x,\xi) = a(x)H(\xi)$ with $a \in L^1(\mathbb{R}^N;[0,\infty[)$ and $H : \mathbb{M}^{m\times N} \to [0,\infty]$ Borel measurable, then such a W satisfies (A$_6$).

Remark 2.29. One always has $\widehat{\mathcal{H}}[\mathcal{G}W] \ge \mathcal{H}[\mathcal{G}W]$. On the other hand, we have:

Lemma 2.30. *If $\mathcal{G}W$ has p-growth, i.e., $\mathcal{G}W(x,\cdot) \le c(1+|\cdot|^p)$ for all $x \in \mathbb{R}^N$, and if $\mathcal{G}W(x,\cdot)$ is upper semicontinuous (usc) for a.a. $x \in \mathbb{R}^N$, then $\widehat{\mathcal{H}}[\mathcal{G}W] \le \mathcal{H}[\mathcal{G}W]$.*

Hence, under the assumption of Lemma 2.30, $\widehat{\mathcal{H}}[\mathcal{G}W] = \mathcal{H}[\mathcal{G}W]$ and so (A$_7$) holds because $\widehat{\mathcal{Z}}[\widehat{\mathcal{H}}[\mathcal{G}W]] \le \widehat{\mathcal{H}}[\mathcal{G}W]$.

Proof of Lemma 2.30. Fix $\xi \in \mathbb{M}^{m\times N}$. Let $L : \mathbb{R}^N \times \mathbb{M}^{m\times N} \to [0,\infty]$ be defined by

$$L(x,\zeta) := c(1+|\xi+\zeta|^p) - \mathcal{G}W(x,\xi+\zeta)\,.$$

Fix any $k \ge 1$ and any $\varphi \in W_0^{1,p}(kY;\mathbb{R}^m)$. As $\mathrm{Aff}_0(kY;\mathbb{R}^m)$ is strongly dense in $W_0^{1,p}(kY;\mathbb{R}^m)$ we can assert that there exists $\{\varphi_n\}_n \subset \mathrm{Aff}_0(kY;\mathbb{R}^m)$ such that:

$$\nabla\varphi_n \to \nabla\varphi \text{ in } L^p(kY;\mathbb{R}^m)\,; \tag{2.12}$$

$$\nabla\varphi_n(x) \to \nabla\varphi(x) \text{ for a.a. } x \in kY\,. \tag{2.13}$$

By Fatou's lemma we have

$$\varliminf_{n\to\infty} \int_{kY} L(x,\nabla\varphi_n(x))\,\mathrm{d}x \ge \int_{kY} \varliminf_{n\to\infty} L(x,\nabla\varphi_n(x))\,\mathrm{d}x\,.$$

But, by using (2.12) and (2.13) we see that:

$$\begin{aligned}\varliminf_{n\to\infty}\int_{kY} L(x,\nabla\varphi_n(x))\,\mathrm{d}x &= \int_{kY} c(1+|\xi+\nabla\varphi(x)|^p)\,\mathrm{d}x\\ &\quad-\varlimsup_{n\to\infty}\int_{kY}\mathcal{G}W(x,\xi+\nabla\varphi_n(x))\,\mathrm{d}x\,;\\ \varliminf_{n\to\infty} L(x,\nabla\varphi_n(x)) &= c(1+|\xi+\nabla\varphi(x)|^p)\\ &\quad-\varlimsup_{n\to\infty}\mathcal{G}W(x,\xi+\nabla\varphi_n(x)) \text{ for a.a. } x\in kY\,,\end{aligned}$$

and consequently

$$\varlimsup_{n\to\infty}\int_{kY}\mathcal{G}W(x,\xi+\nabla\varphi_n(x))\,\mathrm{d}x \le \int_{kY}\varlimsup_{n\to\infty}\mathcal{G}W(x,\xi+\nabla\varphi_n(x))\,\mathrm{d}x. \tag{2.14}$$

As $\mathcal{G}W(x,\cdot\,)$ is usc for a.a. $x\in\mathbb{R}^N$, taking (2.13) into account we have

$$\varlimsup_{n\to\infty}\mathcal{G}W(x,\xi+\nabla\varphi_n(x)) \le \mathcal{G}W(x,\xi+\nabla\varphi(x)) \text{ for a.a. } x\in kY\,. \tag{2.15}$$

From (2.14) and (2.15) it follows that

$$\varlimsup_{n\to\infty}\int_{kY}\mathcal{G}W(x,\xi+\nabla\varphi_n(x))\,\mathrm{d}x \le \int_{kY}\mathcal{G}W(x,\xi+\nabla\varphi(x))\,\mathrm{d}x\,.$$

But, for each $n\ge 1$, $\widehat{\mathcal{H}}[\mathcal{G}W](\xi)\le \fint_{kY}\mathcal{G}W(x,\xi+\nabla\varphi_n(x))\,\mathrm{d}x$, hence

$$\widehat{\mathcal{H}}[\mathcal{G}W](\xi)\le \fint_{kY}\mathcal{G}W(x,\xi+\nabla\varphi(x))\,\mathrm{d}x$$
$$\text{for all } k\ge 1 \text{ and all } \varphi\in W_0^{1,p}(kY;\mathbb{R}^m)\,,$$

which gives the result. □

Remark 2.31. When $W(y,\xi)=a(y)H(\xi)$ with $a\in L^1(\mathbb{R}^N;[0,\infty[)$ and $H:\mathbb{M}^{m\times N}\to[0,\infty]$ Borel measurable, we have

Lemma 2.32. *If a is lower semicontinuous (lsc) a.e., H is upper semicontinuous (usc) and $\widehat{\mathcal{Z}}H$ is finite, then $\mathcal{G}W(x,\cdot\,)$ is usc for a.a. $x\in\mathbb{R}^N$.*

Proof. Let $x\in\mathbb{R}^N$. As $a\in L^1(\mathbb{R}^N;[0,\infty[)$, up to a set of zero measure, we can assert that

$$\lim_{\rho\to 0}\fint_{x+\rho U}|a(y)-a(x)|\,\mathrm{d}y = 0 \tag{2.16}$$

for all bounded open set $U \subset \mathbb{R}^N$. Without loss of generality we can also assume that a is lsc at x. Let $\xi \in \mathbb{M}^{m\times N}$ and let $\{\xi_n\}_n \subset \mathbb{M}^{m\times N}$ be such that $|\xi_n - \xi| \to 0$. We have to prove that

$$\varlimsup_{n\to\infty} \mathcal{G}W(x,\xi_n) \leq \mathcal{G}W(x,\xi)\,. \tag{2.17}$$

Fix any $\delta > 0$. By definition of $\widehat{\mathcal{Z}}H(\xi)$, there exists $\varphi_\delta \in \mathrm{Aff}_0(Y;\mathbb{R}^m)$ such that

$$\int_Y H(\xi + \nabla\varphi_\delta(y))\,\mathrm{d}y \leq \widehat{\mathcal{Z}}H(\xi) + \delta\,,$$

and multiplying by $a(x)$ we obtain

$$a(x)\int_Y H(\xi + \nabla\varphi_\delta(y))\,\mathrm{d}y \leq a(x)\widehat{\mathcal{Z}}H(\xi) + a(x)\delta\,. \tag{2.18}$$

Since a is lsc at x, there exists $\rho_\delta > 0$ such that

$$\inf_{y\in Q_{\rho_\delta}(x)} a(y) \geq a(x) + \delta\,. \tag{2.19}$$

Fix any $\rho \in \,]0,\rho_\delta[$. Taking Proposition 3.1 (a) and Remark 3.2 into account, from (2.19) we see that

$$\begin{aligned}
&\frac{\mathcal{S}^W_\xi(Q_\rho(x))}{|Q_\rho(x)|}\\
&\quad = \inf\left\{ \fint_{Q_\rho(x)} a(y)H(\xi+\nabla\varphi(y))\,\mathrm{d}y : \varphi \in \mathrm{Aff}_0(Q_\rho(x);\mathbb{R}^m)\right\}\\
&\quad \geq (a(x)+\delta)\inf\left\{ \fint_{Q_\rho(x)} H(\xi+\nabla\varphi(y))\,\mathrm{d}y : \varphi \in \mathrm{Aff}_0(Q_\rho(x);\mathbb{R}^m)\right\}\\
&\quad = (a(x)+\delta)\widehat{\mathcal{Z}}H(\xi)\,,
\end{aligned}$$

and, recalling that $\widehat{\mathcal{Z}}H(\xi) < \infty$, it follows that

$$a(x)\widehat{\mathcal{Z}}H(\xi) \leq \frac{\mathcal{S}^W_\xi(Q_\rho(x))}{|Q_\rho(x)|} - \delta\widehat{\mathcal{Z}}H(\xi)\,. \tag{2.20}$$

Combining (2.18) with (2.20) we obtain

$$a(x)\int_Y H(\xi + \nabla\varphi_\delta(y))\,\mathrm{d}y \leq \frac{\mathcal{S}^W_\xi(Q_\rho(x))}{|Q_\rho(x)|} + \delta\left(a(x) - \widehat{\mathcal{Z}}H(\xi)\right). \tag{2.21}$$

Since $\varphi_\delta \in \mathrm{Aff}_0(Y;\mathbb{R}^m)$, there exists a finite family $\{U_i\}_{i\in I}$ of open disjoint subsets of Y such that $|\partial U_i| = 0$ for all $i \in I$, $|Y \setminus \bigcup_{i\in I} U_i| = 0$ and for every $i \in I$, $\nabla\varphi_\delta \equiv \zeta_i$ in U_i with $\zeta_i \in \mathbb{M}^{m\times N}$. For each $i \in I$, set $\theta_i := \overline{\lim}_{n\to\infty} H(\xi_n + \zeta_i)$. As I is finite we can assert that there exists $n_\delta \geq 1$ such that

$$H(\xi_n + \zeta_i) \leq \theta_i + \delta \text{ for all } i \in I \text{ and all } n \geq n_\delta.$$

But H is usc and $|\xi_n - \xi| \to 0$, hence $\theta_i \leq H(\xi + \zeta_i)$ for all $i \in I$, and so

$$H(\xi_n + \zeta_i) \leq H(\xi + \zeta_i) + \delta \text{ for all } i \in I \text{ and all } n \geq n_\delta. \tag{2.22}$$

Fix any $n \geq n_\delta$. Using (2.22) we have

$$\begin{aligned} a(x)\int_Y H(\xi_n + \nabla\varphi_\delta(y))\,\mathrm{d}y &= a(x)\sum_{i\in I} H(\xi_n + \zeta_i)|U_i| \\ &\leq a(x)\sum_{i\in I} H(\xi + \zeta_i)|U_i| + a(x)\delta \\ &= a(x)\int_Y H(\xi + \nabla\varphi_\delta(y))\,\mathrm{d}y + a(x)\delta, \end{aligned}$$

and taking (2.21) into account we get

$$a(x)\int_Y H(\xi_n + \nabla\varphi_\delta(y))\,\mathrm{d}y \leq \frac{\mathcal{S}^W_\xi(Q_\rho(x))}{|Q_\rho(x)|} + \delta\left(2a(x) - \widehat{\mathcal{Z}}H(\xi)\right). \tag{2.23}$$

Thus, by using (2.23), we deduce that

$$\begin{aligned} \frac{\mathcal{S}^W_{\xi_n}(Q_\rho(x))}{|Q_\rho(x)|} &\leq \fint_{Q_\rho(x)} a(y) H\left(\xi_n + \nabla\varphi_\delta\left(\frac{y-x}{\rho}\right)\right)\mathrm{d}y \\ &= \fint_{Q_\rho(x)} (a(y) - a(x)) H\left(\xi_n + \nabla\varphi_\delta\left(\frac{y-x}{\rho}\right)\right)\mathrm{d}y \\ &\quad + a(x)\fint_{Q_\rho(x)} H\left(\xi_n + \nabla\varphi_\delta\left(\frac{y-x}{\rho}\right)\right)\mathrm{d}y \\ &= \int_Y (a(x+\rho y) - a(x)) H(\xi_n + \nabla\varphi_\delta(y))\,\mathrm{d}y \\ &\quad + a(x)\int_Y H(\xi_n + \nabla\varphi_\delta(y))\,\mathrm{d}y \\ &\leq \sum_{i\in I} H(\xi_n + \zeta_i)|x + \rho U_i| \fint_{x+\rho U_i} |a(y) - a(x)|\,\mathrm{d}y \\ &\quad + \frac{\mathcal{S}^W_\xi(Q_\rho(x))}{|Q_\rho(x)|} + \delta\left(2a(x) - \widehat{\mathcal{Z}}H(\xi)\right). \end{aligned}$$

for all $\rho \in]0, \rho_\delta[$ and all $n \geq n_\delta$. Taking (2.16) into account, by letting $\rho \to 0$ and then $n \to \infty$ we conclude that

$$\begin{aligned}\overline{\lim_{n\to\infty}}\, \mathcal{G}W(x,\xi_n) &= \overline{\lim_{n\to\infty}}\, \overline{\lim_{\rho\to 0}}\, \frac{\mathcal{S}^W_{\xi_n}(Q_\rho(x))}{|Q_\rho(x)|} \\ &\leq \overline{\lim_{\rho\to 0}}\, \frac{\mathcal{S}^W_{\xi}(Q_\rho(x))}{|Q_\rho(x)|} + \delta\left(2a(x) - \widehat{\mathcal{Z}}H(\xi)\right) \\ &= \mathcal{G}W(x,\xi) + \delta\left(2a(x) - \widehat{\mathcal{Z}}H(\xi)\right),\end{aligned}$$

and (2.17) follows by letting $\delta \to 0$. $\square$

Here is the third homogenization result of the paper.

Theorem 2.33. *If $W \in \mathcal{K}^p_{\rm per}$ then* (2.2) *holds with $W_{\rm hom} = \mathcal{H}[\mathcal{G}W]$.*

Let us set $\mathcal{H}_{\rm usc} := \{H \in \mathcal{H} : H \text{ is usc}\}$ and let us introduce the following class of integrands.

Definition 2.34. Let $\mathcal{S}_4$ be the class of Borel measurable functions $W : \mathbb{R}^N \times \mathbb{M}^{N\times N} \to [0,\infty]$ for which there exist $H \in \mathcal{H}_{\rm usc}$ and $a \in L^\infty(\mathbb{R}^N; [0,\infty[)$ such that a is lsc and 1-periodic, $a(x) \geq \eta$ for all $x \in \mathbb{R}^N$ and some $\eta > 0$, and W is defined by

$$W(x,\xi) = a(x)H(\xi). \tag{2.24}$$

Here is the link between $\mathcal{S}_4$ and $\mathcal{K}^p_{\rm per}$.

Lemma 2.35. *Let $m = N$. The class $\mathcal{S}_4$ is a subclass of $\mathcal{K}^p_{\rm per}$, i.e., $\mathcal{S}_4 \subset \mathcal{K}^p_{\rm per}$.*

Proof. Let $W \in \mathcal{S}_4$. Then, by definition, W is given by (2.24) with $H \in \mathcal{H}_{\rm usc}$ and $a \in L^\infty(\mathbb{R}^N; [0,\infty[)$ which is lsc and 1-periodic and such that $a(\cdot) \geq \eta > 0$. It is thus clear that (A_0) and (A_1) are verified. So it remains to prove that (A_5), (A_6) and (A_7) hold. Firstly, since $H \in \mathcal{H}_{\rm usc} \subset \mathcal{H}$, by Theorem A.4 we deduce that $\widehat{\mathcal{Z}}H$ has p-growth. Hence (A_5) holds because $a \in L^\infty(\mathbb{R}^N; [0,\infty[)$. Secondly, by Remark 2.28 we can assert that (A_6) is satisfied. Finally, $\mathcal{G}W$ has p-growth because (A_5) is verified and, since a is lsc, H is usc and $\widehat{\mathcal{Z}}H$ is finite, we can assert that $\mathcal{G}W(x,\cdot)$ is usc for a.a. $x \in \mathbb{R}^N$, see Remark 2.31 and Lemma 2.32. Consequently, (A_7) holds by using Remark 2.29 and Lemma 2.30. $\square$

As a direct consequence of Theorem 2.33 and Lemma 2.35 we have the following result.

Corollary 2.36. *Let $m = N$. If $W \in \mathcal{S}_4$ then* (2.2) *holds with $W_{\mathrm{hom}} = \mathcal{H}[\mathcal{G}W]$.*

Remark 2.37 (link between the almost continuous case and the non-continuous one)*.* In Theorems 2.8 and 2.19 and Corollaries 2.13, 2.17 and 2.22, the homogenized density W_{hom} is given by Braides–Müller's homogenization formula, i.e., $W_{\mathrm{hom}} = \mathcal{H}W$. On the other hand, in Theorem 2.33 and Corollary 2.36, the homogenized formula W_{hom} is given by another formula, i.e., $W_{\mathrm{hom}} = \mathcal{H}[\mathcal{G}W]$, which is a priori different from the classical one by Braides–Müller. To make clear the link between these two formulas, we begin with the following proposition whose proof is given below.

