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Andrea Brini

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# $E_8$ SPECTRAL CURVES

ANDREA BRINI

ABSTRACT. I provide an explicit construction of spectral curves for the affine  $E_8$  relativistic Toda chain. Their closed form expression is obtained by determining the full set of character relations in the representation ring of  $E_8$  for the exterior algebra of the adjoint representation; this is in turn employed to provide an explicit construction of both integrals of motion and the action-angle map for the resulting integrable system.

I consider two main areas of applications of these constructions. On the one hand, I consider the resulting family of spectral curves in the context of the correspondences between Toda systems, 5d Seiberg–Witten theory, Gromov–Witten theory of orbifolds of the resolved conifold, and Chern–Simons theory to establish a version of the B-model Gopakumar–Vafa correspondence for the  $sl_N$  Lê–Murakami–Ohtsuki invariant of the Poincaré integral homology sphere to all orders in  $1/N$ . On the other, I consider a degenerate version of the spectral curves and prove a 1-dimensional Landau–Ginzburg mirror theorem for the Frobenius manifold structure on the space of orbits of the extended affine Weyl group of type  $E_8$  introduced by Dubrovin–Zhang (equivalently, the orbifold quantum cohomology of the type- $E_8$  polynomial  $\mathbb{C}P^1$  orbifold). This leads to closed-form expressions for the flat co-ordinates of the Saito metric, the prepotential, and a higher genus mirror theorem based on the Chekhov–Eynard–Orantin recursion. I will also show how the constructions of the paper lead to a generalisation of a conjecture of Norbury–Scott to ADE  $\mathbb{P}^1$ -orbifolds, and a mirror of the Dubrovin–Zhang construction for all Weyl groups and choices of marked roots.

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## Contents

### 1. INTRODUCTION

Spectral curves have been the subject of considerable study in a variety of contexts. These are moduli spaces  $\mathcal{S}$  of complex projective curves endowed with a distinguished pair of meromorphic abelian differentials and a marked symplectic subring of their first homology group; such data define (one or more) families of flat connections on the tangent bundle of the smooth part of moduli space. In particular, a Frobenius manifold structure on the base of the family, a dispersionless integrable hierarchy on its loop space, and the genus zero part of a semi-simple CohFT are then naturally defined in terms of periods of the aforementioned differentials over the marked cycles; a canonical reconstruction of the dispersive deformation (resp. the higher genera of the CohFT) is furthermore determined by  $\mathcal{S}$  through the topological recursion of [49].

The one-line summary of this paper is that I offer two constructions (related to Points (II) and (IV) below) and two isomorphisms (related to Points (III), (V) and (VI)) in the context of spectral curves with exceptional gauge symmetry of type  $E_8$ .

**1.1. Context.** Spectral curves are abundant in several problems in enumerative geometry and mathematical physics. In particular:

- (I) in the spectral theory of finite-gap solutions of the KP/Toda hierarchy, spectral curves arise as the (normalised, compactified) affine curve in  $\mathbb{C}^2$  given by the vanishing locus of the Burchnell–Chaundy polynomial ensuring commutativity of the operators generating two distinguished flows of the hierarchy; the marked abelian differentials here are just the differentials of the two coordinate projections onto the plane. In this case, to each smooth point in moduli space with fibre a smooth Riemann surface  $\Gamma$  there corresponds a canonical theta-function solution of the hierarchy depending on  $g(\Gamma)$  times, and the associated dynamics is encoded into a linear flow on the Jacobian of the curve;

- (II) in many important cases, this type of linear flow on a Jacobian (or, more generally, a principally polarised Abelian subvariety thereof, singled out by the marked basis of 1-cycles on the curve) is a manifestation of the Liouville–Arnold dynamics of an auxiliary, finite-dimensional integrable system. Coordinates in moduli space correspond to Cauchy data – i.e., initial values of involutive Hamiltonians/action variables – and flow parameters are given by linear coordinates on the associated torus;
- (III) all the action has hitherto taken place at a fixed fibre over a point in moduli space; however additional structures emerge once moduli are varied by considering secular (adiabatic) deformations of the integrals of motions via the Whitham averaging method. This defines a dynamics on moduli space which is itself integrable and admits a  $\tau$ -function; remarkably, the logarithm of the  $\tau$ -function satisfies the big phase-space version of WDVV equations, and its restriction to initial data/small phase space defines an almost-Frobenius manifold structure on the moduli space;
- (IV) from the point of view of four dimensional supersymmetric gauge theories with eight supercharges, the appearance of WDVV equations for the Whitham  $\tau$ -function is equivalent to the constraints of rigid special Kähler geometry on the effective prepotential; such equivalence is indeed realised by presenting the Coulomb branch of the theory as a moduli space of spectral curves, the marked differentials giving rise to the the Seiberg–Witten 1-form, the BPS central charge as the period mapping on the marked homology sublattice, and the prepotential as the logarithm of the Whitham  $\tau$ -function;
- (V) in several cases, the Picard–Fuchs equations satisfied by the periods of the SW differential are a reduction of the GKZ hypergeometric system for a toric Calabi–Yau variety, whose quantum cohomology is then isomorphic to the Frobenius manifold structure on the moduli of spectral curves. What is more, spectral curve mirrors open the way to include higher genus Gromov–Witten invariants in the picture through the Chekhov–Eynard–Orantin topological recursion: a universal calculus of residues on the fibres of the family  $\mathcal{S}$ , which is recursively determined by the spectral data. This provides simultaneously a definition of a higher genus topological B-model on a curve, a higher genus version of local mirror symmetry, and a dispersive deformation of the quasi-linear hierarchy obtained by the averaging procedure;
- (VI) in some cases, spectral curves may also be related to multi-matrix models and topological gauge theories (particularly Chern–Simons theory) in a formal  $1/N$  expansion: for fixed 't Hooft parameters, the generating function of single-trace insertion of the gauge field in the planar limit cuts out a plane curve in  $\mathbb{C}^2$ . The asymptotic analysis of the matrix model/gauge theory then falls squarely within the above setup: the formal solution of the Ward identities of the model dictates that the planar free energy is calculated by the special Kähler geometry relations for the associated spectral curve, and the full  $1/N$  expansion of connected multi-trace correlators is computed by the topological recursion.

A paradigmatic example is given by the spectral curves arising as the vanishing locus for the characteristic polynomial of the Lax matrix for the periodic Toda chain with  $N + 1$  particles. In this

case (I) coincides with the theory of  $N$ -gap solutions of the Toda hierarchy, which has a counterpart (II) in the Mumford–van Moerbeke algebro-geometric integration of the Toda chain by way of a flow on the Jacobian of the curves. In turn, this gives a Landau–Ginzburg picture for an (almost) Frobenius manifold structure (III), which is associated to the Seiberg–Witten solution of  $\mathcal{N} = 2$  pure  $SU(N+1)$  gauge theory (IV). The relativistic deformation of the system relates the Frobenius manifold above to the quantum cohomology (V) of a family of toric Calabi–Yau threefolds (for  $N = 1$ , this is  $K_{\mathbb{P}^1 \times \mathbb{P}^1}$ ), which encodes the planar limit of  $SU(M)$  Chern–Simons–Witten invariants on lens spaces  $L(N+1, 1)$  in (VI).

**1.2. What this paper is about.** A wide body of literature has been devoted in the last two decades to further generalising at least part of this web of relations to a wider arena (e.g. quiver gauge theories). A somewhat orthogonal direction, and one where the whole of (I)-(VI) have a concrete generalisation, is to consider the Lie-algebraic extension of the Toda hierarchy and its relativistic counterpart to arbitrary root systems  $R$  associated to semi-simple Lie algebras, the standard case corresponding to  $R = A_N$ . Constructions and proofs of the relations above have been available for quite a while for (II)-(IV) and more recently for (V)-(VI), in complete generality except for one, single, annoyingly egregious example:  $R = E_8$ , whose complexity has put it out of reach of previous treatments in the literature. This paper grows out of the author’s stubbornness to fill out the gap in this exceptional case and provide, as an upshot, some novel applications of Toda spectral curves which may be of interest for geometers and mathematical physicists alike. As was mentioned, the aim of the paper is to provide two main constructions, and prove two isomorphisms, as follows.

**Construction 1:** The first construction fills the gap described above by exhibiting closed-form expressions for arbitrary moduli of the family of curves associated to the relativistic Toda chain of type  $\widehat{E}_8$  for its sole quasi-minuscule representation – the adjoint. This is achieved in two steps: by determining the dependence of the regular fundamental characters of the Lax matrix on the spectral parameter, and by subsequently computing the polynomial character relations in the representation ring of  $E_8$  (viewed as a polynomial ring over the fundamental characters) corresponding to the exterior powers of the adjoint representation. The last step, which is of independent representation-theoretic interest, is the culmination of a computational tour-de-force which in itself is beyond the scope of this paper, and will find a detailed description in [23]; I herein limit myself to announce and condense the ideas of [23] into the 2-page summary given in Appendix C, and accompany this paper with a *Mathematica* package<sup>1</sup> containing the solution thus achieved. As an immediate spin-off I obtain the generating function of the integrable model (in the language of [55]) as a function of the basic involutive Hamiltonians attached to the fundamental weights, and a family of spectral curves as its vanishing locus. In the process, this yields a canonical set of integrals of motion in involution in cluster variables and in Darboux co-ordinates for the integrable

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<sup>1</sup>This is available at <http://tiny.cc/E8SpecCurve>. Part of the complexity is reflected in the size of the compressed data containing the final solution ( $\sim 180$ Mb – should the reader wish to have a closer look at this, they should be aware that this unpacks to binary files and a *Mathematica* notebook that are collectively 5x this size).

system on a special double Bruhat cell of the coextended Poisson–Lie loop group  $\widehat{E}_8^\#$ , which, by analogy with the case of  $\widehat{A}$ -series, I call “the relativistic  $\widehat{E}_8$  Toda chain”, and whose dynamics is solved completely by the preceding construction.

**Construction 2:** The previous construction gives the first element in the description of the spectral curve – a family of plane complex algebraic curves, which are themselves integrals of motion. The next step determines the three remaining characters in the play, namely the two marked Abelian differentials and the distinguished sublattice of the first homology of the curves; this goes hand in hand with the construction of appropriate action–angle variables for the system. The ideology here is classical [37, 60, 68, 87, 117] in the non-relativistic case, and its adaptation to the relativistic setting at hand is straightforward: I identify the phase space of the Toda system with a fibration over the Cartan torus of  $E_8$  (times  $\mathbb{C}^*$ ) by Abelian varieties, which are Prym–Tyurin sub-tori of the spectral curve Jacobian. These are selected by the curve geometry itself, due to an argument going back to Kanev [68], and the Liouville–Arnold flows linearise on them. The Hamiltonian structure inherited from the embedding of the system into a Poisson–Lie–Bruhat cell translates into a canonical choice of symplectic form on the universal family of Prym–Tyurins, and it pins down (up to canonical transformation) a marked pair of Abelian third kind differentials on the curves.

Altogether, the family of curves, the marked 1-forms, and the choice of preferred cycles lead to the assignment of a set of *Dubrovin–Krichever data* (Definition 3.1) to the family of spectral curves. Armed with this, I turn to some of the uses of Toda spectral curves in the context of Fig. 1.

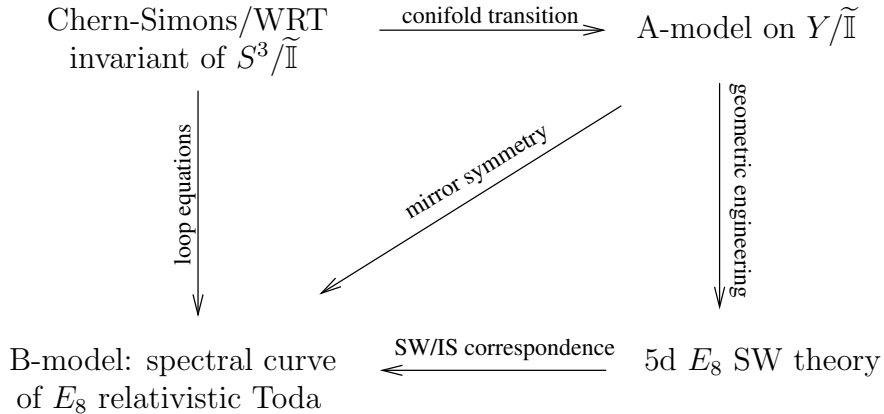


FIGURE 1. Duality web for the B-model on Toda spectral curves

**Isomorphism 1:** Toda spectral curves have long been proposed to encode the Seiberg–Witten solution of  $\mathcal{N} = 2$  pure gluodynamics in four dimensional Minkowski space [59, 85], as well as of its higher dimensional  $\mathcal{N} = 1$  parent theory on  $\mathbb{R}^4 \times S^1$  [94] in the relativistic case. From the physics point of view, Constructions 1-2 provide the Seiberg–Witten solution for minimal, five-dimensional supersymmetric  $E_8$  Yang–Mills theory on  $\mathbb{R}^4 \times S^1$ ; and as the latter should be related to (twisted) curve counts on an orbifold of the resolved conifold

$Y = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$  by the action of the binary icosahedral group  $\tilde{\mathbb{I}}$ , the same construction provides a conjectural 1-dimensional mirror construction for the orbifold Gromov–Witten theory of these targets, as well as to its large  $N$  Chern–Simons dual theory on the Poincaré sphere  $S^3/\tilde{\mathbb{I}} \simeq \Sigma(2, 3, 5)$  [3, 15, 58, 99]. I do not pursue here the proof of either the bottom horizontal (SW/integrable systems correspondence) or the diagonal (mirror symmetry) arrow in the diagram of Fig. 1, although it is highlighted in the text how having access to the global solution on its Coulomb branch allows to study particular degeneration limits of the solution corresponding to superconformal (maximally Argyres–Douglas) points where mutually non-local dyons pop up in the massless spectrum, and limiting versions of mirror symmetry for the Toda curves in Isomorphism 2 below are also considered. What I do prove instead is a version of the vertical arrow, completing results in a previous joint paper with Borot [15]: namely, that the Chern–Simons/Reshetikhin–Turaev–Witten invariant of  $\Sigma(2, 3, 5)$  restricted to the trivial flat connection (the Lê–Murakami–Ohtsuki invariant), as well as the quantum invariants of fibre knots therein in the same limit and for arbitrary colourings, are computed to all orders in  $1/N$  from the Chekhov–Eynard–Orantin topological recursion on a suitable subfamily of  $\widehat{E}_8$  relativistic Toda spectral curves. As in [15], the strategy resorts to studying the trigonometric eigenvalue model associated to the LMO invariant of the Poincaré sphere at large  $N$  and to prove that the planar resolvent is one of the meromorphic coordinate projections of a plane curve in  $(\mathbb{C}^*)^2$ , which is in turn shown to be the affine part of the spectral curve of the  $\widehat{E}_8$  relativistic Toda chain.

**Isomorphism 2:** I further consider two meaningful operations that can be performed on the spectral curve setup of Construction 1-2. The first is to take a degeneration limit to the leaf where the natural Casimir function of the affine Toda chain goes to zero; this corresponds to the restriction to degree-zero orbifold invariants on the top-right corner of Fig. 1, and to the perturbative limit of the 5D prepotentials of the bottom-right corner. The second is to replace one of the marked Abelian integrals with their exponential; this is a version of Dubrovin’s notion of (almost)-duality of Frobenius manifolds [41].

I conjecture and prove that the resulting spectral curve provides a 1-dimensional Landau–Ginzburg mirror for the Frobenius manifold structure constructed on orbits of the extended affine Weyl group of type  $E_8$  by Dubrovin and Zhang [43]. Their construction depends on a choice of simple root, and the canonical choice they take matches with the Frobenius manifold structure on the Hurwitz space determined by our global spectral curve. This extends to the first (and most) exceptional case the LG mirror theorems of [42] for the classical series; and it opens the way to formulate a precise conjecture for how the general case, encompassing general choices of simple roots in the Dubrovin–Zhang construction, should receive an analogous description in terms of Toda spectral curves for the corresponding Poisson–Lie group and twists thereof by the action of a Type I symmetry of WDVV (in the language of [39]). Restricting to the simply-laced case, this gives a mirror theorem for the quantum cohomology of ADE orbifolds of  $\mathbb{P}^1$ ; our genus zero mirror statement then lifts to an all-genus statement by virtue of the equivalence of the topological recursion with

Givental’s quantisation for R-calibrated Frobenius manifolds. This provides a version, for the ADE series, of statements by Norbury–Scott [46, 53, 96] for the Gromov–Witten theory of  $\mathbb{P}^1$ .

The two constructions and two isomorphisms above will find their place in Section 2, 3, 4 and 5 respectively. I have tried to give a self-contained exposition of the material in each of them, and to a good extent the reader interested in a particular angle of the story may read them independently (in particular Sections 4 and 5).

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## 2. THE $E_8$ AND $\widehat{E}_8$ RELATIVISTIC TODA CHAIN

I will provide a succinct, but rather complete account of the construction of Lax pairs for the relativistic Toda chain for both the finite and affine  $E_8$  root system. This is mostly to fix notation and key concepts for the discussion to follow, and there is virtually no new material here until Section 2.4. I refer the reader to [55, 98, 105, 114, 118] for more context, references, and further discussion. I will subsequently move to the explicit construction of spectral curves and the action-angle map for the affine  $E_8$  chain in Sections 2.4 and 3.

**2.1. Notation.** I will start by fixing some basic notation for the foregoing discussion; in doing so I will endeavour to avoid the uncontrolled proliferation of subscripts “8” related to  $E_8$  throughout the text, and stick to generic symbols instead (such as  $\mathcal{G}$  for the  $E_8$  Lie group,  $\mathfrak{g}$  for its Lie algebra, and so on). I wish to make clear from the outset though that whilst many aspects of the discussion are general, the focus of this section is on  $E_8$  alone; the attentive reader will notice that some of its properties, such as simply-lacedness, or triviality of the centre, are implicitly assumed in the formulas to follow.

Let then  $\mathfrak{g} \triangleq \mathfrak{e}_8$  denote the complex simple Lie algebra corresponding to the Dynkin diagram of type  $E_8$  (Fig. 2). I will write  $\mathcal{G} = \exp \mathfrak{g}$  for the corresponding simply-connected complex Lie group,  $\mathcal{T} = \exp \mathfrak{h}$  for the maximal torus (the exponential of the Cartan algebra  $\mathfrak{h} \subset \mathfrak{g}$ ), and  $\mathcal{W} = N_{\mathcal{T}}/\mathcal{T}$  for the Weyl group. I will also write  $\Pi = \{\alpha_1, \dots, \alpha_8\}$  for the set simple roots (see e.g. (B.1)), and  $\Delta$ ,  $\Delta^*$ ,  $\Delta^{(0)}$ ,  $\Delta^\pm$  to indicate respectively the full root system, the non-vanishing roots, the zero roots, and the negative/positive roots; the choice of splitting  $\Delta^\pm$  determines accordingly Borel subgroups



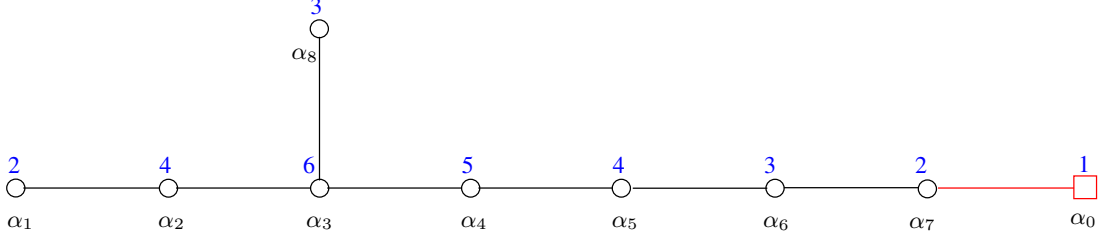


FIGURE 2. The Dynkin diagrams of type  $E_8$  and, superimposed in red, type  $E_8^{(1)}$ ; roots are labelled following Dynkin's convention (left-to-right, bottom-to-top). The numbers in blue are the Dynkin labels for each vertex – for the non-affine roots, these are the components of the highest root in the  $\alpha$ -basis.

$\mathcal{B}^\pm$  intersecting at  $\mathcal{T}$ . Each Borel realises  $\mathcal{G}$  as a disjoint union of double cosets  $\mathcal{G} = \mathcal{B}^\pm \mathcal{W} \mathcal{B}^\pm = \coprod_{w \in \mathcal{W}} \mathcal{B}^\pm w \mathcal{B}^\pm = \coprod_{(w_+, w_-) \in \mathcal{W} \times \mathcal{W}} (\mathcal{B}^+ w_+ \mathcal{B}^+ \cap \mathcal{B}^- w_- \mathcal{B}^-) =: \coprod_{(w_+, w_-) \in \mathcal{W} \times \mathcal{W}} \mathcal{C}_{w_+, w_-}$ , the double Bruhat cells of  $\mathcal{G}$ . The Euclidean vector space  $\text{span}_{\mathbb{R}} \Pi; \langle \cdot, \cdot \rangle \subset \mathfrak{h}^*$  is a vector subspace of  $\mathfrak{h}^*$  with an inner product structure  $\langle \beta, \gamma \rangle$  given by the dual of the Killing form; in particular,  $\langle \alpha_i, \alpha_j \rangle \triangleq \mathcal{C}_{i,j}^{\mathfrak{g}}$  is the Cartan matrix (B.3). For a weight  $\lambda$  in the lattice  $\Lambda_w(\mathcal{G}) \triangleq \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha \rangle \in \mathbb{Z}\}$ , I will write  $\mathcal{W}_\lambda = \text{Stab}_\lambda \mathcal{W}$  for the parabolic subgroup stabilised by  $\lambda$ ; the action of  $\mathcal{W}$  on weights is the restriction of the coadjoint action on  $\mathfrak{h}^*$ ; since  $Z(\mathcal{G}) = e$  in our case, the weight lattice is isomorphic to the root lattice  $\Lambda_r(\mathcal{G}) = \mathbb{Z} \langle \Pi \rangle \simeq \Lambda_w(\mathcal{G})$ . Corresponding to the choice of  $\Pi$ , Chevalley generators  $\{(h_i \in \mathfrak{h}, e_{\pm i} \in \text{Lie}(\mathcal{B}^\pm) \mid i \in \Pi)\}$  for  $\mathfrak{g}$  will be chosen satisfying

$$\begin{aligned}
[h_i, h_j] &= 0, \\
[h_i, e_j] &= \text{sgn}(j) \delta_{i|j} e_j, \\
[e_i, e_{-i}] &= \text{sgn}(i) \mathcal{C}_{i,j}^{\mathfrak{g}} h_j, \\
(\text{ade}_i)^{1 - \mathcal{C}_{ij}^{\mathfrak{g}}} e_j &= 0 \quad \text{for } i + j \neq 0.
\end{aligned} \tag{2.1}$$

Accordingly, the corresponding time- $t$  flows on  $\mathcal{G}$  lead to Chevalley generators  $H_i(t) = \exp t h_i$ ,  $E_i(t) = \exp t e_i$  for the Lie group. Finally, I denote by  $R(\mathcal{G})$  the representation ring of  $\mathcal{G}$ , namely the free abelian group of virtual representations of  $\mathcal{G}$  (i.e. formal differences), with ring structure given by the tensor product; this is a polynomial ring  $\mathbb{Z}[\omega]$  over the integers with generators given by the irreducible  $\mathcal{G}$ -modules having  $\omega_i \in \Lambda_w(G)$  as their highest weights, where  $\langle \omega_i, \alpha_j \rangle = \delta_{ij}$ .

Most of the notions (and notation) above carries through to the setting of the Kac–Moody group<sup>2</sup>  $\widehat{\mathcal{G}} = \exp \mathfrak{g}^{(1)}$  where  $\mathfrak{g}^{(1)} \simeq \mathfrak{g} \otimes \mathbb{C}[\lambda, \lambda^{-1}] \oplus \mathbb{C}c$  is the (necessarily untwisted, for  $\mathfrak{g} \simeq \mathfrak{e}_8$ ) affine Lie

<sup>2</sup>It should be noticed that, while in (2.1) passing from  $h_i$  to  $h'_i = \sum \mathcal{C}_{i,j}^{\mathfrak{g}} h_j$  is an isomorphism of Lie algebras, the same is not true in the affine setting as the Cartan matrix is then degenerate. Our discussion below sticks to the Lie algebra relations as written in (2.1), rather than their more common dualised form; in the affine setting, this substantial difference leads to the centrally coextended loop group instead of the more familiar central extension in Kac–Moody theory. In [55], this is stressed by employing the notation  $G^\#$  for the co-extended group; as I make clear from the outset in (2.1) what side of the duality I am sitting on, I somewhat abuse notation and denote  $\widehat{\mathcal{G}}$  the resulting Poisson–Lie group.

algebra corresponding to  $\mathfrak{e}_8$ . In this case we adjoin the highest (affine) root  $\alpha_0$  as in (B.2), leading to the Dynkin diagram and Cartan matrix in Fig. 2 and (B.4). Elements  $g \in \widehat{\mathcal{G}}$  are linear  $q$ -differential polynomials in the spectral parameter  $\lambda$ ; namely,  $g = M(\lambda)q^{\lambda\partial\lambda}$ , with the pointwise multiplication rule leading to

$$g_1 g_2 = M_1(\lambda)M_2(q_1\lambda) (q_1 q_2)^{\lambda\partial\lambda}. \quad (2.2)$$

The Chevalley generators for the simple Lie group  $\mathcal{G}$  are then lifted to  $\widehat{H}_i(q) \triangleq H_i(q)q^{\mathfrak{d}_i\lambda\partial\lambda}$ , with  $\mathfrak{d}_i$  the Dynkin labels as in Fig. 2, and extended to include  $(H_0, E_0, E_{\bar{0}})$  where

$$H_0(q) = q^{\lambda\partial\lambda}, \quad E_0 = \exp(\lambda e_0), \quad E_{\bar{0}} = \exp(e_{\bar{0}}/\lambda) \quad (2.3)$$

with  $e_0 \in \text{Lie}(\mathcal{B}^+)$  and  $e_{\bar{0}} \in \text{Lie}(\mathcal{B}^-)$  the Lie algebra generators corresponding to the highest (lowest) roots – i.e. the only non-vanishing iterated commutators of order  $h(\mathfrak{g}) = 30$  of  $e_i$  ( $e_{\bar{i}}$ ),  $i = 1, \dots, 8$ .

**2.2. Kinematics.** Consider now the 16-dimensional symplectic algebraic torus

$$\mathcal{P} \simeq ((\mathbb{C}_x^*)^8 \times (\mathbb{C}_y^*)^8, \{, \}_\mathcal{G})$$

with Poisson bracket

$$\{x_i, y_j\}_\mathcal{G} = \mathcal{C}_{ij}^{\mathfrak{g}} x_i y_j. \quad (2.4)$$

Semi-simplicity of  $\mathcal{G}$  amounts to the non-degeneracy of the bracket, so that  $\mathcal{P}$  is symplectic.

There is an injective morphism from  $\mathcal{P}$  to a distinguished Bruhat cell of  $\mathcal{G}$ , as follows. Notice first that  $\mathcal{G}$  carries an adjoint action by the Cartan torus  $\mathcal{T}$  which obviously preserves the Borels, and therefore, descends to an action on the double cosets of the Bruhat decomposition. Consider now Weyl group elements  $w_+ = w_- = \bar{w}$  where  $\bar{w}$  is the ordered product of the eight simple reflections in  $\mathcal{W}$ . The corresponding cell  $\mathcal{P}^{\text{Toda}} \triangleq \mathcal{C}_{\bar{w}, \bar{w}} \subset \mathcal{G}/\mathcal{T}$  has dimension 16 [55], and it inherits a symplectic structure from  $\mathcal{G}$ , as I now describe. Recall that the latter carries a Poisson structure given by the canonical Belavin–Drinfeld–Olive–Turok solution of the classical Yang–Baxter equation [11, 97]:

$$\{g_1 \otimes g_2\}_{\text{PL}} = \frac{1}{2} [r, g_1 g_2], \quad (2.5)$$

with  $r \in \mathfrak{g} \otimes \mathfrak{g}$  given by

$$r = \sum_{i \in \Pi} h_i \otimes h_i + \sum_{\alpha \in \Delta^+} e_\alpha \otimes e_{-\alpha}. \quad (2.6)$$

Since  $\mathcal{T}$  is a trivial Poisson submanifold,  $\mathcal{P}^{\text{Toda}}$  inherits a Poisson structure from the parent Poisson–Lie group. Consider now the (Lax) map

$$\begin{aligned} L_{x,y} &: \mathcal{P} \rightarrow \mathcal{P}^{\text{Toda}} \\ (x, y) &\rightarrow \prod_{i=1}^8 H_i(x_i) E_i H_i(y_i) E_{-i} \end{aligned} \quad (2.7)$$

Then the following proposition holds.

**Proposition 2.1** (Fock–Goncharov, [54]).  *$L$  is an algebraic Poisson embedding into an open subset of  $\mathcal{P}^{\text{Toda}}$ .*

Similar considerations apply to the affine case. In  $(\mathbb{C}^*)^{18} \simeq (\mathbb{C}_x^*)^9 \times (\mathbb{C}_y^*)^9$  with exponentiated linear co-ordinates  $(x_0, x_1, \dots, x_8; y_0, y_1, \dots, y_8)$  and log-constant Poisson bracket

$$\{x_i, y_j\}_{\widehat{\mathcal{G}}} = \mathcal{C}_{ij}^{g(1)} x_i y_j, \quad (2.8)$$

consider the hypersurface  $\widehat{\mathcal{P}} \triangleq \mathbb{V}(\prod_{i=0}^8 (x_i y_i)^{d_i} - 1)$ , where  $\{d_i\}_i$  are the Dynkin labels of Fig. 2. Since  $\text{Ker} C^{\widehat{\mathcal{G}}} = 1$ ,  $\widehat{\mathcal{P}}$  is not symplectic anymore, unlike the simple Lie group case above; in particular, the regular function

$$\mathcal{O}(\widehat{\mathcal{P}}) \ni \mathfrak{K} \triangleq \prod_{i=0}^8 x_i^{d_i} = \prod_{i=0}^8 y_i^{-d_i} \quad (2.9)$$

is a Casimir of the bracket (2.8), and it foliates  $\widehat{\mathcal{P}}$  symplectically. As before, there is a double coset decomposition of  $\widehat{\mathcal{G}}$  indexed by pairs of elements of the affine Weyl group  $\widehat{\mathcal{W}}$ , and a distinguished cell  $\mathcal{C}_{\bar{w}, \bar{w}}$  labelled by the element  $\bar{w}$  corresponding to the longest cyclically irreducible word in the generators of  $\widehat{\mathcal{W}}$ . Projecting to trivial central (co)extension

$$\widehat{G} \ni g = M(\lambda) q^{\lambda d \lambda} \xrightarrow{\pi} M(\lambda) \in \text{Loop}(\mathcal{G}) \quad (2.10)$$

induces a Poisson structure on the projections of the cells  $\mathcal{C}_{w_+, w_-}$  (and in particular  $\mathcal{C}_{\bar{w}, \bar{w}}$ ), as well as their quotients  $\mathcal{C}_{w_+, w_-} / \text{Ad}\mathcal{T}$  by the adjoint action of the Cartan torus, upon lifting to the loop group the Poisson–Lie structure of the non-dynamical r-matrix (2.5). I will write  $\widehat{\mathcal{P}}^{\text{Toda}} \triangleq \pi(\mathcal{C}_{\bar{w}, \bar{w}}) / \text{Ad}\mathcal{T}$  for the resulting Poisson manifold; and we have now that [55]

$$\dim_{\mathbb{C}} \mathcal{P}^{\text{Toda}} = 2 \text{ length}(\bar{w}) - 1 = 2 \times 9 - 1 = 17.$$

Consider now the morphism

$$\begin{aligned} \widehat{L}_{x,y}(\lambda) &: \widehat{\mathcal{P}} \rightarrow \widehat{\mathcal{P}}^{\text{Toda}} \\ (x, y) &\rightarrow \prod_{i=0}^8 \widehat{H}_i(x_i) \widehat{E}_i \widehat{H}_i(y_i) \widehat{E}_{-i}. \end{aligned} \quad (2.11)$$

It is instructive to work out explicitly the form of the loop group element corresponding to  $\widehat{L}_{x,y}$ ; we have

$$\begin{aligned} \widehat{L}_{x,y}(\lambda) &= \prod_{i=0}^8 \widehat{H}_i(x_i) \widehat{E}_i \widehat{H}_i(y_i) \widehat{E}_{-i} \\ &= E_0(\lambda/y_0) E_{\bar{0}}(\lambda) \left[ \prod_{i=0}^8 (x_i y_i)^{d_i} \right]^{\lambda d \lambda} \prod_{i=1}^8 H_i(x_i) E_i H_i(y_i) E_{-i} \\ &= E_0(\lambda/y_0) E_{\bar{0}}(\lambda) \prod_{i=1}^8 H_i(x_i) E_i H_i(y_i) E_{-i} \end{aligned} \quad (2.12)$$

where in moving from the first to the second line we have expanded  $g \in \widehat{G}$  as a linear  $q$ -differential operator and grouped together all the multiplicative  $q$ -shifts, and then used that  $\prod_{i=0}^8 (x_i y_i)^{d_i} = 1$  on  $\widehat{\mathcal{P}}$ , which gives indeed an element with trivial co-extension. The same line of reasoning of Proposition 2.1 shows that  $\widehat{L}$  is a Poisson monomorphism.

**2.3. Dynamics.** For functions  $H_1, H_2 \in \mathcal{O}(\widehat{\mathcal{P}}^{\text{Toda}})$ , the Poisson bracket (2.5) reads, explicitly,

$$\{H_1, H_2\}_{\text{PL}} = -\frac{1}{2} \sum_{\alpha \in \Delta^+} [L_{e_\alpha} H_1 R_{e_{-\alpha}} H_2 - (1 \leftrightarrow 2)] \quad (2.13)$$

where  $L_X$  (resp.  $R_X$ ) denotes the left (resp. right) invariant vector field generated by  $X \in T_e G \simeq \mathfrak{g}$ . Then a complete system of involutive Hamiltonians for (2.5) on  $\mathcal{G}$ , and any Poisson Ad-invariant submanifold such as  $\mathcal{P}^{\text{Toda}}$ , is given by Ad-invariant functions on the group – or equivalently, Weyl-invariant functions on  $\mathcal{T}$ . This is a subring of  $\mathcal{O}(\mathcal{P}^{\text{Toda}})$  generated by the regular fundamental characters

$$H_i(g) = \chi_{\rho_i}(g), \quad i = 1, \dots, 8 \quad (2.14)$$

where  $\rho_i$  is the irreducible representation having the  $i^{\text{th}}$  fundamental weight  $\omega_i$  as its highest weight. In the affine case the same statements hold, with the addition of the central Casimir  $\mathfrak{K}$  in (2.9). The Lax maps (2.7), (2.11) then pull-back this integrable dynamics to the respective tori  $\mathcal{P}$  and  $\widehat{\mathcal{P}}$ . Fixing a faithful representation  $\rho \in R(\mathcal{G})$  (say, the adjoint), the same dynamics on  $\mathcal{P}^{\text{Toda}}$  and  $\widehat{\mathcal{P}}^{\text{Toda}}$  takes the form of isospectral flows [7, Sec. 3.2-3.3]

$$\frac{\partial \rho(L)}{\partial t_i} = \{\rho(L), H_i(L)\}_{\text{PL}} = [\rho(L), (P_i(\rho(L)))_+] \quad (2.15)$$

$$\frac{\partial \rho(\widehat{L})}{\partial t_i} = \{\rho(\widehat{L}), H_i(\widehat{L})\}_{\text{PL}} = [\rho(\widehat{L}), (P_i(\rho(\widehat{L})))_+] \quad (2.16)$$

where  $P_i \in \mathbb{C}[x]$  is the expression of the Weyl-invariant Laurent polynomial  $\chi_{\omega_i} \in \mathcal{O}(\mathcal{T})^{\mathcal{W}}$  in terms of power sums of the eigenvalues of  $\rho(g)$ , and  $(\ )_+ : \mathcal{G} \rightarrow \mathcal{B}^+$  denotes the projection to the positive Borel.

**2.4. The spectral curve.** We henceforth consider the affine case only. Since (2.16) is isospectral, all functions of the spectrum  $\sigma(\rho(\widehat{L}))$  of  $\rho(\widehat{L})$  are integrals of motion. A central role in our discussion will be played by the spectral invariants constructed out of elementary symmetric polynomials in the eigenvalues of  $\widehat{L}$ , for the case in which  $\rho = \mathfrak{g}$  is the adjoint representation, that is, is the minimal-dimensional non-trivial irreducible representation of  $\mathcal{G}$ . I write

$$\Xi_{\mathfrak{g}}(\mu, \lambda) \triangleq \det_{\mathfrak{g}} \left( \widehat{L}(\lambda) - \mu \mathbf{1} \right) \quad (2.17)$$

for the characteristic polynomial of  $\widehat{L}$  in the adjoint, thought of as a 2-parameter family of maps  $\Xi_{\mathfrak{g}}(\mu, \lambda) : \widehat{\mathcal{P}} \rightarrow \mathbb{C}$ . It is clear by (2.16) that  $\Xi_{\mathfrak{g}}(\mu, \lambda)$  is an integral of motion for all  $(\mu, \lambda)$ , and so is therefore the plane curve in  $\mathbb{A}^2$  given by its vanishing locus  $\mathbb{V}(\Xi_{\mathfrak{g}})$ .

We will be interested in expanding out the flow invariant (2.17) as an explicit polynomial function of the basic integrals of motion (2.14). I will do so in two steps: by determining the dependence of (2.14) on the spectral parameter when  $g = \widehat{L}(\lambda)$  in (2.12) and (2.14), and by computing the dependence of  $\Xi_{\mathfrak{g}}(\mu, \lambda)$  on the basic invariants (2.14). We have first the following

**Lemma 2.2.**  $H_i(\widehat{L})$ ,  $i = 1, \dots, 8$  are Laurent polynomials in  $\lambda$ , which are constant except for  $i = 3$ . In particular, there exist functions  $u_i \in \mathcal{O}(\widehat{\mathcal{P}})$  such that

$$H_i(\widehat{L}) = u_i(x, y) - \delta_{i,3} \left( \lambda' + \frac{\aleph^2}{\lambda'} \right) \quad (2.18)$$

with  $\partial_{x_0} u_i(x, y) = \partial_{y_0} u_i(x, y) = 0$  and  $\lambda' = \lambda y_0 \aleph^2$ .

*Proof (sketch).* The proof follows from a lengthy but straightforward calculation from (2.12). Since we are looking at the adjoint representation, explicit matrix expressions for the Chevalley generators (2.1) can be computed by systematically reading off the structure constants in (2.1), the full set of which for all the  $\dim \mathfrak{g} = 248$  generators of the algebra is determined from the canonical assignment of signs to so-called extra-special pairs of roots reflecting the ordering of simple roots within  $\Pi$  (see [34] for details). The resulting  $248 \times 248$  matrix in (2.12), with coefficients depending on  $(\lambda, x, y)$ , is moderately sparse, which allows to compute power sums of its eigenvalues efficiently. We can then show from a direct calculation that (2.18) holds for  $i = 3, \dots, 7$  the relations in  $R(\mathcal{G})$

$$\begin{aligned} \rho_{\omega_7} &= \mathfrak{g}, & \rho_{\omega_6} &= \wedge^2 \mathfrak{g} \ominus \mathfrak{g}, & \rho_{\omega_5} &= \wedge^3 \mathfrak{g} \ominus \rho_{\omega_6} \oplus \wedge^2 \mathfrak{g} \ominus \mathfrak{g} \otimes \mathfrak{g} \\ \rho_{\omega_4} &= \wedge^4 \mathfrak{g} \ominus \rho_{\omega_6} \otimes \mathfrak{g} \oplus \rho_{\omega_6}, & \rho_{\omega_3} &= \wedge^5 \mathfrak{g} \oplus \rho_{\omega_5} \otimes (\mathbf{1} \ominus \mathfrak{g}) \ominus \text{Sym}^3 \mathfrak{g} \end{aligned} \quad (2.19)$$

which are an easy consequence of the decomposition into irreducibles of  $\wedge^n \mathfrak{g}$ ,  $\text{Sym}^n \mathfrak{g}$ , and their tensor powers for  $n \leq 5$ ,<sup>3</sup> and the use of Newton identities relating power sum polynomials (i.e. traces of powers) to elementary/complete symmetric polynomials (i.e. antisymmetric/symmetric traces). A little more work is required to show that  $u_i$  is constant for  $i = 1, 2, 8$ ; this uses the more complicated character relations (2.25)-(2.26) of Appendix C. The final result is (2.18).  $\square$

It is immediately seen from (2.18) that  $u_i(x, y)$  are involutive, independent integrals of motion; they are equal to the fundamental Hamiltonians (2.14) for  $i \neq 3$ , and for  $i = 3$  they are a  $\mathbb{C}[\lambda, \lambda^{-1}]$  linear combination of  $H_3$  and the Casimir  $\aleph$ . Denote now by  $\mathcal{U} = \vec{u}(\widehat{\mathcal{P}}) \subset \mathbb{C}^8$  the image of  $\widehat{\mathcal{P}}$  under the map  $\vec{u} = (u_i)_i : \widehat{\mathcal{P}} \rightarrow \mathbb{C}^8$ . It is clear from (2.17) and (2.18) that  $\Xi_{\mathfrak{g}} : \widehat{\mathcal{P}} \rightarrow \mathbb{C}[\lambda', \aleph^2 \lambda^{-1}, \mu]$  factors through  $\vec{u}$  and a map  $\mathfrak{p} = \sum_k (-)^k \mathfrak{p}_k : \mathcal{U} \rightarrow \mathbb{C}[\lambda', \aleph^2 \lambda^{-1}, \mu]$  given by the decomposition of the characteristic polynomial into fundamental characters:

$$\begin{aligned} \Xi_{\mathfrak{g}}(\lambda, \mu) &= \sum_{k=0}^{248} (-\mu)^k \chi_{\wedge^k \mathfrak{g}} \widehat{L}_{x,y}(\lambda) \\ &= \sum_{k=0}^{124} (-)^k \mathfrak{p}_k(u_1, u_2, u_3 + (\lambda' + \aleph^2/\lambda'), u_4, \dots, u_8) (\mu^k + \mu^{248-k}) \end{aligned} \quad (2.20)$$

where the reality of the adjoint representation has been used. Here  $\mathfrak{p}_k$  is the polynomial relation of formal characters

$$\chi_{\wedge^k \mathfrak{g}} = \mathfrak{p}_k(\chi_{\omega_1}, \dots, \chi_{\omega_8}) \in \mathbb{Z}[\chi_{\omega_1}, \dots, \chi_{\omega_8}] \simeq R(\mathcal{G}) \quad (2.21)$$

evaluated at the group element  $\widehat{L}$ . For fixed  $(u_i)_i \in \mathcal{U}$  and  $\aleph \in \mathbb{C}$ , the vanishing locus  $\mathbb{V}(\Xi_{\mathfrak{g}})$  of the characteristic polynomial is a complex algebraic curve in  $\mathbb{C}^2$ ; I shall write  $\mathcal{B}_{\mathfrak{g}} \triangleq \mathcal{U} \times \mathbb{A}^1$  for

<sup>3</sup>We used *Sage* for the decomposition of the plethysms, and *LieART* for that of the tensor powers.

the variety of parameters this polynomial will depend on. Even though  $\mathfrak{g}$  is irreducible, the curve  $\mathbb{V}(\Xi_{\mathfrak{g}})$  is reducible since  $\Xi_{\mathfrak{g}}$  is. Indeed, conjugating  $\widehat{L}$  to an element  $\exp l \in \mathcal{T}$  in the Cartan torus,  $l \in \mathfrak{h}$ , we have

$$\begin{aligned}\Xi_{\mathfrak{g}}(\lambda, \mu) &= \det_{\mathfrak{g}} \left( \widehat{L} - \mu \mathbf{1} \right) = \prod_{\alpha \in \Delta} (\exp(\alpha(l)) - \mu) \\ &= (\mu - 1)^8 \prod_{\alpha \in \Delta_+} (\exp(\alpha(l)) - \mu) (\exp(-\alpha(l)) - \mu)\end{aligned}\quad (2.22)$$

For a general representation  $\rho$ , we would obtain as many irreducible components as the number of Weyl orbits in the weight system. When  $\rho = \mathfrak{g}$ , and for this case alone, we have only one non-trivial orbit, as well as eight trivial orbits corresponding to the zero roots. I will factor out the trivial component corresponding to zero roots by writing  $\Xi_{\mathfrak{g}, \text{red}} = \Xi_{\mathfrak{g}} / (\mu - 1)^8$ .

**Definition 2.1.** For  $(u, \aleph) \in \mathcal{B}_{\mathfrak{g}}$ , let  $\Gamma_{(u, \aleph)}$  be the normalisation of the projective closure of  $\text{Spec} \frac{\mathbb{C}[\lambda, \mu]}{\langle \Xi_{\mathfrak{g}, \text{red}} \rangle}$ . We call the corresponding family of plane curves  $\pi : \mathcal{S}_{\mathfrak{g}} \rightarrow \mathcal{U} \times \mathbb{C}$ ,

$$\begin{array}{ccc} \Gamma_{u, \aleph} & \hookrightarrow & \mathcal{S}_{\mathfrak{g}} \xrightarrow{(\lambda, \mu)} \mathbb{P}^1 \times \mathbb{P}^1 \\ P_i \uparrow & & \uparrow \Sigma_i \\ & & \downarrow \pi \\ (u, \aleph) & \xrightarrow{\text{pt}} & \mathcal{B}_{\mathfrak{g}} \end{array}\quad (2.23)$$

the family of spectral curves of the  $\widehat{E}_8$  relativistic Toda chain in the adjoint representation. In (2.23),  $P_i$  are the points added in the compactification of  $\mathbb{V}(\Xi_{\mathfrak{g}, \text{red}})$  (see Remark 2.5 and Table 1 below) and  $\Sigma_i$  are the sections marking them.

