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# ON THE LIMIT SOBOLEV REGULARITY FOR DIRICHLET AND NEUMANN PROBLEMS ON LIPSCHITZ DOMAINS

MARTIN COSTABEL

ABSTRACT. We construct a bounded  $C^1$  domain  $\Omega$  in  $\mathbb{R}^n$  for which the  $H^{3/2}$  regularity for the Dirichlet and Neumann problems for the Laplacian cannot be improved, that is, there exists  $f$  in  $C^\infty(\overline{\Omega})$  such that the solution of  $\Delta u = f$  in  $\Omega$  and either  $u = 0$  on  $\partial\Omega$  or  $\partial_n u = 0$  on  $\partial\Omega$  is contained in  $H^{3/2}(\Omega)$  but not in  $H^{3/2+\varepsilon}(\Omega)$  for any  $\varepsilon > 0$ . An analogous result holds for  $L^p$  Sobolev spaces with  $p \in (1, \infty)$ .

## 1. INTRODUCTION

The motivation for this note comes from a question of regularity of the time-harmonic Maxwell equations in Lipschitz domains. In the variational theory of Maxwell's equations, basis for the analysis of many algorithms of numerical electrodynamics, the following two function spaces are fundamental:

$$\begin{aligned} X_N &= H(\operatorname{div}, \Omega) \cap H_0(\operatorname{curl}, \Omega) \\ &= \{u \in L^2(\Omega; \mathbb{C}^3) \mid \operatorname{div} u \in L^2(\Omega), \operatorname{curl} u \in L^2(\Omega; \mathbb{C}^3), u \times n = 0 \text{ on } \partial\Omega\} \end{aligned} \quad (1.1)$$

$$\begin{aligned} X_T &= H_0(\operatorname{div}, \Omega) \cap H(\operatorname{curl}, \Omega) \\ &= \{u \in L^2(\Omega; \mathbb{C}^3) \mid \operatorname{div} u \in L^2(\Omega), \operatorname{curl} u \in L^2(\Omega; \mathbb{C}^3), u \cdot n = 0 \text{ on } \partial\Omega\} \end{aligned} \quad (1.2)$$

Here  $n$  is the outward unit normal vector field on the boundary of the domain  $\Omega \subset \mathbb{R}^3$ .

If  $\Omega$  is a bounded Lipschitz domain, then it has been known for a long time [15, 11] that  $X_N$  and  $X_T$  are compactly embedded subspaces of  $L^2(\Omega; \mathbb{C}^3)$ , and it has been shown more precisely [5, 10] that they are contained in the Sobolev space  $H^{\frac{1}{2}}(\Omega, \mathbb{C}^3) = W^{\frac{1}{2}, 2}(\Omega, \mathbb{C}^3)$ . For large classes of more regular domains,  $X_N$  and  $X_T$  are contained in  $H^1(\Omega, \mathbb{C}^3)$  (see [3] for  $C^{1,1}$  domains, [7] for  $C^{\frac{3}{2}+\varepsilon}$  domains, [12] for  $X_N$  on convex domains, [13] for ‘‘almost convex’’ domains). The regularity is diminished by corner singularities, but one also knows [3] that for every Lipschitz polyhedron or, more generally, piecewise smooth domain  $\Omega$  that is at least  $C^2$ -diffeomorphic to a polyhedron, there exists  $\varepsilon > 0$  such that

$$X_N \cup X_T \subset H^{\frac{1}{2}+\varepsilon}(\Omega; \mathbb{C}^3). \quad (1.3)$$

The additional regularity described by  $\varepsilon$  is of some use in the numerical analysis of Maxwell's equations (see for example [2, 1]). The parameter  $\varepsilon$  can become arbitrarily small, depending on the corner angles of  $\partial\Omega$ , but it depends only on these angles, that is, on the local Lipschitz constant of  $\partial\Omega$ . Based on this observation, one could ask the question whether for any Lipschitz domain  $\Omega$ , there exists such an  $\varepsilon > 0$  for which (1.3) holds. This question is the motivation for the present investigation.

To the best of the author's knowledge, the conjecture that such an  $\varepsilon > 0$  always exists is not incompatible with the currently available regularity results for Maxwell's equations on Lipschitz domains, but we shall show that it is not true. As a corollary of our constructions, we obtain a counterexample that is even  $C^1$ .

**Proposition 1.1.** *There exists a bounded  $C^1$  domain  $\Omega \subset \mathbb{R}^3$ , an  $L^2(\Omega)$  function  $g$  and an  $L^2(\Omega; \mathbb{C}^3)$  function  $h$  such that the solutions  $u \in L^2(\Omega; \mathbb{C}^3)$  of the system*

$$\operatorname{div} u = g, \quad \operatorname{curl} u = h \quad \text{in } \Omega \quad (1.4)$$

and either

$$u \times n = 0 \quad \text{on } \partial\Omega \quad (1.5)$$

or

$$u \cdot n = 0 \quad \text{on } \partial\Omega \quad (1.6)$$

do not belong to  $H^{\frac{1}{2}+\varepsilon}(\Omega; \mathbb{C}^3)$  for any  $\varepsilon > 0$ .

