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# Two-level solitary waves as generalized solutions of the KdV equation

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An analytical class of standing wave solutions to the Korteweg–de Vries (KdV) equation is obtained in the framework of continuous finite generalized functions. This paper shows how this class can be used to describe a great variety of wave shapes, especially bores and jumps. These new solutions are built by appropriately combining parts of two ordinary KdV waves. These represent a system with a steep transition between different energy levels of two potential wells. A number of specific cases of the generalized solutions are identical to those obtained recently within the theory of the KdV equation forced by a Dirac delta function. Numerical simulations of both stationary and transient KdV equations are carried out in a few cases. The weak formulation used in the numerical scheme is equivalent to the analytical generalized functions approach. Simulations of initially perturbed wave fronts prove the high degree of stability of many of these solutions. © 1995 American Institute of Physics.

## I. INTRODUCTION

It is known that free-surface waves with a transition between two different flat levels cannot exist without dissipation of energy. The Korteweg–de Vries (KdV) equation, as a long-wave approximation of dispersive waves in the background of a perfect fluid model, admits no solution of this kind.

One way of generalizing the equation to include such a solution is to add a term to represent a forcing action. This forced KdV (fKdV) equation arises in many situations: moving pressure distribution,<sup>1</sup> one-layer flow,<sup>2</sup> two-layer flow,<sup>3</sup> stratified flow over topography,<sup>4</sup> surface waves in channels of arbitrary shape,<sup>5</sup> and so forth. In this family of works, level-jump flows, hydraulic falls, and cusped solitary waves were derived as steady-state solutions.<sup>2,3</sup>

Another approach is the one that motivates the present work. It consists of looking for weak solutions of the original KdV model in the context of generalized functions. Such a solution has already been obtained in gas dynamics in the form of infinitely narrow solitary wave with infinitely large amplitude.<sup>6</sup> A more realistic result of this idea was achieved by fitting laboratory-generated bores with discontinuities of velocity and acceleration.<sup>7,8</sup>

Many works have been devoted to the numerical behavior of the KdV equation. Most of these deal with the stability of regular solitary or periodic solutions. The stability of generalized solution waves have received much less attention and their numerical simulation is still in its infancy.

In the following, we introduce the generalized formulation of the KdV equation (Sec. II). We deduce its weak solutions by introducing a discontinuity in the first and/or second derivative. The equivalence is outlined between some of these solutions and those obtained for the fKdV. Section III is dedicated to numerical time-dependent simulations. Re-

sults are presented on the stability of certain generalized solutions with respect to perturbed initial conditions. The value of the jump in the second derivative is related to the amplitude of a localized force.

## II. GENERALIZED SOLUTIONS OF KDV EQUATION

When the reduced KdV equation

$$\frac{\partial u(x,t)}{\partial t} + u(x,t) \frac{\partial u(x,t)}{\partial x} + \frac{\partial^3 u(x,t)}{\partial x^3} = 0 \quad (1)$$

is written in the coordinate system  $X = x - ct$ , it becomes the equation of the stationary KdV wave:

$$-c \frac{du}{dX} + u \frac{du}{dX} + \frac{d^3 u}{dX^3} = 0. \quad (2)$$

Let  $U(X)$  be a continuous finite generalized solution of (2), which is infinitely differentiable everywhere except at one point  $X = X_0$ , where derivatives may be discontinuous:  $U(X) = u(X)$  at  $X \neq X_0$ . Use  $U_0^{(i)}$  to designate the jump of the  $i$ th-order derivative of  $U(X)$  at the point  $X = X_0$ :

$$U_0^{(i)} = U^{(i)}(X_0 + 0) - U^{(i)}(X_0 - 0), \quad (3)$$

where  $u(X)$  is the solution of the steady-state KdV in the ordinary sense. The derivative is undefined at the point  $X = X_0$ , but it is well defined in the generalized sense. A generalized function is usually defined on a space of functions  $\phi(X)$  as a linear continuous functional:

$$(U(X), \phi(X)) = \int_{-\infty}^{\infty} U(X) \phi(X) dX. \quad (4)$$

From the definition, one gets the expression for the  $k$ th-order derivative of  $U(X)$ :

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$$\left( \frac{d^k U}{dX^k}, \phi(X) \right) = \left( \left[ \frac{d^k u}{dX^k} + \sum_{i=0}^{k-1} U_0^{(i)} \delta^{(k-i-1)}(X-X_0) \right], \phi(X) \right), \quad (5)$$

where  $\delta^{(i)}(X)$  is the  $i$ th-order derivative of the Dirac delta function. The basic function  $\phi(X)$  belongs to a space of nontrivial infinitely differentiable functions on a compact support, so that a generalized form is easily obtained for the dispersive term of Eq. (2). The nonlinear term requires some extra algebra.

