



**HAL**  
open science

# Blocking and invasion for reaction-diffusion equations in periodic media

Romain Ducasse, Luca Rossi

► **To cite this version:**

Romain Ducasse, Luca Rossi. Blocking and invasion for reaction-diffusion equations in periodic media. 2017. hal-01637967v1

**HAL Id: hal-01637967**

**<https://hal.science/hal-01637967v1>**

Preprint submitted on 18 Nov 2017 (v1), last revised 3 Jan 2018 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Blocking and invasion for reaction-diffusion equations in periodic media

Romain DUCASSE\* and Luca ROSSI\*

\*Ecole des Hautes Etudes en Sciences Sociales, PSL Research University, Centre d'Analyse et Mathématiques Sociales, 54 boulevard Raspail 75006 Paris, France

**Keywords:** reaction-diffusion equations, invasion, propagation, spreading, domains with holes.

**MSC:** 35A08, 35B30, 35K05, 35K57, 35B40

## Abstract

We investigate the large time behavior of solutions of reaction-diffusion equations with general reaction terms in periodic media. We first derive some conditions which guarantee that solutions with compactly supported initial data invade the domain. In particular, we relate such solutions with front-like solutions such as pulsating traveling fronts. Next, we focus on the homogeneous equation set in a domain with periodic holes, and specifically in the cases where fronts are not known to exist. We show how the geometry of the domain can block or allow invasion. We finally exhibit a periodic domain on which the propagation takes place in an asymmetric fashion, in the sense that the invasion occurs in a direction but is blocked in the opposite one.

## 1 Introduction

### 1.1 Large time behavior for the Cauchy problem

Reaction-diffusion equations classically arise in the study of biological phenomena (propagation of genes, epidemics), in physics (combustion) and more recently in social sciences (rioting models). They have been extensively studied since the seminal papers of Fisher [13] and Kolmogorov, Petrovski and Piskunov [19] who dealt with the *homogeneous* equation

$$\partial_t u = \Delta u + f(u), \quad t > 0, \quad x \in \mathbb{R}^N. \quad (1.1)$$

A crucial progress in the study of (1.1) is due to Aronson and Weinberger [1]. The basic assumption there is that  $f(0) = f(1) = 0$ . Then the authors consider three different sets of hypotheses. With the terminology commonly employed in the literature, they are:

$$\textit{monostable} \quad f > 0 \quad \text{in } (0, 1);$$

$$\textit{combustion} \quad \exists \theta \in (0, 1), \quad f = 0 \quad \text{in } [0, \theta], \quad f > 0 \quad \text{in } (\theta, 1);$$

$$\textit{bistable} \quad \exists \theta \in (0, 1), \quad f < 0 \quad \text{in } (0, \theta), \quad f > 0 \quad \text{in } (\theta, 1).$$

The monostable case is the one considered in [13], [19] and includes the logistic equation  $f(u) = u(1-u)$ ; the prototype of the bistable term is  $f(u) = u(1-u)(u-\theta)$ , which reduces to the Allen-Cahn nonlinearity when  $\theta = 1/2$ .

Two key features of the equation (1.1) are exhibited in [19, 1]. First, the existence of a special type of solutions named *traveling fronts* (or *traveling waves*). These are solutions of the form  $u(t, x) = \phi(x \cdot e - ct)$ , with  $e \in \mathbb{S}^{N-1}$ ,  $c \in \mathbb{R}$  and  $\phi(z) \rightarrow 1$  as  $z \rightarrow -\infty$  and  $\phi(z) \rightarrow 0$  as  $z \rightarrow +\infty$ . The second feature is the *invasion* property: for any “large enough” compactly supported non-negative initial datum, the solution of (1.1) converges locally uniformly to 1 as time goes to infinity. This result requires the hypothesis  $\int_0^1 f > 0$ , which is automatically fulfilled in the monostable and combustion cases. How large the initial datum needs to be depends on the type of nonlinearity: in the monostable case it is sufficient to be larger than a positive constant on a large ball, in the combustion or bistable case the constant needs to be larger than  $\theta$ . Actually, in the monostable case, there exists a critical exponent  $\beta > 1$  such that if  $f(u) \geq u^\beta$  for  $u \sim 0$  (hence in particular if  $f'(0) > 0$ ) then all solutions with non-negative, not identically equal to 0 initial data converge to 1 as  $t$  goes to  $+\infty$ . This is known as the *hair-trigger effect*, c.f. [1]. Observe instead that in the combustion case, if the initial datum lies below  $\theta$ , then (1.1) reduces to the heat equation and thus the solution converges uniformly to 0 as  $t \rightarrow +\infty$ . In the bistable case the situation is even worse; moreover if  $\int_0^1 f = 0$  then all solutions with compactly supported initial data  $\leq 1$  stay bounded away from 1, and they all converge to 0 if  $\int_0^1 f < 0$ .

The aim of this paper is to investigate the question of invasion for reaction-diffusion equations in periodic media. Specifically, we consider the problem

$$\begin{cases} \partial_t u = \nabla \cdot (A(x)\nabla u) + q(x) \cdot \nabla u + f(x, u), & t > 0, \quad x \in \Omega \\ \nu \cdot A(x)\nabla u = 0, & t > 0, \quad x \in \partial\Omega. \end{cases} \quad (1.2)$$

Here  $\Omega \subset \mathbb{R}^N$  is an unbounded smooth domain which is periodic in the directions of the canonical basis. Typical examples are domains with “holes” which are periodically arranged, that is,  $\Omega = (K + L\mathbb{Z}^N)^c$  (here and in the sequel, for a given subset  $A$  of  $\mathbb{R}^N$ , we denote its complement by  $A^c := \mathbb{R}^N \setminus A$ ) where  $K \subset \mathbb{R}^N$  is a compact set and  $L > 0$ . In the case  $\Omega = \mathbb{R}^N$ , we neglect the second equation in (1.2). The diffusion matrix  $A$ , the drift term  $q$  and the nonlinearity  $f$  are also assumed to be periodic with respect to the  $x$  variable, with the same period as  $\Omega$ .

We will describe the large time behavior of solutions in terms of the following properties.

**Definition 1.1.** We say that a solution  $u$  satisfies the properties of

- blocking*      if  $\limsup_{t \rightarrow +\infty} \left( \sup_{x \in \Omega} u(t, x) \right) < 1$ ;
- persistence*    if  $\liminf_{t \rightarrow +\infty} \left( \min_{x \in K} u(t, x) \right) > 0$  for all compact  $K \subset \bar{\Omega}$ ;
- invasion*      if  $u(t, x) \rightarrow 1$  as  $t \rightarrow +\infty$ , locally uniformly in  $x \in \bar{\Omega}$ .

Persistence is compatible with both blocking and invasion, whereas the latter two properties are mutually exclusive.

We shall say that a function  $u : \Omega \mapsto [0, 1]$  is *front-like* if

$$\exists e \in \mathbb{S}^{N-1}, \quad \lim_{\substack{x \cdot e \rightarrow -\infty \\ x \in \Omega}} u(x) = 1, \quad \lim_{\substack{x \cdot e \rightarrow +\infty \\ x \in \Omega}} u(x) = 0. \quad (1.3)$$

The kind of questions we address here are:

- Does the validity of the invasion property for *all* front-like data implies that the invasion property holds for *some* compactly supported data ?
- Does the validity of the invasion property for a *single* front-like datum implies that the invasion property holds for *some* compactly supported data ?

We will show in Theorem 1.3 that the answer to the first question is yes, specifying also how “large” the compactly supported data need to be. Then we will exhibit an example providing a negative answer to the second question, see Theorem 1.10.