Proposition 2.38. *If W satisfies* (A_4) *then:*

(a) $\mathcal{S}_\xi^W(U) \le \displaystyle\int_U \widehat{\mathcal{Z}}W(y,\xi)\,\mathrm{d}y$ *for all $\xi \in \mathbb{M}^{m\times N}$ and all $U \in \mathcal{O}_{\mathrm{b}}(\mathbb{R}^N)$;*

(b) $\widehat{\mathcal{Z}}W(x,\xi) \le \mathcal{G}W(x,\xi)$ *for a.a. $x \in \mathbb{R}^N$ and all $\xi \in \mathbb{M}^{m\times N}$,*

where $\widehat{\mathcal{Z}}W : \mathbb{R}^N \times \mathbb{M}^{m\times N} \to [0,\infty]$ is defined by

$$\widehat{\mathcal{Z}}W(x,\xi) := \inf\left\{\int_Y W(x,\xi+\nabla\varphi(y))\,\mathrm{d}y : \varphi \in \mathrm{Aff}_0(Y;\mathbb{R}^m)\right\}. \tag{2.25}$$

As a consequence, we have

Corollary 2.39. *Assume that W satisfies* (A_4) *and the following condition:*

($\widehat{\mathrm{A}}_3$) *$\widehat{\mathcal{Z}}W$ has p-growth, i.e., $\widehat{\mathcal{Z}}W(x,\xi) \le c(1+|\xi|^p)$ for all $(x,\xi) \in \mathbb{R}^N \times \mathbb{M}^{m\times N}$ and some $c > 0$.*

Then $\mathcal{G}W(x,\xi) = \widehat{\mathcal{Z}}W(x,\xi)$ for all $\xi \in \mathbb{M}^{m\times N}$ and a.a. $x \in \mathbb{R}^N$.

Proof. Taking Proposition 2.38 (b) into account, we see that it suffices to prove that $\mathcal{G}W(x,\xi) \le \widehat{\mathcal{Z}}W(x,\xi)$ for all $\xi \in \mathbb{M}^{m\times N}$ and a.a. $x \in \mathbb{R}^N$. First of all, by ($\widehat{\mathrm{A}}_3$), $\widehat{\mathcal{Z}}W$ has p-growth, and so $\widehat{\mathcal{Z}}W(\,\cdot\,,\xi) \in L^1(\mathbb{R}^N)$ for all $\xi \in \mathbb{M}^{m\times N}$. Hence, for every $\xi \in \mathbb{M}^{m\times N}$ and a.a. $x \in \mathbb{R}^N$,

$$\lim_{\rho\to 0} \fint_{Q_\rho(x)} \widehat{\mathcal{Z}}W(y,\xi)\,\mathrm{d}y = \widehat{\mathcal{Z}}W(x,\xi). \tag{2.26}$$

On the other hand, by Proposition 2.38 (a) we see that for every $x \in \mathbb{R}^N$ and every $\xi \in \mathbb{M}^{m\times N}$,

$$\mathcal{G}W(x,\xi) = \overline{\lim_{\rho\to 0}} \frac{\mathcal{S}^W_\xi(Q_\rho(x))}{|Q_\rho(x)|} \leq \lim_{\rho\to 0} \fint_{Q_\rho(x)} \widehat{\mathcal{Z}}W(y,\xi)\,\mathrm{d}y\,,$$

which gives the desired inequality when combining with (2.26). □

Definition 2.40. We denote by $\widehat{\mathcal{I}}^p$ the class of Borel measurable functions $W : \mathbb{R}^N \times \mathbb{M}^{m\times N} \to [0,\infty]$ satisfying (A_0), ($\widehat{\mathrm{A}}_3$) and (A_4). (Note that ($\widehat{\mathrm{A}}_3$) implies (A_3) and so $\widehat{\mathcal{I}}^p \subset \mathcal{I}^p$.)

Proposition 2.41. *If $W \in \widehat{\mathcal{I}}^p$ then $\mathcal{H}[\mathcal{G}W] = \mathcal{H}W$.*

Proof. From Corollary 2.39 we have $\mathcal{G}W(x,\xi) = \widehat{\mathcal{Z}}W(x,\xi)$ for all $\xi \in \mathbb{M}^{m\times N}$ and a.a. $x \in \mathbb{R}^N$, and so $\mathcal{H}[\mathcal{G}W] = \mathcal{H}[\widehat{\mathcal{Z}}W]$. But $\mathcal{H}[\widehat{\mathcal{Z}}W] = \mathcal{H}W$ by Lemma 4.4, and the result follows. □

Let us set $\widehat{\mathcal{I}}^p_{\mathrm{per}} := \{W \in \widehat{\mathcal{I}}^p : W \text{ satisifes } (\mathrm{A}_0)\}$. From the above we have

Lemma 2.42. *The class $\widehat{\mathcal{I}}^p_{\mathrm{per}}$ is a subclass of $\mathcal{K}^p_{\mathrm{per}}$, i.e., $\widehat{\mathcal{I}}^p_{\mathrm{per}} \subset \mathcal{K}^p_{\mathrm{per}}$.*

Proof. Let $W \in \widehat{\mathcal{I}}^p_{\mathrm{per}}$. It suffices to show that W satisfies (A_5), (A_6) and (A_7). Since W verifies ($\widehat{\mathrm{A}}_3$), $\widehat{\mathcal{Z}}W$ has p-growth, and so (A_5) holds by Proposition 2.38 (a). By Corollary 2.39 we have $\mathcal{G}W = \widehat{\mathcal{Z}}W$, hence ($\mathrm{A}_6$) is satsified because $\widehat{\mathcal{Z}}W \leq W$. From Proposition 3.1 (c) and Remark 3.2 we can assert that $\widehat{\mathcal{Z}}W = \mathcal{G}W$ is continuous with respect to the matrix variable. Hence $\widehat{\mathcal{H}}[\mathcal{G}W] = \mathcal{H}[\mathcal{G}W]$ by Lemma 2.30, which implies that (A_7) holds. □

As a direct consequence of Theorem 2.33 and Proposition 2.41 we have

Corollary 2.43. *If $W \in \widehat{\mathcal{I}}^p_{\mathrm{per}}$ then* (2.2) *holds with $W_{\mathrm{hom}} = \mathcal{H}W$.*

To conclude Remark 2.37, here is the proof of Proposition 2.38.

Proof of Proposition 2.38. Assertion (a) is just Lemma 3.14 (see Remark 3.12). So, we only need to prove (b). Let $\lambda \in \mathcal{L}$ be given by (A_4), where the class $\mathcal{L}$ is defined at p. 142. Let $(x,\xi) \in \mathbb{R}^N \times \mathbb{M}^{m\times N}$. By Lemma 2.5, up to a set of zero measure, we can assert that λ is continuous at x. So, there exists $\{\rho_\delta\}_{\delta>0} \subset\,]0,\infty[$ such that $\rho_\delta \to 0$ as $\delta \to 0$ and

$$\text{for each } \delta > 0,\ \text{if } y \in Q_{\rho_\delta}(x) \text{ then } |\lambda(x) - \lambda(y)| \leq \delta\,. \tag{2.27}$$

Fix any $\delta > 0$ and set $\theta_\delta := \frac{\mathcal{S}_\xi^W(Q_{\rho_\delta}(x))}{|Q_{\rho_\delta}(x)|}$. Then

$$\varlimsup_{\delta\to 0}\theta_\delta \leq \varlimsup_{\rho\to 0}\frac{\mathcal{S}_\xi^W(Q_\rho(x))}{|Q_\rho(x)|} = \mathcal{G}W(x,\xi). \tag{2.28}$$

Let $\varphi_\delta \in \mathrm{Aff}_0(Q_{\rho_\delta}(x);\mathbb{R}^m)$ be such that

$$\fint_{Q_{\rho_\delta}(x)} W(y,\xi+\nabla\varphi_\delta(y))\,\mathrm{d}y \leq \theta_\delta + \delta. \tag{2.29}$$

Taking (2.27) into account, from (A_4) we see that

$$\begin{aligned} &W(x,\xi+\nabla\varphi_\delta(y)) \\ &\qquad\leq |\lambda(x)-\lambda(y)|(1+W(y,\xi+\nabla\varphi_\delta(y)) + W(y,\xi+\nabla\varphi_\delta(y)) \\ &\qquad\leq \delta(1+W(y,\xi+\nabla\varphi_\delta(y)) + W(y,\xi+\nabla\varphi_\delta(y)) \end{aligned}$$

for all $y \in Q_{\rho_\delta}(x)$, and so, by using (2.29), we get

$$\fint_{Q_{\rho_\delta}(x)} W(x,\xi+\nabla\varphi_\delta(y))\,\mathrm{d}y \leq \delta^2+\delta+(\delta+1)\theta_\delta.$$

Moreover, by using Proposition 3.1 (a) and Remark 3.2, we have

$$\begin{aligned} \widehat{\mathcal{Z}}W(x,\xi) &= \inf\left\{\fint_{Q_{\rho_\delta}(x)} W(x,\xi+\nabla\varphi(y))\,\mathrm{d}y : \varphi\in\mathrm{Aff}_0\left(Q_{\rho_\delta}(x);\mathbb{R}^m\right)\right\} \\ &\leq \fint_{Q_{\rho_\delta}(x)} W(x,\xi+\nabla\varphi_\delta(y))\,\mathrm{d}y. \end{aligned}$$

Hence, for every $\delta > 0$,

$$\widehat{\mathcal{Z}}W(x,\xi) \leq \delta^2+\delta+(\delta+1)\theta_\delta,$$

which gives $\widehat{\mathcal{Z}}W(x,\xi) \leq \mathcal{G}W(x,\xi)$ by letting $\delta\to 0$ and using (2.28). □

3. Auxiliary relaxation theorems

Given a Borel measurable function $W : \mathbb{R}^N \times \mathbb{M}^{m\times N} \to [0,\infty]$, where $\mathbb{M}^{m\times N}$ denotes the space of real $m\times N$ matrices with $m, N \geq 1$ two integers, we consider $\mathcal{Z}W : \mathbb{R}^N \times \mathbb{M}^{m\times N} \to [0,\infty]$ defined by

$$\mathcal{Z}W(x,\xi) := \inf\left\{\int_Y W(x,\xi+\nabla\varphi(y))\,\mathrm{d}y : \varphi \in W_0^{1,\infty}(Y;\mathbb{R}^m)\right\}$$

with $Y :=]-\frac{1}{2}, \frac{1}{2}[^N$. The following result is due to Fonseca (see [22, lemma 2.16, Theorem 2.17 and Proposition 2.3]).

Proposition 3.1. *The function $\mathcal{Z}W$ satisfies the following properties.*

(a) *For every bounded open set $U \subset \mathbb{R}^N$ with $|\partial U| = 0$ and every $(x,\xi) \in \mathbb{R}^N \times \mathbb{M}^{m\times N}$,*

$$\mathcal{Z}W(x,\xi) = \inf\left\{ \fint_U W(x, \xi + \nabla\varphi(y))\,\mathrm{d}y : \varphi \in W_0^{1,\infty}(U;\mathbb{R}^m)\right\} := \mathcal{Z}_U W(x,\xi).$$

More precisely, we have $\mathcal{Z}_U W \le \mathcal{Z}W$ for all bounded open set $U \subset \mathbb{R}^N$, and $\mathcal{Z}W \le \mathcal{Z}_U W$ for all bounded open set $U \subset \mathbb{R}^N$ with $|\partial U| = 0$.

(b) *For every $x \in \mathbb{R}^N$, if $\mathcal{Z}W(x,\cdot)$ is finite then $\mathcal{Z}W(x,\cdot)$ is rank-one convex, i.e., for every $\xi, \xi' \in \mathbb{M}^{m\times N}$ with* $\operatorname{rank}(\xi - \xi') \le 1$,

$$\mathcal{Z}W(x, \lambda\xi + (1-\lambda)\xi') \le \lambda \mathcal{Z}W(x,\xi) + (1-\lambda)\mathcal{Z}W(x,\xi').$$

(c) *For every $x \in \mathbb{R}^N$, if $\mathcal{Z}W(x,\cdot)$ is finite then $\mathcal{Z}W(x,\cdot)$ is continuous, i.e., $\mathcal{Z}W$ is a Carathéodory integrand*[(1)] *whenever $\mathcal{Z}W$ is finite.*

(d) *For every bounded open set $U \subset \mathbb{R}^N$ with $|\partial U| = 0$, every $(x,\xi) \in \mathbb{R}^N \times \mathbb{M}^{m\times N}$ and every $\varphi \in \mathrm{Aff}_0(U;\mathbb{R}^m)$,*

$$\mathcal{Z}W(x,\xi) \le \fint_U \mathcal{Z}W(x, \xi + \nabla\varphi(y))\,\mathrm{d}y.$$

Remark 3.2. Proposition 3.1 is also valid with "$\widehat{\mathcal{Z}}W$" instead of "$\mathcal{Z}W$" (see [5, Proposition 2.3]) where $\widehat{\mathcal{Z}}W : \mathbb{R}^N \times \mathbb{M}^{m\times N} \to [0,\infty]$ is given by

$$\widehat{\mathcal{Z}}W(x,\xi) := \inf\left\{ \int_Y W(x, \xi + \nabla\varphi(y))\,\mathrm{d}y : \varphi \in \mathrm{Aff}_0(Y;\mathbb{R}^m)\right\}.$$

In particular, Proposition 3.1 (d) can be rewritten as $\widehat{\mathcal{Z}}[\mathcal{Z}W] = \mathcal{Z}W$.

---

(1) A function $W : \mathbb{R}^N \times \mathbb{M}^{m\times N} \to [0,\infty]$ is called a Carathéodory integrand if $W(x,\xi)$ is measurable in x and continuous in ξ.

Given $x \in \mathbb{R}^N$ we say that $W(x,\cdot)$ is quasiconvex (in the sense of Morrey [23]) if for every $\xi \in \mathbb{M}^{m\times N}$, every bounded open set $U \subset \mathbb{R}^N$ with $|\partial U| = 0$ and every $\varphi \in W_0^{1,\infty}(U;\mathbb{R}^m)$,

$$W(x,\xi) \leq \fint_U W(x,\xi + \nabla\varphi(y))\,\mathrm{d}y.$$

By the quasiconvex envelope of $W(x,\cdot)$, that we denote by $\mathcal{Q}W(x,\cdot)$, we mean the greatest quasiconvex function which less than or equal to $W(x,\cdot)$. (Clearly, $W(x,\cdot)$ is quasiconvex if and only if $\mathcal{Q}W(x,\cdot) = W(x,\cdot)$.) The concept of quasiconvex envelope was introduced by Dacorogna (see [17]) who proved the following theorem (see [18, Theorem 6.9]).

Theorem 3.3. *If W is finite then $\mathcal{Q}W = \widehat{\mathcal{Z}}W = \mathcal{Z}W$.*

The following result is a slight generalization of Theorem 3.3 (see [2]).

Theorem 3.4. *If $\mathcal{Z}W$ is finite then $\mathcal{Q}W = \mathcal{Z}W$. In particular, $\mathcal{Z}W(x,\cdot)$ is quasiconvex for all $x \in \mathbb{R}^N$.*

Remark 3.5. Theorem 3.3 can be also generalized as follows: if $\widehat{\mathcal{Z}}W$ is finite then $\mathcal{Q}W = \widehat{\mathcal{Z}}W = \mathcal{Z}W$ (see [5, Corollaire 2.17]).

Let $p > 1$ and let $U \subset \mathbb{R}^N$ be a bounded open set such that $|\partial U| = 0$. Let us define $E : W^{1,p}(U;\mathbb{R}^m) \to [0,\infty]$ by

$$E(\phi) := \int_U W(x,\nabla\phi(x))\,\mathrm{d}x$$

and let us consider the relaxed functionals $\overline{E}, \overline{E}_0 : W^{1,p}(U;\mathbb{R}^m) \to [0,\infty]$ given by:

$$\overline{E}(\phi) := \inf\left\{\varliminf_{n\to\infty} E(\phi_n) : \phi_n \to \phi \text{ in } L^p(U;\mathbb{R}^m)\right\};$$

$$\overline{E}_0(\phi) := \inf\left\{\varliminf_{n\to\infty} E(\phi_n) : W_0^{1,p}(U;\mathbb{R}^m) \ni \phi_n \to \phi \text{ in } L^p(U;\mathbb{R}^m)\right\}$$

with $W_0^{1,p}(U;\mathbb{R}^m) := \{\phi \in W^{1,p}(U;\mathbb{R}^m) : \phi = 0 \text{ on } \partial U\}$. As $\overline{E}$ and $\overline{E}_0$ are not given by explicit formulas, it is of interest to know under which conditions on W we have:

$$\overline{E}(\phi) = \int_U \overline{W}(x,\nabla\phi(x))\,\mathrm{d}x \text{ for all } \phi \in W^{1,p}(U;\mathbb{R}^m); \tag{3.1}$$

$$\overline{E}_0(\phi) = \begin{cases}\overline{E}(\phi) & \text{if } \phi \in W_0^{1,p}(U;\mathbb{R}^m)\\ \infty & \text{otherwise}\end{cases} \tag{3.2}$$

with $\overline{W}:\mathbb{R}^N\times\mathbb{M}^{m\times N}\to[0,\infty]$ whose we wish to give a representation formula. When W has p-growth, such integral representation problems was studied by Dacorogna (see [17, Theorem 5], see also [18, Theorem 9.1]) and Acerbi and Fusco (see [1, Statement III.7]) who proved the following theorem.