The problem defining a direct product of distributions has no general solution. In the current case we can write for the convective term

$$\left( U \frac{dU}{dX}, \phi \right) = \int_{-\infty}^{\infty} U(X) \left[ \frac{du}{dX} + U_0^{(0)} \delta(X-X_0) \right] \phi(X) dX, \quad (6a)$$

$$\lim_{\epsilon \rightarrow 0^+} \left\{ \int_{-\infty}^{X_0-\epsilon} u(X) \frac{du}{dX} \phi(X) dX + \int_{X_0+\epsilon}^{\infty} u(X) \frac{du}{dX} \phi(X) dX + \int_{X_0-\epsilon}^{X_0+\epsilon} U(X) \frac{du}{dX} \phi(X) dX \right\}.$$

The integrand of the last integral (6a) is finite, so we get

$$\left( U \frac{dU}{dX}, \phi \right) = \int_{-\infty}^{\infty} u(X) \frac{du}{dX} \phi(X) dX = \left( u \frac{du}{dX}, \phi \right). \quad (6b)$$

By multiplying Eq. (2) by  $\phi(X)$  and integrating using (6b) and (5) for  $k=3$ , we get the stationary KdV equation in the generalized sense:

$$\int_{-\infty}^{\infty} \left[ -c \frac{du}{dX} + u(X) \frac{du}{dX} + \frac{d^3 u}{dX^3} \right] \phi(X) dX = 0, \quad (7a)$$

or

$$\left( \left[ -c \frac{dU}{dX} + U(X) \frac{dU}{dX} + \frac{d^3 U}{dX^3} - U_0^{(2)} \delta(X-X_0) - U_0^{(1)} \delta^{(1)}(X-X_0) \right], \phi \right) = 0. \quad (7b)$$

The main property of (7b) is that it not only has infinitely differentiable solutions, but also cusped solutions with two free parameters ( $U_0^{(2)}$  and  $U_0^{(1)}$ ). Such a wave propagates without change of form.

The vanishing boundary conditions are usually applied at infinity for solitary waves:

$$U \rightarrow h \rightarrow 0 \quad |X| \rightarrow \infty, \quad (8a)$$

where  $h$  is the level at infinity. For bores, the limiting values of the solution differ at plus and minus infinity:

$$U \rightarrow H \quad X \rightarrow -\infty, \quad U \rightarrow h \quad X \rightarrow \infty. \quad (8b)$$

In the case of undular bores, one can impose the limits on the wavelength  $\lambda$ :

$$U \leq H, \quad U(X+\lambda) = U(X) \quad X \rightarrow -\infty, \quad U \rightarrow h \quad X \rightarrow \infty. \quad (8c)$$

In the general case of a double-sided undulation we have to set different undulation modes at plus and minus infinity:

$$U \leq H_-, \quad U(X+\lambda_+) = U(X), \quad X \rightarrow -\infty \quad U \leq H_+, \quad (8d)$$

$$U(X+\lambda_+) = U(X), \quad X \rightarrow \infty.$$

The formalism developed further on is applied here to find a finite solution of problem (7b) with appropriate boundary conditions. The suggested form for  $U(X)$  is

$$U(X) = u_-(X) \theta_-(X-X_0) + u_+(X) \theta_+(X-X_0), \quad (9)$$

where  $u_-(X)$ ,  $u_+(X)$  are smooth enough functions for convergence of the integral (7a);  $\theta_- \equiv \theta_-(X-X_0)$ ,  $\theta_+ \equiv \theta_+(X-X_0)$  are asymmetrical Heaviside functions.

Inserting (9) in (7b) and collecting terms in  $u_-$  and  $u_+$  leads to the following equation:

$$\int_{-\infty}^{X_0} \left[ -c \frac{du_-}{dX} + u_-(X) \frac{du_-}{dX} + \frac{d^3 u_-}{dX^3} \right] \phi(X) dX + \int_{X_0}^{\infty} \left[ -c \frac{du_+}{dX} + u_+(X) \frac{du_+}{dX} + \frac{d^3 u_+}{dX^3} \right] \phi(X) dX + [U_0^{(2)} - u_0^{(2)}] \phi(X_0) - [U_0^{(1)} - u_0^{(1)}] \frac{d\phi}{dX}(X_0) = 0. \quad (10)$$

The constants  $u_0^{(i)}$  denote the jump between  $i$ -order derivatives of  $u_-(X)$  and  $u_+(X)$  at point  $X=X_0$

$$u_0^{(i)} = \frac{d^i u_+}{dX^i}(X_0) - \frac{d^i u_-}{dX^i}(X_0).$$

Thus  $U(X)$  in (9) gives the finite continuous generalized solution of Eq. (7b) if  $u_-(X)$  and  $u_+(X)$  satisfy the steady-state KdV wave equation in the ordinary sense (2):

$$-c \frac{du_-}{dX} + u_-(X) \frac{du_-}{dX} + \frac{d^3 u_-}{dX^3} = 0, \quad (11a)$$

$$-c \frac{du_+}{dX} + u_+(X) \frac{du_+}{dX} + \frac{d^3 u_+}{dX^3} = 0,$$

and if the following conditions are fulfilled at the point of discontinuity  $X=X_0$ :

$$u_+(X_0) = u_-(X_0), \quad u_0^{(1)} = U_0^{(1)}, \quad u_0^{(2)} = U_0^{(2)}. \quad (11b)$$

Relationships (11a) and (11b) show that the generalized solution sought can be obtained by an appropriate matching of two ordinary standing KdV waves at the point  $X=X_0$ . The situation is similar to that of shock waves in one-dimensional gas dynamics where two ordinary solutions corresponding to our  $u_-(X)$ ,  $u_+(X)$  are matched at the point of discontinuity. The Hugoniot conditions correspond to the conditions (11b).