In the system (1.4), the field  $h$  can be chosen to be zero and  $g$  can be chosen to be continuous on  $\overline{\Omega}$ .

As we will see in the following, analogous results are true in dimension 2 and in higher dimensions, and also for non-Hilbert Sobolev spaces over  $L^p$  with  $p$  different from 2.

Non-regular solutions of the div-curl system (1.4) are typically sought as gradients of solutions of the inhomogeneous Laplace (Poisson) equation with either Dirichlet (for (1.5)) or Neumann (for (1.6)) boundary conditions. A non-regularity result for these Laplace boundary value problems is the main result of this paper, see Theorem 1.2 below. It will be proved in Section 3 for dimension  $d = 2$  and in Section 4 for higher dimensions.

We use the standard notation  $W^{s,p}(\Omega)$  for the Sobolev-Slobodeckij spaces on  $\Omega \subset \mathbb{R}^d$ , and we recall that for  $0 < s < 1$  the seminorm

$$|u|_{s,p;\Omega} = \left( \int_{\Omega} \int_{\Omega} \frac{|u(y) - u(x)|^p}{|y - x|^{d+sp}} dx dy \right)^{\frac{1}{p}} \quad (1.7)$$

defines the norm  $\|u\|_{W^{s,p}(\Omega)} = \|u\|_{L^p(\Omega)} + |u|_{s,p;\Omega}$ , that  $W^{0,p}(\Omega) = L^p(\Omega)$ , and that for any  $s$  there holds

$$u \in W^{s+1,p}(\Omega) \iff u \in W^{s,p}(\Omega) \text{ and } \nabla u \in W^{s,p}(\Omega; \mathbb{C}^d).$$

As usual, we write  $W_0^{s,p}(\Omega)$  for the closure of  $C_0^\infty(\Omega)$  in  $W^{s,p}(\Omega)$ . In view of an interesting property of the domain we are going to construct (see equations (1.10) and (4.1)), we recall that for  $\frac{1}{p} < s < 1 + \frac{1}{p}$  the subspace  $W_0^{s,p}(\Omega)$  is characterized by the condition that the boundary trace vanishes, whereas for  $1 + \frac{1}{p} < s < 2 + \frac{1}{p}$  the condition is that both the trace and the normal derivative vanish on  $\partial\Omega$ .

In order to describe known regularity results, we also need the Bessel potential spaces  $H^{s,p}(\Omega)$ , which are different from  $W^{s,p}(\Omega)$  if  $p \neq 2$ . For the main properties of these spaces, see [14]. In Triebel's notation  $W^{m,p}(\Omega) = F_{p,2}^m(\Omega)$  for  $m \in \mathbb{N}$  and

$$H^{s,p}(\Omega) = F_{p,2}^s(\Omega), \quad \text{and for } s \notin \mathbb{Z} : W^{s,p}(\Omega) = B_{p,p}^s(\Omega).$$

Note that the trace space for both  $W^{s,p}(\Omega)$  and  $H^{s,p}(\Omega)$  on a sufficiently smooth boundary is  $W^{s-\frac{1}{p},p}(\partial\Omega)$  if  $s > \frac{1}{p}$ .

Comprehensive regularity results in the  $H^{s,p}$  spaces for the Dirichlet problem on Lipschitz domains were given by Jerison and Kenig [8]. They had previously studied the homogeneous Laplace equation with inhomogeneous Neumann conditions [9], and corresponding results for the homogeneous Neumann problem of the inhomogeneous Laplace equation were obtained by Fabes, Mendez and Mitrea [6] and Zanger [16]. In particular, there exist precise answers to the question for which  $s$  and  $p$  the condition  $g \in H^{s-2,p}(\Omega)$  implies  $v \in H^{s,p}(\Omega)$  for the solutions  $v$  of the problems

$$\Delta v = g \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega \quad (1.8)$$

$$\Delta v = g \quad \text{in } \Omega, \quad \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega \quad (1.9)$$

For the maximal regularity one finds a limit at  $s = 1 + \frac{1}{p}$ . We summarize the main results pertaining to the question of maximal regularity (here formulated for the Dirichlet problem, see [8, Thms 1.1–1.3], where  $H^{s,p}$  is written  $L_s^p$ ; the results for the Neumann problem are similar):

For any bounded Lipschitz domain  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 2$ , there exists  $p_0 \in [1, 2)$  such that for  $p_0 < p < \frac{p_0}{p_0-1}$  and  $\frac{1}{p} < s < 1 + \frac{1}{p}$  the solution  $v$  of the Dirichlet problem (1.8) with  $g \in H^{s-2,p}(\Omega)$  belongs to  $H^{s,p}(\Omega)$ . The following 4 points indicate the known borders of this result.