Theorem 3.6. *Under* (A_0) *if in addition W is a Carathéodory integrand having p-growth then* (3.1) *and* (3.2) *hold with $\overline{W}=\mathcal{Z}W=\mathcal{Q}W$. If moreover $W(x,.)$ is quasiconvex for all $x\in\mathbb{R}^N$ then $\overline{W}=W$.*

Because of the p-growth assumption on the integrand W, Acerbi–Dacorogna–Fusco's relaxation theorem cannot handle integrands having a singular behavior of type (1.2) when $m=N$.

3.1. Relaxation with singular integrands which are continuous almost everywhere with respect to the space variable

In [2] it was proved the following relaxation theorem whose distinguishing feature is to be consistent with (1.2).

Theorem 3.7. *Under* (A_0) *if in addition W satisfies* (A_2) *and* (A_3) *then* (3.1) *and* (3.2) *hold with $\overline{W}=\mathcal{Z}W=\mathcal{Q}W$.*

Theorem 3.7 was used in [2] to establish Theorem 2.3. However, due to the assumption (A_2), in Theorem 3.7 the integrand W is necessarily continuous with respect to its first variable, and so this latter theorem cannot be used to prove Theorem 2.8. The following relaxation theorem improves Theorem 3.7 by allowing to the integrand W not to be necessarily continuous with respect to its first variable and will play an essential role in the proof of Theorem 2.8.

Theorem 3.8. *If $W\in\mathcal{I}^p$ then* (3.1) *and* (3.2) *hold with $\overline{W}=\mathcal{Z}W=\mathcal{Q}W$.*

Proof. Let $\mathcal{Z}E:W^{1,p}(U;\mathbb{R}^m)\to[0,\infty]$ be defined by

$$\mathcal{Z}E(\phi):=\int_U \mathcal{Z}W(x,\nabla\phi(x))\,\mathrm{d}x \tag{3.3}$$

and let $\overline{\mathcal{Z}E}, \overline{\mathcal{Z}E}_0 : W^{1,p}(U;\mathbb{R}^m) \to [0,\infty]$ be given by:

$$\overline{\mathcal{Z}E}(\phi) := \inf\left\{\varliminf_{n\to\infty} \mathcal{Z}E(\phi_n) : \phi_n \to \phi \text{ in } L^p(U;\mathbb{R}^m)\right\}; \tag{3.4}$$

$$\overline{\mathcal{Z}E}_0(\phi) := \inf\left\{\varliminf_{n\to\infty} \mathcal{Z}E(\phi_n) : W_0^{1,p}(U;\mathbb{R}^m) \ni \phi_n \to \phi \text{ in } L^p(U;\mathbb{R}^m)\right\}. \tag{3.5}$$

We need the following lemma whose proof is given below.

Lemma 3.9. *Under* (A$_4$) *if* $\phi \in \mathrm{Aff}(U;\mathbb{R}^m)$ *(resp.* $\phi \in \mathrm{Aff}_0(U;\mathbb{R}^m)$*) then*

$$\overline{E}(\phi) \le \int_U \mathcal{Z}W\left(x, \nabla\phi(x)\right) \mathrm{d}x$$
$$\left(\textit{resp. } \overline{E}_0(\phi) \le \int_U \mathcal{Z}W(x, \nabla\phi(x))\, \mathrm{d}x\right). \tag{3.6}$$

As $\mathcal{Z}W$ has p-growth and $\mathrm{Aff}(U;\mathbb{R}^m)$ (resp. $\mathrm{Aff}_0(U;\mathbb{R}^m)$) is strongly dense in $W^{1,p}(U;\mathbb{R}^m)$ (resp. $W_0^{1,p}(U;\mathbb{R}^m)$), from Lemma 3.9 we deduce that (3.6) holds for all $\phi \in W^{1,p}(U;\mathbb{R}^m)$ (resp. $\phi \in W_0^{1,p}(U;\mathbb{R}^m)$). Thus $\overline{E} \le \overline{\mathcal{Z}E}$ (resp. $\overline{E}_0 \le \overline{\mathcal{Z}E}_0$). Moreover, $\overline{\mathcal{Z}E} \le \overline{E}$ (resp. $\overline{\mathcal{Z}E}_0 \le \overline{E}_0$), hence

$$\overline{E} = \overline{\mathcal{Z}E} \quad \left(\text{resp. } \overline{E}_0 = \overline{\mathcal{Z}E}_0\right). \tag{3.7}$$

Since W is p-coercive, also is $\mathcal{Z}W$. Moreover, as $\mathcal{Z}W$ is finite (because (A$_3$) holds) we see that $\mathcal{Z}W$ is a Carathéodory integrand by Proposition 3.1 (c) and $\mathcal{Z}W(x,\cdot\,)$ is quasiconvex for all $x \in \mathbb{R}^N$ by Theorem 3.4. From Acerbi–Dacorogna–Fusco's relaxation theorem (see Theorem 3.6) it follows that

$$\overline{\mathcal{Z}E} = \mathcal{Z}E \quad \left(\text{resp. } \overline{\mathcal{Z}E}_0 = \begin{cases} \mathcal{Z}E & \text{on } W_0^{1,p}(U;\mathbb{R}^m) \\ \infty & \text{elsewhere} \end{cases}\right)$$

which gives the theorem when combined with (3.7). □

Proof of Lemma 3.9. Let $\phi \in \mathrm{Aff}(U;\mathbb{R}^m)$ (resp. $\phi \in \mathrm{Aff}_0(U;\mathbb{R}^m)$). Without loss of generality we can assume that

$$\int_U \mathcal{Z}W(x, \nabla\phi(x))\, \mathrm{d}x < \infty. \tag{3.8}$$

By definition, there exists a finite family $\{U_i\}_{i\in I}$ of open disjoint subsets of U such that $|\partial U_i| = 0$ for all $i \in I$, $|U \setminus \bigcup_{i\in I} U_i| = 0$ and, for every

$i \in I$, $\nabla\phi \equiv \xi_i$ in U_i with $\xi_i \in \mathbb{M}^{m\times N}$. Thus

$$\int_U \mathcal{Z}W(x, \nabla\phi(x))\,\mathrm{d}x = \sum_{i\in I}\int_{U_i} \mathcal{Z}W(x,\xi_i)\,\mathrm{d}x\,. \tag{3.9}$$

In particular, from (3.8) and Remark 2.7 (d) we see that $\mathcal{Z}W(x,\xi_i) < \infty$ for all $i \in I$ and all $x \in \mathbb{R}^N$. Let $\lambda \in \mathcal{L}$ be given by (A_4). Then, for each $i \in I$ and each $\delta \in\,]0,\delta_0]$ with $\delta_0 > 0$ small enough, there exists a compact $K_{i,\delta} \subset \overline{U_i}$ such that

$$\begin{cases} |\partial K_{i,\delta}| = 0 \\ |\overline{U_i} \setminus K_{i,\delta}| < \delta \\ \lambda|_{K_{i,\delta}} \text{ is continuous.}\end{cases}$$

Fix any $\delta \in\,]0,\delta_0]$. By Remark 2.7 (d) we see that for every $i \in I$, $\mathcal{Z}W(\,\cdot\,,\xi_i)|_{K_{i,\delta}}$ is continuous and so, since $\overline{\mathrm{int}(K_{i,\delta})} \subset K_{i,\delta}$, $\mathcal{Z}W(\,\cdot\,,\xi_i)|_{\overline{\mathrm{int}(K_{i,\delta})}}$ is Riemann integrable, where $\mathrm{int}(K_{i,\delta})$ denotes the interior of $K_{i,\delta}$ (2). Hence, for each $i \in I$ and each $k \geq 1$, there exists a finite family $\{U_{i,j}^k\}_{j\in J_i^k}$ of disjoint subsets of $\mathrm{int}(K_{i,\delta})$ with $|\partial U_{i,j}^k| = 0$ for all $j \in J_i^k$ and $|\,\mathrm{int}(K_{i,\delta}) \setminus \bigcup_{j\in J_i^k} U_{i,j}^k| = 0$ such that:

$$\mathrm{diam}(U_{i,j}^k) < \frac{1}{k} \text{ for all } j \in J_i^k\,; \tag{3.10}$$

$$\lim_{k\to\infty}\sum_{j\in J_i^k} |U_{i,j}^k|\mathcal{Z}W(x_{i,j}^k,\xi_i) = \int_{\mathrm{int}(K_{i,\delta})} \mathcal{Z}W(x,\xi_i)\,\mathrm{d}x\,, \tag{3.11}$$

where $x_{i,j}^k \in U_{i,j}^k$ and, for $X \subset \mathbb{R}^N$, $\mathrm{diam}(X) := \sup\{|x_1 - x_2| : x_1, x_2 \in X\}$. On the other hand, as for every $i \in I$, $\lambda|_{K_{i,\delta}}$ is continuous with I finite and $K_{i,\delta}$ compact, we deduce that there exists $\eta > 0$ such that for every $i \in I$,

$$\text{if } x, y \in K_{i,\delta} \text{ and } |x-y| < \eta \text{ then } |\lambda(x) - \lambda(y)| < \delta\,. \tag{3.12}$$

(2) For every $i \in I$, $\mathrm{int}(K_{i,\delta}) \neq \emptyset$ because $|\partial K_{i,\delta}| = 0$ and $|\overline{U_i} \setminus K_{i,\delta}| < \delta$ where without loss of generality we can assume that $\delta < \min_{i\in I} |\overline{U}_i|$.

Fix any $k > \frac{1}{\eta}$. As $\mathcal{Z}W(x, \xi_i) < \infty$ for all $i \in I$ and all $x \in \mathbb{R}^N$, for each $i \in I$ and each $j \in J_i^k$, there exist $\varphi_{i,j}, \widehat{\varphi}_i \in W_0^{1,\infty}(Y; \mathbb{R}^m)$ such that:

$$\int_Y W(x_{i,j}^k, \xi_i + \nabla\varphi_{i,j}(y))\,\mathrm{d}y \leq \mathcal{Z}W(x_{i,j}^k, \xi_i) + \delta\,; \tag{3.13}$$

$$\int_Y W(0, \xi_i + \nabla\widehat{\varphi}_i(y))\,\mathrm{d}y \leq \mathcal{Z}W(0, \xi_i) + \delta\,. \tag{3.14}$$

For every $n \geq 1$, from Vitali's covering theorem we can assert that:

- there exists a finite or countable family $\{Y_{i,j,\ell} := a_{i,j,\ell} + \alpha_{i,j,\ell} Y\}_{\ell \in L_{i,j}}$ of disjoint subsets of $U_{i,j}^k$ with $a_{i,j,\ell} \in \mathbb{R}^N$ and $0 < \alpha_{i,j,\ell} < \frac{1}{n}$ such that $|U_{i,j}^k \setminus \bigcup_{\ell \in L_{i,j}} Y_{i,j,\ell}| = 0$ (and so $\sum_{\ell \in L_{i,j}} \alpha_{i,j,\ell}^N = |U_{i,j}^k|$);

- there exists a finite or countable family $\{\widehat{Y}_{i,q} := \widehat{a}_{i,q} + \widehat{\alpha}_{i,q} Y\}_{q \in Q_i}$ of disjoint subsets of $U_i \setminus K_{i,\delta}$ with $\widehat{a}_{i,q} \in \mathbb{R}^N$ and $0 < \widehat{\alpha}_{i,q} < \frac{1}{n}$ such that $|(U_i \setminus K_{i,\delta}) \setminus \bigcup_{q \in Q_i} \widehat{Y}_{i,\ell}| = 0$ (and so $\sum_{q \in Q_i} \widehat{\alpha}_{i,q}^N = |U_i \setminus K_{i,\delta}| = |\overline{U_i} \setminus K_{i,\delta}| < \delta$).

Since $|\partial K_{i,\delta}| = 0$ for all $i \in I$, we can define $\{\psi_n\}_{n \geq 1} \subset W_0^{1,\infty}(U; \mathbb{R}^m)$ by

$$\psi_n(x) := \begin{cases} \alpha_{i,j,\ell}\varphi_{i,j}\left(\dfrac{x - a_{i,j,\ell}}{\alpha_{i,j,\ell}}\right) & \text{if } x \in Y_{i,j,\ell} \\ \widehat{\alpha}_{i,q}\widehat{\varphi}_i\left(\dfrac{x - \widehat{a}_{i,q}}{\widehat{\alpha}_{i,q}}\right) & \text{if } x \in \widehat{Y}_{i,q}. \end{cases} \tag{3.15}$$

It is then easy to see that

$$\|\psi_n\|_{L^\infty(U;\mathbb{R}^m)} \leq \frac{1}{n} \max_{\substack{i \in I \\ j \in J_i^k}} \left\{ \|\varphi_{i,j}\|_{L^\infty(Y;\mathbb{R}^m)}, \|\widehat{\varphi}_i\|_{L^\infty(Y;\mathbb{R}^m)} \right\}$$

for all $n \geq 1$, and so $\psi_n \to 0$ in $L^\infty(U; \mathbb{R}^m)$. Thus $\{\phi + \psi_n\}_{n \geq 1} \subset W^{1,p}(U; \mathbb{R}^m)$ (resp. $\{\phi + \psi_n\}_{n \geq 1} \subset W_0^{1,p}(U; \mathbb{R}^m)$) and $\phi + \psi_n \to \phi$ in $L^p(U; \mathbb{R}^m)$. Hence, to prove (3.6) it is sufficient to show that for every $n \geq 1$,

$$\int_U W(x, \nabla\phi(x) + \nabla\psi_n(x))\,\mathrm{d}x \leq \int_U \mathcal{Z}W(x, \nabla\phi(x))\,\mathrm{d}x\,. \tag{3.16}$$

Let $n \geq 1$. Using the fact that $|\partial K_{i,\delta}| = 0$ for all $i \in I$ we see that

$$\begin{aligned}
\int_U W(x, \nabla\phi(x)+\nabla\psi_n(x))\,\mathrm{d}x
&= \sum_{i\in I}\int_{U_i} W(x,\xi_i+\nabla\psi_n(x))\,\mathrm{d}x\\
&= \sum_{i\in I}\int_{U_i\setminus K_{i,\delta}} W(x,\xi_i+\nabla\psi_n(x))\,\mathrm{d}x\\
&\quad + \sum_{i\in I}\int_{\mathrm{int}(K_{i,\delta})} W(x,\xi_i+\nabla\psi_n(x))\,\mathrm{d}x\\
&= \sum_{i\in I}\int_{U_i\setminus K_{i,\delta}} W(x,\xi_i+\nabla\psi_n(x))\,\mathrm{d}x\\
&\quad + \sum_{i\in I}\sum_{j\in J_i^k}\int_{U_{i,j}^k} W(x,\xi_i+\nabla\psi_n(x))\,\mathrm{d}x. \qquad (3.17)
\end{aligned}$$

Using (A_4) we see that for every $i \in I$,

$$\begin{aligned}
W(x,\xi_i+\nabla\psi_n(x)) &\leq |\lambda(x)-\lambda(0)|\big(1+W(0,\xi_i+\nabla\psi_n(x))\big)\\
&\quad + W(0,\xi_i+\nabla\psi_n(x))\\
&\leq 2\|\lambda\|_{L^\infty} + (2\|\lambda\|_{L^\infty}+1)\,W(0,\xi_i+\nabla\psi_n(x))
\end{aligned}$$

for all $x \in \mathbb{R}^N$, and taking (3.15) and (3.14) into account we deduce that

$$\begin{aligned}
&\int_{U_i\setminus K_{i,\delta}} W(x,\xi_i+\nabla\psi_n(x))\,\mathrm{d}x\\
&\quad\leq 2\|\lambda\|_{L^\infty}|U_i\setminus K_{i,\delta}|\\
&\qquad + (2\|\lambda\|_{L^\infty}+1)\sum_{q\in Q_i}\int_{\widehat{Y}_{i,q}} W\left(0,\xi_i+\nabla\widehat{\varphi}_i\left(\frac{x-\widehat{a}_{i,q}}{\widehat{\alpha}_{i,q}}\right)\right)\mathrm{d}x\\
&\quad\leq 2\|\lambda\|_{L^\infty}|U_i\setminus K_{i,\delta}|\\
&\qquad + (2\|\lambda\|_{L^\infty}+1)\left(\sum_{q\in Q_i}\widehat{\alpha}_{i,q}^N\right)\int_Y W(0,\xi_i+\nabla\widehat{\varphi}_i(y))\,\mathrm{d}y
\end{aligned}$$

$$\le 2\|\lambda\|_{L^\infty}|U_i \setminus K_{i,\delta}| \\ + (2\|\lambda\|_{L^\infty}+1)\left(\sum_{q\in Q_i}\widehat{\alpha}^N_{i,q}\right)(\mathcal{Z}W(0,\xi_i)+\delta)\,.$$