The generalized standing KdV wave (9) can now be constructed using two elliptic cosines. These are defined by the values of four different parameters, from the left side ( $X < X_0$ :  $c_-, \chi_-, \omega_-, e_-$ ) to the right side ( $X > X_0$ :  $c_+, \chi_+, \omega_+, e_+$ ) of the discontinuity:

$$U = \left\{ c - \gamma_-^2 (1 - 2m_-) \omega_-^2 + 3\gamma_-^2 m_- \omega_-^2 c n^2 \left[ \omega_- \frac{\gamma_-}{2} (X - \chi_-) |m_- \right] \right\} \theta_-(X - X_0) + \left\{ c - \gamma_+^2 (1 - 2m_+) \omega_+^2 + 3\gamma_+^2 m_+ \omega_+^2 c n^2 \left[ \omega_+ \frac{\gamma_+}{2} (X - \chi_+) |m_+ \right] \right\} \theta_+(X - X_0), \quad (12a)$$

where

$$\gamma_-^2 = \frac{2}{\sqrt{3}} \cos\left(\frac{1}{3} \arcsin e_-\right),$$

$$\gamma_+^2 = \frac{2}{\sqrt{3}} \cos\left(\frac{1}{3} \arcsin e_+\right),$$

$$m_- = \frac{\cos\left(\frac{1}{3}(\pi - \arcsin e_-)\right)}{\cos\left(\frac{1}{3} \arcsin e_-\right)},$$

$$m_+ = \frac{\cos\left(\frac{1}{3}(\pi - \arcsin e_+)\right)}{\cos\left(\frac{1}{3} \arcsin e_+\right)}.$$

This shows that the system jumps from a phase potential well, described by parameter  $\omega_-$ , to another potential well, determined by  $\omega_+$ , at the "phase time"  $X_0$ . The values of  $c$ ,  $\omega_-$ ,  $\omega_+$ ,  $\chi_-$ ,  $\chi_+$ ,  $e_-$ ,  $e_+$  in (12a) can be found by solving the system of nonlinear equations consisting of the conditions at the point of discontinuity (11b) and of boundary conditions (8a)–(8c). In the general case this system should be solved numerically.

The energy levels ( $e_-$ ,  $e_+$ ) of the system may differ. Here are three typical examples of such constructions.

When the initial energy level equals one ( $e_- = 1$ ), the system is in the state of a solitary wave [Figs. 1 and 2(d)]. Then, if the phase energy of the system does not change at  $X_0$  ( $e_+ = 1$ ), the system performs the same type of phase motion described by another parameter, since the potential well is changed:

$$U = \left[ c - \omega_-^2 + 3\omega_-^2 \operatorname{sech}^2 \frac{\omega_-}{2} (X - \chi_-) \right] \theta_-(X - X_0) + \left[ c - \omega_+^2 + 3\omega_+^2 \operatorname{sech}^2 \frac{\omega_+}{2} (X - \chi_+) \right] \theta_+(X - X_0). \quad (12b)$$

In the physical space, such a solution describes the family of asymmetrical cusped waves which, in the symmetrical case, coincides with that of Miles<sup>2</sup> and Shen.<sup>3</sup>

When the energy in the initial potential well equals zero, i.e., the wave is calm [Figs. 1 and 2(a)], and in the final potential well it equals one, the wave appears in the physical space as a family of bores.<sup>8</sup>

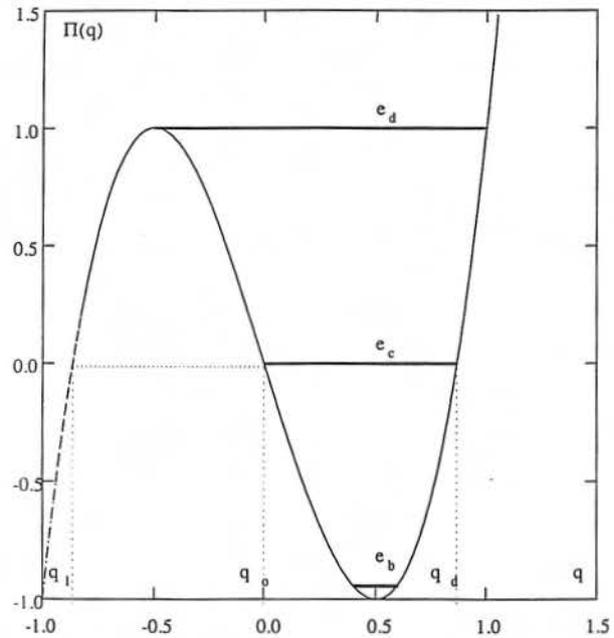


FIG. 1. Phase potential curve of the KdV integral of motion  $\Pi(q) = 4q^3 - 3q$  plotted as a function of  $q: q_l, q_0, q_d$  roots of the equation  $\Pi(q) = 0$ . Energy levels of the stationary KdV wave as a hydrodynamic conservative system are shown in bold:  $e_b$  the level of symmetrical harmonic vibrations;  $e_c$  the level of anharmonic cnoidal vibrations;  $e_d$  the level of aperiodic phase motion.

