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On moving thick layer approach for graded damage modelling

Claude Stolz

Abstract We propose to examine analytical solutions of propagation of graded damage for particular geometries in the context of the formulation of a damage law based on a continuous transition between a sound material and a totally or partially broken material. The evolution of damage is associated with a moving layer of finite thickness, defined in the frame of a moving surface Γ_o . The dissipation and the driven forces for the motion of the layer depend on the local shape, in particular on the curvature of the surface Γ_o . This study is then focussed on the effect of this curvature on the stability of the propagation of the layer. Comparison with description of damage with sharp interface is also presented.

Keywords Nucleation of defects · Moving layers · Elastic brittle material

1 Introduction

The full scenario of the degradation of solids under mechanical loading is described generally by a damage description modelling the gradual loss of stiffness.

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Recently a new approach have been proposed based on a moving layer associated with the propagation of level set surface (Moës et al. 2011). For elastic brittle material (Bui and Ehrlacher 1980) or elastic quasi-brittle material, the evolution of the interface separating the undamaged material $d = 0$ from the damaged material $d = 1$ have been studied using an energetic description of the propagation (Pradeilles-Duval and Stolz 1991, 1995). In this description the parameter d jumps instantaneously from 0 to 1. The interface has no thickness and some mechanical quantities present discontinuities (Abeyaratne and Knowles 1990). Criteria of uniqueness and of stability of the propagation have been also established. In this approach, nucleation of defects can be considered as a bifurcation of equilibrium solution (Stolz 2007).

Descriptions of moving interfaces and of moving layers are also useful for studying loss of matter. Such a thermodynamical description of thin or thick layers permits to describe complex process of wear between two bodies in relative motion (Stolz 2010).

In the article Stolz and Moës (2012), the model studied the case of the propagation of damage in a bar. The initial material and the damaged material are separated by a surface Γ_o . This boundary is a moving interface. A surface is considered as an equipotential or a level-set. This surface is an interface where the material begins to change its properties. Two cases have been considered : the propagation of a sharp interface and the propagation of a moving layer.

Sharp interface If the transition zone is very thin, the transition from sound material to damaged material can be discontinuous and the interface is called sharp interface.

The driving force associated to the propagation of the interface is the local energy release rate $G(s)$, as proposed (Bui and Ehrlacher 1980; Abeyaratne and Knowles 1990). Moreover stability and non-bifurcation conditions for the problem of evolution are given when a normality rule is associated with a generalized Griffith's criterion like $G(s) \leq G_c$.

In this case the addition of a surface energy density is a factor of stabilization of the propagation of the surface due to the influence of the curvature on the energy release rate (Son et al. 1989)

Moving layer If the interface is thick, the transition should be more regular. The evolution of damage is then associated to a moving layer. The thickness of the transition zone provides a length scale in the model.

The purpose of this article is to use the notion of moving layer to describe a graded damage as in Stolz and Moës (2012). The analysis of dissipation provides the definition of the energy release rate $G(s)$ associated to the motion of the layer. A normality rule based on a criterion function of $G(s)$ is then proposed.

To emphasize the rule of the curvature of the layer, we study three particular systems: the longitudinal shear of a cylinder, a cylinder and a sphere under radial expansion.

2 The sharp interface

The sound material 0 is transformed into partially damaged material 1 when the boundary between the two materials is moving when some threshold is reached as proposed in Pradeilles-Duval and Stolz (1991, 1995). The system is a two phase composite, when the boundary is moving the potential energy changes, the associated dissipation is a function of the velocity of the boundary. This defines the driving force associated to the motion of the interface as an energy release rate $G(s)$. The evolution of the interface is then governed by a kinetic relation (Gurtin and Podio-Guiugli 1996) or by a normality rule as proposed in Pradeilles-Duval and Stolz (1995).

We analyse in this framework, the response of a cylinder under longitudinal shear, a cylinder and a sphere under radial loading.

2.1 The longitudinal shear on a cylinder

We consider a cylinder with circular section of internal radius R_i and external radius R_e . On the surface $r = R_i$, the displacement is null. On $r = R_e$ the displacement or the shear stress is imposed. In linear elasticity, the displacement solution satisfies

$$\mathbf{u} = u(r)\mathbf{e}_z, \quad \nabla \mathbf{u} = \frac{\partial u}{\partial r} \mathbf{e}_z \otimes \mathbf{e}_r. \quad (1)$$

The shear stress satisfies the constitutive law and the conservation of the momentum

$$\tau = \mu(r) \frac{du}{dr}, \quad \frac{d}{dr}(r\tau) = \frac{d}{dr} \left(\mu(r)r \frac{du}{dr} \right) = 0. \quad (2)$$

Then $r\tau = C$ where the constant C is given by the imposed displacement or the prescribed shear at $r = R_e$. We choose to impose the displacement $u(R_e) = E$ then

$$E = u(R_e) = C \int_{R_i}^{R_e} \frac{dr}{r\mu(r)}; \quad \Sigma = \frac{C}{R_e}. \quad (3)$$

The cylinder is now a composite cylinder, the surface $r = \Gamma_o$ is the boundary between sound material 0 and partially damaged material 1. Then the total potential energy is

$$\mathcal{E}(u, \Gamma_o) = \pi C^2 \int_{R_i}^{R_e} \frac{dr}{r\mu(r)} = \pi C^2 J(\Gamma_o). \quad (4)$$

When the boundary moves with velocity $\dot{\Gamma}_o = a$ the potential energy changes. In the case of sharp interface

$$J(\Gamma_o) = \int_{R_i}^{\Gamma_o} \frac{dr}{r\mu_1} + \int_{\Gamma_o}^{R_e} \frac{dr}{r\mu_o}, \quad (5)$$

in this case, the dissipation is

$$D_m = -\frac{\partial \mathcal{E}}{\partial \Gamma_o} a = \frac{\pi C^2}{\Gamma_o} \frac{\mu_o - \mu_1}{\mu_o \mu_1} a = 2\pi \Gamma_o G_\Gamma a. \quad (6)$$