As $\sum_{q\in Q_i}\widehat{\alpha}^N_{i,q} = |U_i \setminus K_{i,\delta}| = |\overline{U_i}\setminus K_{i,\delta}| < \delta$ for all $i\in I$, it follows that

$$\sum_{i\in I}\int_{U_i\setminus K_{i,\delta}} W(x,\xi_i+\nabla\psi_n(x))\,\mathrm{d}x \le \Delta(\delta) \tag{3.18}$$

with $\Delta(\delta) := \delta\left(2\|\lambda\|_{L^\infty} + (2\|\lambda\|_{L^\infty}+1)\left(\max_{i\in I}\mathcal{Z}W(0,\xi_i)+\delta\right)\right)$ where

$$\lim_{\delta\to 0}\Delta(\delta) = 0\,. \tag{3.19}$$

Using again (A_4) we can assert that for every $i\in I$, every $j\in J^k_i$ and every $x\in U^k_{i,j}$,

$$W(x,\xi_i+\nabla\psi_n(x)) \le \left|\lambda(x)-\lambda(x^k_{i,j})\right|(1+W(x^k_{i,j},\xi_i+\nabla\psi_n(x))) \\ + W(x^k_{i,j},\xi_i+\nabla\psi_n(x))\,.$$

But $x^k_{i,j}\in U^k_{i,j}$ and $U^k_{i,j}\subset K_{i,\delta}$ and by (3.10) we have $\mathrm{diam}(U^k_{i,j}) < \frac{1}{k}$ where $\frac{1}{k}<\eta$, hence $|\lambda(x)-\lambda(x^k_{i,j})|<\delta$ for all $x\in U^k_{i,j}$ by (3.12). So, for every $x\in U^k_{i,j}$,

$$W(x,\xi_i+\nabla\psi_n(x)) \le \delta + (\delta+1)W(x^k_{i,j},\xi_i+\nabla\psi_n(x))\,,$$

and taking (3.15) and (3.13) into account we deduce that

$$\begin{aligned}
&\int_{U^k_{i,j}} W(x,\xi_i+\nabla\psi_n(x))\,\mathrm{d}x \\
&\quad\le \delta|U^k_{i,j}| + (\delta+1)\sum_{l\in L_{i,j}}\int_{Y_{i,j,l}} W\left(x^k_{i,j},\xi_i+\nabla\varphi_{i,j}\left(\frac{x-a_{i,j,\ell}}{\alpha_{i,j,\ell}}\right)\right)\mathrm{d}x \\
&\quad\le \delta|U^k_{i,j}| + (\delta+1)\left(\sum_{\ell\in L_{i,j}}\alpha^N_{i,j,\ell}\right)\int_Y W\left(x^k_{i,j},\xi_i+\nabla\varphi_{i,j}(y)\right)\mathrm{d}y \\
&\quad\le \delta|U^k_{i,j}| + (\delta+1)\left(\sum_{\ell\in L_{i,j}}\alpha^N_{i,j,\ell}\right)\left(\mathcal{Z}W(x^k_{i,j},\xi_i)+\delta\right).
\end{aligned}$$

As $\sum_{\ell\in L_{i,j}} \alpha^N_{i,j,\ell} = |U^k_{i,j}|$ for all $i \in I$ and all $j \in J^k_i$ and $\sum_{i\in I}\sum_{j\in J^k_i} |U^k_{i,j}| \leq |U|$ because $\sum_{j\in J^k_i} |U^k_{i,j}| = |\operatorname{int}(K_{i,\delta})| \leq |U_i|$ and $\sum_{i\in I} |U_i| = |U|$, it follows that

$$\sum_{i\in I}\sum_{j\in J^k_i}\int_{U^k_{i,j}} W(x, \xi_i + \nabla\psi_n(x))\,\mathrm{d}x \leq \left(2\delta+\delta^2\right)|U| + (\delta+1)\sum_{i\in I}\sum_{j\in J^k_i}|U^k_{i,j}|\mathcal{Z}W(x^k_{i,j},\xi_i)\,. \tag{3.20}$$

Combining (3.18) and (3.20) with (3.17) we obtain

$$\int_U W(x, \nabla\phi(x) + \nabla\psi_n(x))\,\mathrm{d}x \leq \Delta(\delta) + \left(2\delta+\delta^2\right)|U| + (\delta+1)\sum_{i\in I}\sum_{j\in J^k_i}|U^k_{i,j}|\mathcal{Z}W(x^k_{i,j},\xi_i)\,.$$

Letting $k \to \infty$ and using (3.11) and (3.9) we conclude that

$$\begin{aligned}\int_U W(x, \nabla\phi(x) + \nabla\psi_n(x))\,\mathrm{d}x &\leq \Delta(\delta) + \left(2\delta+\delta^2\right)|U| + (\delta+1)\sum_{i\in I}\int_{\operatorname{int}(K_{i,\delta})}\mathcal{Z}W(x,\xi_i)\,\mathrm{d}x\\ &\leq \Delta(\delta) + \left(2\delta+\delta^2\right)|U| + (\delta+1)\sum_{i\in I}\int_{U_i}\mathcal{Z}W(x,\xi_i)\,\mathrm{d}x\\ &= \Delta(\delta) + \left(2\delta+\delta^2\right)|U| + (\delta+1)\int_U\mathcal{Z}W(x,\nabla\phi(x))\,\mathrm{d}x\,,\end{aligned}$$

and (3.16) follows by letting $\delta \to 0$ and using (3.19). $\square$

Remark 3.10. By analysing the proof of Lemma 3.9 we see that this lemma is also valid with "$\widehat{\mathcal{Z}}W$", defined in (2.25), instead of "$\mathcal{Z}W$". Thus, by the same method we can also establish the following analogue of Theorem 3.8.

Theorem 3.11. *If $W \in \widehat{\mathcal{I}}^p$, with $\widehat{\mathcal{I}}^p$ defined in Remark 2.37, then* (3.1) *and* (3.2) *hold with $\overline{W} = \widehat{\mathcal{Z}}W = \mathcal{Q}W$. (In fact, since $\widehat{\mathcal{I}}^p \subset \mathcal{I}^p$, we have $\overline{W} = \mathcal{Z}W = \widehat{\mathcal{Z}}W = \mathcal{Q}W$.)*

Remark 3.12. It is easily seen that from the proof of Lemma 3.9 we can extract the following result.

Lemma 3.13. *Under* (A_4), *for every* $\xi \in \mathbb{M}^{m\times N}$ *we have*

$$\mathfrak{S}_\xi^W(U) \le \int_U \mathcal{Z}W(x,\xi)\,\mathrm{d}x$$

for all bounded open set $U \subset \mathbb{R}^N$ *with*

$$\mathfrak{S}_\xi^W(U) := \inf\left\{\int_U W(x,\xi+\nabla\psi(x))\,\mathrm{d}x : \psi \in W_0^{1,\infty}(U;\mathbb{R}^m)\right\}.$$

By the same reasoning, in replacing "$W_0^{1,\infty}$" by "Aff_0" and "$\mathcal{Z}W$" by "$\widehat{\mathcal{Z}}W$" defined in (2.25), we can also prove the following result.

Lemma 3.14. *Under* (A_4), *for every* $\xi \in \mathbb{M}^{m\times N}$ *we have*

$$\mathcal{S}_\xi^W(U) \le \int_U \widehat{\mathcal{Z}}W(x,\xi)\,\mathrm{d}x$$

for all bounded open set $U \subset \mathbb{R}^N$ *with*

$$\mathcal{S}_\xi^W(U) := \inf\left\{\int_U W(x,\xi+\nabla\psi(x))\,\mathrm{d}x : \psi \in \mathrm{Aff}_0(U;\mathbb{R}^m)\right\}.$$

3.2. **Relaxation with a sum of singular integrands**

The following relaxation theorem is a variant of Theorem 3.8.

Theorem 3.15. *If* $W \in \mathcal{J}^p$ *then* (3.1) *and* (3.2) *hold with* $\overline{W} = \mathcal{Z}W = \mathcal{Q}W$.

Proof. Let $\mathcal{Z}E, \overline{\mathcal{Z}E}, \overline{\mathcal{Z}E}_0 : W^{1,p}(U;\mathbb{R}^m) \to [0,\infty]$ be defined by (3.3), (3.4) and (3.5) respectively. The proof of Theorem 3.15 follows from Lemma 3.16 below by using the same arguments as in the proof of Theorem 3.8. □

Lemma 3.16. *If* W *is given by* (2.11) *then for each* $\phi \in \mathrm{Aff}(U;\mathbb{R}^m)$ *(resp.* $\phi \in \mathrm{Aff}_0(U;\mathbb{R}^m)$*) we have*

$$\overline{E}(\phi) \le \int_U \mathcal{Z}W\left(x,\nabla\phi(x)\right)\mathrm{d}x \qquad \left(\textit{resp. } \overline{E}_0(\phi) \le \int_U \mathcal{Z}W\left(x,\nabla\phi(x)\right)\mathrm{d}x\right). \tag{3.21}$$

Proof. Let $\phi \in \mathrm{Aff}(U;\mathbb{R}^m)$ (resp. $\phi \in \mathrm{Aff}_0(U;\mathbb{R}^m)$). By definition, there exists a finite family $\{U_i\}_{i\in I}$ of open disjoint subsets of U such that $|\partial U_i| = 0$ for all $i \in I$, $|U \setminus \bigcup_{i\in I} U_i| = 0$ and, for every $i \in I$, $\nabla\phi \equiv \xi_i$ in U_i with $\xi_i \in \mathbb{M}^{m\times N}$. Thus

$$\int_U \mathcal{Z}W(x,\nabla\phi(x))\,\mathrm{d}x = \sum_{i\in I}\int_{U_i} \mathcal{Z}W(x,\xi_i)\,\mathrm{d}x\,.$$

By assumption, there exist a finite family $\{V_j\}_{j\in J}$ of open disjoint subsets of $\mathbb{R}^N$, with $|\partial V_j| = 0$ for all $j \in J$ and $|\mathbb{R}^N \setminus \bigcup_{j\in J} V_j| = 0$, and a finite family $\Big\{H_j : \mathbb{M}^{m\times N} \to [0,\infty]\Big\}_{j\in J}$ of Borel measurable functions such that W is given by

$$W(x,\xi) = \sum_{j\in J} \mathbb{1}_{V_j}(x) H_j(\xi)\,.$$

Then, we have $\mathcal{Z}W(x,\xi) = \sum_{j\in J} \mathbb{1}_{V_j}(x)\mathcal{Z}H_j(\xi)$ for all $x \in \mathbb{R}^N$ and all $\xi \in \mathbb{M}^{m\times N}$. Hence

$$\int_U \mathcal{Z}W(x,\nabla\phi(x))\,\mathrm{d}x = \sum_{i\in I}\sum_{j\in J} |U_{i,j}|\mathcal{Z}H_j(\xi_i) \tag{3.22}$$

with $U_{i,j} := U_i \cap V_j$. Fix any $\delta > 0$. Given any $i \in I$ and any $j \in J$ we consider $\varphi_{i,j} \in W_0^{1,\infty}(Y;\mathbb{R}^m)$ such that

$$\int_Y H_j(\xi_i + \nabla\varphi_{i,j}(y))\,\mathrm{d}y \le \mathcal{Z}H_j(\xi_i) + \frac{\delta}{|U_i|}\,. \tag{3.23}$$

For every $n \ge 1$, by Vitali's covering theorem, there exists a finite or countable family $\{a_{i,j,\ell} + \alpha_{i,j,\ell}Y\}_{\ell\in L_{i,j}}$ of disjoint subsets of $U_{i,j}$, where $a_{i,j,\ell} \in \mathbb{R}^N$ and $0 < \alpha_{i,j,\ell} < \frac{1}{n}$, such that $|U_{i,j} \setminus \bigcup_{\ell\in L_{i,j}} (a_{i,j,\ell} + \alpha_{i,j,\ell}Y)| = 0$ (and so $\sum_{\ell\in L_{i,j}} \alpha_{i,j,\ell}^N = |U_{i,j}|$). Define $\{\phi_n\}_{n\ge1} \subset W_0^{1,\infty}(U;\mathbb{R}^m)$ by

$$\psi_n(x) := \alpha_{i,j,\ell}\varphi_{i,j}\left(\frac{x - a_{i,j,\ell}}{\alpha_{i,j,\ell}}\right) \text{ if } x \in a_{i,j,\ell} + \alpha_{i,j,\ell}Y\,.$$

It is then easy to see that

$$\|\psi_n\|_{L^\infty(U;\mathbb{R}^m)} \le \frac{1}{n}\max_{\substack{i\in I\\ j\in J}} \|\varphi_{i,j}\|_{L^\infty(Y;\mathbb{R}^m)}$$

for all $n \ge 1$, and so $\psi_n \to 0$ in $L^\infty(U;\mathbb{R}^m)$. Thus $\{\phi + \psi_n\}_{n\ge1} \subset W^{1,p}(U;\mathbb{R}^m)$ (resp. $\{\phi + \psi_n\}_{n\ge1} \subset W_0^{1,p}(U;\mathbb{R}^m)$) and $\phi + \psi_n \to \phi$ in

$L^p(U;\mathbb{R}^m)$. Moreover, using (3.23) and (3.22), for every $n \geq 1$, we have

$$\begin{aligned}
\int_U W(x, \nabla\psi(x) + \nabla\psi_n(x))\,\mathrm{d}x &= \sum_{i\in I}\int_{U_i} W(x, \xi_i + \nabla\psi_n(x))\,\mathrm{d}x \\
&= \sum_{i\in I}\sum_{j\in J}\int_{U_{i,j}} H_j(\xi_i + \nabla\psi_n(x))\,\mathrm{d}x \\
&= \sum_{i\in I}\sum_{j\in J}|U_{i,j}|\int_Y H_j(\xi_i + \nabla\varphi_{i,j}(y))\,\mathrm{d}y \\
&\leq \sum_{i\in I}\sum_{j\in J}|U_{i,j}|\mathcal{Z}H_j(\xi_i) + \delta \\
&= \int_U \mathcal{Z}W(x, \nabla\phi(x))\,\mathrm{d}x + \delta\,.
\end{aligned}$$

It follows that

$$\overline{E}(\phi) \leq \int_U \mathcal{Z}W(x, \nabla\phi(x))\,\mathrm{d}x + \delta$$

$$\left(\text{resp. } \overline{E}_0(\phi) \leq \int_U \mathcal{Z}W(x, \nabla\phi(x))\,\mathrm{d}x + \delta\right),$$

and (3.21) follows by letting $\delta \to 0$. □

4. Proof of the homogenization theorems

4.1. Singular integrands which are continuous almost everywhere with respect to the space variable

In this section we prove Theorem 2.8 by following the same lines as in the proof of [2, Theorem 3.4]. We will need Theorems 3.8 and 2.2 and the following classical property of the Γ-convergence.

Proposition 4.1. *The* Γ*-limit is stable by substituting* I_ε *by its relaxed functional* $\overline{I}_\varepsilon$*, i.e.,*

$$\Gamma\text{-}\varliminf_{\varepsilon\to 0} I_\varepsilon = \Gamma\text{-}\varliminf_{\varepsilon\to 0} \overline{I}_\varepsilon \quad \text{and} \quad \Gamma\text{-}\varlimsup_{\varepsilon\to 0} I_\varepsilon = \Gamma\text{-}\varlimsup_{\varepsilon\to 0} \overline{I}_\varepsilon\,,$$

where, for each $\varepsilon > 0$, $\overline{I}_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,\infty]$ *is given by*

$$\overline{I}_\varepsilon(\phi) := \inf\left\{\varliminf_{n\to\infty} I_\varepsilon(\phi_n) : \phi_n \to \phi \text{ in } L^p(\Omega;\mathbb{R}^m)\right\}.$$

Proof of Theorem 2.8. By Proposition 4.1 it suffices to prove Theorem 2.8 with "$\overline{I}_\varepsilon$" instead of "I_ε". Fix any $\varepsilon > 0$ and consider $W_\varepsilon : \mathbb{R}^N \times \mathbb{M}^{m\times N} \to [0,\infty]$ given by $W_\varepsilon(x,\xi) := W(\frac{x}{\varepsilon},\xi)$. As $W \in \mathcal{I}^p_{\rm per}$ and $\mathcal{Z}W_\varepsilon(x,\xi) = \mathcal{Z}W(\frac{x}{\varepsilon},\xi)$ for all $(x,\xi) \in \mathbb{R}^N \times \mathbb{M}^{m\times N}$ it is easy to see that $W_\varepsilon \in \mathcal{I}^p$. Applying Theorem 3.8 to W_ε we deduce that for every $\varepsilon > 0$,

$$\overline{I}_\varepsilon(\phi) = \int_\Omega \mathcal{Z}W\left(\frac{x}{\varepsilon}, \nabla\phi(x)\right) {\rm d}x\,,$$

where $\mathcal{Z}W$ is clearly p-coercive, 1-periodic and has p-growth. From Braides–Müller's homogenization theorem (see Theorem 2.2) it follows that $I_{\rm hom} = \Gamma\text{-}\lim_{\varepsilon\to 0} \overline{I}_\varepsilon$ with $I_{\rm hom}$ defined by (2.2) and $W_{\rm hom} : \mathbb{M}^{m\times N} \to [0,\infty]$ given by

$$W_{\rm hom}(\xi) = \inf_{k\geq 1} \inf\left\{ ⨍_{kY} \mathcal{Z}W(x,\xi+\nabla\varphi(x))\,{\rm d}x : \varphi \in W^{1,p}_0(kY;\mathbb{R}^m)\right\} =: \mathcal{H}[\mathcal{Z}W](\xi)\,.$$

Fix any $k \geq 1$, any $\xi \in \mathbb{M}^{m\times N}$ and consider $W_\xi : \mathbb{R}^N \times \mathbb{M}^{m\times N} \to [0,\infty]$ given by $W_\xi(x,\zeta) := W(x,\xi+\zeta)$. As $W \in \mathcal{I}^p_{\rm per}$ and $\mathcal{Z}W_\xi(x,\zeta) = \mathcal{Z}W(x,\xi+\zeta)$ for all $(x,\zeta) \in \mathbb{R}^N \times \mathbb{M}^{m\times N}$ it is easy to see that $W_\xi \in \mathcal{I}^p$. Applying again Theorem 3.8 to W_ξ with $U = kY$ we see that (3.2) holds with $\overline{W} = \mathcal{Z}W_\xi$. Consequently, for every $k \geq 1$ and every $\xi \in \mathbb{M}^{m\times N}$, we have

$$\begin{aligned}\mathcal{H}W(\xi) := \inf_{\varphi\in W^{1,p}_0(kY;\mathbb{R}^m)} ⨍_{kY} W(x,\xi+\nabla\varphi(x))\,{\rm d}x \\ &= \inf_{\varphi\in W^{1,p}_0(kY;\mathbb{R}^m)} ⨍_{kY} W_\xi(x,\nabla\varphi(x))\,{\rm d}x \\ &= \inf_{\varphi\in W^{1,p}_0(kY;\mathbb{R}^m)} ⨍_{kY} \mathcal{Z}W_\xi(x,\nabla\varphi(x))\,{\rm d}x \\ &= \inf_{\varphi\in W^{1,p}_0(kY;\mathbb{R}^m)} ⨍_{kY} \mathcal{Z}W(x,\xi+\nabla\varphi(x))\,{\rm d}x \\ &= \mathcal{H}[\mathcal{Z}W](\xi)\,,\end{aligned}$$

and the proof of Theorem 2.8 is complete. $\square$

Remark 4.2. From the proof of Theorem 2.8 we can extract the following result.