As is known in the more familiar setting of  $\widehat{\mathcal{G}} = \widehat{\mathfrak{sl}}_N$ , and as we will discuss in Section 3, spectral curves are a key ingredient in the integration of the Toda flows. Knowledge of the spectral curves is encoded into knowledge of the character relations (2.21), which grant access to the explicit form of the polynomial  $\Xi_{\mathfrak{g}, \text{red}}$  to spectral curves for arbitrary moduli  $(u, \aleph)$ : the description of the spectral curves is then reduced to the purely representation-theoretic problem of determining these relations.

In view of this, denote  $\theta_{\bullet} \triangleq \chi_{\rho_{\omega_{\bullet}}}$ ,  $\phi_{\bullet} \triangleq \chi_{\Lambda^{\bullet} \mathfrak{g}}$ . What we are looking for are explicit polynomials

$$\mathfrak{p}_k(\theta) = \sum_{I \in M} n_{I, k} \prod_{j=1}^8 \theta_j^{d_j^{(I)}} \quad (2.24)$$

where the index  $I$  runs over a suitable finite set  $M \ni (d_1^{(I)}, \dots, d_8^{(I)})$ ,  $M \subset \mathbb{N}^8$ , and  $n_{I, k} \in \mathbb{Z}$ . Since what we are ultimately interested in is the reduced characteristic curve  $\Gamma_{u, \aleph}$ , it suffices to compute  $\{n_{I, k}\}$  (and hence  $\mathfrak{p}_k$ ) for  $k \leq 120$ .

**Claim 2.3** ([23]). We determine  $\{n_{I, k} \in \mathbb{Z}\}$  for all  $I \in M$ ,  $k \leq 120$ .

This is the result of a series of computer-assisted calculations, of independent interest and whose details will appear elsewhere [23], but for which I provide a fairly comprehensive summary in Appendix C. For the sake of example, we obtain for the first few values of  $k$ ,

$$\begin{aligned} \mathfrak{p}_6 &= \theta_7\theta_1^2 - \theta_1^3 - \theta_6\theta_1^2 - \theta_1^2 + 2\theta_7^2\theta_1 + 2\theta_2\theta_1 - \theta_4\theta_1 + \theta_5\theta_1 - \theta_6\theta_1 + \theta_6\theta_7\theta_1 - 2\theta_7\theta_1 - \theta_8\theta_1 - \theta_6^2 \\ &+ \theta_6\theta_7^2 - \theta_7^2 - \theta_3 + \theta_2\theta_6 + \theta_5\theta_6 + \theta_2\theta_7 + \theta_4\theta_7 - 2\theta_6\theta_7 + \theta_2\theta_8 - \theta_6\theta_8 - \theta_7\theta_8, \end{aligned} \quad (2.25)$$

$$\begin{aligned} \mathfrak{p}_7 &= \theta_7^4 + 2\theta_1\theta_7^3 - 4\theta_7^3 + \theta_1^2\theta_7^2 - 6\theta_1\theta_7^2 + 2\theta_2\theta_7^2 + 2\theta_5\theta_7^2 - 2\theta_6\theta_7^2 + \theta_7^2 - 2\theta_1^3\theta_7 - \theta_1^2\theta_7 + 4\theta_1\theta_7 \\ &+ 4\theta_1\theta_2\theta_7 - \theta_3\theta_7 + \theta_4\theta_7 + 2\theta_1\theta_5\theta_7 - 4\theta_5\theta_7 + \theta_1\theta_6\theta_7 + 4\theta_6\theta_7 - \theta_1\theta_8\theta_7 - \theta_6\theta_8\theta_7 \\ &+ \theta_1^3 + 2\theta_1^2 + \theta_2^2 + \theta_5^2 + \theta_1\theta_6^2 + \theta_6^2 + \theta_1\theta_8^2 + \theta_1 - \theta_1\theta_2 - 2\theta_1\theta_3 - \theta_3 + \theta_1\theta_4 + \theta_4 - 2\theta_1^2\theta_5 \\ &+ 2\theta_2\theta_5 - \theta_5 + 2\theta_1^2\theta_6 + 3\theta_1\theta_6 - \theta_2\theta_6 + \theta_4\theta_6 - 2\theta_5\theta_6 + \theta_6 - \theta_1^2\theta_8 - \theta_1\theta_8 + \theta_2\theta_8 \\ &- 2\theta_2\theta_7 - \theta_8\theta_7 + 2\theta_7 - \theta_4\theta_8 - \theta_1\theta_6\theta_8 - 3\theta_1\theta_5 \end{aligned} \quad (2.26)$$

$$\begin{aligned} \mathfrak{p}_8 &= \theta_8 - \theta_1^4 - \theta_6\theta_1^3 + 2\theta_7^2\theta_1^2 + 3\theta_2\theta_1^2 - \theta_4\theta_1^2 + \theta_5\theta_1^2 + \theta_6\theta_1^2 - \theta_7\theta_1^2 - 2\theta_7\theta_8\theta_1^2 - 2\theta_7^3\theta_1 + \theta_6^2\theta_1 \\ &+ 3\theta_6\theta_7^2\theta_1 - 3\theta_3\theta_1 + 2\theta_4\theta_1 - \theta_5\theta_1 + 2\theta_2\theta_6\theta_1 + \theta_5\theta_6\theta_1 + 2\theta_6\theta_1 + 2\theta_2\theta_7\theta_1 + \theta_4\theta_7\theta_1 \\ &+ 2\theta_6\theta_7\theta_1 + 5\theta_7\theta_1 + \theta_7^2\theta_8\theta_1 + \theta_2\theta_8\theta_1 - 2\theta_5\theta_8\theta_1 - 3\theta_7\theta_8\theta_1 + \theta_8\theta_1 - 2\theta_7^4 + \theta_6\theta_7^3 + \theta_7^3 \\ &- 2\theta_5^2 + \theta_6^2 - \theta_2\theta_7^2 + 2\theta_4\theta_7^2 - 4\theta_5\theta_7^2 + 3\theta_7^2 - \theta_8^2 + \theta_2 + \theta_3 + \theta_2\theta_4 - \theta_4 - 2\theta_2\theta_5 \\ &- 2\theta_3\theta_6 + \theta_4\theta_6 - \theta_5\theta_6 + \theta_6^2\theta_7 + \theta_2\theta_7 - 3\theta_3\theta_7 + \theta_5\theta_7 + 2\theta_2\theta_6\theta_7 + \theta_5\theta_6\theta_7 + \theta_6\theta_7 - 2\theta_7 \\ &- \theta_2\theta_8 - 3\theta_3\theta_8 + 2\theta_4\theta_8 - 3\theta_5\theta_8 + 2\theta_6\theta_8 + 3\theta_2\theta_7\theta_8 + 2\theta_6\theta_7\theta_8 + 4\theta_7\theta_8, \\ &- \theta_7^2\theta_1 + 2\theta_4\theta_5 + 2\theta_5 - 3\theta_5\theta_7\theta_1 + \theta_8^3 - \theta_2^2 - 4\theta_7^2\theta_8 \end{aligned} \quad (2.27)$$

$$\begin{aligned} \mathfrak{p}_9 &= 2\theta_1^2\theta_7^3 - 2\theta_7^4 - 7\theta_1\theta_7^3 - 3\theta_6\theta_7^3 + 4\theta_7^3 - 2\theta_1^3\theta_7^2 - \theta_1^2\theta_7^2 + \theta_6^2\theta_7^2 + \theta_8^2\theta_7^2 + 10\theta_1\theta_7^2 + 4\theta_1\theta_2\theta_7^2 \\ &- 2\theta_3\theta_7^2 + \theta_4\theta_7^2 + 2\theta_1\theta_5\theta_7^2 - 5\theta_5\theta_7^2 + 2\theta_1\theta_6\theta_7^2 + 2\theta_6\theta_7^2 + \theta_6\theta_8\theta_7^2 - \theta_8\theta_7^2 - 2\theta_7^2 + 2\theta_1^3\theta_7 + \theta_1^2\theta_7 \\ &+ \theta_2^2\theta_7 + 2\theta_1\theta_6^2\theta_7 + 6\theta_6^2\theta_7 + \theta_1\theta_8^2\theta_7 - 2\theta_8^2\theta_7 - \theta_1\theta_7 - 3\theta_1\theta_2\theta_7 + 5\theta_2\theta_7 - 4\theta_1\theta_3\theta_7 + 2\theta_3\theta_7 \\ &+ 2\theta_1\theta_4\theta_7 - 2\theta_1^2\theta_5\theta_7 - 6\theta_1\theta_5\theta_7 + 2\theta_2\theta_5\theta_7 + 5\theta_5\theta_7 + \theta_1^2\theta_6\theta_7 + 5\theta_1\theta_6\theta_7 + 2\theta_2\theta_6\theta_7 + 3\theta_4\theta_6\theta_7 \\ &- 4\theta_5\theta_6\theta_7 + 4\theta_6\theta_7 - 2\theta_1\theta_8\theta_7 + \theta_4\theta_8\theta_7 - 2\theta_5\theta_8\theta_7 - 2\theta_1\theta_6\theta_8\theta_7 - \theta_6\theta_8\theta_7 + \theta_8\theta_7 - \theta_1^3 + 2\theta_6^3 \\ &+ 2\theta_4^2 - \theta_5^2 + 2\theta_1\theta_6^2 + \theta_2\theta_6^2 + 3\theta_6^2 - \theta_1\theta_8^2 + \theta_2\theta_8^2 + \theta_5\theta_8^2 + 2\theta_1\theta_2 + \theta_2 + \theta_1^2\theta_3 + \theta_1\theta_3 - 2\theta_2\theta_3 \\ &- \theta_1\theta_4 - 2\theta_2\theta_4 - \theta_4 + \theta_1^2\theta_5 + 2\theta_1\theta_5 - 2\theta_2\theta_5 - 3\theta_3\theta_5 - \theta_1^2\theta_6 + \theta_1\theta_6 + \theta_1\theta_2\theta_6 + 4\theta_2\theta_6 - 2\theta_3\theta_6 \\ &+ \theta_1\theta_4\theta_6 - \theta_4\theta_6 - 2\theta_1\theta_5\theta_6 + \theta_5\theta_6 + \theta_6 - \theta_1^3\theta_8 + 2\theta_1\theta_2\theta_8 - \theta_3\theta_8 - 2\theta_1\theta_4\theta_8 + \theta_4\theta_8 + \theta_1\theta_5\theta_8 \\ &- \theta_1^2\theta_6\theta_8 + \theta_2\theta_6\theta_8 - \theta_5\theta_6\theta_8 - \theta_1^2 + \theta_1^2\theta_4 - 4\theta_2\theta_7^2 - \theta_4\theta_7. \end{aligned} \quad (2.28)$$

2.4.1. *Genus, ramification points, and points at infinity.* The curves  $\Gamma_{u,\aleph}$  have two obvious involutions, coming from the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry (2.20) of the reduced characteristic polynomial  $\Xi_{\mathfrak{g},\text{red}}$ ,

$$\Xi_{\mathfrak{g},\text{red}}(\lambda', \mu) = \Xi_{\mathfrak{g},\text{red}}(\aleph/\lambda', \mu), \quad \Xi_{\mathfrak{g},\text{red}}(\lambda', \mu) = \mu^{240} \Xi_{\mathfrak{g},\text{red}}(\lambda', 1/\mu). \quad (2.29)$$

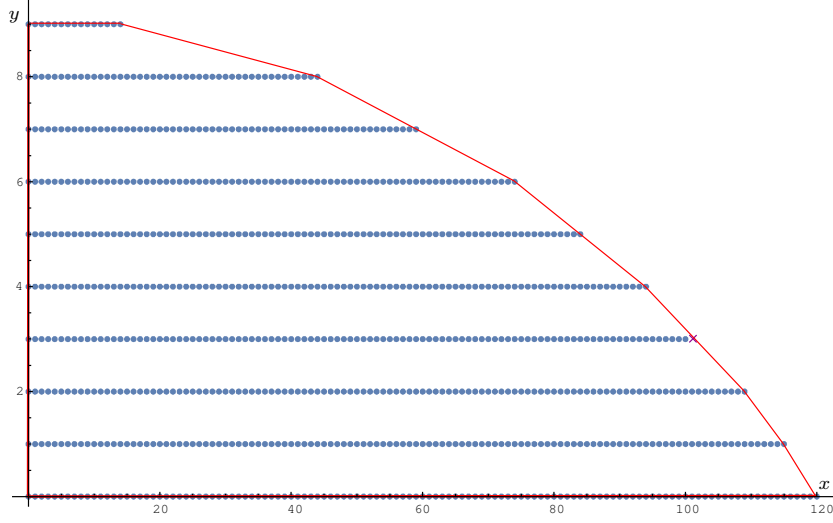


FIGURE 3. The Newton polygon of  $\Xi''_{g,\text{red}}$  (in red); blue spots depict monomials in  $\Xi''_{g,\text{red}}$  with non-zero coefficients; the purple cross marks the vanishing of the coefficient of  $x^{101}y^3$  on the boundary of the polygon.

This realises  $\Gamma_{u,\aleph} \xrightarrow{y} \Gamma'_{u,\aleph} \xrightarrow{x} \Gamma''_{u,\aleph}$ , where  $x = \mu + \mu^{-1}$ ,  $y = \lambda + \aleph\lambda^{-1}$ , as a branched fourfold cover of a curve  $\Gamma''_{u,\aleph} \triangleq \{\Xi''_{g,\text{red}}(y, x) = 0\}$ , so that

$$\Xi_{g,\text{red}}(\lambda', \mu) =: \mu^{120} \Xi''_{g,\text{red}} \left( \lambda' + \frac{\aleph}{\lambda'}, \mu + \frac{1}{\mu} \right) \quad (2.30)$$

We see from (2.20) and (C.2) that  $\deg_y \Xi''_{g,\text{red}}(y, x) = 9$ ,  $\deg_x \Xi''_{g,\text{red}} = 120$ . The Newton polygon of  $\Xi''_{g,\text{red}}$  is depicted in Figure 3. By way of example, some of the simplest coefficients on the boundary are given by:

$$[y^9] \Xi''_{g,\text{red}} = (x+1)^3(x+2)(-1+x+x^2)^5, \quad (2.31)$$

$$\left\{ \left[ x^{\deg_x [y^i] \Xi''_{g,\text{red}}} \right] \Xi''_{g,\text{red}} \right\}_{i=0}^8 = \{1, -1, -1, -3u_7 - 5, 1, 2, 1, -2, 1\}. \quad (2.32)$$

Let us now compute the genus of  $\Gamma''_u$ ,  $\Gamma'_u$  and  $\Gamma_{u,\aleph}$ .

**Proposition 2.4.** *We have, for generic  $(u, \aleph) \in \mathcal{B}_g$ ,*

$$g(\Gamma''_u) = 61, \quad g(\Gamma'_u) = 128, \quad g(\Gamma_{u,\aleph}) = 495. \quad (2.33)$$

*Proof.* Since Lemma 2.2 and Claim 2.3 determine the polynomial  $\Xi''_{g,\text{red}}$  completely, the calculation of the genus can be turned into an explicit calculation of discriminants of  $\Xi''_{g,\text{red}}$ ; and because  $\deg_y \Xi''_{g,\text{red}} \ll \deg_x \Xi''_{g,\text{red}}$ , it is much easier to start from the  $y$ -discriminant. This is computed to be

$$\text{Discr}_y \Xi''_{g,\text{red}} = (x+2)^4 \Delta_1(x) \Delta_2(x)^2 \Delta_3(x)^2 \quad (2.34)$$

where  $\deg \Delta_1 = 133$ ,  $\deg \Delta_2 = 215$  and  $\deg \Delta_3 = 392$ . Call  $r_i^k$ ,  $i = 1, 2, 3$ ,  $k = 1, \dots, \deg \Delta_i$  the roots of  $\Delta_i$ . We can verify directly by substitution into  $\Xi''_{g,\text{red}}$  that the roots  $x = r_2^k$  and  $x = r_3^k$  correspond to images on the  $x$ -line of exactly one point with  $\partial_y \Xi''_g = 0$ , which is always an ordinary



double point. Similarly, we get that the roots  $x = -2$  and  $x = r_1^k$  correspond in all cases to degree 2 ramification points; there are four of them lying over  $x = -2$ . On the desingularised projective curve  $\Gamma''_u$ , the nodes are resolved into pairs of unramified points; and Puiseux expansions of  $\Xi''_{g,\text{red}}$  at infinity show that we have one extra point with degree 2 ramification above  $x = \infty$  (see below). By Riemann–Hurwitz, this give

$$g(\Gamma''_u) = 1 - \deg_y \Xi''_{g,\text{red}} + \frac{1}{2} \sum_{P|\text{dx}(P)=0} e_x(P) = 1 - 9 + \frac{133 + 1 + 4}{2} = 61. \quad (2.35)$$

The genera of the branched double covers  $x : \Gamma'_u \rightarrow \Gamma''_u$ ,  $y : \Gamma_{u,\aleph} \rightarrow \Gamma'_u$  follow from an elementary Riemann–Hurwitz calculation.  $\square$

**Remark 2.5.** It can readily be deduced from (2.31) that the smooth completion  $\Gamma''_u$  is obtained topologically by adding 12 points at infinity  $P''_i$ ; their relevant properties are shown in Table 1. Their pre-images in  $\Gamma'_u$  and  $\Gamma_{u,\aleph}$  will be labelled  $P'_k$  and  $P_j$  respectively,  $k = 1, \dots, 23$  (notice that  $P''_1$  is a branch point of  $x : \Gamma'_u \rightarrow \Gamma''_u$ ),  $j = 1, \dots, 46$ .

$i$	$x(P''_i)$	$e_y(P''_i)$	$-\text{ord}_y P''_i$
1	-2	1	1
2	-1	1	3
3	$-\phi$	1	5
4	$\phi^{-1}$	1	5
5	$\infty$	1	5
6	$\infty$	1	6
7	$\infty$	1	10
8	$\infty$	1	10
9	$\infty$	2	15
10	$\infty$	1	15
11	$\infty$	1	15
12	$\infty$	1	30

TABLE 1. Points at infinity in  $\Gamma''_u$ . I indicate the value of their  $x$ -projection, their degree of ramification in  $y$ , and the order of the poles of  $y$  in the second, third, and fourth column respectively. Here  $\phi = \frac{\sqrt{5}+1}{2}$  is the golden ratio.

**2.5. Spectral vs parabolic vs cameral cover.** The construction of  $\Gamma_{u,\aleph}$  as the non-trivial irreducible component of the vanishing locus of (2.17)-(2.22) realises it as a “curve of eigenvalues”: it is a branched cover of the space of spectral parameters  $\lambda \in \mathbb{P}^1 \setminus \{0, \infty\}$  of the Lax matrix  $\widehat{L}_{x,y}(\lambda)$ ; the fibre over a  $\lambda$ -unramified point is given by the eigenvalues  $\mu_\alpha$  of  $\widehat{L}_{x,y}(\lambda)$  that are different from 1. By (2.22), each sheet  $\mu_\alpha$  is labeled by a non-trivial root  $\alpha \in \Delta^*$ , and there is an action of the Weyl group  $\mathcal{W}$  on  $\Gamma_{u,\aleph}$  given by the interchange of sheets corresponding to the Coxeter action of  $\mathcal{W}$  on the root space  $\Delta$ .

Away from the ramification locus, this structure can be understood as follows. Let

$$\mathcal{G}^{\text{red}} = \{g \in \mathcal{G} \mid \dim_{\mathbb{C}} C_{\mathcal{G}}(g) = \text{rank } \mathcal{G} = 8\}$$

be the Zariski open set of regular elements of  $\mathcal{G}$ ; I'll similarly append a superscript  $\mathcal{T}^{\text{red}}$  for the regular elements of  $\mathcal{T}$ . Then the projection

$$\begin{aligned} \pi : \mathcal{G}/\mathcal{T} \times \mathcal{T}^{\text{red}} &\rightarrow \mathcal{G}^{\text{red}} \\ (g\mathcal{T}, t) &\rightarrow \text{Ad}_g t \end{aligned} \tag{2.36}$$

is a principal  $\mathcal{W}$ -bundle on  $\mathcal{G}^{\text{red}}$ , the fibre over a regular element  $g'$  being  $N_{\mathcal{T}}/\mathcal{T} \simeq \mathcal{W}$ . We can pull this back via  $\widehat{L}_{x,y}$  to a  $\mathcal{W}$ -bundle

$$\Theta_{x,y} \triangleq \widehat{L}_{x,y}^* (\mathcal{G}/\mathcal{T} \times \mathcal{T}^{\text{red}})$$

over  $\mathbb{P}^1 \setminus D$ , where  $D = \widehat{L}_{x,y}^{-1}(\mathcal{G} \setminus \mathcal{G}^{\text{red}})$ . This is a regular  $\mathcal{W}$ -cover and each weight  $\omega \in \Lambda_w(\mathcal{G})$  determines a subcover  $\Theta_{x,y}^{\omega} \simeq \Theta_{x,y}/\mathcal{W}_{\omega}$ , where we quotient by the action of the stabiliser of  $\omega$  by deck transformations. Write  $\overline{\Theta_{x,y}}$  and  $\overline{\Theta_{x,y}^{\omega}}$  for the pull-back to  $\mathbb{C}^* \simeq \mathbb{P}^1 \setminus \{0, \infty\}$  of the closure of (2.36) in  $\mathcal{G}/\mathcal{T} \times \mathcal{T} \rightarrow \mathcal{G}$ . As in [37], we call  $\overline{\Theta_{x,y}}$  (resp.  $\overline{\Theta_{x,y}^{\omega}}$ ) the *cameral* (resp. the  $\omega$ -*parabolic*) cover associated to  $\widehat{L}_{x,y}$ .

Notice that when  $\omega = \omega_7 = \alpha_0$  is the highest weight of the adjoint representation, i.e. the highest (affine) root  $\alpha_0$ ,  $\mathcal{W}/\mathcal{W}_{\alpha_0}$  is set-theoretically the root system of  $\mathfrak{g}$ , minus the set of zero roots; the residual  $\mathcal{W}$  action is just the restriction to  $\Delta$  of the Coxeter action on  $\mathfrak{h}^*$ . In particular, we have that  $\overline{\Theta_{x,y}^{\omega}}$  is a degree  $|W/\mathcal{W}_{\alpha_0}| = |\text{Weyl}(\mathfrak{e}_8)/\text{Weyl}(\mathfrak{e}_7)| = \frac{696729600}{2903040} = 240$  branched cover of  $\mathbb{P}^1$ , with sheets labelled by non-zero roots  $\alpha \in \Delta^*$ .

**Proposition 2.6.** *There is a birational map  $\iota : \Gamma_{u,\aleph} \dashrightarrow \overline{\Theta_{x,y}^{\omega_7}}$  given by an isomorphism*

$$\begin{aligned} \iota : \Gamma_{u,\aleph} \setminus \{d\mu = 0\} &\xrightarrow{\sim} \Theta_{x,y}^{\omega_7} \\ (\lambda, \mu_{\alpha}(\lambda)) &\rightarrow (\lambda, \alpha) \end{aligned} \tag{2.37}$$

*away from the ramification locus of the  $\lambda$ -projection.*

*Proof.* The proof is nearly verbatim the same as that of [86, Thm. 13]. □

From the proposition, we learn that a possible source of ramification  $\lambda : \Gamma_{u,\aleph} \rightarrow \mathbb{P}^1$  comes from the spectral values  $\lambda$  such that  $\widehat{L}_{x,y}(\lambda)$  is an irregular element of  $\mathcal{G}$ ; and from (2.22), we see that this happens if and only if  $\alpha(l) = 0$  for some  $\alpha \in \Delta$ .

**Proposition 2.7.** *For generic  $(u, \aleph)$ , there are exactly 18 values of  $\lambda$ ,*

$$b_i^{\pm} \triangleq \lambda(Q_i^{\pm}), \quad i = 1, \dots, 9, \tag{2.38}$$

*such that  $\widehat{L}_{x,y}(\lambda)$  is irregular, i.e.  $\alpha(\log \widehat{L}_{x,y}(\lambda)) = 0$  for some  $\alpha \in \Delta$ . Furthermore,  $\alpha \in \Pi$  is a simple root in each of these cases.*

*Proof.* To see this, look at the base curve  $\Gamma''_u$ . It is obvious that  $\Xi_{\mathfrak{g},\text{red}}$  has only double zeroes at  $x = 2$ , since  $\Xi_{\mathfrak{g}}$  has only double zeroes at  $\mu = 1$  as roots come in (positive/negative) pairs in (2.22). For each of the nine points

$$\{Q''_i\}_{i=1}^9 \triangleq x^{-1}(2) \subset \Gamma''_u,$$

we compute from Lemma 2.2 and Claim 2.3 that

$$e_x(Q''_i) = 28 \tag{2.39}$$

for all  $i$ . Calling  $\alpha_i \in \Delta^+$  the positive root such that  $\alpha_i \cdot l(\lambda(Q_i)) = 0$ , we see from (2.22) that

$$e_x(Q''_i) = \text{card} \{ \beta \in \Delta^+ \mid \beta - \alpha_i \in \Delta^+ \}. \tag{2.40}$$

It can be immediately verified that the right hand side is less than or equal to 28, with equality iff  $\alpha_i$  is simple. It is also clear that there are no other points of ramification in the affine part of the curve<sup>4</sup>; indeed, from Table 1, we have that  $e_x(\infty) = 120 - 12 = 108$ , and from (2.33) we see that

$$60 = g(\Gamma''_u) - 1 = -\text{deg}_x \Xi''_{\mathfrak{g},\text{red}} + \frac{1}{2} \sum_{dx(P)=0} e_x(P) = -120 + \frac{9 \times 28 + 108}{2}. \tag{2.41}$$

As the covering map  $x : \Gamma'_u \rightarrow \Gamma''_u$  is ramified at  $x = 2$ , and  $y : \Gamma_{u,\aleph} \rightarrow \Gamma'_u$  is generically unramified therein for generic  $\aleph$ , we have two preimages  $Q_{i,\pm}$  on  $\Gamma_{u,\aleph}$  for each  $Q''_i \in \Gamma''_u$ .  $\square$

### 3. ACTION-ANGLE VARIABLES AND THE PREFERRED PRYM–TYURIN

Since (2.14) are a complete set of Hamiltonians in involution on the leaves of the foliation of  $\widehat{\mathcal{P}}$  by level sets of  $\aleph$ , the compact fibres of the map  $(u, \aleph) : \widehat{\mathcal{P}} \rightarrow \mathbb{C}^9$  are isomorphic to a  $\text{rank}(\mathfrak{g}) = 8$ -dimensional torus by the (holomorphic) Liouville–Arnold–Moser theorem. A central feature of integrable systems of the form (2.16) is an algebraic characterisation of their Liouville–Arnold dynamics, the torus in question being an Abelian sub-variety of the Jacobian of  $\Gamma_{u,\aleph}$ .

I determine in this section the action-angle integration explicitly for the  $\widehat{\mathbb{E}}_8$  relativistic Toda chain, which results in endowing  $\mathcal{S}_g$  with extra data [38, 76], as per the following

**Definition 3.1.** *We call Dubrovin–Krichever data a  $n$ -tuple  $(\mathcal{F}, \mathcal{B}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{D}, \Lambda, \Lambda^L)$ , with*

- $\pi : \mathcal{F} \rightarrow \mathcal{B}$  a family of (smooth, proper) curves over an  $n$ -dimensional variety  $\mathcal{B}$ ;
- $\mathcal{D}$  a smooth normal crossing divisor intersecting the fibres of  $\pi$  transversally;
- meromorphic sections  $\mathcal{E}_i \in H^0(\mathcal{F}, \omega_{\mathcal{F}/\mathcal{B}}(\log \mathcal{D}))$  of the relative canonical sheaf having logarithmic poles along  $\mathcal{D}$ ;
- $(\Lambda^L, \Lambda)$  a locally-constant choice of a marked subring  $\Lambda$  of the first homology of the fibres, and a Lagrangian sublattice  $\Lambda^L$  thereof.

<sup>4</sup>In principle, from (2.22), this would be the case if  $\alpha(l(\lambda)) = \beta(l(\lambda))$  for  $\alpha - \beta \notin \Delta$ , leading to a double zero at  $\mu \neq 1$  in (2.22), which we can't *a priori* rule out without appealing to (2.33) and (2.39) as we do below.

Definition 3.1 isolates the extra data attached to spectral curves that were identified in [38, 76] (see also [39, 77]) to provide the basic ingredients for the construction of a Frobenius manifold structure on  $\mathcal{B}$  and a dispersionless integrable dynamics on its loop space given by the Whitham deformation of the isospectral flows (2.16); the logarithm of those  $\tau$ -functions respects the type of constraints that arise in theory with eight global supersymmetries (rigid special Kähler geometry). These will be key aspects of the story to be discussed in Sections 4 and 5; in the language of [38], when the pull-back of  $\mathcal{E}_1$  to the fibres of the family is exact, the associated potential is the superpotential of the Frobenius manifold, and  $\mathcal{E}_2$  its associated primitive differential. Now, Claim 2.3 and Definition 2.1 gave us  $\mathcal{F} = \mathcal{S}_{\mathfrak{g}}$ ,  $\mathcal{B} = \mathcal{B}_{\mathfrak{g}}$  already. We'll see, following [77], how the remaining ingredients are determined by the Hamiltonian dynamics of (2.16): this will culminate with the content of Theorem 3.6. I wish to add from the outset that the process leading up to Theorem 3.6 relies on both common lore and results in the literature that are established and known to the *cognoscenti* at least for the non-relativistic limit; the gist of this section is to unify several of these scattered ideas and adapt them to the setting at hand. For the sake of completeness, I strived to provide precise pointers to places in the literature where similar arguments have been employed.

**3.1. Algebraic action-angle integration.** From now until the end of this section, I will be sitting at a generic point  $(x, y) \in \widehat{\mathcal{P}}$ , and correspondingly, smooth moduli point  $(u, \mathfrak{N}) \in \mathcal{B}_{\mathfrak{g}}$ . As is the case for the ordinary periodic Toda chain with  $N$  particles, and for initial data specified by  $(u, \mathfrak{N})$ , the compact orbits of (2.16) are geometrically encoded into a linear flow on the Jacobian variety  $\text{Pic}^{(0)}(\Gamma_{u, \mathfrak{N}})$  [2, 60, 75, 117]; I recall here why this is the case. The eigenvalue problem<sup>5</sup> at time- $t$ ,

$$\widehat{L}_{x,y}(\lambda)\Psi_{x,y} = \mu\Psi_{x,y} \quad (3.1)$$

with  $x = x(\vec{t})$ ,  $y = y(\vec{t})$ , endows the spectral curve with an eigenvector line bundle  $\mathcal{L}_{x,y} \rightarrow \Gamma_{u, \mathfrak{N}}$  and a section  $\Psi : \Gamma_{u, \mathfrak{N}} \rightarrow \mathcal{L}_{x,y}$  given as follows. We have an eigenspace morphism

$$\mathcal{E}_{x,y} : \Gamma_{u, \mathfrak{N}} \rightarrow \mathbb{P}^{\dim \mathfrak{g} - 1} = \mathbb{P}^{247} \quad (3.2)$$

that, away from ramification points of the  $\lambda : \Gamma_{u, \mathfrak{N}} \rightarrow \mathbb{P}^1$  projection, assigns to a point  $(\lambda, \mu) \in \Gamma_{u, \mathfrak{N}}$  the (time-dependent) eigenline of (3.1) with eigenvalue  $\mu$ ; this in fact extends to a locally free rank one sheaf on the whole of  $\Gamma_{u, \mathfrak{N}}$  [7, Ch. 5, II Proposition on p.131]. We write

$$\mathcal{L}_{x,y} \triangleq \mathcal{E}_{x,y}^* \mathcal{O}_{\mathbb{P}^{247}}(1) \in \text{Pic}(\Gamma_{u, \mathfrak{N}}) \quad (3.3)$$

for the pullback of the hyperplane bundle on  $\mathbb{P}^{\dim \mathfrak{g} - 1}$  via the eigenline map  $\mathcal{E}_{x,y}$ , and fix (non-canonically) a section of the latter by

$$\Psi_j(\lambda, \mu_i(\lambda)) = \frac{\Delta_{j1} \left( \widehat{L}_{x,y}(\lambda) - \mu_i(\lambda) \right)}{\Delta_{11} \left( \widehat{L}_{x,y}(\lambda) - \mu_i(\lambda) \right)}, \quad (3.4)$$

---

<sup>5</sup>For ease of notation, and since we've fixed  $\rho = \mathfrak{g}$  in the previous section, I am dropping here any reference to the representation  $\rho$  of the Lax operator.

where  $\mu_i(\lambda) = \exp(\alpha_i(l(\lambda)))$  (cfr. (2.22)) and we denoted by  $\Delta_{ij}(M)$  the  $(i, j)^{\text{th}}$  minor of a matrix  $M$ . As  $t$  and  $x(t)$ ,  $y(t)$  vary, so will  $\mathcal{L}_{x(t),y(t)}$ , and

$$\mathcal{B}_{x,y}(t) \triangleq \mathcal{L}_{x,y}(t) \otimes \mathcal{L}_{x,y}^*(0) \in \text{Pic}^{(0)}(\Gamma_{u,\aleph}) \simeq \frac{H^1(\Gamma_{u,\aleph}, \mathcal{O})}{H^2(\Gamma_{u,\aleph}, \mathbb{Z})} \quad (3.5)$$

is a time-dependent degree zero line bundle on  $\Gamma_{u,\aleph}$ .

The flows (2.16) thus determine a flow  $t \rightarrow \mathcal{B}_{x(t),y(t)}$  in the Jacobian of  $\Gamma_{u,\aleph}$ , which is actually *linear* in Cartesian coordinates for the torus  $\text{Pic}^{(0)}(\Gamma_{u,\aleph})$ . Indeed, let  $\{\omega_k\}_k$  be a basis for the  $\mathbb{C}$ -vector space of holomorphic differentials on  $\Gamma_{u,\aleph}$ ,  $\mathbb{C}\langle\{\omega_k\}_k\rangle = H^1(\Gamma_{u,\aleph}, \mathcal{O})$ , and let

$$\begin{aligned} \psi : \text{Sym}^g \Gamma_{u,\aleph} &\rightarrow \text{Pic}^{(0)}(\Gamma_{u,\aleph}) \\ (\gamma_1 + \dots + \gamma_g) &\rightarrow \sum_{i=1}^g \mathcal{A}(p_i) \end{aligned} \quad (3.6)$$

be the surjective, degree one morphism from the  $g^{\text{th}}$ -symmetric power of  $\Gamma_{u,\aleph}$  to its Jacobian, given by taking the Abel sums of  $g$  unordered points on  $\Gamma_{u,\aleph}$ ; here

$$\begin{aligned} \mathcal{A} : \Gamma_{u,\aleph} &\rightarrow \text{Pic}^{(0)}(\Gamma_{u,\aleph}) \\ \gamma &\rightarrow \left( \int^\gamma d\omega_1, \dots, \int^\gamma d\omega_g \right) \end{aligned} \quad (3.7)$$

denotes the Abel map for some fixed choice of base point. Writing

$$\text{Sym}^g \ni \gamma(t) = (\gamma_1(t), \dots, \gamma_g(t)) = \psi^{-1}(\mathcal{B}_{x(t),y(t)})$$

for the inverse of  $\mathcal{B}_{x(t),y(t)}$ , which is unique for generic time  $t$  by Jacobi's theorem, we have that [117, Thm. 4]

$$\Omega_{ik} \triangleq \frac{\partial}{\partial t_i} \sum_{j=1}^g \int^{\gamma_j(t)} \omega_k = \sum_{p \in \lambda^{-1}(0) \cup \lambda^{-1}(\infty)} \text{Res}_p \left[ \omega_k P_i(\widehat{L}_{x,y}(\lambda)) \right] \quad \forall k = 1, \dots, g \quad (3.8)$$

The left hand side is the derivative of the flow on the Jacobian (its angular frequencies) in the chart on  $\text{Pic}^{(0)}(\Gamma_{u,\aleph})$  determined by the linear coordinates  $H^1(\Gamma_{u,\aleph}, \mathcal{O})$  w.r.t the chosen basis  $\{\omega_k\}_k$ . The right hand side shows that this is independent of time, and hence the flow is linear in these coordinates, since  $\omega_k$  and  $P_i(\widehat{L}_{x,y}(\lambda))$  are: the former since it only feels the dynamical phase space variables  $\{x_i, y_i\}_{i=0}^8$  in  $\widehat{L}_{x,y}(\lambda)$  via  $\Gamma_{u,\aleph}$ , itself an integral of motion, and the latter by (2.16).

**3.2. The Kanev–McDaniel–Smolinsky correspondence.** The story above is common to a large variety of systems (the Zakharov–Shabat systems with spectral-parameter-dependent Lax pairs), and the  $\widehat{\text{E}}_8$  relativistic Toda fits entirely into this scheme. In particular, in the better known examples of the periodic relativistic and non-relativistic Toda chain with  $N$ -particles (i.e.  $\mathfrak{g} = \mathfrak{sl}_N; \rho = \square$  in (2.16)), where the spectral curves have genus  $g = N - 1$ , the action-angle map  $\{x_i, y_i\} \rightarrow (\Gamma_{u,\aleph}, \mathcal{L}_{x,y})$  gives a family of  $\text{rank } \mathfrak{g} = N - 1$  commuting flows on their  $N - 1$ -dimensional Jacobian. A question that does *not* arise in these ordinary examples, however, is the following: in our case, we have way more angles than we have actions, as the genus of the spectral curve is

much higher than the rank of  $\mathfrak{g} = \mathfrak{e}_8$ . Indeed, the Jacobian is 495-complex dimensional in our case by (2.33); but the (compact) orbits of (2.33) only span an 8-dimensional Abelian subvariety of the Jacobian.

How do we single out this subvariety geometrically? In the non-relativistic case, pinning down the dynamical subtorus from the geometry of the spectral curve has been the subject of intense study since the early studies of Adler and van Moerbeke [2] for  $\mathfrak{g} = \mathfrak{b}_n, \mathfrak{c}_n, \mathfrak{d}_n, \mathfrak{g}_2$ , and the fundamental works of Kanev [68], Donagi [37] and McDaniel–Smolinsky [87, 88] in greater generality. We now work out how these ideas can be applied to our case as well.

Recall from Proposition 2.6 that we have a  $\mathcal{W}$ -action on  $\Gamma_{u, \mathbb{N}}$  by deck transformations given by

$$\begin{aligned} \phi : \mathcal{W} \times \Gamma_{u, \mathbb{N}} &\rightarrow \Gamma_{u, \mathbb{N}} \\ (w, \lambda, \mu_\alpha(\lambda)) &\rightarrow (\lambda, \mu_{w(\alpha)}(\lambda)) \end{aligned} \quad (3.9)$$

which is just the residual action of the vertical transformations on the cameral cover. Write  $\phi_w \triangleq \phi(w, -) \in \text{Aut}(\Gamma_{u, \mathbb{N}})$  for the automorphism corresponding to  $w \in \mathcal{W}$ . Extending by linearity,  $\phi_w$  induces an action on  $\text{Div}(\Gamma_{u, \mathbb{N}})$  which obviously descends to give actions on the Picard group  $\text{Pic}(\Gamma_{u, \mathbb{N}})$ , the Jacobian  $\text{Pic}^{(0)}(\Gamma_{u, \mathbb{N}}) \simeq \text{Jac}(\Gamma_{u, \mathbb{N}})$  (since  $\phi_w$  is compatible with degree and linear equivalence), and the  $\mathbb{C}$ -space of holomorphic 1-forms  $H^1(\Gamma_{u, \mathbb{N}}, \mathcal{O})$ . At the divisorial level we have furthermore an action of the group ring

$$\begin{aligned} \varphi : \mathbb{Z}[\mathcal{W}] \times \text{Div}(\Gamma_{u, \mathbb{N}}) &\rightarrow \text{Div}(\Gamma_{u, \mathbb{N}}), \\ \left( \sum_i a_i w_i, \sum_j b_j (\lambda_j, \mu_\alpha(\lambda_j)) \right) &\rightarrow \sum_{i, j} a_i b_j (\lambda_j, \mu_{w_i(\alpha)}(\lambda_j)). \end{aligned} \quad (3.10)$$

Recall from Proposition 2.6 that, since the group of deck transformations of the cover  $\Gamma_{u, \mathbb{N}} \setminus \{d\mu = 0\}$  is isomorphic to the Coxeter action of  $\mathcal{W}$  on the root space  $\Delta \simeq \mathcal{W}/\mathcal{W}_{\alpha_0}$ , the map (3.10) factors through the coset projection map  $\mathcal{W} \rightarrow \Delta$ , i.e.

$$\varphi(w, -) = |\mathcal{W}_{\alpha_0}| \sum_{\alpha \in \Delta} \tilde{a}_\alpha w_\alpha, \quad (3.11)$$

for some  $\{\tilde{a}_\alpha \in \mathbb{Z}\}_{\alpha \in \Delta}$ . Restrict now to elements  $\varphi(w, -) \in \mathbb{Z}[\mathcal{W}]$  such that  $\varphi(w, -) : \mathbb{Z}[\mathcal{W}] \rightarrow \mathbb{Z}[\text{Aut}(\Gamma_{u, \mathbb{N}})]$  is a ring homomorphism. Then the action (3.10) is the pull-back of an action of the maximal subgroup of  $\mathbb{Z}[\Delta]$  which respects the product structure induced from  $\mathbb{Z}[\mathcal{W}]$ : this is the Hecke ring  $H(\mathcal{W}, \mathcal{W}_{\alpha_0}) \simeq \mathbb{Z}[\mathcal{W}_{\alpha_0} \setminus \mathcal{W}/\mathcal{W}_{\alpha_0}] \simeq \mathbb{Z}[\Delta]^{\mathcal{W}_{\alpha_0}}$ . Its additive structure is given by the free  $\mathbb{Z}$ -module structure on the space of double cosets of  $\mathcal{W}$  by  $\mathcal{W}_{\alpha_0}$ , and its product is defined as the push-forward<sup>6</sup> of the product on  $\mathbb{Z}[\mathcal{W}]$ . In practical terms, this forces the integers  $a_\alpha$  in the sum over roots in  $\Delta^*$  (i.e. right cosets of  $\mathcal{W}/\mathcal{W}_{\alpha_0}$ ) to be constant over left cosets  $\mathcal{W}_{\alpha_0} \setminus \mathcal{W}$  in (3.11).

The Weyl-symmetry action is the key to single out the Liouville–Arnold algebraic torus that is home to the flows (2.16). We first start from the following

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<sup>6</sup>That is, the image under the double-quotient projection of the product of the pullback functions on  $\mathcal{W}$ , which is well-defined on the double quotient even when  $\mathcal{W}_{\alpha_0}$  is not normal, as in our case.

**Definition 3.2.** Let  $\mathcal{D} \in \text{Div}(\Gamma \times \Gamma)$  be a self-correspondence of a smooth projective irreducible curve  $\Gamma$  and let  $\mathcal{C} \in \text{End}(\Gamma)$  be the map

$$\begin{aligned} \mathcal{C} : \text{Jac}(\Gamma) &\rightarrow \text{Jac}(\Gamma) \\ \gamma &\rightarrow (p_2)_*(p_1^*(\gamma) \cdot \mathcal{D}), \end{aligned} \quad (3.12)$$

where  $p_i$  denotes the projection to the  $i^{\text{th}}$  factor in  $\Gamma \times \Gamma$ . The Abelian subvariety

$$\text{PT}_{\mathcal{C}}(\Gamma) \triangleq (\text{id} - \mathcal{C}) \text{Jac}(\Gamma) \quad (3.13)$$

is called a Prym–Tyurin variety iff

$$(\text{id} - \mathcal{C})(\text{id} - \mathcal{C} - q_{\mathcal{C}}) = 0 \quad (3.14)$$

for  $q_{\mathcal{C}} \in \mathbb{Z}$ ,  $q_{\mathcal{C}} \geq 2$ .