1. In general,  $p_0 > 1$  and there are counterexamples as soon as  $p$  or  $s$  are outside of the given bounds, but when  $\Omega$  is a  $C^1$  domain, one can choose  $p_0 = 1$ , so that the result is true for any  $p \in (1, \infty)$ .
2. When  $p > 2$ , there are Lipschitz counterexamples with  $g \in C^\infty(\overline{\Omega})$  and  $v \notin W^{1+\frac{1}{p},p}(\Omega)$ . There is a  $C^1$  counterexample for  $p = 1$  with  $g \in C^\infty(\overline{\Omega})$  and  $v \notin W^{2,1}(\Omega)$ .
3. In the optimal regularity-shift result for  $C^1$  domains, the condition on  $s$  cannot be weakened, because for any  $p > 1$  there exists a bounded  $C^1$  domain  $\Omega$  and a  $g \in H^{-1+\frac{1}{p},p}(\Omega)$  such that  $v \notin H^{1+\frac{1}{p},p}(\Omega)$ .
4. On the other hand, if  $g$  is more regular, for example  $g \in H^{-1+\frac{1}{p}+\varepsilon,p}(\Omega)$  for some  $\varepsilon > 0$  and  $p > 1$ , then  $v \in H^{1+\frac{1}{p},p}(\Omega)$  follows. The latter result is obtained by subtracting from  $v$  a solution  $v_0 \in H^{1+\frac{1}{p}+\varepsilon,p}(\Omega)$  of  $\Delta v_0 = g$  without boundary conditions and observing that a harmonic function with trace in  $W^{1,p}(\partial\Omega)$  belongs to  $H^{1+\frac{1}{p},p}(\Omega)$ .

We will prove that one will have  $v \notin H^{1+\frac{1}{p}+\varepsilon,p}(\Omega)$  for any  $\varepsilon > 0$ , in general, even for more regular  $g$ . Because of the mutual inclusions  $H^{s+\varepsilon,p} \subset W^{s,p} \subset H^{s-\varepsilon,p}$  for any  $\varepsilon > 0$ , the result is equivalently formulated in the scale of  $W^{s,p}$  spaces.

**Theorem 1.2.** *In  $\mathbb{R}^d$ ,  $d \geq 2$ , there exists a bounded  $C^1$  domain  $\Omega$  and for both the Dirichlet problem (1.8) and the Neumann problem (1.9) functions  $g \in L^\infty(\Omega)$  such that the solutions  $v \in H^1(\Omega)$  do not belong to  $W^{1+\frac{1}{p}+\varepsilon,p}(\Omega)$  for any  $p \in [1, \infty)$  and any  $\varepsilon > 0$ .*

*Remark 1.3.* It will follow from the proof that in dimension  $d = 2$ , there are functions  $g \in C^\infty(\overline{\Omega})$  that provide examples, even  $g = 1$  is possible for the Dirichlet problem and a second degree polynomial  $g$  for the Neumann problem. See also Remark 3.3. In dimension  $d \geq 3$ , there is still an example with  $g = 1$  for the Dirichlet problem, and examples with  $g \in C^\alpha(\overline{\Omega})$ ,  $\alpha > 0$ , for the Neumann problem.

*Remark 1.4.* Not all of this is new: For  $p = 1$ , the counterexample from [8, Theorem 1.2(b)] shows that the result for the Dirichlet problem holds even with  $\varepsilon = 0$ . Moreover, for  $p > 2$  the result of Theorem 1.2 is not interesting in the class of Lipschitz domains, because singularities at

conical points provide a limit of regularity that is strictly below  $s = 1 + \frac{1}{p}$ . This follows from the well-known singular asymptotic behavior  $O(r^\alpha)$  near a straight  $d - 2$  dimensional facet of the boundary (corner in dimension  $d = 2$  or “edge” in dimension  $\geq 3$ ) of opening angle  $\frac{\pi}{\alpha}$  of generic solutions of the Dirichlet and Neumann problems with smooth right hand sides, where  $r$  is the distance to the corner or edge. Such functions are in  $W^{s,p}(\Omega)$  only for  $s < \alpha + \frac{2}{p}$ , hence not in  $W^{1+\frac{1}{p},p}(\Omega)$  as soon as the opening angle exceeds  $\frac{p}{p-1}\pi$ . But for  $C^1$  domains the result still seems to be new even for  $p > 2$ . We provide a proof that works for any  $p \geq 1$ , because there is no extra cost with respect to the proof for  $p = 2$ . One just has to be careful to observe that the same domain  $\Omega$  and the same function  $g$  give an example valid for all  $p$  and all  $\varepsilon$ .