This defines the energy release rate G_Γ by unit of area $G_\Gamma = G(\Gamma_o, C)$, the evolution of the boundary is now governed by a kinetic equation between a and G_Γ . The proposed normality rule takes the form

$$G_\Gamma \leq G_c, \quad a \geq 0, \quad (G_\Gamma - G_c) a = 0. \quad (7)$$

During the propagation of the interface, the relation $G_\Gamma = G(\Gamma_o, C) = G_c$ defines implicitly the position of the interface. We obtain

$$\frac{C^2}{2\Gamma_o^2} \frac{\mu_o - \mu_1}{\mu_o \mu_1} = G_c. \quad (8)$$

At the initiation of the damaged zone $\Gamma_o = R_i$, we define some constant

$$C = C_c, \quad \Sigma_c = \tau_c \frac{R_i}{R_e}, \quad \tau_c^2 \frac{\mu_o - \mu_1}{\mu_o \mu_1} = 2G_c. \quad (9)$$

The dissipation induces that during the initiation of the layer, $C/\Gamma_o = C_c/R_i = \tau_c$.

The response of the cylinder under loading is then decomposed in three parts for an increasing E ,

Step I From initial state. From the initial state, the first step is an elastic response with modulus μ_o

$$\Sigma = \frac{C}{R_e}, \quad E = u(R_e) = \frac{C}{\mu_o} \log \frac{R_e}{R_i}, \quad (10)$$

until $\tau(R_i) = \tau_c$ or $C = C_c$.

Step II Propagation of the interface. From the state $C = C_c$ and increasing E or C , the damage evolves accordingly with

$$\Sigma = \tau_c \frac{\Gamma_o}{R_e},$$

$$E = \tau_c \Gamma_o \left(\frac{1}{\mu_o} \log \frac{R_e}{R_i} + \frac{\mu_o - \mu_1}{\mu_o \mu_1} \log \frac{\Gamma_o}{R_i} \right),$$

until $\tau(R_e) = \tau_c$

State III Final state after total transformation. From the state, $\tau(R_e) = \tau_c$ and an increasing E , the evolution is that of a cylinder of modulus μ_1 .

$$\Sigma = \frac{C}{R_e}, \quad E = \frac{C}{\mu_1} \log \frac{R_e}{R_i}. \quad (11)$$

2.2 The cylinder and the sphere under radial loading

In the same spirit, we consider now the case of a composite cylinder and a composite sphere under radial loading, the displacement solution of the problem of equilibrium is assumed to be radial

$$\mathbf{u} = u(r)\mathbf{e}_r, \quad (12)$$

and the strains are:

$$\text{for the cylinder: } n = 2, \quad \varepsilon_r = \frac{du}{dr}, \quad \varepsilon_\theta = \frac{u}{r},$$

$$\text{for the sphere: } n = 3, \quad \varepsilon_r = \frac{du}{dr}, \quad \varepsilon_\theta = \varepsilon_\phi = \frac{u}{r}.$$

In linear elasticity, with Lamé's moduli (λ, μ) the conservation of the momentum, when $\Lambda(r) = \lambda(r) + 2\mu$ with μ uniform, implies the differential equation on u

$$\frac{d}{dr} \left(\Lambda \left(\frac{du}{dr} + (n-1) \frac{u}{r} \right) \right) = 0. \quad (13)$$

The displacement solution is then obtained in the kernel $R_i \leq r \leq R_e$

$$r^{n-1}u(r) = AI_n(r) + R_i^{n-1}u_i, \quad I_n(r) = \int_{R_i}^r \frac{r^{n-1}dr}{\Lambda}.$$

The radial displacement for the global response is then determined by

$$u(R_e) = E R_e, \quad R_e^n E = AI_n(R_e) + R_i^{n-1}u_i. \quad (14)$$

The total energy of composite is now evaluated

$$\mathcal{E}(u) = \pi 2^{n-1} \int_{R_i}^{R_e} \frac{1}{2} \boldsymbol{\sigma} : \boldsymbol{\epsilon} r^{n-1} dr$$

$$= \pi 2^{n-1} \left[\sigma_{rr}(r) u(r) r^{n-1} \right]_{R_i}^{R_e}.$$

As the radial stress is

$$\sigma_{rr} = \lambda \left(\frac{du}{dr} + (n-1) \frac{u}{r} \right) + 2\mu \frac{du}{dr}$$

$$= \Lambda \left(\frac{du}{dr} + (n-1) \frac{u}{r} \right) - 2\mu(n-1) \frac{u}{r}$$

$$= A - 2\mu(n-1) \frac{u}{r},$$

then

$$\mathcal{E} = \pi 2^{n-2} A^2 I_n(R_e)$$

$$- 2^{n-1} \pi \left(R_e^{n-2} u^2(R_e) - R_i^{n-2} u^2(R_i) \right)$$

We consider a composite cylinder or a composite sphere with external radius R_e . To complete the solution, a boundary condition at $r = R_i$ must be introduced. For $R_i \neq 0$ we can imposed $\sigma \cdot \mathbf{e}_r(R_i) = 0$.

We consider here that $R_i = 0$, then the displacement at this point satisfies $u(R_i) = 0 = u_i$.