By (3.14), the tangent fibre at the identity  $T_e(\text{Jac}(\Gamma))$  splits into eigenspaces  $T_e(\text{Jac}(\Gamma)) = \mathfrak{t}_{\text{PT}} \oplus \mathfrak{t}_{\text{PT}}^{\vee}$  of  $\mathcal{C}$  with eigenvalues 1 and  $1 - q_{\mathcal{C}}$ . Because  $q_{\mathcal{C}} \in \mathbb{Z}$ , these exponentiate to subtori  $\mathcal{T}_{\text{PT}} = \exp \mathfrak{t}_{\text{PT}}$ ,  $\mathcal{T}_{\text{PT}}^{\vee} = \exp \mathfrak{t}_{\text{PT}}^{\vee}$ , with  $\mathcal{T}_{\text{PT}} = \text{PT}_{\mathcal{C}}(\Gamma)$ , such that  $\text{Jac}(\Gamma) = \mathcal{T}_{\text{PT}} \times \mathcal{T}_{\text{PT}}^{\vee}$ . In particular, in terms of the linear spaces  $V_{\text{PT}} \simeq \widetilde{\mathcal{T}_{\text{PT}}}$ ,  $V_{\text{PT}}^{\vee} \simeq \widetilde{\mathcal{T}_{\text{PT}}^{\vee}}$  which are the universal covers of the two factor tori, we have

$$\text{PT}_{\mathcal{C}}(\Gamma) \simeq V_{\text{PT}} / \Lambda_{\text{PT}} \quad (3.15)$$

where  $\Lambda_{\text{PT}} = H_1(\Gamma, \mathbb{Z}) \cap V_{\text{PT}}$ . Furthermore [68], there is a natural principal polarisation  $\Xi$  on  $\text{PT}_{\mathcal{C}}(\Gamma)$  given by the restriction of the Riemann form  $\Theta$  on  $H^1(\Gamma, \mathcal{O}) \simeq V_{\text{PT}} \oplus V_{\text{PT}}^{\vee}$  to  $V_{\text{PT}}$ ; we have  $\Theta = q_{\mathcal{C}} \Xi$ , with  $\Xi$  unimodular on  $\Lambda_{\text{PT}}$ . In particular,  $\text{id} - \mathcal{C}$  acts as a projector on the space of 1-holomorphic differentials, and, dually, 1-homology cycles on  $\Gamma$ , which

- selects a symplectic vector space  $V_{\text{PT}} \subset H_1(\Gamma, \mathcal{O})$  and dual subring  $\Lambda_{\text{PT}} \in H_1(\Gamma, \mathbb{Z})$ ; 1-forms in  $V_{\text{PT}}$  have zero periods on cycles in  $\Lambda_{\text{PT}}^{\vee}$ ;
- bases  $\{\omega_1, \dots, \omega_{\dim V_{\text{PT}}}\}$ ,  $\{(A_i, B_i)\}_{i=1}^{\dim V_{\text{PT}}}$  can be chosen such that corresponding minors of the period matrix of  $\Gamma$  satisfy

$$\int_{A_j} \omega_i = q_{\mathcal{C}} \delta_{ij}, \quad \int_{B_j} \omega_i = \tau_{ij} \quad (3.16)$$

with  $\tau_{ij}$  non-degenerate positive definite.

There is a canonical element of  $H(\mathcal{W}, \mathcal{W}_{\alpha_0})$  which has particular importance for us, and which will eventually act as a projector on a distinguished Prym–Tyurin subvariety of  $\text{Jac}(\Gamma_{u, \mathbb{N}})$ . This is the Kanev–McDaniel–Smolinsky self-correspondence<sup>7</sup> [68, 87, 88]

$$\mathcal{P}_{\mathfrak{g}} \triangleq \sum_{w \in \mathcal{W} / \mathcal{W}_{\alpha_0}} \langle w^{-1}(\alpha_0), \alpha_0 \rangle w. \quad (3.17)$$

I summarise here some of its key properties, some of which are easily verifiable from the definition (3.17), with others having been worked out in meticulous detail in [87, Sec. 3–5]. Some further

<sup>7</sup>This has also been considered in the gauge theory literature, implicitly in [64, 85] and more diffusely in [82].

explicit results that are relevant to our case, but that did not fit in the discussion of [87], are presented below.

**Proposition 3.1.** *In the root space  $(\mathfrak{h}^*, \langle, \rangle)$  consider the hyperplanes*

$$H_i = \{\beta \in \mathfrak{h}^* \mid \langle \beta, \alpha_0 = i \rangle\}. \quad (3.18)$$

*Then, set-theoretically,  $\mathcal{W}_{\alpha_0} \backslash \mathcal{W} / \mathcal{W}_{\alpha_0} \simeq \{\delta_i \triangleq H_i \cap \Delta\}_{i=-2}^2$ . Letting  $\mathcal{W} \xrightarrow{\pi_1} \mathcal{W} / \mathcal{W}_{\alpha_0} \xrightarrow{\pi_2} \mathcal{W}_{\alpha_0} \backslash \mathcal{W} / \mathcal{W}_{\alpha_0}$  be the projection to the double coset space and  $\{s_i\}_{i=-2}^2 = \pi(\Delta)$ , we furthermore have*

$$\mathcal{P}_{\mathfrak{g}} = \pi_2^* \left[ \sum_{\delta_i \in \mathcal{W}_{\alpha_0} \backslash \mathcal{W} / \mathcal{W}_{\alpha_0}} i s_i \in H(\mathcal{W}, \mathcal{W}_{\alpha_0}) \right]. \quad (3.19)$$

*Proof.* The fact that  $\mathcal{P}_{\mathfrak{g}} \in \mathbb{Z}[\Delta]^{\mathcal{W}_{\alpha_0}} = H(\mathcal{W}, \mathcal{W}_{\alpha_0})$  follows immediately from its definition in (3.17) and the constancy of  $\langle w^{-1}(\alpha_0), \alpha_0 \rangle$  on left cosets. The rest of the proof follows from explicit identification of the elements of  $H(\mathcal{W}, \mathcal{W}_{\alpha_0})$  in terms of the hyperplanes of (3.18), and evaluation of (3.17) on them. The proof is somewhat lengthy and the reader may find the details in Appendix A.  $\square$

**Corollary 3.2.**  *$\mathcal{P}_{\mathfrak{g}}$  satisfies the quadratic equation in  $H(\mathcal{W}, \mathcal{W}_{\alpha_0})$  with integral roots*

$$\mathcal{P}_{\mathfrak{g}}^2 = q_{\mathfrak{g}} \mathcal{P}_{\mathfrak{g}} \quad (3.20)$$

*with*

$$q_{\mathfrak{g}} = 60. \quad (3.21)$$

*In particular, the correspondence  $\mathcal{C} = 1 - \mathcal{P}_{\mathfrak{g}}$  defines a Prym–Tyurin variety  $\text{PT}_{1-\mathcal{P}_{\mathfrak{g}}}(\Gamma_{u, \mathbb{N}}) \subset \text{Jac}(\Gamma_{u, \mathbb{N}})$ .*

*Proof.* This is a straightforward calculation from Eq. (3.19).  $\square$

In the following, I will simply write  $\text{PT}(\Gamma_{u, \mathbb{N}})$ , dropping the  $1 - \mathcal{P}_{\mathfrak{g}}$  subscript which will be implicitly assumed.

The main statement about  $\text{PT}(\Gamma_{u, \mathbb{N}})$  is the subject of the next Theorem. Note that this bears a large intellectual debt to previous work in [68, 88]; the modest contribution of this paper is a combination of the results of this and the previous Section with [68, 88] to prove that the Liouville–Arnold torus (the image of the flows (2.16) on the Jacobian) is indeed isomorphic to the full Kanev–McDaniel–Smolinsky Prym–Tyurin, rather than being just a closed subvariety thereof.

**Theorem 3.3.** *The flows (2.16), (3.8) of the  $\widehat{E}_8$  relativistic Toda chain linearise on the Prym–Tyurin variety  $\text{PT}(\Gamma_{u, \mathbb{N}})$  and they fill it for generic initial data  $(u, \mathbb{N})$ .*

*Proof.* The linearisation of the flows on  $\text{PT}(\Gamma_{u, \mathbb{N}})$  amounts to say that

$$\sum_{p \in \lambda^{-1}\{0, \infty\}} \text{Res}_p \left[ \omega P_i(\widehat{L_{x,y}}(\lambda)) \right] \neq 0 \quad \Rightarrow \quad \mathcal{P}_{\mathfrak{g}}^* \omega = \omega \quad (3.22)$$



in (3.8). This is essentially the content of [68, Theorem 8.5] and especially [88, Theorem 29], to which the reader is referred. The latter paper greatly relaxes an assumption on the spectral dependence of  $\widehat{L}_{x,y}(\lambda)$  [68, Condition 8.4] which renders incompatible [68, Theorem 8.5] with (2.12); this restriction is entirely lifted in [88, Theorem 29], where the fact that (2.12) depends rationally on  $\lambda$  is sufficient for our purposes. While [68, 88] deal with the non-relativistic counterpart of the system (2.16), it is easy to convince oneself that replacing their Lie-algebraic setting with the Lie-group arena we are playing in in this paper amounts to a purely notational redefinition of  $\mathfrak{g}$  to  $\mathcal{G}$  in the arguments leading up to [88, Theorem 29].

Since the first part of the statement has been settled in [88], I now move on to prove that the Prym–Tyurin *is* the Liouville–Arnold torus. Denoting  $\phi_t^{(i)} : \widehat{\mathcal{P}} \rightarrow \widehat{\mathcal{P}}$  be the time- $t$  flow of (2.16), and for fixed  $(x, y) \in \widehat{\mathcal{P}}$ , the above proves that

$$\begin{aligned} \phi_{t_1}^{(1)} \cdots \phi_{t_8}^{(8)} : \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 &\rightarrow \widehat{\mathcal{P}} \\ (x, y) &\rightarrow (x(\vec{t}), y(\vec{t})) \end{aligned} \quad (3.23)$$

surjects to an eight-dimensional subtorus of  $\text{PT}(\Gamma_{u,\mathbb{N}})$ . To see the resulting torus is the Prym–Tyurin, we use the dimension formula of [87, Theorem 17]. Let  $C_\star \triangleq \mathbb{P}^1 \setminus \{b_i^\pm\}_{i=1}^9$ ,  $\mathfrak{M} : \pi_1(C_\star) \rightarrow \mathcal{W}$  be the Galois map of the spectral cover  $\Gamma_{u,\mathbb{N}}$ , and for  $P \in \Gamma_{u,\mathbb{N}}$  write  $S^{(P)}$  for the stabiliser of  $P$  in the group of deck transformations of  $\Gamma_{u,\mathbb{N}}$ , and  $\mathfrak{h}_P^*$  for the fixed point eigenspace of  $S^{(P)} \subset \mathcal{W}$ . Then [87, Theorem 17]

$$\dim_{\mathbb{C}} \text{PT}(\Gamma_{u,\mathbb{N}}) = \frac{1}{2} \sum_{\lambda(p)|d\mu(p)=0} (8 - \dim_{\mathbb{C}} \mathfrak{h}_p^*) - 8 + \langle \mathfrak{h}, \mathbb{C}[\mathcal{W}/\mathfrak{M}(\pi_1(\mathbb{P}_\star^1))] \rangle \quad (3.24)$$

where one representative  $p$  is chosen in each fibre of  $\lambda : \Gamma_{u,\mathbb{N}} \rightarrow \mathbb{P}^1$ . In our case,  $\mathfrak{M}(\pi_1(\mathbb{P}_\star^1)) = \mathcal{W}$  by Proposition 2.6 and the fact that the  $\alpha_0$ -parabolic cover is irreducible (hence a connected covering space of  $\mathbb{P}^1$ ), so the last term vanishes. Then

$$\dim_{\mathbb{C}} \text{PT}(\Gamma_{u,\mathbb{N}}) = \frac{1}{2} \sum_{i=1,\dots,9, j=\pm} (8 - \dim_{\mathbb{C}} \mathfrak{h}_{Q_{i,j}}^*) + \frac{1}{2} \sum_{j=\pm} (8 - \dim_{\mathbb{C}} \mathfrak{h}_{Q_{\infty,j}}^*) - 8 \quad (3.25)$$

Since  $\alpha_{k(i)} \cdot \mu(Q_{i,\pm}) = 0$  for some permutation  $k : \{1, \dots, 8\} \rightarrow \{1, \dots, 8\}$ , the deck transformations in  $S^{(Q_{i,\pm})}$  are simple reflections that stabilise the hyperplane orthogonal to the root  $\alpha_{k(i)}$ , so that  $\dim_{\mathbb{C}} \mathfrak{h}_{Q_{i,j}}^* = 7$ . As far as  $Q_{\infty,\pm}$  are concerned, the deck transformation associated to a simple loop around them corresponds to the product of the Coxeter element of  $\mathcal{W}$  times a simple root, as this is the lift under the projection to the base curve of a loop around all branch points on the affine part of the curve<sup>8</sup>. Then  $\dim_{\mathbb{C}} \mathfrak{h}_{Q_{\infty,j}}^* = 1$ ,  $\dim_{\mathbb{C}} \text{PT}(\Gamma_{u,\mathbb{N}}) = 8$ , and the flows (3.23) surject on the latter.  $\square$

An explicit construction of Kanev’s Prym–Tyurin  $\text{PT}(\Gamma_{u,\mathbb{N}})$ , after [85, Section 3], can be given as follows. With reference to Figure 4, let  $\gamma_i^\pm$  be a simple counterclockwise loop around the branch

<sup>8</sup>The root in question is the one that is repeated in the sequence  $\{k(i)\}_{i=1}^9$ . There could be more of them in principle, but this would be in contrast with  $\mathfrak{M}(\pi_1(\mathbb{P}_\star^1)) = \mathcal{W}$ ; equivalently, *a posteriori*, this would lead to  $\dim_{\mathbb{C}} \text{PT}(\Gamma_{u,\mathbb{N}}) < 8$ , contradicting the independence of the flows (2.16), which in turn is a consequence of the algebraic independence of the fundamental characters  $\theta_i$  in  $R(\mathcal{G})$ .

point  $b_i^\pm$ . I will similarly write  $\gamma_0^-$  (resp.  $\gamma_0^+$ ) for analogous loops around  $\lambda = 0$  (resp.  $\lambda = \infty$ ). For  $\alpha \in \Delta^*$  and  $i = 1, \dots, 8$ , I define  $C_i^\alpha, D_i^\alpha \in C^1(\Gamma_{u, \mathbb{N}}, \mathbb{Z})$  to be the lifts of the contours in red (respectively in blue) to the cover  $\Gamma_{u, \mathbb{N}}$ , where we fix arbitrarily a base point  $r \in \gamma_i^\pm$  and we look at the path in  $\Gamma_{u, \mathbb{N}}$  lying over  $\gamma_i^\pm$  with starting point on the  $\lambda$ -preimage of  $r$  labelled by  $\alpha$ . In other words,

$$\begin{aligned} C_i^\alpha &\triangleq \lambda_{\sigma_i(\alpha)}^{-1}(\gamma_i^+) \cdot \lambda_\alpha^{-1}(\gamma_0^-), \\ D_i^\alpha &\triangleq \lambda_{\sigma_i(\alpha)}^{-1}(\gamma_i^+) \cdot \lambda_\alpha^{-1}(\gamma_i^-). \end{aligned} \quad (3.26)$$

Let now

$$A_i \triangleq \frac{1}{q_{\mathfrak{g}}} (\mathcal{P}_{\mathfrak{g}})_* C_i^{\alpha_0}, \quad B_i \triangleq \frac{1}{2} (\mathcal{P}_{\mathfrak{g}})_* D_i^{\alpha_0} \quad (3.27)$$

where the normalisation factor for  $A_i, B_i$  will be justified momentarily. Notice that  $A_i, B_i \in Z_1(\Gamma_{u, \mathbb{N}}, \mathbb{Q})$  are closed paths on the cover: every summand  $C_i^\alpha$  and  $D_i^\alpha$  is indeed always accompanied by a return path  $C_i^{\sigma_i(\alpha)}$  and  $D_i^{\sigma_i(\alpha)}$ , which has opposite weight in (3.27). Denoting by the same letters  $A_i, B_i$  their conjugacy classes in homology, we identify  $H_1(\Gamma_{u, \mathbb{N}}, \mathbb{Q}) \supset \Lambda_{\text{PT}} \triangleq \mathbb{Z} \langle \{A_i, B_i\}_{i=1}^8 \rangle$ . If  $\{\omega_1, \dots, \omega_8\}$  is any choice of 1-holomorphic differentials such that  $\dim \mathcal{P}_{\mathfrak{g}}^* \mathbb{C} \langle \omega_1, \dots, \omega_8 \rangle = 8$ , then

$$\text{PT}(\Gamma_{u, \mathbb{N}}) = \frac{\mathcal{P}_{\mathfrak{g}}^* \mathbb{C} \langle \omega_1, \dots, \omega_8 \rangle}{\mathbb{Z} \langle \{A_i, B_i\}_{i=1}^8 \rangle} \quad (3.28)$$

by construction. It is instructive to compute the intersection index of the cycles (3.27): we have, from (3.26), that

$$(A_i, B_j) = \frac{1}{2q_{\mathfrak{g}}} \sum_{\beta, \gamma \in \Delta^*} (C_i^\beta, D_j^\gamma) = \frac{\delta_{ij}}{2q_{\mathfrak{g}}} \sum_{\beta, \gamma \in \Delta^*} \delta_{\beta\gamma} \langle \alpha_0, \beta \rangle^2 = \delta_{ij}, \quad (3.29)$$

$$(A_i, A_j) = (B_i, B_j) = 0, \quad (3.30)$$

so that they are a symplectic basis for  $\Lambda_{\text{PT}}$ ; the normalisation factor (3.27) has been chosen to ensure both that this is so and to render the period integrals on  $\{A_i, B_i\}$  compatible with the usual form of special geometry relations.

**3.3. Hamiltonian structure and the spectral curve differential.** The fact that the isospectral flows (2.16) turn into straight line motions on  $\text{PT}(\Gamma_{u, \mathbb{N}})$  is the largest bit in the proof of the algebraic complete integrability of the  $\widehat{\text{E}}_8$  relativistic Toda. We conclude it now by working out in detail a choice of Darboux co-ordinates  $\{S_i, \vartheta^i\}_{i=1}^8$ , with  $\vartheta^i \in S^1$ , such that the Hamiltonians (2.18) are functions of  $S_i$  alone. In the process, this will complete the construction of the Dubrovin–Krichever data of Definition 3.1.

Composing the surjection (3.17) with the Abel–Jacobi map gives an Abel–Prym–Tyurin map

$$\begin{aligned} \mathcal{A}_{\text{PT}} : \Gamma_{u, \mathbb{N}} &\rightarrow \text{PT}(\Gamma_{u, \mathbb{N}}) \\ p &\rightarrow \mathcal{P}_{\mathfrak{g}} \cdot \mathcal{A}(p). \end{aligned} \quad (3.31)$$

Since  $\text{PT}(\Gamma_{u, \mathbb{N}})$  is principally polarised, an analogue of the Jacobi theorem holds for  $\mathcal{A}_{\text{PT}}$  [67, Lemma 2.1], and the Abel–Prym–Tyurin map (3.31) is an embedding of  $\Gamma_{u, \mathbb{N}}$  into  $\text{PT}(\Gamma_{u, \mathbb{N}})$  as

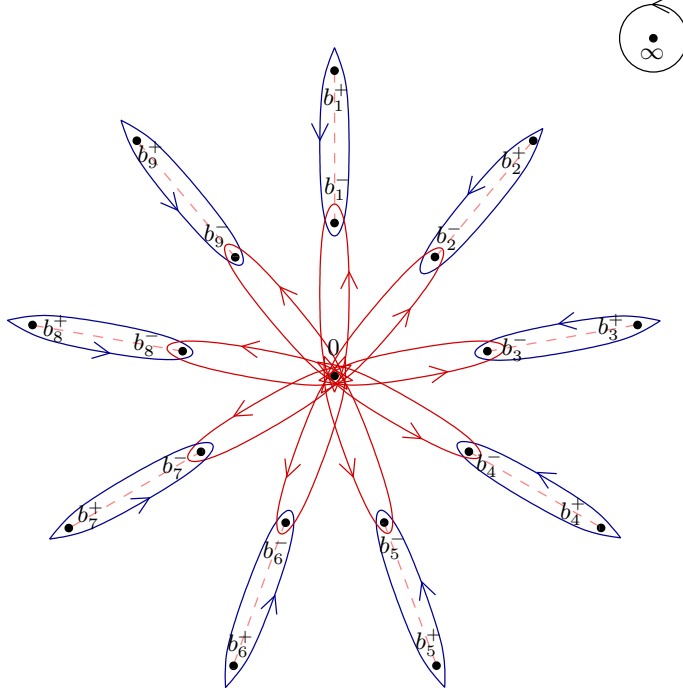


FIGURE 4. Contours on  $\mathbb{P}_*^1 = \mathbb{C}^* \setminus \{b_i^\pm\}_{i=1}^9$ . Projections of the  $A$ - and  $B$ -cycles are depicted in red and blue respectively.

a  $q_{\mathfrak{g}} = 60$ -multiple of its minimal curve  $\frac{\Xi^7}{7!}$ . Then, taking Abel sums of 8 points on  $\Gamma_{u,\aleph}$  and projecting their image to PT,

$$\begin{aligned} \mathcal{A}_{\text{PT}} : \text{Sym}^8 \Gamma_{u,\aleph} &\rightarrow \text{PT}(\Gamma_{u,\aleph}) \\ (\gamma_1 + \cdots + \gamma_8) &\rightarrow (\mathcal{P}_{\mathfrak{g}})_* \sum_{i=1}^8 \mathcal{A}(\gamma_i) \end{aligned} \quad (3.32)$$

gives a finite, degree  $q_{\mathfrak{g}}^8 = 2^{16}3^{85}5^8$  surjective morphism<sup>9</sup> from the 8-fold symmetric product of  $\Gamma_{u,\aleph}$  to  $\text{PT}(\Gamma_{u,\aleph})$  which maps the fundamental class  $[\text{Sym}^8(\Gamma_{u,\aleph})] \rightarrow q_{\mathfrak{g}}^8[\text{PT}(\Gamma_{u,\aleph})]$  to  $q_{\mathfrak{g}}^8$  the fundamental class of the Prym–Tyurin. The fibre  $\mathcal{A}_{\text{PT}}^{-1}(\xi)$  of a point  $\xi \in \text{PT}(\Gamma_{u,\aleph})$  is given by  $q_{\mathfrak{g}}^8$  unordered 8-tuples of points  $\gamma_1 + \cdots + \gamma_8$  on  $\Gamma_{u,\aleph}$  satisfying

$$\begin{aligned} \xi &= \mathcal{A}_{\text{PT}} \left( \sum_i \gamma_i \right) = (\mathcal{P}_{\mathfrak{g}})_* \sum_{i=1}^8 \mathcal{A}(\gamma_i) = (\mathcal{P}_{\mathfrak{g}})_* \sum_{i=1}^8 \left( \int^{\gamma_i} d\omega_1, \dots, \int^{\gamma_i} d\omega_{495} \right), \\ &= \sum_{i=1}^8 \left( \int^{\mathcal{P}_{\mathfrak{g}}(\gamma_i)} d\omega_1, \dots, \int^{\mathcal{P}_{\mathfrak{g}}(\gamma_i)} d\omega_{495} \right), \\ &= \sum_{i=1}^8 \left( \int^{\gamma_i} \mathcal{P}_{\mathfrak{g}}^* d\omega_1, \dots, \int^{\gamma_i} \mathcal{P}_{\mathfrak{g}}^* d\omega_{495} \right). \end{aligned} \quad (3.33)$$

<sup>9</sup>I slightly abuse notation here and call it with the same symbol of (3.31).

Let us now reconsider the action-angle map  $\{x_i, y_i\} \rightarrow (\Gamma_{u, \mathfrak{N}}, \mathcal{B}_{x,y})$  of (2.18), (3.5) and (3.8) in light of Theorem 3.3. By the above reasoning, the flows  $(x(t), y(t))$  are encoded into the motion of  $\mathcal{B}_{x,y}(t)$ , or equivalently, any of the pre-images  $\mathcal{A}_{\text{PT}}^{-1}\mathcal{B}(t) = (\gamma_1(t) + \cdots + \gamma_8(t))$ . I want to study the motion in terms of the latter, and argue that the Cartesian projections of  $\gamma_i$  provide logarithmic Darboux coordinates for (2.5). I begin with the following

**Theorem 3.4.** *Write  $\omega_{\text{PL}}$  for the symplectic 2-form on an  $\mathfrak{N}$ -leaf of  $\mathcal{P}_{\text{Toda}}$  and let  $\delta : \Omega^\bullet(\widehat{\mathcal{P}}) \rightarrow \Omega^{\bullet+1}(\widehat{\mathcal{P}})$  denote exterior differentiation on  $\widehat{\mathcal{P}}$ . Then*

$$\omega_{\text{PL}} = \mathcal{P}_{\mathfrak{g}}^* \sum_{i=1}^8 \frac{\delta\lambda(\gamma_i)}{\lambda(\gamma_i)} \wedge \frac{\delta\mu(\gamma_i)}{\mu(\gamma_i)}. \quad (3.34)$$

*Proof.* Recall that (see e.g. [7, Section 3.3]) any Lax system of the type (2.16) with rational spectral parameter and with  $L(\lambda) \in \mathfrak{g}$  can be interpreted as a flow on a coadjoint orbit of  $\mathfrak{g}^*$  which is Hamiltonian with respect to the Kostant–Kirillov bracket. More in detail, the pull-back of the Kostant–Kirillov symplectic 2-form reads [7, Sections 3.3, 5.9, 14.2]

$$\omega_{\text{KK}} = \frac{1}{2 \dim \mathfrak{g}} \sum_{\lambda_k=0, \infty} \text{Res}_{\lambda_k} \text{Tr} \left( (A_k)_{-} g_k^{-1} \delta g_k \wedge g_k^{-1} \delta g_k \right), \quad (3.35)$$

where we diagonalise<sup>10</sup>  $L(\lambda) = g_k^{-1} A_k g_k$  locally around the poles at  $\lambda = 0, \infty$ , we denote  $M_-(\lambda_0)$  the projection to the Laurent tail around  $\lambda = \lambda_0$ , and  $\delta$  indicates exterior differentiation on  $\widehat{\mathcal{P}}$ . This can be rewritten in terms of the Baker–Akhiezer eigenvector line bundle (3.3) and its marked section (3.4) as an instance of the Krichever–Phong universal symplectic form  $\omega_{\text{KP}}$  [35, 77]. Let  $\Psi = (\Psi_j)$  be the  $248 \times 248$  matrix whose  $j^{\text{th}}$  column is the normalised eigenvector (3.4). Then [35, Section 5]

$$\omega_{\text{KK}} = \omega_{\text{KP}}^{(1)} \triangleq \frac{1}{\dim \mathfrak{g}} \sum_{\lambda_k=0, \infty} \text{Res}_{\lambda_k} \text{Tr} \left( \Psi_{x,y}^{-1} \delta \widehat{L}_{x,y} \wedge \delta \Psi_{x,y} \right) d\lambda. \quad (3.36)$$

where  $d : \Omega(\mathbb{P}^1) \rightarrow \Omega(\mathbb{P}^1)$  is the exterior differential on the spectral parameter space.

This is pretty close to what need, and it would recover the results of obtained in [1] in a related context, but it actually requires two extra tweaks to get the symplectic form we are after,  $\omega_{\text{PL}}$ . First off, as explained in [7, Section 6.5], if we are interested in the  $r$ -matrix solution (2.6) for the Toda lattice, what we need to consider is rather a version  $\omega_{\text{KP}}^{(1)}$  of the universal symplectic form which is logarithmic in  $\lambda$ , i.e.

$$\omega_{\text{KP}}^{(1)} \triangleq \frac{1}{\dim \mathfrak{g}} \sum_{\lambda_k=0, \infty} \text{Res}_{\lambda_k} \text{Tr} \left( \Psi_{x,y}^{-1} \delta \widehat{L}_{x,y} \wedge \delta \Psi_{x,y} \right) \frac{d\lambda}{\lambda}. \quad (3.37)$$

Secondly and more importantly, since we are dealing with an integrable system on a Poisson–Lie Kac–Moody *group*, rather than a Lie algebra,  $\omega_{\text{PL}}$  is given by a different<sup>11</sup> Poisson bracket, as

<sup>10</sup>Note that the eigenvalue 1 of  $\widehat{L}_{x,y}(\lambda)$  has full geometric multiplicity 8, and the other eigenvalues are all distinct when  $\lambda$  is in a punctured neighbourhood of 0 or  $\infty$ .

<sup>11</sup>But, non-trivially, compatible: the resulting system is then bihamiltonian.

explained in [35, Section 5.3]. This is the logarithmic Krichever–Phong Poisson bracket  $\omega_{\text{KP}}^{(2)}$

$$\omega_{\text{PL}} = \omega_{\text{KP}}^{(2)} \triangleq \frac{1}{\dim \mathfrak{g}} \sum_{\lambda_k=0, \infty} \text{Res}_{\lambda_k} \text{Tr} \left( \Psi_{x,y}^{-1} \widehat{L}_{x,y}^{-1} \delta \widehat{L}_{x,y} \wedge \delta \Psi_{x,y} \right) \frac{d\lambda}{\lambda}. \quad (3.38)$$

The calculation of the residues of (3.38) is straightforward (see [7, Section 5.9] for a completely analogous calculation in the context of the Kostant–Kirillov form (3.35)). From the general theory of Baker–Akhiezer functions<sup>12</sup> and (3.8),  $\ln \Psi_{x,y}$  has simple poles, with residue equal to the identity, at a divisor  $D(t) \in \text{Div}(\Gamma_{u,\mathbb{N}})$  such that

$$\frac{\mathcal{A}(D(t)) - \mathcal{A}(D(0))}{t} = \sum_{p \in \lambda^{-1}(0) \cup \lambda^{-1}(\infty)} \text{Res}_p \left[ \omega_k P_i(\widehat{L}_{x,y}(\lambda)) \right], \quad (3.39)$$

and by Theorem 3.3, the l.h.s. is actually in the Prym–Tyurin variety  $\text{PT}(\Gamma_{u,\mathbb{N}})$ . This means that  $\Psi_{x(t),y(t)}$  has simple poles at  $(\mathcal{P}_{\mathfrak{g}})_*(\gamma_1(t) + \dots + \gamma_8(t))$  for some  $\gamma = \gamma_1(t) + \dots + \gamma_8(t) \in \text{Sym}^8(\Gamma_{u,\mathbb{N}})$ ; different  $\gamma$  have the same image under  $(\mathcal{P}_{\mathfrak{g}})_*$ . Write

$$\sum r_k \epsilon_k \triangleq \gamma = \sum_{i,\alpha} \langle w(\alpha_0), \alpha_0 \rangle w_* \gamma_i \quad (3.40)$$

for some  $r_k \in \mathbb{Z}$ ,  $\epsilon_k \in \Gamma_{u,\mathbb{N}}$ . Near  $\epsilon_k$  we have then

$$\delta \Psi_{x,y} = \frac{\Psi \delta \lambda(\epsilon_k)}{\lambda - \lambda(\epsilon_k)} (1 + \mathcal{O}(\lambda - \lambda(\epsilon_k))). \quad (3.41)$$

It turns out that the rest of the expression (3.38) is regular at  $\epsilon_k$ . Indeed, exterior differentiation of the eigenvalue equation (3.1) yields

$$\delta \left[ \ln \widehat{L}_{x,y} - \ln \mu \Psi_{x,y} \right] = \left( \widehat{L}_{x,y} \right)^{-1} \delta \widehat{L}_{x,y} - \frac{\delta \mu}{\mu} \Psi_{x,y} - \ln \mu \delta \Psi_{x,y} = 0 \quad (3.42)$$

Multiplying by  $(\lambda \Psi_{x,y})^{-1}$  and exploiting the fact that  $\Psi_{x,y}^{-1} (\widehat{L}_{x,y} - \mu) = 0$  for the dual section of  $\mathcal{L}_{x,y}$ , we get

$$\begin{aligned} \text{Res}_{\lambda(\epsilon_k)} \text{Tr} \left( \Psi_{x,y}^{-1} \widehat{L}_{x,y}^{-1} \delta \widehat{L}_{x,y} \wedge \delta \Psi_{x,y} \right) \frac{\delta \lambda}{\lambda} &= \text{Res}_{\lambda(\epsilon_k)} \text{Tr} \left( \Psi_{x,y}^{-1} \widehat{L}_{x,y}^{-1} \delta \widehat{L}_{x,y} \Psi_{x,y} \right) \wedge \frac{\delta \lambda(\epsilon_k)}{\lambda - \lambda(\epsilon_k)} \frac{d\lambda}{\lambda}, \\ &= \text{Tr} \left( \Psi_{x,y}^{-1} \widehat{L}_{x,y}^{-1} \delta \widehat{L}_{x,y} \Psi_{x,y} \right) \wedge \frac{\delta \lambda(\epsilon_k)}{\lambda(\epsilon_k)}, \\ &= 248 \delta \ln \mu(\epsilon_k) \wedge \delta \ln \lambda(\epsilon_k). \end{aligned} \quad (3.43)$$

Swapping orientation in the contour giving the sum over residues (3.38) amounts to picking up residues over the affine part of  $\Gamma_{u,\mathbb{N}} \setminus \lambda^{-1}(0)$ . We have two possible contributions here: one is the sum over residues at the Baker–Akhiezer poles  $\epsilon_k$  that we have just computed. Another is given by the branch points of the  $\lambda$  projection, since  $\det \Psi_{x,y}(b_i^\pm) = \mathcal{O}(\sqrt{\lambda - b_i^\pm})$ : hence both  $\Psi_{x,y}^{-1}$  and  $\delta|_{\lambda=\text{const}} \Psi_{x,y}$  develop a (simple) pole there. Whilst the residues are individually non-zero, their sum is: it is a simple observation that adding a contribution of the form

$$\Delta_{\text{KP}} = \sum_{\lambda_k=0, \infty} \text{Res}_{\lambda_k} \text{Tr} \left( \Psi_{x,y}^{-1} \delta \ln \mu \Psi_{x,y} \right) \wedge \frac{\delta \lambda}{\lambda} \quad (3.44)$$

<sup>12</sup>See for example the discussion in [73, Section 2].

to (3.38) exactly offsets the aforementioned non-vanishing residues at the branch points, and it has opposite residues at  $\lambda = 0$  and  $\infty$ . Taking into account sign changes and summing over poles, images of the Weyl action as in (3.40), and pre-images of the Abel–Prym–Tyurin map returns (3.34). □

We are now ready to write down explicitly the action-angle integration variables for the system. Let  $\pi^{\text{sym}} : \mathcal{S}_{\mathfrak{g}}^{\text{sym}} \rightarrow \mathcal{B}_{\mathfrak{g}}$  be the family of Abelian varieties obtained by replacing  $\Gamma_{u, \mathfrak{N}}$  with its eightfold symmetric product in the top left corner of (2.23); this is a  $q_{\mathfrak{g}}^8$ -cover of  $\widehat{\mathcal{P}}$ . Let  $\mathcal{D}_{\mathfrak{g}} \in \text{Div} \mathcal{S}_{\mathfrak{g}}$  be the sum of  $\Sigma_i(\mathcal{B}_{\mathfrak{g}})$  in (2.23).

On the open set where the Prym–Tyurin does not degenerate,

$$\mathcal{B}_{\mathfrak{g}}^{\text{reg}} \triangleq \{(u, \mathfrak{N}) \in \mathcal{B}_{\mathfrak{g}} \mid \dim_{\mathbb{C}} \text{PT}(\Gamma_{u, \mathfrak{N}}) = 8\}, \quad (3.45)$$

introduce the (vertical) 1-forms on  $\mathcal{S}_{\mathfrak{g}}^{\text{sym}}$  written, on a bundle chart  $((u, \mathfrak{N}); (\gamma_i)_i)$ , as

$$\begin{aligned} \mathcal{L} &\triangleq \sum_{i=1}^8 \frac{d\lambda(\gamma_i)}{\lambda(\gamma_i)} \in \omega_{\mathcal{S}_{\mathfrak{g}}^{\text{sym}}/\mathcal{B}_{\mathfrak{g}}}(\mathcal{B}_{\mathfrak{g}}^{\text{reg}}), \\ \mathcal{M} &\triangleq \sum_{i=1}^8 \frac{d\mu(\gamma_i)}{\mu(\gamma_i)} \in \omega_{\mathcal{S}_{\mathfrak{g}}^{\text{sym}}/\mathcal{B}_{\mathfrak{g}}}(\mathcal{B}_{\mathfrak{g}}^{\text{reg}}), \\ d\mathcal{S} &\triangleq \sum_{i=1}^8 d\sigma(\gamma_i) \triangleq \sum_{i=1}^8 \log \lambda(\gamma_i) \frac{d\mu(\gamma_i)}{\mu(\gamma_i)} \in \omega_{\mathcal{S}_{\mathfrak{g}}^{\text{sym}}/\mathcal{B}_{\mathfrak{g}}}(\mathcal{B}_{\mathfrak{g}}^{\text{reg}}). \end{aligned} \quad (3.46)$$

The same notation  $d\mathcal{S}$  and  $d\sigma$  will indicate the pullbacks to fibres  $\text{Sym}^8(\Gamma_{u, \mathfrak{N}})$  and  $\Gamma_{u, \mathfrak{N}}$  of the respective families; in (3.46)  $d\sigma$  is an arbitrary choice of branch of the log-meromorphic differential  $\log \lambda d \log \mu$  on  $\Gamma_{u, \mathfrak{N}}$ . Notice that

$$\omega_{\text{PL}} = \omega_{\text{KP}}^{(1)} = \mathcal{L} \wedge \mathcal{D}_{\mathfrak{g}}^* \mathcal{M}.$$

**Lemma 3.5.** *We have that*

$$\frac{\partial d\sigma}{\partial u_i} \in H^0(\Gamma_{u, \mathfrak{N}}, \Omega^1) \quad \forall i = 1, \dots, 8, \quad (3.47)$$

where the moduli derivative in (3.47) is taken at fixed  $\mu : \Gamma_{u, \mathfrak{N}} \rightarrow \mathbb{P}^1$ .

*Proof.* By definition,

$$d\kappa_i \triangleq \frac{\partial d\sigma}{\partial u_i} = \frac{\partial_{u_i} \lambda d\mu}{\lambda \mu} = \frac{\lambda^9 \partial_{u_i} \Xi_{\mathfrak{g}, \text{red}} d\mu}{\partial_{\lambda} (\lambda^9 \Xi_{\mathfrak{g}, \text{red}}) \lambda \mu} \quad (3.48)$$

Recall that, for a generic polynomial  $P(x, y)$ , the 1-forms

$$d\omega_{ij} = \frac{x^{i-1} y^{j-1} dx}{\partial_y P}$$

with  $(i, j)$  in the strict interior of the Newton polygon of  $P(x, y)$  are holomorphic 1-forms on  $\overline{\{P(x, y) = 0\}}$ . The expression (3.48) is a linear combination of terms that are exactly of this form: notice that the doubly logarithmic form of  $d\sigma$  in (3.46) is crucial to ensure the presence of

the product  $\lambda\mu$  at the denominator which makes this statement true. However  $\lambda^9\Xi_{\mathfrak{g},\text{red}}$  is highly non-generic, and by the way  $\Gamma_{u,\aleph}$  was introduced in Definition 2.1 the 1-forms in (3.48) may still have simple poles with opposite residues at the strict transform of the nodes in (2.34). A direct computation however shows that

$$\frac{\partial\Xi_{\mathfrak{g},\text{red}}}{\partial u_i}(p) = 0 \quad \text{if } d\mu(p) = 0, \quad \mu(p) + \frac{1}{\mu(p)} = r_i^k, i = 2, 3. \quad (3.49)$$

which entails the vanishing of the residues on the normalization, and thus  $d\kappa_j \in H^1(\Gamma_{u,\aleph}, \mathcal{O}) \forall j = 1, \dots, 8$ .

□

As a consequence, an algebraic Liouville–Arnold-type statement can be made as follows. Locally on  $\Gamma_{u,\aleph}$  and its 8-fold symmetric product, consider the Abelian integral

$$\sigma(p) = \int^p d\sigma \quad (3.50)$$

and correspondingly  $\mathcal{S}(p_i) = \sum_i \sigma(p_i)$ . Define the  $A_i$ -periods of  $d\sigma$  as

$$\alpha_i \triangleq \oint_{A_i} d\sigma. \quad (3.51)$$

By Corollary 3.2 and Theorem 3.4, these are phase-space areas (action variables) for the angular motion on  $\text{PT}(\Gamma_{u,\aleph})$ . Indeed,

$$\begin{aligned} \alpha_i &= \oint_{A_i} d\sigma = \frac{1}{q_{\mathfrak{g}}} \oint_{(\mathcal{P}_{\mathfrak{g}})_*(A_i)} d\sigma \\ &= \frac{1}{q_{\mathfrak{g}}} \oint_{A_i} \mathcal{P}_{\mathfrak{g}}^* d\sigma = \frac{1}{q_{\mathfrak{g}}} \oint_{D^2|_{\partial D^2=A_i}} \omega_{\text{PL}}. \end{aligned} \quad (3.52)$$

Define the normalised basis of holomorphic 1-forms

$$d\vartheta_i = \sum_j \left( \frac{\partial\alpha}{\partial u} \right)_{ij}^{-1} d\kappa_j \in H^1(\Gamma_{u,\aleph}, \mathcal{O}), \quad (3.53)$$

so that  $\oint_{A_j} d\vartheta_i = \delta_{ij}$ . This is the normalised  $\mathbb{C}$ -basis of the vector space  $V_-$  of  $\mathcal{P}_{\mathfrak{g}}$ -invariant forms on  $\Gamma_{u,\aleph}$  with respect to our choice of  $A$  and  $B$  cycles in (3.27).

**Theorem 3.6.** *We have*

$$\omega_{\text{PL}} = \frac{1}{q_{\mathfrak{g}}} \sum_{i=1}^8 du_i \wedge d\kappa_i = \frac{1}{q_{\mathfrak{g}}} \sum_{i=1}^8 d\alpha_i \wedge d\vartheta_i \quad (3.54)$$

and in the action-angle coordinates  $(\alpha_i, \vartheta_i)_{i=1}^8$  the flows (2.16), (3.8) are Hamiltonian with respect to  $\omega_{\text{PL}}$ , with Hamiltonian  $u_i = u_i(\alpha)$ . In particular, the angular frequencies in (3.8) are given by the Jacobian

$$\Omega_{ij} = \left( \frac{\partial\alpha}{\partial u} \right)_{ij}^{-1}. \quad (3.55)$$

*Proof.* The statement just follows from writing down the symplectic change-of-variables given by looking at  $\mathcal{S}$  as a type II generating function of canonical transformation, first in  $u$  and then in  $\alpha$ ,

$$\lambda(\gamma_i) = \exp(\mu(\gamma_i)\partial_{\mu(\gamma_i)}\mathcal{S}), \quad \kappa_i = \frac{\partial\mathcal{S}}{\partial u_i} = \int d\kappa_i, \quad \vartheta_i = \frac{\partial\mathcal{S}}{\partial\alpha_i} = \int d\theta_i, \quad (3.56)$$

and use of Lemma 3.5. □

Keeping in mind the discussion below Definition 3.1, the constructions of this section bestow on  $\mathcal{S}_g$  a canonical choice of Dubrovin–Krichever data as follows

$$\begin{aligned} \mathcal{F} &\longleftrightarrow \mathcal{S}_g \\ \mathcal{B} &\longleftrightarrow \mathcal{B}_g \\ \mathcal{D} &\longleftrightarrow \sum_i \Sigma_i(\mathcal{B}_g) \\ \mathcal{E}_1 &\longleftrightarrow \mathcal{L} \longleftrightarrow \frac{d\lambda}{\lambda} \\ \mathcal{E}_2 &\longleftrightarrow \mathcal{M} \longleftrightarrow \frac{d\mu}{\mu} \\ \Lambda &\longleftrightarrow \Lambda_{\text{PT}} \end{aligned} \quad (3.57)$$

which is complete but for the choice of the Lagrangian sublattice  $\Lambda^L$ . The latter is left unspecified by the Toda dynamics – and, in the applications of the next two Sections its choice will vary depending on the circumstances.

## 4. APPLICATION I: GAUGE THEORY AND TODA

### 4.1. Seiberg–Witten, Gromov–Witten and Chern–Simons.

4.1.1. *Gauge theory.* From the physics point of view, the first object of interest for us is the minimal supersymmetric five-dimensional gauge theory on the product  $M_5 = \mathbb{R}^4 \times S_R^1$  of four-dimensional Minkowski space times a circle of radius  $R$  with gauge group the compact real form  $E_8^{\mathbb{R}}$ . On shell, and at  $R \rightarrow \infty$ , its gauge/matter content consists of one  $E_8^{\mathbb{R}}$  vector multiplet  $(A, \lambda, \varphi)$  with real scalar  $\varphi$ , gluino  $\lambda$ , and gauge field  $A$ ; upon compactification this is enriched by an extra scalar  $\vartheta$ , which is the Wilson loop around the fifth-dimensional  $S_R^1$ . The infrared dynamics of the compactified theory is characterised by a dynamical holomorphic scale in four dimensions  $\Lambda_4$  [100, 107], which is (perturbatively) a renormalisation group invariant. For generic vacua parametrised by the complexified scalar vev  $\phi = \langle \varphi + i\vartheta \rangle$  and  $R \in \mathbb{R}^+$ , and assuming that the latter is much higher than the non-perturbative scale  $|\phi| \gg \Lambda_4$ , the massless modes are those of a theory of  $\text{rank}_{\mathfrak{e}_8} = 8$  weakly coupled photons in four dimensions, whose Wilsonian effective action is given by integrating out both perturbative (electric) and non-perturbative (dyonic) contributions of BPS saturated Kaluza–Klein states. This is expressed (up to two derivatives in the  $U(1)$  gauge superfield



strenghts  $W_\alpha^i$ , and in four-dimensional  $\mathcal{N} = 1$  superspace coordinates  $(x, \theta)$  by the Wilsonian effective Lagrangian

$$\mathcal{L} = \frac{1}{4\pi} \text{Im} \left[ \int d^4\theta \frac{\partial F_0^{\text{SYM}}}{\partial A_i} \bar{A}^i + \frac{1}{2} \int d^2\theta \frac{\partial^2 F_0^{\text{SYM}}}{\partial A_i \partial A_j} W_\alpha^i \bar{W}^{\alpha, j} \right], \quad (4.1)$$

which is entirely encoded by the prepotential  $F_0^{\text{SYM}}$ ; in particular, the Hessian  $\frac{\partial^2 F_0^{\text{SYM}}}{\partial A_i \partial A_j}$  returns the gauge coupling matrix for the low-energy photons.