$$U = (c + \omega_-^2) \theta_-(X - X_0) + \left[ c - \omega_+^2 + 3\omega_+^2 \operatorname{sech}^2 \frac{\omega_+}{2} (X - \chi_+) \right] \theta_+(X - X_0). \quad (12c)$$

Finally, if the energy in the initial potential well takes an intermediate level  $-1 < e_- < 1$ , the system performs vibrations [Figs. 1, 2(b), and 2(c)], which transform into the sech-profile of a family of undular KdV bores:

$$U = \left\{ c - \gamma_-^2 (1 - 2m_-) \omega_-^2 + 3\gamma_-^2 m_- \omega_-^2 c n^2 \times \left[ \omega_- \frac{\gamma_-}{2} (X - \chi_-) |m_- \right] \right\} \theta_-(X - X_0) + \left[ c - \omega_+^2 + 3\omega_+^2 \operatorname{sech}^2 \frac{\omega_+}{2} (X - \chi_+) \right] \theta_+(X - X_0). \quad (12d)$$

The potential energy presents the surface  $\Pi = \Pi(q, \tau)$  in the phase space  $(\Pi, q, \tau)$ . Figure 1 shows only the transverse cross section of this surface for a fixed phase time. For  $\tau < X_0$  the system in the state  $e_-$  moves along axis  $\tau$  in the potential well  $\omega_-$ . At the moment  $\tau = X_0$  both the total phase energy of the system and the form of the potential well  $\omega_+$  change instantaneously, while the continuity of the phase trajectory is provided by the appropriate matching of the phase potential curved surfaces.

The differential operator of Eq. (7b), in comparison with that of the ordinary sense (2), has two new terms. These

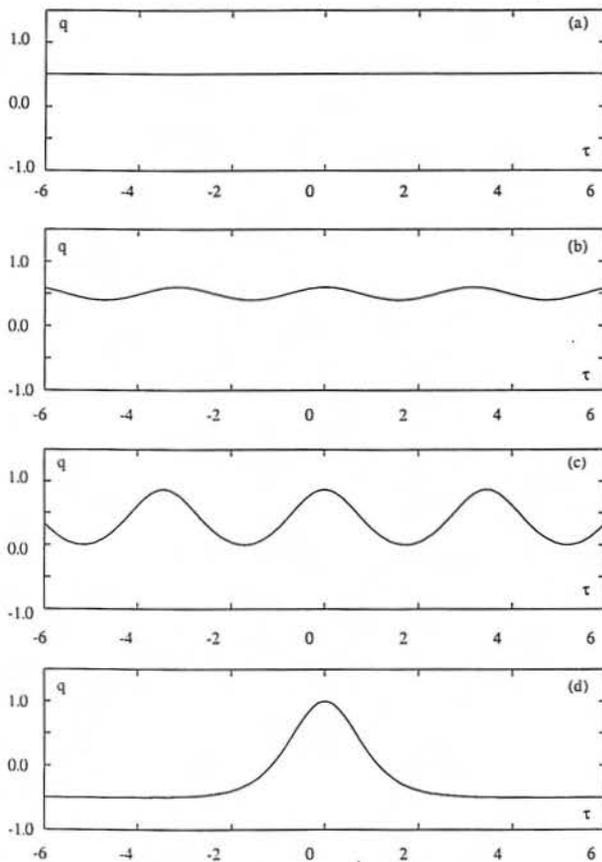


FIG. 2. Phase trajectories of the hydrodynamic system at different energy levels as functions of the phase time  $\tau=X$ . (a)  $-e_a=0$  corresponds to the calm surface, (b) to the sinusoidal wave  $e_b=-0.94$ , (c) to the cnoidal wave  $e_c=0$ , (d) to the solitary wave  $e_d=0$ . The phase mass of the system is  $\mu=3$ .

contain the Dirac delta function and its first derivative, which are functions of both coordinate space and time. Thus the generation of a standing generalized KdV wave can be performed both by an instantaneous and localized process (in comparison with the characteristic scales of the wave propagation). This kind of process is also used to recover the forced KdV equation.

If the first derivative of an external force  $f$  is added to the right-hand part of the KdV equation (1), the equation is called the forced KdV equation<sup>3</sup> (fKdV). Usually the term  $f$  is assumed to be distributed over a finite interval, which makes the fKdV equation valid in the ordinary sense. When "the length of this interval tends to zero and the amplitude of the force tends to infinity," or strictly speaking  $(f, \phi) \rightarrow (\delta, \phi)$ , the fKdV equation can be written only in the generalized sense, which leads to equation of the generalized stationary KdV wave (7b).