For imposed displacement $u(R_e) = E R_e$ the condition (14) determines A and the system is solved for any distribution of $\lambda(r)$. With the definition of the displacement at the boundary $r = R_e$, the potential energy is reduced to

$$\mathcal{E} = \pi 2^{n-2} R_e^n E^2 \left(\frac{R_e^n}{I_n(R_e)} - 2\mu \right) \quad (15)$$

2.3 The evolution of the sharp interface

We consider now a two phases-composite, the boundary between the sound and partially damaged material is noted $\Gamma_o > 0$ as previously. When the interface is moving with velocity $\dot{\Gamma}_o = a$ the potential energy changes and the dissipation is determined. For this system, the integral $I_n(R_e)$ is a function of Γ_o

$$I_n = \int_0^{\Gamma_o} \frac{r^{n-1} dr}{\Lambda_1} + \int_{\Gamma_o}^{R_e} \frac{r^{n-1} dr}{\Lambda_o}. \quad (16)$$

For given E , A and I_n satisfies

$$E R_e^n = A I_n. \quad (17)$$

Combining the derivation of I_n and E relatively to Γ_o , we find

$$D_m = -\frac{\partial \mathcal{E}}{\partial \Gamma_o} a = 2^{n-2} \pi A^2 \frac{\partial I_n}{\partial \Gamma_o} a = 2^{n-1} \pi G_\Gamma \Gamma_o^{n-1} a. \quad (18)$$

The release rate of energy by unit of area is then defined as

$$G_\Gamma = G(E, \Gamma_o) = \frac{1}{2} A^2 \frac{\partial I_n}{\partial \Gamma_o}. \quad (19)$$

The normality rule (7) imposes the constant value of G_Γ during the motion and then

$$A^2 \frac{\partial I_n}{\partial \Gamma_o} = 2G_c. \quad (20)$$

In the case of sharp interface, we obtain that A is the constant A_c such that

$$G(E, \Gamma_o) = \frac{1}{2} A_c^2 \frac{\Lambda_o - \Lambda_1}{\Lambda_o \Lambda_1} = G_c, \\ A_c^2 = 2G_c \frac{\Lambda_o - \Lambda_1}{\Lambda_o \Lambda_1}. \quad (21)$$

As previously, for increasing E , the global response of the system is decomposed in three parts.

State I Elastic behaviour, $A \leq A_c$. The global response of the two-phases composite is purely linear elastic, with modulus λ_o, μ .

$$\Sigma = \sigma_{rr}(R_e) = A - 2^{n-1} \mu E, \\ E = \frac{A}{R_e^n} I_n,$$

where $I_n = \frac{R_e^n}{n \Lambda_o}$, and $A \leq A_c$.

State II Propagation of the interface. Now that the interface is moving and during the propagation $G = G_c$, then $A = A_c$ is constant, and I_n takes the value

$$I_n(R_e, \Gamma_o) = \frac{R_e^n}{n} \left(\frac{1}{\Lambda_o} + \left(\frac{\Gamma_o}{R_e} \right)^n \frac{\Lambda_o - \Lambda_1}{\Lambda_o \Lambda_1} \right). \quad (22)$$

Then during the propagation, the global response of the system is

$$E R_e^n = A_c I_n(R_e, \Gamma_o), \\ \Sigma = \Lambda_o A_c - 2^{n-1} \mu E.$$

The relation between Σ and E is linear, and Σ is a decreasing function of E .

When $\Gamma_o \rightarrow 0^+$, the results proposed in Stolz (2010) for the sphere ($n = 3$) are recovered. For $n = 2$ we obtain the critical state (E_c, Σ_c) for the initiation of an infinitesimal defect as a bifurcation point in the case of the cylinder:

$$E_c R_e^2 = A_c I_2(R_e, 0^+), \quad \Sigma_c = \Lambda_o A_c - 2\mu E_c. \quad (23)$$

We consider now a diffuse interface with graded damage.

3 A level set based damage model

Classical constitutive equations for describing the stress-strain relations are considered and the formulation for the damage evolution based on level-set model

is used. The model is defined accordingly to the paper (Stolz and Moës 2012).

The main idea of the model is to consider that damage occurs in a layer of finite thickness l evolving from 0 to l_c . Damage is described by a parameter d which varies from 0 to 1. The boundary between the sound material and the damaged material is denoted by Γ_o which is moving accordingly to the damage evolution. The surface Γ_o is the level-set $\phi(\mathbf{M}, t) = 0$, where $\phi(\mathbf{M}, t)$ is a signed distance function of point \mathbf{M} to Γ_o . The damage d is considered as a function of the level-set ϕ . In the case of cylinder and sphere, the surface Γ_o is defined by a radius R_o , and the level set z is defined by the value of the radius $R_o - z$.

For a damage constitutive law, the free energy w per unit of volume depends on the strain ϵ and on a scalar variable d for the damage: $w = w(\epsilon, d)$. The state law is given by

$$\boldsymbol{\sigma} = \frac{\partial w}{\partial \boldsymbol{\epsilon}}, \quad Y = -\frac{\partial w}{\partial d}, \quad (24)$$

where $\boldsymbol{\sigma}$ is the stress and Y the local energy release rate, they are functions of the state variables $(\boldsymbol{\epsilon}, d)$. For instance, we may use a potential of the form

$$w(\boldsymbol{\epsilon}, d) = \frac{1}{2} \boldsymbol{\epsilon} : \mathcal{C}(d) : \boldsymbol{\epsilon}, \quad (25)$$

then the equations of state are given as

$$\boldsymbol{\sigma} = \mathcal{C}(d) : \boldsymbol{\epsilon}, \quad Y = -\frac{1}{2} \boldsymbol{\epsilon} : \frac{\partial \mathcal{C}}{\partial d} : \boldsymbol{\epsilon}. \quad (26)$$

The driven force $Y(\boldsymbol{\epsilon}, d)$ must be finite for any $d \in [0, 1]$, this condition of regularity ensures that the dissipation is defined.

The simple form of $\mathcal{C}(d)$ is given by choosing $\mathcal{C} = d\mathcal{C}_1 + (1-d)\mathcal{C}_o$ or $\mathcal{C}^{-1} = d\mathcal{C}_1^{-1} + (1-d)\mathcal{C}_o^{-1}$. When d increases monotonically from 0 to 1, this describes the transition between a material of elastic properties \mathcal{C}_o to a partially damaged material of elastic properties \mathcal{C}_1 . For total damage description $\mathcal{C}_1 = 0$. Such behaviours are generalizations of behaviour considered in Moës et al. (2011).