## 1.2 Statement of the main results

Throughout the paper,  $\Omega$  is a domain with  $C^3$  boundary,  $\nu : \partial\Omega \rightarrow \mathbb{S}^{N-1}$  denotes its exterior normal field and  $\partial_\nu$  the associated directional derivative. We further assume  $\Omega$  to be periodic, i.e., there are  $L_1, \dots, L_N > 0$  such that

$$\forall k \in L_1\mathbb{Z} \times \dots \times L_N\mathbb{Z}, \quad \Omega + \{k\} = \Omega.$$

We let  $\mathcal{C}$  denote the periodicity cell:

$$\mathcal{C} := \bar{\Omega} \cap \prod_{i=1}^N [0, L_i).$$

We assume, unless otherwise stated, that the coefficients are also periodic with respect to  $x$  with the same periodicity as  $\Omega$ , i.e.,

$$\forall k \in L_1\mathbb{Z} \times \dots \times L_N\mathbb{Z}, \quad x \in \bar{\Omega}, \quad A(x+k) = A(x), \quad q(x+k) = q(x),$$

$$\forall k \in L_1\mathbb{Z} \times \dots \times L_N\mathbb{Z}, \quad x \in \bar{\Omega}, \quad s \in [0, 1], \quad f(x+k, s) = f(x, s).$$

The following regularity and ellipticity hypotheses will always be understood in the paper:  $A \in C^3(\bar{\Omega})$  is a symmetric, uniformly elliptic matrix field,  $q \in C^{1,\alpha}(\bar{\Omega})$ , for

some  $\alpha \in (0, 1)$  is a vector field and  $f : \bar{\Omega} \times [0, 1] \mapsto \mathbb{R}$  is globally Lipschitz-continuous. We shall denote  $0 < \lambda \leq \Lambda$  the ellipticity constants of the matrix field  $A$ , so that

$$\forall x \in \bar{\Omega}, \quad \lambda I_N \leq A(x) \leq \Lambda I_N, \quad (1.4)$$

where  $I_N$  is the identity matrix and the order is the usual one on symmetric matrices. In the whole paper, initial data and solutions are always understood to be bounded in order to avoid non-uniqueness problems.

We shall further assume that  $f$  satisfies the following properties:

$$\begin{cases} \forall x \in \bar{\Omega}, f(x, 0) = f(x, 1) = 0 \\ \exists S \in (0, 1), \forall x \in \bar{\Omega}, s \mapsto f(x, s) \text{ is strictly decreasing for } s \in (S, 1). \end{cases} \quad (1.5)$$

The first condition implies that the constant functions 0 and 1 are solutions of the problem (1.2). We extend  $f(x, s)$  by a negative function for  $s \notin [0, 1]$  such that the resulting function is globally Lipschitz-continuous. Thanks to (1.5), we can define

$$\theta := \max \{s \in [0, 1) : \exists x \in \bar{\Omega}, f(x, s) = 0\}. \quad (1.6)$$

Thus  $\theta \in [0, 1)$  is the smallest quantity for which  $f > 0$  in  $\bar{\Omega} \times (\theta, 1)$ .

This paper deals with the phenomena of invasion, persistence and blocking - formulated in Definition 1.1 - for equation (1.2), and is divided into two main parts. In the first part, we give sufficient conditions for persistence and invasion to occur for “large enough” compactly supported initial data. In a second part, we study how the geometry of the domain  $\Omega$  can influence the invasion in the bistable case.

### 1.2.1 Persistence and invasion

Our first result provides a sufficient condition for the persistence of solutions to (1.2) with “large enough” initial data, in the absence of the drift term  $q$ .

**Theorem 1.2.** *Assume that  $q \equiv 0$  and that  $f$  satisfies (1.5) together with*

$$\int_{\mathcal{C}} \int_0^1 f(x, s) ds dx > 0. \quad (1.7)$$

*Then, for all  $\eta \in (\theta, 1)$ , where  $\theta$  is defined by (1.6), there is  $r > 0$  such that the persistence property holds for every solution to (1.2) arising from a non-negative initial datum satisfying*

$$u_0 > \eta \quad \text{in } \Omega \cap B_r.$$

Let us make one comment about the condition  $q \equiv 0$ . It comes from the fact that our proof of Theorem 1.2 relies on an energy method. This condition is presumably not optimal, but it guarantees some control on the drift : indeed, a drift that is “too strong” would lead to extinction for all compactly supported initial data, no matter how large they are.

Our second main result is the equivalence of the invasion property for initial data which are “large enough on a large set” and for front-like initial data, i.e., satisfying (1.3).

**Theorem 1.3.** *Assume that  $f$  satisfies (1.5). Then, the following properties are equivalent:*

- (i) *Invasion occurs for all solutions of (1.2) with non-negative front-like initial data.*
- (ii) *Invasion occurs for “large enough” initial data, in the sense that for all  $\eta \in (\theta, 1)$ , where  $\theta$  is defined by (1.6), there is  $r > 0$  such that the invasion property holds for every solution to (1.2) arising from a non-negative initial datum satisfying*

$$u_0 > \eta \quad \text{in } \Omega \cap B_r.$$

The fact that (ii) implies (i) in Theorem 1.3 is an immediate consequence of the parabolic comparison principle. The interest is in the other implication.

We mention that Weinberger derives in [27] a related result of invasion. Roughly speaking, he shows that, if invasion occurs for an initial datum  $u_0$  whose support is in a half-plane such that  $\sup_{x \in \Omega} u_0(x) < 1$  (this is not front-like in the sense of (1.3)), and if this invasion occurs with a strictly positive speed, then invasion occurs for “large enough” compactly supported initial data. We mention that, up to some technicalities, our Theorem 1.3 allows to recover this result. The proof of [27] uses a discrete system approach. In particular, the rather involved argument that allows the author to handle compactly supported initial data cannot be directly performed in the continuous PDE setting.

As we mentioned already, it may be complicated to check that condition (i) of Theorem 1.3 holds. Our next result - Theorem 1.5 - provides a sufficient condition for the invasion property expressed in terms of *pulsating traveling fronts*. These are particular entire solutions of (1.2) satisfying some structural properties that generalize to the periodic framework the notion of traveling front. They were first introduced in dimension  $N = 1$  by Shigesada, Kawasaki and Teramoto in [24].

**Definition 1.4.** A *pulsating traveling front* in the direction  $e \in \mathbb{S}^{N-1}$  with speed  $c \in \mathbb{R} \setminus \{0\}$  connecting 1 to 0 is an entire (i.e., for all times) solution of (1.2)  $0 < v < 1$  such that

$$\begin{cases} \forall z \in \prod_{i=1}^N L_i \mathbb{Z}, \quad x \in \Omega, \quad v\left(t + \frac{z \cdot e}{c}, x\right) = v(t, x - z) \\ v(t, x) \xrightarrow{x \cdot e \rightarrow -\infty} 1 \quad \text{and} \quad v(t, x) \xrightarrow{x \cdot e \rightarrow +\infty} 0. \end{cases} \quad (1.8)$$

A pulsating traveling front with speed  $c = 0$  is a front-like stationary solution of (1.2), in the sense of (1.3).

Observe that, if  $v$  is a pulsating traveling front, then  $x \mapsto v(t, x)$  is front-like for all  $t \in \mathbb{R}$ . The use of traveling fronts to study the large time behavior of solutions of the Cauchy problem in the combustion and bistable cases is quite a natural approach, already used in the pioneering paper [1] for the homogeneous equation (1.1).