Proposition 1.1 follows from Theorem 1.2 for  $p = 2$ ,  $d = 3$  if we take  $u = \nabla v$  (“electrostatic field”). The Laplace equation for  $v$  implies the div-curl system (1.4) for  $u$  with  $h = 0$ , and the Dirichlet and Neumann conditions in (1.8) and (1.9) for  $v$  imply the vanishing of the tangential component (1.5) or of the normal component (1.6), respectively. Finally,  $v \in W^{1+\frac{1}{p}+\varepsilon,p}(\Omega)$  is equivalent to  $u \in W^{\frac{1}{p}+\varepsilon,p}(\Omega; \mathbb{C}^3)$ .

The construction of our counterexample uses the ideas of Filonov in the paper [7], where he considers a related question for  $\varepsilon = \frac{1}{2}$  and constructs a  $C^{\frac{3}{2}}$  domain  $\Omega$  that satisfies, among other interesting properties

$$H^2(\Omega) \cap H_0^1(\Omega) = H_0^2(\Omega),$$

that is, the homogeneous Dirichlet condition for  $H^2$  functions implies the homogeneous Neumann condition, see also [4]. Generalizing this, the  $C^1$  domain  $\Omega$  that we will construct satisfies

$$W^{1+\frac{1}{p}+\varepsilon,p}(\Omega) \cap W_0^{1,p}(\Omega) = W_0^{1+\frac{1}{p}+\varepsilon,p}(\Omega) \quad \forall 1 \leq p < \infty, 0 < \varepsilon < 1. \quad (1.10)$$

## 2. GENERALIZING FILONOV’S SEPARATING FUNCTION

We construct a continuous real-valued function  $f$  on  $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$  with the following property: If  $a$  and  $b$  belong to  $W^{\varepsilon,p}(\mathbb{T})$  for some  $\varepsilon > 0$ ,  $p \geq 1$ , and  $af = b$ , then  $a = b = 0$ .

The construction and proof are modeled after Filonov’s construction of a  $C^{\frac{1}{2}}$  function that has the above separation property for  $\varepsilon = \frac{1}{2}$  and  $p = 2$ . It is in the lineage of Weierstrass’ example of a continuous nowhere differentiable function.

We define  $f$  via a lacunary Fourier series

$$f(x) = \sum_{k=1}^{\infty} a_k \sin(b_k x) = \sum_{k=1}^{\infty} f_k(x) \quad (2.1)$$

where the sequences  $a_k > 0$  and  $b_k \in \mathbb{N}$  are chosen so that they satisfy  $\sum a_k < \infty$  and  $b_k \geq 2$ ,  $b_{k+1} \geq 2b_k$ ,  $k \geq 1$ , and the following properties for a given small constant  $\gamma > 0$  to be fixed later

on (see (2.7)):

$$\sum_{k=1}^{m-1} a_k b_k \leq \gamma a_m b_m \quad \forall m \geq 2 \quad (2.2)$$

$$\sum_{k=m+1}^{\infty} a_k \leq \gamma a_m \quad \forall m \geq 1 \quad (2.3)$$

$$\sum_{m=1}^{\infty} a_m^p b_m^{p\varepsilon} = +\infty \quad \forall \varepsilon > 0, p \geq 1. \quad (2.4)$$

We first show that for sufficiently large  $q \in \mathbb{N}$  the sequences  $a_k = q^{-k}$ ,  $b_k = 2^{q^k}$  have the properties (2.2)–(2.4), and we shall keep this choice from now on<sup>1</sup>.

For (2.2), let  $s_m = \frac{1}{a_m b_m} \sum_{k=1}^{m-1} a_k b_k$ . Noting that for  $q \geq 7$  we have  $q^2 2^{1-q} < 1$ , we show by induction that then  $s_m < \frac{1}{q-1}$  for all  $m \geq 2$ , which implies (2.2) for  $q$  large enough. Indeed,

$$s_2 = \frac{a_1 b_1}{a_2 b_2} = q 2^{(1-q)q} < q 2^{1-q} < \frac{1}{q} < \frac{1}{q-1},$$

and if  $s_m < \frac{1}{q-1}$  it follows that

$$s_{m+1} = (s_m + 1) \frac{a_m b_m}{a_{m+1} b_{m+1}} = (s_m + 1) q 2^{(1-q)q^m} < (s_m + 1) q 2^{(1-q)} < \left(\frac{1}{q-1} + 1\right) \frac{1}{q} = \frac{1}{q-1}.$$

For (2.3), we have

$$\sum_{k=m+1}^{\infty} \frac{a_k}{a_m} = \sum_{k=1}^{\infty} q^{-k} = \frac{1}{q-1}$$

which again is less than  $\gamma$  for  $q$  large enough.