The evolution of damage is determined by a dissipation potential $D(\dot{d})$ convex function of \dot{d} such that $D(0) = 0$. The driving force Y satisfies the normality rule

$$D(\dot{d}) + Y(d^* - \dot{d}) \leq D(d^*), \quad \forall d^*, \quad (27)$$

which is equivalent to the relation $Y = \frac{\partial D}{\partial \dot{d}}$ in the case of a regular potential D . These properties ensure positive dissipation.

For a domain of reversibility defined by $Y - Y_c \leq 0$ and a time independent constitutive relation, the normality rule is expressed as

$$\dot{d} \geq 0, \quad Y - Y_c \leq 0, \quad (Y - Y_c)\dot{d} = 0. \quad (28)$$

In the case of graded damage this law is modified, to take into account of the change of the internal parameter. The internal parameter if now the level set ϕ and the damage d is an explicit function of ϕ .

3.1 The graded damage model

The level set ϕ is introduced as the internal parameter. The level set $\phi(M, t) = 0$ separates the domain Ω into a undamaged and a damaged zone, this surface is denoted Γ_o . The motion of the surface is defined by the normal velocity $V = a\nabla\phi = a\mathbf{N}$

The whole body Ω is then decomposed into three parts, the undamaged body Ω_o , the zone Ω_c of transition, the damaged body Ω_1 when damage is established. In this case, the boundary $\partial\Omega_c$ is decomposed into Γ_o where $d = 0$, $\phi(\mathbf{M}_o^t, t) = 0$, and Γ_1 where $d = 1$, $\phi(\mathbf{M}^t, t) = l_c$.

In the damaged zone Ω_c the variable d is an explicit continuous function of the level set ϕ . The damage increases as the level set value rises :

$$\begin{cases} d(\phi) = 0, & \phi \leq 0, & \mathbf{X} \in \Omega_o, \\ d'(\phi) \geq 0, & 0 \leq \phi \leq l_c, & \mathbf{X} \in \Omega_c, \\ d(\phi) = 1, & \phi \geq l_c, & \mathbf{X} \in \Omega_1, \end{cases} \quad (29)$$

$d'(\phi)$ is the derivative of d relatively to ϕ .

As the damage is a given continuous function of the signed distance ϕ to Γ_o , the evolution of the layer is then associated to the motion of the surface Γ_o . This implies that

$$\begin{aligned} \phi(M, t) = z &= \phi(M_o + z\mathbf{N}, t) = \phi(M_o, t) + z, \\ M_o &\in \Gamma_o, \\ 0 &= \dot{\phi} + a\nabla\phi \cdot \mathbf{N}, \quad \forall z. \end{aligned}$$

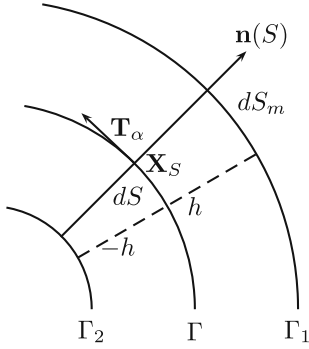


Fig. 1 Local geometry of the layer

3.2 The problem of equilibrium

Consider a body Ω . The external boundary $\partial\Omega$ is decomposed into two parts: $\partial\Omega_u$ where the displacement \mathbf{u} is prescribed $\mathbf{u}^d(t)$ and $\partial\Omega_T$ where the loading $\mathbf{T}^d(t)$ is applied; $\partial\Omega = \partial\Omega_u \cup \partial\Omega_T$, $\emptyset = \partial\Omega_u \cap \partial\Omega_T$.

The total potential energy \mathcal{E} is a function of (\mathbf{u}, ϕ)

$$\mathcal{E}(\mathbf{u}, \phi) = \int_{\Omega} w(\boldsymbol{\epsilon}(\mathbf{u}), d(\phi)) \, d\Omega - \int_{\partial\Omega_T} \mathbf{T}^d \cdot \mathbf{u} \, ds, \quad (30)$$

$\boldsymbol{\epsilon}(\mathbf{u})$ is the strain, symmetric part of $\nabla\mathbf{u}$. The displacement \mathbf{u} satisfies the boundary conditions on $\partial\Omega_u$: $\mathbf{u} = \mathbf{u}^d$.

Continuity conditions (CC) The displacement \mathbf{u} , the stress vectors $\boldsymbol{\sigma} \cdot \mathbf{n}$, the modulus $\mathcal{C}(d)$ are continuous functions on the surface Γ_o . Then the gradient of the displacement and the free energy are continuous functions along each iso- ϕ surface and inside the layer. These properties imply that the velocity $\dot{\mathbf{u}}$ is also continuous. Therefore virtual variations of the displacement and of the moving surface $\phi = 0$ are uncoupled (Fig. 1).

The solution of the problem of equilibrium is determined by the solution of minimization of the potential energy. Then, for a state of equilibrium, the variations of the total potential with respect to \mathbf{u} are zero

$$\frac{\partial \mathcal{E}}{\partial \mathbf{u}} \cdot \delta \mathbf{u} = \int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\epsilon}(\delta \mathbf{u}) \, d\Omega - \int_{\partial\Omega_T} \mathbf{T}^d \cdot \delta \mathbf{u} \, ds = 0, \quad (31)$$

where the variations $\delta \mathbf{u}$ satisfy the boundary conditions $\delta \mathbf{u} = 0$ over $\partial\Omega_u$. By integration by parts equations of equilibrium are recovered

$$\operatorname{div} \boldsymbol{\sigma} = 0, \quad \mathbf{n} \cdot \boldsymbol{\sigma} = \mathbf{T}^d, \quad \text{over } \partial\Omega_T. \quad (32)$$

Evolution of the layer The variations of \mathcal{E} are then reduced to those with respect to ϕ

$$\begin{aligned} \frac{\partial \mathcal{E}}{\partial \phi} \cdot \delta a &= - \int_{\Omega_c} Y d'(\phi) \delta a \, d\Omega \\ &= - \int_{\Gamma_o} G_{\Gamma}(s) \delta a(s) \, ds, \end{aligned}$$

$$d\Omega = \det(I - z\kappa) dz \, ds = j dz \, ds,$$

where κ is the curvature. The release rate of energy G_{Γ} appears associated to the motion of the layer

$$G_{\Gamma}(s) = \int_o^l Y d'(\phi) j \, d\phi. \quad (33)$$

This defines the variation of the energy when the layer is moving. The thickness at time t is $l(t)$ less than l_c . The description is valid from the initiation of the layer from zero-thickness to a finite thickness l_c .