**Hypothesis 1.** For every direction  $e \in \mathbb{S}^{N-1}$ , there is a pulsating traveling front with speed  $c > 0$ .

The requirement that the speeds of the fronts are positive in Hypothesis 1 is necessary for the invasion property to hold, in the sense that, owing to the parabolic comparison principle, the existence of – even a single – pulsating traveling front with speed  $c \leq 0$  prevents the invasion of all solutions with compactly supported initial data smaller than 1.

**Theorem 1.5.** *Assume that  $f$  satisfies (1.5) and that there is  $\delta > 0$  such that*

$$s \mapsto f(x, s) \text{ is nonincreasing in some interval } (0, \delta).$$

*Then, Hypothesis 1 implies that properties (i)-(ii) of Theorem 1.3 hold.*

The usefulness of this result lies in the fact that there is a huge literature about pulsating traveling fronts. In particular, their existence and the positivity of their speed is proved by Berestycki and Hamel in [3, Theorems 1.13, 1.14] when  $f$  is of the monostable or combustion type (cf. the definition given in Section 1.1 in the homogeneous case) under the following additional assumptions on  $q$ :

$$\begin{cases} \nabla \cdot q = 0 & \text{in } \bar{\Omega}, \\ \int_{\mathcal{C}} q = 0, \\ q \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.9)$$

Observe that in the monostable case the monotonicity hypothesis of Theorem 1.5 is not fulfilled, but since the larger the nonlinearity, the more luckily the invasion property holds, the idea is to consider a term  $\tilde{f} \leq f$  of combustion type. This allows to derive the following result concerning non-negative reaction terms, such as monostable and combustion.

**Corollary 1.6.** *Assume that  $q$  satisfies (1.9) and that  $f$  satisfies (1.5) together with*

$$\forall x \in \Omega, \forall s \in (0, 1), \quad f(x, s) \geq 0. \quad (1.10)$$

*Then, properties (i)-(ii) of Theorem 1.3 hold.*

We point out that Corollary 1.6 could be also derived by combining the results of [3] with those of Weinberger [27] mentioned above.

Under the generality of our assumptions (1.5), which include the bistable case, the question of the existence of fronts is still widely open, in particular if the domain  $\Omega$  is not  $\mathbb{R}^N$ . We are not aware of any result in such case. If the equation is set on  $\mathbb{R}^N$ , then Xin [30, 29] and Ducrot [11] derive the existence of fronts with positive speed under some conditions on the coefficients. Theorem 1.5 then yields the invasion result in such cases, see Corollaries 3.2, 3.3 below.

One may wonder if the Hypothesis 1 is necessary in Theorem 1.5. This is not the case. Indeed Zlatoš exhibits in [31] a bistable periodic nonlinearity such that the 1-dimensional equation

$$\partial_t u = \partial_{xx} u + f(x, u), \quad t > 0, \quad x \in \mathbb{R},$$

admits no pulsating traveling fronts (actually, he shows that there are no *transition fronts*, a notion generalizing pulsating traveling fronts), but such that the invasion

property holds for sufficiently large compactly supported initial data. On the other hand, we exhibit in Theorem 1.9 below a situation where there are no fronts at all, and where invasion does not occur. We mention that other examples of reaction-diffusion equations that do not admit fronts are known in dimension 1, see [28] and in cylindrical domains (with drift-term), see [4].

Once we know that invasion occurs for a solution with compactly supported initial datum, it is not hard to see that this happens with a strictly positive speed, see Remark 1 below. In all the situations where Hypothesis 1 holds, we can actually compute this speed. Indeed, a formula analogous to the one derived by Freidlin and Gärtner in [15, 14] in the KPP case holds in our case. This was first proved by the second author in [23] when  $\Omega = \mathbb{R}^N$  and then by the first author in [10] for periodic domains  $\Omega$ . We mention that Weinberger also derives a similar formula in [27].

Our next result applies to equations with non-negative nonlinearities – such as monostable or combustion – set in the whole space. It provides an explicit sufficient condition for invasion to occur, as well as a lower bound on the asymptotic speed at which this takes place. This is expressed in terms of the following quantity:

$$R(f) := \sup_{0 < K < H < 1} \left( (H - K) \min_{\substack{x \in \bar{\Omega} \\ s \in [K, H]}} f(x, s) \right),$$

which is the area of the largest rectangle one can fit under the graph of  $s \mapsto \inf_{x \in \bar{\Omega}} f(x, s)$  in the upper half-plane. For this result, it is convenient to consider equations in non-divergence form, that is,

$$\partial_t u = \text{Tr}(A(x)D^2u) + q(x) \cdot \nabla u + f(x, u), \quad \forall t > 0, \forall x \in \mathbb{R}^N, \quad (1.11)$$

under the usual standing assumption of Section 1.2 on the terms  $A$ ,  $q$ ,  $f$ .

**Theorem 1.7.** *Assume that  $f$  satisfies (1.5), (1.10), that  $q$  is continuous and satisfies*

$$\limsup_{|x| \rightarrow +\infty} q(x) \cdot \frac{x}{|x|} < \frac{\lambda}{\sqrt{\Lambda}} \sqrt{R(f)},$$

where  $\lambda, \Lambda$  are given by (1.4). Then, properties (i)-(ii) of Theorem 1.3 hold for the equation (1.11).

Moreover, calling  $w^* := \frac{\lambda}{\sqrt{\Lambda}} \sqrt{R(f)} - \limsup_{|x| \rightarrow +\infty} q(x) \cdot \frac{x}{|x|}$ , there holds

$$\forall c < w^*, \quad \inf_{|x| \leq ct} u(t, x) \xrightarrow{t \rightarrow +\infty} 1.$$

Let us state some comments about this result. First, we point out that it holds without any assumption on  $q$  besides boundedness, unlike Corollary 1.6 which requires (1.9). Next, the theorem holds true without the periodicity assumption, provided (1.5), (1.10) are fulfilled by the function  $\inf_{x \in \mathbb{R}^N} f(x, \cdot)$ . This is a non-degeneracy hypothesis, without which the result could not hold (if  $f(x, \cdot) \equiv 0$  for  $x$  large for instance, there is no way invasion could occur). The proof will actually use only this non-degeneracy hypothesis, which is guaranteed by the periodicity. Finally, one can observe that the more negative  $q(x) \cdot \frac{x}{|x|}$  is, the larger  $w^*$  becomes. This is a



bit counter-intuitive because  $q(x) \cdot \frac{x}{|x|} < 0$  roughly means that the drift “gathers the mass” instead of scattering it. Hence, such a drift should slow down the invasion. This suggests that our bound  $w^*$  on the invasion speed may not be optimal in general. On the other hand, because this drift prevents the mass to scatter, it is natural that the more negative  $q(x) \cdot \frac{x}{|x|}$  is, the more likely invasion should occur. In any case, though probably not optimal, our estimate on the invasion speed has the advantage of being explicit and easy to compute. While an estimate in the same spirit is derived in [7] under the assumption  $\partial_s f(\cdot, 0) > 0$ , we are not aware of analogous results of this type in the combustion case.

The proof of Theorem 1.2 is carried out in Section 2. Theorem 1.3 and Theorem 1.5 are derived in Section 3. In Section 3.2, we derive Corollary 1.6. Theorem 1.7 is proved in Section 4.

### 1.2.2 Influence of the geometry of the domain

The results presented above guarantee some sufficient conditions ensuring the persistence or invasion property. It is known that, if  $f$  is of the bistable type, the geometry of the domain can produce “obstacles” which may prevent propagation. This is observed by Berestycki, Hamel and Matano in [5] for an exterior domain (i.e., the complement of a compact set) and by Berestycki, Bouhours and Chapuisat in [2] for a cylindrical-type domain with a bottleneck. If such obstacles repeat periodically in the domain, one could expect that the blocking property holds. We will show that this is actually the case, but also that other scenarios are possible.