Mathematically, this gauge theory prepotential should coincide with the equivariant limit of a suitable generating function of instanton numbers. Let  $\text{Bun}_k(\mathcal{G})$  be the the moduli space of principal  $E_8$ -bundles on the projective compactification  $\mathbb{P}^2 \simeq \mathbb{C}^2 \cup \mathbb{P}_\infty^1$  of  $\mathbb{R}^4 \simeq \mathbb{C}^2$  with second Chern class  $k$ , which I assume to be positive in the following; here “framed” means that we fix a trivialisation of the projective line at infinity.  $\text{Bun}_k(\mathcal{G})$  is an irreducible smooth quasi-affine variety of dimension  $2h(\mathfrak{g})k = 60k$ , and it admits an irreducible, affine partial compactification given by the Uhlenbeck stratification

$$\mathcal{U}_k(\mathcal{G}) = \text{Bun}_k(\mathcal{G}) \sqcup (\text{Bun}_{k-1}(\mathcal{G}) \times \mathbb{C}^2) \sqcup (\text{Bun}_{k-2}(\mathcal{G}) \times \text{Sym}^2 \mathbb{C}^2) \cdots \sqcup (\text{Sym}^k \mathbb{C}^2). \quad (4.2)$$

There is an algebraic  $\mathbb{T} \times (\mathbb{C}^*)^2 \simeq (\mathbb{C}^*)^{10}$  torus action on  $\text{Bun}_k(\mathcal{G})$ , where the two factors act by scaling the trivialisation at infinity and the linear coordinates of  $\mathbb{C}^2$  respectively, and which extends to the whole of  $\mathcal{U}_k(\mathcal{G})$  [22], and leads to a ten-dimensional torus action on the vector space  $H^0(\mathcal{U}_k(\mathcal{G}), \mathcal{O})$  of regular functions on  $\mathcal{U}_k(\mathcal{G})$ . Denoting the characters of  $\mathbb{T} \times (\mathbb{C}^*)^2$  by  $\mu$ , the latter decomposes

$$H^0(\mathcal{U}_k(\mathcal{G}), \mathcal{O}) = \bigoplus_{\mu} (H^0(\mathcal{U}_k(\mathcal{G}), \mathcal{O}))_{\mu} \quad (4.3)$$

as a direct sum of weight spaces which are, non-trivially, finite-dimensional over  $\mathbb{C}$  (see e.g. [92]). Let  $a_i = c_1(\mathbb{L}_i)$ ,  $i = 1, \dots, 8$  for the first Chern class of the dual of the  $i^{\text{th}}$  tautological line bundle  $\mathbb{L}_i \rightarrow B\mathcal{T}$ , and likewise write  $\epsilon_{1,2} = c_1(\mathbb{L}_{1,2})$  for the equivariant parameters of the right (spacetime)  $(\mathbb{C}^*)^2$  factor in  $\mathbb{T} \times (\mathbb{C}^*)^2 \curvearrowright \mathcal{U}_k(\mathcal{G})$ . The instanton partition function of the 5-dimensional gauge theory<sup>13</sup> is then defined to be the Hilbert sum

$$\mathcal{Z}_{\mathcal{G}}^{\text{inst}}(a_1, \dots, a_8; \epsilon_1, \epsilon_2, Q) = \sum_{n \in \mathbb{Z}^+} \left( QR^{q_8} e^{-q_8 R(\epsilon_1 + \epsilon_2)/2} \right)^n \sum_{\mu} \dim_{\mathbb{C}} (H^0(\mathcal{U}_k(\mathcal{G}), \mathcal{O}))_{\mu} \exp[\mu \cdot (\vec{a}, \vec{\epsilon})]. \quad (4.4)$$

The prepotential (4.1) is recovered as the sum of the non-equivariant limit of  $\epsilon_1 \epsilon_2 \ln \mathcal{Z}_{\mathcal{G}}^{\text{inst}}$ , which is well-defined [91, 93], plus a classical+one-loop perturbative contribution. Namely,

$$F_0^{\text{SYM}} \triangleq F_{\text{cl}} + F_{1\text{-loop}} + F_{\text{inst}}, \quad (4.5)$$

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<sup>13</sup>Physically this should be thought to be in an  $\Omega$ -background, with equivariant parameters  $(\epsilon_1, \epsilon_2)$  corresponding to the torus weights of the  $(\mathbb{C}^*)^2$  factor acting on  $\mathbb{C}^2$  above.

where

$$\begin{aligned}
F_{\text{cl}} &= \frac{\tau_{ij} a_i a_j}{2}, \\
F_{1\text{-loop}} &= \sum_{\alpha \in \Delta^*} \left[ \frac{(\alpha \cdot a)^2 \log(R\Lambda_4)}{2} - (\alpha \cdot a)^3 \frac{R}{12} + \text{Li}_3(e^{-R\alpha \cdot a}) \right], \\
F_{\text{inst}} &= \lim_{\epsilon_i \rightarrow 0} \epsilon_1 \epsilon_2 \log \mathcal{Z}_{\mathcal{G}}^{\text{inst}}(a_1, \dots, a_8; \epsilon_1, \epsilon_2, \Lambda_4),
\end{aligned} \tag{4.6}$$

where  $\tau$  is the bare gauge coupling matrix. In the following, I am going to measure energies in units of  $\Lambda_4$  and thus set  $\Lambda_4 = 1$ ; the dependence on  $\Lambda_4$  can be restored by appropriate rescaling of the dimensionful quantities  $a_i$  and  $1/R$ .

**4.1.2. Topological strings.** It has long been argued that the prepotential (4.6) might also be regarded as the generating function of genus zero Gromov–Witten invariants on a suitable non-compact Calabi–Yau threefold [69, 78]. Let

$$X = \text{Tot}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 2}(-1)) = \{(A, v) \in \text{Mat}(2, \mathbb{C}) \times \mathbb{P}^1 \mid Av = 0\} \tag{4.7}$$

be the minimal toric resolution of the singular quadric  $\det A = 0$  in  $\mathbb{A}^4$ ; columns of  $A$  give trivialisations of  $\mathcal{O}(-1)$  over the North/South affine patches of  $\mathbb{P}^1$ . Any  $\Gamma \triangleleft \text{SL}_2(\mathbb{C})$ ,  $|\Gamma| < \infty$  gives an action  $\Gamma \curvearrowright X$  by left multiplication,  $(\gamma \in \Gamma, A) \rightarrow \gamma \cdot A$ , which is trivial on the canonical bundle of  $X$ , and covers the trivial action on the base  $\mathbb{P}^1$ . The quotient space is thus a singular Calabi–Yau fibration over  $\mathbb{P}^1$  by surface singularities of the same ADE type of  $\Gamma$  [103]; and type  $E_8$  corresponds to taking  $\Gamma \simeq \tilde{\mathbb{I}}$  the binary icosahedral group (see Appendix B.1).

There are two distinguished chambers in the stringy Kähler moduli space of  $\mathcal{X} \triangleq [X/\tilde{\mathbb{I}}]$  that are of importance in our discussion. One is the large radius chamber: in this case we take the minimal crepant resolution  $\pi : Y \rightarrow X/\tilde{\mathbb{I}}$ , which corresponds to fibrewise resolving the  $E_8$  singularities on a chain of rational curves whose intersection matrix equates  $-\mathcal{C}_{ij}^g$  [103]. In particular,  $H_2(Y, \mathbb{Z}) = \mathbb{Z} \langle \widehat{H}; E_1, \dots, E_8 \rangle$  where  $H$  (resp.  $E_i$ ) is the the pull-back to  $Y$  from the base  $\mathbb{P}^1$  (resp. the blow-up  $\mathbb{C}^2/\tilde{\mathbb{I}}$  of the fibre singularity) of the fundamental class of the base curve (resp. of the class of the  $i^{\text{th}}$  exceptional curve). The Gromov–Witten potential of  $Y$  is the generating sum

$$F^{\text{GW}}(Y; \epsilon; t_B, t_1, \dots, t_8) = \sum_{d \in H_2(Y, \mathbb{Z}), g \in \mathbb{Z}^+} \epsilon^{2g-2} e^{-d \cdot t} N_{g,d}^Y = \sum_{g \in \mathbb{Z}^+} \epsilon^{2g-2} F_g^{\text{GW}}(Y; t_B, t_1, \dots, t_8) \tag{4.8}$$

where

$$N_{g,d}^Y = \int_{[\mathcal{M}_g(Y,d)]^{\text{vir}}} 1 \tag{4.9}$$

is the genus- $g$ , degree  $d$  Gromov–Witten invariant of  $Y$ , and we write  $d = d_B[H] + d_1[E_1] + \dots + d_8[E_8]$  for the degrees of stable maps from  $\mathbb{P}^1$  to  $Y$ . Owing to the non-compactness of  $Y$ , what is really meant by  $N_{g,d}^Y$  is a sum of degrees of the localised virtual fundamental classes at the fixed loci w.r.t. a suitable  $\mathbb{C}^*$  action: notice that  $Y$  supports a rank two torus action given by a diagonal scaling the fibres (which commutes with  $\Gamma$ , and w.r.t. which the resolution map  $\pi$  is equivariant) and a  $\mathbb{C}^*$  rotation of the base  $\mathbb{P}^1$ . In particular we can always cut out a 1-torus action which is trivial on  $K_Y$  (the equivariantly Calabi–Yau case) by tuning the weights of the two factors appropriately,

and this is the choice that is picked<sup>14</sup> in (4.9). Furthermore, natural Lagrangian A-branes  $L \hookrightarrow Y$  and a counting theory of open stable maps can be constructed (at least operatively) via localisation [15, 25, 70]; in a vein similar to (4.8)-(4.9), one defines

$$W_{g,h}^{\text{GW}}(Y, L; \lambda; t_B, t_1, \dots, t_8, \lambda) = \sum_{d \in H_2(Y, L, \mathbb{Z})} \sum_{w_1, \dots, w_h \in H_1(L, \mathbb{Z})} e^{-d \cdot t} \prod_i \lambda_i^{w_i} N_{g,h,d,w}^{Y,L} \quad (4.10)$$

where

$$N_{g,h,d,w}^{Y,L} = \int_{[\mathcal{M}_{g,h}(Y, L, d, w)]^{\text{virt}}} 1 \quad (4.11)$$

is the genus- $g$ ,  $h$ -holed open Gromov–Witten invariant of  $(X, L)$  of relative degree  $d$  and winding numbers  $\{w_i\}_{i=1}^h$ ,  $\partial d = \sum w_i$ .

The relation of these curve-counting generating functions and the instanton prepotentials of the previous section is given by the so-called *geometric engineering of gauge theories*, a (partial) statement of which can be given as follows

**Claim 4.1** ([69, 78]). *The genus zero Gromov–Witten potential of  $Y$  equates the five-dimensional gauge theory prepotential/instanton generating function*

$$F_0^{\text{SYM}}(R, a_1, \dots, a_8) = F_0^Y(t_B, t_1, \dots, t_8) + \text{cubic} \quad (4.12)$$

under the identification

$$a_i = t_i R, \quad R = e^{-t_B/4}. \quad (4.13)$$

Claim 4.1 has an extension to higher genera wherein gravitational corrections to  $F_0^{\text{SYM}}$  are considered [5, 12], or equivalently, the gauge theory is placed in the  $\Omega$ -background (without taking the limit (4.6)) and one restricts to the self-dual background  $\epsilon_1 = -\epsilon_2 = \epsilon$  [95]. The open string potentials (4.10) have similarly a counterpart in terms of surface operators in the gauge theory [4, 72].

The second chamber is the orbifold chamber: here we consider the stack quotient  $\mathcal{X} = [\mathcal{O}(-1)^{\oplus 2}/\tilde{\mathbb{I}}]$ , which has a  $\mathbb{P}^1$  worth of  $\tilde{\mathbb{I}}$ -stacky points. Open and closed Gromov–Witten invariants of  $\mathcal{X}$  can be defined, if only computationally, along the same lines as before by virtual localisation on moduli of twisted stable maps [25]; I refer the reader to [15, Sec. 3.3-3.4] where this is more amply discussed.

**4.1.3. Chern–Simons theory.** The previous Calabi–Yau geometry has been argued in [15], following the earlier work [3], to be related to the large  $N$  limit of  $U(N)$  Chern–Simons theory on the Poincaré sphere. This is a real three-manifold  $\Sigma$  obtained from  $S^2 \times S^1$  after rational surgery with exponents  $1/2$ ,  $1/3$  and  $1/5$  on a 3-component unlink wrapping the fibre direction of  $S^1 \times S^2 \rightarrow S^2$ , and it is the only  $\mathbb{Z}$ -homology sphere, other than  $S^3$ , to have a finite fundamental group. Equivalently, it

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<sup>14</sup>This is the choice that is picked, for toric targets, by the topological vertex; this is consistent with the fact that, by the equivariant CY condition, this is a rational number (rather than an element of  $H_{BC^*}(\text{pt}, \mathbb{Q})$ ).

can be realised as the quotient  $S^3/\tilde{\mathbb{I}} \simeq \mathbb{R}\mathbb{P}^3/\mathbb{I}$  of the three-sphere by the left-action of the binary icosahedral group [120].

I will very succinctly present the statement we are after, referring the reader to the beautiful review [84] or the presentation of [15] for more details. Let  $k \in \mathbb{Z}^+$ ,  $\mathcal{A}$  a smooth gauge connection on the trivial  $U(N)$  bundle on  $\Sigma$ . The  $U(N)$  Chern–Simons partition function of  $\Sigma$  at level  $k$  is the functional integral

$$Z_{\text{CS}}(\Sigma, k, N) = \langle 1 \rangle_{\text{CS}} = \int_{\mathcal{A}/\mathcal{G}} [\mathcal{D}\mathcal{A}] \exp\left(\frac{ik}{2\pi} \text{CS}[\mathcal{A}]\right), \quad (4.14)$$

$$\text{CS}[\mathcal{A}] = \int_{\Sigma} \text{Tr}_{\square} \left( \mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A}^3 \right), \quad (4.15)$$

where (4.15) is the Chern–Simons action. For  $\mathcal{K} \hookrightarrow \Sigma$  a link in  $\Sigma$  and  $\rho \in R(U(N))$ , we will also consider the expectation value under the measure (4.14)–(4.15) of the  $\rho$ -character of the holonomy around  $\mathcal{K}$ ,

$$W_{\text{CS}}(\Sigma, \mathcal{K}, k, N, \rho) = \frac{\langle \text{Tr}_{\rho} \text{Hol}_{\mathcal{K}}(A) \rangle_{\text{CS}}}{Z_{\text{CS}}} = Z_{\text{CS}}^{-1} \int_{\mathcal{A}/\mathcal{G}} [\mathcal{D}\mathcal{A}] \exp\left(\frac{ik}{2\pi} \text{CS}[\mathcal{A}]\right) \text{Tr}_{\rho} \text{Hol}_{\mathcal{K}}(A). \quad (4.16)$$

(4.14)–(4.16) were proposed by Witten [119] to be smooth<sup>15</sup> invariants of  $\Sigma$  and  $(\Sigma, \mathcal{K})$ , reflecting the near metric independence of (4.14) at the quantum level [104]; when  $\Sigma$  is replaced by  $S^3$ , (4.16) is the HOMFLY polynomial of  $\mathcal{K}$  coloured in the representation  $\rho$ .

We will be looking at (4.14) in two ways, which are both essentially disentangled with the question of giving a rigorous treatment of the path integral (4.14). One is in Gaussian perturbation theory at large  $N$ , where we take (4.14) as a formal expansion in ribbon graphs [84, 115]. Writing

$$g_{\text{YM}} = \frac{2\pi i}{k + N}, \quad t = g_{\text{YM}} N, \quad (4.17)$$

the perturbative free energy takes the form

$$\begin{aligned} \mathcal{F}^{\text{CS}}(\Sigma, g_{\text{YM}}, t) &= \ln Z^{\text{CS}}(\Sigma, k, N) \\ &= \sum_{g \geq 0} \mathcal{F}_g^{\text{CS}}(\Sigma, t) g_{\text{YM}}^{2g-2} \in g_{\text{YM}}^{-2} \mathbb{Q}[[t, g_{\text{YM}}^2]]. \end{aligned} \quad (4.18)$$

Similarly, for  $h > 0$ ,  $l \in \mathbb{N}^h$  and  $\mathcal{K} \in \Sigma$  a link in  $\Sigma$ , we get for the connected Chern–Simons average of a Wilson loop around  $\mathcal{K}$  that

$$\begin{aligned} W_{\text{CS}}^{(h)}(\Sigma, \mathcal{K}, k, N, \{\lambda_i\}_{i=1}^h) &\triangleq \sum_{l \in \mathbb{N}^h} \prod_i \lambda_i^{l_i} \frac{1}{|l|!} \frac{\partial^{|l|} \log \left\langle e^{\sum_i q_i \text{Tr}(\text{Hol}_{\mathcal{K}}(A))^i} \right\rangle_{\text{CS}}}{\partial q_1^{l_1} \dots \partial q_s^{l_s}} \Bigg|_{q_i=0}, \\ &= \sum_{g \geq 0} W_{g,h}(\Sigma, \mathcal{K}, t, \{\lambda_i\}) g_{\text{YM}}^{2g-2+h} \in g_{\text{YM}}^{h-2} \mathbb{Q}[[t, g_{\text{YM}}^2]]. \end{aligned} \quad (4.19)$$

<sup>15</sup>More precisely,  $Z_{\text{CS}}$  is only invariant under diffeomorphisms of  $\Sigma$  that preserve a given framing of its tangent bundle, and changes in a definite way under change-of-framing; the same applies for  $W_{\text{CS}}$  and a choice of framing on  $\mathcal{K}$ . In the following I implicitly work in canonical framing for both  $\Sigma$  and  $\mathcal{K}$ ; also the change of framing won't affect the large  $N$  behaviour of  $\mathcal{F}_{\text{CS}}$  but for a constant in  $t$ ,  $\mathcal{O}(N^0)$  (unstable) term.

The second way of looking at (4.18) and (4.19) comes from their independent mathematical life as the  $U_q(\mathfrak{sl}_N)$  Reshetikhin–Turaev–Witten invariants of  $\Sigma$  and  $\mathcal{K} \hookrightarrow \Sigma$  respectively [104]. Recall that  $\Sigma$  has a Hopf-like realisation as a circle bundle over the orbifold projective line  $\mathbb{P}_{2,3,5}^1$  with three orbifold points with isotropy group  $\mathbb{Z}_{\mathfrak{s}(n)}$ , with  $\mathfrak{s}(1) = 2$ ,  $\mathfrak{s}(2) = 3$ ,  $\mathfrak{s}(3) = 5$ . I will write  $\mathfrak{s} = \prod_i \mathfrak{s}(i) = 30$ , and  $\mathcal{K}_n \simeq S^1$  for the knots wrapping the exceptional fibre labelled by  $n$ . Then the RTW invariants of  $\Sigma$  and  $(\Sigma, \mathcal{K}_n)$  can be computed explicitly from a rational surgery formula [62] (or equivalently, Witten’s surgery prescription for Chern–Simons vevs [119]), leading to closed-form expressions for (4.18) and (4.19) alike in terms of Weyl-group sums [83]. Denote by  $\mathcal{F}_l$  the set of dominant weights  $\omega$  of  $SU(N)$  such that, if  $\omega = \sum a_i \omega_i$  in terms of the fundamental weights  $\omega_i$ , then  $\sum_i a_i < l$ . Then,

$$\begin{aligned} Z_{\text{CS}}(\Sigma, k, N) &= \mathcal{N}(\Sigma) \sum_{\beta \in \mathcal{F}_{k+N}} \frac{1}{\prod_{\alpha > 0} \sin\left(\frac{\pi \beta \cdot \alpha}{k+N}\right)} \prod_{i=1}^3 \sum_{f_i \in \Lambda_r / \mathfrak{s}(i) \Lambda_r} \sum_{w_i \in S_N} \epsilon(w_i) \times \\ &\times \exp\left\{ \frac{i\pi}{(k+N)\mathfrak{s}(i)} \left( -\beta^2 - 2\beta \left( (k+N)f_i + w(\rho) \right) + \left( (k+N)f_i + w(\rho) \right)^2 \right) \right\}, \end{aligned} \quad (4.20)$$

where  $\rho$  is the Weyl vector of  $\mathfrak{sl}_N$ ,  $\rho = \sum_{i=1}^{N-1} \omega_i$ ,  $\Lambda_r$  is the  $\mathfrak{sl}_N$  root lattice, and  $\mathcal{N}(\Sigma)$  is an explicit multiplicative factor involving the surgery data and the Casson–Walker–Lescop invariant of  $\Sigma$ . A similar expression holds for the (un-normalised) Chern–Simons vevs of the Wilson loops around fibre knots: this is obtained by replacing  $\rho \rightarrow \rho + \Lambda$  for  $\Lambda$  a dominant weight in Section 4.1.3, after which (4.19) can be recovered by expressing the representation-basis colouring by the connected power sum colouring of (4.19), and powers multiple of  $\mathfrak{s}_i$  for  $i = 1, 2, 3$  single out the holonomies around the  $i^{\text{th}}$  exceptional fibre (see [15, 17] and the discussion of Section 4.2.1 below).

Two remarks are in order about (4.20). Firstly, unlike (4.18)–(4.19), (4.20) is an *exact* expression at finite  $N$ ; among its virtues however, as first emphasised in [83], is the possibility to express it as a matrix-like integral, and thus use standard asymptotic methods in random matrix theory to study its large  $N$ , finite  $t$  regime: this fact will be used extensively in the next Section. Secondly, as pointed out in [83] and further confirmed in [10, 13] by a functional integral analysis, the sum over  $f_i$  in (4.20) may be interpreted as a sum over critical points of the Chern–Simons functional (4.15),

$$\text{Crit}_{\text{CS}}^N = \{A \in \text{Conn}(\Sigma, U(N)) \mid F_A = 0\} = \text{Hom}(\pi_1(\Sigma), U(N)) / U(N) \quad (4.21)$$

namely, flat  $U(N)$ -connections on  $\Sigma$ ; this is a finite set at finite  $N$  since  $|\pi_1(\Sigma)| < \infty$ . In the monodromy representation of the latter equality in (4.21), these can be labelled by integers  $(f_0, f_1, \dots, f_8)$  satisfying

$$\sum_i \mathfrak{d}_i f_i = N \quad (4.22)$$

where  $\mathfrak{d}_i$ ,  $i = 0, \dots, 8$  is the dimension of the  $i^{\text{th}}$  irreducible representation of  $\pi_1(\Sigma) = \tilde{\mathbb{I}}$  (see Table 6; equivalently, the  $i^{\text{th}}$  Dynkin label in Figure 2), the trivial connection contribution to (4.20) being given by  $f_i = 0$ ,  $i > 0$ . The latter is the exponentially dominant summand in the limit  $g_{\text{YM}} \rightarrow 0$ , as the classical Chern–Simons functional attains there its minimum value (equal to zero), and it

leads to a quantum invariant of 3-manifolds in its own right: this is the *Lê–Murakami–Ohtsuki* (LMO) invariant, which is a derivation of the universal Vassiliev–Kontsevich by taking its Kirby–move-invariant part [79].

In landmark papers by Gopakumar and Vafa [58] and Ooguri–Vafa [99], it was proposed that the large  $N$  expansion of the Chern–Simons invariants of  $S^3$  and  $\mathcal{K} = \mathbb{O}$  yield the genus expansion of the topological A-model on the resolved conifold  $X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1$ . Following in this direction and that of its generalisation of [3] for lens spaces<sup>16</sup>, it was proposed amongst other things in [15] that the large  $N$  limit of the connected averages (4.18) and (4.19) should be interpreted (respectively) as the generating functions of closed and open Gromov–Witten invariants (4.8) and (4.10) of the orbifold-of-the-conifold  $\mathcal{X}$ , as I now recall. Let  $\text{Crit}_{\text{CS}}^\infty = \lim_{N \rightarrow \infty} \text{Crit}_{\text{CS}}^N$  be the direct limit of the finite critical point sets (4.21) w.r.t. the composition of morphisms of sets induced by  $U(N) \hookrightarrow U(N+1)$ . A point in  $\text{Crit}_{\text{CS}}^\infty$  consists of a flat background  $[A]_{\mathfrak{t}}$  parameterised by  $\mathfrak{t}_i \triangleq N_i g_{\text{YM}}$  for  $i \in \{0, \dots, 8\}$ . Write now  $\mathcal{F}_g(\Sigma, \mathfrak{t})$  and  $\mathcal{W}_{g,n}(\Sigma, \mathfrak{t}; \vec{x})$  for the contribution of each  $[A]_{\mathfrak{t}}$  at large  $N$  to the perturbative free energies and correlators of  $U(N)$  Chern–Simons theory (4.18) and (4.19).

**Claim 4.2** ([15]). *There is an affine linear change-of-variables  $(\tau, t_B) = \mathcal{L}(\mathfrak{t})$  such that*

$$F_g^{\text{CS}}(\Sigma, \mathfrak{t}) = F_g^{\text{GW}}(\mathcal{X}, \mathcal{L}(\mathfrak{t})), \quad W_{g,h}^{\text{CS}}(\Sigma, \mathcal{K}; \mathfrak{t}, \lambda_1, \dots, \lambda_h) = W_{g,h}^{\text{GW}}(\mathcal{X}, \mathcal{L}, \mathcal{L}(\mathfrak{t}); \lambda_1, \dots, \lambda_h). \quad (4.23)$$

Consequently, the LMO contribution to the Chern–Simons free energy ( $f_i = 0$ ) is obtained as the corresponding restriction of GW potentials:

$$\begin{aligned} F_g^{\text{CS}}(\Sigma, \mathfrak{t}_i = \mathfrak{t}_0 \delta_{i0}) &= F_g^{\text{GW}}(\mathcal{X}, \mathcal{L}(\mathfrak{t})) \Big|_{\mathfrak{t}_i = \mathfrak{t}_0 \delta_{i0}}, \\ W_{g,h}^{\text{CS}}(\Sigma, \mathfrak{t}_i = \mathfrak{t}_0 \delta_{i0}, \lambda_1, \dots, \lambda_h) &= W_{g,h}^{\text{GW}}(\mathcal{X}, \mathcal{L}, \mathcal{L}(\mathfrak{t}); \lambda_1, \dots, \lambda_h) \Big|_{\mathfrak{t}_i = \mathfrak{t}_0 \delta_{i0}}. \end{aligned} \quad (4.24)$$

I will refer to (4.23) and (4.24) as, respectively, the *strong* and *weak A-model Gopakumar–Vafa correspondence* for  $\Sigma$ .

**4.1.4. Toda spectral curves and the topological recursion.** A major point of the foregoing discussion is to argue that there exist completions of the Dubrovin–Krichever data (3.57) of the  $E_8$  relativistic Toda spectral curves in the form Lagrangian sublattices  $\Lambda_{\text{PT}}^L \subset \Lambda_{\text{PT}}$  leading to the existence of genus zero prepotentials  $F_0^{\text{Toda}}$  from rigid special Kähler geometry relations<sup>17</sup> on  $\mathcal{B}_g$ , as well as higher genus open/closed potentials  $F_g^{\text{Toda}}, W_{g,h}^{\text{Toda}}$  from the Chekhov–Eynard–Orantin topological recursion [49], which are purported to be the all-genus solutions of the open/closed topological B-model with  $\mathcal{S}_g$  as its target geometry [19]. Following completely analogous statements [3, 19, 51, 61,

<sup>16</sup>Some of these arguments require extra care when one considers non-SU(2) quotients of the three-sphere; see e.g. [28].

<sup>17</sup>This type of relations, which condense the fact there exists a prepotential for the periods on the mirror curve, have different names and tasks in different communities: in gauge theory, they are a manifestation of  $\mathcal{N} = 2$  super-Ward identities; and in Whitham theory, they codify the existence of a  $\tau$ -structure for the underlying hierarchy.

94] for the  $SU(N)$  case, and in [15] for ADE types other than  $E_8$ , it will be proposed that the open and closed B-model theory on the relativistic Toda spectral curves  $\mathcal{S}_g$  with Dubrovin–Krichever data specified by (3.57) give in one go the Seiberg–Witten solution of the five-dimensional  $E_8$  gauge theory in a self-dual  $\Omega$ -background, the mirror theory of the A-model on  $(Y, L)$  and  $(\mathcal{X}, \mathcal{L})$ , and a large- $N$  dual of Chern–Simons theory on  $\Sigma$ .

For definiteness, let's put again ourselves at a generic moduli point  $(u, \aleph)$ . The first step to define a prepotential from the assignment (3.57) to  $\mathcal{S}_g$  is to consider periods of  $d\sigma = \log \mu d \log \lambda$  on  $\Lambda_{PT}$  [38, 76, 113]. At genus zero, define

$$\Pi_{A_i}(d\sigma) = \frac{1}{2\pi i} \alpha_i = \frac{1}{2\pi i} \oint_{A_i} d\sigma, \quad \Pi_{B_i}(d\sigma) = \frac{1}{2} \oint_{B_i} d\sigma, \quad (4.25)$$

for the set of  $(A_i, B_i)_{i=1}^8$  cycles generating the  $\mathcal{P}_g$ -invariant part of  $H_1(\Gamma_{u, \aleph}, \mathbb{Z})$ . I am first of all going to fix  $\Lambda_{PT}^I \triangleq \mathbb{Z} \langle \{A_i\}_i \rangle$ ; what this means is that, locally around  $a_i = \infty$ , the A-periods (4.25) will define a map

$$\begin{aligned} a_i : \mathcal{B}_g &\rightarrow \mathbb{C} \\ (u, \aleph) &\rightarrow \Pi_{A_i}(d\sigma). \end{aligned} \quad (4.26)$$

with the B-periods (4.25) being further subject to the *rigid special Kähler relations* [38, 76, 113]

$$\Pi_{B_i}(d\sigma) = \frac{\partial F^{\text{Toda}}}{\partial a_i} \quad (4.27)$$

for a locally defined analytic function  $F^{\text{Toda}}(a)$  in a punctured neighbourhood of  $a_i = \infty$ .

**Conjecture 4.3.** *We have*

$$F_0^{\text{Toda}} = F_0^{\text{SYM}} = F_0^Y \quad (4.28)$$

locally around  $a_i = \infty = t_i$ , under the identifications of (4.13), and after setting  $\aleph = R = e^{-t_B/4}$ . Furthermore, let  $\tilde{A}_i \triangleq -B_i$ ,  $\tilde{B}_i \triangleq A_i$  and define

$$\tilde{a}_i \triangleq \frac{1}{2\pi i} \Pi_{\tilde{A}_i}(d\sigma), \quad \frac{\partial \tilde{F}_0^{\text{Toda}}}{\partial a_i} \triangleq \frac{1}{2} \Pi_{\tilde{B}_i}(d\sigma) \quad (4.29)$$

Then there exist linear change-of-variables  $\tilde{a} = \mathcal{L}_1(\tau) = \mathcal{L}_2(\mathbf{t})$  such that

$$\tilde{F}_0^{\text{Toda}}(\tilde{a}) = F_0^{X_I}(\mathcal{L}_1^{-1}\tau) = F_0^{\text{CS}}(\mathcal{L}_2^{-1}\mathbf{t}). \quad (4.30)$$

For the reader familiar with Figure 1 in the  $SU(N)$  case, this all by and large expected provided we show that our choice of  $A$  and  $B$  cycles in (3.29)-(3.30) reflects the corresponding choice of SW cycles in the weakly coupled (electric) duality frame in the gauge theory, and of mirror B-model cycles for the smooth chamber in the stringy Kähler moduli space of  $Y$ : that would justify the first part of the claim, with the second following by composing with the  $S$ -duality transformation  $(A_i, B_i) \rightarrow (\tilde{A}_i, \tilde{B}_i)$  to the orbifold/Chern–Simons chamber. For the first bit, I re-introduce  $\Lambda_4$  everywhere on the gauge theory side by dimension counting and take the limit  $\Lambda_4 \rightarrow 0$  holding fixed  $a_i$  and  $R$ , which corresponds to switching off the non-perturbative part of (4.5). At the level

of the Toda chain variables this is  $\aleph \rightarrow 0$  with  $u_i$  kept fixed. Recall that the branch points  $b_i^\pm$  of  $\lambda : \Gamma_{u,\aleph}$  come in pairs related by

$$b_i^- = \frac{\aleph}{b_i^+}. \quad (4.31)$$

In particular, in the degeneration limit  $\aleph \rightarrow 0$ , where  $\Gamma_{u,0} \simeq \Gamma'_{u,0}$ , the branch points  $b_i^-$  in Figure 4 all collapse to zero, and therefore, the contours  $C_i^\alpha$  are given by the difference of the lifts to the sheet labelled by  $\alpha$  and  $\sigma_i(\alpha)$  of a simple loop around the origin in the  $\lambda$ -plane. In other words, and in terms of the Cartan torus element  $\exp(l)$  in Section 2.4, we find

$$\begin{aligned} \lim_{\aleph \rightarrow 0} \oint_{A_i} d\sigma &= \lim_{\aleph \rightarrow 0} \frac{1}{2q_{\mathfrak{g}}} \sum_{\alpha \in \Delta^*} \langle \alpha, \alpha_i \rangle \oint_{C_i^\alpha} \log \mu \frac{d\lambda}{\lambda}, \\ &= \frac{1}{2q_{\mathfrak{g}}} \sum_{\alpha \in \Delta^*} \langle \alpha, \alpha_i \rangle \oint_{\lambda=0} \lim_{\aleph \rightarrow 0} \frac{(\sigma_i(\alpha)(l) - \alpha(l))d\lambda}{\lambda}, \\ &= \frac{1}{2q_{\mathfrak{g}}} \sum_{\alpha \in \Delta^*} (\langle \alpha, \alpha_i \rangle)^2 \alpha_i(l)|_{\aleph=\lambda=0} = \alpha_i(l)|_{\aleph=\lambda=0}, \end{aligned} \quad (4.32)$$

where we have used (see [68, 82])

$$\frac{1}{2q_{\mathfrak{g}}} \sum_{\alpha \in \Delta^*} (\langle \alpha, \alpha_i \rangle)^2 = 1. \quad (4.33)$$

The r.h.s. of (4.32) is just the semi-classical Higgs vev  $(a_i)_{\Lambda_4=0}$  for the complexified scalar  $\phi = \varphi + i\vartheta$  [85]. This pins down  $A_i$  as the correct choice of an electric cycle for the  $i^{\text{th}}$   $U(1)$  factor in the IR theory, with logarithmic monodromy around the weakly coupled/maximally unipotent monodromy point  $a_i = \infty$ , and  $B_i$  (up to monodromy) as their doubly-logarithmic counterpart.

The identifications in Conjecture 4.3 pave the way to an extension to the higher genus theory upon appealing to the remodelled-B-model recursive scheme of [19]. Let  $\Psi$  be a sub-lattice of  $H_1(\Gamma_{u,\aleph}, \mathbb{Z})$  containing  $\{A_i\}_i$  which is maximally isotropic w.r.t. the intersection pairing. Denote  $B^{\text{Toda}} \in H^0(\text{Sym}^2 \Gamma_{u,\aleph} \setminus \text{diagonal}, \mathcal{K}_{\Gamma_{u,\aleph}}^{\boxtimes 2})$  the unique (up to scale) meromorphic bidifferential on  $\Gamma_{u,\aleph}$  with double pole on the diagonal, vanishing residues thereon, and vanishing periods on all cycles  $C \in \Psi$ ; we fix the scaling ambiguity by imposing the coefficient of the double pole to be 1 in a local coordinate patch given by the  $\lambda$  projection. I further write

$$B^{\text{Toda}}(p, q) \triangleq \mathcal{P}_{\mathfrak{g}}^* E_{\Psi}(p, q) \quad (4.34)$$

whose definition, by the nature of  $(\mathcal{P}_{\mathfrak{g}})_*$  as a projection on  $\text{PT}(\Gamma_{u,\aleph})$ , is independent of the choice of the particular Lagrangian extension  $\Psi \supset \Lambda_{\text{PT}}$ . Further write, for  $\lambda(q)$  locally near  $b_i^\pm$ ,

$$\mathcal{K}_{0,2}^{\text{Toda}}(p, q) \triangleq \frac{1}{2} \frac{\int_{\bar{q}}^q B^{\text{Toda}}(p, q)}{\log \mu(p) - \log \mu(q)}, \quad (4.35)$$

where locally around each ramification point  $\lambda^{-1}(b_i^\pm)$ ,  $\bar{q}$  is the local deck transformation  $\mu(q) = \alpha \cdot l \rightarrow \alpha \cdot l + \langle \alpha_i, \alpha \rangle \alpha_i \cdot l$ . We call  $B^{\text{Toda}}$  and  $\mathcal{K}^{\text{Toda}}$  respectively the *symmetrised Bergmann kernel and recursion kernel for the DK data* (3.57).

**Remark 4.4.** In terms of the Dubrovin–Krichever data (3.57), notice that the family of differentials  $B^{\text{Toda}}$  is determined by  $\mathcal{S}_{\mathfrak{g}}$  and  $\Lambda_{\text{PT}}^L \subset \Lambda_{\text{PT}}$  alone – that is, by the curves themselves, the invariant



periods  $\Lambda_{\text{PT}}$ , and the specific marking of the “A” cycles in  $\Lambda_{\text{PT}}^L$  to be those with vanishing periods for  $B^{\text{Toda}}$ . On the other hand,  $\mathcal{K}^{\text{Toda}}$  feels on top of that the specific choice of relative differential  $\mathcal{M} \leftrightarrow d \ln \mu$  in (3.57), which is reflected by the presence of the logarithm of the universal map  $\mu$  to  $\mathbb{P}^1$  of (2.23) in the denominator of (4.35). The further choice of  $\mathcal{L} \leftrightarrow d \ln \lambda$  will play a role momentarily in the definition of the topological recursion.

**Definition 4.1.** For  $g, h \in \mathbb{N}$ ,  $2g - 2 + h > 0$ , the Chekhov–Eynard–Orantin generating functions [32, 49] for the Toda spectral curve  $\mathcal{S}_{\mathfrak{g}}$  with DK data (3.57) are recursively defined as

$$W_{0,2}^{\text{Toda}}(p, q) \triangleq \frac{B^{\text{Toda}}(p, q)}{dpdq} - \frac{\lambda'(p)\lambda'(q)}{(\lambda(p) - \lambda(q))^2}, \quad (4.36)$$

$$\begin{aligned} W_{g,h+1}^{\text{Toda}}(p_0, p_1, \dots, p_h) &= \sum_{b_i^\pm} \text{Res}_{\lambda(p)=b_i^\pm} \mathcal{K}_{0,2}^{\text{Toda}}(p_0, p) \left( W_{g-1,h+2}^{\text{Toda}}(p, \bar{p}, p_1, \dots, p_h) \right. \\ &\quad \left. + \sum_{l=0}^g \sum_{J \subset H} W_{g-l,|J|+1}^{\text{Toda}}(p, p_J) W_{(l),|H|-|J|+1}^{\text{Toda}}(\bar{p}, p_{H \setminus J}) \right). \end{aligned} \quad (4.37)$$

where  $I \cup J = \{p_1, \dots, p_h\}$ ,  $I \cap J = \emptyset$ , and  $\sum'$  denotes omission of the terms  $(h, I) = (0, \emptyset)$  and  $(g, J)$ . Furthermore, for  $g > 0$  we define the higher genus free energies

$$\begin{aligned} F_1^{\text{Toda}} &\triangleq \frac{1}{2} \left[ -\log \tau_{\text{KK}}(\Xi_{\mathfrak{g},\text{red}}) + \frac{1}{12} \log \det \Omega \right], \\ F_g^{\text{Toda}} &\triangleq \frac{1}{2-2g} \sum_{b_i^\pm} \text{Res}_{\lambda(p)=b_i^\pm} \sigma(p) W_{g,1}^{\text{Toda}}, \end{aligned} \quad (4.38)$$

where  $\tau_{\text{KK}}$  is the Kokotov–Korotkin  $\tau$ -function of the branched cover  $\Xi_{\mathfrak{g},\text{red}}$  [71],  $\Omega$  is the Jacobian matrix of angular frequencies (3.55), and  $\sigma$  is the Poincaré action (3.50).

(4.37) is the celebrated topological recursion of [49], which inductively defines generating functions  $\{W_{g,h}^{\text{Toda}}\}_{g,h}$  purely in terms of the Dubrovin–Krichever data (3.57). The root motivation of Definition 4.1, which arose in the formal study of random matrix models, is that the generating functions thus constructed provide a solution of Virasoro constraints whenever the spectral curve setup arises as the genus zero solution of the planar loop equation for the 1-point function; it was put forward in [19], and further elaborated upon in [36], that the very same recursion solves  $\mathcal{W}$ -algebra constraints for the the open/closed Kodaira–Spencer theory of gravity/holomorphic Chern–Simons theory on local Calabi–Yau threefolds of the form

$$\nu \xi = \Phi(\lambda, \mu),$$

with B-branes wrapping either of the lines  $\nu = 0$  or  $\xi = 0$ . We follow the same path of [15, 19] by setting  $\Phi = \Xi_{\mathfrak{g},\text{red}}$ , taking (4.37)-(4.38) as the *definition* of the higher genus/open string completion of the Toda prepotential (4.27), and submit the following

**Conjecture 4.5.** We have

$$F_g^{\text{Toda}} = F_g^{\text{SYM}} = F_g^{\text{GW}} \quad (4.39)$$

locally around  $a_i = \infty = t_i$  and under the same identifications of Conjecture 4.3; here we defined the gravitational correction

$$F_g^{\text{SYM}} = [\epsilon^{2g}] F^{\text{SYM}}(\epsilon, -\epsilon), \quad (4.40)$$

as the  $\mathcal{O}((\epsilon_1 = -\epsilon_2)^{2g})$  coefficient in an expansion of the  $\Omega$  background around the flat space limit. Furthermore, denote by  $(\widetilde{W}_{g,h}, \widetilde{F}_g)$  the Toda/CEO generating functions obtained upon applying (4.37)-(4.38) to the Toda spectral curves with zero  $\tilde{A}_i$ -period normalisation for (4.34) and (4.35). Then, with the same notation as in Conjecture 4.3, we have that

$$\begin{aligned} \widetilde{F}_g^{\text{Toda}}(\tilde{a}) &= F_g^{\text{GW}}(\mathcal{X}; \mathcal{L}_1^{-1}t) = F_g^{\text{CS}}(\mathcal{L}_2^{-1}\mathfrak{t}), \\ \widetilde{W}_{g,h}^{\text{Toda}}(\tilde{a}, \lambda_1, \dots, \lambda_h) &= W_{g,h}^{\text{GW}}(\mathcal{X}, \mathcal{L}; \mathcal{L}_1^{-1}t, \lambda_1, \dots, \lambda_h) = W_{g,h}^{\text{CS}}(\mathcal{L}_2^{-1}\mathfrak{t}, \lambda_1, \dots, \lambda_h), \end{aligned} \quad (4.41)$$

where we have identified  $\lambda_i = \lambda(p_i)$ .

As in Claim 4.2, I will refer to the equality of Toda and Chern–Simons generating functions as the *strong/weak B-model Gopakumar–Vafa correspondence* for  $\Sigma$ , according to whether the restriction to the trivial connection  $\mathfrak{t}_i = t_0 \delta_{i0}$  is taken or not.

**Remark 4.6.** The two claims above are slightly asymmetrical between  $Y$  and  $\mathcal{X}$  in that they do not include the open string sector in the latter. On the GW side, exactly by the same token as for the orbifold chamber and in keeping with the toric cases [19], the same type of statement should hold, namely that the topological recursion potentials  $W_{g,h}^{\text{Toda}}$  equate to  $W_{g,h}^{Y,L}$ ; for the gauge theory, the extension one is after requires the introduction to surface defects in the gauge theory [4, 72]. I do not further discuss these here, and refer the reader to [15, 72] for more details.

**4.2. On the Gopakumar–Vafa correspondence for the Poincaré sphere.** After much conjecturing I will prove at least one of the correspondences of the previous section. In the next section, I will show that the weak version of the B-model Gopakumar–Vafa correspondence holds for all genera, colourings, and degrees of expansion in the 't Hooft parameter.