To define the evolution of the system, we need a link between the driven force $G(s)$ and the velocity $a(s)$.

On the evolution law We propose now an evolution law to determine the value of velocity a . Consider the normality rule (28). From integration of $(Y - Y_c) \dot{d} = 0$ over the thickness $l \neq 0$ we get

$$\left(\int_o^l Y d'(\phi) j \, d\phi - Y_c \int_o^l d'(\phi) j \, d\phi \right) a = 0. \quad (34)$$

This provides a normality law on $G(s)$

$$(G_{\Gamma}(s) - G_c(s)) a = 0, \quad a \geq 0, \quad G_{\Gamma}(s) - G_c(s) \leq 0. \quad (35)$$

For $l > 0$, $\int_o^l d'(\phi) j \, d\phi$ is positive, we can introduce the notation $\langle F \rangle$:

$$\langle F \rangle = \frac{\int_o^l F d'(\phi) j \, d\phi}{\int_o^l d'(\phi) j \, d\phi}, \quad (36)$$

with this driving force Y the normality rule is rewritten as

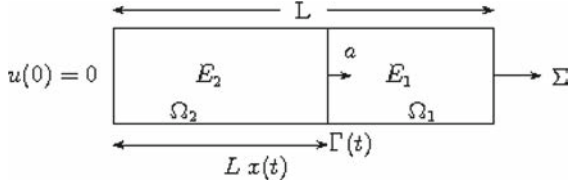


Fig. 2 A sharp interface

$$\langle Y \rangle - Y_c a = 0, \quad a \geq 0, \quad \langle Y \rangle - Y_c \leq 0. \quad (37)$$

As shown in Stolz and Moës (2012), taking account of (36) and (37) the initiation of the layer occurs when $Y(\mathbf{X}, t) = Y_c$. After initiation, the thickness l of the layer evolves from 0 to l_c . The surface $\phi = l_c$ is the surface defined by $d(x, t) = 1$. The transition from $d = 0$ to $d = 1$ along the normal vector $\nabla\phi$ is smooth. This approach is essentially non-local, but here the non-locality is restricted to the layer. Any classical constitutive law described by $w(\epsilon, d)$ can be used.

To study the evolution of a system under a loading history, we need two functions $\mathcal{C}(d)$ and $d(\phi)$ or $\mathcal{C}(\phi)$ and $d(\phi)$, because $d(\phi)$ is a continuous increasing function of ϕ .

4 Comparison of sharp and layer model

4.1 Some results on a straight bar

To understand the effect of the curvature, we recall previous results obtained on a bar in Stolz and Moës (2012). A bar of length L is considered. This bar is made of two linear elastic materials and is submitted to pure tension. The Young modulus of the sound material is E_1 and for the damaged material the Young modulus is E_2 (Fig. 2).

At point $X = L$, the applied tension is Σ . At point $X = 0$, displacement is $u(0) = 0$. The position of the interface is denoted Γ . We consider that a fraction $x = \Gamma/L$ of the bar is transformed. The equilibrium and constitutive laws give:

$$\sigma = E_1 \epsilon_1 = E_2 \epsilon_2 = \Sigma. \quad (38)$$

And the total potential energy is

$$W(\Sigma, x) = -\frac{1}{2} L \Sigma^2 \left(\frac{x}{E_2} + \frac{1-x}{E_1} \right). \quad (39)$$

Differentiation of the energy at equilibrium with respect to the motion of interface is

$$G_\Gamma(\Sigma) = -\frac{\partial W}{\partial \Gamma} = \frac{1}{2} \Sigma^2 \left(\frac{1}{E_2} - \frac{1}{E_1} \right), \quad (40)$$

consequently the dissipation is

$$D_m = G_\Gamma(\Sigma) \dot{\Gamma}. \quad (41)$$

The speed $\dot{\Gamma} = a$ is determined by the normality law

$$a \geq 0, \quad G_\Gamma - G_c \leq 0, \quad (G_\Gamma - G_c) a = 0. \quad (42)$$

The velocity a will be positive if the consistency condition $G_\Gamma - G_c = 0$ is satisfied. This condition determines the critical value of Σ from (40):

$$\Sigma_c = \sqrt{\frac{2G_c E_1 E_2}{E_1 - E_2}}. \quad (43)$$

Initially the bar is composed of material 1 only. Assuming that initial conditions are $x(0^+) = 0$ and $u(L, 0^+) = 0$. Prescribing the displacement at point $X = L$, the total strain of the bar is $\Delta(t)$ and satisfies the global constitutive law

$$\Delta(t) = \frac{u(L, t) - u(0, t)}{L} = \Sigma \left(\frac{x}{E_2} + \frac{1-x}{E_1} \right). \quad (44)$$

The response of the bar is given by

$$\begin{cases} \Delta \leq \Delta_c = \frac{\Sigma_c}{E_1}, & \Sigma = E_1 \Delta, \quad x = 0, \\ \frac{\Sigma_c}{E_2} \geq \Delta \geq \Delta_c, & \Sigma = \Sigma_c, \quad \dot{x} = \dot{\Delta} \frac{E_1 E_2}{\Sigma_c (E_1 - E_2)}, \\ \Delta \geq \frac{\Sigma_c}{E_2}, & \Sigma = E_2 \Delta, \quad x = 1. \end{cases}$$

4.2 A model with a moving layer

The free energy w for uniaxial response is

$$w(\epsilon, d) = \frac{1}{2} E(d) \epsilon^2, \quad (45)$$

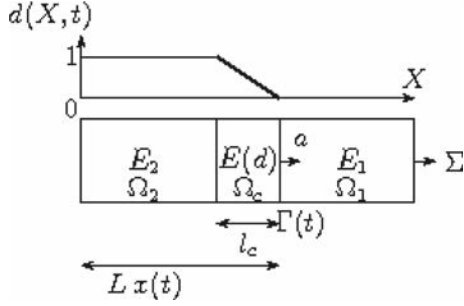


Fig. 3 The propagation of a layer

where d varies from 0 to 1, the Young modulus $E(d)$ is a continuous function of d , then there is no discontinuity at $d = 0$.