We consider the simplest problem set in a periodic domain:

$$\begin{cases} \partial_t u = \Delta u + f(u), & t > 0, x \in \Omega, \\ \partial_\nu u = 0, & t > 0, x \in \partial\Omega, \end{cases} \quad (1.12)$$

where  $f$  is an *unbalanced bistable* nonlinearity, i.e.,

$$\exists \theta \in (0, 1), \quad f < 0 \quad \text{in } (0, \theta), \quad f > 0 \quad \text{in } (\theta, 1), \quad \int_0^1 f(s) ds > 0. \quad (1.13)$$

Notice that, under the assumptions (1.5) and (1.13), Theorem 1.2 ensures that the persistence property for “large enough” initial data holds for problem (1.12), whatever the domain  $\Omega$  is. Actually our results could be carried out for the more general equation (1.2), but we have chosen to emphasize here the role of the geometry of the domain. We shall show that the latter can affect the propagation in three different ways. Namely, we construct three periodic domains  $\Omega_1, \Omega_2, \Omega_3$  that exhibit, respectively, invasion, blocking (in the sense of Definition 1.1) and a new phenomenon that we call *oriented invasion*.

The domain  $\Omega_1$  is given by the whole space with a star-shaped hole  $K$  repeated periodically, with a period  $L$  sufficiently large.

**Theorem 1.8.** *Let  $f$  satisfy (1.5), (1.13) and let  $K \subset \mathbb{R}^N$  be a star-shaped compact set. Then, for  $L > 0$  sufficiently large, properties (i)-(ii) of Theorem 1.3 hold for the problem (1.12) set on*

$$\Omega_1 := (K + LZ^N)^c.$$

Next, building on the result of [5], we exhibit a domain  $\Omega_2$  where the propagation is always blocked.

**Theorem 1.9.** *Let  $f$  satisfy (1.5), (1.13). There exists a periodic domain  $\Omega_2$  such that the problem (1.12) set on  $\Omega_2$  satisfies the following:*

- (i) *Any solution arising from a compactly supported initial datum  $u_0 \leq 1$  is blocked in the sense of Definition 1.1.*
- (ii) *There exists a solution with compactly supported initial datum which converges (increasingly) to a periodic non-constant stationary solution as  $t$  goes to  $+\infty$ .*
- (iii) *Invasion fails for every front like initial datum and moreover there exist no pulsating traveling fronts.*

The lack of the invasion property is in strong contrast with the homogeneous case  $\Omega = \mathbb{R}^N$ , where such property is guaranteed by the condition  $\int_0^1 f(s)ds > 0$ , at least for large enough initial data. We point out that Theorem 1.9 part (ii) implies the existence of an intermediate periodic steady state between 0 and 1 which is *stable from below*. So, it turns out that the geometry of  $\Omega_2$  alters the bistable character of the nonlinearity  $f$  making some non-trivial stable steady states appear. This is the reason for which there are no fronts in such case, but one should rather expect the existence of *propagating terraces* instead, see [11, 18].

We finally construct a domain  $\Omega_3$  which exhibits a new phenomenon that we call *oriented invasion*. Namely, invasion occurs in a direction  $e$ , with a positive linear speed, whereas the propagation is blocked in the opposite direction  $-e$ . We state and prove the theorem in  $\mathbb{R}^2$ , but it can be easily generalized to higher dimensions. We denote by  $e_1, e_2$  the unit vectors of the canonical basis of  $\mathbb{R}^2$ .

**Theorem 1.10.** *Let  $f$  satisfy (1.5), (1.13). There exists a periodic domain  $\Omega_3 \subset \mathbb{R}^2$  and a positive constant  $w$  such that, for every  $\eta > \theta$ , there is  $r > 0$  for which the following properties hold for every solution to (1.12) arising from a compactly supported initial datum satisfying*

$$0 \leq u_0 \leq 1, \quad u_0 > \eta \quad \text{in } \Omega \cap B_r.$$

- (i) *Invasion in the direction  $e_1$ :*

$$\forall 0 < c < c' < w, \quad \forall a > 0, \quad \min_{\substack{x \in \overline{\Omega}_3 \\ ct \leq x \cdot e_1 \leq c't \\ |x \cdot e_2| \leq a}} |u(t, x) - 1| \xrightarrow{t \rightarrow +\infty} 0.$$

- (ii) *Blocking in the direction  $-e_1$ :*

$$u(t, x) \xrightarrow{x \cdot e_1 \rightarrow -\infty} 0 \quad \text{uniformly in } t > 0.$$

Theorem 1.10 provides an example of a periodic domain on which invasion takes place in an *asymmetric* way. In the KPP case, it is known that the speed of invasion in a given direction and in the opposite one coincide. The validity of such property for other reaction terms was an open question. Our result provides a counter-example in the bistable case.

From Theorem 1.10, it is easy to see that the invasion property is verified for the front-like initial datum  $u_0(x) := H(x_1)$ , where  $H(z)$  is the Heaviside function equal to 1 if  $z \geq 0$  and equal to 0 if  $z < 0$ . Hence, this shows that it is not enough to have invasion for a single front-like initial datum to have invasion for every “large enough” compactly supported non-negative initial datum. Theorem 1.3 states that, if we have invasion for every front-like initial datum, then we have invasion for large enough compactly supported initial datum. It is natural to wonder if it is sufficient to have invasion for at least one pleating-front initial datum in each direction to ensure invasion for “large enough” compactly supported initial datum. We leave it as an open question.

The proofs of Theorems 1.8 and 1.9 are given in Sections 5.1 and 5.2 respectively. Section 6 is dedicated to the proof of Theorem 1.10.

## 2 Persistence

This section is devoted to the proof of the persistence result, Theorem 1.2. In this whole section, we assume that  $q \equiv 0$  and that  $f$  satisfies (1.5) and (1.7). The proof relies on the study of the stationary problem

$$\begin{cases} \nabla \cdot (A(x)\nabla u) + f(x, u) = 0, & x \in \Omega \\ \nu \cdot A(x)\nabla u = 0, & x \in \partial\Omega. \end{cases} \quad (2.14)$$

The main tool is the construction of a family of solutions in truncated domains. This will be achieved using an energy method, in the same spirit of Berestycki, Lions [8], where the authors study the existence of positive solutions for homogeneous bistable equations in the whole space  $\mathbb{R}^N$ .

We consider the primitive of  $f$ , defined by

$$F(x, s) := \int_0^s f(x, \sigma) d\sigma.$$

For  $r > 0$ , we introduce the energy functional  $\mathcal{E}_r$  associated with (2.14) in the truncated domain  $\Omega \cap B_r$ :

$$\mathcal{E}_r(\phi) := \int_{\Omega \cap B_r} \left( \frac{1}{2} (A(x)\nabla\phi, \nabla\phi) - F(x, \phi) \right),$$

acting on the space  $H_0^1(\overline{\Omega} \cap B_r)$ . We study the existence of minimizers for this energy. In order to do so, we first derive a geometrical lemma ensuring that  $\Omega$  and  $B_r$  are not tangent for a.e.  $r > 0$ . We recall that  $\nu(x)$  stands for the exterior normal derivative at point  $x \in \partial\Omega$ .