4.2.1. *LMO invariants and matrix models.* I will set

$$\begin{aligned} F_g^{\text{LMO}}(\Sigma; \tau) &= F_g^{\text{CS}}(\Sigma; \mathfrak{t}_i = \tau \delta_{i0}, x) \\ W_{g,h}^{\text{LMO}}(\Sigma, \mathcal{K}; \tau, \lambda_1, \dots, \lambda_h) &= W_{g,h}^{\text{CS}}(\Sigma, \mathcal{K}; \mathfrak{t}_i = \tau \delta_{i0}, \lambda_1, \dots, \lambda_h) \end{aligned} \quad (4.42)$$

to designate the LMO contribution ( $f_i = 0$ ) to the Chern–Simons partition function (4.20) of  $\Sigma$ , and quantum invariants of the fibre knot  $\mathcal{K}$  respectively; similarly I will use  $Z^{\text{LMO}}$  for the restricted partition function. The first step to relate the latter to spectral curves, as in [3], is to re-write (4.20) as a matrix model as first pointed out in [83] (see also [8, 13]): this follows from taking a Gaussian integral representation of the exponential in (4.20) and using Weyl’s denominator formula. The upshot [83] is that the restriction of (4.20) to its summand at  $f_i = 0$  is the total mass of an eigenvalue model

$$Z_{\text{LMO}}(\Sigma, k, N) = \mathcal{N}(\Sigma) \mathbb{E}_{d\mu}(1) = \mathcal{N}(\Sigma) \int_{\mathbb{R}^N} d\mu, \quad (4.43)$$

with measure given by a Gaussian 1-body potential, and a trigonometric Coulomb 2-body interaction,

$$d\mu \triangleq d^N \kappa \prod_{i < j} \frac{\prod_{l=1}^3 \sinh \frac{\kappa_i - \kappa_j \phi}{s(l)}}{\sinh(\kappa_i - \kappa_j)} e^{-\frac{N\kappa \cdot \kappa}{2\tau}}. \quad (4.44)$$

with  $\tau = g_{\text{YM}} N$ ,  $g_{\text{YM}} = 2\pi i(k + N)^{-1}$ . The integral of (4.43) is by *fiat* a convergent matrix (eigenvalue) model, and it takes the form of a perturbation of the ordinary (gauged) Gaussian matrix model by double-trace insertions, owing to the sinh-type 2-body interaction of the eigenvalues (see [3, Sec. 6]). The Chern–Simons knot invariants (4.19) are similarly computed as

$$W_h^{\text{LMO}}(\Sigma, \mathcal{K}, k, N, \lambda_1, \dots, \lambda_h) = \mathbb{E}_{d\mu}^{\text{conn}} \left( \prod_{i=1}^h \sum_{j=1}^N \frac{x_i}{x_i - e^{\kappa_i}} \right) \quad (4.45)$$

where the coefficients of degree  $k_i$  in  $\lambda_i$ , for  $k_i = (30/s(l))j_i$  and  $j_i \in \mathbb{Z}$ , gives the perturbative quantum invariant (in colouring given by the  $j^{\text{th}}$  connected power sum) of the knot going along the fiber of order  $s(l)$  in  $s$ ,  $l = 1, 2, 3$ .

This type of eigenvalue measures falls squarely under the class of  $N$ -dimensional eigenvalue models considered in [18], for which the authors rigorously prove that a topological expansion of the form (4.18) and (4.19) applies to the asymptotic expansion of (4.43) and (4.45) respectively. What is more, in [16] the authors prove that the topological recursion (4.37)-(4.38) with initial data for the induction given by

$$W_{0,1}^{\text{LMO}}(x) \triangleq \lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E}_{d\mu} \left( \sum_{i=1}^N \frac{x}{x - e^{\kappa_i}} \right), \quad (4.46)$$

$$\begin{aligned} W_{0,2}^{\text{LMO}}(x_1, x_2) &\triangleq \lim_{N \rightarrow \infty} \left[ \mathbb{E}_{d\mu} \left( \sum_{i_1, i_2=1}^N \frac{x_1 x_2}{(x_1 - e^{\kappa_{i_1}})(x_2 - e^{\kappa_{i_2}})} \right) \right. \\ &\quad \left. - \mathbb{E}_{d\mu} \left( \sum_{i_1=1}^N \frac{x_1}{x_1 - e^{\kappa_{i_1}}} \right) \mathbb{E}_{d\mu} \left( \sum_{i_2=1}^N \frac{x_2}{x_2 - e^{\kappa_{i_2}}} \right) \right]. \end{aligned} \quad (4.47)$$

computes the all-order, higher genus, all-colourings quantum invariants of fibre knots  $\mathcal{K}$ . As is typical in most settings where the topological recursion applies, the planar two point function (4.47) can be written as a section  $W_{0,2}^{\text{CS}} \in K_{\Gamma_{\text{LMO}}^{\boxtimes 2}}(\text{Sym}^2 \Gamma_{\tau}^{\text{LMO}} \setminus \Delta(\Gamma_{\tau}^{\text{LMO}}))$  on the double symmetric product (minus the diagonal) of the smooth completion  $\Gamma_{\tau}^{\text{LMO}}$  of the algebraic<sup>18</sup> plane curve  $y = W_{0,1}^{\text{LMO}}(x)$ : the *LMO spectral curve*. A strategy to determine the family of Riemann surfaces  $\Gamma_{\tau}^{\text{LMO}}$  as the base parameter  $\tau$  is varied was put forward in the extensive analysis of Chern–Simons-type matrix models of [17], and is summarised in the next Section.

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<sup>18</sup>From the discussion above this does not need to be more than just analytic; it turns however that  $e^y = e^{W_{0,1}^{\text{CS}}(x)}$  is algebraic, as follows from the proof of [17, Prop. 1.1], and as we will review in Section 4.2.2.

4.2.2. *The planar solution, after Borot–Eynard.* The LMO spectral curve can be expressed as the solution of the singular integral equation describing the equilibrium density for the eigenvalues in (4.43) [17]. Introduce the density distribution

$$\varrho(x) \triangleq \lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E}_{d\mu} \left( \sum_{i=1}^N \delta(x - e^{\kappa_i}) \right). \quad (4.48)$$

As in the case of the Wigner distribution, Borot–Eynard in [17] prove that, for  $\tau \in \mathbb{R}^+$ , the large  $N$  eigenvalue density  $\varrho \in C_c^0(\mathbb{R})$  is a continuous function with compact support  $\mathcal{C}_\varrho = [-b(\tau), b(\tau)]$  given by a single segment, symmetric around the origin, at whose ends  $\pm b(\tau)$   $\varrho$  has square-root vanishing,  $\varrho = \mathcal{O}(\sqrt{x \pm b(\tau)})$ . Furthermore, by (4.43),  $\varrho$  satisfies the saddle-point equation

$$\frac{\kappa}{\tau} = \sum_{l=1}^3 \text{pv} \int_{\mathbb{R}} \varrho(\kappa') \left[ \coth \frac{\kappa - \kappa'}{2s(l)} - \coth \frac{\kappa - \kappa'}{2} \right]. \quad (4.49)$$

By the Plémely lemma, this is equivalent to a Riemann–Hilbert problem for the planar 1-point function (4.46),

$$W_{0,1}^{\text{LMO}}(x + i0) + W_{0,1}^{\text{LMO}}(x - i0) - \sum_{\ell=1}^s W_{0,1}^{\text{LMO}}(\zeta^\ell x) + \sum_{m=1}^3 \sum_{\ell_m=1}^{s/s(m)-1} W_{0,1}^{\text{LMO}}(\zeta_{s/s(m)}^{\ell_m} x) = (s^2/\kappa) \ln x + s \quad (4.50)$$

with  $\zeta_k$  a primitive  $k$ -th root of unity; note that  $W_{0,1}^{\text{LMO}}(x)$  has a cut for  $x \in \mathcal{C}_\varrho \triangleq \text{supp} \varrho$ , with jump equal to  $2\pi i \varrho$ . Following [27], and setting

$$c \triangleq \exp(\tau/2s). \quad (4.51)$$

the exponentiated resolvent

$$\mathcal{Y}(x) \triangleq -cx \exp \left( \frac{\tau W_{0,1}^{\text{LMO}}(x)}{s^2} \right), \quad (4.52)$$

is holomorphic on  $\mathbb{C} \setminus \mathcal{C}_\varrho$ , it asymptotes to

$$\begin{aligned} \mathcal{Y}(x) &\sim -cx, & x = 0, \\ \mathcal{Y}(x) &\sim -c^{-1}x, & x = \infty, \end{aligned} \quad (4.53)$$

and further satisfies

$$\mathcal{Y}(x + i0)\mathcal{Y}(x - i0) \left[ \prod_{\ell=1}^{s-1} \mathcal{Y}(\zeta_s^\ell x) \right]^{-1} \times \prod_{m=1}^3 \left[ \prod_{\ell_m=1}^{s/s(m)-1} \mathcal{Y}(\zeta_{s/s(m)}^{\ell_m} x) \right] = 1. \quad (4.54)$$

Furthermore, the  $\mathbb{Z}_2$ -symmetry  $\{\kappa_i \rightarrow -\kappa_i\}$  of (4.43) entails that

$$\mathcal{Y}(x)\mathcal{Y}(1/x) = 1. \quad (4.55)$$

Every time we cross the cut  $\mathcal{C}_\varrho$ , the exponentiated resolvent is subject to the monodromy transformation (4.54). An approach to solve the monodromy problem (4.54) together the asymptotic

conditions at 0 and  $\infty$  was systematically developed in [17] following in the direction of [27], and it goes as follows. Fix  $v \in \mathbb{Z}^s$  and let

$$\mathcal{Y}_v(x) \triangleq \prod_{j=0}^{s-1} [\mathcal{Y}(\zeta_s^j x)]^{v_j}, \quad (4.56)$$

Here  $\mathcal{Y}_v(x)$  inherits a cut on the rotation  $\mathcal{C}_\varrho^{(j)} = \zeta_s^{-j} \mathcal{C}_\varrho$  for all  $j$  such that  $v_j \neq 0$ ; in particular, the jump on each of these cuts returns the spectral density  $\varrho$ , and thus  $W_{0,1}^{\text{LMO}}(x)$ .

By definition,  $\mathcal{Y}_v(x)$  is a single-valued function on the universal cover  $\widehat{\Gamma}$  of  $\mathbb{P}^1 \setminus \{\zeta_s^j b^\pm(\tau)\}_{j=1}^s$ . We want to ask whether there is a clever choice of  $v$  such that this factors through a *finite degree* covering map  $\Gamma_{\text{LMO}} \rightarrow \mathbb{P}^1$  branched at  $\{\zeta_s^j b^\pm(\tau)\}_{j=1}^s$  such that  $\mathcal{Y}_v(x)$  is single-valued on  $\Gamma_{\text{LMO}}$ . This was answered in the affirmative in [17], as follows. A direct consequence of (4.54), as in the study of the torus knots matrix model of [27], is that the change-of-sheet transition given by crossing the cut  $\mathcal{C}_\varrho^{(j)}$  results in a lattice automorphism  $T_j \in \text{GL}(s, \mathbb{Z})$  such that

$$\mathcal{Y}_v(x + i0) = \mathcal{Y}_{T_j(v)}(x - i0). \quad (4.57)$$

The monodromy group of the local system determined by  $\mathcal{Y}_v(x)$  is then (a subgroup of) the group of lattice transformations  $T_j$  for  $j = 0, \dots, (s-1)$ . This is beautifully characterised by the following

**Proposition 4.7** ([17]). *There is a  $\mathbb{Z}$ -linear monomorphism*

$$\iota : \Lambda_r \rightarrow \mathbb{Z}^s \quad (4.58)$$

*embedding  $\Lambda_r(\mathfrak{e}_8)$  as a rank 8 sublattice of  $\mathbb{Z}^s$ . Its image  $\iota(\Lambda_r)$  is invariant under the  $\{T_j\}_j$ -action, and the pullback of the monodromy (4.54) to  $\Lambda_r$  is isomorphic to the Coxeter action of  $\mathcal{W} = \text{Weyl}(\mathfrak{e}_8)$ .*

By Proposition 4.7, picking  $v$  to lie in  $\iota(\Lambda_r)$  does exactly the trick of returning a finite degree covering of the complex line by the affine curve

$$y : \mathbb{V} \left[ \prod_{\varpi \in \iota(\mathcal{W})v} (y - \mathcal{Y}_\varpi(x)) \right] \rightarrow \mathbb{A}^1, \quad (4.59)$$

with sheets labelled by elements of a  $\mathcal{W}$ -orbit on  $\Lambda_r$ . Our freedom in the choice of the initial element  $v$  in the orbit is given by the number of semi-simple, 7-vertex Dynkin subdiagrams of the black part of Figure 2 [56], which classify the stabilisers of any given element in the orbit; in other words, by the choice of a fundamental weight  $\omega_i$  of  $\mathfrak{g}$ . The natural choice here is to pick the minimal orbit, corresponding to the largest stabilising group, by choosing to delete the node  $\alpha_7$  in Figure 2, so that  $v = \omega_7 = \alpha_0$ : in this case, obviously,  $\mathcal{W}v = \Delta^*$ , the set of non-zero roots. I refer the reader to Appendix B.0.1 for further details on the orbit, and give the following

**Definition 4.2.** *We call the normalisation of the closure in  $\mathbb{C}\mathbb{P}^2$  of (4.59) with  $v = \iota(\alpha_0)$  the LMO curve of type  $E_8$ .*

This places us in the same setup of the Toda spectral curves of Sections 2.4, 2.4.1 and 2.5 (see in particular (2.22) and Definition 2.1), by realising the LMO curve as a curve of eigenvalues for a  $\mathcal{G}$ -valued Lax operator with rational spectral parameter; at this stage, of course, it is still unclear whether this rational dependence has anything to do with that of Section 2.2. The upshot of the discussion above is that there exists a degree-240, monic polynomial  $\mathcal{P}_{\alpha_0} \in \mathbb{C}[x, y]$  with  $y$ -roots given exactly by the branches of the  $\mathbb{Z}_s$ -symmetrised, exponentiated resolvent  $\mathcal{Y}(x)$ :

$$\mathcal{P}_{\alpha_0}(x, y) = \prod_{\alpha \in \Delta^*}^{240} (y - \mathcal{Y}_{\varpi_\alpha}(x)). \quad (4.60)$$

where we wrote  $\varpi_\alpha \triangleq \iota(\alpha)$ . As we point out in Appendix B.0.1, the rescaling  $x \rightarrow \zeta_s^{-1}x$  corresponds to an action on  $\mathbb{Z}^s$  given by the image of the action of the Coxeter element on  $\Lambda_r$ , under which the orbit  $\Delta^*$  is obviously invariant. The resulting  $\mathbb{Z}_s$ -symmetry implies that  $\mathcal{P}_{\alpha_0}(x, y)$  is in fact a polynomial in  $\lambda = x^s$ , and we define

$$\Xi^{\text{LMO}}(\lambda, \mu) \triangleq \mathcal{P}_{\alpha_0}(\lambda^{1/s}, \mu) \in \mathbb{C}[\lambda, \mu]. \quad (4.61)$$

Vanishing of  $\Xi^{\text{LMO}}$  defines a family  $\pi : \mathcal{S}_{\text{LMO}} \rightarrow \mathcal{B}_{\text{LMO}} \simeq \mathbb{A}^1$  algebraically varying over a one-dimensional base  $\mathcal{B}_{\text{LMO}}$  parametrised by the 't Hooft parameter  $\tau$ ; the same picture of (2.23) then holds over this smaller dimensional base.

**4.2.3. Hunting down the Toda curves.** We are now ready to show the weak B-model Gopakumar–Vafa correspondence, Conjecture 4.5. This will follow from establishing that the LMO spectral curves are a subfamily of Toda curves with canonical Dubrovin–Krichever data matching with (3.57).

**Theorem 4.8.** *There exists an embedding*

$$\begin{aligned} \mathcal{B}_{\text{LMO}} &\hookrightarrow \mathcal{B}_{\mathfrak{g}} \\ \tau &\longrightarrow (u(c), \aleph(c)), \end{aligned} \quad (4.62)$$

such that

$$\mathcal{S}_{\text{LMO}} = \mathcal{S}_{\mathfrak{g}} \times_{\mathcal{B}_{\mathfrak{g}}} \mathcal{B}_{\text{LMO}}. \quad (4.63)$$

Explicitly, this is realised by the existence of algebraic maps  $u_i = u_i(c)$ ,  $\aleph = c^{-q_{\mathfrak{g}}}$  such that

$$\Xi_{\text{LMO}} = \Xi_{\mathfrak{g}, \text{red}} \Big|_{u=u(c), \aleph=c^{-q_{\mathfrak{g}}}}. \quad (4.64)$$

Furthermore, the full  $1/N$  asymptotic expansion of (4.45) is computed by the topological recursion (4.37)-(4.38) with induction data (4.46)-(4.47), and the  $\mathcal{O}(\prod_i x_i^{k_i})$  coefficients with  $k_i = (\mathfrak{s}/\mathfrak{s}(m))j_i$ ,  $j_i \in \mathbb{Z}$ ,  $m = 1, 2, 3$ , return the  $1/N$  expansion of the perturbative quantum invariants of the knot  $\mathcal{K}_m$  going along the singular fibre of order  $\mathfrak{s}(m)$  with colouring given by the virtual connected power sum representation specified by  $\{j_i\}$ .

*Proof.* The statement of the first part of the theorem condenses what were called “Step A” and “Step B” in the construction of LMO spectral curves that was offered in our previous paper [15],

where we stated that Step B could not be completed due to its computational complexity. I am going to show how the stumbling blocks we found there can be overcome here.

Let me first recall the strategy of [15]. As in [27], the first thing we do is to use the asymptotic conditions (4.53) for the un-symmetrised resolvent on the physical sheet (the eigenvalue plane), to read off the asymptotics of the symmetrised resolvent  $\mathcal{Y}_{\iota(\alpha)}$  on the sheet labelled by  $\alpha$ . Let  $\varpi = (\varpi_j)_j = \iota(\alpha)$  as displayed in Table 5, and further write

$$n_0(\varpi) \triangleq \sum_{j=1}^s \varpi_j, \quad n_1(v) = \sum_{j=1}^{s-1} j \varpi_j. \quad (4.65)$$

Then, from (4.53), we have

$$x \rightarrow 0, \quad \mathcal{Y}_{\varpi}(x) \sim (-cx)^{n_0(\varpi)} \zeta_s^{n_1(\varpi)}, \quad (4.66)$$

$$x \rightarrow \infty, \quad \mathcal{Y}_{\varpi}(x) \sim (-x/c)^{n_0(\varpi)} \zeta_s^{n_1(\varpi)}. \quad (4.67)$$

which in one shot gives both the Puiseux slopes of the Newton polygon of  $\mathcal{P}_{\alpha_0}$  as  $(\pm 1, n_0(\varpi))$ , and the coefficients of its boundary lattice points up to scale; in view of the comparison with  $\Xi_{\mathfrak{g}, \text{red}}$  we set the normalisation for the latter by fixing the coefficient of  $y^0$  to be equal to one. Taking into account the symmetries of  $\mathcal{P}_{\alpha_0}$  and plugging in the data of Table 5 on the minimal orbit, this is seen to return exactly the Newton polygon and the boundary coefficients of  $\Xi_{\mathfrak{g}, \text{red}}$  (see Figure 3).

The remaining part is to prove the existence of the map  $u_i(c)$  such that all the interior coefficients match as well. As was done in [15], I set out to prove it by working out the constraints due to the global nature of  $\mathcal{Y}$  as a meromorphic function on  $\Gamma_{\tau}^{\text{LMO}}$ . Write

$$\frac{\tau}{s^2} W(x) = \sum_{k \geq 1} \mathfrak{m}_k x^{k+1}, \quad (4.68)$$

for the expansion of the 1-point function (4.46) in terms of the planar moments

$$\mathfrak{m}_k = \lim_{N \rightarrow \infty} \mathbb{E}_{\text{d}\mu} \left( \sum_{i=1}^N e^{k\lambda_i} \right). \quad (4.69)$$

Then, by (4.52) and (4.54), we have that

$$\mathcal{Y}_{\varpi_{\alpha}}(x) = (-cx)^{n_0(\varpi_{\alpha})} s_a^{n_1(\varpi_{\alpha})} \exp \left[ \sum_{k > 0} \mathfrak{m}_{k-1} (\widehat{\varpi}_{\alpha})_k \text{ mod } s x^k \right], \quad (4.70)$$

where, as in [15], we wrote

$$(\widehat{\varpi}_{\alpha})_k \triangleq \sum_{j=0}^{s-1} \zeta_s^{jk} (\varpi_{\alpha})_j. \quad (4.71)$$

for the discrete Fourier transform of  $\varpi_{\alpha}$ . There are only eight Fourier modes that are non-vanishing: these are<sup>19</sup>

$$\exists \alpha | (\widehat{\varpi}_{\alpha})_k \neq 0 \Rightarrow k \in \{6, 10, 12, 15, 18, 20, 24, 30\} =: \mathfrak{k}. \quad (4.72)$$

<sup>19</sup>In [15], versions 1 and 2, these were erroneously listed as being in the complement of the r.h.s. of (4.72).

In particular, the only moments  $\mathbf{m}_k$  that may be found when Taylor-expanding  $\mathcal{Y}$  at one of the pre-images of  $x = 0$  satisfy

$$(k + 1) \bmod \mathfrak{s} \in \mathfrak{k}.$$

Consider now inserting the Taylor expansion (4.70) into the r.h.s. (4.60). Without any further constraints on the surviving momenta  $\mathbf{m}_k$ , we have no guarantee a priori that (4.70) is indeed (a) the Taylor expansion of a branch of an algebraic function and (b) that it gives the roots of a polynomial  $\mathcal{P}_{\alpha_0}$  as presented in (4.60). This means that if we expand up to power  $\mathcal{O}(x^{L+1})$  the product

$$\prod_{\alpha \in \Delta^*}^{240} (y - \mathcal{Y}_{\varpi_\alpha}(x)) = \sum_{i=1}^L B_i(y)x^i + \mathcal{O}(x^{L+1}) \quad (4.73)$$

then the polynomial  $\sum_{i=1}^L B_i(y)x^i$  may well have non-vanishing coefficients outside the Newton polygon of  $\mathcal{P}^{\text{LMO}}$ ; imposing that these are zero, and that those at the boundary return the slope coefficients of (4.66) and (4.67), gives a set of algebraic conditions on  $\{\mathbf{m}_k\}_{k \bmod \mathfrak{s} \in \mathfrak{k}}$ . In [15] we pointed out that the complexity of the calculations to solve for these conditions is unworkable if taken at face value, and refrained to pursue their solution; however I am going to show here that it is possible to carve out a sub-system of these equations which pins down uniquely an 8-parameter family of solutions, provides a solution to *all* these constraints for arbitrary  $L$ , and simultaneously leads exactly to the full family of Toda spectral curves (2.20)-(2.22). Take

$$L = 540 = \deg_x \Xi_{\mathfrak{g}, \text{red}}(\mu, x^{\mathfrak{q}}) = q_{\mathfrak{g}} \deg_\lambda \Xi_{\mathfrak{g}, \text{red}}(\mu, \lambda)$$

and expand (4.70) up to  $L$ . Plugging this into (4.60) and equating to zero leads to an algebraic equation for each coefficient of  $x^m y^n$  with  $(m, n)$  once we impose that  $\Xi_{\text{LMO}}(\lambda, \mu) = \lambda^{18} \Xi_{\text{LMO}}(\lambda^{-1}, \mu)$ . Consider now the subsystem of equations given by  $(m, n)$  in the region

$$\mathfrak{v} \triangleq \{(m, n) \in \mathbb{Z}^2 \mid 1 \leq n \leq 10, 1 \leq m \leq [L - 6n, \mathfrak{s}]\} \quad (4.74)$$

depicted in Section 4.2.3: imposing that  $B_n(y) = B_{L-n}(y)$  for the list of exponents in  $\mathfrak{v}$  gives an algebraic system of 165 equations for the 144 moments  $\mathbf{m}_k$ ,  $L > k \in \mathfrak{k}$ . Recall that the planar moments  $\mathbf{m}_k(c)$  are analytic in  $c$  around  $c = 1$  [18, 83] (i.e., small 't Hooft parameter), and they vanish in that limit

$$\mathbf{m}_k(c) = \sum_{n \geq 1} \mathbf{m}_k^{(n)}(c-1)^n. \quad (4.75)$$

Since the constraints on  $\mathbf{m}_k$  are analytic, we can solve them order by order in a Taylor expansion around  $c = 1$ . It is easy to figure out from (4.70) that, at  $\mathcal{O}(c-1)^n$ , we find a linear inhomogeneous system of the form

$$\mathcal{A} \mathbf{m}^{(n)} = \mathcal{B}_n \quad (4.76)$$

with  $\mathcal{A}$  independent of  $n$ ,  $\mathcal{B}_1 = 0$ , and  $\mathcal{B}_n$  a polynomial in  $\mathbf{m}_k^{(m)}$  for  $m < n$ . From the explicit form of  $\mathcal{A}$  we calculate that  $\dim \text{Ker } \mathcal{A} = 8$ ; the solutions of the system (4.76) must then take the form

$$\mathbf{m}_k^{(n)}(c) = \mathbf{m}_k \left( \{\mathbf{m}_l^{(m)}\}_{l \in \mathfrak{k}, m \leq n} \right) \quad (4.77)$$

and they are *uniquely determined* by solving recursively (4.76) order by order in  $n$ . A priori, the Taylor coefficients  $\mathbf{m}_{l+1 \in \mathfrak{k}}^{(m)}$  of the basic moments  $\mathbf{m}_5, \mathbf{m}_9, \mathbf{m}_{11}, \mathbf{m}_{14}, \mathbf{m}_{17}, \mathbf{m}_{19}, \mathbf{m}_{23}$  and  $\mathbf{m}_{29}$  are subject



to two further sets of closed conditions: the first stems from imposing that the inhomogeneous system (4.76) has solutions for  $n > 1$  (i.e.  $\dim \text{Ker}[\mathcal{A}|\mathcal{B}_n] = 9$ ), and the second from coefficients of the expansion (4.73) which are outside of the region  $\mathbf{v}$ : this may lead to the solution manifold of (4.76) having positive codimension in  $\mathbb{C}\langle \mathbf{m}_{l+1 \in \mathfrak{k}}^{(m)} \rangle$ . This is however not the case: let us *impose* that

$$[\lambda^9 \mu^i] \Xi_{\text{LMO}}(\lambda, \mu) = \mathbf{p}_i(u_1, \dots, u_8), \quad i = 1, \dots, 120. \quad (4.78)$$

where  $\mathbf{p}_i$  are the decomposition of the antisymmetric characters defined as in (2.21), (2.24) and Claim 2.3. (4.78) gives invertible polynomial maps  $\mathbf{m}_l \in c^{a_l} \mathbb{C}[u_1, \dots, u_8]$ ; for  $l+1 \in \mathfrak{k}$  we find

$$\begin{aligned} \mathbf{m}_5 &= \frac{c^6}{30}(u_7 + 2), & \mathbf{m}_9 &= \frac{c^{10}}{30}(u_1 + 2u_7 + 3), & \mathbf{m}_{11} &= \frac{c^{12}}{300}(-3u_7^2 + 38u_7 + 20u_1 + 10u_6 + 38), \\ \mathbf{m}_{14} &= \frac{c^{15}}{30}(5u_1 + 2u_6 + 8u_7 + u_8 + 7), \\ \mathbf{m}_{17} &= \frac{c^{18}}{2250} \left( 7u_7^3 + 117u_7^2 + 90u_1u_7 - 30u_6u_7 + 1284u_7 + 855u_1 + 75u_5 + 315u_6 \right. \\ &\quad \left. + 225u_8 + 1031 \right), \\ \mathbf{m}_{19} &= -\frac{c^{20}}{180} (u_1^2 - 26u_7u_1 - 114u_1 - 26u_7^2 - 6u_2 - 12u_5 - 48u_6 - 180u_7 - 30u_8 - 129), \\ \mathbf{m}_{23} &= \frac{c^{24}}{15000} \left( 60u_6u_7^2 - 11u_7^4 - 88u_7^3 - 80u_1u_7^2 + 12686u_7^2 + 12280u_1u_7 - 100u_5u_7 + 2240u_6u_7 \right. \\ &\quad \left. + 1200u_8u_7 + 45448u_7 + 1300u_1^2 - 50u_6^2 + 27880u_1 + 1000u_2 + 500u_4 + 2800u_5 + 800u_1u_6 \right. \\ &\quad \left. + 10740u_6 + 7400u_8 + 28374 \right), \\ \mathbf{m}_{29} &= \frac{c^{30}}{30} \left( 14u_7^3 + 16u_1u_7^2 + 233u_7^2 + 3u_1^2u_7 + 238u_1u_7 + 2u_2u_7 + 7u_5u_7 + 65u_6u_7 + 35u_8u_7 \right. \\ &\quad \left. + 499u_7 + 44u_1^2 + 3u_6^2 + 287u_1 + 9u_2 + u_3 + 2u_4 + 2u_1u_5 + 23u_5 + 29u_1u_6 + 108u_6 \right. \\ &\quad \left. + 11u_1u_8 + 3u_6u_8 + 65u_8 + 259 \right), \end{aligned} \quad (4.79)$$

which are easily seen to have polynomial inverses  $u_k \in \mathbb{C}[\{\mathbf{m}_l\}_l; c^{-1}]$ . As we know that  $\Xi_{\mathfrak{g}, \text{red}}$  and  $\Xi_{\text{LMO}}$  share the same Newton polygon with the same boundary coefficients by (2.31), (2.32) and Table 1, postulating (4.78) is the same as giving an eight parameter family of polynomial solutions of the constraints (4.78) which furthermore satisfies all our constraints  $B_n(y) = B_{L-n}(y)$  for all  $n \in [[0, 18]]$ . The first part of the claim, (4.64), follows then from the uniqueness of the solution of (4.78) above.

To prove the second part, we show that the two-point functions (4.36) and (4.47) coincide. We have just shown that  $\Xi_{\mathfrak{g}, \text{red}} = \Xi_{\text{LMO}}$  under the change-of-variables (4.79), and we know that the symmetrised Bergmann kernel of (4.34) is completely determined by  $\Gamma_{u, \mathfrak{N}}$  and the choice of  $\tilde{A}_i$  cycles in Conjecture 4.5: by its definition in (4.34), it is the unique bidifferential on  $\Gamma_{u, \mathfrak{N}}$  with vanishing  $\tilde{A}$ -periods and double poles with zero residues at the  $240 \times 240$  components of the image of the diagonal in  $\Gamma_{u, \mathfrak{N}}^{[2]}$  under the correspondence  $\mathcal{P}_{\mathfrak{g}} \times \mathcal{P}_{\mathfrak{g}}$ , the coefficients of the double poles

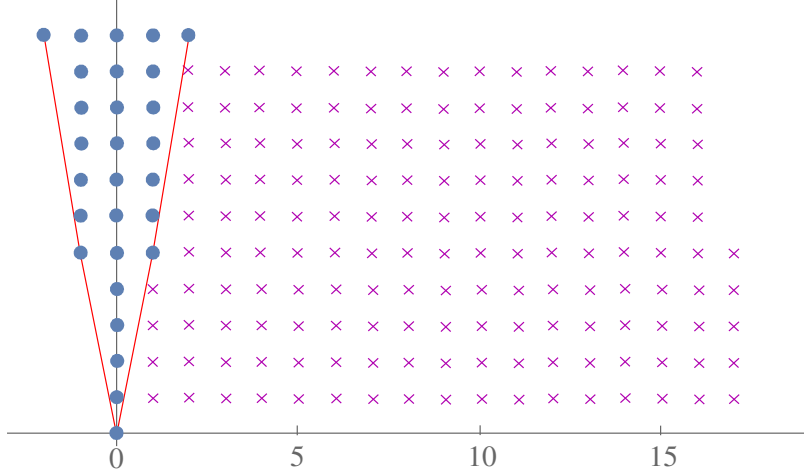


FIGURE 5. Points in  $\mathbb{Z}^2$  corresponding to the region  $\mathfrak{v}$  of non-vanishing coefficients of the Taylor expansion of  $\mathcal{Y}_{\varpi_\alpha}$  around zero; those indicated with a purple cross lie outside the Newton polygon of  $\Xi_{\text{LMO}}$ .

being specified by (3.17) in terms of a  $240 \times 240$  matrix of integers  $B_{ij}^{\text{Toda}}$ . As was proved in [17], the regularised two-point function,

$$\frac{B_{\text{LMO}}(p, q)}{dpdq} \triangleq W_{0,2}^{\text{LMO}}(p, q) + \frac{\lambda'(p)\lambda'(q)}{(\lambda(p) - \lambda(q))^2}, \quad (4.80)$$

has precisely the same properties: its matrix of singularities in [17, Sec. 6.6.3] can be shown to coincide with  $B_{ij}^{\text{Toda}}$  above, and the vanishing of the  $\tilde{A}$ -periods can be proven exactly as in the case of the ordinary Hermitian 1-matrix model to be a consequence of the planar loop equation for the 2-point function [50]; we conclude by uniqueness that

$$W_{0,2}^{\text{LMO}} = W_{0,2}^{\text{Toda}} \quad (4.81)$$

under the identification (4.79). This suffices to reach the conclusion of the second part of the claim: on the Toda side, the higher generating functions satisfy the topological recursion relations (4.37)-(4.38) by definition<sup>20</sup>. On the LMO side, the higher generating functions (4.43)-(4.45) fall within the class of integrals studied in [16], for which the authors prove that the Chekhov–Eynard–Orantin recursion determines the ribbon graph expansion in  $1/N$  via (4.37)-(4.38). Since both sides satisfy the recursion, the recursion kernels coincide from (4.81), and so do the initial data  $W_{0,1}$  and  $W_{0,2}$ , the statement of the theorem follows by induction on  $(g, h)$ .  $\square$

**Remark 4.9.** The proof of the existence of the embedding  $\mathcal{B}_{\text{LMO}} \hookrightarrow \mathcal{B}_{\mathfrak{g}}$  is only constructive up to the point where the fibres of the LMO family are shown to be determined by the planar moments,

<sup>20</sup>From a physics point of view, a first-principles heuristic argument to prove straight from the Kodaira–Spencer theory of gravity that these are genuine open/closed B-model amplitudes may be found in [36]. Also note that, as  $Y$  is non-toric, it momentarily lies outside the scope of existing proofs of the remodelling-the-B-model approach of [19], which rely either on the existence of a topological vertex formalism [51] or of a torus action on the target with zero-dimensional fixed loci [52]. I nonetheless believe these obstructions to be merely of a technical nature.

and in turn by the Toda Hamiltonians and Casimirs via (4.79). Providing explicit algebraic equations for the restriction  $u_i(c)$  to the codimension 8 locus  $\mathcal{B}_{\text{LMO}}$  is however a separate problem. It is worth pointing out that a direct way of calculating the restriction exists in perturbation theory around  $c = 1$  using the Gaussian perturbation theory methods of [83], which allow to determine  $\mathfrak{m}_k^{(n)}$  for arbitrary order in  $n$ ; it would be however desirable to present a closed-form algebraic solution by alternative methods, such as the one provided for spherical 3-manifolds of type D in [17] and type  $E_6$  in [15].

**4.3. Some degeneration limits.** To conclude this section, I will highlight three degeneration limits of the  $E_8$  relativistic Toda spectral curves which have a neat geometrical interpretation on each of the other three corners of Figure 1. This is summarised in the following table, and discussed in detail in the next three Sections.

Limit	$\aleph$	$u_1$	$u_2$	$u_3$	$u_4$	$u_5$	$u_6$	$u_7$	$u_8$
I	1	1	3	0	3	-3	3	-2	-2
II	0	$\gg 1$	$\gg 1$	$\gg 1$	$\gg 1$	$\gg 1$	$\gg 1$	$\gg 1$	$\gg 1$
III	$\mathcal{O}(\epsilon)$	$\dim \rho_{\omega_1}$ $+\mathcal{O}(\epsilon^2)$	$\dim \rho_{\omega_2}$ $+\mathcal{O}(\epsilon^2)$	$\dim \rho_{\omega_3}$ $+\mathcal{O}(\epsilon^2)$	$\dim \rho_{\omega_4}$ $+\mathcal{O}(\epsilon^2)$	$\dim \rho_{\omega_5}$ $+\mathcal{O}(\epsilon^2)$	$\dim \rho_{\omega_6}$ $+\mathcal{O}(\epsilon^2)$	$\dim \rho_{\omega_7}$ $+\mathcal{O}(\epsilon^2)$	$\dim \rho_{\omega_8}$ $+\mathcal{O}(\epsilon^2)$
Limit	5d gauge theory			GW target			CS		
I	maximal Argyres– –Douglas SCFT			$\tilde{\mathbb{I}}$ -orbifold of the singular conifold			zero 't Hooft limit		
II	perturbative limit			$\widehat{\mathbb{C}^2/\tilde{\mathbb{I}}} \times \mathbb{C}$			?		
III	4d limit			?			?		

TABLE 2. Notable degeneration limits of the Toda spectral curves.

**4.3.1. Limits I: the maximal Argyres–Douglas point and geometry.** I am gonna start with the case in which we set

$$\aleph = 1, \quad u_1 = 1, \quad u_2 = 3 = u_4 = u_6 = -u_5, \quad u_7 = u_8 = 2. \quad (4.82)$$

In terms of the LMO variables (4.79), this corresponds to  $c = 1$ ,  $\mathfrak{m}_i = 0$  for all  $i$ : this is the limit of zero 't Hooft coupling of the measure (4.44), in which the support of the eigenvalue density shrinks to a point. In this limit the  $\mathcal{Y}$ -branches of the LMO curve are given by the slope asymptotics of (4.66), which is in turn entirely encoded by the orbit data of Table 5. From (4.73) and Table 5, we get

$$\begin{aligned} \Xi_{\mathfrak{g},\text{red}}(\lambda, \mu) \Big|_{u(\mathfrak{m}=0), \aleph=1} &= (\mu + 1)^2 (\mu^2 + \mu + 1)^3 (\mu^4 + \mu^3 + \mu^2 + \mu + 1)^5 (\lambda + \mu^5) (\lambda \mu^5 + 1) \\ &\times (\lambda \mu^6 - 1) (\lambda - \mu^{10})^2 (\lambda \mu^{10} - 1)^2 (\lambda^2 - \mu^{15}) (\lambda + \mu^{15})^2 (\lambda \mu^{15} + 1)^2 \\ &\times (\lambda^2 \mu^{15} - 1) (\lambda - \mu^{30}) (\lambda \mu^{30} - 1) (\lambda - \mu^6). \end{aligned} \quad (4.83)$$

I'll call (4.83) the *super-singular limit* of the  $E_8$  Toda curves: in this limit,  $\text{Spec} \mathbb{C}[\lambda, \mu] / \langle \Xi_{\mathfrak{g},\text{red}} \rangle$  is a reducible, non-reduced scheme with the radicals of its 19 distinct non-reduced components given

by lines or plane cusps. In particular, denoting by  ${}^h\Xi$  the homogenisation of  $\Xi$ , the Picard group of the corresponding reduced scheme is trivial,

$$\mathrm{Pic}^{(0,\dots,0)}\left(\mathrm{Proj}\frac{\mathbb{C}[\lambda,\mu,\nu]}{\sqrt{\langle {}^h\Xi_{\mathfrak{g},\mathrm{red}}\rangle}}\right)\simeq 0, \quad (4.84)$$

the resolution of singularities  $\Gamma_{u(\mu=0),1}$  is a disjoint union of 19  $\mathbb{P}^1$ 's, and the whole Prym–Tyurin  $\mathrm{PT}(\Gamma_{u_{\mu=0},1})$  collapses to a point in the super-singular limit. This is more tangibly visualised by what happens to Figure 4 when we approach (4.82): since  $\aleph = 1$ , the branch points of the  $\lambda$ -projection satisfy  $b_i^+ b_i^- = 1$  from (4.31), and from Proposition 2.6 and the discussion that follows it, they correspond to  $\alpha_i(l) = 0$  for some simple root  $\alpha \in \Pi$ . The corresponding ramification points on the curve are then at  $\mu = \exp(\alpha_i(l)) = 1$ , and substituting into (4.83) we get

$$\Xi_{\mathfrak{g},\mathrm{red}}(\lambda, 1)\Big|_{u(\mathfrak{m}=0), \aleph=1} = 337500(\lambda - 1)^8(\lambda + 1)^{10}, \quad (4.85)$$

which means that the branch points collide together in four pairs with  $b_i^+ = b_- = 1$ , and five with  $b_i^+ = b_- = -1$ . It is immediate to see that the  $A/\tilde{B}$ -periods of  $d\sigma$  vanish in the limit (as the corresponding cycles shrink), as do the  $B/\tilde{A}$  periods upon performing the elementary cycle integration explicitly.

This degeneration limit should have a meaningful physical counterpart in the dynamics of the corresponding compactified 5d theory at this particular point on its Coulomb branch, and in particular it should correspond to the UV fixed point of [65, Section 7-8] (see also the recent works [66, 121]). I won't pursue the details here, but I will give some comments on the resulting A- and B-model geometries, and on the broad type of physics implications it might lead to. The first comment is on the geometrical character of (4.83): it is clearly expected that singularities in the Wilsonian 4d action should arise from vanishing cycles in the family of Seiberg–Witten curves [109], and in turn from the development of nodes as we approach its discriminant; and furthermore, more exotic phenomena related typically related to superconformal symmetry arise whenever these vanishing cycles have non-trivial intersection [6], leading to the appearance in the low energy spectrum of mutually non-local BPS solitons, and cusp-like singularities in the SW geometry (see [110] for a review). (4.83) provides a limiting version of this phenomenon whereby *all SW periods* vanish<sup>21</sup>. I will refer to (4.83) as the *maximal Argyres–Douglas point* of the  $E_3$  gauge theory, and as in the more classical cases of Argyres–Douglas theories, it presents several hallmarks of a theory at a superconformal fixed point. Besides the vanishing of the central charges of its BPS saturated states, we see that the way we reach the super-singular vacuum is akin to the mechanism of [65, 108] to engineer fixed points from five-dimensional gauge theories: since the engineering dimension of the five-dimensional gauge coupling  $1/g_{\mathrm{YM}}^{(5)}$  is that of mass, the theory is non-renormalisable and quantising it requires a cutoff; in the  $M$ -theoretic version [65, 78] of the geometric engineering of [69],

<sup>21</sup>There is no room for cusp-like singularities like this in the simpler setting of pure  $SU(2)$   $\mathcal{N} = 2$  pure Yang–Mills with SW curve  $y^2 = (x^2 - u)^2 - \Lambda_4^4$ , unless we put ourselves in the physically degenerate situation where we sit at the point of classically unbroken gauge symmetry  $u = 0$  and take the classical limit  $\Lambda_4 \rightarrow 0$ : the theory is then *classical* pure  $\mathcal{N} = 2$  gluodynamics, where we have essentially imposed by *fiat* to discard the quantum corrections that give a gapped vacuum and the breaking of superconformal symmetry.

this is naturally given in terms of the inverse of the radius of the eleventh-dimensional circle  $R$  in (4.13). Considerations about brane dynamics in [108] allow to conclude that the limit in which the bare gauge coupling diverges leads to a sensible quantum field theory at an RG fixed point with enhanced global symmetry; and notice that, under the identifications (4.13), setting the Casimir  $\aleph = 1$  amounts to taking precisely that limit. Indeed, upon reintroducing the four-dimensional scale  $\Lambda_4$  and identifying  $\Lambda_{UV} = 1/R$  as the cutoff scale, the second equality in (4.13) reads

$$\aleph = \frac{\Lambda_4}{\Lambda_{UV}} = e^{-t_B/4}. \quad (4.86)$$

Recall that  $\Lambda_4 = \Lambda_{UV} e^{-\frac{1}{g_{UV}}}$ , the RG invariant scale in four dimensions; the Seiberg limit  $g_{UV} \rightarrow \infty$  for the fixed point theory is given then by  $\aleph = 1$ , with the vanishing of the masses of BPS modes being realised by (4.82).

In light of Theorem 4.8, there is an A-model/Gromov–Witten take on this as well, which also allows us to reconnect the above to the work of [65, 121]. Let us put ourselves in the appropriate duality frame for (4.82)-(4.83), which corresponds to the choice of  $\tilde{A}_i$  as the cycles whose  $d\sigma$ -periods serve as flat coordinates around (4.82). By Claim 4.1 this corresponds to the maximally singular chamber in the extended Kähler moduli space of  $Y$  given by the orbifold GW theory of  $\mathcal{X}$ . Notice first that  $\aleph = 1$  corresponds to the shrinking limit of the Kähler volume of the base  $\mathbb{P}^1$ ,  $t_B = 0$ . Furthermore, as remarked in our earlier paper [15], the Bryan–Graber Crepant Resolution Conjecture [29] for the  $E_8$  singularity prescribes that the orbifold point in its stringy Kähler moduli space should be given by a vector  $OP \in \mathfrak{h}^* \simeq H_2\left(\widehat{\mathbb{C}^2/\tilde{\mathbb{I}}}, \mathbb{Z}\right)$  such that

$$\tau_i(OP) = \left( \frac{2\pi i \alpha_0}{|\tilde{\mathbb{I}}|} \right)_i = \frac{\pi i \mathfrak{d}_i}{15} \quad (4.87)$$

the second equality being taken w.r.t. the root basis for  $\mathfrak{h}^*$ . The values (4.82) for the Toda Hamiltonians correspond exactly to the values of the fundamental traces of a Cartan torus element corresponding to (4.87): (4.83) is then the spectral curve mirror of the A-model at the  $E_8$ -orbifold-of-the-conifold singularity, that is, the tip of the Kähler cone of  $Y$ .

The above, together with the constructions in Sections 2 and 3 provides some preliminary take, in this specific  $E_8$  case, to a few of the questions raised at the end of [121] regarding the Seiberg–Witten geometry, Coulomb branch and prepotential of 5d SCFT corresponding to Gorenstein singularities. A detailed study and the determination of some of the relevant quantities for the 5d SCFT (such as the superconformal index) is left for future study, and will be pursued elsewhere.