On Fig. 3, $\phi(X, t) = \Gamma(t) - X$ and the value of the damage parameter is given by

$$\begin{cases} \phi(X, t) \leq 0, & d(\phi) = 0, \\ 0 \leq \phi(X, t) \leq l_c, & d(\phi) = \phi/l_c, \\ \phi(X, t) \geq l_c, & d(\phi) = 1. \end{cases} \quad (46)$$

The damage parameter d is an increasing function of the distance ϕ to the boundary Γ separating the sound material to the damaged one. Generally, the function $d(\phi)$ is a given continuous function of ϕ . As the Young modulus E is a continuous function of d and for comparison with the sharp interface, it is natural to consider the matching conditions $E(0) = E_1$ and $E(1) = E_2$.

Initially $\Gamma(t) = 0$ and the propagation of the layer begins at the origin of the bar, so the thickness $\Gamma(t) = l(t)$ is smaller than l_c . The thickness increases to l_c and after this step of initialization, the thickness is kept constant.

For the given constitutive laws, the dissipation is local and only due to damage $d_m = Y\dot{d}$. Integrating over the layer we get the total dissipation due to damage inside the bar:

$$D_m = \int_0^l Y \dot{d} d\phi. \quad (47)$$

Assuming that d is a continuous function of ϕ .

Using the constitutive law, the local driving force Y is

$$Y = -\frac{1}{2} \frac{dE}{dd} \epsilon^2 = -\frac{\Sigma^2}{2E^2(d)} \frac{dE}{dd}. \quad (48)$$

The function $E(d)$ must ensure that Y is finite. Then $\frac{dE}{dd}$ is finite.

The motion of the level-set $\phi(X, t) = 0$ is given by the velocity $\dot{\phi} = a(s)$, the total dissipation is finally expressed as

$$D_m = \frac{a}{2} \int_0^l \left(-\frac{\Sigma^2}{2E^2(d)} \right) \frac{dE}{dd} d'(\phi) d\phi = G(l, \Sigma) a, \quad (49)$$

where the release rate of energy $G(l, \Sigma)$ is

$$G(l, \Sigma) = \frac{1}{2} \Sigma^2 \left(\frac{1}{E(d(l))} - \frac{1}{E(0)} \right). \quad (50)$$

When $l = l_c$, we recover the expression of the sharp interface. The value of the energy release rate $G(l_c, \Sigma)$ doesn't depend on the distribution of $E(d)$.

We study now the case of cylinder under longitudinal shear and cylinder and sphere in radial expansion.

4.3 The longitudinal shear

We consider a damage law defined as

$$\frac{1}{\mu} = \frac{d}{\mu_1} + \frac{1-d}{\mu_0}. \quad (51)$$

The local driving force is then

$$Y = -\frac{\partial}{\partial d} \left(\frac{1}{2} \frac{\tau^2}{\mu} \right) = \frac{1}{2} \frac{C^2}{r^2} \frac{\mu_0 - \mu_1}{\mu_0 \mu_1} \quad (52)$$

Here we consider the damage law $d(\phi) = \phi/l_c$.

Initially there is no damaged zone, the local response is purely elastic. The maximum value of Y is obtained at surface R_i . When the critical value Y_c is reached, the damage zone appears, the thickness of the layer increases to attain the final value l_c and then the layer propagates inside the volume Ω_o with the constant thickness l_c .

To simplify the analysis, we consider C as the loading parameter, the damage zone appears when $Y(R_i, C) = Y_c$ this defines the critical value τ_c

$$\tau_c^2 = 2Y_c R_i^2 / R_e^2 \left(\frac{\mu_0 - \mu_1}{\mu_0 \mu_1} \right)^{-1} \quad (53)$$

We define now the dissipation during the different state of the evolution, we consider the damage function

$$d(\phi) = \frac{\phi}{l_c}, \quad d' = \frac{1}{l_c} \quad (54)$$

During initiation of the layer Then damage zone appears at $r = R_i$ and the dissipation is given by the variation of the potential energy relatively to the motion of Γ_o , then $\phi = \Gamma_o - r$, $R_i \leq r \leq \Gamma_o$. The value of J is

$$J(\Gamma_o) = \int_{R_i}^{\Gamma_o} \frac{dr}{r\mu(r)} + \int_{\Gamma_o} \frac{dr}{r\mu_o}, \quad (55)$$

and the dissipation is given by

$$D_m = \pi \frac{C^2}{l_c} \frac{\mu_o - \mu_1}{\mu_o\mu_1} \log\left(\frac{\Gamma_o}{R_i}\right) a = 2\pi \Gamma_o G_\Gamma a. \quad (56)$$

For this state the normality rule implies

$$D_m = 2\pi Y_c \int_{R_i}^{\Gamma_o} d'r dr a = \frac{\pi Y_c}{l_c} (\Gamma_o^2 - R_i^2). \quad (57)$$