**Lemma 2.1.** *For a.e.  $r > 0$ , there holds*

$$\forall x \in (\partial\Omega) \cap \partial B_r, \quad \nu(x) \neq \pm \frac{x}{|x|}. \quad (2.15)$$

*Proof.* Let  $d$  be a regularised signed distance from  $\partial\Omega$ , that is, a smooth function on  $\mathbb{R}^N$  coinciding with the signed distance from  $\partial\Omega$  in a neighbourhood of  $\partial\Omega$ , positive inside  $\Omega$ . Consider the pair of functions

$$\rho_{\pm}(x) := |x| \pm d(x).$$

These functions are smooth outside the origin. It follows from the Morse-Sard theorem [22] that the inverse images  $\rho_{\pm}^{-1}(r)$  do not contain any critical points of  $\rho_{\pm}$ , except for  $r$  belonging respectively to some sets  $\mathcal{R}_{\pm}$  with zero Lebesgue measure. Hence, for  $r \in \mathbb{R}^+ \setminus (\mathcal{R}_- \cup \mathcal{R}_+)$ , that is, for a.e.  $r > 0$ , any  $x \in \partial\Omega \cap \partial B_r$  satisfies  $\rho_{\pm}(x) = |x| = r$  and thus, for such  $x$ , we have

$$0 \neq \nabla \rho_{\pm}(x) = \frac{x}{|x|} \mp \nu(x).$$

□

For all  $r$  for which (2.15) holds, the set  $\Omega \cap B_r$  satisfies an interior and exterior cone condition. Also, it is not hard to check that, for all  $r$  for which (2.15) holds, the set  $\Omega \cap B_r$  has a finite number of connected components

**Proposition 2.2.** *For all  $r > 0$  such that (2.15) holds, the functional  $\mathcal{E}_r$  admits a global minimiser  $\underline{u}_r \in H_0^1(\overline{\Omega} \cap B_r)$  such that  $0 \leq \underline{u}_r \leq 1$  a.e. in  $\Omega \cap B_r$ .*

*Proof.* First, observe that, because we have extended  $f(x, s)$  by a negative function for  $s > 1$ , we have

$$\forall x \in \overline{\Omega}, \quad \forall s \geq 0, \quad F(x, s) \leq \int_0^{\min\{s, 1\}} f(x, \sigma) d\sigma \leq \max_{\overline{\Omega} \times [0, 1]} f.$$

We deduce that

$$\mathcal{E}_r \geq - \max_{\overline{\Omega} \times [0, 1]} f |B_r|,$$

that is,  $\mathcal{E}_r$  is bounded from below. Consider a minimizing sequence  $(u_n)_{n \in \mathbb{N}}$  of elements of  $H_0^1(\overline{\Omega} \cap B_r)$ . We can assume without loss of generality that  $\text{ess sup } u_n \leq 1$ , because, defining  $v_n = \min(u_n, 1)$ , we have that  $v_n \in H_0^1(\overline{\Omega} \cap B_r)$  and  $\mathcal{E}_r(v_n) \leq \mathcal{E}_r(u_n)$  again because  $f(x, s) < 0$  for  $s > 1$ . Likewise, we can take  $\text{ess inf } u_n \geq 0$ . Let us check that the sequence  $(u_n)_{n \in \mathbb{N}}$  is bounded in  $H^1(\Omega \cap B_r)$ . Indeed, on the one hand, the sequence is bounded in  $L^2(\Omega \cap B_r)$ , and on the other hand, remembering that  $\lambda$  denote the ellipticity constant of  $A$ ,

$$\frac{1}{2} \lambda \|\nabla u_n\|_{L^2(\Omega \cap B_r)}^2 - \max_{\overline{\Omega} \times [0, 1]} f |B_r| \leq \mathcal{E}_r(u_n).$$

Now, we cannot directly apply the Rellich-Kondrachev theorem to the sequence  $(u_n)_{n \in \mathbb{N}}$  because  $\Omega \cap B_r$  is not necessarily smooth. However, it is a Lipschitz domain thanks to (2.15), hence we can apply the usual Sobolev extension theorem (see

[26, Section 6] or [9]) for  $(u_n)_{n \in \mathbb{N}}$  and then apply the Rellich-Kondrachov theorem to the sequence of extended functions, getting then that, up to extraction,  $(u_n)_{n \in \mathbb{N}}$  converges in the  $L^2$  norm (up to subsequences) to some  $u \in L^2(\Omega \cap B_r)$ .

Let us show that the convergence actually holds in  $H_0^1(\overline{\Omega} \cap B_r)$ . To do so, we show that  $(u_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in this space. For  $m, n \in \mathbb{N}$ , we have

$$\begin{aligned} \inf_{H_0^1(\overline{\Omega} \cap B_r)} \mathcal{E}_r &\leq \mathcal{E}_r\left(\frac{u_m + u_n}{2}\right) \\ &\leq \frac{1}{2}\mathcal{E}_r(u_m) + \frac{1}{2}\mathcal{E}_r(u_n) - \frac{\lambda}{4}\|\nabla(u_n - u_m)\|_{L^2(\Omega \cap B_r)}^2 \\ &\quad + \frac{1}{2}\int_{\Omega \cap B_r} \left(F(x, u_m) + F(x, u_n) - 2F\left(x, \frac{u_m + u_n}{2}\right)\right). \end{aligned}$$

Using the fact that  $s \mapsto F(x, s)$  is Lipschitz-continuous, uniformly in  $x$ , and that  $(u_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $L^2(\Omega \cap B_r)$ , and therefore in  $L^1(\Omega \cap B_r)$ , we see that the above integral goes to 0 as  $m, n$  go to  $+\infty$ . On the other hand,  $\frac{1}{2}\mathcal{E}_r(u_m) + \frac{1}{2}\mathcal{E}_r(u_n) \rightarrow \inf_{H_0^1(\overline{\Omega} \cap B_r)} \mathcal{E}_r$  as  $m, n$  go to  $+\infty$ . It follows that  $(\nabla u_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $L^2(\Omega \cap B_r)$ . Now, by the continuity of  $\mathcal{E}_r$  in  $H_0^1(\overline{\Omega} \cap B_r)$ , we conclude that the limit  $u$  is a global minimiser for  $\mathcal{E}_r$ . Finally, the fact that  $0 \leq u \leq 1$  a.e. follows from the same argument as before.  $\square$

We know that  $\underline{u}_r$  satisfies the Euler-Lagrange equation associated with  $\mathcal{E}_r$ , together with mixed boundary conditions. Namely, it is a solution of the problem (2.14) inside  $B_r$  and vanishes on  $\Omega \cap \partial B_r$  in the sense of the trace. However, at this stage, we cannot exclude that the minimizer  $\underline{u}_r$  is the trivial solution identically equal to zero. Owing to Lemma 2.1, we can infer that  $\underline{u}_r$  is continuous on  $\overline{\Omega} \cap \overline{B_r}$ , as shown in the following.