### III. STABILITY SIMULATION OF CERTAIN GENERALIZED SOLUTIONS

Many methods have been used to explore the numerical behavior of the KdV equation. For instance, in a previous work we used a splitting scheme<sup>9</sup> for the initial value problem with a pseudospectral method for the dispersive term and a shape preserving spline, or Bernstein's polynomial in-

terpolation, for the nonlinear one.<sup>10</sup> Here, we will keep on with the idea of cubic splines. The most appropriate numerical approaches in determining nonsmooth solutions of partial differential equations are based on integral formulations. More precisely, the weighted residual method is equivalent to the generalized formulation, from the point of view of computation.

If one looks for a spatial approximation  $\bar{u}(x,t)$  of the solution  $u(x,t)$ , then the residual is defined as

$$R(\bar{u};x,t) \equiv \frac{\partial \bar{u}(x,t)}{\partial t} + \bar{u}(x,t) \frac{\partial \bar{u}(x,t)}{\partial x} + \frac{\partial^3 \bar{u}(x,t)}{\partial x^3}.$$

A similar equation, with  $-\partial \bar{u}/\partial t$ , will be examined for waves propagating in the opposite direction. The approximate solution is determined in such a way that the residual is orthogonal to a Sobolev set of functions, denoted  $\varphi_j(x)$ . The problem is then equivalent to finding the  $\bar{u}(x,t)$  that minimizes this residual, weighted with functions  $\varphi_j$  over the domain  $\Omega$ .

The classical solution of the KdV equation is at least  $C^{(3)}(\Omega)$ , and an integral formulation was necessary in order to "weaken the equation" and obtain classes of generalized solutions. An analogous procedure is used in the computational method, and the third term in  $R(\bar{u},x,t)$  is integrated twice

$$\begin{aligned} \int_{\Omega} dx \varphi_j(x) \frac{\partial^3 \bar{u}(x,t)}{\partial x^3} &= \left[ \varphi_j(x) \frac{\partial^2 \bar{u}(x,t)}{\partial x^2} \right]_{\Gamma} \\ &\quad - \left[ \frac{d\varphi_j(x)}{dx} \frac{\partial \bar{u}(x,t)}{\partial x} \right]_{\Gamma} \\ &\quad + \int_{\Omega} dx \frac{d^2 \varphi_j(x)}{dx^2} \frac{\partial \bar{u}(x,t)}{\partial x}, \end{aligned} \quad (13)$$

where  $\Gamma$  is the set of boundary points. The right-hand side (RHS) of (13) is well defined if the  $\varphi_j(x)$ 's are  $C^{(2)}(\Omega)$ . Nevertheless, our former experience with collocation methods for classical KdV and Vlasov equations has proved that the convective term is better computed with a third degree polynomial interpolation like cubic splines. A monotonic partition of  $\Omega$  is defined with  $(N+1)$  nodes of abscissa  $\{x_0=L^-, \dots, x_i, \dots, x_N=L^+\}$ . The approximation  $\bar{u}(x,t)$  is taken as

$$\bar{u}(x,t) = v_i(t) \psi_i(x),$$

where Einstein's convention is used. Since only the first derivative of  $u$  appears in the left-hand side (LHS) of (13),  $\bar{u}(x,t)$  needs only to be continuously differentiable in  $x$  and the  $\psi_i(x)$ 's may be taken to be  $C^{(1)}(\Omega)$ . One simple way is to use piecewise first order splines in each interval  $\Omega_e = [x_e, x_{e+1}]$ . The advantage is that the coefficients are the nodal values of  $\bar{u}(x,t): v_i(t) = \bar{u}(x_i, t)$  for any  $i$ . As the number of the time-dependent unknowns  $v_i(t)$  equals the number of interpolating nodes  $x_i$ , we need  $(N+1)$  conditions. Inserting  $\bar{u}(x,t)$  into the KdV operator we get, for convective form,

$$\begin{aligned}
& \pm \frac{du_i(t)}{dt} \int_{\Omega_e} \psi_i(x) \varphi_j(x) dx + u_k(t) u_i(t) \\
& \times \int_{\Omega_e} \psi_k(x) \frac{d\psi_i(x)}{dx} \varphi_j(x) dx \quad (j=0 \dots N) \\
& + \Omega_e u_i(t) \int_{\Omega_e} \frac{d^2\psi_i(x)}{dx^2} \varphi_j(x) dx \\
& = \left[ \frac{d\varphi_j(x)}{dx} \frac{du_i(x)}{dx} - \varphi_j(x) \frac{d^2u_i(x)}{dx^2} \right]_{\Gamma_e} \quad (14a)
\end{aligned}$$

and conservative form in the frame moving with velocity  $c$

$$\begin{aligned}
& \left[ \frac{u_i^2}{2} - c \cdot u_i \right] \int_{\Omega^i} \varphi_j \frac{d\psi_i}{dX} dX + u_i \int_{\Omega^i} \frac{d^2\varphi_j}{dX^2} \frac{d\psi_i}{dX} dX \\
& = \left[ \varphi_j(X) \frac{d^2u(X)}{dX^2} \right]_{\Gamma} - \left[ \frac{d\varphi_j(X)}{dX} \frac{du(X)}{dX} \right]_{\Gamma} \quad (14b)
\end{aligned}$$