*4.3.2. Limits II: orbifold quantum cohomology of the  $E_8$  singularity.* Since the correspondence of the left vertical line of Figure 1 was shown to hold in the context of Theorem 4.8, I will offer here some calculations giving plausibility (other than the expectation from the underlying physics) for the lower horizontal and the diagonal arrow in the diagram. This will be done in a second interesting limit, given by taking  $\aleph \rightarrow 0$  while keeping all the other parameters finite (but possibly large). By (4.13) and Conjecture 4.3, this corresponds to a partial decompactification limit in which we send the Kähler parameter of the base  $\mathbb{P}^1$  in  $Y \rightarrow \mathbb{P}^1$  to infinity; the resulting A-model theory has thus

the resolution of the threefold transverse  $E_8$  singularity  $\widehat{\mathbb{C}^2/\mathbb{I}} \times \mathbb{C}$  as its target, or equivalently, by [29], the orbifold  $[\mathbb{C}^2/\mathbb{I} \times \mathbb{C}]$  upon analytic continuation in the Kähler parameters. Accordingly, on the gauge theory side, this corresponds to sending  $\Lambda_4 \rightarrow 0$  while keeping the classical order parameters  $u_i$  constant, and it singles out the perturbative part in the prepotential (4.5). And finally, in the Toda context, this type of limit was considered in [21, 74] for the non-relativistic type  $A$  chain, where it was shown to recover, after a suitable change-of-variables, the non-periodic Toda chain.

To bolster the claim, let me show that special geometry on the space of  $E_8$  Toda curves does indeed reproduce correctly the degree-zero part of the genus zero GW potential<sup>22</sup> of  $\widehat{\mathbb{C}^2/\mathbb{I}} \times \mathbb{C}$  in the sector where we have at least one insertion of  $\mathbf{1}_Y$ : by the string equation, this is the  $tt$ -metric on the Frobenius manifold  $QH(\widehat{\mathbb{C}^2/\mathbb{I}} \times \mathbb{C}) \simeq QH(\widehat{\mathbb{C}^2/\mathbb{I}})$  (see Section 5.1.1 for more details on this). As vector spaces, we have

$$QH(\widehat{\mathbb{C}^2/\mathbb{I}}) = H(\widehat{\mathbb{C}^2/\mathbb{I}}) = H^0(\widehat{\mathbb{C}^2/\mathbb{I}}) \oplus H^2(\widehat{\mathbb{C}^2/\mathbb{I}}) \simeq \mathbb{C} \oplus \mathfrak{h}.$$

Let us use linear coordinates  $\{l_i\}_{i=0}^8$  for the decomposition in the last two equalities, where we write  $H(\widehat{\mathbb{C}^2/\mathbb{I}}) \ni v = l_0 \mathbf{1}_Y \oplus l_i [E_i]$ , with  $E_i$  the  $i^{\text{th}}$  exceptional curve in the canonical resolution of singularities  $\pi : \widehat{\mathbb{C}^2/\mathbb{I}} \rightarrow \mathbb{C}^2/\mathbb{I}$ , and likewise  $\{l_i\}_{i=1}^8$  in the second isomorphism are taken w.r.t. the  $\alpha$ -basis of  $\mathfrak{h}^*$ . On the GW side, the McKay correspondence implies that

$$\eta_{ij} = (E_i, E_j)_Y = -\mathcal{C}_{ij}^g \quad (4.88)$$

On the other hand, by (4.25)-(4.27) (see [39, Lecture 5], and Section 5.1.1 below), the  $tt$ -metric on the Frobenius manifold on the base of the family of Toda spectral curves is

$$\eta_{ij} = - \sum_{d\mu(p)=0} \text{Res}_p \frac{\partial_{l_i} \mu \partial_{l_j} \mu}{\mu \partial_{\lambda} \mu} \frac{d\lambda}{q_g \lambda^2}. \quad (4.89)$$

where, in the language of [39, Lecture 5] and [45] and as will be reviewed more in detail in Sections 5.1.1 and 5.1.3, we view the family of Toda spectral curves as a closed set in a Hurwitz space with  $\mu$ ,  $\ln \mu$  and  $d \ln \lambda$  identified with the covering map, the superpotential, and the prime form respectively (see Section 5.1.3); this identification follows straight from the special Kähler relations (4.25). The argument of the residue has poles at  $\partial_{\lambda} \mu = 0$ ,  $\lambda, \mu = 0, \infty$ . Swapping sign and orientation in the contour integral we pick up the residues at the poles and zeroes of  $\lambda$  and  $\mu$ . Let me start from the zeroes of  $\lambda$ . Note that

$$\begin{aligned} \Xi_{g,\text{red}}(0, \mu) \Big|_{\mathbb{N}=0} &= \Xi'_{g,\text{red}}(0, \mu) = \prod_{\alpha \in \Delta^*} (\mu - e^{\alpha \cdot l}) \\ &= \prod_{\alpha \in \Delta^*} (\mu - e^{\sum_j \alpha^{[j]} l_j}), \end{aligned} \quad (4.90)$$

<sup>22</sup>Physically, this is  $g_s \rightarrow 0$ ,  $\alpha' \rightarrow 0$ .

so that  $\lambda = 0$  amounts to  $\mu = e^{\sum_j \alpha^{[j]l_j}$  for some non-zero root  $\alpha$ . Then,

$$\begin{aligned}
\text{Res}_{\mu=e^{\sum_j \alpha^{[j]l_j}} \frac{\partial_{l_i} \mu}{\mu} \frac{\partial_{l_j} \mu}{\partial \lambda \mu} \frac{d\lambda}{q_{\mathfrak{g}} \lambda} &= -\text{Res}_{\mu=e^{\sum_j \alpha^{[j]l_j}} \frac{\partial_{l_i} \lambda \partial_{l_j} \lambda}{\lambda \partial \mu \lambda} \frac{d\mu}{q_{\mathfrak{g}} \mu^2} \\
&= -\frac{\partial_{l_i} \Xi'_{\mathfrak{g},\text{red}} \partial_{l_j} \Xi'_{\mathfrak{g},\text{red}}}{q_{\mathfrak{g}} \mu^2 (\partial_{\mu} \Xi'_{\mathfrak{g},\text{red}})^2} \Big|_{\mu=e^{\sum_j \alpha^{[j]l_j}} \\
&= -\frac{e^{2\sum_j \alpha^{[j]l_j} \alpha^{[i]}\alpha^{[j]}} \prod_{\beta, \gamma \neq \alpha} (e^{\sum_j \alpha^{[j]l_j} - e^{\sum_j \beta_j l_j}}) (e^{\sum_j \alpha^{[j]l_j} - e^{\sum_j \gamma_j l_j}})}{q_{\mathfrak{g}} e^{2\sum_j \alpha^{[j]l_j} \alpha^{[i]}\alpha^{[j]}} \prod_{\beta, \gamma \neq \alpha} (e^{\sum_j \alpha^{[j]l_j} - e^{\sum_j \beta_j l_j}}) (e^{\sum_j \alpha^{[j]l_j} - e^{\sum_j \gamma_j l_j}})} \\
&= -\frac{\alpha^{[i]}\alpha^{[j]}}{q_{\mathfrak{g}}} \tag{4.91}
\end{aligned}$$

where we have used the ‘‘thermodynamic identity’’<sup>23</sup> of [39, Lemma 4.6] to switch  $\mu \leftrightarrow \lambda$  at the cost of a swap of sign in the first line, the implicit function theorem for the derivatives  $\partial_{\bullet} \lambda$  in the second line, and finally (4.90). It is easy to see that the poles at  $\mu = 0, \infty$  have vanishing residues; summing over the pre-images of  $\lambda = 0$  then gives

$$\eta_{ij} = \sum_{\lambda(p)=0} \text{Res}_p \frac{\partial_{l_i} \mu}{\mu} \frac{\partial_{l_j} \mu}{\partial \lambda \mu} \frac{d\lambda}{q_{\mathfrak{g}} \lambda^2} = - \sum_{\alpha \in \Delta^*} \frac{\alpha^{[i]}\alpha^{[j]}}{q_{\mathfrak{g}}} = -\mathcal{C}_{ij}^{\mathfrak{g}}, \tag{4.92}$$

where we used [82, Appendix E]

$$\sum_{\alpha \in \Delta^*} \langle \alpha, \alpha_i \rangle \langle \alpha, \alpha_j \rangle = q_{\mathfrak{g}} \mathcal{C}_{ij}^{\mathfrak{g}}, \tag{4.93}$$

and we find precise agreement with (4.88). The calculation of the Frobenius product (namely, the 3-point function  $\partial_{ijk}^3 F \eta^{il}$ )

$$c_{ijk} = \sum_{\substack{p \in \Gamma'_{u(l),0} \\ d\mu(p)=0}} \text{Res}_p \frac{\partial_{l_i} \mu}{\mu} \frac{\partial_{l_j} \mu}{\partial \lambda \mu} \frac{\partial_{l_k} \mu}{q_{\mathfrak{g}} \lambda^2} \frac{d\lambda}{q_{\mathfrak{g}} \lambda^2}, \tag{4.94}$$

is slightly more involved due to the necessity to expand the integrand in (4.91) to higher order at  $\lambda = 0$ ; in other words, and unsurprisingly, the product does depend on the expression of the higher order terms in  $\lambda$  of  $\Xi_{\mathfrak{g},\text{red}}$ , unlike  $\eta_{ij}$  for which all we needed to know was  $\Xi_{\mathfrak{g},\text{red}}(\lambda = 0, \mu)$  in (4.90). Let us content ourselves with noticing, however, that by the same token of the preceding calculation for  $\eta_{ij}$ , the r.h.s. of (4.94) is necessarily a rational function in exponentiated flat variables  $t_j$ : this is in keeping with the trilogarithmic nature of the 1-loop correction (4.6), whose triple derivatives have precisely such functional dependence on the flat variables  $a_j$ .

<sup>23</sup>Namely, the fact that the exchange  $\mu \leftrightarrow \lambda$  is an anti-canonical transformation of the symplectic algebraic torus  $((\mathbb{C}^*)^2, d \ln \mu \wedge d \ln \lambda)$  the curve  $\mathbb{V}(\Xi_{\mathfrak{g},\text{red}})$  embeds into, leading to  $F \rightarrow -F$  in the expression of the Frobenius prepotential, and thus  $\eta \rightarrow -\eta$ .

4.3.3. *Limits III: the 4d/non-relativistic limit.* The last limit we consider involves the fibres of  $\pi : \mathcal{S}_{\mathfrak{g}} \rightarrow \mathcal{B}_{\mathfrak{g}}$ . We take

$$\mu = e^{\epsilon\chi}, \quad \mathfrak{l}(\lambda) \rightarrow \epsilon\mathfrak{l}(\lambda) \quad (4.95)$$

and take the  $\epsilon \rightarrow 0$  limit while holding  $\chi$ ,  $\lambda$  and  $\mathfrak{l}$  fixed; note that rescaling the Cartan torus representative  $\mathfrak{l}(\lambda)$  of the conjugacy class of  $\widehat{L}_{x,y}$  and taking  $\epsilon \rightarrow 0$  corresponds to the limits in row III of Table 2 at the level of  $u_i$  and  $\mathfrak{N}$ . Then (2.22) becomes

$$\begin{aligned} \epsilon^{-d_{\mathfrak{g}}}\Xi_{\mathfrak{g}}(\lambda, \mu) &= \epsilon^{-d_{\mathfrak{g}}}(\mu - 1)^8 \prod_{\alpha \in \Delta^*} (e^{\alpha \cdot \mathfrak{l}} - \mu) = \chi^8 \prod_{\alpha \in \Delta^*} (\alpha \cdot \mathfrak{l} - \chi) + \mathcal{O}(\epsilon), \\ &= \det_{\mathfrak{g}} \left( \log \widehat{L}_{x,y} - \chi 1 \right) + \mathcal{O}(\epsilon) \end{aligned} \quad (4.96)$$

so in this limit the curve  $\mathbb{V}(\Xi_{\mathfrak{g}}(\lambda, \mu))$  degenerates to the spectral curve of the family of Lie-algebra elements  $\log \widehat{L}_{x,y}$ . These coincide with the spectral-parameter dependent Lax operators of the  $\widehat{E}_8$  *non-relativistic* Toda chain [14], to which (2.7) reduce upon taking  $\epsilon \rightarrow 0$ . As the picture of (4.96) as a curve-of-eigenvalues carries through to this setting<sup>24</sup>, so does the construction of the preferred Prym–Tyurin; on the other hand, the  $\epsilon \rightarrow 0$  degenerate limit of Theorem 3.4, which amounts in its proof to pick up the Lie-algebraic Krichever–Poisson Poisson bracket  $\omega_{\text{KP}}^{(1)}$ , leads to a non-relativistic spectral differential of the form

$$d\sigma_{\epsilon \rightarrow 0} \rightarrow \chi \frac{d\lambda}{q_{\mathfrak{g}}\lambda}. \quad (4.97)$$

As the non-relativistic limit is equivalent to the shrinking limit of the five-dimensional circle in  $\mathbb{R}^4 \times S^1$ , the corresponding limit on the gauge theory side leads to pure  $E_8$   $\mathcal{N} = 2$  super Yang–Mills theory in four dimensions, with (4.97) being the appropriate Seiberg–Witten differential in that limit. Then Claim 2.3 solves the problem of giving an explicit Seiberg–Witten curve for this theory; it is instructive to present what the polynomial (4.96) looks like more in detail. We have

$$\lim_{\epsilon \rightarrow 0} \epsilon^{-d_{\mathfrak{g}}}\Xi_{\mathfrak{g}}(\lambda, \mu) = \chi^8 \sum_{i=0}^{120} \mathfrak{q}_{120-k}(v_1, \dots, v_8) \chi^{2k} \quad (4.98)$$

where the  $\chi \rightarrow -\chi$  parity operation reflects the reality of  $\mathfrak{g}$ , and  $v_1, \dots, v_8$  is a set of generators of  $\mathbb{C}[\mathfrak{h}]^{\mathcal{W}}$ . Taking the power sum basis  $v_1 = \text{Tr}_{\mathfrak{g}}(\mathfrak{l}^2)$ ,  $v_i = \text{Tr}_{\mathfrak{g}}(\mathfrak{l}^{2i+6})$ , we get

$$\mathfrak{q}_0 = 1, \quad \mathfrak{q}_1 = -\frac{v_1}{2}, \quad \mathfrak{q}_2 = -\frac{7}{40}v_1, \quad \mathfrak{q}_3 = -\frac{49}{240}v_1, \quad \mathfrak{q}_4 = -\frac{1697}{9600}v_1 - \frac{v_2}{8}, \dots \quad (4.99)$$

## 5. APPLICATION II: THE $\widehat{E}_8$ FROBENIUS MANIFOLD

### 5.1. Dubrovin–Zhang Frobenius manifolds and Hurwitz spaces.

<sup>24</sup>A less mathematically dishonest way of putting it would be to point out that the original setting of [37, 68, 87, 88] dealt precisely with Lie algebra-valued systems of this type; since  $\mathcal{G}$  is simply-laced, the construction of the PT variety dates back to [68]; and since  $\log \widehat{L}_{x,y}$  depends rationally on  $\lambda$ , Theorem 29 [37] applies despite  $\mathfrak{g}$  not being minuscule.



5.1.1. *Generalities on Frobenius manifolds.* I gather here the basic definitions about Frobenius manifolds for the appropriate degree of generality that is needed here. The reader is referred to the classical monograph [39] for more details.

**Definition 5.1.** *An  $n$ -dimensional complex manifold  $X$  is a semi-simple Frobenius manifold if it supports a pair  $(\eta, \star)$ , with  $\eta$  a non-degenerate, holomorphic symmetric  $(0, 2)$ -tensor with flat Levi-Civita connection  $\nabla$ , and a commutative, associative, unital, fibrewise  $\mathcal{O}_X$ -algebra structure on  $TX$  satisfying*

**Compatibility:**

$$\eta(A \star B, C) = \eta(A, B \star C) \quad \forall A, B, C \in \mathcal{X}(X); \quad (5.1)$$

**Flatness:** *the 1-parameter family of connections*

$$\nabla_A^{(\hbar)} B \triangleq \nabla_A B + \hbar A \star B \quad \hbar \in \mathbb{C} \quad (5.2)$$

*is flat identically in  $\hbar \in \mathbb{C}$ ;*

**String equation:** *the unit vector field  $e \in \mathcal{X}(X)$  for the product  $\star$  is  $\nabla$ -parallel,*

$$\nabla e = 0 ;$$

**Conformality:** *there exists a vector field  $E \in \mathcal{X}(X)$  such that  $\nabla E \in \Gamma(\text{End}(TX))$  is diagonalisable,  $\nabla$ -parallel, and the family of connections Eq. (5.2) extends to a holomorphic connection  $\nabla^{(\hbar)}$  on  $X \times \mathbb{C}^*$  by*

$$\nabla^{(\hbar)} \frac{\partial}{\partial \hbar} = 0 \quad (5.3)$$

$$\nabla_{\partial/\partial \hbar}^{(\hbar)} A = \frac{\partial}{\partial \hbar} A + E \star A - \frac{1}{\hbar} \widehat{\mu} A \quad (5.4)$$

*where  $\widehat{\mu}$  is the traceless part of  $-\nabla E$ ;*

**Semi-simplicity:** *the product law  $\star|_x$  on the tangent fibres  $T_x X$  has no nilpotent elements for generic  $x \in X$ .*

**Definition 5.2.**  *$X$  is a semi-simple Frobenius manifold iff there exists an open set  $X_0$ , a coordinate chart  $t^1, \dots, t^n$  on  $X_0$ , and a regular function  $F \in \mathcal{O}(X_0)$  called the Frobenius prepotential such that, defining  $c_{ijk} \triangleq \partial_{ijk}^3 F$ , we have*

$$(1) \quad \partial_{1jk}^3 F = \eta_{jk} = \text{const}, \quad \det \eta \neq 0$$

(2) *Letting  $\eta^{ij} = (\eta^{-1})_{ij}$  and summing over repeated indices, the Witten–Dijkgraaf–Verlinde–Verlinde equations hold*

$$c_{ijk} \eta^{kl} c_{lmn} = c_{imk} \eta^{kl} c_{ljn} \quad \forall i, j, m, n \quad (5.5)$$

(3) *there exists a linear vector field and numbers  $d_i, r_i, d_F$*

$$E = \sum_i d_i t^i \partial_i + \sum_{i|d_i=0} r^i \partial_i \in \mathcal{X}(X_0) \quad (5.6)$$

such that

$$\mathcal{L}_E F = d_F F + \text{quadratic in } t \quad (5.7)$$

(4) there is a positive co-dimension subset  $X_0^* \subset X_0$  and coordinates  $u^1, \dots, u^n$  on  $X_0^*$  such that for all  $m$

$$\partial_i u^m \eta^{ij} c_{jkl} = \partial_k u^m \partial_l u^m. \quad (5.8)$$

Upon defining  $\partial_{t_i} \star \partial_{t_j} = \eta^{kl} c_{lij} \partial_{t_k}$ ,  $e = \partial_{t_1}$  the latter definition is easily seen to be equivalent to the previous one. Point 1) ensures non-degeneracy of the metric<sup>25</sup>  $\eta$ , its flatness and the String Equation; Point 2) and the fact that the structure constants come from a potential function implies the restricted flatness condition, with the extension due to conformality coming from Point 3); and Point 4) establishes that  $\partial_{u_i}$  are idempotents of the  $\star$  product on  $X_0^*$ ; the reverse implications can be worked out similarly [40].

The Conformality property has an important consequence, related to the existence of a bihamiltonian structure on the loop space of the  $X$ . Define a second metric  $g$  by

$$g(E \star A, B) = \eta(A, B) \quad (5.9)$$

which makes sense on all tangent fibres  $T_p X$  where  $E$  is in the group of units of  $\star|_p$ . In flat coordinates  $t_i$ , this reads

$$g_{ij} = E^k c_{kij}. \quad (5.10)$$

A central result in the theory of Frobenius manifolds is that this second metric is flat, and that it forms a non-trivial<sup>26</sup> flat pencil of metrics with  $\eta$ , namely  $g + \lambda \eta$  is a flat metric  $\forall \lambda \in \mathbb{C}$ . Knowledge of the second metric in flat coordinates for the first is sufficient to reconstruct the full prepotential: indeed, the induced metric on the cotangent bundle (the *intersection form*) reads

$$g^{\alpha\beta} = (2 - d_F + d_\alpha + d_\beta) \eta^{\alpha\lambda} \eta^{\beta\mu} \partial_{\lambda,\mu}^2 F \quad (5.11)$$

from which the Hessian of the prepotential can be read off.

5.1.2. *Extended affine Weyl groups and Frobenius manifolds.* A classical construction of Dubrovin [39, Lecture 4], proved to be complete in [63], gives a classification of all Frobenius manifolds with *polynomial* prepotential: these are in bijection with the finite Euclidean reflection groups (Coxeter groups). I'll recall briefly here their construction in the case in which the group is a Weyl group  $\mathcal{W}$  of a simple Lie algebra  $\mathfrak{g}$  of dimension  $d_{\mathfrak{g}}$ . Let  $(\mathfrak{h}, \langle, \rangle)$  be the Cartan subalgebra with  $\langle, \rangle$  being the  $\mathbb{C}$ -linear extension of the Euclidean inner product given by the Cartan–Killing form, and let  $\{x_i\}_i$  be orthonormal coordinates on  $(\mathfrak{h}^*, \langle, \rangle)$ . It is well-known [20] that the  $\mathcal{W}$ -invariant part  $S(\mathfrak{h}^*)^{\mathcal{W}}$  of the polynomial algebra  $S(\mathfrak{h}^*) = H^0(\mathfrak{h}, \mathcal{O})$  is a graded polynomial ring in  $r_{\mathfrak{g}} = \dim_{\mathbb{C}}(\mathfrak{h})$  homogeneous variables  $y_1, \dots, y_{r_{\mathfrak{g}}}$ ; the degrees of the basic invariants  $d_i \triangleq \deg_x y_i$ , which are distinct and ordered

<sup>25</sup>As is customary in the subject, I use the word “metric” without assuming any positivity of the symmetric bilinear form  $\eta$ .

<sup>26</sup>I.e., it does not share a flat co-ordinate frame with  $\eta$ .

so that  $d_i > d_{i+1}$ , are the *Coxeter exponents* of the Weyl group<sup>27</sup>; also  $d_1 = h(\mathfrak{g}) = \frac{\dim \mathfrak{g}}{\text{rank } \mathfrak{g}} - 1$ , the Coxeter number. Let now

$$\text{Discr}_{\mathcal{W}}(\mathfrak{h}) = \text{Spec} \frac{\mathbb{C}[x_1, \dots, x_{r_{\mathfrak{g}}}]}{\langle \{\alpha_i \cdot x\}_{i=1}^{r_{\mathfrak{g}}} \rangle} = \bigcup_i H_i \quad (5.12)$$

where  $H_i$  are root hyperplanes in  $\mathfrak{h}$ : the open set

$$\mathfrak{h}^{\text{reg}} \triangleq \mathfrak{h} \setminus \text{Discr}_{\mathcal{W}}(\mathfrak{h}) \quad (5.13)$$

is the set of regular Cartan algebra elements (i.e.  $\text{Stab}_{\mathcal{W}}(h) = e$  for  $h \in \mathfrak{h}^{\text{reg}}$ ). We will be interested in the unstable and stable quotients

$$\begin{aligned} X_{\mathfrak{g}}^{\text{us}} &\triangleq \mathfrak{h}/\mathcal{W} = \text{Spec} \mathbb{C}[x_1, \dots, x_{r_{\mathfrak{g}}}]^{\mathcal{W}} = \text{Spec} \mathbb{C}[y_1, \dots, y_{r_{\mathfrak{g}}}], \\ X_{\mathfrak{g}}^{\text{st}} &\triangleq \mathfrak{h} // \mathcal{W} = \mathfrak{h}^{\text{reg}} / \mathcal{W} = \text{Spec} (\mathcal{O}_{\mathfrak{h}}(\mathfrak{h}^{\text{reg}}))^{\mathcal{W}} \end{aligned} \quad (5.14)$$

Notice that  $\pi : \mathfrak{h}^{\text{reg}} \rightarrow X_{\mathfrak{g}}^{\text{st}}$  is a regular cover (a principal  $\mathcal{W}$ -bundle) on  $X_{\mathfrak{g}}^{\text{st}}$ , and linear co-ordinates on  $\mathfrak{h}^{\text{reg}}$  can serve as a set of local co-ordinates on  $X^{\text{st}}$ .

Dubrovin constructs a polynomial Frobenius structure on  $X_{\mathfrak{g}}^{\text{st}}$  as follows. First off, the Coxeter exponents are used to define a vector field

$$E \triangleq \frac{1}{d_1} \sum_i x^i \partial_{x_i} = \partial_{y_1} + \sum_{i=1}^{r_{\mathfrak{g}}} \frac{d_i}{d_1} \partial_{y_i}. \quad (5.15)$$

Also, view the Cartan–Killing pairing on  $\mathfrak{h}$  as giving a flat metric  $\xi$  on  $T\mathfrak{h}$ , i.e.  $\xi(\partial_{x_i}, \partial_{x_j}) = \delta_{ij}$ . If  $V = \pi^{-1}(U) = V_1 \sqcup \dots \sqcup V_{r_{\mathfrak{g}}}$  for  $U \subset X_{\mathfrak{g}}^{\text{st}}$ , and for  $i = 1, \dots, |\mathcal{W}|$ , let  $\sigma_i : U \rightarrow \mathfrak{h}^{\text{reg}}$  be a section of  $\pi : V \rightarrow U$  lifting  $U$  isomorphically to the  $i^{\text{th}}$  sheet of the cover, so that  $\sigma_i(U) \simeq V_i$ , and define

$$g \triangleq (\sigma_i)^* \xi. \quad (5.16)$$

By the Weyl invariance of  $\xi$  and  $\{y_j\}_j$ , it is immediately seen that  $g$  defines a well-defined pairing on  $T^*X_{\mathfrak{g}}^{\text{st}}$  (i.e., the r.h.s. is invariant under deck transformations of the cover  $\mathfrak{h}^{\text{reg}}$ , see [39, Lemma 4.1]). Armed with this, a Frobenius structure with unit  $\partial_{y_1}$ , Euler vector field  $E$ , intersection form  $g$  and flat pairing  $\eta = \mathcal{L}_{\partial_{y_1}} g$  is defined on  $X^{\text{st}}$  upon proving that  $g + \lambda \eta$  thus defined give a flat pencil of metrics on  $T^*X_{\mathfrak{g}}^{\text{st}}$  [39, Theorem 4.1]. In the same paper, it is further proved that such Frobenius structure is polynomial in flat co-ordinates for  $\eta$ , semi-simple, and unique given  $(e, E, g)$ .

In a subsequent paper [43], Dubrovin–Zhang consider a group theory version of the above construction, as follows. Fix a node  $\bar{i} \in \{1, \dots, r_{\mathfrak{g}}\}$  in the Dynkin diagram of  $\mathfrak{g}$ , and let  $\alpha_{\bar{i}}, \omega_{\bar{i}}$  be the corresponding simple root and fundamental weight. The  $\mathcal{W}$ -action on  $\mathfrak{h}$  can be lifted to an action of the affine Weyl group  $\widehat{\mathcal{W}} \simeq \mathcal{W} \rtimes \Lambda_r(\mathcal{G})$  by affine transformations on  $\mathfrak{h}$ ,

$$\begin{aligned} \widehat{w} : \widehat{\mathcal{W}} \times \mathfrak{h} &\longrightarrow \mathfrak{h} \\ ((w, \alpha), l) &\longrightarrow w(l) + \alpha, \end{aligned} \quad (5.17)$$

<sup>27</sup>A parallel and somewhat more common convention is to call  $d_i - 1$  the exponents of the group (the eigenvalue of a Coxeter element), rather than the degrees  $d_i$  themselves.

which is further covered by a  $\widetilde{W}_{\bar{i}} \triangleq \widehat{W} \times \mathbb{Z}$ -action on  $\mathfrak{h} \times \mathbb{C}$  given by

$$\begin{aligned} \tilde{w} : \widetilde{W} \times \mathfrak{h} \times \mathbb{C} &\longrightarrow \mathfrak{h} \times \mathbb{C} \\ ((w, \alpha, r_{\mathfrak{g}}), (l, v)) &\longrightarrow (w(l) + \alpha + r_{\mathfrak{g}}\omega_{\bar{i}}, x_{r_{\mathfrak{g}}+1} - r_{\mathfrak{g}}). \end{aligned} \quad (5.18)$$

$\widetilde{W}_{\bar{i}}$  is called *the extended affine Weyl group* with marked root  $\alpha_{\bar{i}}$ . In [43], the authors give a characterisation of the ring of invariants of  $\widetilde{W}_{\bar{i}}$ , which may be reformulated as follows. Set  $\mathfrak{g} = e^{2\pi i l} \in \mathcal{G}$  and let  $u_i = \chi_{\rho_i}(\mathfrak{g})$  be as in (2.14) the regular fundamental characters of  $\mathfrak{g}$ ; also define  $d_j \triangleq \langle \omega_j, \omega_{\bar{i}} \rangle$ . Then<sup>28</sup> [43, Theorem 1.1],

$$\mathbb{C}[t_1, \dots, t_{r_{\mathfrak{g}}}; t_{r_{\mathfrak{g}}+1}]^{\widetilde{W}_{\bar{i}}} \simeq \mathbb{C}[e^{2\pi i d_1 t_{r_{\mathfrak{g}}+1}} u_1, \dots, e^{2\pi i d_{r_{\mathfrak{g}}} t_{r_{\mathfrak{g}}+1}} u_{r_{\mathfrak{g}}}; u_{r_{\mathfrak{g}}+1} e^{2\pi i t_{r_{\mathfrak{g}}+1}}] \quad (5.21)$$

As before, define

$$\widetilde{X}_{\mathfrak{g}, \bar{i}}^{\text{us}} \triangleq \text{Spec} \mathbb{C}[u_1, \dots, u_{r_{\mathfrak{g}}+1}] \simeq (\mathfrak{h} \times \mathbb{C}) / \widetilde{W}_{\bar{i}} \simeq \mathcal{T} / \mathcal{W} \times \mathbb{C}^* \quad (5.22)$$

$$\widetilde{X}_{\mathfrak{g}, \bar{i}}^{\text{st}} \triangleq (\mathfrak{h}^{\text{reg}} \times \mathbb{C}) // \widetilde{W}_{\bar{i}} = \text{Spec} (\mathcal{O}_{\mathfrak{h} \times \mathbb{C}}(\mathfrak{h}^{\text{reg}} \times \mathbb{C}))^{\widetilde{W}_{\bar{i}}} \simeq \mathcal{T}^{\text{reg}} / \mathcal{W} \times \mathbb{C}^* \quad (5.23)$$

with  $\mathcal{T}^{\text{reg}} = \exp(\mathfrak{h}^{\text{reg}})$  and  $\mathcal{T}^{\text{reg}} / \mathcal{W}$  being the set of regular elements of  $\mathcal{T}$  and regular conjugacy classes of  $\mathcal{G}$  respectively. A Frobenius structure polynomial in  $u_1, \dots, u_{r_{\mathfrak{g}}+1}$  can be constructed along the same lines as for the classical case of finite Coxeter groups: adding a further linear coordinate  $x_{r_{\mathfrak{g}}+1}$  for the right summand in  $\mathfrak{h} \oplus \mathbb{C}$ , we define a metric  $\xi$  with signature  $(r_{\mathfrak{g}}, 1)$  on  $\mathfrak{h} \times \mathbb{C}$  by orthogonal extension of  $4\pi^2$  times the Cartan–Killing pairing on  $\mathfrak{h}$ , and normalising  $\|\partial_{x_{r_{\mathfrak{g}}+1}}\|^2 = -4\pi^2$ :

$$\xi(\partial_{x_i}, \partial_{x_j}) = \begin{cases} 4\pi^2 \delta_{ij}, & i, j < r_{\mathfrak{g}} + 1, \\ -4\pi^2, & i = j = r_{\mathfrak{g}} + 1, \\ 0 & \text{else.} \end{cases} \quad (5.24)$$

Exactly as in the previous discussion of the finite Weyl groups, we have a  $\mathcal{W}$ -principal bundle

$$\begin{array}{c} \mathcal{T}^{\text{reg}} \times \mathbb{C}^* \\ \tilde{\sigma}_i \left( \begin{array}{c} \uparrow \\ \downarrow u \\ \downarrow \\ \widetilde{X}_{\mathfrak{g}, \bar{i}}^{\text{st}} \end{array} \right. \end{array} \quad (5.25)$$

with sections  $\tilde{\sigma}_i$ ,  $i = 1, \dots, |\mathcal{W}|$  defined as before. Then the following theorem holds [43, Theorem 2.1]:

<sup>28</sup>The reader familiar with [43] will notice the slight difference between what we call  $u_i$  here and the basic Laurent polynomial invariants  $\tilde{y}_i$  in [43], the latter being defined as the Weyl-orbit sums

$$\tilde{y}_i(t) \triangleq e^{2\pi i d_1 t_{r_{\mathfrak{g}}+1}} \sum_{w \in \mathcal{W}} e^{2\pi i \langle w(\omega_i), t \rangle}. \quad (5.19)$$

It is immediate from the definition that there exists a linear, triangular change-of-variables with rational coefficients

$$u_i = \sum_j \frac{\text{Mult}_{\rho_{\omega_i}}(\omega_j)}{|\mathcal{W}_{\omega_j}|} \tilde{y}_j(t) + \text{Mult}_{\rho_{\omega_i}}(0). \quad (5.20)$$

with  $\text{Mult}_{\rho}(\omega)$  being the multiplicity of  $\omega \in \Lambda_w$  in the weight system of  $\rho \in R(\mathcal{G})$ , so that [43, Theorem 1.1] holds as in (5.21).

**Theorem 5.1.** *There is a unique semi-simple Frobenius manifold structure  $(\widetilde{X}_{\mathfrak{g},i}^{\text{st}}, e, E, \xi, g, \star)$  on  $\widetilde{X}_{\mathfrak{g},i}^{\text{st}}$  such that*

- (1) *in flat co-ordinates  $t^1, \dots, t^{r_{\mathfrak{g}}}, t^{r_{\mathfrak{g}}+1}$  for  $\xi$ , the prepotential is polynomial in  $t^1, \dots, t^{r_{\mathfrak{g}}}$  and  $e^{t^{r_{\mathfrak{g}}+1}}$ ;*
- (2)  $e = \partial_{u_{\bar{i}}} = \partial_{t^{\bar{i}}}$ ;
- (3)  $E = \frac{1}{2\pi i d_{\bar{i}}} \partial_{x_{r_{\mathfrak{g}}+1}} = \sum_j \frac{d_j}{d_{\bar{i}}} t^j \partial_{t_j} + \frac{1}{d_{\bar{i}}} \partial_{t^{r_{\mathfrak{g}}+1}}$ ;
- (4)  $g = \widetilde{\sigma}_{\bar{i}}^* \xi$ .

5.1.3. *Hurwitz spaces and Frobenius manifolds.* As was already hinted at in Section 4.3.2, a further source of semi-simple Frobenius manifolds is given by Hurwitz spaces [39, Lecture 5]. For  $r \in \mathbb{N}_0$ ,  $\mathfrak{m} \in \mathbb{N}_0^r$ , these are moduli spaces  $\mathcal{H}_{g,\mathfrak{m}} = \mathcal{M}_g(\mathbb{P}^1, \mathfrak{m})$  of isomorphism classes of degree  $|\mathfrak{m}|$  covers  $\lambda$  of the complex projective line by a smooth genus- $g$  curve  $C_g$ , with marked ramification profile over  $\infty$  specified by  $\mathfrak{m}$ ; in other words,  $\lambda$  is a meromorphic function on  $C_g$  with pole divisor  $(\lambda)_- = -\sum_i \mathfrak{m}_i P_i$  for points  $P_i \in C_g$ ,  $i = 1, \dots, r$ . Denoting as in Definition 2.1 by  $\pi$ ,  $\lambda$  and  $\Sigma_i$  respectively the universal family, the universal map, and the sections marking the  $i$ -th point in  $(\lambda)_-$ , this is

$$\begin{array}{ccc}
 C_g & \hookrightarrow & \mathcal{C} \xrightarrow{\lambda} \mathbb{P}^1 \\
 \uparrow P_i & \downarrow \Sigma_i & \downarrow \pi \\
 & & [\lambda] \xrightarrow{\text{pt}} \mathcal{H}
 \end{array} \tag{5.26}$$

As a result,  $\mathcal{H}_{g,\mathfrak{m}}$  is a reduced, irreducible complex variety with  $\dim_{\mathbb{C}} \mathcal{H}_{g,\mathfrak{m}} = 2g + \sum_i \mathfrak{m}_i + r - 1$ , which is typically smooth (i.e. so long as the ramification profile is incompatible with automorphisms of the cover).

Dubrovin provides in [39] a systematic way of constructing a semi-simple Frobenius manifold structure on  $\mathcal{H}_{g,\mathfrak{m}}$ , for which I here provide a simplified account. As in Section 2.4, let  $d = d_{\pi}$  denote the relative differential with respect to the universal family (namely, the differential in the fibre direction), and let  $p_i^{\text{cr}} \in C_g \simeq \pi^{-1}([\lambda])$  be the critical points  $d\lambda = 0$  of the universal map (i.e., the ramification points of the cover). By the Riemann existence theorem, the critical values

$$\mathbf{u}^i = \lambda(p_i^{\text{cr}}) \tag{5.27}$$

are local co-ordinates on  $\mathcal{H}_{g,\mathfrak{m}}$  away from the discriminant  $\mathbf{u}^i = \mathbf{u}^j$ . We then locally define an  $\mathcal{O}_{\mathcal{H}_{g,\mathfrak{m}}}$ -algebra structure on the space of vector fields  $\mathcal{X}(\mathcal{H}_{g,\mathfrak{m}})$  by imposing that the co-ordinate vector fields  $\partial_{\mathbf{u}^i}$  are idempotents for it:

$$\partial_{\mathbf{u}^i} \star \partial_{\mathbf{u}^j} = \delta_{ij} \partial_{\mathbf{u}^i}. \tag{5.28}$$

The algebra is obviously unital with unit  $e = \sum_i \partial_{\mathbf{u}^i}$ ; a linear (in these co-ordinates) vector field  $E$  is further defined as  $\sum_i \mathbf{u}^i \partial_{\mathbf{u}^i}$ . The one missing ingredient in the definition of a Frobenius manifold is a flat pairing of the vector fields, which is provided by specifying some auxiliary data. Let

then  $\phi \in \Omega_C^1(\log(\lambda))$  be an *exact* meromorphic one form having *simple* poles<sup>29</sup> at the support of  $(\lambda)_-$  with constant residues; the pair  $(\lambda, \phi)$  are called respectively *superpotential* and the *primitive differential* of  $\mathcal{H}_{g,m}$ . A non-degenerate symmetric pairing  $\eta(X, Y)$  for vector fields  $X, Y \in \mathcal{X}(\mathcal{H}_{g,m})$  is defined by

$$\eta(X, Y) \triangleq \sum_i \operatorname{Res}_{p_i^{\text{cr}}} \frac{X(\lambda)Y(\lambda)}{q_{\mathfrak{g}} d\lambda} \phi^2, \quad (5.29)$$

where, for  $p$  locally around  $p_i^{\text{cr}}$ , the Lie derivatives  $X(\lambda), Y(\lambda)$  are taken at constant  $\mu(p) = \int^p \phi$ . It turns out that  $\eta$  thus defined is flat, compatible with  $\star$ , with  $E$  being linear in flat co-ordinates, and it further satisfies

$$\eta(X, Y \star Z) = \sum_i \operatorname{Res}_{p_i^{\text{cr}}} \frac{X(\lambda)Y(\lambda)Z(\lambda)}{q_{\mathfrak{g}} d\mu d\lambda} \phi^2, \quad (5.30)$$

$$g(X, Y) = \sum_i \operatorname{Res}_{p_i^{\text{cr}}} \frac{X(\log \lambda)Y(\log \lambda)}{q_{\mathfrak{g}} d \log \lambda} \phi^2. \quad (5.31)$$

**Remark 5.2.** There is a direct link between the prepotential of the Frobenius manifold structure above on  $\mathcal{H}_{g,m}$  and the special Kähler prepotential of families of spectral curves (see (4.27) in the Toda case), whenever the latter is given by moduli of a generic cover of the line with ramification profile  $\mathfrak{m}$ : the two things coincide upon identifying the superpotential and primitive Abelian integral  $(\lambda, \mu)$  on the Hurwitz space side with the marked meromorphic functions  $(\lambda, \mu)$  on the spectral curve end [39, 76]. It is a common situation, however, that the  $\lambda$ -projection is highly non-generic: the Toda spectral curves of Section 2.4.1 are an obvious example in this sense. One might still ask, however, what type of geometric conditions ensure that a semi-simple, conformal Frobenius manifold structure exists on the base of the family  $\mathcal{B} \xrightarrow{\iota} \mathcal{H}_{g,m}$ : an obvious sufficient condition is that, away from the discriminant and locally on an open set  $\Omega \subset \mathcal{H}_{g,m}$  with a chart  $\mathfrak{t} : \Omega \rightarrow \mathbb{C}^{\dim \mathcal{H}_{g,m}}$  given by flat co-ordinates for  $\eta$ ,

- (1)  $\mathcal{B}$  embeds as a linear subspace of  $H \subset \mathbb{C}^{\dim \mathcal{H}_{g,m}}$ ;
- (2)  $H \simeq T_0 H \simeq \mathbb{C} \langle e \rangle \oplus H'$  contains the line through  $e$ ;
- (3) the minor corresponding to the restriction to  $H$  of the Gram matrix of  $\eta$  is non-vanishing.

In this case, (5.29)-(5.30) define a semi-simple, conformal Frobenius manifold structure with flat identity on the base  $\mathcal{B}$  of the family of spectral curves, with all ingredients obtained being projected down from the parent Frobenius manifold. We will see in the next section that the family of  $\widehat{E}_8$  Toda spectral curves falls precisely within this class.

## 5.2. A 1-dimensional LG mirror theorem.

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<sup>29</sup>Exactness and simplicity of the poles can be disposed of by looking instead at suitably normalised Abelian differentials w.r.t. a chosen symplectic basis of 1-homology cycles on  $C_g$ ; a fuller discussion, with a classification of the five types of differentials that are compatible with the existence of flat structures on the resulting Frobenius manifold, is given in the discussion preceding [39, Theorem 5.1]. The generality considered here however suits our purposes in the next section.

5.2.1. *Saito co-ordinates.* I will now elaborate on the previous Remark 5.2 in the case of the degenerate limit  $\aleph \rightarrow 0$  of the family of Toda curves over  $\mathcal{U} \times \mathbb{C}$ . Recall from Section 2.4.1 that there is an intermediate branched double cover  $\Gamma'_u \simeq \Gamma_{u,0}$  of the base curve  $\Gamma''_u$ , defined as

$$\Gamma'_u = \mathbb{V} \left[ \overline{\Xi''_{\mathfrak{g},\text{red}} \left( \mu + \frac{1}{\mu}, \lambda \right)} \right] \quad (5.32)$$

For future convenience, rescale  $\lambda \rightarrow \frac{\lambda}{u_0}$  in the following. Looking at  $\lambda$  as our marked covering map gives, by (2.33) and Table 1, we have an embedding of

$$\iota : X_{\mathfrak{g}}^{\text{Toda}} \hookrightarrow \mathcal{H}_{g,\mathfrak{m}} \quad (5.33)$$

of  $X_{\mathfrak{g}}^{\text{Toda}} \simeq \mathbb{C}_{u_0} \times \mathcal{U}$  into the Hurwitz space  $\mathcal{H}_{g,\mathfrak{m}}$  with  $g = 128$  and, letting  $\varepsilon_k = e^{2\pi i/k}$ ,

$$\mathfrak{m} = \left( \underbrace{\mu=-1}_{2}, \underbrace{\mu=\varepsilon_3^j \neq 1}_{3,3}, \underbrace{\mu=\varepsilon_5^j \neq 1}_{5,5,5,5}, \underbrace{\mu=0}_{5,6,10,10,15,15,15,30}, \underbrace{\mu=\infty}_{5,6,10,10,15,15,15,30} \right). \quad (5.34)$$

Mindful of Remark 5.2 I am going to declare  $(\lambda, \phi)$  with  $\phi \triangleq d \ln \mu$  to be the superpotential and primary differential and proceed to examine the pull-back of  $\eta$  to  $\mathbb{C}_{u_0} \times \mathcal{U}$ . An important point to stress here is that *this will not be a repetition of what was done in Section 4.3.2*: in that case, we were looking at (5.29) with  $\log \lambda$  as the superpotential (up to  $\mu \leftrightarrow \lambda$ ,  $F \leftrightarrow -F$ ); this means that the computation leading up to the flat metric (4.92) was rather computing *the intersection form*  $g$  of  $X_{\mathfrak{g}}^{\text{Toda}}$ , by (5.31). The relation between the Frobenius manifold structure on  $X^{\text{Toda}}$  defined by (5.29)-(5.31) and  $\widehat{QH}(\mathbb{C}/\tilde{I})$  is indeed a non-trivial instance of Dubrovin's notion of almost-duality of Frobenius manifolds [41], with the almost-dual product being given by (4.94).