Lemma 4.3. *If $W \in \mathcal{I}^p$ then $\mathcal{H}[\mathcal{Z}W] = \mathcal{H}W$.*

In the same way, by using Theorem 3.11 (see Remark 3.10) instead of Theorem 3.8, we can also establish the following result.

Lemma 4.4. *If $W \in \widehat{\mathcal{I}}^p$, with $\widehat{\mathcal{I}}^p$ defined in Remark 2.37, then $\mathcal{H}[\widehat{\mathcal{Z}}W] = \mathcal{H}W$.*

4.2. Sum of singular integrands

In this section we prove Theorem 2.33 by using Theorems 3.15 and 2.2 and Proposition 4.1.

Proof of Theorem 2.19. It is the same proof than the one of Theorem 2.8 where we replace "Theorem 3.8" by "Theorem 3.15", "$\mathcal{I}^p$" by "$\mathcal{J}^p$" and "$\mathcal{I}^p_{\mathrm{per}}$" by "$\mathcal{J}^p_{\mathrm{per}}$". □

4.3. Singular integrands which are not continuous with respect to the space variable

In this section we prove Theorem 2.33.

Proof of Theorem 2.33. Using (A_6) we deduce that $\Gamma\text{-}\underline{\lim}_{\varepsilon\to 0} I_\varepsilon \geq \Gamma\text{-}\underline{\lim}_{\varepsilon\to 0} \mathcal{G}I_\varepsilon$ where, for each $\varepsilon > 0$, $\mathcal{G}I_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,\infty]$ is given by

$$\mathcal{G}I_\varepsilon(\phi) := \int_\Omega \mathcal{G}W\left(\frac{x}{\varepsilon}, \nabla\phi(x)\right) \mathrm{d}x\,.$$

But, by (A_5) we see that $\mathcal{G}W$ has p-growth, and so, by Braides–Müller's homogenization theorem (see Theorem 2.2), it follows that

$$\left(\Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{G}I_\varepsilon\right)(\phi) = \int_\Omega \mathcal{H}[\mathcal{G}W](\nabla\phi(x))\,\mathrm{d}x$$

for all $\phi \in W^{1,p}(\Omega;\mathbb{R}^m)$. Consequently

$$\left(\Gamma\text{-}\underline{\lim}_{\varepsilon\to 0} I_\varepsilon\right)(\phi) \geq \int_\Omega \mathcal{H}[\mathcal{G}W](\nabla\phi(x))\,\mathrm{d}x$$

for all $\phi \in W^{1,p}(\Omega;\mathbb{R}^m)$. Taking ($A_7$) into account, it remains to prove that for every $\phi \in W^{1,p}(\Omega;\mathbb{R}^m)$,

$$\left(\Gamma\text{-}\overline{\lim_{\varepsilon\to 0}}\, I_\varepsilon\right)(\phi) \le \int_\Omega \widehat{\mathcal{Z}}\big[\widehat{\mathcal{H}}[\mathcal{G}W]\big](\nabla\phi(x))\,\mathrm{d}x\,. \tag{4.1}$$

Since $\mathcal{G}W$ has p-growth, so is $\widehat{\mathcal{H}}[\mathcal{G}W]$. Hence $\widehat{\mathcal{Z}}\big[\widehat{\mathcal{H}}[\mathcal{G}W]\big]$ has p-growth and, by Proposition 3.1 and Remark 3.2, we can assert that $\widehat{\mathcal{Z}}\big[\widehat{\mathcal{H}}[\mathcal{G}W]\big]$ is continuous. Thus, as $\mathrm{Aff}(\Omega;\mathbb{R}^m)$ is strongly dense in $W^{1,p}(\Omega;\mathbb{R}^m)$, we are reduced to prove (4.1) for all $\phi \in \mathrm{Aff}(\Omega;\mathbb{R}^m)$.

If we prove that

$$\overline{I}_\varepsilon(u) \le \mathcal{G}I_\varepsilon(u) \text{ for all } \varepsilon > 0 \text{ and all } u \in \mathrm{Aff}(\Omega;\mathbb{R}^m) \tag{4.2}$$

then (4.1) holds for all $\phi \in \mathrm{Aff}(\Omega;\mathbb{R}^m)$. Indeed, fix $\phi \in \mathrm{Aff}(\Omega;\mathbb{R}^m)$. By definition, there exists a finite family $\{U_i\}_{i\in I}$ of open disjoint subsets of Ω such that $|\partial U_i| = 0$ for all $i \in I$, $|\Omega \setminus \bigcup_{i\in I} U_i| = 0$ and, for every $i \in I$, $\nabla\phi \equiv \xi_i$ in U_i with $\xi_i \in \mathbb{M}^{m\times N}$. Then

$$\int_\Omega \widehat{\mathcal{Z}}\big[\widehat{\mathcal{H}}[\mathcal{G}W]\big](\nabla\phi(x))\,\mathrm{d}x = \sum_{i\in I} |U_i|\widehat{\mathcal{Z}}\big[\widehat{\mathcal{H}}[\mathcal{G}W]\big](\xi_i)\,.$$

Fix any $i \in I$. Using Proposition A.5 with $L = \widehat{\mathcal{H}}[\mathcal{G}W]$ and $A = U_i$ we can assert that there exists $\{\varphi_{i,k}\}_k \subset \mathrm{Aff}_0(U_i;\mathbb{R}^m)$ such that:

$$\lim_{k\to\infty} \|\varphi_{i,k}\|_{L^\infty(U_i;\mathbb{R}^m)} = 0\,; \tag{4.3}$$

$$\lim_{k\to\infty} \int_{U_i} \widehat{\mathcal{H}}[\mathcal{G}W](\xi_i + \nabla\varphi_{i,k}(x))\,\mathrm{d}x = |U_i|\widehat{\mathcal{Z}}\big[\widehat{\mathcal{H}}[\mathcal{G}W]\big](\xi_i)\,. \tag{4.4}$$

Fix any $k \ge 1$. By definition, there exists a finite family $\{V_j\}_{j\in J}$ of open disjoint subsets of U_i such that $|\partial V_j| = 0$ for all $j \in J$, $|U_i \setminus \bigcup_{j\in J} V_j| = 0$ and, for every $j \in J$, $\nabla\varphi_{i,k} \equiv \zeta_j$ in V_j with $\zeta_j \in \mathbb{M}^{m\times N}$. Then

$$\int_{U_i} \widehat{\mathcal{H}}[\mathcal{G}W](\xi_i + \nabla\varphi_{i,k}(x))\,\mathrm{d}x = \sum_{j\in J} |V_j|\widehat{\mathcal{H}}[\mathcal{G}W](\xi_i + \zeta_j)\,.$$

Fix any $j \in J$. Recalling that $\mathcal{G}W$ has p-growth, by using Proposition A.8 with $L = \mathcal{G}W$ and $A = V_j$ we can assert that there exists $\{\psi_{i,j,k,\varepsilon}\}_\varepsilon \subset$

$\mathrm{Aff}_0(V_j;\mathbb{R}^m)$ such that:

$$\lim_{\varepsilon\to 0}\|\psi_{i,j,k,\varepsilon}\|_{L^\infty(V_j;\mathbb{R}^m)}=0\,; \tag{4.5}$$

$$\lim_{\varepsilon\to 0}\int_{V_j}\mathcal{G}W\left(\frac{x}{\varepsilon},\xi_i+\zeta_j+\nabla\psi_{i,j,k,\varepsilon}(x)\right)\mathrm{d}x=|V_j|\widehat{\mathcal{H}}[\mathcal{G}W](\xi_i+\zeta_j)\,. \tag{4.6}$$

For each $\varepsilon>0$, defining $\psi_{i,k,\varepsilon}\in\mathrm{Aff}_0(U_i;\mathbb{R}^m)$ by $\psi_{i,k,\varepsilon}(x)=\psi_{i,j,k,\varepsilon}(x)$ if $x\in V_j$, from (4.5) and (4.6) we deduce that:

$$\lim_{\varepsilon\to 0}\|\psi_{i,k,\varepsilon}\|_{L^\infty(U_i;\mathbb{R}^m)}=0\,; \tag{4.7}$$

$$\lim_{\varepsilon\to 0}\int_{U_i}\mathcal{G}W\left(\frac{x}{\varepsilon},\xi_i+\nabla\varphi_{i,k}(x)+\nabla\psi_{i,k,\varepsilon}(x)\right)\mathrm{d}x =\int_{U_i}\widehat{\mathcal{H}}[\mathcal{G}W](\xi_i+\nabla\varphi_{i,k}(x))\,\mathrm{d}x\,. \tag{4.8}$$

For each $k\geq 1$ and each $\varepsilon>0$, we define $\varphi_k,\psi_{k,\varepsilon}\in\mathrm{Aff}_0(\Omega;\mathbb{R}^m)$ by $\varphi_k(x)=\varphi_{i,k}(x)$ if $x\in U_i$ and $\psi_{k,\varepsilon}(x)=\psi_{i,k,\varepsilon}(x)$ if $x\in U_i$. Then, from (4.3), (4.4), (4.7) and (4.8) we see that:

$$\lim_{k\to\infty}\|\varphi_k\|_{L^\infty(\Omega;\mathbb{R}^m)}=0; \tag{4.9}$$

$$\lim_{k\to\infty}\int_\Omega\widehat{\mathcal{H}}[\mathcal{G}W](\nabla\phi(x)+\nabla\varphi_k(x))\,\mathrm{d}x=\int_\Omega\widehat{\mathcal{Z}}\big[\widehat{\mathcal{H}}[\mathcal{G}W]\big](\nabla\phi(x))\,\mathrm{d}x. \tag{4.10}$$

$$\lim_{\varepsilon\to 0}\|\psi_{k,\varepsilon}\|_{L^\infty(\Omega;\mathbb{R}^m)}=0\quad\text{ for all } k\geq 1; \tag{4.11}$$

$$\lim_{\varepsilon\to 0}\int_\Omega\mathcal{G}W\left(\frac{x}{\varepsilon},\nabla\phi(x)+\nabla\varphi_k(x)+\nabla\psi_{k,\varepsilon}(x)\right)\mathrm{d}x =\int_\Omega\widehat{\mathcal{H}}[\mathcal{G}W](\nabla\phi(x)+\nabla\varphi_k(x))\,\mathrm{d}x\quad\text{for all } k\geq 1. \tag{4.12}$$

Define $\{\phi_{k,\varepsilon}\}_{k,\varepsilon}\subset\mathrm{Aff}(\Omega;\mathbb{R}^m)$ by $\phi_{k,\varepsilon}:=\phi+\varphi_k+\psi_{k,\varepsilon}$. Combining (4.9) with (4.11) and (4.10) with (4.12) we deduce that:

$$\lim_{k\to\infty}\lim_{\varepsilon\to 0}\|\phi_{k,\varepsilon}-\phi\|_{L^\infty(\Omega;\mathbb{R}^m)}=0\,;$$

$$\lim_{k\to\infty}\lim_{\varepsilon\to 0}\mathcal{G}I_\varepsilon(\phi_{k,\varepsilon})=\int_\Omega\widehat{\mathcal{Z}}\big[\widehat{\mathcal{H}}[\mathcal{G}W]\big](\nabla\phi(x))\,\mathrm{d}x\,.$$

By diagonalization there exists a mapping $\varepsilon \mapsto k_\varepsilon$ with $k_\varepsilon \to \infty$ as $\varepsilon \to 0$ such that:

$$\lim_{\varepsilon\to 0} \|\phi_\varepsilon - \phi\|_{L^\infty(\Omega;\mathbb{R}^m)} = 0; \tag{4.13}$$

$$\lim_{\varepsilon\to 0} \mathcal{G}I_\varepsilon(\phi_\varepsilon) = \int_\Omega \widehat{\mathcal{Z}}[\widehat{\mathcal{H}}[\mathcal{G}W]](\nabla\phi(x))\,\mathrm{d}x, \tag{4.14}$$

where $\phi_\varepsilon = \phi_{k_\varepsilon,\varepsilon}$. But, by (4.2) we have $\overline{I}_\varepsilon(\phi_\varepsilon) \le \mathcal{G}I_\varepsilon(\phi_\varepsilon)$ for all $\varepsilon > 0$, and so $\overline{\lim}_{\varepsilon\to 0} \overline{I}_\varepsilon(\phi_\varepsilon) \le \lim_{\varepsilon\to 0} \mathcal{G}I_\varepsilon(\phi_\varepsilon)$. Then, $\left(\Gamma\text{-}\overline{\lim}_{\varepsilon\to 0} I_\varepsilon\right)(\phi) \le \lim_{\varepsilon\to 0} \mathcal{G}I_\varepsilon(\phi_\varepsilon)$ because $\phi_\varepsilon \to \phi$ in $L^p(\Omega;\mathbb{R}^m)$ by (4.13), and (4.1) follows by using (4.14).

Finally, to complete the proof of Theorem 2.33, we have to prove (4.2).

Proof of (4.2). Fix $\varepsilon > 0$ and $u \in \mathrm{Aff}(\Omega;\mathbb{R}^m)$. Denote the class of open subsets of Ω by $\mathcal{O}(\Omega)$ and define $\mathrm{m}_u : \mathcal{O}(\Omega) \to [0,\infty]$ by

$$\mathrm{m}_u(A) := \inf\left\{\int_A W\left(\frac{y}{\varepsilon}, \nabla u(y) + \nabla\varphi(y)\right)\mathrm{d}y : \varphi \in \mathrm{Aff}_0(A;\mathbb{R}^m)\right\}.$$

For each $\delta > 0$ and each $A \in \mathcal{O}(\Omega)$, denote the class of countable families $\{Q_i\}_{i\in I}$ of disjoint open cubes such that $\mathrm{diam}(Q_i) \in]0,\delta[$ for all $i \in I$ and $|A \setminus \bigcup_{i\in I} Q_i| = 0$ by $\mathcal{V}_\delta(A)$, consider $\mathrm{m}_u^\delta : \mathcal{O}(\Omega) \to [0,\infty]$ given by

$$\mathrm{m}_u^\delta(A) := \inf\left\{\sum_{i\in I} \mathrm{m}_u(Q_i) : \{Q_i\}_{i\in I} \in \mathcal{V}_\delta(A)\right\},$$

and define $\mathrm{m}_u^* : \mathcal{O}(\Omega) \to [0,\infty]$ by

$$\mathrm{m}_u^*(A) := \sup_{\delta>0} \mathrm{m}_u^\delta(A) = \lim_{\delta\to 0} \mathrm{m}_u^\delta(A).$$

The set function m_u^* is called the Vitali envelope of m_u, see §A.4 for more details.