At last the final time differential equations are assembled to give system

$$\pm \mathbf{M} \frac{d\mathbf{U}(t)}{dt} + \mathbf{C}(\mathbf{U})\mathbf{U}(t) + \mathbf{D}\mathbf{U}(t) = \mathbf{B} \quad (15)$$

The RHS vector  $\mathbf{B}$  is zero everywhere except eventually at the discontinuity points and the boundaries;  $\mathbf{U}$  is the vector of nodal unknowns and  $\mathbf{M}$ ,  $\mathbf{C}$ , and  $\mathbf{D}$  are 5-diagonal matrices. The boundary values will be those of the predicted analytical solution. The actual scheme involves only Neumann boundary conditions, but another integration by parts may be performed to incorporate boundary conditions of the first kind. The global system (15) is quadratic relative to the unknowns ( $u_i$ ). It is solved by means of a Newton-Raphson algorithm which is iterative and therefore has to start from a seed  $U_0(X)$ . The high precision of the algorithm also requires that  $U_0(X)$  not be too far from the final solution. Furthermore, Eq. (15) is not a convex problem, and the scheme may not converge under certain circumstances.

The steady-state problem is solved first. There are at least three reasons for proceeding this way. First, we have to check the stability of the numerical scheme. Second, certain effects like that of the discontinuities on the selection of the solution are easier to obtain in the steady case. Last, as the equation is nonlinear and permits many different solutions, it is also a faster way of investigating the result obtained from a given initial set of data. Computations show that different initial conditions with the same value  $U_0^{(2)}$  of the jump in the second derivative lead to different steady-state solutions. These simulations agree with analytical expressions (12a)–(12c) up to four figures. Other nonobvious waves have been found that are all combinations of parts of two classical solutions.

One may consider the value  $U_0^{(2)}$  of the discontinuity  $[u'']$  as an independent parameter. In the forced KdV approach, this value may be understood as the amplitude of a Dirac force applied at  $x=X_0$ . The question is then to find the wave profile under the effect of a given force. For  $X_0=0$  and  $U_0(X)$  given by (12c) with amplitude  $3\omega^2=3$  and celerity  $c=1$ , the analytical value is

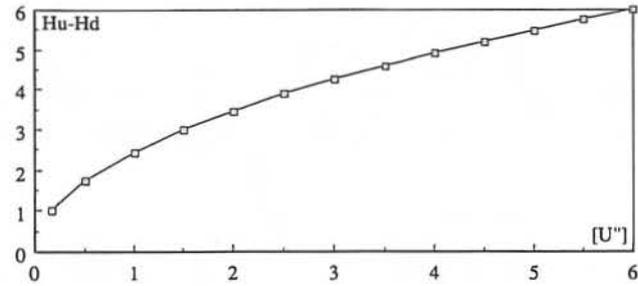


FIG. 3. Numerical (squares) and analytical (solid) elevation of the bore as function of the jump  $U_0^{(2)}$  in  $u''(x)$ .

$$U_0^{(2)} = -\frac{3}{2} \omega^4 = -\frac{3}{2}.$$

If  $U_0^{(2)}$  is changed to  $-1$ , the amplitude of the asymptotic solution is decreased from 3 to 2.4495. More precisely, the upstream value is 2.6330 and the downstream at 0.1835. If we denote the upstream and downstream elevations of the fluid as  $H_u$  and  $H_d$  and their difference as  $\Delta H$ , the three values are then very close to the theoretical predictions. These predictions can be deduced from system (12c). For  $\text{sech}^2$  profiles we have

$$\Delta H = 3\omega^2 \text{sech}^2\left(\frac{\omega}{2} \cdot (X - X_+)\right) \quad (16a)$$

Solving for  $X=X_0$ , we get

$$c \equiv \omega^2 = 3 \frac{U_0^{(1)2}}{\Delta H^2} - 2 \frac{U_0^{(2)}}{\Delta H} \quad (16b)$$

In the case of a bore,  $U_0^{(1)}=0$  and  $X_0$  always coincides with  $X_+$ . This simplification gives

$$H_u - H_d = \pm \sqrt{-6U_0^{(2)}} \quad \text{and} \quad c = \pm \frac{1}{3} \sqrt{-6U_0^{(2)}} \quad (16c)$$

Figure 3 summarizes this influence for certain values of the jump  $[u'']$ . In these examples with continuous first derivative, the determining parameters of the bore are now  $c$  and  $U_0^{(2)}$ , and no longer the amplitude  $A$ , as for the usual KdV solitary wave equation.