**Lemma 5.3.** *Let  $X, Y \in \mathcal{X}(X_{\mathfrak{g}}^{\text{Toda}})$  be holomorphic vector fields on  $X_{\mathfrak{g}}^{\text{Toda}}$ . Then (5.29) defines a flat non-degenerate pairing on  $TX_{\mathfrak{g}}^{\text{Toda}}$ , with flat co-ordinates given by*

$$\begin{aligned} \mathfrak{t}_0(u) &= \frac{\ln u_0}{30}, \\ \mathfrak{t}_i(u) &= M_i \mathfrak{m}_i (c = u_0^{1/30}, u_1, \dots, u_8), \quad i > 0. \end{aligned} \quad (5.35)$$

where  $\{\mathfrak{m}_i\}_i$  are the planar moments (4.79) and  $M_i \in \mathbb{C}$ . Furthermore, the metric has constant anti-diagonal form in these coordinates.

*Proof.* As in Section 4.3.2, let's reverse orientation in the residue formula (5.29) and pick up residues on  $\Gamma'_u \setminus \{b_i^{\text{cf}}\}_i$ ; these are all located at the poles  $P_i$  of  $\lambda$ . As in [39], I define local coordinates  $\nu_i$  centred around  $P_i$  such that  $\lambda = \nu_i^{-\mathfrak{m}_i} + \mathcal{O}(1)$ , as well as functions  $r_i^j$  with  $(i, j) \in \mathbb{R} = \{(k, l) | 1 \leq k \leq 23 = l(\mathfrak{m}), 1 \leq j \leq \mathfrak{m}_i\}$  by

$$r_i^j \triangleq \begin{cases} \text{pv} \int_{P_8}^{P_i} \frac{d\mu}{\mu} & \text{for } j = 0 \\ \text{Res}_{P_i} \nu_i^j \text{pv} \ln \mu d\lambda & \text{for } j = 1, \dots, \mathfrak{m}_i - 1, \\ \text{Res}_{P_i} \lambda \frac{d\mu}{\mu} & \text{for } j = \mathfrak{m}_i, \end{cases} \quad (5.36)$$

and where

$$\text{pv} \ln \mu(P_i) = \text{pv} \int_{P_8}^{P_i} \frac{d\mu}{\mu}.$$

Here  $P_8$ , in the numbering of marked points of (5.34), is the lowest (fifth) order pole of  $\lambda$  at  $\mu = 0$ . A remarkable fact, that can be proven straightforwardly from the Puiseux expansion of  $\lambda$  near  $P_i$  using (2.20)-(2.24) and Claim 2.3, is that  $\mathbf{r}_i^j$  is in all cases a multiple of one of the planar moments (4.79):

$$\mathbf{r}_i^j(u) = \mathcal{N}_i^j \mathbf{t}_{\ell(i,j)}(u) \quad (5.37)$$

for some map of finite sets  $\ell : \mathbb{R} \rightarrow [[0, 8]]$  and complex constants  $\mathcal{N}_i^j \in \mathbb{C}$ . For  $0 < j < m_i$  the result is collected in Table 3, where we denoted

$$\begin{aligned} v_1 &= 5 - \sqrt{5} - i\sqrt{10 + 2\sqrt{5}}, & v_2 &= -5 - \sqrt{5} + i\sqrt{10 - 2\sqrt{5}}, \\ v_3 &= \left(1 + i\sqrt{5 + 2\sqrt{5}}\right) v_1/2, & v_4 &= -(-1)^{3/5} v_1. \end{aligned}$$

We furthermore have  $r_i^0 = 0$  and  $r_i^{m_i} = 0$  for  $\mu(P_i) \neq 0, \infty$ , and  $r_i^0 = 1 = -r_i^{m_i} = -r_{i+8}^0 = r_{i+8}^{m_i}$  for  $\mu(P_i) = 0$ . It turns out that (5.37) suffices to prove constancy of  $\eta$  in the coordinate chart given by  $\mathbf{t}_i$  above; indeed, from (5.36) and Table 3 we obtain

$$\begin{aligned} \eta(\partial_{\mathbf{t}_i}, \partial_{\mathbf{t}_j}) &= - \sum_l \operatorname{Res}_{P_l} \frac{\partial_{\mathbf{t}_i} \lambda \partial_{\mathbf{t}_j} \lambda}{\mu \partial_\mu \lambda} d\mu, = - \sum_l \operatorname{Res}_{P_l} \frac{\partial_{\mathbf{t}_i} \ln \mu d\lambda \partial_{\mathbf{t}_j} \ln \mu d\lambda}{d\lambda} \\ &= - \sum_l \sum_{k=0}^{m_l} \partial_{\mathbf{t}_i} \mathbf{r}_l^k \partial_{\mathbf{t}_j} \mathbf{r}_l^{m_l-k} = - \sum_i \sum_{k=0}^{m_i} \delta_{i,\ell(l,k)} \delta_{j,\ell(l,m_l-k)} \mathcal{N}_l^k \mathcal{N}_l^{m_l-k} \\ &= \delta_{i,8-j} \eta_i, \end{aligned} \quad (5.38)$$

for numbers  $\eta_i$ , so that the Gram matrix of  $\eta$  is constant and anti-diagonal in these coordinates. These can be scaled away by an appropriate rescaling of  $M_i$  in the definition of  $\mathbf{t}_i$  in (5.35). □

Before we carry on to examine the product structure on  $X_{\mathfrak{g}}^{\text{Toda}}$  let us first check that the unit  $e$  and Euler vector field  $E$  satisfy indeed the String Equation and (part of the) Conformality properties of Definition 5.1 by verifying that  $\nabla e = 0$ ,  $\nabla \nabla E = 0$  with  $\nabla = \nabla^{(\eta)}$ . It is easy to verify the following

**Proposition 5.4.**

$$e = \frac{\partial}{\partial \mathbf{t}_8}; \quad E = \sum_j \frac{d_j}{d_8} t^j \partial_{\mathbf{t}_j} + \frac{1}{d_8} \frac{\partial}{\partial \mathbf{t}_0}. \quad (5.39)$$

with

$$d_1 = 6, \quad d_2 = 10, \quad d_3 = 12, \quad d_4 = 15, \quad d_5 = 18, \quad d_6 = 20, \quad d_7 = 24, \quad d_8 = 30. \quad (5.40)$$

*Proof.* The easiest way to see this is to realise that, by their definition in the Hurwitz space setting,  $e$  and  $E$  generate an affine subgroup of  $\operatorname{PSL}(2, \mathbb{Z})$  on the target  $\mathbb{P}^1$  in (5.26) by

$$\mathcal{L}_e \lambda = \text{const}, \quad \mathcal{L}_E \lambda = \lambda, \quad (5.41)$$



$i$	$j$	$\ell(i, j)$	$\mathcal{N}_i^j$	$i$	$j$	$\ell(i, j)$	$\mathcal{N}_i^j$	$i$	$j$	$\ell(i, j)$	$\mathcal{N}_i^j$
1	1	5	$i$	10	5	3	$-\frac{5\sqrt[3]{-1}}{2}$	13	6	4	$\frac{3\phi}{10}$
2	1	3	$-\frac{\sqrt[6]{-1}}{\sqrt{3}}$	10	6	4	$\frac{3}{20}(-1)^{2/5}$	13	9	6	$-\frac{\phi}{25}$
2	2	7	$\frac{(-1)^{5/6}}{6\sqrt{3}}$	10	9	6	$\frac{1}{50}(-1)^{3/5}$	13	10	7	$-\frac{5}{6}$
4	1	2	$\frac{v_1}{20}$	10	10	7	$\frac{5}{12}(-1)^{2/3}$	13	12	8	$-\frac{3}{500\phi}$
4	2	4	$\frac{v_2}{200}$	10	12	8	$\frac{3(-1)^{4/5}}{1000}$	13	else	—	0
4	3	6	$\frac{v_3}{1500}$	10	else	—	0	14	3	2	$-3\phi$
4	4	8	$\frac{u_4}{10000}$	11	2	2	$-2\sqrt[5]{-1}\phi$	14	5	3	$-5$
5	1	2	$-\frac{v_3}{20}$	11	4	4	$\frac{(-1)^{2/5}}{5\phi}$	14	6	4	$-\frac{3}{10\phi}$
5	2	4	$\frac{v_1}{200}$	11	5	5	$5i$	14	9	6	$\frac{1}{25\phi}$
5	3	6	$\frac{u_4}{1500}$	11	6	6	$\frac{2(-1)^{3/5}}{75\phi}$	14	10	7	$-\frac{5}{6}$
5	4	8	$-\frac{v_2}{10000}$	11	8	8	$-\frac{(-1)^{4/5}\phi}{250}$	14	12	8	$\frac{3\phi}{500}$
8	1	2	1	11	else	—	0	14	else	—	0
8	2	4	$-\frac{1}{10}$	12	2	2	$\frac{2\sqrt[3]{-1}}{\phi}$	15	6	2	$-6\sqrt[5]{-1}$
8	3	6	$\frac{1}{75}$	12	4	4	$-\frac{(-1)^{2/5}\phi}{5}$	15	10	3	$-10\sqrt[3]{-1}$
8	4	8	$-\frac{1}{500}$	12	5	5	$5i$	15	12	4	$-\frac{3(-1)^{2/5}}{5}$
9	2	3	$2\sqrt[3]{-1}$	12	6	6	$-\frac{2(-1)^{3/5}}{75}\phi$	15	15	5	$-15i$
9	3	5	$-3i$	12	8	8	$\frac{(-1)^{4/5}}{250\phi}$	15	18	6	$-\frac{2(-1)^{3/5}}{25}$
9	4	7	$-\frac{(-1)^{2/3}}{3}$	12	else	—	0	15	20	7	$\frac{5(-1)^{2/3}}{3}$
9	else	—	0	13	3	2	$\frac{3}{\phi}$	15	24	8	$-\frac{3(-1)^{4/5}}{250}$
10	3	2	$\frac{3\sqrt[3]{-1}}{2}$	13	5	3	$-5$	15	else	—	0

TABLE 3. Residues (5.36) in terms of the planar moments (4.79); poles of  $\lambda$  not appearing in the table are related to those listed above by  $\mu \rightarrow 1/\mu$  and a sign-flip in the residue.

Now, recall first of all that in the natural Hurwitz (B-model) co-ordinates  $u_i$ , a shift in  $u_3$  keeping all other variables fixed gives a constant shift in  $\lambda/u_0$  by (2.2) with  $\lambda \rightarrow \lambda/u_0$  and  $\aleph \rightarrow 0$ , as we are considering here. Moreover, from (4.79) we see that in flat co-ordinates for  $\eta$ , and since  $\partial_{\mathbf{t}_i} u_j = \delta_{j3} e^{-\mathbf{t}_0} \mathbf{t}_8$ , a constant shift in  $\mathbf{t}_8$  leaving all other flat co-ordinates constant gives a constant shift in the rescaled  $\lambda \rightarrow \lambda/u_0$ :

$$\frac{\partial \lambda}{\partial \mathbf{t}_8} = \text{const}, \quad (5.42)$$

Then  $L_e \lambda \propto L_{\partial_{\mathbf{t}_8}} \lambda$ , from which we deduce  $e \propto \partial_{\mathbf{t}_8}$  as there is no continuous symmetry on  $\lambda$  that holds up identically in  $\mu$ ; the proportionality can be turned into an equality upon appropriate choice of  $M_8$ , which gives in any case an isomorphism of Frobenius manifolds. As far as the Euler vector field is concerned, recall similarly that a rescaling in  $u_0$  at constant  $u_i$  gives a rescaling of  $\lambda$  with the same scaling factor, so that  $E = u_0 \partial_{u_0}$ . Writing down  $u_0 \partial_{u_0}$  in flat co-ordinates using (4.79) concludes the proof.  $\square$

5.2.2. *The mirror theorem.* Let us now dig deeper into the Frobenius manifold structure of  $X_{\mathfrak{g}}^{\text{Toda}}$ . By (5.11), the  $\star$ -structure on  $TX_{\mathfrak{g}}^{\text{Toda}}$  can be retrieved from knowledge of the intersection form  $g$  in flat co-ordinates for  $\eta$ ; whilst theoretically this would only require the calculation of a Jacobian from the flat co-ordinates for  $g$  in Section 4.3.2 to (5.35), such calculation is however unviable due to the difficulty in inverting the Laurent polynomials  $u_i(t)$ . We proceed instead from an analysis of (5.30), and prove the following

**Theorem 5.5.** *There is an isomorphism of Frobenius manifolds*

$$\widetilde{X}_{\mathfrak{g},3}^{\text{us}} \simeq X_{\mathfrak{g}}^{\text{Toda}} \quad (5.43)$$

*Proof.* We have already come a long way proving (5.43): by Proposition 5.4 the pairs  $(e, E)$  match on the nose already, since  $d_i = \langle \omega_j, \omega_3 \rangle$  by (5.40). Also, by (4.88) and since

$$\begin{aligned} g_{k0} &= - \sum_{\lambda(p)=\infty} \text{Res}_p \frac{\partial_{x_k} \lambda}{q_{\mathfrak{g}} \mu^2 \partial_{\mu} \lambda} d\mu = 0, \\ g_{00} &= - \sum_{\lambda(p)=\infty} \text{Res}_p \frac{\lambda}{q_{\mathfrak{g}} \mu^2 \partial_{\mu} \lambda} d\mu = \frac{1}{q_{\mathfrak{g}}} \sum_{\lambda(p)=\infty} \text{ord}_p \lambda = \frac{107}{900}, \end{aligned} \quad (5.44)$$

so do the two intersection forms up to a linear change-of-variables. By Theorem 5.1, the remaining and largest bit in the proof resides then in the proof of the polynomiality of the  $\star$ -product in flat co-ordinates (5.35), which I am now going to show. This would be achieved once we show that

$$c_{i,j,k} = \sum_i \text{Res}_{b_i^{\pm}} \frac{\partial_{\mathbf{t}_i} \lambda d\mu \partial_{\mathbf{t}_j} \lambda d\mu \partial_{\mathbf{t}_k} \lambda d\mu}{q_{\mathfrak{g}} \mu^2 d\mu d\lambda} \in \mathbb{C}[\mathbf{t}_1, \dots, \mathbf{t}_8, e^{\mathbf{t}_0}] \quad (5.45)$$

for all  $i, j, k$ . As in the proof of Lemma 5.3, because of the difficulty in controlling the moduli dependence of  $b_i^{\pm}$  in either  $u_i$  or  $\mathbf{t}_i$ , we turn the contour around and pick up residues in the complement of  $\{b_i^{\pm}\}$ . However, one major difference here with the case of the calculation of  $\eta$  is that not only do poles of  $\lambda$  contribute, but also the 240 ramification points of the  $\mu$ -projection  $q_i$  (counted with multiplicity) satisfying either  $\mu(q_i) = -1$  or  $\{\mu(q_i) + \mu^{-1}(q_i) = r_1^k\}_{k=1}^{133}$  (see (2.34)), whose dependence on  $u_i$  is even more involved. Indeed, near one of those points, the superpotential behaves like

$$\lambda(p) = \lambda_0(\mathbf{t}) + \lambda_1(\mathbf{t}) \sqrt{\mu - q_i} + \mathcal{O}(\mu - q_i), \quad (5.46)$$

and therefore the moduli derivatives of  $\lambda$  at  $\mu = \text{const}$  which appear in (5.45) develop a simple pole as soon as  $\partial_{\mathbf{t}_i} \lambda_1 \neq 0$ . This leads to a non-vanishing contribution to the residue as the triple pole resulting from them in (5.45) is now, unlike for  $\eta$ , only partially offset by the vanishing of  $d\mu$  and  $1/\partial_{\mu} \lambda$  at the branch points. It is a straightforward calculation to check that these contributions *do contribute*: ideally, it would be best to avoid considering them directly, as their dependence on the moduli is essentially intractable. Luckily, there's a workaround to do precisely so, as follows. Instead of (5.45), consider the 3-point function in *B-model* co-ordinates  $\tilde{u}_0 = \ln u_0, \tilde{u}_1 = u_1, \dots, \tilde{u}_8 = u_8$ ,

$$\tilde{c}_{i,j,k} = - \sum_{d\mu(p)=0 \text{ or } \lambda(p)=\infty} \text{Res}_p \frac{\partial_{u_i} \lambda d\mu \partial_{u_j} \lambda d\mu \partial_{u_k} \lambda d\mu}{q_{\mathfrak{g}} \mu^2 d\mu d\lambda} \quad (5.47)$$

sticking to the case  $i = 0$  to begin with. Now,  $\partial_{\tilde{u}_0}$  is the Euler vector field,  $\partial_{\tilde{u}_0}\lambda = \lambda$ , and we have  $\partial_{\tilde{u}_0}\lambda_1 = 0$  at all ramification points of  $\mu$ : this means that the problematic residues at  $d\mu = 0$  give individually vanishing contributions to (5.47), unlike for the flat 3-point functions  $c_{0,i,j}$ . For this restricted set of correlators and in this particular set of co-ordinates, the only contribution to the LG formula (5.47) may come from the poles  $P_i$ : here, a direct calculation from the Puiseux expansion of  $\lambda$  at its pole divisor immediately shows that the Puiseux coefficients of  $\lambda$  are polynomial in  $u_0, u_1, \dots, u_8$  at  $\mu = 0, \infty$ . Furthermore, while the Puiseux coefficients at  $\mu = -1$ ,  $\mu^3 = 1$  and  $\mu^5 = 1$  are only Laurent polynomials in  $t_i$  with denominators given by powers of  $t_4, t_2$  and  $t_1$  respectively, these powers turn out to delicately cancel from the final answer in (5.47). All in all, we find

$$\tilde{c}_{0,j,k} \in \mathbb{Q}[e^{u_0/30}, u_1, \dots, u_8]. \quad (5.48)$$

To consider the case  $i > 0$  in (5.47), we use the WDVV equation in these co-ordinates:

$$\tilde{c}_{ijk}\tilde{\eta}^{kl}\tilde{c}_{lmn} = \tilde{c}_{imk}\tilde{\eta}^{kl}\tilde{c}_{ljn}. \quad (5.49)$$

Setting  $n = 0$ , and letting  $i, j, m$  go for the ride, (5.49) gives a linear inhomogeneous system with unknowns  $\tilde{c}_{ijk}$ ,  $i > 0$  with coefficients being given by (complicated) polynomials in  $e^{u_0/30}, u_1, \dots, u_8$  with rational coefficients. One way to circumvent the complexity of solving it explicitly is as follows: firstly, it is immediate to prove that the system has maximal rank, which is an open condition, by evaluating the coefficients at a generic moduli point, so that  $\tilde{c}_{ijk}$  are uniquely determined rational functions in  $e^{u_0/30}, u_1, \dots, u_8$ . To check that the solution is indeed polynomial, we just plug a general polynomial ansatz into (5.49) satisfying the degree conditions of Proposition 5.4 and solve for its coefficients, and find that such an ansatz does indeed solve (5.49). The claim follows by uniqueness, the polynomiality of the inverse of (4.79), and Theorem 5.1. □

One immediate bonus of Theorem 5.5, and a further vindication of taking great pains to give a closed-form calculation of the mirror in Claim 2.3, is that both the Saito–Sekiguchi–Yano coordinates (4.79) and the prepotential of  $\tilde{X}_{\mathfrak{g},3}$ , for which an explicit form was unavailable to date<sup>30</sup>, can now be computed straightforwardly: the reader may find an expression for the latter in Appendix B.2. A further bonus is a mirror theorem for the Gromov–Witten theory of the polynomial  $\mathbb{P}^1$ -orbifold of type  $E_8$  [106, 122]:

**Corollary 5.6.** *Let  $C_{\mathfrak{g}} \simeq \mathbb{P}_{2,3,5}$  denote the orbifold base of the Seifert fibration of the Poincaré sphere  $\Sigma$  (see Section 4.1.3). Then,*

$$QH_{\text{orb}}(C_{\mathfrak{g}}) \simeq X_{\mathfrak{g}}^{\text{Toda}}. \quad (5.50)$$

*as Frobenius manifolds.*

This follows from composing the isomorphism  $QH_{\text{orb}}(C_{\mathfrak{g}}) \simeq \tilde{X}_{\mathfrak{g},3}$  (see [106]) with Theorem 5.5.

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<sup>30</sup>This is a private communication from Boris Dubrovin and Youjin Zhang.

**Remark 5.7.** In [106], a different type of mirror theorem was proved in terms of a polynomial three-dimensional Landau–Ginzburg model; it would be interesting to deduce directly a relation between the two mirror pictures, along the lines of what was done in a related context in [81]. As in [81], the two mirror pictures have complementary virtues: the threefold mirror of [106] has a considerably simpler form than the Toda/spectral curve mirror. On the other hand, having a spectral curve mirror pays off two important dividends: firstly, at genus zero, the calculation of flat co-ordinates for the Dubrovin connection (5.2) is simplified down to 1-dimensional (as opposed to 3-dimensional) oscillating integrals. Furthermore, and more remarkably, Givental’s formalism and the topological recursion might allow one to foray into the higher genus theory, recursively to all genera. This second aspect of the story will indeed be the subject of Section 5.4.

**5.3. General mirrors for Dubrovin–Zhang Frobenius manifolds.** There’s a fairly compelling picture emerging from Theorem 5.5 and the constructions of Sections 2 and 3 relating the Dubrovin–Zhang Frobenius manifolds of Section 5.1.2 to relativistic Toda spectral curves. I am going to propose here what the most general form of the conjecture should be. For this section only, the symbols  $\mathfrak{g}$ ,  $\mathfrak{h}$ ,  $\mathcal{G} = \exp(\mathfrak{g})$ ,  $\mathcal{W}$  will refer to an arbitrary simple, not necessarily simply-laced, complex Lie algebra, the corresponding Cartan subalgebra, simple simply-connected complex Lie group, Cartan torus and Weyl group. As in Section 2, let  $\rho \in R(\mathcal{G})$  be an irreducible representation of  $\mathcal{G}$  and for  $\mathfrak{g} \in \mathcal{G}$  consider the characteristic polynomial

$$\Xi_\rho(\theta_1, \dots, \theta_{r_{\mathfrak{g}}}; \mu) = \det_\rho(\mu \mathbf{1} - \mathfrak{g}) = \sum_{k=0}^{\dim \rho} \mu^{\dim \rho - k} \mathfrak{p}_k(\theta_1, \dots, \theta_{r_{\mathfrak{g}}}) \in \mathbb{Z}[\theta, \mu], \quad (5.51)$$

with  $\mathfrak{p}_k \in \mathbb{Z}[\theta]$  and  $\theta_i$  defined as in Section 2.4. Recall that  $\Xi_\rho$  reduces to a product over Weyl orbits  $W_i^\rho$

$$\Xi_\rho(\theta; \mu) = \prod_k \Xi_\rho^{(k)}(\theta; \mu) = \prod_k \prod_{j=1}^{|W_k^\rho|} \left( \mu - e^{\omega_j^{(k)} \cdot l} \right) \quad (5.52)$$

where  $e^l$  with  $[e^l] = [\mathfrak{g}]$  is any conjugacy class representative in the Cartan torus, and  $l \in \mathfrak{h}$ ; for example, when  $\rho = \mathfrak{g}$ , we have two factors  $\Xi_\rho^{(0)} = (\mu - 1)^{r_{\mathfrak{g}}}$  ( $W_0^\mathfrak{g} = \Delta^{(0)}$ ) and  $\Xi_{\mathfrak{g}, \text{red}} \triangleq \Xi_\rho^{(1)}$  irreducible of degree  $\dim_{\mathfrak{g}} - r_{\mathfrak{g}}$  ( $W_1^\mathfrak{g} = \Delta^+ \cup \Delta^-$ ); this was the case we considered for  $\mathcal{G} = E_8$ . In general, let  $\bar{k}$  be any integer in the product over  $k$  in (5.52) such that  $W_{\bar{k}}^\rho$  is non-trivial and define  $\Xi_{\rho, \text{red}} \triangleq \Xi_\rho^{(\bar{k})}$ , fix  $\alpha_{\bar{i}} \in \Pi$  a simple root, and define

$$\Gamma_u^{(\bar{i})} = \overline{\mathbb{V} \left( \Xi_{\rho, \text{red}} \left( \theta_j = u_j - \delta_{\bar{i}j} \frac{\lambda}{u_0} \right) \right)} \quad (5.53)$$

where the overline sign once again indicates taking the normalisation of the projective closure. Furthermore, letting  $\omega_1^{(\bar{k})}$  be the dominant weight in  $W_{\bar{k}}^\rho$ , denote

$$q_\rho = \frac{1}{2} \sum_{j=1}^{|W_{\bar{k}}^\rho|} \left\langle \omega_j^{(\bar{i})}, \omega_1^{(\bar{i})} \right\rangle^2. \quad (5.54)$$

Writing  $X_{\mathfrak{g},\bar{i}}^{\text{Toda}} = \mathbb{C}^\star \times (\mathbb{C}^\star)^{r_{\mathfrak{g}}}$  with co-ordinates  $u_0; u_1, \dots, u_{r_{\mathfrak{g}}}$ , define pairings  $(\eta, g)$  and product structure  $\partial_{u_i} \star \partial_{u_j}$  on  $TX_{\mathfrak{g}}^{\text{Toda}}$  by

$$\eta(\partial_{u_i}, \partial_{u_j}) = \sum_l \text{Res}_{p_l^{\text{cf}}} \frac{\partial_{u_i} \lambda \partial_{u_j} \lambda}{q_\rho \mu^2 \partial_\mu \lambda} d\mu, \quad (5.55)$$

$$\eta(\partial_{u_i}, \partial_{u_j} \star \partial_{u_k}) = \sum_l \text{Res}_{p_l^{\text{cf}}} \frac{\partial_{u_i} \lambda \partial_{u_j} \lambda \partial_{u_k} \lambda}{q_\rho \mu^2 \partial_\mu \lambda} d\mu, \quad (5.56)$$

$$g(\partial_{u_i}, \partial_{u_j}) = \sum_l \text{Res}_{p_l^{\text{cf}}} \frac{\partial_{u_i} \lambda \partial_{u_j} \lambda \partial_{u_k} \lambda}{q_\rho \mu^2 \partial_\mu \lambda} d\mu, \quad (5.57)$$

where  $\{p_l^{\text{cf}}\}_l$  are the ramification points of  $\lambda : \Gamma_u^{(i)} \rightarrow \mathbb{P}^1$ .

**Conjecture 5.8** (Mirror symmetry for DZ Frobenius manifolds). *The Landau–Ginzburg formulas (5.55)–(5.57) define a semi-simple, conformal Frobenius manifold  $(X_{\mathfrak{g},\bar{i}}^{\text{Toda}}, \eta, e, E, \star)$ , which is independent of the choice of irreducible representation  $\rho$  and non-trivial Weyl orbit  $W_k^\rho$ . In particular, (5.55) and (5.57) define flat non-degenerate metrics on  $TX_{\mathfrak{g},\bar{i}}^{\text{Toda}}$ , and the identity and Euler vector fields read, in curved coordinates  $u_0, \dots, u_{r_{\mathfrak{g}}}$ ,*

$$e = u_0^{-1} \partial_{u_{\bar{i}}}, \quad E = u_0 \partial_{u_0}. \quad (5.58)$$

Moreover,

$$X_{\mathfrak{g},\bar{i}}^{\text{Toda}} \simeq \tilde{X}_{\mathfrak{g},\bar{i}}. \quad (5.59)$$

There is a fair amount of circumstantial evidence in favour of the validity of the mirror conjecture in the form and generality proposed.

- (1) Firstly, the independence on the choice of representation should be a consequence of the work of [86–88] on the “hierarchy” of Jacobians of spectral curves for the periodic Toda lattice and associated, isomorphic preferred Prym–Tyurins. The very same calculation of Section 4.3.2 of the intersection form (5.57) in this case does indeed show that the sum-of-residues in different representations  $\rho, \rho'$  coincide up to an overall factor of  $q_\rho/q_{\rho'}$ , which in (5.55)–(5.57) is accounted for by the explicit inclusion of  $q_\rho$  at the denominator.
- (2) Isomorphisms of the type (5.59) have already appeared in the literature, and they all fit in the framework of Conjecture 5.8. In their original paper [43], Dubrovin–Zhang formulate a mirror theorem for the  $A$ -series which is indeed the specialisation of Conjecture 5.8 to  $\mathfrak{g} = \mathfrak{sl}_N$  and  $\rho = \square = \rho_{\omega_1}$  the fundamental representation. Their mirror theorem was extended to the other classical Lie algebras  $\mathfrak{b}_N, \mathfrak{c}_N$  and  $\mathfrak{d}_N$  by the same authors with Strachan and Zuo in [42]: the Toda mirrors of Conjecture 5.8 specialise to their LG models for  $\mathfrak{g} = \mathfrak{so}_{2N+1}, \mathfrak{sp}_N$  and  $\mathfrak{so}_{2N}$  with  $\rho$  being in all cases the defining vector representation  $\rho = \rho_{\omega_1}$ .

$\mathcal{G}$	$\rho$	$q_\rho$	$\dim \rho$	$\deg_\mu \Xi_{\rho,\text{red}}$	$\deg_{u_i} \Xi_{\rho,\text{red}}$	$g(\Gamma_u^{(\bar{i})})$
E <sub>6</sub>	$\rho_{\omega_1}$	6	27	27	5, 3, 2, 3, 5, 3	5
E <sub>7</sub>	$\rho_{\omega_6}$	12	56	56	6, 4, 3, 4, 5, 10, 6	33
E <sub>8</sub>	$\rho_{\omega_7}$	60	248	240	23, 13, 9, 11, 14, 19, 29, 17	128
F <sub>4</sub>	$\rho_{\omega_4}$	6	26	24	3, 2, 3, 5	4
G <sub>2</sub>	$\rho_{\omega_1}$	6	7	7	2, 1	0

TABLE 4. Degree and genera of minimal spectral curve (putative) mirrors of  $\tilde{X}_{\mathfrak{g},\bar{i}}$  for the exceptional series EFG; for simplicity I only indicate the genus for the original choice of marked node  $\bar{i}$  in [43, Table I].

- (3) At the opposite end of the simple Lie algebra spectrum, Theorem 5.5 gives an affirmative answer to Conjecture 5.8 for the most exceptional example of  $\mathcal{G} = \text{E}_8$ ; it is only natural to speculate that the missing exceptional cases should fit in as well.
- (4) Some further indication that Conjecture 5.8 should hold true comes from the study of Seiberg–Witten curves in the same limit considered for Section 4.3.3, together with  $\Lambda_4 \rightarrow 0$ . It was speculated already in [81] that the perturbative limit of 4d SW curves with ADE gauge group should be related to ADE topological Landau–Ginzburg models (and hence the finite Coxeter Frobenius manifolds of (5.14)) via an operation foreshadowing the notion of almost-duality in [41]; this was further elaborated upon in [48] for  $\mathfrak{g} = \mathfrak{e}_6$ , and [47] for  $\mathfrak{g} = \mathfrak{e}_7$ .
- (5) Finally, our way of accommodating the extra datum of the choice of simple root  $\alpha_{\bar{i}}$  is not only consistent with the results [42], but also with the general idea that these Frobenius manifolds should be related to each other by a Type I symmetry of WDVV (a Legendre-type transformation) in the language of [39, Appendix B]. Indeed, different choices of fundamental characters  $u_{\bar{i}}$  shifting the value of the superpotential correspond precisely to a symmetry of WDVV where the new unit vector field is one of the old non-unital coordinate vector fields. This parallels precisely the general construction of [42, 43].

It should be noticed that, away from the classical ABCD series and the exceptional case G<sub>2</sub>,  $\Gamma_u^{(\bar{i})}$  is typically not a rational curve, not even for the “minimal” case in which  $\alpha_{\bar{i}}$  is chosen as the root corresponding to the attaching node of the external root in the Dynkin diagram, and  $\rho$  is a minimal non-trivial irreducible representation. For the time being, I’ll content myself to provide some data on the exceptional cases in Table 4, and defer a proof of Conjecture 5.8 to a separate publication.

**5.4. Polynomial  $\mathbb{P}^1$  orbifolds at higher genus.** As a final application, I restrict my attention to  $\mathcal{G}$  being simply-laced. In this case, Conjecture 5.8 and [106] would imply the following

**Conjecture 5.9.** *With notation as in Conjecture 5.8, let  $\bar{i}$  be an arbitrary node of the Dynkin diagram for  $\mathfrak{g}$  of type A, or the node corresponding to the highest dimensional fundamental representation<sup>31</sup> for type D and E:*

$$\bar{i} = \begin{cases} i = 1, \dots, n, & \mathfrak{g} = A_n \\ n - 2, & \mathfrak{g} = D_n \\ 3, & \mathfrak{g} = E_n \end{cases} \quad (5.60)$$

Then,

$$X_{\mathfrak{g}, \bar{i}}^{\text{Toda}} \simeq QH_{\text{orb}}(C_{\mathfrak{g}}) \quad (5.61)$$

where  $C_{\mathfrak{g}}$  is the polynomial  $\mathbb{P}^1$ -orbifold of type  $\mathfrak{g}$ :

$$C_{\mathfrak{g}} = \begin{cases} \mathbb{P}(\bar{i}, n - \bar{i} + 1), & \mathfrak{g} = A_n \\ \mathbb{P}(2, 2, n - 2) & \mathfrak{g} = D_n \\ \mathbb{P}(2, 3, n - 3) & \mathfrak{g} = E_n \end{cases} \quad (5.62)$$

There are two noteworthy implications of such a statement. The first is that the LG model of the previous section would provide a dispersionless Lax formalism for the integrable hierarchy of topological type on the loop space of the Frobenius manifold  $QH_{\text{orb}}(C_{\mathfrak{g}})$  [39, Lecture 6]; for type A, this is well-known to be the extended bi-graded Toda hierarchy of [30] (see also [90]), and for all ADE types, a construction was put forward in [89] for these hierarchies in the form of Hirota quadratic equations. The zero-dispersion Lax formulation of the hierarchy could be a key to relate such remarkable, yet obscure hierarchy to a well-understood parent 2+1 hierarchy such as 2D-Toda, as was done in a closely related context in [24]. A more direct consequence is a Givental-style, genus-zero-controls-higher-genus statement, as follows. On the Gromov–Witten side, and as a vector space, the Chen–Ruan co-homology of  $C_{\mathfrak{g}}$  is the co-homology of the inertia stack  $IC_{\mathfrak{g}}$  [33, 122], which is generated by the identity class  $\phi_{r_{\mathfrak{g}}} \triangleq \mathbf{1}_0$ , the Kähler class  $\phi_0 \triangleq p$ , and twisted cohomology classes concentrated at the stacky points of  $C_{\mathfrak{g}}$ ,

$$\phi_{\nu(i,r)} \triangleq \mathbf{1}_{\left(\frac{i}{s_r}, r\right)} \in H^{\left(\frac{i}{s_r}, r\right)}(C_{\mathfrak{g}}) \simeq H(B\mathbb{Z}_{s_r}), \quad i = 1, \dots, r - 1 \quad (5.63)$$

where  $r = 1, 2$  for type A and  $r = 1, 2, 3$  for type D and E label the orbifold points of  $C_{\mathfrak{g}}$ ,  $s_r$  is the order of the respective isotropy groups, we label components of the  $IC_{\mathfrak{g}}$  by  $\left(\frac{i}{r}, s_r\right)$ , and  $\nu(i, r)$  is a choice of a map to  $[[1, r_{\mathfrak{g}}]]$  increasingly sorting the sets of pairs  $(i, r)$  by the value of  $i/s_r$ .<sup>32</sup> Define now the genus- $g$  full-descendent Gromov–Witten potential of  $C_{\mathfrak{g}}$  as the formal power series

$$\mathcal{F}_g^{C_{\mathfrak{g}}} = \sum_{n \geq 0} \sum_{d \in \text{Eff}(C_{\mathfrak{g}})} \sum_{\substack{\alpha_1, \dots, \alpha_n \\ k_1, \dots, k_n}} \frac{\prod_{i=1}^n t_{\alpha_i, k_i}}{n!} \langle \tau_{k_1}(\phi_{\alpha_1}) \cdots \tau_{k_n}(\phi_{\alpha_n}) \rangle_{g, n, d}^{C_{\mathfrak{g}}}, \quad (5.64)$$

<sup>31</sup>Equivalently, this is the attaching node of the external root(s) in the Dynkin diagram.

<sup>32</sup>For the case  $\mathfrak{g} = \mathfrak{e}_8$ , since  $\text{gcd}(2, 3, 5) = 1$ , there is no ambiguity in the choice of  $\nu$ , and the choice of labelling of  $\phi_{\alpha}$  here was made to match that of the Saito vector fields  $\partial_{\mathfrak{t}_{\alpha}}$  of Lemma 5.3: up to scale, we have  $\phi_{\alpha} = \partial_{\mathfrak{t}_{\alpha}}$ ,  $\partial_{\mathfrak{t}_0} = p$ ,  $\partial_{\mathfrak{t}_8} = \mathbf{1}_0$ ,  $\partial_{\mathfrak{t}_1} = \mathbf{1}_{\left(\frac{1}{5}, 3\right)}$ ,  $\partial_{\mathfrak{t}_2} = \mathbf{1}_{\left(\frac{1}{3}, 2\right)}$ ,  $\partial_{\mathfrak{t}_3} = \mathbf{1}_{\left(\frac{2}{5}, 3\right)}$ ,  $\partial_{\mathfrak{t}_4} = \mathbf{1}_{\left(\frac{1}{3}, 2\right)}$ ,  $\partial_{\mathfrak{t}_5} = \mathbf{1}_{\left(\frac{1}{2}, 1\right)}$ ,  $\partial_{\mathfrak{t}_6} = \mathbf{1}_{\left(\frac{3}{5}, 3\right)}$ ,  $\partial_{\mathfrak{t}_7} = \mathbf{1}_{\left(\frac{2}{3}, 2\right)}$ ,  $\partial_{\mathfrak{t}_8} = \mathbf{1}_{\left(\frac{4}{5}, 2\right)}$ .

where  $\text{Eff}(C_g) \subset H^2(C_g, \mathbb{Z})/H_{\text{tor}}^2(C_g, \mathbb{Z})$  is the set of degrees of twisted stable maps to  $C_g$ , and the usual correlator notation for multi-point descendent Gromov–Witten invariants was employed,

$$\langle \tau_{k_1}(\phi_{\alpha_1}) \cdots \tau_{k_n}(\phi_{\alpha_n}) \rangle_{g,n,d}^{C_g} \triangleq \int_{[\mathcal{M}_{g,n}(C_g, d)]^{\text{vir}}} \prod_{i=1}^n \text{ev}_i^* \phi_{\alpha_i} \psi_i^{k_i}. \quad (5.65)$$

Since  $QH(C_g)$  is semi-simple, the Givental–Teleman Reconstruction theorem applies [116]. I will refer the reader to [26, 57, 80] for the relevant background material, context, and detailed explanations of origin and inner workings of the formula; symbolically and somewhat crudely, this is, for a general target  $X$  with semi-simple quantum cohomology,

$$\exp \left( \sum_g \epsilon^{2g-2} \mathcal{F}_g^X(t) \right) = \widehat{S}_{\text{GW},X}^{-1} \widehat{\psi}_{\text{GW},X} \widehat{R}_{\text{GW},X} \prod_{i=1}^{r_g+1} \tau_{\text{KdV}}(u) \quad (5.66)$$

where the *calibrations*  $S_{\text{GW},X}$  and  $R_{\text{GW},X}$  are elements of the linear symplectic loop group of  $QH(X) \otimes \mathbb{C}[\hbar, \hbar^{-1}]$  given by flat coordinate frames for the restricted Dubrovin connection to the internal direction of the Frobenius manifold (5.2), which are respectively analytic in  $\hbar$  and formal in  $1/\hbar$ . The hat symbol signifies normal-ordered quantisation of the corresponding linear symplectomorphism (namely, an exponentiated quantised quadratic Hamiltonian),  $\psi_{\text{GW},X}$  is the Jacobian matrix of the change-of-variables from flat to normalised canonical frame, and  $\tau_{\text{KdV}}$  is the Witten–Kontsevich Kortweg–de-Vries  $\tau$ -function, that is, the exponentiated generating function of GW invariants of the point. The essence of (5.66) is that there exists a judicious composition of explicit, exponentiated quadratic differential operators in  $t_{\alpha,k}$  and changes of variables  $u_k^{(i)} \rightarrow t_{\alpha,k}$  from the  $k^{\text{th}}$  KdV time of the  $i^{\text{th}}$   $\tau$ -function in (5.66) which returns the full-descendent, all-genus GW partition function of  $X$ . In our specific case  $X = C_g$  (and in general, whenever we consider non-equivariant GW invariants), by the Conformality axiom of Definition 5.1, both  $\widehat{S}_{\text{GW},X}$  and  $\widehat{R}_{\text{GW},X}$  are determined by the Frobenius manifold structure of  $QH(X)$  alone, without any further input [116]: the grading condition given by the flatness in the  $\mathbb{C}_\hbar^*$  direction of the Dubrovin connection fixes uniquely the normalisation of the canonical flat frames  $S$  and  $R$  at  $\hbar = 0, \infty$  respectively. For reference, the  $R$ -action on the Witten–Kontsevich  $\tau$ -functions gives the *ancestor potential* in the normalised canonical frame

$$\exp \left( \sum_g \epsilon^{2g-2} \mathcal{A}_g^X(t) \right) \triangleq \widehat{R}_{\text{GW},X} \prod_{i=1}^{r_g+1} \tau_{\text{KdV}}(u) \quad (5.67)$$

to which the descendent generating function (5.66) is related by a linear change of variables (via  $\psi$ ) and a triangular transformation of the full set of time variables (via  $S^{-1}$ ); see [80, Chapter 2].

On the Toda/spectral curve side, a similar higher genus reconstruction theorem exists in light of its realisation as a Frobenius submanifold of a Hurwitz space: this is, as in Section 4.1.4, the Chekhov–Eynard–Orantin (CEO) topological recursion procedure, giving a sequence  $(F_g^{\text{CEO}}(\mathcal{S}), W_{g,h}^{\text{CEO}}(\mathcal{S}))$  of generating functions (4.37)–(4.38) specified by the Dubrovin–Krichever data of Definition 3.1. Having proved, or taking for granted the isomorphism of the underlying Frobenius manifolds as in Conjecture 5.8, it is natural to ask whether the two higher genus theories are related at all. A precise answer comes from the work of [46], where the authors show that there exists an explicit



change of variables  $t_{\alpha,k} \rightarrow v_{i,j}$  and an  $R$ -calibration of the Hurwitz space Frobenius (sub)manifold associated to  $\mathcal{S}_{\mathfrak{g}}$  such that

$$\exp \left( \epsilon^{2g-2} \sum_{g,d} W^{\text{CEO}}(\mathcal{S})_{g,d}(v) \right) = \widehat{R_{\text{CEO}}}(\mathcal{S}) \prod_{i=1}^{r_{\mathfrak{g}}+1} \tau_{\text{KdV}}(u) \quad (5.68)$$

where the independent variables  $v_{i,j}$  on the l.h.s. are obtained from the arguments of the CEO multi-differentials upon expansion around the  $i^{\text{th}}$  branch point of the spectral curve (see [46, Theorem 4.1] and the discussion preceding it for the exact details). In other words, the topological recursion reconstructs the *ancestor* potential of a two-dimensional semi-simple cohomological field theory, with  $R$ -calibration  $R_{\text{CEO}}(\mathcal{S})$  entirely specified by the spectral curve geometry via a suitable Laplace transform of the Bergman kernel. One upshot of this is that, up to a further change-of-variables and a (non-trivial) shift by a quadratic term, (5.68) can be put in the form of (5.66).

So, in a situation where  $\mathcal{S}_X$  is a spectral curve mirror to  $X$ , we have two identical reconstruction theorems for the higher genus ancestor potential starting from genus zero CohFT data, both being unambiguously specified in terms of  $R$ -actions  $R_{\text{GW},X}$  and  $R_{\text{CEO}}(\mathcal{S}_X)$ . If these agree, then the full higher genus potentials agree, and the higher genus ancestor invariants of  $X$  are computed by the topological recursion on  $\mathcal{S}_X$  by (5.68). Happily, it is a result of Shramchenko that in non-equivariant GW theory this is always precisely the case [111] (see also [44, Theorem 7]):

$$R_{\text{GW},X} = R_{\text{CEO}}(\mathcal{S}_X). \quad (5.69)$$

In other words, the  $R$ -calibration  $R_{\text{CEO}}(\mathcal{S}_X)$ , which is uniquely specified by the Bergmann kernel of a family of spectral curves  $\mathcal{S}_X$  whose prepotential coincides with the genus zero GW potential of a projective variety<sup>33</sup>  $X$ , coincides with the  $R$ -calibration  $R_{\text{GW},X}$  uniquely picked by the de Rham grading in the (non-equivariant) quantum cohomology of  $X$ . We get to the following

**Corollary 5.10.** *Suppose that Conjecture 5.8 holds. Then the ancestor higher genus potential of  $C_{\mathfrak{g}}$  equates to the higher genus topological recursion potential*

$$\mathcal{A}_g^{C_{\mathfrak{g}}} = \sum_h W_{g,h}^{\mathcal{S}_{\mathfrak{g}}} \quad (5.70)$$

*up to the change-of-variables of [46, Theorem 4.1].*

In particular, such all-genus full-ancestor statements hold in type A by [43, Theorem 3.1], type D by [42, Theorem 5.6] and type  $E_8$  by Theorem 5.5. The two remaining exceptional cases can be treated along the same lines of Theorem 5.5, and far more easily than the case of  $E_8$ , and are left as an exercise to the reader.