For (2.4) we use that  $2^t \geq t \log 2$  for all  $t > 0$ , so that  $a_m^p b_m^{p\varepsilon} = (2^{\varepsilon q^m} / q^m)^p \geq (\varepsilon \log 2)^p$  for all  $m$ .

**Lemma 2.1.** *The function  $f$  defined by (2.1) is continuous on  $\mathbb{T}$  and satisfies*

$$\int_0^{2\pi} \frac{|f(y) - f(x)|^p}{|y - x|^{1+p\varepsilon}} dy = +\infty \quad \text{for all } x \in [0, 2\pi], \varepsilon > 0, 1 \leq p < \infty. \quad (2.5)$$

*Proof.* We first note that we have  $f(2\pi - x) = -f(x)$ , so that it is sufficient to prove (2.5) for  $x \in [0, \pi]$ . In this case  $[x, x+1] \subset [0, 2\pi]$ , and therefore with the disjoint intervals  $I_m = [\frac{1}{b_m}, \frac{2}{b_m}]$  we have

$$\int_0^{2\pi} \frac{|f(y) - f(x)|^p}{|y - x|^{1+p\varepsilon}} dy \geq \sum_{m=1}^{\infty} \int_{I_m} \frac{|f(x+h) - f(x)|^p}{|h|^{1+p\varepsilon}} dh \quad (2.6)$$

Now for  $h \in I_m$  we estimate

$$\left( \int_{I_m} \frac{|f(x+h) - f(x)|^p}{|h|^{1+p\varepsilon}} dh \right)^{\frac{1}{p}} \geq J_1 - J_2$$

$$\text{with } J_1 = \left( \int_{I_m} \frac{|f_m(x+h) - f_m(x)|^p}{|h|^{1+p\varepsilon}} dh \right)^{\frac{1}{p}} \text{ and } J_2 = \sum_{k \neq m} \left( \int_{I_m} \frac{|f_k(x+h) - f_k(x)|^p}{|h|^{1+p\varepsilon}} dh \right)^{\frac{1}{p}}.$$

<sup>1</sup>By plotting approximate values of the integral in (2.7) against the variable  $z$  and visual inspection of the graph, one can obtain a rough numerical approximation of  $\gamma$  that indicates that  $\gamma \geq 0.0154$ . In view of the condition  $1/(q-1) < \gamma$ , this suggests that a value of  $q = 66$  should be ‘‘sufficiently large’’.

To estimate  $J_1$ , we assume that  $0 < \varepsilon < 1$  and make the change of variables  $t = b_m h$  to obtain

$$J_1 = a_m b_m^\varepsilon \left( \int_1^2 |\sin(b_m x + t) - \sin(b_m x)|^p t^{-(1+p\varepsilon)} dt \right)^{\frac{1}{p}} \geq 5 \gamma a_m b_m^\varepsilon,$$

where we defined

$$\gamma = \frac{1}{5} \min_{z \in \mathbb{T}} \int_1^2 |\sin(z+t) - \sin(z)| t^{-2} dt > 0. \quad (2.7)$$

Here we used Hölder's inequality,

$$\begin{aligned} \int_1^2 \frac{|\sin(z+t) - \sin(z)|}{t^2} dt &\leq \int_1^2 \frac{|\sin(z+t) - \sin(z)|}{t^{1+\varepsilon}} dt \\ &\leq \left( \int_1^2 |\sin(z+t) - \sin(z)|^p t^{-(1+p\varepsilon)} dt \right)^{\frac{1}{p}} \left( \int_1^2 \frac{dt}{t} \right)^{1-\frac{1}{p}}. \end{aligned}$$

To estimate  $J_2$ , we use for  $k \leq m-1$

$$|f_k(x+h) - f_k(x)| \leq a_k b_k |h| \leq 2a_k b_k \frac{1}{b_m}$$

and for  $k \geq m+1$

$$|f_k(x+h) - f_k(x)| \leq 2a_k$$

so that we obtain with (2.2)

$$\sum_{k=1}^{m-1} \left( \int_{I_m} \frac{|f_k(x+h) - f_k(x)|^p}{|h|^{1+p\varepsilon}} dh \right)^{\frac{1}{p}} \leq 2\gamma a_m \left( \int_{I_m} \frac{dh}{|h|^{1+p\varepsilon}} \right)^{\frac{1}{p}} \leq 2\gamma a_m b_m^\varepsilon$$

and with (2.3)

$$\sum_{k=m+1}^{\infty} \left( \int_{I_m} \frac{|f_k(x+h) - f_k(x)|^p}{|h|^{1+p\varepsilon}} dh \right)^{\frac{1}{p}} \leq 2\gamma a_m \left( \int_{I_m} \frac{dh}{|h|^{1+p\varepsilon}} \right)^{\frac{1}{p}} \leq 2\gamma a_m b_m^\varepsilon,$$

hence  $J_2 \leq 4\gamma a_m b_m^\varepsilon$ .