**Lemma 2.3.** *If (2.15) holds then  $\underline{u}_r \in C^0(\overline{\Omega} \cap \overline{B_r})$  and  $\max_{\overline{\Omega} \cap \overline{B_r}} \underline{u}_r < 1$ .*

*Proof.* Let  $r$  be such that (2.15) holds. Then, the weak bounded solution  $\underline{u}_r$  is actually continuous up to the boundary of  $\Omega \cap B_r$ , see [25, Theorem 14.5]. Because  $\underline{u}_r$  is continuous and vanishes on  $\Omega \cap \partial B_r$ , it attains its maximum  $M \leq 1$  at some  $\bar{x} \in \overline{\Omega} \cap \overline{B_r}$ . Assume by contradiction that  $M = 1$ . The function  $v := M - \underline{u}_r$  is non-negative, vanishes at  $\bar{x}$  and satisfies  $\nabla(A(x)\nabla v) = f(x, M - v)$  in  $\Omega \cap B_r$ . Because  $s \mapsto f(x, s)$  is Lipschitz-continuous uniformly in  $x$ , we see that  $\nabla \cdot (A(x)\nabla v) = f(x, M - v)$  can be rewritten as a linear equation with bounded coefficients. It follows from Hopf's lemma and  $\nu A \nabla v = 0$  on  $(\partial\Omega) \cap B_r$  that  $\bar{x} \notin \partial\Omega$ . Thus, the strong maximum principle implies that  $v \equiv 0$  on the connected component  $\mathcal{O}$  of  $\Omega \cap B_r$  containing  $\bar{x}$ . Observe that  $\overline{\mathcal{O}} \cap \partial B_r \neq \emptyset$  because  $\Omega$  is connected. As a consequence, since  $v$  is continuous up to the boundary, there exists  $x \in \Omega \cap \partial B_r$  such that  $v(x) = 0$ , i.e.,  $\underline{u}_r(x) = M$ . We have reached a contradiction because  $\underline{u}_r = 0$  on  $\Omega \cap \partial B_r$ .  $\square$

The next lemma states that  $\underline{u}_r \not\equiv 0$ , provided  $r > 0$  is large enough. That is, we have built non-trivial solutions of (2.14) set on truncated domains.

**Lemma 2.4.** *Assume that  $f$  satisfies (1.7). Then, there is  $R^*$  such that, for  $r > R^*$ , there holds*

$$\underline{u}_r \not\equiv 0.$$

*Proof.* Since  $\underline{u}_r$  minimises  $\mathcal{E}_r$ , we can get an upper bound for  $\mathcal{E}_r(\underline{u}_r)$  by estimating  $\mathcal{E}_r(\phi_r)$  on a suitable function  $\phi_r \in H_0^1(\overline{\Omega} \cap B_r)$ . For  $r > 1$ , we define  $\phi_r : \overline{\Omega} \cap B_r \rightarrow \mathbb{R}$  as follows:

$$\phi_r(x) = \begin{cases} 1 & \text{if } |x| \leq r-1, \\ r-|x| & \text{if } r-1 < |x| < r. \end{cases}$$

Observe that  $|\nabla \phi_r| \leq 1$ . We compute

$$\begin{aligned} \mathcal{E}_r(\phi_r) &= \int_{\Omega \cap B_r} \left( \frac{1}{2} (A \nabla \phi_r) \cdot \nabla \phi_r - F(x, \phi_r) \right) \\ &\leq \frac{1}{2} \Lambda |B_r \setminus B_{r-1}| - \int_{\Omega \cap B_{r-1}} F(x, 1) - \left( \min_{\overline{\Omega} \times [0,1]} F \right) |B_r \setminus B_{r-1}|, \end{aligned}$$

where  $\Lambda$  is given by (1.4). We eventually infer the existence of a constant  $C$  independent of  $r$  such that

$$\forall r > 1, \quad \mathcal{E}_r(\underline{u}_r) \leq C r^{N-1} - \int_{\Omega \cap B_{r-1}} F(x, 1).$$

Observing that, for any measurable periodic function  $g$  in  $L_{loc}^1(\mathbb{R}^N)$ , we have  $\int_{B_r} g \rightarrow \int_{[0, L_1] \times \dots \times [0, L_n]} g$  as  $r$  goes to  $+\infty$ , where  $\int$  stands for the integral average, we have

$$|\Omega \cap B_r| \sim_{r \rightarrow +\infty} \frac{|\mathcal{C}|}{\prod L_i} |B_r| = \frac{|\mathcal{C}|}{\prod L_i} |B_1| r^N,$$

and then

$$\frac{1}{r^N} \int_{\Omega \cap B_{r-1}} F(x, 1) \xrightarrow{r \rightarrow +\infty} \frac{|\mathcal{C}|}{\prod L_i} |B_1| \int_{\mathcal{C}} F(x, 1).$$

Then, because the latter term is positive by hypothesis (1.7), we have

$$\limsup_{r \rightarrow +\infty} \frac{\mathcal{E}_r(\underline{u}_r)}{r^N} \leq \limsup_{r \rightarrow +\infty} \left( C \frac{1}{r} - \frac{1}{r^N} \int_{\Omega \cap B_{r-1}} F(x, 1) \right) < 0,$$

whence  $\mathcal{E}_r(\underline{u}_r) < 0$  if  $r > 0$  is sufficiently large. Therefore,  $\underline{u}_r \not\equiv 0$  because  $\mathcal{E}_r(0) = 0$ , and the result follows.  $\square$

Now, we can prove Theorem 1.2.

*Proof of Theorem 1.2.* Thanks to Lemma 2.4, we can take  $r > 0$  such that  $\underline{u}_r \not\equiv 0$ . This function, extended by 0 on  $\Omega \setminus \overline{B}_r$ , is a generalized subsolution of (1.2). Let  $\underline{u}$  denote the solution of (1.2) arising from such initial datum. Using the parabolic comparison principle, it is classical to get that  $\underline{u}(t, x)$  is increasing with respect to  $t$  and converges locally uniformly in  $\overline{\Omega}$  to a stationary solution of (1.2) as  $t$  goes to  $+\infty$ . This stationary solution is strictly positive, thanks to the elliptic strong maximum principle and Hopf principle, and then  $\underline{u}(t, x)$  satisfies the persistence property. By the parabolic comparison principle, we can infer that every solution of (1.2) with initial datum  $u_0$  satisfying  $u_0 \geq \underline{u}_r$  satisfies the persistence property.

Now, take  $\eta \in (\theta, 1)$ , and, for  $n \in \mathbb{N}$ , let  $u_0^n$  be a function with compact support in  $\overline{\Omega} \cap B_n$  such that  $u_0^n \leq \eta$  and

$$u_0^n = \eta \quad \text{in } \overline{\Omega} \cap B_{n-1}.$$

Then, we have that  $\nu \cdot (A(x)\nabla u_0^n) = 0$ ,  $\forall x \in \partial\Omega \cap B_{n-1}$ , i.e.,  $u_0^n$  satisfies the boundary conditions of (1.2). This is necessary to have the usual parabolic estimates up to time  $t = 0$ , see, for instance [20, Theorems 5.2, 5.3]. Let  $u^n(t, x)$  denote the solution of (1.2) arising from the initial datum  $u_0^n$ . By the parabolic estimates,  $u^n$  converges locally uniformly in  $[0, +\infty) \times \overline{\Omega}$  to the solution  $v$  of (1.2) with constant initial datum  $v(0, \cdot) \equiv \eta$ . Because  $\eta > \theta$ ,  $v(t, x)$  converges uniformly to 1 as  $t$  goes to  $+\infty$ : indeed, we can define  $z(t)$  to be the solution of the ODE  $\dot{z}(t) = \min_{x \in \overline{\Omega}} f(x, z(t))$  with initial value  $z(0) = \eta$ . It is easy to see that  $z(t)$  is a subsolution of (1.2), and the parabolic comparison principle yields  $z(t) \leq v(t, x)$ ,  $\forall t \geq 0$ ,  $\forall x \in \Omega$ . Observe that  $z(t)$  goes to 1 as  $t$  goes to  $+\infty$ , because  $z(0) > \theta$  and  $\theta$  is defined by (1.6) as the largest  $s \in [0, 1)$  such that  $f(x, s)$  vanishes. Combining this with the fact that  $v \leq 1$ , we obtain the uniform convergence of  $v(t, \cdot)$  to 1 as  $t$  goes to  $+\infty$ . Because  $\underline{u}_r < 1$  and is compactly supported in  $\overline{\Omega} \cap B_r$ , there exists  $T > 0$  such that  $v(T, \cdot) > \underline{u}_r$ . We can then find  $\bar{n}$  so that

$$\forall x \in \overline{\Omega} \cap B_r, \quad u^{\bar{n}}(T, x) \geq \underline{u}_r(x).$$

The parabolic comparison principle implies that  $u^{\bar{n}}(T + t, \cdot) \geq \underline{u}(t, \cdot)$  and therefore  $u^{\bar{n}}$  satisfies the persistence property. By comparison, the same holds true for any solution of (1.2) with initial datum  $u_0$  larger than  $u_0^{\bar{n}}$ , and in particular if  $u_0 > \eta$  on  $\overline{\Omega} \cap B_{\bar{n}}$ , hence the result.  $\square$

Now that Theorem 1.2 is proved, and before turning to the proof of Theorem 1.3, we show that, under an additional assumption on  $f$ , we have ‘‘almost invasion’’.