**Remark 5.11** (On an ADE Norbury–Scott theorem). For the case of the Gromov–Witten theory of  $\mathbb{P}^1$ , it was proposed by Norbury–Scott in [96], supported by a low-genus proof and a heuristic all-genus argument, and later proved in full generality by the authors of [46] using (5.68), that the

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<sup>33</sup>More generally, a Gorenstein orbifold with projective coarse moduli space.

residue at infinity of the CEO differentials  $W_{g,n}^{\mathcal{S}_a}$  gives the  $n$ -point, genus- $g$  stationary GW invariant of  $\mathbb{P}^1$ ,

$$\prod_{j=1}^n \operatorname{Res}_{z_j=\infty} \frac{z_j^{m_j+1}}{(m_j+1)!} W_{g,n}^{\operatorname{Toda}}(z_1, \dots, z_n) = (-)^n \left\langle \prod_{i=1}^n \tau_{m_i}(p) \right\rangle. \quad (5.71)$$

I fully expect that a completely analogous ADE orbifold version of (5.71), which allows for a very efficient way to compute GW invariants at higher genera, would hold for all polynomial  $\mathbb{P}^1$ -orbifolds. For type A and D, where the curve is rational, the statement of (5.71) would probably carry forth verbatim, with the r.h.s. being given by  $n$ -pointed, non-stationary, untwisted GW invariants. For type E, it will perhaps be necessary to sum over all branches above  $\infty$  (8, in the case of  $E_8$ ) to obtain the desired result. I also expect that in type D and E, poles of  $\lambda$  at finite  $\mu$  will presumably compute twisted invariants, with twisted insertions being labelled by the location of the poles. In particular, in type  $E_8$ , the poles at  $\frac{\ln \mu}{2\pi i} \in \{1/5, 1/3, 2/5, 1/2, 3/5, 1/3, 4/5\}$  should correspond to insertions of  $\mathbf{1}_{i/s_r, r}$  for the corresponding value of  $i/s_r$ .

## APPENDIX A. PROOF OF PROPOSITION 3.1

I am going to prove Proposition 3.1 by first establishing the following

**Lemma A.1.** *The number of double cosets of  $\mathcal{W}$  by  $\mathcal{W}_{\alpha_0}$  is*

$$|\mathcal{W}_{\alpha_0} \backslash \mathcal{W} / \mathcal{W}_{\alpha_0}| = 5. \quad (A.1)$$

*Proof.* The order of the double coset space  $\mathcal{W}_{\alpha_0} \backslash \mathcal{W} / \mathcal{W}_{\alpha_0}$  is the square norm of the character of the trivial representation of  $\mathcal{W}_{\alpha_0}$ , induced up to  $\mathcal{W}$  [112, Ex. 7.77a],

$$|\mathcal{W}_{\alpha_0} \backslash \mathcal{W} / \mathcal{W}_{\alpha_0}| = \left\langle \operatorname{ind}_{\mathcal{W}_{\alpha_0}}^{\mathcal{W}} 1, \operatorname{ind}_{\mathcal{W}_{\alpha_0}}^{\mathcal{W}} 1 \right\rangle. \quad (A.2)$$

Now,  $\operatorname{ind}_{\mathcal{W}_{\alpha_0}}^{\mathcal{W}} 1$  is just the permutation representation  $\mathbb{C} \langle \Delta^* \rangle$  on the free vector space on the set of non-zero roots  $\Delta^* \simeq \mathcal{W} / \mathcal{W}_{\alpha_0}$ . Suppose that

$$\mathbb{C} \langle \Delta^* \rangle = \bigoplus m_i R_i \quad (A.3)$$

for irreducible representations  $R_i \in R(\mathcal{W})$  and  $m_i \in \mathbb{Z}$ . Then, by (A.2),

$$|\mathcal{W}_{\alpha_0} \backslash \mathcal{W} / \mathcal{W}_{\alpha_0}| = \sum_i m_i^2. \quad (A.4)$$

The multiplicity of  $R_i$  in  $\mathbb{C} \langle \Delta^* \rangle$  is easily computed as follows. Let  $c \in \mathcal{W}$  and  $[c]$  its conjugacy class. Then its  $\mathbb{C} \langle \Delta^* \rangle$ -character

$$\chi_{\mathbb{C} \langle \Delta^* \rangle}([c]) = \dim_{\mathbb{C}} \{v \in \mathfrak{h}^* | cv = v\} \quad (A.5)$$

is equal to the dimension of the eigenspace of fixed points of  $c$ . In the standard labelling [31, 101] of conjugacy classes of  $\mathcal{W} = \text{Weyl}(\mathfrak{e}_8)$ , we compute the r.h.s. of (A.5) to be

$$\chi_{\mathcal{C}(\Delta^*)}([c]) = \begin{cases} 2 & [c] \in \{2b, 4c, 6a, 12a, 4e, 4g, 30b, 10d, 6m, 3d, 24c, \\ & 6s, 18c, 6x, 12n, 7a, 14a, 6y, 6z, 6ab\}, \\ 4 & [c] \in \{4h, 2e, 10b, 30c, 6i, 30d, 12p, 12r\}, \\ 6 & [c] \in \{8c, 4b, 8e, 6c, 2c, 12h, 8h, 6q, 14b, 12q\}, \\ 8 & [c] \in \{4d, 4m\}, \\ 12 & [c] \in \{12f, 2f, 5a, 6g, 10e\}, \\ 14 & [c] = 12m, \\ 20 & [c] = 18d, \\ 24 & [c] \in \{8a, 24b, 20b\}, \\ 26 & [c] = 6b, \\ 30 & [c] = 12d, \\ 40 & [c] = 6h, \\ 60 & [c] = 12e, \\ 72 & [c] = 4f, \\ 126 & [c] = 6e, \\ 240 & [c] = 1a, \\ 0 & \text{else.} \end{cases}$$

From (A.6), we obtain<sup>34</sup>

$$\begin{aligned} m_i &= \langle \chi_{R_i}, \chi_{\mathcal{C}(\Delta^*)} \rangle \\ &= \begin{cases} 1 & R_i \in \{\mathbf{1}_+ \simeq 1, \mathbf{8}_+ \simeq \mathfrak{h}^*, \mathbf{35}_-, \mathbf{84}_-, \mathbf{112}_+\}, \\ 0 & \text{else,} \end{cases} \end{aligned} \quad (\text{A.6})$$

from which the claim follows.  $\square$

Proposition 3.1 is an easy consequence of the Lemma A.1: by  $\mathcal{W}_{\alpha_0}$ -invariance, (3.17) defines an element of the Hecke ring, and it is immediately seen to assume exactly five constant values  $-2, -1, 0, 1, 2$  on the hyperplanes  $H_i$ , which are then in bijection with the elements of  $H(\mathcal{W}, \mathcal{W}_{\alpha_0})$ .

## APPENDIX B. SOME FORMULAS FOR THE $\mathfrak{e}_8$ AND $\mathfrak{e}_8^{(1)}$ ROOT SYSTEM

I gather here some reference material for the finite and affine  $E_8$  root systems. Let  $\{e_i\}$ ,  $i = 1, \dots, 8$  be an orthonormal basis for  $\mathbb{R}^8$ . The simple roots  $\{\alpha_i\}$ ,  $i = 1, \dots, 8$  have components in this basis

<sup>34</sup>I have labelled here irreps of  $\mathcal{W}$  as  $\dim R_{\text{sgn}\chi_R(\mathfrak{8d})}$ . A rather standard use in the literature is to label irreps of exceptional Weyl groups by how they are stored in the **GAP** library; for reference, here the summands in the decomposition (A.6) would be called **X.1** (trivial), **X.3** (Coxeter), **X.8**, **X.15** and **X.16**.

given by

$$\begin{aligned}
\alpha_1 &= \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right), \\
\alpha_2 &= (0, 0, 0, 0, 0, -1, 1, 0), \\
\alpha_3 &= (0, 0, 0, 0, -1, 1, 0, 0), \\
\alpha_4 &= (0, 0, 0, -1, 1, 0, 0, 0), \\
\alpha_5 &= (0, 0, -1, 1, 0, 0, 0, 0), \\
\alpha_6 &= (1, 1, 0, 0, 0, 0, 0, 0), \\
\alpha_7 &= (0, -1, 1, 0, 0, 0, 0, 0), \\
\alpha_8 &= (-1, 1, 0, 0, 0, 0, 0, 0)
\end{aligned} \tag{B.1}$$

The affine root system is obtained from (B.1) upon adding the affine root

$$\alpha_0 = (0, 0, 0, 0, 0, 0, -1, -1). \tag{B.2}$$

The respective Cartan matrices are given, from (B.1)–(B.2), by

$$\mathcal{C}^{\epsilon_8} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 2 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \tag{B.3}$$

$$\mathcal{C}^{\epsilon_8^{(1)}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}. \tag{B.4}$$

The resulting simple Lie algebra for  $\epsilon_8$  has rank 8 and dimension 240. In the  $\alpha$ -basis, (B.1), the affine root (B.2) reads

$$\alpha_0 = \sum_i \mathfrak{d}_i \alpha_i = 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + 3\alpha_8. \tag{B.5}$$

Since  $\det \mathcal{C}^{\epsilon_8} = 1$ , we have  $\Lambda_r(\epsilon_8) \simeq \Lambda_w(\epsilon_8)$ .

B.0.1. *On the minimal orbit of  $\mathcal{W}$ .* I group here the details of the minimal orbit of  $\mathcal{W}$  in  $\Lambda_r \subset \mathbb{Z}^s$  generated by the adjoint weight  $\omega_7$ , in terms of 240 vectors in a  $s = 30$ -dimensional lattice. Since the orbit is in bijection with the set of non-zero roots of  $\mathfrak{g}$ ,  $\omega$  is in the orbit iff  $-\omega$  is; also the cyclic shift of the components in  $\mathbb{Z}^s$  corresponds to the action of the Coxeter element on the orbit, which

is thus preserved if we send  $\omega \rightarrow (\omega_{j+1 \bmod s})_j$ . The resulting  $\mathbb{Z}_2 \times \mathbb{Z}_{30}$  action breaks up the orbit into sub-orbits, representatives for which are displayed<sup>35</sup> in Table 5.

$n_0$	$\pm 6$	$\pm 5$	$\pm 4$	$\pm 3$	$\pm 3$	$\pm 2$	$\pm 2$	$\pm 1$	0	0	0	0
card	10	12	30	20	20	30	30	60	10	10	6	2
$(\omega)_1$	1	-1	-1	-1	1	-1	1	2	-1	1	1	1
$(\omega)_2$	0	1	1	1	-1	1	-1	-1	0	0	-1	-1
$(\omega)_3$	0	0	0	0	1	1	0	0	0	-1	0	1
$(\omega)_4$	0	0	0	1	-1	-1	1	0	0	0	1	-1
$(\omega)_5$	0	0	1	-1	1	0	-1	0	1	0	-1	1
$(\omega)_6$	1	1	-1	0	0	0	1	0	-1	1	0	-1
$(\omega)_7$	0	-1	0	0	0	0	0	1	0	0	1	1
$(\omega)_8$	0	1	1	1	0	1	-1	-1	0	-1	-1	-1
$(\omega)_9$	0	0	0	0	0	0	1	0	0	0	0	1
$(\omega)_{10}$	0	0	0	0	0	-1	0	0	1	0	1	-1
$(\omega)_{11}$	1	0	0	-1	1	0	0	1	-1	1	-1	1
$(\omega)_{12}$	0	1	0	1	-1	1	0	-1	0	0	0	-1
$(\omega)_{13}$	0	-1	0	0	1	0	0	1	0	-1	1	1
$(\omega)_{14}$	0	1	1	1	-1	0	0	-1	0	0	-1	-1
$(\omega)_{15}$	0	0	0	-1	1	0	0	0	1	0	0	1
$(\omega)_{16}$	1	0	-1	0	0	-1	1	1	-1	1	1	-1
$(\omega)_{17}$	0	0	1	0	0	1	-1	0	0	0	-1	1
$(\omega)_{18}$	0	1	0	1	0	1	0	-1	0	-1	0	-1
$(\omega)_{19}$	0	-1	0	0	0	-1	1	1	0	0	1	1
$(\omega)_{20}$	0	1	1	0	0	0	-1	-1	1	0	-1	-1
$(\omega)_{21}$	1	0	-1	-1	1	0	1	1	-1	1	0	1
$(\omega)_{22}$	0	0	0	1	-1	0	0	0	0	0	1	-1
$(\omega)_{23}$	0	0	1	0	1	1	-1	0	0	-1	-1	1
$(\omega)_{24}$	0	1	0	1	-1	0	1	-1	0	0	0	-1
$(\omega)_{25}$	0	-1	0	-1	1	-1	0	1	1	0	1	1
$(\omega)_{26}$	1	1	0	0	0	0	0	0	-1	1	-1	-1
$(\omega)_{27}$	0	0	0	0	0	1	0	0	0	0	0	1
$(\omega)_{28}$	0	0	0	1	0	0	0	0	0	-1	1	-1
$(\omega)_{29}$	0	0	1	0	0	0	0	0	0	0	-1	1
$(\omega)_{30}$	0	1	0	0	0	0	0	-1	1	0	0	-1

TABLE 5.  $\mathbb{Z}_2 \times \mathbb{Z}_{30}$  sub-orbits of the minimal orbit of  $\mathcal{W}$ .

<sup>35</sup>The entries in the 7<sup>th</sup> and 8<sup>th</sup> column of row  $(\omega)_7$ , as well as those of the 26<sup>th</sup> and 27<sup>th</sup> column of row  $(\omega)_{11}$  correct four typos in the table of [15, Appendix F.1].

	1 <sub>1</sub>	1 <sub>-1</sub>	30 <sub>1</sub>	20 <sub>1</sub>	20 <sub>-1</sub>	12 <sub>ϕ</sub>	12 <sub>ϕ<sup>-1</sup></sub>	12 <sub>-ϕ</sub>	12 <sub>-ϕ<sup>-1</sup></sub>
χ <sub>id</sub>	1	1	1	1	1	1	1	1	1
χ <sub>1</sub>	2	-2	0	1	-1	ϕ	ϕ <sup>-1</sup>	-ϕ	-ϕ <sup>-1</sup>
χ <sub>2</sub>	4	-4	0	-1	1	1	-1	-1	1
χ <sub>3</sub>	6	-6	0	0	0	-1	1	1	-1
χ <sub>5</sub>	4	4	0	1	1	-1	-1	-1	-1
χ <sub>6</sub>	3	3	-1	0	0	-ϕ <sup>-1</sup>	ϕ	-ϕ <sup>-1</sup>	ϕ
χ <sub>4</sub>	5	5	1	-1	-1	0	0	0	0
χ <sub>7</sub>	2	-2	0	1	-1	-ϕ <sup>-1</sup>	-ϕ	ϕ <sup>-1</sup>	ϕ
χ <sub>8</sub>	3	3	-1	0	0	ϕ	-ϕ <sup>-1</sup>	ϕ	-ϕ <sup>-1</sup>

TABLE 6. The character table of  $\tilde{\mathbb{I}}$ . Conjugacy classes  $[g]$  are denoted by their order and sign of the SU(2) character  $\chi_1$  as  $||[g]||_{\chi_1([g])}$ .

**B.1. The binary icosahedral group  $\tilde{\mathbb{I}}$ .** The binary icosahedral group  $\tilde{\mathbb{I}}$  is the preimage of the symmetry group of a regular icosahedron in  $\mathbb{E}^3$  by the degree two covering map  $SU(2) \rightarrow SO(3)$ . It has a presentation as the group generated by the unit quaternions

$$s = \frac{1}{2}(1 + i + j + k), \quad t = \frac{1}{2}(\phi + \phi^{-1} + i + j), \quad (\text{B.6})$$

whose full set of relations is  $s^3 = t^5 = (st)^2$ . The resulting group  $\tilde{\mathbb{I}}$  has order 120, exponent 60, and class order 9. Its character table is given in Table 6.

**B.2. The prepotential of  $\tilde{X}_{e_8,3}$ .**

$$\begin{aligned}
F_{\tilde{X}_{e_8,3}} = & -\frac{7t_1^{10}}{135000000} + \frac{521e^{6t_0}t_1^9}{2700000} + \frac{5117e^{12t_0}t_1^8}{100000} + \frac{e^{2t_0}t_2t_1^8}{300000} + \frac{7t_3t_1^8}{15000000} + \frac{243}{250}e^{18t_0}t_1^7 \\
& + \frac{136e^{8t_0}t_2t_1^7}{1875} + \frac{67e^{6t_0}t_3t_1^7}{75000} + \frac{e^{3t_0}t_4t_1^7}{1875} - \frac{t_5t_1^7}{3937500} + \frac{1954}{375}e^{24t_0}t_1^6 + \frac{7}{600}e^{4t_0}t_2^2t_1^6 \\
& - \frac{13t_3^2t_1^6}{7500000} + \frac{2401}{750}e^{14t_0}t_2t_1^6 + \frac{2151e^{12t_0}t_3t_1^6}{25000} + \frac{e^{2t_0}t_2t_3t_1^6}{25000} + \frac{243}{500}e^{9t_0}t_4t_1^6 + \frac{19e^{6t_0}t_5t_1^6}{56250} \\
& - \frac{e^{4t_0}t_6t_1^6}{1000} + \frac{t_7t_1^6}{18750000} + 12e^{30t_0}t_1^5 + \frac{43}{12}e^{10t_0}t_2^2t_1^5 + \frac{71e^{6t_0}t_3^2t_1^5}{50000} \\
& + \frac{43}{2}e^{20t_0}t_2t_1^5 + \frac{189}{250}e^{18t_0}t_3t_1^5 + \frac{4}{25}e^{8t_0}t_2t_3t_1^5 + 7e^{15t_0}t_4t_1^5 \\
& + \frac{1}{2}e^{5t_0}t_2t_4t_1^5 + \frac{1}{250}e^{3t_0}t_3t_4t_1^5 + \frac{67e^{12t_0}t_5t_1^5}{6250} + \frac{e^{2t_0}t_2t_5t_1^5}{18750} + \frac{11t_3t_5t_1^5}{4687500} \\
& - \frac{1}{12}e^{10t_0}t_6t_1^5 + \frac{e^{6t_0}t_7t_1^5}{25000} + \frac{159}{10}e^{36t_0}t_1^4 + \frac{117}{100}e^{6t_0}t_2^3t_1^4 + \frac{7t_3^3t_1^4}{2500000} + \frac{592}{15}e^{16t_0}t_2^2t_1^4 \\
& + \frac{2557e^{12t_0}t_3^2t_1^4}{50000} + \frac{7e^{2t_0}t_2^2t_3^2t_1^4}{50000} + \frac{88}{25}e^{6t_0}t_4^2t_1^4 - \frac{t_5^2t_1^4}{703125} + \frac{507}{10}e^{26t_0}t_2t_1^4 + \frac{229}{125}e^{24t_0}t_3t_1^4 \\
& + \frac{23}{600}e^{4t_0}t_2^2t_3t_1^4 + \frac{343}{125}e^{14t_0}t_2t_3t_1^4 + \frac{98}{5}e^{21t_0}t_4t_1^4 + \frac{1}{300}e^{t_0}t_2^2t_4t_1^4 + \frac{1331}{50}e^{11t_0}t_2t_4t_1^4 \\
& + \frac{459}{500}e^{9t_0}t_3t_4t_1^4 + \frac{9}{250}e^{18t_0}t_5t_1^4 + \frac{76e^{8t_0}t_2t_5t_1^4}{1875} + \frac{11e^{6t_0}t_3t_5t_1^4}{9375} + \frac{17e^{3t_0}t_4t_5t_1^4}{3750}
\end{aligned}$$

$$\begin{aligned}
& - \frac{4}{15}e^{16t_0}t_6t_1^4 - \frac{27}{100}e^{6t_0}t_2t_6t_1^4 - \frac{19e^{4t_0}t_3t_6t_1^4}{3000} - \frac{1}{300}e^{t_0}t_4t_6t_1^4 + \frac{e^{12t_0}t_7t_1^4}{25000} + \frac{e^{2t_0}t_2t_7t_1^4}{25000} \\
& - \frac{t_3t_7t_1^4}{1250000} + 10e^{42t_0}t_1^3 + \frac{1}{45}e^{2t_0}t_2^4t_1^3 + \frac{159}{5}e^{12t_0}t_2^3t_1^3 + \frac{19e^{6t_0}t_3^3t_1^3}{25000} + \frac{484}{5}e^{22t_0}t_2^2t_1^3 \\
& + \frac{9}{50}e^{18t_0}t_3^2t_1^3 + \frac{54}{625}e^{8t_0}t_2t_3^2t_1^3 + \frac{248}{5}e^{12t_0}t_4^2t_1^3 + \frac{4}{5}e^{2t_0}t_2t_4^2t_1^3 + \frac{2e^{6t_0}t_5^2t_1^3}{5625} + \frac{1}{45}e^{2t_0}t_6^2t_1^3 \\
& + 48e^{32t_0}t_2t_1^3 + 2e^{30t_0}t_3t_1^3 + \frac{19}{6}e^{10t_0}t_2^2t_3t_1^3 + 9e^{20t_0}t_2t_3t_1^3 + 18e^{27t_0}t_4t_1^3 + \frac{343}{15}e^{7t_0}t_2^2t_4t_1^3 \\
& + \frac{4}{625}e^{3t_0}t_3^2t_4t_1^3 + \frac{578}{5}e^{17t_0}t_2t_4t_1^3 + 6e^{15t_0}t_3t_4t_1^3 + e^{5t_0}t_2t_3t_4t_1^3 + \frac{4}{375}e^{24t_0}t_5t_1^3 \\
& + \frac{7}{375}e^{4t_0}t_2^2t_5t_1^3 - \frac{t_3^2t_5t_1^3}{187500} + \frac{98}{375}e^{14t_0}t_2t_5t_1^3 + \frac{51e^{12t_0}t_3t_5t_1^3}{3125} + \frac{e^{2t_0}t_2t_3t_5t_1^3}{3125} \\
& + \frac{36}{125}e^{9t_0}t_4t_5t_1^3 - \frac{2}{45}e^{2t_0}t_2^2t_6t_1^3 - 3e^{12t_0}t_2t_6t_1^3 - \frac{1}{6}e^{10t_0}t_3t_6t_1^3 - \frac{49}{15}e^{7t_0}t_4t_6t_1^3 \\
& - \frac{1}{125}e^{4t_0}t_5t_6t_1^3 + \frac{2}{625}e^{8t_0}t_2t_7t_1^3 + \frac{3e^{6t_0}t_3t_7t_1^3}{12500} + \frac{53e^{12t_0}t_3^4}{100000} + 90e^{9t_0}t_2^3t_4t_1 \\
& + \frac{2}{625}e^{3t_0}t_4t_7t_1^3 + \frac{t_5t_7t_1^3}{468750} + \frac{15}{2}e^{48t_0}t_1^2 + \frac{34}{3}e^{8t_0}t_2^4t_1^2 - \frac{9t_3^4t_1^2}{5000000} + 99e^{18t_0}t_2^3t_1^2 \\
& + \frac{153e^{12t_0}t_3^3t_1^2}{25000} + \frac{3e^{2t_0}t_2t_3^2t_1^2}{25000} + 6e^{3t_0}t_4^3t_1^2 + 105e^{28t_0}t_2^2t_1^2 + \frac{81}{250}e^{24t_0}t_3^2t_1^2 + \frac{29e^{4t_0}t_2^2t_3^2t_1^2}{1000} \\
& + \frac{147}{250}e^{14t_0}t_2t_3^2t_1^2 + 108e^{18t_0}t_4^2t_1^2 + 96e^{8t_0}t_2t_4^2t_1^2 + \frac{78}{25}e^{6t_0}t_3t_4^2t_1^2 + \frac{9e^{12t_0}t_5^2t_1^2}{3125} \\
& + \frac{2e^{2t_0}t_2t_5^2t_1^2}{9375} + \frac{t_3t_5^2t_1^2}{234375} + \frac{5}{6}e^{8t_0}t_6^2t_1^2 - \frac{3t_7^2t_1^2}{1250000} + 30e^{38t_0}t_2t_1^2 + \frac{3}{5}e^{36t_0}t_3t_1^2 \\
& + \frac{56}{5}e^{16t_0}t_2^2t_3t_1^2 + \frac{39}{5}e^{26t_0}t_2t_3t_1^2 + 3e^{3t_0}t_2^3t_4t_1^2 + 169e^{13t_0}t_2^2t_4t_1^2 + \frac{27}{100}e^{9t_0}t_2^2t_4t_1^2 \\
& + 138e^{23t_0}t_2t_4t_1^2 + \frac{42}{5}e^{21t_0}t_3t_4t_1^2 + \frac{1}{50}e^{t_0}t_2^2t_3t_4t_1^2 + \frac{363}{25}e^{11t_0}t_2t_3t_4t_1^2 \\
& + \frac{7}{15}e^{10t_0}t_2^2t_5t_1^2 + \frac{e^{6t_0}t_3^2t_5t_1^2}{1250} + \frac{2}{5}e^{20t_0}t_2t_5t_1^2 + \frac{3}{125}e^{18t_0}t_3t_5t_1^2 + \frac{4}{125}e^{8t_0}t_2t_3t_5t_1^2 \\
& + \frac{4}{5}e^{15t_0}t_4t_5t_1^2 + \frac{2}{5}e^{5t_0}t_2t_4t_5t_1^2 + \frac{1}{125}e^{3t_0}t_3t_4t_5t_1^2 - \frac{14}{3}e^{8t_0}t_2^2t_6t_1^2 - \frac{1}{200}e^{4t_0}t_2^2t_6t_1^2 \\
& - 3e^{18t_0}t_2t_6t_1^2 - \frac{2}{5}e^{16t_0}t_3t_6t_1^2 - \frac{21}{50}e^{6t_0}t_2t_3t_6t_1^2 - 13e^{13t_0}t_4t_6t_1^2 - 3e^{3t_0}t_2t_4t_6t_1^2 \\
& - \frac{1}{50}e^{t_0}t_3t_4t_6t_1^2 - \frac{1}{15}e^{10t_0}t_5t_6t_1^2 + \frac{3}{500}e^{4t_0}t_2^2t_7t_1^2 + \frac{3t_3^2t_7t_1^2}{1250000} - \frac{1}{50}e^{6t_0}t_2t_6t_7 \\
& + \frac{3e^{12t_0}t_3t_7t_1^2}{12500} + \frac{3e^{2t_0}t_2t_3t_7t_1^2}{12500} + \frac{e^{12t_0}t_3^2t_5t_1}{6250} + \frac{51}{50}e^{6t_0}t_2^3t_3t_1^2 + \frac{1}{50}e^{t_0}t_2^2t_4t_7 \\
& + \frac{9}{250}e^{9t_0}t_4t_7t_1^2 + \frac{e^{6t_0}t_5t_7t_1^2}{3125} - \frac{3}{500}e^{4t_0}t_6t_7t_1^2 + \frac{5}{6}e^{4t_0}t_2^5t_1 + \frac{245}{6}e^{14t_0}t_2^4t_1 + \frac{3e^{6t_0}t_3^4t_1}{100000} \\
& + 60e^{24t_0}t_2^3t_1 + \frac{3}{250}e^{18t_0}t_3^3t_1 + \frac{9}{625}e^{8t_0}t_2t_3^3t_1 + 60e^{9t_0}t_4^3t_1 - \frac{2t_5^3t_1}{2109375} + 30e^{34t_0}t_2^2t_1 \\
& + \frac{7}{20}e^{10t_0}t_2^2t_3^2t_1 + \frac{3}{10}e^{20t_0}t_2t_2^2t_1 + 60e^{24t_0}t_4^2t_1 + 40e^{4t_0}t_2^2t_4^2t_1 + 210e^{14t_0}t_2t_4^2t_1
\end{aligned}$$

$$\begin{aligned}
& + \frac{42}{5}e^{12t_0}t_3t_4^2t_1 + \frac{6}{5}e^{2t_0}t_2t_3t_4^2t_1 + \frac{4e^{8t_0}t_2t_5^2t_1}{1875} + \frac{2e^{6t_0}t_3t_5^2t_1}{9375} + 6e^{27t_0}t_3t_4t_1 \\
& + \frac{4e^{3t_0}t_4t_5^2t_1}{1875} + \frac{5}{6}e^{14t_0}t_6^2t_1 + \frac{5}{6}e^{4t_0}t_2t_6^2t_1 + \frac{1}{30}e^{2t_0}t_3t_6^2t_1 + 80e^{15t_0}t_2^3t_4 \\
& + \frac{3e^{6t_0}t_7^2t_1}{25000} + \frac{1}{30}e^{2t_0}t_2^4t_3t_1 + \frac{41}{5}e^{12t_0}t_2^3t_3t_1 + \frac{66}{5}e^{22t_0}t_2^2t_3t_1 + 6e^{32t_0}t_2t_3t_1 \\
& + \frac{3e^{3t_0}t_3^3t_4t_1}{1250} + 190e^{19t_0}t_2^2t_4t_1 + \frac{3}{5}e^{15t_0}t_3^2t_4t_1 + \frac{3}{10}e^{5t_0}t_2t_3^2t_4t_1 + 60e^{29t_0}t_2t_4t_1 \\
& + \frac{49}{5}e^{7t_0}t_2^2t_3t_4t_1 + \frac{102}{5}e^{17t_0}t_2t_3t_4t_1 + \frac{4}{25}e^{6t_0}t_2^3t_5t_1 + \frac{t_3^3t_5t_1}{312500} + \frac{8}{15}e^{16t_0}t_2^2t_5t_1 \\
& + \frac{e^{2t_0}t_2t_3^2t_5t_1}{6250} + \frac{14}{25}e^{6t_0}t_4^2t_5t_1 + \frac{2}{125}e^{24t_0}t_3t_5t_1 + \frac{7}{375}e^{4t_0}t_2^2t_3t_5t_1 + \frac{14}{125}e^{14t_0}t_2t_3t_5t_1 \\
& + \frac{4}{5}e^{21t_0}t_4t_5t_1 + \frac{2}{75}e^{t_0}t_2^2t_4t_5t_1 + \frac{44}{25}e^{11t_0}t_2t_4t_5t_1 + \frac{12}{125}e^{9t_0}t_3t_4t_5t_1 - \frac{5}{3}e^{4t_0}t_2^3t_6t_1 \\
& - \frac{35}{3}e^{14t_0}t_2^2t_6t_1 - \frac{1}{20}e^{10t_0}t_3^2t_6t_1 - 10e^{4t_0}t_4^2t_6t_1 - \frac{1}{15}e^{2t_0}t_2^2t_3t_6t_1 - e^{12t_0}t_2t_3t_6t_1 \\
& - 10e^{19t_0}t_4t_6t_1 - 30e^{9t_0}t_2t_4t_6t_1 - \frac{7}{5}e^{7t_0}t_3t_4t_6t_1 - \frac{2}{15}e^{16t_0}t_5t_6t_1 - \frac{4}{25}e^{6t_0}t_2t_5t_6t_1 \\
& - \frac{1}{375}e^{4t_0}t_3t_5t_6t_1 - \frac{2}{75}e^{t_0}t_4t_5t_6t_1 + \frac{1}{50}e^{10t_0}t_2^2t_7t_1 + \frac{3e^{6t_0}t_3^2t_7t_1}{25000} + \frac{3}{625}e^{8t_0}t_2t_3t_7t_1 \\
& + \frac{3}{25}e^{5t_0}t_2t_4t_7t_1 + \frac{3}{625}e^{3t_0}t_3t_4t_7t_1 + \frac{e^{12t_0}t_5t_7t_1}{3125} + \frac{e^{2t_0}t_2t_5t_7t_1}{3125} - \frac{t_3t_5t_7t_1}{156250} \\
& - \frac{1}{50}e^{10t_0}t_6t_7t_1 + \frac{3}{125}t_7t_8t_1 + e^{60t_0} - \frac{t_6^2}{324} + \frac{23}{3}e^{10t_0}t_2^5 + \frac{t_3^5}{5000000} + \frac{185}{6}e^{20t_0}t_2^4 \\
& + \frac{3e^{2t_0}t_2t_3^4}{100000} - \frac{5t_4^4}{8} + 20e^{30t_0}t_2^3 + \frac{e^{4t_0}t_2^2t_3^3}{1000} + 20e^{15t_0}t_4^3 + 60e^{5t_0}t_2t_4^3 + 2e^{3t_0}t_3t_4^3 \\
& - \frac{5t_6^3}{324}15e^{40t_0}t_2^2 + \frac{1}{10}e^{36t_0}t_2^3 + \frac{11}{100}e^{6t_0}t_2^3t_3^2 + \frac{3}{5}e^{16t_0}t_2^2t_3^2 + \frac{3}{10}e^{26t_0}t_2t_3^2 + 30e^{30t_0}t_4^2 \\
& + 130e^{10t_0}t_2^2t_4^2\frac{9}{25}e^{6t_0}t_3^2t_4^2 + 60e^{20t_0}t_2t_4^2 + 6e^{18t_0}t_3t_4^2 + 12e^{8t_0}t_2t_3t_4^2 + \frac{1}{375}e^{24t_0}t_5^2 \\
& - \frac{t_3^2t_5^2}{468750} + \frac{2}{375}e^{14t_0}t_2t_5^2 + \frac{5}{6}e^{20t_0}t_6^2 - \frac{5}{108}t_2^2t_6^2 + \frac{5}{3}e^{10t_0}t_2t_6^2 + \frac{5}{3}e^{5t_0}t_4t_6^2 + \frac{1}{90}e^{2t_0}t_5t_6^2 \\
& + \frac{3e^{12t_0}t_7^2}{25000} + \frac{3e^{2t_0}t_2t_7^2}{25000} + \frac{3t_3t_7^2}{1250000} + 30t_0t_8^2 + e^{8t_0}t_2^4t_3 + 3e^{18t_0}t_2^3t_3 + \frac{35}{3}e^{5t_0}t_2^4t_4 \\
& + \frac{3}{500}e^{9t_0}t_3^3t_4 + 60e^{25t_0}t_2^2t_4 + \frac{1}{100}e^{t_0}t_2^2t_3^2t_4 + \frac{33}{50}e^{11t_0}t_2t_3^2t_4 + e^{3t_0}t_2^3t_3t_4 \\
& + 6e^{23t_0}t_2t_3t_4 + \frac{1}{90}e^{2t_0}t_2^4t_5 + \frac{2}{5}e^{12t_0}t_2^3t_5 + \frac{2}{5}e^{22t_0}t_2^2t_5 + \frac{1}{250}e^{18t_0}t_3^2t_5 \\
& + \frac{4}{5}e^{12t_0}t_4^2t_5 + \frac{2}{5}e^{2t_0}t_2t_4^2t_5 + \frac{1}{75}e^{10t_0}t_2^2t_3t_5 + \frac{14}{15}e^{7t_0}t_2^2t_4t_5 + \frac{e^{3t_0}t_3^2t_4t_5}{1250} \\
& + \frac{2}{25}e^{5t_0}t_2t_3t_4t_5 - \frac{5}{324}t_2^4t_6 - \frac{10}{3}e^{10t_0}t_2^3t_6 - \frac{e^{4t_0}t_3^3t_6}{1000} - \frac{5}{3}e^{20t_0}t_2^2t_6 - \frac{1}{100}e^{6t_0}t_2t_3^2t_6 \\
& - 10e^{10t_0}t_4^2t_6 - e^{8t_0}t_2^2t_3t_6 - e^{18t_0}t_2t_3t_6 - \frac{40}{3}e^{5t_0}t_2^2t_4t_6 - \frac{1}{100}e^{t_0}t_3^2t_4t_6 - 20e^{15t_0}t_2t_4t_6
\end{aligned}$$



$$\begin{aligned}
& - e^{13t_0} t_3 t_4 t_6 - e^{3t_0} t_2 t_3 t_4 t_6 - \frac{1}{45} e^{2t_0} t_2^2 t_5 t_6 - \frac{1}{75} e^{10t_0} t_3 t_5 t_6 - \frac{2}{15} e^{7t_0} t_4 t_5 t_6 \\
& - \frac{t_3^3 t_7}{1250000} + \frac{3e^{12t_0} t_3^2 t_7}{25000} + \frac{3e^{2t_0} t_2 t_3^2 t_7}{25000} + \frac{3}{25} e^{6t_0} t_4^2 t_7 + \frac{t_5^2 t_7}{468750} + \frac{1}{500} e^{4t_0} t_2^2 t_3 t_7 \\
& + \frac{3}{25} e^{11t_0} t_2 t_4 t_7 + \frac{3}{250} e^{9t_0} t_3 t_4 t_7 + \frac{1}{625} e^{8t_0} t_2 t_5 t_7 + \frac{1}{625} e^{3t_0} t_4 t_5 t_7 \\
& - \frac{1}{500} e^{4t_0} t_3 t_6 t_7 - \frac{1}{50} e^{t_0} t_4 t_6 t_7 + 15 t_4^2 t_8 + \frac{2}{125} t_3 t_5 t_8 - \frac{10}{3} t_2 t_6 t_8 + \frac{1}{375} e^{4t_0} t_2^2 t_5^2 \\
& + \frac{2e^{6t_0} t_5^3}{84375} + \frac{1}{50} e^{6t_0} t_2^3 t_7 + \frac{3}{625} e^{8t_0} t_2 t_3^2 t_5 + 13e^{13t_0} t_2^2 t_3 t_4 + \frac{4}{5} e^{17t_0} t_2 t_4 t_5 . \tag{B.7}
\end{aligned}$$

APPENDIX C.  $\wedge^\bullet \mathfrak{e}_8$  AND RELATIONS IN  $R(\mathbb{E}_8)$ : AN OVERVIEW OF THE RESULTS OF [23]

I provide here a summary of the computer-aided proof of Claim 2.3. In principle, a direct approach to the determination of  $\{\mathfrak{p}_k\}$  is as follows

- (1) decompose  $\wedge^k \mathfrak{g} = \bigoplus R_i^{(k)}$  into irreducibles;
- (2) letting  $\lambda_j = \sum_{i=1}^8 m_{i,j} \omega_j$  be the highest weight of  $R_j^{(k)}$ , consider the tensor product decomposition of  $\otimes_i \rho(m_{i,j} \omega_j)$ : this will contain  $R_j^{(k)}$  as a summand (with coefficient 1), plus extra terms;
- (3) iterate the operation until all virtual summands (possibly with negative coefficients) have been replaced by tensor products of fundamental representations; taking the character and summing over  $j$  gives  $\mathfrak{p}_k$ .

This however turns out to be computationally unfeasible already for  $k \sim 7$ ; not only is the decomposition of  $\wedge^k \mathfrak{g}$  a daunting (and still unsolved, see however [9, 102]) task; the decomposition into irreducibles of even simple products such as  $\rho_{\omega_3} \otimes \rho_{\omega_3}$  would take hundreds of Gigabytes of RAM to compute. Instead, we proceed as follows:

- (1) For a given Cartan torus element  $\exp(l) \in \mathcal{T}$  with  $l = \sum_i l_i \alpha_i^* \in \mathfrak{h}$ , we can compute explicit Laurent polynomials  $\theta_j(\exp(l)) \in \mathbb{Z}[(e^{l_i})_i, (e^{-l_i})_i]$ ,  $\phi_k(l) \in \mathbb{Z}[(e^{l_i})_i, (e^{-l_i})_i]$  for  $i = 1, \dots, 8$  and  $k = 1, \dots, 120$  from the sole knowledge of the weight systems of  $\rho_{\omega_i}$  with  $i = 1, 7, 8$ , which have manageably small cardinality 2401, 241 and 26401 respectively (not taking into account weight multiplicities): for  $\theta_i$  with  $3 \leq i \leq 7$  and all  $\phi_k$  this follows from using Newton's identities applied to the power sums  $\theta_7(\exp(kl))$ , and then use of (2.19);  $\theta_2$  is similarly computed from  $\theta_j$  with  $j = 1, 7, 8$  by the Adams operation on  $\theta_1$ :

$$\theta_1(\exp(2l)) = \theta_2(\exp(l)) - \theta_1(\exp l) + \theta_7(\exp l) \theta_1(\exp l) - \theta_8(\exp l). \tag{C.1}$$

- (2) We may then impose *a priori* constraints on the exponents  $d_j^{(l)}$  appearing by inspection of the weight systems as well as on the dimensions of the tensor powers appearing on the

monomials of the r.h.s. of (2.24). We find

$$\{\max_{I \in M} d_j^{(I)}\}_{j=1}^8 = \{23, 13, 9, 11, 14, 19, 29, 17\}, \quad \max_{I \in M} \prod_{j=1}^8 (\dim \rho_{\omega_j})^{d_j^{(I)}} = 1.25366 \times 10^{96}. \quad (\text{C.2})$$

This truncates the sum on the r.h.s. to a finite, if large ( $|M| = \mathcal{O}(10^6)$ ), number of monomials.

- (3) In principle, imposing the identity of polynomials  $\phi_k = \mathbf{p}_k(\theta_1, \dots, \theta_8)$  determines uniquely all  $n_{I,k}$ ; however the range of sum and the complexity of the polynomials involved renders this entirely unwieldy. A more sensible alternative is to solve (2.24) for  $n_{I,k}$  by sampling the relation (2.24) at  $|M|$  random generic points  $\exp(l) \in \mathbb{Q}^8$  of the torus<sup>36</sup>, leading to a generically non-singular  $|M| \times |M|$  linear system with rational entries for  $n_{I,k}$ , which can be solved exactly. Due to the sheer size of the monomial set and the density of the resulting linear system, however, this is unfeasible both for memory and time constraints.
- (4) There is however a non-generic choice of sampling points which, with some preparation, does the trick of reducing the problem to a large number of smaller problems of manageable size. First we subdivide the monomial set  $M$  into slices  $M_n = h^{-1}(n)$ ,  $n = 0, \dots, 8$  given by level sets of the function

$$h(d^{(I)} \in M) = \sum_i \zeta(d_j^{(I)}), \quad \zeta(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases} \quad (\text{C.3})$$

It is clear then that sampling  $\exp(l)$  at values such that  $\theta_i(\exp(l)) = 0$  except for  $n$  values of  $i$  truncates the r.h.s. of (2.24) to one of  $\binom{8}{n}$  subsets of  $M_n$ . The strategy here is to solve numerically for  $\{\theta_i = u_i\}$  with some  $u_i \in \mathbb{Q} + I\mathbb{Q}$ ; there is a clever choice of the sampling set here such that with sufficient floating point precision, we obtain a reliable – and in fact exact, with suitable analytical bounds – rounding to rational expressions for both the l.h.s. and the r.h.s. of (2.24), for all values of  $k$ . This simultaneously bypasses the ill-conditioning problem for (2.24), since we can then use exact arithmetic methods to solve it, and moreover breaks it up into subsystems of size in the range  $\mathcal{O}(10) - \mathcal{O}(10^4)$ .

- (5) The latter point is not satisfactory yet since in the worst case scenario we deal with dense rational matrices of rank in the tens of thousands. However there's a refinement of the sets  $M_k$  by considering a further slicing by one (or more) of the  $d_j^{(I)}$  (i.e. level sets of the projections  $p_j(d^{(I)} \in M) = d_j^{(I)}$ ); these refined monomial sets are just selected by taking derivatives of (2.24) w.r.t.  $\theta_j$  of order  $d_j^{(I)}$ . Since we have closed-form expressions for  $\phi_k$  and  $\theta_j$  as functions of  $e^{l_i}$ , these can be computed using Faa' di Bruno type formulas; while the complexity of the latter grows factorially, it turns out that derivative slicings of order up to five are both computable in finite time, compatible with the rounding of  $\phi_k$  with

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<sup>36</sup>One might in principle pick generic, numerical random values with fixed precision and then hope to get an accurate integer truncation for the resulting  $n_{I,k} \in \mathbb{Z}$ . Such hope is however misplaced, as the resulting numerical matrix of monomial values (a matrix minor of a multi-variate Vandermonde matrix) is extremely ill-conditioned and leads to uncontrollable numerical errors.

$8 \times$  machine precision, and they allow to break up the size of the resulting linear systems down to a maximum<sup>37</sup> of  $\mathcal{O}(4 \times 10^3)$ .

- (6) We are then left with a large number ( $\mathcal{O}(3.10^3)$ ) of relatively small linear systems and a large ( $\mathcal{O}(3.10^5)$ ) number of sampling points to evaluate  $\phi_k$ ,  $\theta_i$ , and their derivatives in  $l_j$ ; this would lead to a total runtime in the hundreds of months (about 120). However the numerical inversion, evaluation and calculation of derivatives at one sampling point is independent from that at another; this means that the calculation can be easily distributed over several CPU cores just by segmentation of the sampling set. Similar considerations apply, *mutatis mutandis*, to the solution of the linear subsystems. With  $N \simeq 75$  processor cores<sup>38</sup>, the absolute runtime gets reduced to about six weeks.

The full result of the calculation is available at <http://tiny.cc/E8SpecCurve>, and the original C source code is available upon request.

It should be noted that, despite the innocent-looking appearance of (2.25)-(2.28), both the number of terms and the size of the coefficients grow extremely quickly with  $k$ . The monomial set  $M$  turns out to have cardinality  $|M| = 949468$ , with the matrix  $n_{I,k}$  growing more and more dense for high  $k$  up to a maximum of 949256 non-zero coefficients for  $k = 118$ , and  $\max_{I,k} n_{I,k} \simeq 1.7025 \times 10^{10}$ ,  $\min_{I,k} n_{I,k} \simeq -1.5403 \times 10^{10}$ .

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<sup>37</sup>A linear system of this rank, for the type of Vandermonde-type matrices we consider, required typically around 90Gb of RAM and half a day to terminate when solved using exact arithmetics (in our case,  $p$ -adic expansions and Dixon’s method).

<sup>38</sup>In my specific case, this involved an average of about nine entry-level 64-bit cluster machines with dual 4-core CPUs.

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