Together, this gives

$$\left( \int_{I_m} \frac{|f(x+h) - f(x)|^p}{|h|^{1+p\varepsilon}} dh \right)^{\frac{1}{p}} \geq \gamma a_m b_m^\varepsilon,$$

and finally with (2.6) and (2.4)

$$\int_0^{2\pi} \frac{|f(y) - f(x)|^p}{|y-x|^{1+p\varepsilon}} dy \geq \sum_{m=1}^{\infty} \gamma^p a_m^p b_m^{p\varepsilon} = +\infty.$$

□

**Proposition 2.2.** *The function  $f$  defined by (2.1) has the following separation property: Let  $0 < \varepsilon < 1$ ,  $p \geq 1$  and  $a, b \in W^{\varepsilon,p}(0, 2\pi)$ . If  $af = b$ , then  $a = b = 0$ .*

*Proof.* Write the  $W^{\varepsilon,p}$  seminorm as in (1.7)

$$|b|_{\varepsilon,p} = \left( \int_0^{2\pi} \int_0^{2\pi} \frac{|b(y) - b(x)|^p}{|y-x|^{1+p\varepsilon}} dy dx \right)^{\frac{1}{p}}.$$

Using

$$b(y) - b(x) = (f(y) - f(x))a(x) + f(y)(a(y) - a(x))$$

and the triangle inequality, we find for  $a, b \in W^{\varepsilon,p}(0, 2\pi)$

$$\left( \int_0^{2\pi} \int_0^{2\pi} \frac{|a(x)|^p |f(y) - f(x)|^p}{|y - x|^{1+p\varepsilon}} dy dx \right)^{\frac{1}{p}} \leq |b|_{\varepsilon,p} + \|f\|_{L^\infty(\mathbb{T})} |a|_{\varepsilon,p} < \infty.$$

Because of (2.5) from Lemma 2.1, this implies  $a(x) = 0$  for almost all  $x \in \mathbb{T}$  and then  $b = af = 0$ .  $\square$

### 3. 2D DOMAIN WITH LIMITED REGULARITY

Let  $F(x) = 1 + \int_0^x f(t)dt$ . Then  $F \in C^1(\mathbb{T})$ ,  $F' = f$ , and  $\frac{1}{2} < F(x) < \frac{3}{2}$ .

The latter estimate follows easily from

$$|F(x) - 1| = \left| \sum_{k=1}^{\infty} a_k \frac{1 - \cos(b_k x)}{b_k} \right| \leq 2^{-q} \sum_{k=1}^{\infty} 2^q q^{-k} = 2^{1-q} \frac{1}{q-1} \leq \frac{1}{2}.$$

We define now the  $C^1$  domain  $\omega \subset \mathbb{R}^2$  using polar coordinates  $(r, \theta)$

$$\omega = \{(r, \theta) \mid r < F(\theta)\}.$$

**Proposition 3.1.** *Let  $p \geq 1$ ,  $\varepsilon > 0$  and  $u \in W^{\frac{1}{p} + \varepsilon, p}(\omega; \mathbb{C}^2)$  be such that its normal trace  $n \cdot u$  vanishes on  $\partial\omega$ . Then  $u = 0$  on  $\partial\omega$ . The same conclusion is valid when the tangential trace  $n \times u$  vanishes on  $\partial\omega$ .*

*Proof.* (Following Filonov [7, §5]) The unit normal  $n$  on  $\partial\omega$  has the Cartesian components

$$n_1 = (F^2 + f^2)^{-\frac{1}{2}}(F \cos \theta + f \sin \theta), \quad n_2 = (F^2 + f^2)^{-\frac{1}{2}}(F \sin \theta - f \cos \theta).$$

Therefore the condition  $n_1 u_1 + n_2 u_2 = 0$  implies  $af = b$  if we define

$$a = u_2 \cos \theta - u_1 \sin \theta, \quad b = (u_1 \cos \theta + u_2 \sin \theta)F$$

Now, since the traces  $u_j$  on  $\partial\omega$ , understood as functions  $\theta \mapsto u_j(F(\theta), \theta)$  on  $\mathbb{T}$ , belong to  $W^{\varepsilon,p}(\mathbb{T})$ , we also have  $a, b \in W^{\varepsilon,p}(\mathbb{T})$ . According to Proposition 2.2 we find  $a = b = 0$ , which implies  $u_1 = u_2 = 0$  on  $\partial\omega$ . The result using vanishing tangential trace follows by a rotation by  $\pi/2$ .  $\square$

**Corollary 3.2.** (i) *There exists  $g \in C^\infty(\bar{\omega})$  such that the solution  $v_D \in H_0^1(\omega)$  of the Dirichlet problem*

$$\Delta v_D = g \text{ in } \omega; \quad v_D = 0 \text{ on } \partial\omega$$

*does not belong to  $W^{1+\frac{1}{p}+\varepsilon,p}(\omega)$  for any  $\varepsilon > 0$ ,  $p \geq 1$ .*