**Proposition 2.5.** *Assume that  $f$  satisfies (1.5), (1.7) and that*

$$\forall x \in \overline{\Omega}, \quad \forall s \in [0, 1), \quad F(x, s) < F(x, 1). \quad (2.16)$$

*Then, for every compact  $K \subset \overline{\Omega}$ , for every  $\varepsilon > 0$  and  $\eta > \theta$ , where  $\theta$  is defined by (1.6), there is  $r > 0$  such that any solution to (1.2) with a non-negative initial datum  $u_0$  satisfying*

$$u_0 > \eta \quad \text{in } \Omega \cap B_r$$

*satisfies*

$$\liminf_{t \rightarrow +\infty} \left( \min_{x \in K} u(t, x) \right) > 1 - \varepsilon.$$

*Proof.* The proof is divided into five steps.

*Step 1. Estimate on  $\mathcal{E}_r(\underline{u}_r)$ .*

Recalling that  $\underline{u}_r$  minimises  $\mathcal{E}_r$ , we have that  $\mathcal{E}_r(\underline{u}_r) \leq \mathcal{E}_r(\phi)$ , for any  $\phi \in H_0^1(\overline{\Omega} \cap B_r)$ . As in the proof of Lemma 2.4, for  $r > 1$ , we define  $\phi_r : \overline{\Omega} \cap B_r \rightarrow \mathbb{R}$  as follows:

$$\phi_r(x) = \begin{cases} 1 & \text{if } |x| \leq r - \sqrt{r}, \\ \sqrt{r} - \frac{|x|}{\sqrt{r}} & \text{if } r - \sqrt{r} < |x| < r. \end{cases}$$

We have  $|\nabla\phi_r| \leq \frac{1}{\sqrt{r}}$ , and then

$$\begin{aligned}\mathcal{E}_r(\phi_r) &= \int_{\Omega \cap B_r} \left( \frac{1}{2} (A \nabla \phi_r) \cdot \nabla \phi - F(x, \phi_r) \right) \\ &\leq \frac{1}{2} \Lambda |B_r \setminus B_{r-\sqrt{r}}| r^{-1} - \int_{\Omega \cap B_{r-\sqrt{r}}} F(x, 1) - \left( \min_{\bar{\Omega} \times [0,1]} F \right) |B_r \setminus B_{r-\sqrt{r}}|,\end{aligned}$$

where  $\Lambda$  is given by (1.4). We eventually infer the existence of a constant  $C$  independent of  $r$  such that

$$\forall r > 1, \quad \mathcal{E}_r(\underline{u}_r) \leq Cr^{N-1/2} - \int_{\Omega \cap B_{r-\sqrt{r}}} F(x, 1). \quad (2.17)$$

*Step 2. Lower bound for the average of  $F(x, \underline{u}_r)$ .*

First, observe that we have

$$\mathcal{E}_r(\underline{u}_r) \geq - \int_{\Omega \cap B_{r-\sqrt{r}}} F(x, \underline{u}_r) - |B_r \setminus B_{r-\sqrt{r}}| \max_{\bar{\Omega} \times [0,1]} f.$$

Combining this with (2.17), we get

$$\int_{\Omega \cap B_{r-\sqrt{r}}} (F(x, 1) - F(x, \underline{u}_r)) - |B_r \setminus B_{r-\sqrt{r}}| \max_{\bar{\Omega} \times [0,1]} f \leq Cr^{N-1/2}. \quad (2.18)$$

The inequality (2.18) holds for all  $r > 1$ , hence, using the fact that  $|\Omega \cap B_r| \sim \frac{|c|}{\prod L_i} |B_r| = \frac{|c|}{\prod L_i} |B_1| r^N$  as  $r \rightarrow +\infty$ , dividing (2.18) by  $|\Omega \cap B_{r-\sqrt{r}}|$  and taking the limit as  $r \rightarrow +\infty$ , we eventually infer that

$$\liminf_{r \rightarrow +\infty} \int_{\Omega \cap B_{r-\sqrt{r}}} (F(x, \underline{u}_r) - F(x, 1)) \geq 0. \quad (2.19)$$

*Step 3. Convergence of the maxima to 1.*

We show now that (2.19) implies that

$$\sup_{\Omega \cap B_{r-\sqrt{r}}} \underline{u}_r \xrightarrow{r \rightarrow +\infty} 1. \quad (2.20)$$

We proceed by contradiction : assume that there is a diverging sequence  $(r_n)_{n \in \mathbb{N}}$  such that (we recall that  $\underline{u}_r$  satisfies  $\max \underline{u}_r < 1$ )

$$M := \sup_{\substack{n \in \mathbb{N} \\ x \in \Omega \cap B_{r_n - \sqrt{r_n}}}} \underline{u}_{r_n}(x) < 1.$$

Then, (2.19) implies that

$$0 \leq \liminf_{n \rightarrow +\infty} \int_{\Omega \cap B_{r_n - \sqrt{r_n}}} (F(x, \underline{u}_{r_n}) - F(x, 1)) \leq \max_{\bar{\Omega} \times [0, M]} (F(x, s) - F(x, 1)).$$



This contradicts the hypothesis (2.16).

*Step 4.  $\underline{u}_r$  is large on a large set.*

Consider a sequence of radii  $(r_n)_{n \in \mathbb{N}}$  diverging to  $+\infty$  and satisfying (2.15). For  $n \in \mathbb{N}$ , let  $x_n \in \overline{\Omega \cap B_{r_n - \sqrt{r_n}}}$  be such that  $\underline{u}_{r_n}(x_n) = \max_{\overline{\Omega \cap B_{r_n - \sqrt{r_n}}}} \underline{u}_{r_n}$ . Then let  $z_n \in \prod_{i=1}^N L_i \mathbb{Z}$  be such that  $x_{r_n} - z_n \in \mathcal{C}$ . Finally, define  $u^n(x) := \underline{u}_{r_n}(x + z_n)$ . For any compact set  $K \subset \overline{\Omega}$ , these functions are well defined in  $K$ , for  $n$  large enough, because  $r_n - |z_n| > \sqrt{r_n} - \sqrt{\sum_{i=1}^N L_i^2}$ . Hence, owing to the partial boundary estimates (see, e.g., [17, Theorem 6.30]) they converge (up to subsequences) locally uniformly in  $\overline{\Omega}$  to a solution  $u^* \leq 1$  of (2.14). Furthermore, by the choice of  $x_n$  and (2.20), we have that  $\max_{\overline{\Omega}} u^* = 1$ . Proceeding exactly as in the proof of Lemma 2.3, by means of Hopf's lemma and strong maximum principle, we eventually infer that  $u^* \equiv 1$ . This shows that  $\underline{u}_{r_n}(\cdot + z_n)$  converges to 1 locally uniformly in  $\overline{\Omega}$  as  $n$  goes to  $+\infty$ .