During propagation of the layer The damage is defined in the layer ($\Gamma_o - l_c \leq r \leq \Gamma_o$) and $\phi = \Gamma_o - r$, the value of J becomes

$$J = \int_{R_i}^{\Gamma_o - l_c} \frac{dr}{r\mu_1} + \int_{\Gamma_o - l_c}^{\Gamma_o} \frac{dr}{r\mu(r)} + \int_{\Gamma_o}^{R_e} \frac{dr}{r\mu_o}. \quad (58)$$

and the dissipation is reduced to

$$D_m = \pi \frac{C^2}{l_c} \frac{\mu_o - \mu_1}{\mu_o\mu_1} \log\left(\frac{\Gamma_o}{\Gamma_o - l_c}\right) a. \quad (59)$$

And the normality rule implies

$$\begin{aligned} D_m &= 2\pi Y_c \int_{\Gamma_o - l_c}^{\Gamma_o} d' dr a \\ &= \frac{\pi Y_c}{l_c} \left(\Gamma_o^2 - (\Gamma_o - l_c)^2 \right) a. \end{aligned}$$

When $R_e + l_c \geq \Gamma_o \geq R_e$. In this case

$$D_m = \pi C^2 \frac{\partial J}{\partial \Gamma_o} = \frac{\pi C^2}{l_c} \frac{\mu_o - \mu_1}{\mu_o\mu_1} \log \frac{R_e}{\Gamma_o - l_c}, \quad (60)$$

and the normality rule implies during for state

$$D_m = \pi Y_c \frac{1}{l_c} \left(R_e^2 - (\Gamma_o - l_c)^2 \right) \quad (61)$$

Final state When the transformation is complete, the behaviour is those of a elastic system with modulus μ_1 .

Taking account of the normality rule, the global response is now determined. For increasing E or C we have successively

State I The response is purely elastic until the value $Y(R_i) \leq Y_c$, then

$$\tau(R_e) = \frac{C}{R_e}, \quad u(R_e) = \frac{C}{\mu_o} \log \frac{R_e}{R_i}. \quad (62)$$

This is the solution until $\tau(R_e) \leq \tau_c$.

State II Initiation of the layer. The damage zone evolves accordingly to the normality rule, then

$$C^2 \frac{\mu_o - \mu_1}{\mu_o\mu_1} \log \frac{\Gamma_o}{R_i} = Y_c (\Gamma_o^2 - R_i^2), \quad (63)$$

and the global response is

$$\begin{aligned} \tau(R_e) &= C/R_e, \\ u(R_e) &= \frac{\mu_o - \mu_1}{\mu_o\mu_1} \frac{C}{l_c} \left(\Gamma_o \log \frac{\Gamma_o}{R_i} - \Gamma_o + R_i \right) \\ &\quad + \frac{C}{\mu_o} \log \frac{R_e}{R_i}. \end{aligned}$$

Knowing the position of the layer, the response of the system is determined.

State III Propagation of the layer. In this case $R_e \geq \Gamma_o \geq R_i + l_c$, the normality rule gives the relation

$$\begin{aligned} C^2 \frac{\mu_o - \mu_1}{\mu_o\mu_1} \log\left(\frac{\Gamma_o}{\Gamma_o - l_c}\right) \\ = Y_c \left(\Gamma_o^2 - (\Gamma_o - l_c)^2 \right), \end{aligned}$$

and during this step

$$\begin{aligned} \tau(R_e) &= C/R_e, \\ u(R_e) &= \frac{C}{\mu_o} \log \frac{R_e}{R_i} + C \frac{\mu_o - \mu_1}{\mu_o\mu_1} \left(\log \frac{\Gamma_o - l_c}{R_i} - 1 \right) \\ &\quad - C \frac{\mu_o - \mu_1}{\mu_o\mu_1} \frac{\Gamma_o}{l_c} \log \frac{\Gamma_o - l_c}{\Gamma_o}. \end{aligned}$$

It can be noticed that when l_c/Γ_o tends to 0, the response of the sharp interface is recover. This can be

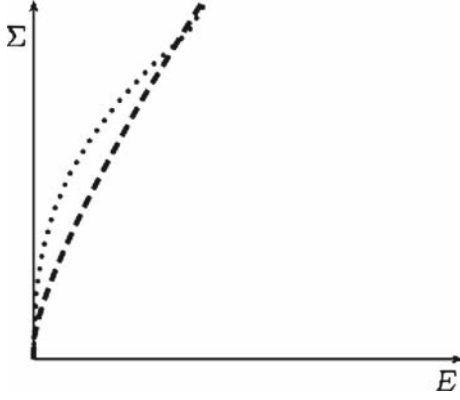


Fig. 4 The response of the cylinder under longitudinal shear: sharp interface (*dashed curve*) and graded damage (*dotted curve*)

interpreted into two manners, for fixed Γ_o when the thickness of the layer tends to zero, the diffuse interface tends to the sharp interface, when l_c is fixed, if the radius Γ_o of the moving layer grows, the sharp interface is recovered (Fig. 4).

There is a great difference with the response of the bar, the dissipation and the response depends on the relation $\mu(d)$, $d(\phi)$. For the bar it has been shown that the dissipation does not depend on the distribution of the Young modulus in the layer, when the thickness l_c is reached. This point emphasizes the influence of the curvature on the response in the case of diffuse damage.