(i) *There exists  $g \in C^\infty(\bar{\omega})$  such that any solution  $v_N \in H^1(\omega)$  of the Neumann problem*

$$\Delta v_N = g \text{ in } \omega; \quad \partial_n v_N = 0 \text{ on } \partial\omega$$

*does not belong to  $W^{1+\frac{1}{p}+\varepsilon,p}(\omega)$  for any  $\varepsilon > 0$ ,  $p \geq 1$ .*

*Proof.* For  $v_D$  one can take  $g = 1$ . Set  $u = \nabla v_D$ . If  $v_D \in W^{1+\frac{1}{p}+\varepsilon,p}(\omega)$ , then  $u$  satisfies the hypotheses of Proposition 3.1 with vanishing tangential trace. Hence also the normal trace of  $u$  vanishes, i.e.  $\partial_n v_D = 0$  on  $\partial\omega$ . Then Green's formula implies  $\int_\omega g = 0$ , which is not the case.



For  $v_N \in W^{1+\frac{1}{p}+\varepsilon,p}(\omega)$  one obtains similarly that the tangential derivative on the boundary vanishes, hence the trace of  $v_N$  on  $\partial\omega$  is constant, without loss of generality equal to zero. Thus  $v_N$  is also solution of the Dirichlet problem. That there exists  $g \in L^2(\omega)$  for which this is impossible can be seen as follows:

Let  $g$  be a non-zero harmonic polynomial such that  $\int_{\omega} g = 0$ , for example  $g(x_1, x_2) = \alpha x_1 x_2 + \beta(x_1^2 - x_2^2)$  with suitably chosen coefficients  $\alpha, \beta \in \mathbb{R}$ . Then  $v_N$  exists, and Green's formula gives the contradiction

$$0 = \int_{\partial\omega} (\partial_n v_N g - v_N \partial_n g) ds = \int_{\omega} (\Delta v_N g - v_N \Delta g) dx = \int_{\omega} g^2 dx.$$

□

*Remark 3.3.* No eigenfunction of the Laplacian with Dirichlet conditions on  $\omega$  can belong to  $W^{1+\frac{1}{p}+\varepsilon,p}(\omega)$  with  $\varepsilon > 0$ , because it would also have vanishing normal derivative. Its extension by zero outside  $\omega$  would then be a Dirichlet eigenfunction with the same eigenvalue on any domain containing  $\omega$ . This contradicts for example the well known behavior of Dirichlet eigenvalues on disks or squares with varying size. It contradicts also the well known interior analyticity of Dirichlet eigenfunctions.

#### 4. EXAMPLE IN HIGHER DIMENSIONS

From  $\omega \subset \mathbb{R}^2$  one can construct  $\Omega \subset \mathbb{R}^d$  as follows (see [7], for  $n = 3$  also [4, §6]). In cylindrical coordinates  $(r, \theta, z)$ ,  $z \in \mathbb{R}^{d-2}$ :

$$\Omega = \{(r, \theta, z) \mid \frac{r^2}{F(\theta)^2} + |z|^2 < 1\}$$

The intersection with the plane  $z = z_0$  gives for  $|z_0| < 1$  the scaled domain  $\sqrt{1 - |z_0|^2} \omega$ . One can still prove that for this domain  $\Omega$  and  $0 < \varepsilon < 1$  there holds

$$W^{1+\frac{1}{p}+\varepsilon,p}(\Omega) \cap W_0^{1,p}(\Omega) = W_0^{1+\frac{1}{p}+\varepsilon,p}(\Omega), \quad (4.1)$$

that is, for functions in  $W^{1+\frac{1}{p}+\varepsilon,p}(\Omega)$  the vanishing of the boundary trace implies that also the normal derivative is zero on the boundary.

Indeed, suppose that  $v \in W^{1+\frac{1}{p}+\varepsilon,p}(\Omega)$ ,  $v = 0$  on  $\partial\Omega$  and let  $u = \nabla v$ . Then the tangential components of  $u$  are zero on the boundary, and we have to show that the normal component of  $u$  vanishes, too, on  $\partial\Omega$ . Define

$$\tilde{u}(r, \theta, z) = u(\sqrt{1 - |z|^2} r, \theta, z).$$

Then  $\tilde{u}$  is defined on the product domain

$$\tilde{\Omega} = \omega \times B_1 = \{(r, \theta, z) \mid (r, \theta) \in \omega, |z| < 1\}.$$