The effect of  $[u']$  on the asymptotic solution can be explained by an analysis similar to the one above. According to (16b), the value of  $U_0^{(1)}$  is such that  $|2U_0^{(1)}/\omega U_0^{(2)}| < 1$ . In the two cases discussed below, we again use  $X_0=0$  and a bore  $U_0(X)$ . Without loss of generality,  $U_0^{(2)}$  may be a non-zero constant. The action of  $U_0^{(1)}$  consists in shifting the profile of the wave horizontally in such a way that at  $X_0$ ,  $[u'(X_0)] = U_0^{(1)}$ . The value of this shift is obtained from (16b). Therefore the amplitude of the bore is no longer fitted by  $U_0(X)$ : the process described in the former paragraph is performed and adjusts the amplitude. The final steady-state profiles may be compared to Figs. 3(a) and 3(b) in Shen,<sup>11</sup> obtained for locally forced critical surface waves in channels of arbitrary cross section.

The next step is to find the stability of the solutions obtained from the time-dependent gKdV. One significant reason for this is that it is quite impossible to generate experi-

mental waves with exact analytical shapes. On the other hand, the initial profile will always be close to the solutions of the steady-state problem<sup>7</sup> and then will be taken as  $u(x,0) = ag(x/\lambda)$ , where  $g$  is a Gaussian function,  $\lambda$  a characteristic scale, and  $a$  the constant amplitude. The upper bound for  $\lambda$  is known from the inverse scattering theory applied to the KdV equation. Its value is  $\lambda_1 = 12(2^{1/4}/\pi^{1/2}) \cong 8.0513$  for an initially Gaussian function. The computational grid has  $N=160$  points from  $L_- = -40$  to  $L_+ = 40$ . The time step is equal to a third of the C.F.L. criterion. A first-order finite difference is used to discretize the time derivative in (16b). The scheme is completely implicit and thereby depends heavily on the boundary limits. Since we still set only the first and second derivatives at the edges, the artefact due to the numerical reflection of the waves limits the validity of the numerical model in time.

The evolution of three typical two-level waves is investigated: secant bore, truncated bore, truncated soliton. The reason for this choice is that these shapes were fitted to a real free surface, in a previous experimental work.<sup>8</sup> In this experiment, the initial condition is obtained by applying a Lorentz force to a volume of a conducting liquid. At the initial time, the magnetic field is cut off and the wave propagates freely.

The numerical initial condition is a Gauss function with given  $\lambda$  close to  $\lambda_c = \lambda_l/2$ , linked upstream to a flat level. The point of the junction is  $x=0$  for the sech bore, and is shifted  $4\Delta x$  for the truncated bore and  $-4\Delta x$  for the truncated soliton. The values of  $[u']$  and  $[u'']$  are those of the expected solution. Computations are stopped either after a sufficiently long time or when the relative error on the first three invariants becomes greater than 1%. The invariants are defined here, by analogy with the classical ones, as the sum of the invariants on each side of  $X_0$ . As the wave fronts are trapped at  $X_0$ , the depicted results are referenced to the moving discontinuities.

The approach towards the asymptotic solution may be very complex and take a very long time. However, some properties of this process may be seen from examining the simplest cases:

(i) Waves with discontinuous second derivative have been already considered by Benjamin.<sup>12</sup> His conclusion is that "the effects of an initial discontinuity in  $u_{xx}$  are instantly manifested over a range of  $x$  extending to  $-\infty$ ." Our computations confirm this result for a bore (Fig. 4). However, we show that this instability disappears for a KdV equation with negative time derivative. In other words, the jump (obtained from a bore with an  $x$ -axis inversion) is stable for the usual equation (Fig. 5).

(ii) A wide class of waves are emitted from the initial profile, depending on its extent. These waves may be faster or slower than the bore, and may move up or downstream. Their amplitudes are naturally smallest when  $\lambda = \lambda_c$ . For sufficiently large  $\lambda$ , the Gaussian gives a bore and a solitary wave (KdV with  $-u_t$ ), the latter is always traveling faster than the bore. When, for numerical reasons, they are reflected by the right boundaries, they never intersect the bore. More generally, only waves coming from upstream can cross the bore.

(iii) Nearly the same conclusions may be deduced for

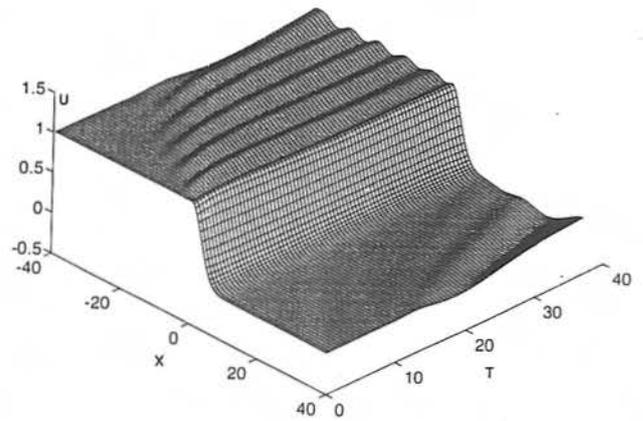


FIG. 4. Oscillations on the plateau of the bore solution of the KdV equation  $u_t + uu_x + u_{xxx} = 0$ . These disappear when the time derivative term is changed to  $-u_t$ .

truncated solitons which are stable within the preceding restrictions [Figs. 6(a) and 6(b)]. As the initial profiles are all increasing functions, a bump has to be created by the system in order to produce a truncated soliton. The specific feature in this example is an oscillation of the peak of the wave, which propagates up- and downstream. These oscillations damp out more or less rapidly as the time increases.