First of all, it is easy to see that m_u is subadditive. On the other hand, as $u \in \mathrm{Aff}(\Omega;\mathbb{R}^m)$, there exists a finite family $\{U_j\}_{j\in J}$ of open disjoint subsets of Ω such that $|\partial U_j| = 0$ for all $j \in J$, $|\Omega \setminus \bigcup_{j\in J} U_j| = 0$ and, for every $j \in J$, $\nabla u \equiv \xi_j$ in U_j with $\xi_j \in \mathbb{M}^{m\times N}$. Hence, given any $A \in \mathcal{O}(\Omega)$, we have $|A \setminus \bigcup_{j\in J}(A \cap U_j)| = 0$, and so, by subadditivity of m_u, it follows that

$$
\begin{aligned}
&\mathrm{m}_u(A)\\
&\leq \sum_{j\in J}\mathrm{m}_u(A\cap U_j)\\
&=\sum_{j\in J}\inf\left\{\int_{A\cap U_j}W\left(\frac{y}{\varepsilon},\xi_j+\nabla\varphi(y)\right)\mathrm{d}y:\varphi\in\mathrm{Aff}_0(A\cap U_j;\mathbb{R}^m)\right\}\\
&=\sum_{j\in J}\varepsilon^N\inf\left\{\int_{\frac{1}{\varepsilon}(A\cap U_j)}W\left(y,\xi_j+\nabla\varphi(y)\right)\mathrm{d}y:\varphi\in\mathrm{Aff}_0\left(\frac{1}{\varepsilon}(A\cap U_j);\mathbb{R}^m\right)\right\}.
\end{aligned}
$$

Using (A_5) we see that

$$
\begin{aligned}
\mathrm{m}_u(A)\leq\sum_{j\in J}\varepsilon^N\mathcal{S}^W_{\xi_j}\left(\frac{1}{\varepsilon}(A\cap U_j)\right)&\leq\sum_{j\in J}\varepsilon^N c\left|\frac{1}{\varepsilon}(A\cap U_j)\right|(1+|\xi_j|^p)\\
&=\sum_{j\in J}c\,|A\cap U_j|\,(1+|\xi_j|^p)\\
&=c\int_A(1+|\nabla u(y)|^p)\,\mathrm{d}y=:\nu(A)\,.
\end{aligned}
$$

Thus $\mathrm{m}_u(A)\leq\nu(A)$ for all $A\in\mathcal{O}(\Omega)$ where ν is a finite Radon measure on Ω which is absolutely continuous with respect to the Lebesgue measure. Applying Theorem A.16 with $\Theta=\mathrm{m}_u$, we deduce that m_u is differentiable, i.e., see Definition A.12,

$$
\mathrm{dm}_u(x):=\lim_{\rho\to0}\frac{\mathrm{m}_u(Q_\rho(x))}{|Q_\rho(x)|}\text{ exists and is finite for a.a. }x\in\Omega\,,
$$

$\mathrm{dm}_u\in L^1(\Omega)$ and

$$
\mathrm{m}^*_u(A)=\int_A\mathrm{dm}_u(x)\,\mathrm{d}x \tag{4.15}
$$

for all $A\in\mathcal{O}(\Omega)$. But, given any $x\in\Omega$ such that $\lim_{\rho\to0}\frac{\mathrm{m}_u(Q_\rho(x))}{|Q_\rho(x)|}$ exists and $x\in\bigcup_{j\in J}U_j$, we have $x\in U_{j_0}$ for some $j_0\in J$ and $Q_{\rho_0}(x)\subset U_{j_0}$ for all $\rho\in\,]0,\rho_0[$ and some $\rho_0>0$, and so, for each $\rho\in\,]0,\rho_0[$, $\nabla u\equiv\xi_{j_0}$ in

$Q_\rho(x)$. Hence, by a change of variable, we see that for every $\rho \in]0, \rho_0[$,

$$\begin{aligned}
&\frac{\mathcal{S}^W_{\nabla u(x)}\left(Q_\rho\left(\frac{x}{\varepsilon}\right)\right)}{\left|Q_\rho\left(\frac{x}{\varepsilon}\right)\right|}\\
&= \inf\left\{ \fint_{Q_\rho\left(\frac{x}{\varepsilon}\right)} W\left(y, \nabla u(x) + \nabla\varphi(y)\right) \mathrm{d}y : \varphi \in \mathrm{Aff}_0\left(Q_\rho\left(\frac{x}{\varepsilon}\right); \mathbb{R}^m\right)\right\}\\
&= \inf\left\{ \fint_{Q_\rho\left(\frac{x}{\varepsilon}\right)} W\left(y, \xi_{j_0} + \nabla\varphi(y)\right) \mathrm{d}y : \varphi \in \mathrm{Aff}_0\left(Q_\rho\left(\frac{x}{\varepsilon}\right); \mathbb{R}^m\right)\right\}\\
&= \inf\left\{ \fint_{Q_{\varepsilon\rho}(x)} W\left(\frac{y}{\varepsilon}, \xi_{j_0} + \nabla\varphi(y)\right) \mathrm{d}y : \varphi \in \mathrm{Aff}_0\left(Q_{\varepsilon\rho}(x); \mathbb{R}^m\right)\right\}\\
&= \inf\left\{ \fint_{Q_{\varepsilon\rho}(x)} W\left(\frac{y}{\varepsilon}, \nabla u(y) + \nabla\varphi(y)\right) \mathrm{d}y : \varphi \in \mathrm{Aff}_0\left(Q_{\varepsilon\rho}(x); \mathbb{R}^m\right)\right\}\\
&= \frac{\mathrm{m}_u(Q_{\varepsilon\rho}(x))}{|Q_{\varepsilon\rho}(x)|}.
\end{aligned}$$

Consequently

$$\begin{aligned}
\mathrm{dm}_u(x) = \lim_{\rho\to 0} \frac{\mathrm{m}_u(Q_\rho(x))}{|Q_\rho(x)|} = \lim_{\rho\to 0} \frac{\mathrm{m}_u(Q_{\varepsilon\rho}(x))}{|Q_{\varepsilon\rho}(x)|} &= \lim_{\rho\to 0} \frac{\mathcal{S}^W_{\nabla u(x)}\left(Q_\rho\left(\frac{x}{\varepsilon}\right)\right)}{\left|Q_\rho\left(\frac{x}{\varepsilon}\right)\right|}\\
&= \mathcal{G}W\left(\frac{x}{\varepsilon}, \nabla u(x)\right)
\end{aligned}$$

for a.a. $x \in \Omega$. From (4.15) we deduce that $\mathrm{m}^*_u(\Omega) = \mathcal{G}I_\varepsilon(u)$. (So, in particular, we have $\mathrm{m}^*_u(\Omega) < \infty$ because $\mathcal{G}W$ has p-growth.) Thus, to establish (4.2) it remains to prove that

$$\overline{I}_\varepsilon(u) \le \mathrm{m}^*_u(\Omega). \tag{4.16}$$

Fix any $\delta > 0$. By definition of $\mathrm{m}^\delta_u(\Omega)$ there exists $\{Q_i\}_{i\in I} \in \mathcal{V}_\delta(\Omega)$ such that

$$\sum_{i\in I} \mathrm{m}_u(Q_i) \le \mathrm{m}^\delta_u(\Omega) + \frac{\delta}{2}. \tag{4.17}$$

Given any $i \in I$, by definition of $\mathrm{m}_u(Q_i)$ there exists $\varphi_i \in \mathrm{Aff}_0(Q_i; \mathbb{R}^m)$ such that

$$\int_{Q_i} W\left(\frac{y}{\varepsilon}, \nabla u(y) + \nabla\varphi_i(y)\right) \mathrm{d}y \le \mathrm{m}_u(Q_i) + \frac{\delta|Q_i|}{2|\Omega|}. \tag{4.18}$$

Define $\psi_\delta \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ by $\psi_\delta = u + \varphi_i$ in Q_i. Then $\psi_\delta - u \in W_0^{1,\infty}(\Omega;\mathbb{R}^m)$. From (4.17) and (4.18) we see that

$$\int_\Omega W\left(\frac{y}{\varepsilon}, \nabla\psi_\delta(y)\right) \mathrm{d}y \leq \mathrm{m}_u^\delta(\Omega) + \delta. \tag{4.19}$$

On the other hand, we have

$$\|\psi_\delta - u\|_{L^p(\Omega;\mathbb{R}^m)} = \int_\Omega |\psi_\delta(y) - u(y)|^p \,\mathrm{d}y = \sum_{i\in I} \int_{Q_i} |\varphi_i(y)|^p \,\mathrm{d}y. \tag{4.20}$$

But, since $\mathrm{diam}(Q_i) \in]0,\delta[$ for all $i \in I$, by using Poincaré's inequality we deduce that there exists $K > 0$, which depends only on p and N, such that for every $i \in I$,

$$\int_{Q_i} |\varphi_i(y)|^p \,\mathrm{d}y \leq K\delta^p \int_{Q_i} |\nabla\varphi_i(y)|^p \,\mathrm{d}y,$$

and so, taking (4.20) into account, we get

$$\begin{aligned}\|\psi_\delta - u\|_{L^p(\Omega;\mathbb{R}^m)} &\leq K\delta^p \sum_{i\in I} \int_{Q_i} |\nabla\varphi_i(y)|^p \,\mathrm{d}y \\ &= K\delta^p \int_\Omega |\nabla\psi_\delta(y) - \nabla u(y)|^p \,\mathrm{d}y \\ &\leq 2^p K\delta^p \left(\int_\Omega |\nabla\psi_\delta(y)|^p \,\mathrm{d}y + \int_\Omega |\nabla u(y)|^p \,\mathrm{d}y\right). \end{aligned} \tag{4.21}$$

Using (A_0) and (4.19), from (4.21) we deduce that

$$\|\psi_\delta - u\|_{L^p(\Omega;\mathbb{R}^m)} \leq 2^p K\delta^p \left(\frac{1}{C}(\mathrm{m}_u^\delta(\Omega) + \delta) + \int_\Omega |\nabla u(y)|^p \,\mathrm{d}y\right),$$

which shows that $\psi_\delta \to u$ in $L^p(\Omega;\mathbb{R}^m)$ because $\lim_{\delta\to 0} \mathrm{m}_u^\delta(\Omega) = \mathrm{m}_u^*(\Omega) < \infty$, and (4.15) follows from (4.19) by letting $\delta \to 0$ (and by noticing that $\overline{I}_\varepsilon(u) \leq \underline{\lim}_{\delta\to 0} \int_\Omega W\left(\frac{y}{\varepsilon}, \nabla\psi_\delta(y)\right) \mathrm{d}y$). This completes the proof of Theorem 2.33. $\square$

Acknowledgements

This work was initiated during the stage of Master of the second author, during the period from April 2016 to July 2016, in the Laboratory MIPA of the University of Nîmes.

Appendix

A.1. **Relaxation of singular integrands**

Here we give the proof of Theorem 2.12. For this, we need the following two lemmas. The first is a special case of a theorem due to Dacorogna and Ribeiro (see [19, Theorem 1.3], see also [18, Theorem 10.29]) and the second is a special case of a theorem due to Ben Belgacem (see [11], see also [5, Théorème 3.25] for a proof).

Lemma A.1. *Given $t_1 < t_2$ and $\xi \in \mathbb{M}^{N\times N}$ with $t_1 < \det\xi < t_2$ there exists $\varphi \in W_0^{1,\infty}(Y;\mathbb{R}^N)$ such that $\det(\xi+\nabla\varphi(y)) \in \{t_1, t_2\}$ for a.a. $y \in Y$.*

Lemma A.2. *Let $W : \mathbb{R}^N \times \mathbb{M}^{N\times N} \to [0,\infty]$ be a Borel measurable function. If W satisfies $(\widehat{\mathrm{H}})$ then $\mathcal{R}W$ has p-growth, where for every $x \in \mathbb{R}^N$, $\mathcal{R}W(x,\cdot)$ denotes the rank-one convex envelope of $W(x,\cdot)$, i.e., the greatest rank-one convex function which less than or equal to $W(x,\cdot)$.*

Proof of Theorem 2.12. Fix any $x \in \mathbb{R}^N$ and any $\xi \in \mathbb{M}^{N\times N}$. Clearly, if $|\det\xi| \geq \widehat{\alpha}$ then $\mathcal{Z}W(x,\xi) < \infty$. On the other hand, if $|\det\xi| < \widehat{\alpha}$ then, by Lemma A.1, there exists $\varphi \in W_0^{1,\infty}(Y;\mathbb{R}^N)$ such that $|\det(\xi + \nabla\varphi(y)| = \widehat{\alpha}$ for a.a. $y \in Y$, and using $(\widehat{\mathrm{H}})$ we see that

$$\begin{aligned}\mathcal{Z}W(x,\xi) &\leq \int_Y W(x,\xi+\nabla\varphi(y))\,\mathrm{d}y \\ &\leq 2^p\widehat{\beta}\left(1+|\xi|^p+\|\nabla\varphi\|^p_{L^p(Y;\mathbb{R}^N)}\right) < \infty\,.\end{aligned}$$

Thus $\mathcal{Z}W(x,\xi) < \infty$ for all $\xi \in \mathbb{M}^{N\times N}$, i.e., $\mathcal{Z}W(x,\cdot)$ is finite. From Proposition 3.1 (b) we deduce that $\mathcal{Z}W(x,\cdot)$ is rank-one convex. Hence $\mathcal{Z}W(x,\cdot) \leq \mathcal{R}W(x,\cdot)$ for all $x \in \mathbb{R}^N$, i.e., $\mathcal{Z}W \leq \mathcal{R}W$, and the theorem follows from Lemma A.2. ☐

Remark A.3. By the same method, in using [4, Lemma 4.2] (instead of Lemma A.1) to establish that $\widehat{\mathcal{Z}}H$ is finite when $H \in \mathcal{H}$, we can also prove the following result.

Theorem A.4. *If $H \in \mathcal{H}$ then $\widehat{\mathcal{Z}}H$ has p-growth.*

A.2. Approximation of the relaxation formula

Given a Borel measurable function $L : \mathbb{M}^{m\times N} \to [0,\infty]$ we consider $\widehat{\mathcal{Z}}L : \mathbb{M}^{m\times N} \to [0,\infty]$ defined by

$$\widehat{\mathcal{Z}}L(\xi) := \inf\left\{\int_Y L(\xi + \nabla\varphi(y))\,\mathrm{d}y : \varphi \in \mathrm{Aff}_0(Y;\mathbb{R}^m)\right\}.$$

The following proposition can be found in [6, Proposition 3.17] (see also [3, 4]).

Proposition A.5. *Given $\xi \in \mathbb{M}^{m\times N}$ and a bounded open set $A \subset \mathbb{R}^N$ there exists $\{\varphi_k\}_k \subset \mathrm{Aff}_0(A;\mathbb{R}^m)$ such that:*

- $\lim\limits_{k\to\infty} \|\varphi_k\|_{L^\infty(A;\mathbb{R}^m)} = 0$;
- $\lim\limits_{k\to\infty} \fint_A L(\xi + \nabla\varphi_k(x))\,\mathrm{d}x = \widehat{\mathcal{Z}}L(\xi)$.

Proof. Given $\xi \in \mathbb{M}^{m\times N}$ there exists $\{\varphi_n\}_n \subset \mathrm{Aff}_0(Y;\mathbb{R}^m)$ such that

$$\lim_{n\to\infty} \int_Y L(\xi + \nabla\varphi_n(y))\,\mathrm{d}y = \widehat{\mathcal{Z}}L(\xi). \tag{A.1}$$

Fix any $n \geq 1$ and $k \geq 1$. By Vitali's covering theorem there exists a finite or countable family $\{a_i + \alpha_i Y\}_{i\in I}$ of disjoint subsets of A, where $a_i \in \mathbb{R}^N$ and $0 < \alpha_i < \frac{1}{k}$, such that $|A \setminus \bigcup_{i\in I}(a_i + \alpha_i Y)| = 0$ (and so $\sum_{i\in I} \alpha_i^N = |A|$). Define $\varphi_{n,k} \in \mathrm{Aff}_0(A;\mathbb{R}^m)$ by

$$\varphi_{n,k}(x) := \alpha_i \varphi_n\left(\frac{x - a_i}{\alpha_i}\right) \quad \text{if } x \in a_i + \alpha_i Y.$$

Clearly $\|\varphi_{n,k}\|_{L^\infty(A;\mathbb{R}^m)} = \frac{1}{k}\|\varphi_n\|_{L^\infty(Y;\mathbb{R}^m)}$, hence

$$\lim_{k\to\infty} \|\varphi_{n,k}\|_{L^\infty(A;\mathbb{R}^m)} = 0$$

for all $k \geq 1$, and consequently

$$\lim_{n\to\infty}\lim_{k\to\infty} \|\varphi_{n,k}\|_{L^\infty(A;\mathbb{R}^m)} = 0. \tag{A.2}$$

On the other hand, we have

$$\begin{aligned}\int_A L(\xi + \nabla\varphi_{n,k}(x))\,\mathrm{d}x &= \sum_{i\in I} \alpha_i^N \int_Y L(\xi + \nabla\varphi_n(y))\,\mathrm{d}y \\ &= |A| \int_Y L(\xi + \nabla\varphi_n(y))\,\mathrm{d}y\end{aligned}$$

for all $n \geq 1$ and all $k \geq 1$. Using (A.1) we deduce that

$$\lim_{n\to\infty}\lim_{k\to\infty} \fint_A L(\xi + \nabla\varphi_{n,k}(x))\,\mathrm{d}x = \widehat{\mathcal{Z}}L(\xi)\,, \tag{A.3}$$

and the result follows from (A.2) and (A.3) by diagonalization. □

Remark A.6. By the same method, in replacing "Aff_0" by "$W_0^{1,\infty}$", we can establish the same approximation for $\mathcal{Z}L : \mathbb{M}^{m\times N} \to [0,\infty]$ given by

$$\mathcal{Z}L(\xi) := \inf\left\{\int_Y L(\xi + \nabla\varphi(y))\,\mathrm{d}y : \varphi \in W_0^{1,\infty}(Y;\mathbb{R}^m)\right\}.$$

Proposition A.7. *Given $\xi \in \mathbb{M}^{m\times N}$ and a bounded open set $A \subset \mathbb{R}^N$ there exists $\{\varphi_k\}_k \subset W_0^{1,\infty}(A;\mathbb{R}^m)$ such that:*

- $\displaystyle\lim_{k\to\infty} \|\varphi_k\|_{L^\infty(A;\mathbb{R}^m)} = 0\,;$
- $\displaystyle\lim_{k\to\infty} \fint_A L(\xi + \nabla\varphi_k(x))\,\mathrm{d}x = \mathcal{Z}L(\xi)\,.$

A.3. Approximation of the homogenization formula

Given a Borel measurable function $L : \mathbb{R}^N \times \mathbb{M}^{m\times N} \to [0,\infty]$ which is 1-periodic with respect to its first variable and for which there exists $c > 0$ such that

$$L(x,\xi) \leq c(1 + |\xi|^p) \tag{A.4}$$

for all $(x,\xi) \in \mathbb{R}^N \times \mathbb{M}^{m\times N}$, we consider $\widehat{\mathcal{H}}L : \mathbb{M}^{m\times N} \to [0,\infty]$ defined by

$$\widehat{\mathcal{H}}L(\xi) := \inf_{k\geq 1}\inf\left\{\fint_{kY} L(x, \xi + \nabla\varphi(x))\,\mathrm{d}x : \varphi \in \mathrm{Aff}_0(kY;\mathbb{R}^m)\right\}.$$

The following proposition is essentially due to [24, Lemma 2.1 (a)] (see also [6, Proposition 3.18]).