For any  $\delta \in (0, 1)$ , let  $\tilde{\Omega}_\delta = \omega \times B_\delta$ . On this product domain, there holds the inclusion

$$W^{s,p}(\tilde{\Omega}_\delta) \subset L^p(B_\delta; W^{s,p}(\omega)),$$

as can be seen first for integer  $s$  directly from the definition of the Sobolev space  $W^{s,p}$  and then for all  $s \geq 0$  by interpolation. Thus  $u \in W^{\frac{1}{p}+\varepsilon,p}(\Omega; \mathbb{C}^d)$  implies that  $\tilde{u}$  restricted to  $\tilde{\Omega}_\delta$  belongs to

$L^p(B_\delta; W^{\frac{1}{p}+\varepsilon,p}(\omega; \mathbb{C}^d))$ , and for almost every  $z_0 \in B_\delta$ , the restriction  $w_{z_0}$  of  $\tilde{u}$  to the plane  $z = z_0$  belongs to  $W^{\frac{1}{p}+\varepsilon,p}(\omega, \mathbb{C}^d)$ . The vanishing of the tangential components of  $u$  on  $\partial\Omega$  implies that the component of  $w_{z_0}$  that is parallel to the plane  $z = 0$  and tangential to  $\partial\omega$  vanishes on  $\partial\omega$ . Then Proposition 3.1 tells us that the component of  $w_{z_0}$  that is parallel to the plane  $z = 0$  and normal to  $\partial\omega$  vanishes on  $\partial\omega$ , too. This means that at such a point  $(r, \theta, z) \in \partial\Omega$  with  $(\sqrt{1 - |z|^2} r, \theta) \in \partial\omega$ ,  $z = z_0$ , in addition to the tangential components a component of  $u$  vanishes that is not tangential, and hence all components of  $u$  vanish there. Since this is true for almost all  $z_0$  satisfying  $|z_0| < \delta$  and for all  $0 < \delta < 1$ , we see that the trace of  $u$  on  $\partial\Omega$  is zero, which proves (4.1).

The non-regularity result of Theorem 1.2 for the Dirichlet problem in  $\Omega$  then follows in the same way as in the two-dimensional case. In particular, one can take  $g = 1$  for the counterexample.

For the Neumann problem, a slightly different variant of adding  $d - 2$  variables works, and this variant could also be used for the Dirichlet problem, giving a counterexample with a somewhat less regular right hand side  $g$ . For this variant, (4.1) still holds. We redefine the domain  $\Omega$  so that it contains a cylindrical part (see also [7, §5.2]). This is done by modifying the function  $1 - |z|^2$  in the previous example. Choose a decreasing  $C^\infty$  function  $\mu$  on  $\mathbb{R}_+$  satisfying

$$\mu(t) = 1 \text{ for } t \leq 1; \quad \mu(t) \leq 0 \text{ for } t \geq 4; \quad \mu'(t) < 0 \text{ for } t \geq 2,$$

and define

$$\Omega = \{(r, \theta, z) \mid r^2 < \mu(|z|^2) F(\theta)^2\}. \quad (4.2)$$

It is not hard to see that  $\Omega$  has a  $C^1$  boundary.

We now use the two-dimensional example presented in the previous section and denote by  $v_0$  the function found there that satisfies the Neumann problem on  $\omega$  with right hand side  $g_0 \in C^\infty(\bar{\omega})$  and that does not belong to any  $W^{1+\frac{1}{p}+\varepsilon,p}(\omega)$  for  $\varepsilon > 0$ ,  $p \geq 1$ . In addition, we choose a function  $\chi \in C_0^\infty(\overline{\mathbb{R}_+})$  satisfying  $\chi(t) = 1$  for  $t < \frac{1}{2}$ ,  $\chi(t) = 0$  for  $t \geq 1$ . Then we define

$$v(x, z) = v_0(x) \chi(|z|); \quad g(x, z) = g_0(x) \chi(|z|) + v_0(x) \Delta_z \chi(|z|); \quad (x \in \omega, |z| < 1).$$

Initially,  $v$  and  $g$  are defined on the cylinder  $\omega \times B_1 \subset \Omega$ , and we extend them by zero on the rest of  $\Omega$ .

One easily verifies that  $v$  satisfies

$$\Delta v = g \text{ in } \Omega; \quad \partial_n v = 0 \text{ on } \partial\Omega.$$

Noting that both  $\chi(|z|)$  and  $\Delta_z \chi(|z|)$  define  $C^\infty(\overline{\Omega})$  functions and using the regularity of  $v_0 \in W^{1+\frac{1}{p},p}(\omega)$  for all  $p > 1$ , so that  $v_0$  is Hölder continuous on  $\bar{\omega}$ , one finds that  $g$  is Hölder continuous on  $\overline{\Omega}$ . Finally the non-regularity of  $v_0$  implies clearly that also  $v \notin W^{1+\frac{1}{p}+\varepsilon,p}(\Omega)$  for  $\varepsilon > 0$ ,  $p \geq 1$ .

This concludes the proof of Theorem 1.2.

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