#### IV. CONCLUSION

It appears from numerical simulations that generalized solutions are highly stable with respect to the perturbations of the wave front. In this approach, both first and second derivatives may be discontinuous. The jump in the second derivative is clearly interpreted as the amplitude of a singular external force. The appearance of a discontinuity in the first derivative extends the fKdV to include a term with a Dirac derivative. The different solutions are obtained only by combining different values of the parameters  $[u']$  and  $[u'']$ . Of course, the method has been tested for the strong solutions, i.e.  $[u'] = [u''] = 0$ . It follows then that the "weak" form of

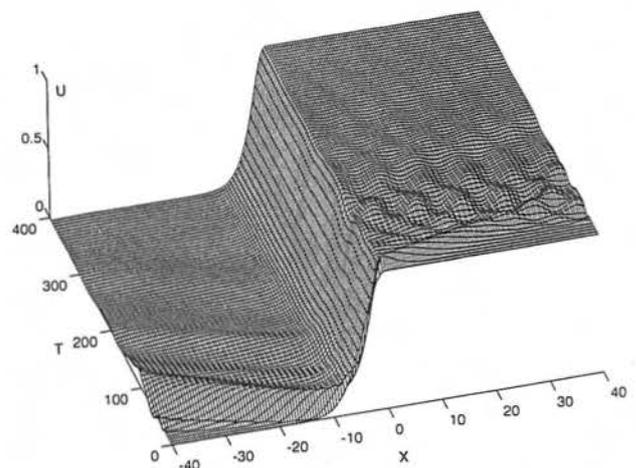


FIG. 5. Nonsolitary contribution to a jump with discontinuous second derivative for initial profile  $u(x,0) = h \exp(-(x-X_0)^2/\lambda^2)$ . Parameters are  $h=1$ ,  $\lambda=10$ ,  $X_0=0$  and  $U_0^2=1.5$ .

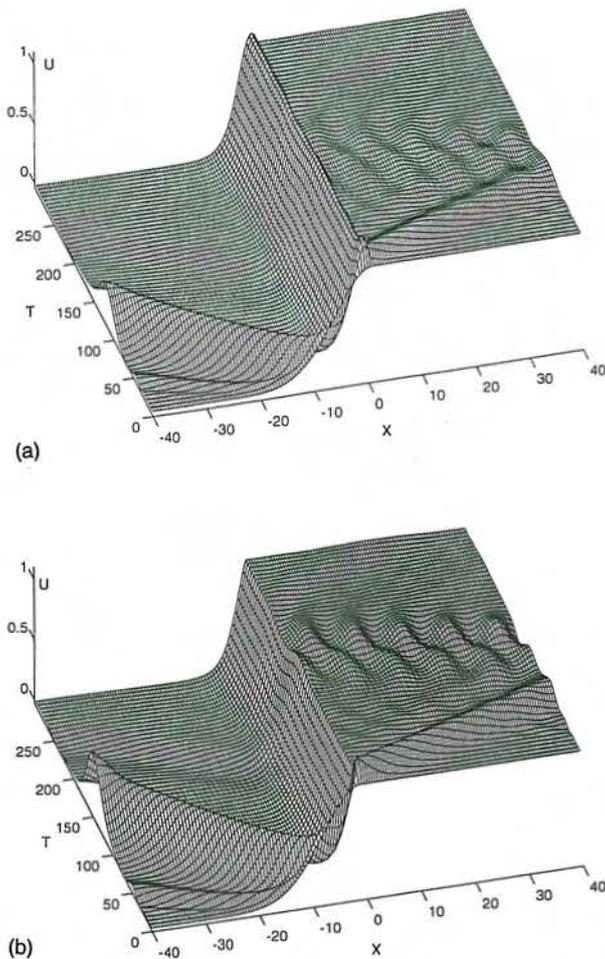


FIG. 6. Nonsolitary contribution to a truncated soliton with discontinuous first and second derivatives for initial profile  $u(x,0) = h \exp(-(x-X_0)2/\lambda^2)$ . Parameters are  $h=1$ ,  $\lambda=10$ ,  $X_0=1$  for (a) and  $X_0=-1$  for (b). The amplitudes of the discontinuities are determined from the steady-state solution.

the KdV equation is a unifying model for a larger class of solitary waves, including solitons, falls, and jumps. Concerning bores, their shape and elevation are uniquely defined by the velocity  $c$  and the discontinuity  $U_0^{(2)}$ , provided the direction of propagation is appropriate. The current numerical algorithm is restricted in time by boundary effects. Neither the Neumann nor the Dirichlet boundary conditions are well

suitable to simulating long-time behavior of solitary waves in unbounded domain. The boundaries defined must not interact with radiating waves. The implementation of "transparent" or "absorbing" conditions, although unnecessary in the present calculations, would enable the simulation of the system (with many discontinuity points for instance) on longer times and reasonable space grids.

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