Proposition A.8. *Under* (A.4)*, given $\xi \in \mathbb{M}^{m\times N}$ and a bounded open set $A \subset \mathbb{R}^N$ there exists $\{\varphi_\varepsilon\}_\varepsilon \subset \mathrm{Aff}_0(A;\mathbb{R}^m)$ such that:*

- $\displaystyle\lim_{\varepsilon\to 0} \|\varphi_\varepsilon\|_{L^\infty(A;\mathbb{R}^m)} = 0\,;$
- $\displaystyle\lim_{\varepsilon\to 0} \fint_A L\left(\frac{x}{\varepsilon}, \xi + \nabla\varphi_\varepsilon(x)\right)\mathrm{d}x = \widehat{\mathcal{H}}L(\xi)\,.$

Proof. Given $\xi \in \mathbb{M}^{m\times N}$ there exists $\{k_n;\hat{\varphi}_n\}_n$ such that:

$$\hat{\varphi}_n \in \mathrm{Aff}_0(k_nY;\mathbb{R}^m) \text{ for all } n\geq 1;$$
$$\lim_{n\to\infty} \fint_{k_nY} L(x,\xi+\nabla\hat{\varphi}_n(x))\,\mathrm{d}x = \widehat{\mathcal{H}}L(\xi)\,. \tag{A.5}$$

For each $n\geq 1$ and $\varepsilon>0$, denote the k_nY-periodic extension of $\hat{\varphi}_n$ by φ_n, consider $A_{n,\varepsilon}\subset A$ given by

$$A_{n,\varepsilon} := \bigcup_{z\in I_{n,\varepsilon}} \varepsilon(z+k_nY)$$

with $I_{n,\varepsilon} := \big\{z\in\mathbb{Z}^N : \varepsilon(z+k_nY)\subset A\big\}$, where $\mathrm{card}(I_{n,\varepsilon})<\infty$ because A is bounded, and define $\varphi_{n,\varepsilon}\in \mathrm{Aff}_0(A;\mathbb{R}^m)$ by

$$\varphi_{n,\varepsilon}(x) := \varepsilon\varphi_n\left(\frac{x}{\varepsilon}\right) \text{ if } x\in A_{n,\varepsilon}\,.$$

Fix any $n\geq 1$. It is easy to see that

$$\|\varphi_{n,\varepsilon}\|_{L^\infty(A;\mathbb{R}^m)} = \varepsilon\|\hat{\varphi}_n\|_{L^\infty(k_nY;\mathbb{R}^m)}$$

for all $\varepsilon>0$, and consequently $\lim_{\varepsilon\to 0}\|\varphi_{n,\varepsilon}\|_{L^\infty(A;\mathbb{R}^m)}=0$ for all $n\geq 1$. It follows that

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0}\|\varphi_{n,\varepsilon}\|_{L^\infty(A;\mathbb{R}^m)}=0. \tag{A.6}$$

Remark A.9. We also have $\lim_{n\to\infty}\lim_{\varepsilon\to 0}\|\phi_{n,\varepsilon}\|_{L^p(A;\mathbb{R}^m)}=0$. Indeed, it is easy to see that

$$\begin{aligned}
\|\phi_{n,\varepsilon}\|^p_{L^p(A;\mathbb{R}^m)} &= \int_{A_{n,\varepsilon}} |\phi_{n,\varepsilon}(x)|^p\,\mathrm{d}x \\
&= \varepsilon^p \sum_{z\in I_{n,\varepsilon}} \int_{\varepsilon(z+k_nY)} \left|\phi_n\left(\frac{x}{\varepsilon}\right)\right|^p \mathrm{d}x \\
&\leq \varepsilon^p \frac{|A|}{k_n^N}\|\hat{\phi}_n\|^p_{L^p(k_nY;\mathbb{R}^m)}
\end{aligned}$$

for all $\varepsilon>0$, and consequently $\lim_{\varepsilon\to 0}\|\phi_{n,\varepsilon}\|_{L^p(A;\mathbb{R}^m)}=0$ for all $n\geq 1$, hence the result.

On the other hand, for every $n \geq 1$ and every $\varepsilon > 0$, we have

$$\int_A L\left(\frac{x}{\varepsilon}, \xi + \nabla\varphi_{n,\varepsilon}(x)\right) \mathrm{d}x = \int_{A_{n,\varepsilon}} L\left(\frac{x}{\varepsilon}, \xi + \nabla\varphi_{n,\varepsilon}(x)\right) \mathrm{d}x + \int_{A\setminus A_{n,\varepsilon}} L\left(\frac{x}{\varepsilon}, \xi\right) \mathrm{d}x\,.$$

But

$$\begin{aligned}\int_{A_{n,\varepsilon}} L\left(\frac{x}{\varepsilon}, \xi + \nabla\varphi_{n,\varepsilon}(x)\right) \mathrm{d}x &= \sum_{z\in I_{n,\varepsilon}} \int_{\varepsilon(z+k_nY)} L\left(\frac{x}{\varepsilon}, \xi + \nabla\varphi_n\left(\frac{x}{\varepsilon}\right)\right) \mathrm{d}x \\ &= |A_{n,\varepsilon}| \fint_{k_nY} L(x, \xi + \nabla\hat{\varphi}_n(x))\,\mathrm{d}x\,,\end{aligned}$$

and consequently

$$\begin{aligned}|A_{n,\varepsilon}|\widehat{\mathcal{H}}L(\xi) &\leq \int_A L\left(\frac{x}{\varepsilon}, \xi + \nabla\varphi_{n,\varepsilon}(x)\right) \mathrm{d}x \\ &\leq |A| \fint_{k_nY} L(x, \xi + \nabla\hat{\varphi}_n(x))\,\mathrm{d}x + c|A \setminus A_{n,\varepsilon}|(1 + |\xi|^p)\end{aligned}$$

by (A.4). As $\lim_{\varepsilon\to 0} |A \setminus A_{n,\varepsilon}| = 0$ for any $n \geq 1$, and by using (A.5), we see that:

$$\lim_{\varepsilon\to 0} |A \setminus A_{n,\varepsilon}|\widehat{\mathcal{H}}L(\xi) = 0;$$

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0}\left(\fint_{k_nY} L\left(x, \xi + \nabla\hat{\varphi}_n(x)\right) \mathrm{d}x - \widehat{\mathcal{H}}L(\xi) + \frac{|A \setminus A_{n,\varepsilon}|}{|A|}(1+|\xi|^p)\right) = 0\,.$$

Hence

$$\lim_{n\to\infty}\overline{\lim_{\varepsilon\to 0}}\left|\fint_A L\left(\frac{x}{\varepsilon}, \xi + \nabla\varphi_{n,\varepsilon}(x)\right) \mathrm{d}x - \widehat{\mathcal{H}}L(\xi)\right| = 0\,, \tag{A.7}$$

and the result follows from (A.6) and (A.7) by diagonalization. □

Remark A.10. By the same method, in using Remark A.9, we can establish the same approximation for $\mathcal{H}L : \mathbb{M}^{m\times N} \to [0, \infty]$ given by

$$\mathcal{H}L(\xi) := \inf_{k\geq 1}\inf\left\{\fint_{kY} L(x, \xi + \nabla\varphi(x))\,\mathrm{d}x : \varphi \in W_0^{1,p}(kY; \mathbb{R}^m)\right\}.$$

Proposition A.11. *Under* (A.4), *given* $\xi \in \mathbb{M}^{m\times N}$ *and a bounded open set* $A \subset \mathbb{R}^N$ *there exists* $\{\varphi_\varepsilon\}_\varepsilon \subset W_0^{1,p}(A;\mathbb{R}^m)$ *such that:*

- $\lim\limits_{\varepsilon\to 0} \|\varphi_\varepsilon\|_{L^p(A;\mathbb{R}^m)} = 0\,;$
- $\lim\limits_{\varepsilon\to 0} \fint_A L\left(\frac{x}{\varepsilon}, \xi + \nabla\varphi_\varepsilon(x)\right) \mathrm{d}x = \mathcal{H}L(\xi)\,.$

A.4. Integral representation of the Vitali envelope of a set function

What follows was first developed in [13, 12] (see also [8, 9]). Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $\mathcal{O}(\Omega)$ be the class of open subsets of Ω. We begin with the concept of differentiability of a set function.

Definition A.12. We say that a set function $\Theta : \mathcal{O}(\Omega) \to \mathbb{R}$ is differentiable (with respect to the Lebesgue measure) if

$$\mathrm{d}\Theta(x) := \lim_{\rho\to 0} \frac{\Theta(Q_\rho(x))}{|Q_\rho(x)|} \tag{A.8}$$

exists and is finite for a.a. $x \in \Omega$, where $Q_\rho(x) := x + \rho Y$ with $Y := \left]-\frac{1}{2}, \frac{1}{2}\right[^N$.

Remark A.13. It is easy to see that the limit in (A.8) exists and is finite if and only if $-\infty < \mathrm{d}^+\Theta \leq \mathrm{d}^-\Theta < \infty$, where $\mathrm{d}^-\Theta : X \to [-\infty, \infty[$ and $\mathrm{d}^+\Theta : X \to]-\infty, \infty]$ are given by:

$$\begin{aligned} \mathrm{d}^-\Theta(x) &:= \lim_{\rho\to 0} \mathrm{d}^-\Theta(x,\rho) \\ &\qquad \text{with } \mathrm{d}^-\Theta(x,\rho) := \inf\left\{\frac{\Theta(Q)}{|Q|} : Q \in \mathrm{Cub}(\Omega, x, \rho)\right\}; \end{aligned} \tag{A.9}$$

$$\begin{aligned} \mathrm{d}^+\Theta(x) &:= \lim_{\rho\to 0} \mathrm{d}^+\Theta(x,\rho) \\ &\qquad \text{with } \mathrm{d}^+\Theta(x,\rho) := \sup\left\{\frac{\Theta(Q)}{|Q|} : Q \in \mathrm{Cub}(\Omega, x, \rho)\right\}, \end{aligned} \tag{A.10}$$

where $\mathrm{Cub}(\Omega, x, \rho)$ denotes the class of open cubes Q of Ω such that $x \in Q$ and $\mathrm{diam}(Q) \in]0, \rho[$. We then have $d\Theta = \mathrm{d}^-\Theta = \mathrm{d}^+\Theta$.

Remark A.14. In (A.9) and (A.10) we can replace $\mathrm{Cub}(\Omega, x, \rho)$ by $\mathrm{Cub}(A, x, \rho)$ whenever $A \in \mathcal{O}(\Omega)$ and $x \in A$.

For each $\delta > 0$ and each $A \in \mathcal{O}(\Omega)$, we denote the class of countable families $\{Q_i\}_{i \in I}$ of disjoint open cubes such that $\operatorname{diam}(Q_i) \in \,]0, \delta[$ for all $i \in I$ and $|A \setminus \bigcup_{i \in I} Q_i| = 0$ by $\mathcal{V}_\delta(A)$.

Definition A.15. Given $\Theta : \mathcal{O}(\Omega) \to \mathbb{R}$, for each $\delta > 0$ we define $\Theta^\delta : \mathcal{O}(\Omega) \to [-\infty, \infty]$ by

$$\Theta^\delta(A) := \inf \left\{ \sum_{i \in I} \Theta(Q_i) : \{Q_i\}_{i \in I} \in \mathcal{V}_\delta(A) \right\}. \tag{A.11}$$

By the Vitali envelope of Θ we call the set function $\Theta^* : \mathcal{O}(\Omega) \to [-\infty, \infty]$ defined by

$$\Theta^*(A) := \sup_{\delta > 0} \Theta^\delta(A) = \lim_{\delta \to 0} \Theta^\delta(A). \tag{A.12}$$

The interest of Definition A.15 comes from the following integral representation result whose proof is given below.

Theorem A.16. *Let $\Theta : \mathcal{O}(\Omega) \to \mathbb{R}$ be a set function satisfying the following two conditions:*

(a) *there exists a finite Radon measure ν on Ω which is absolutely continuous with respect to the Lebesgue measure such that $|\Theta(A)| \leq \nu(A)$ for all $A \in \mathcal{O}(\Omega)$;*

(b) *Θ is subadditive, i.e., $\Theta(A) \leq \Theta(B) + \Theta(C)$ for all $A, B, C \in \mathcal{O}(\Omega)$ with $B, C \subset A$, $B \cap C = \emptyset$ and $|A \setminus B \cup C| = 0$.*

Then Θ is differentiable, $\mathrm{d}\Theta \in L^1(\Omega)$ *and*

$$\Theta^*(A) = \int_A \mathrm{d}\Theta(x)\, \mathrm{d}x$$

for all $A \in \mathcal{O}(\Omega)$.

Proof. First of all, From (a) we see that $-\mathrm{d}\nu \leq \mathrm{d}^-\Theta \leq \mathrm{d}^+\Theta \leq \mathrm{d}\nu$. Hence $\mathrm{d}^-\Theta, \mathrm{d}^+\Theta \in L^1(\Omega)$ because ν is a finite Radon measure which is absolutely continuous with respect to the Lebesgue measure. So $\lambda^-(A)$, $\lambda^+(A) \in \mathbb{R}$ for all $A \in \mathcal{O}(\Omega)$, where λ^-, $\lambda^+ : \mathcal{O}(\Omega) \to \mathbb{R}$ are given by:

$$\lambda^-(A) := \int_A \mathrm{d}^-\Theta(x)\, \mathrm{d}x\,;$$

$$\lambda^+(A) := \int_A \mathrm{d}^+\Theta(x)\, \mathrm{d}x\,.$$

In what follows, we consider $\overline{\Theta}^* : \mathcal{O}(\Omega) \to \mathbb{R}$ defined by

$$\overline{\Theta}^*(A) := \inf_{\delta>0} \sup \left\{ \sum_{i\in I} \Theta(Q_i) : \{Q_i\}_{i\in I} \in \mathcal{V}_\delta(A) \right\}. \tag{A.13}$$

(It is clear that $\Theta^* \le \overline{\Theta}^*$. In fact, we are going to prove that under the assumptions (a) and (b) of Theorem A.16 we have $\Theta^*(A) = \overline{\Theta}^*(A) = \int_A \mathrm{d}\Theta(x)\,\mathrm{d}x$ for all $A \in \mathcal{O}(\Omega)$.) We divide the proof into three steps.

Step 1: proving that $\Theta^* = \lambda^-$ and $\overline{\Theta}^* = \lambda^+$.

Define $\theta^-, \theta^+ : \mathcal{O}(\Omega) \to \mathbb{R}$ by:

$$\theta^-(A) := \Theta(A) - \lambda^-(A);$$
$$\theta^+(A) := \Theta(A) - \lambda^+(A).$$

In what follows, θ^* (resp. $\overline{\theta}^*$) is defined by (A.12) (resp. (A.13)) with Θ replaced by θ^- (resp. θ^+).

Lemma A.17. *Under the assumption* (a) *of Theorem A.16 we have* $\theta^* = \overline{\theta}^* = 0$.

Proof. We only prove that $\theta^* = 0$. (The proof of $\overline{\theta}^* = 0$ follows from similar arguments and is left to the reader.)

First of all, from the assumption (a) it is clear that

$$|\theta^-(A)| \le \hat{\nu}(A) \tag{A.14}$$

for all $A \in \mathcal{O}(\Omega)$, where $\hat{\nu} := \nu + |\nu|$ is absolutely continuous with respect to Lebesgue measure (with $|\nu|$ denoting the total variation of ν).