4.4 The cylinder and the sphere under radial loading

We consider the same damage law $d(\phi) = \phi/l_c$ and the distribution of the modulus of the form

$$\frac{1}{\Lambda(r)} = \frac{d}{\Lambda_1} + \frac{1-d}{\Lambda_o}. \quad (64)$$

During the initiation of the layer, the integral I_n must be evaluated

$$I_n = \int_0^{R_e} \frac{r^{n-1} dr}{\Lambda(r)} = \frac{R_e^n}{\Lambda_o n} + \frac{\Lambda_o - \Lambda_1}{\Lambda_o \Lambda_1} \frac{\Gamma_o^{n+1}}{n(n+1)l_c}, \quad (65)$$

and then during this step the dissipation must satisfies the relation

$$D_m = 2^{n-1} \pi \frac{\Lambda_o - \Lambda_1}{\Lambda_o \Lambda_1} \frac{A^2 \Gamma_o^n}{n l_c} = \pi 2^{n-1} Y_c \int_0^{\Gamma_o} \frac{r^{n-1} dr}{l_c}.$$

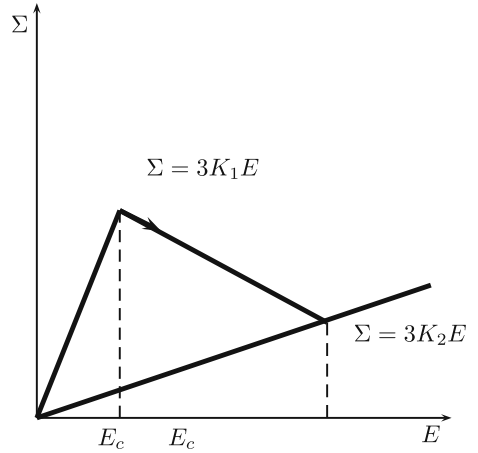


Fig. 5 The response of the cylinder and the sphere under radial loading

and then

$$\frac{\Lambda_o - \Lambda_1}{\Lambda_o \Lambda_1} A^2 = Y_c, \quad (66)$$

the parameter A is constant. The value A_c determines the critical strain E_c

$$A_c = \sqrt{\frac{Y_c}{\frac{\Lambda_o - \Lambda_1}{\Lambda_o \Lambda_1}}}, \quad E_c = A_c I_n(R_e, 0)/R_e^n. \quad (67)$$

With this relation, the global response is determined.

$$\Sigma = A - 2^{n-1} \mu E, \\ E = A \frac{I_n(\Gamma_o)}{R_e^n}.$$

After an elastic response until $E \leq E_c$, the global stress decreases linearly with the imposed strain E as plotted on Fig. 5.

When the layer reaches the thickness l_c , the layer moves inside Ω_o keeping this thickness constant.

$$I_n(\Gamma_o) = \frac{R_e^n}{n \Lambda_o} + \frac{\Lambda_o - \Lambda_1}{\Lambda_o \Lambda_1} \frac{\Gamma_o^{n+1} - (\Gamma_o - l_c)^{n+1}}{l_c n(n+1)}. \quad (68)$$

The relation on the dissipation gives the constrain

$$D_m = A^2 \frac{\Lambda_o - \Lambda_1}{\Lambda_o \Lambda_1} 2^{n-2} \pi \frac{\Gamma_o^n - (\Gamma_o - l_c)^n}{l_c n} \\ = \pi 2^{n-1} Y_c \int_{\Gamma_o - l_c}^{\Gamma_o} \frac{r^{n-1} dr}{l_c}. \quad (69)$$

Then after simplification

$$A^2 \frac{\Lambda_o - \Lambda_1}{\Lambda_o \Lambda_1} = 2Y_c. \quad (70)$$

During this phase, the parameter A keeps a constant value. We recover exactly the response of the sharp interface. For the definition of $\Lambda(d)$ and whatever is $d(\phi)$ it is obvious that we have the same solution when the layer is moving with constant thickness l_c . It is due on the fact that

$$\frac{\partial I_n}{\partial \Gamma_o} = \frac{\Lambda_o - \Lambda_1}{\Lambda_o \Lambda_1} \int_{\Gamma_o - l_c}^{\Gamma_o} d'(\phi) r^{n-1} dr, \quad (71)$$

and the dissipation during the propagation is

$$\begin{aligned} D_m &= 2^{n-2} \pi A^2 \frac{\partial I_n}{\partial \Gamma_o} a \\ &= 2^{n-1} \pi Y_c \int_{\Gamma_o - l_c}^{\Gamma_o} d'(\phi) r^{n-1} dr a. \end{aligned}$$

The relation (70) is recovered.

This result is essentially due to the choice of $\Lambda(d)$. If, for the same function $d(\phi)$, we choose $\Lambda(d) = \Lambda_1 d + \Lambda_o(1 - d)$, during the propagation of the layer when $l = l_c$, we have

$$\Lambda(r) = \Lambda_1 \frac{\Gamma_o - r}{l_c} + \Lambda_o \left(1 - \frac{\Gamma_o - r}{l_c} \right) = \Lambda_M + \Delta \Lambda \frac{r}{l_c}. \quad (72)$$

In this case, for the cylinder

$$I_2 = \frac{(\Gamma_o - l_c)^2}{2\Lambda_1} + \frac{R_e^2 - \Gamma_o^2}{2\Lambda_o} + \frac{l_c^2}{\Delta \Lambda} - \frac{\Lambda_M l_c^2}{\Delta \Lambda^2} \log \frac{\Lambda_o}{\Lambda_1}.$$

The relation taking account of the normality rule is changed into

$$A^2 \left(\frac{\Lambda_o - \Lambda_1}{\Lambda_o \Lambda_1} \Gamma_o + \frac{l_c}{\Lambda_1} \frac{\Lambda_o}{\Delta \Lambda} \right) = 2Y_c \Gamma_o. \quad (73)$$

Then the influence of the thickness of the layer is emphasised and the sharp interface is recovered when $\frac{l_c}{\Gamma_o}$ tends to zero.

This shows that on contrary to the bar, when the layer is moving with a finite thickness the curvature $1/\Gamma_o$ have a influence.

5 Conclusion

We have analysed a new approach on damage evolution based on the notion of level set and moving layer. The initiation of damage and its evolution are presented in the same framework as it has been proved in Pradeilles-Duval and Stolz (1995), Stolz and Moës (2012).

Comparisons with the evolution of a sharp interface are made. For particular distribution of the modulus with respect to the level-set and particular laws of damage, the global response of the system could be the same as those obtained by considering a sharp interface.

The dissipation and the thermodynamical forces for motion of the layer depend on the local shape of the layer, in particular on its curvature.

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