Secondly, we can assert that

$$\mathrm{d}^-\theta^- = 0, \tag{A.15}$$

where for any set function $\mathrm{s} : \mathcal{O}(\Omega) \to \mathbb{R}$, the function $\mathrm{d}^-\mathrm{s} : \Omega \to [-\infty, \infty[$ (resp. $\mathrm{d}^+\mathrm{s} : \Omega \to]-\infty, \infty]$) is defined by (A.9) (resp. (A.10)) with Θ replaced by s. Indeed, for any $x \in \Omega$, it is easily seen that

$$\mathrm{d}^-\Theta(x,\rho) - \mathrm{d}^+\lambda^-(x,\rho) \le \mathrm{d}^-\theta^-(x,\rho) \le \mathrm{d}^-\Theta(x,\rho) - \mathrm{d}^-\lambda^-(x,\rho).$$

for all $\rho > 0$, and letting $\rho \to 0$, we obtain

$$\mathrm{d}^-\Theta(x) - \mathrm{d}^+\lambda^-(x) \le \mathrm{d}^-\theta^-(x) \le \mathrm{d}^-\Theta(x) - \mathrm{d}^-\lambda^-(x).$$

But $\mathrm{d}^-\lambda^-(x) = \mathrm{d}^+\lambda^-(x) = \mathrm{d}^-\Theta(x)$, hence $\mathrm{d}^-\theta^-(x) = 0$.

Finally, to conclude we prove that (A.14) and (A.15) imply $\theta^* = 0$. For this, we are going to prove the following two assertions:

$$\text{if } \mathrm{d}^-\theta^- \leq 0 \text{ then } \theta^* \leq 0\,; \tag{A.16}$$

$$\text{under (A.14), if } \mathrm{d}^-\theta^- \geq 0 \text{ then } \theta^* \geq 0\,. \tag{A.17}$$

Proof of (A.16). Fix $A \in \mathcal{O}(\Omega)$. Fix any $\delta > 0$. Then $\mathrm{d}^-\theta^- < \delta$, and so in particular $\lim_{\rho\to 0} \mathrm{d}^-\theta^-(x,\rho) < \delta$ for all $x \in A$. Hence, for each $x \in A$ there exists $\{\rho_{x,n}\}_n \subset]0,\delta[$ with $\rho_{x,n} \to 0$ as $n \to \infty$ such that $\mathrm{d}^-\theta^-(x,\rho_{x,n}) < \delta$ for all $n \geq 1$. Taking Remark A.14 into account, it follows that for each $x \in A$ and each $n \geq 1$ there is $Q_{x,n} \in \mathrm{Cub}(A,x,\rho_{x,n})$ such that for each $x \in A$ and each $n \geq 1$,

$$\frac{\theta^-(Q_{x,n})}{|Q_{x,n}|} < \delta\,. \tag{A.18}$$

Moreover, since $\mathrm{diam}\,(\overline{Q}_{x,n}) = \mathrm{diam}(Q_{x,n}) \leq \rho_{x,n}$ for all $x \in A$ and all $n \geq 1$, we have $\inf\big\{\mathrm{diam}\,(\overline{Q}_{x,n}) : n \geq 1\big\} = 0$ (where $\overline{Q}_{x,n}$ denotes the closed cube corresponding to the open cube $Q_{x,n}$). Let $\mathcal{F}_0$ be the family of closed cubes of Ω given by

$$\mathcal{F}_0 := \Big\{\overline{Q}_{x,n} : x \in A \text{ and } n \geq 1\Big\}.$$

By the Vitali covering theorem, from the above we deduce that there exists a disjointed countable subfamily $\{\overline{Q}_i\}_{i\in I_0}$ of closed cubes of $\mathcal{F}_0$ (with $Q_i \subset A$ and $\mathrm{diam}(Q_i) \in]0,\delta[$) such that $|A \setminus \bigcup_{i\in I_0} \overline{Q}_i| = 0$, which means that $\{Q_i\}_{i\in I_0} \in \mathcal{V}_\delta(A)$. From (A.18) we see that $\theta^-(Q_i) < \delta|Q_i|$ for all $i \in I_0$, hence

$$\sum_{i\in I_0} \theta^-(Q_i) \leq \delta \sum_{i\in I_0} |Q_i| = \delta|A|\,.$$

Consequently $\theta^{-,\delta}(A) \leq \delta|A|$ for all $\delta > 0$, where $\theta^{-,\delta}$ is defined by (A.11) with Θ replaced by θ^-, and letting $\delta \to 0$ we obtain $\theta^*(A) \leq 0$.

Proof of (A.17). Fix $A \in \mathcal{O}(\Omega)$. By Egorov's theorem, there exists a sequence $\{B_n\}_n$ of Borel subsets of A such that:

$$\lim_{n\to\infty} |A \setminus B_n| = 0\,; \tag{A.19}$$

$$\lim_{\delta\to 0} \sup_{x\in B_n} \big|\mathrm{d}^-\theta^-(x) - \mathrm{d}^-\theta^-(x,\delta)\big| = 0 \quad \text{for all } n \geq 1\,. \tag{A.20}$$

As $\hat{\nu}$ is absolutely continuous with respect to the Lebesgue measure, by (A.19) we have

$$\lim_{n\to\infty} \hat{\nu}(A \setminus B_n) = 0\,. \tag{A.21}$$

Moreover, as $\mathrm{d}^-\theta^- \geq 0$, from (A.20) we deduce that

$$\varliminf_{\delta\to 0} \inf_{x\in B_n} \mathrm{d}^-\theta^-(x,\delta) \geq 0 \quad \text{for all } n \geq 1\,. \tag{A.22}$$

Fix any $n \geq 1$ and any $\delta > 0$. By definition of $\theta^{-,\delta}$, there exists $\{Q_i\}_{i\in I} \in \mathcal{V}_\delta(A)$ such that

$$\theta^{-,\delta}(A) > \sum_{i\in I} \theta^-(Q_i) - \delta\,. \tag{A.23}$$

Set $I_n := \big\{i \in I : Q_i \cap B_n \neq \emptyset\big\}$. Using (A.14) we have

$$\begin{aligned}
\sum_{i\in I} \theta^-(Q_i) &= \sum_{i\in I_n} \theta^-(Q_i) + \sum_{i\in I\setminus I_n} \theta^-(Q_i) \\
&\geq \sum_{i\in I_n} \theta^-(Q_i) - \sum_{i\in I\setminus I_n} \hat{\nu}(Q_i) \\
&\geq \sum_{i\in I_n} \frac{\theta^-(Q_i)}{|Q_i|}|Q_i| - \hat{\nu}\left(\bigcup_{i\in I\setminus I_n} Q_i\right),
\end{aligned}$$

and, choosing $x_i \in Q_i \cap B_n$ for each $i \in I_n$ and noticing that $\bigcup_{i\in I\setminus I_n} Q_i \subset A \setminus B_n$, it follows that

$$\begin{aligned}
\sum_{i\in I} \theta^-(Q_i) &\geq \sum_{i\in I_n} \mathrm{d}^-\theta^-(x_i,\delta)|Q_i| - \hat{\nu}(A \setminus B_n) \\
&\geq \inf_{x\in B_n} \mathrm{d}^-\theta^-(x,\delta) \sum_{i\in I_n} |Q_i| - \hat{\nu}(A \setminus B_n)\,.
\end{aligned}$$

Taking (A.23) into account, we conclude that

$$\theta^{-,\delta}(A) \geq \inf_{x\in B_n} \mathrm{d}^-\theta^-(x,\delta) \sum_{i\in I_n} |Q_i| - \hat{\nu}(A \setminus B_n) - \delta$$

for all $\delta > 0$ and all $n \geq 1$, which gives $\theta^*(A) \geq 0$ by letting $\delta \to 0$ and using (A.22) and then by letting $n \to \infty$ and using (A.21). $\square$

As λ^- and λ^+ are absolutely continuous with respect to the Lebesgue measure, it is easy to see that:

$$\theta^* = \Theta^* - \lambda^- ;$$
$$\overline{\theta}^* = \overline{\Theta}^* - \lambda^+ .$$

Hence $\Theta^* = \lambda^-$ and $\overline{\Theta}^* = \lambda^+$ by Lemma A.17.

Step 2: proving that $\Theta^* = \overline{\Theta}^*$.

We only need to prove that $\overline{\Theta}^* \leq \Theta^*$. For this, it is sufficient to show that for each open cube Q of Ω, one has

$$\Theta(Q) \leq \Theta^*(Q) . \tag{A.24}$$

Fix any $\delta > 0$. By definition of Θ^δ, there exists $\{Q_i\}_{i\in I} \in \mathcal{V}_\delta(Q)$ such that

$$\sum_{i\in I} \Theta(Q_i) \leq \Theta^\delta(Q) + \delta . \tag{A.25}$$

Since $|Q \setminus \bigcup_{i\in I} Q_i| = 0$ there is a sequence $\{I_n\}_n$ of finite subsets of I such that

$$\lim_{n\to\infty} \left| Q \setminus \bigcup_{i\in I_n} Q_i \right| = \lim_{n\to\infty} \left| \bigcup_{i\in I\setminus I_n} Q_i \right| = 0 . \tag{A.26}$$

Fix any $n \geq 1$. As Θ is subadditive, see the assumption (b), we have

$$\Theta\left(\bigcup_{i\in I_n} Q_i \right) \leq \sum_{i\in I_n} \Theta(Q_i) .$$

Moreover, $\left| Q \setminus [(\bigcup_{i\in I_n} Q_i) \cup (Q \setminus \overline{\bigcup_{i\in I_n} Q_i})] \right| = 0$ because $|\overline{Q}_i \setminus Q_i| = 0$ for all $i \in I_n$, hence

$$\Theta(Q) \leq \Theta\left(\bigcup_{i\in I_n} Q_i \right) + \Theta\left(Q \setminus \overline{\bigcup_{i\in I_n} Q_i} \right)$$

by using again the subadditivity of Θ, and consequently

$$\sum_{i\in I_n} \Theta(Q_i) \geq \Theta(Q) - \Theta\left(Q \setminus \overline{\bigcup_{i\in I_n} Q_i} \right) .$$

Thus, using the assumption (a), we get

$$\begin{aligned}\sum_{i\in I}\Theta(Q_i) &= \sum_{i\in I\setminus I_n}\Theta(Q_i)+\sum_{i\in I_n}\Theta(Q_i)\\ &\geq \sum_{i\in I\setminus I_n}\Theta(Q_i)+\Theta(Q)-\Theta\left(Q\setminus\overline{\bigcup_{i\in I_n}Q_i}\right)\\ &\geq \Theta(Q)-\nu\left(\bigcup_{i\in I\setminus I_n}Q_i\right)-\nu\left(Q\setminus\overline{\bigcup_{i\in I_n}Q_i}\right).\end{aligned}$$

But, $\nu(\overline{Q}_i\setminus Q_i)=0$ for all $i\in I_n$ because ν is absolutely with respect to the Lebesgue measure, hence

$$\nu\left(Q\setminus\overline{\bigcup_{i\in I_n}Q_i}\right)=\nu\left(Q\setminus\bigcup_{i\in I_n}Q_i\right)=\nu\left(\bigcup_{i\in I\setminus I_n}Q_i\right),$$

and so

$$\sum_{i\in I}\Theta(Q_i)\geq\Theta(Q)-2\nu\left(\bigcup_{i\in I\setminus I_n}Q_i\right). \tag{A.27}$$

Combining (A.25) with (A.27) we conclude that

$$\Theta(Q)\leq\Theta^\delta(Q)+2\nu\left(\bigcup_{i\in I\setminus I_n}Q_i\right)+\delta,$$

and (A.24) follows by letting $n\to\infty$ and using (A.26) and then by letting $\delta\to 0$.

Step 3: end of the proof.

From steps 1 and 2 we have

$$\int_\Omega \mathrm{d}^-\Theta(x)\,\mathrm{d}x=\Theta^*(\Omega)=\overline{\Theta}^*(\Omega)=\int_\Omega \mathrm{d}^+\Theta(x)\,\mathrm{d}x.$$

Thus $\int_\Omega(\mathrm{d}^+\Theta(x)-\mathrm{d}^-\Theta(x))\,\mathrm{d}x=0$. But $\mathrm{d}^+\Theta\geq\mathrm{d}^-\Theta$, i.e., $\mathrm{d}^+\Theta-\mathrm{d}^-\Theta\geq 0$, hence $\mathrm{d}^+\Theta-\mathrm{d}^-\Theta=0$, i.e., $\mathrm{d}^+\Theta=\mathrm{d}^-\Theta$, and the proof of Theorem A.16 is complete. $\square$

References

[1] E. Acerbi & N. Fusco – “Semicontinuity problems in the calculus of variations”, *Arch. Ration. Mech. Anal.* **86** (1984), no. 2, p. 125–145.

[2] O. Anza Hafsa, M. L. Leghmizi & J.-P. Mandallena – “On a homogenization technique for singular integrals”, *Asymptotic Anal.* **74** (2011), no. 3-4, p. 123–134.

[3] O. Anza Hafsa & J.-P. Mandallena – “Relaxation of variational problems in two-dimensional nonlinear elasticity”, *Ann. Mat. Pura Appl.* **186** (2007), no. 1, p. 185–196.

[4] ———, “Relaxation theorems in nonlinear elasticity”, *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **25** (2008), no. 1, p. 135–148.

[5] ———, “Relaxation et passage 3d-2d avec contraintes de type déterminant”, `https://arxiv.org/abs/0901.3688`, 2009.

[6] ———, “Homogenization of nonconvex integrals with convex growth”, *J. Math. Pures Appl.* **96** (2011), no. 2, p. 167–189.

[7] ———, “Homogenization of unbounded singular integrals in $W^{1,\infty}$”, *Ric. Mat.* **61** (2012), no. 2, p. 185–217.

[8] ———, “Γ-limits of functionals determined by their infima”, *J. Convex Anal.* **23** (2016), no. 1, p. 103–137.

[9] ———, “Relaxation of nonconvex unbounded intergals with general growth conditions in cheeger-sobolev spaces”, à paraître dans *Bull. Sci. Math.*, 2016.

[10] O. Anza Hafsa, J.-P. Mandallena & H. Zorgati – “Homogenization of unbounded integrals with quasiconvex growth”, *Ann. Mat. Pura Appl.* **194** (2015), no. 6, p. 1619–1648.

[11] H. Ben Belgacem – “Modélisation de structures minces en élasticité non linéaire”, Thèse, Université Pierre et Marie Curie (France), 1996.

[12] G. Bouchitté & M. Bellieud – “Regularization of a set function – application to integral representation”, *Ric. Mat.* **49 (suppl.)** (2000), p. 79–93.

[13] G. Bouchitté, I. Fonseca & L. Mascarenhas – “A global method for relaxation”, *Arch. Ration. Mech. Anal.* **145** (1998), no. 1, p. 51–98.

[14] A. Braides – "Homogenization of some almost periodic coercive functional", *Rend. Accad. Naz. Sci. Detta XL, Mem. Mat.* **9** (1985), no. 1, p. 313–322.

[15] ______, Γ*-convergence for beginners*, Oxford Lecture Series in Mathematics and its Applications, vol. 22, Oxford University Press, 2002.

[16] A. Braides & A. Defranceschi – *Homogenization of multiple integrals*, Oxford Lecture Series in Mathematics and its Applications, vol. 12, Clarendon Press, 1998.

[17] B. Dacorogna – "Quasiconvexity and relaxation of nonconvex problems in the calculus of variations", *J. Funct. Anal.* **46** (1982), p. 102–118.

[18] ______, *Direct methods in the calculus of variations*, 2^{e}éd., Applied Mathematical Sciences, vol. 78, Springer, 2008.

[19] B. Dacorogna & A. M. Ribeiro – "Existence of solutions for some implicit partial differential equations and applications to variational integrals involving quasi-affine functions", *Proc. R. Soc. Edinb., Sect. A, Math.* **134** (2004), no. 5, p. 907–921.

[20] G. Dal Maso – *An introduction to* Γ*-convergence*, Progress in Nonlinear Differential Equations and their Applications, vol. 8, Birkhäuse, 1993.

[21] M. Duerinckx & A. Gloria – "Stochastic Homogenization of Nonconvex Unbounded Integral Functionals with Convex Growth", *Arch. Ration. Mech. Anal.* **221** (2016), no. 3, p. 1511–1584.

[22] I. Fonseca – "The lower quasiconvex envelope of the stored energy function for an elastic crystal", *J. Math. Pures Appl.* **67** (1988), no. 2, p. 175–195.

[23] C. B. Morrey – "Quasi-convexity and the lower semicontinuity of multiple integrals", *Pac. J. Math.* **2** (1952), p. 25–53.

[24] S. Müller – "Homogenization of nonconvex integral functionals and cellular elastic materials", *Arch. Ration. Mech. Anal.* **99** (1987), no. 3, p. 189–212.

HOMOGENIZATION OF SINGULAR INTEGRALS

OMAR ANZA HAFSA
Université de Nîmes
Laboratoire MIPA, Site des Carmes
Place Gabriel Péri
30021 Nîmes, France
and
LMGC, UMR-CNRS 5508
Place Eugène Bataillon
34095 Montpellier, France
omar.anza-hafsa@unimes.fr

JEAN-PHILIPPE MANDALLENA
Université de Nîmes
Laboratoire MIPA, Site des Carmes
Place Gabriel Péri
30021 Nîmes, France
jean-philippe.mandallena@unimes.fr

NICOLAS CLOZEAU
École Normale Supérieure de Cachan
61 avenue du président Wilson
94230 Cachan, France
nicolas.clozeau@ens-cachan.fr