*Step 5. Conclusion.*

We are now in position to conclude the proof. Take a compact set  $K \subset \overline{\Omega}$  and  $\varepsilon > 0$  and  $\eta > \theta$ . First, owing to the fourth step, we can find  $\tilde{r} > 0$  and  $z \in \prod_{i=1}^N L_i \mathbb{Z}$  such that

$$\forall x \in K, \quad \underline{u}_{\tilde{r}}(x + z) > 1 - \varepsilon.$$

Now, arguing as in the proof of Theorem 1.2, we can find  $r > 0$  such that, if  $u(t, x)$  is the solution of (1.2) arising from an initial datum  $u_0 \geq \eta$  on  $\overline{\Omega} \cap B_r$ , then there is  $T > 0$  such that

$$u(T, \cdot) \geq \underline{u}_{\tilde{r}}(\cdot + z).$$

Because  $\underline{u}_{\tilde{r}}(\cdot + z)$  extended by 0 out of  $B_r(-z)$  is a generalized stationary subsolution of (1.2), we have that  $u(T + t, \cdot) \geq \underline{u}_{\tilde{r}}(\cdot + z)$ , for every  $t \geq 0$ . The result follows.  $\square$

## 3 Invasion

This section is dedicated to the proof of the invasion results Theorems 1.3, 1.5 and their Corollaries 1.6, 3.2 and 3.3.

### 3.1 Proofs of Theorems 1.3 and 1.5

The idea to prove Theorem 1.3 is mainly geometrical.

*Proof of Theorem 1.3.* It is straightforward to see that property (ii) implies (i): every front-like datum (in the sense of (1.3)) is, up to a suitable translation, larger than any  $\eta < 1$  in any bounded subset of  $\Omega$ . Let us show the reverse implication.

Using the fact that  $f(\cdot, s) < 0$  for  $s > 1$ , one readily infers that any solution to (1.2) satisfies

$$\limsup_{t \rightarrow +\infty} \left( \sup_{x \in \overline{\Omega}} u(t, x) \right) \leq 1. \quad (3.21)$$

Indeed, set  $m := \max\{1, \sup_{x \in \overline{\Omega}} u(0, x)\}$  and let  $v(t)$  be the solution of the ODE  $\dot{v} = \max_{x \in \overline{\Omega}} f(x, v)$  with initial datum  $v(0) = m$ . From one hand, the function  $v$

satisfies  $v(+\infty) = 1$ , and from the other it is a supersolution of (1.2). Hence, (3.21) follows from the parabolic comparison principle.

The rest of the proof of the invasion property (ii) is split into four steps.

*Step 1. Reducing to an equivalent property.*

Take  $\eta, \eta'$  satisfying  $\theta < \eta' < \eta < 1$ , where  $\theta$  is defined by (1.6). For  $n \in \mathbb{N}$ ,  $n \geq 2$ , we let  $u_0^n$  denote a non-negative function compactly supported in  $\overline{\Omega} \cap B_n$  such that  $u_0^n = \eta$  on  $\overline{\Omega} \cap B_{n-1}$ . We call  $u^n$  the solution of (1.2) arising from the initial datum  $u_0^n$ . We claim that there exists  $n > 0$  such that

$$\text{for every compact } K \subset \overline{\Omega}, \quad \liminf_{t \rightarrow +\infty} \left( \min_{x \in K} u^n(t, x) \right) \geq \eta'. \quad (3.22)$$

Before proving this claim, let us show how it entails property (ii). We argue by contradiction, assuming that (3.22) holds but the invasion property fails. By (3.21) and the uniform continuity of  $u^n$ , we can find  $y \in \Omega$  and a sequence  $(t_k)_{k \in \mathbb{N}}$  diverging to  $+\infty$  and such that  $\lim_{k \rightarrow +\infty} u^n(t_k, y) < 1$ . Defining  $\tilde{u}_k(t, x) := u^n(t + t_k, x)$ , we can use the usual parabolic estimates to deduce that  $(\tilde{u}_k)_{k \in \mathbb{N}}$  converges (up to subsequences) locally uniformly to an entire solution  $\tilde{u}_\infty$  of (1.2) such that  $\tilde{u}_\infty(0, y) < 1$ . Moreover, by (3.22),  $\tilde{u}_\infty(t, x) \geq \eta'$  for all  $t \in \mathbb{R}$  and  $x \in \Omega$ .

Let  $v_M$  be to be the solution of the ODE

$$\dot{v}_M(t) = \min_{x \in \overline{\Omega}} f(x, v_M(t)),$$

with initial datum  $v_M(-M) = \eta'$ . Because  $\eta' \in (\theta, 1)$  and  $\min_{x \in \overline{\Omega}} f(x, s) > 0$  for  $s \in (\theta, 1)$ , we see that  $v_M(t) \rightarrow 1$  as  $t$  goes to  $+\infty$ , or, equivalently,  $v_M(t) = v_0(t+M) \rightarrow 1$  as  $M$  goes to  $+\infty$  for any  $t \in \mathbb{R}$ . Since  $v_M$  is a subsolution of (1.2) for  $t > -M$ , with  $v_M(-M) = \eta' \leq \inf \tilde{u}_\infty(-M, \cdot)$ , the parabolic comparison principle implies that  $v_M(t) \leq \inf_{x \in \Omega} \tilde{u}_\infty(t, x)$  for  $t > -M$ . Letting  $M$  go to  $+\infty$  yields  $\tilde{u}_\infty \geq 1$  everywhere. This is a contradiction. We have then derived the invasion property for the initial datum  $u_0^n$ , provided (3.22) holds, and then by comparison for all initial data larger than  $u_0^n$ , that is property (ii).

It remains to prove that (3.22) holds for  $n$  sufficiently large. We argue again by contradiction, assuming that for any  $n \geq 2$ ,

$$\exists K_n \text{ compact subset of } \overline{\Omega}, \quad \liminf_{t \rightarrow +\infty} \left( \min_{x \in K_n} u^n(t, x) \right) < \eta'. \quad (3.23)$$

*Step 2. Lower bound on the expansion of the level sets.*

For  $n \geq 2$  we define

$$T_n := \inf \{ t \geq 0 : \exists x \in \overline{\Omega} \cap B_{\sqrt{t}}, u^n(t, x) \leq \eta' \}.$$

Observe that (3.23) implies that the above set is nonempty because  $K_n \subset \overline{\Omega} \cap B_{\sqrt{t}}$  for  $t$  large. We have that

$$\forall t \in (0, T_n), \quad \forall x \in \overline{\Omega} \cap B_{\sqrt{t}}, \quad u^n(t, x) > \eta',$$

and there exists  $x_n \in \overline{\Omega \cap B_{\sqrt{T_n}}}$  such that  $u^n(T_n, x_n) = \eta'$ .

Let us show that  $T_n \rightarrow +\infty$  as  $n \rightarrow \infty$ . Because the initial datum  $u_0^n$  satisfies the boundary condition of (1.2) on  $\overline{\Omega} \cap B_{n-1}$ , we can apply the parabolic estimates (see [20, Theorems 5.2, 5.3]) to get that  $u^n$  converges locally uniformly in  $[0, +\infty) \times \overline{\Omega}$  to a solution  $v$  of (1.2) satisfying  $v(0, x) \geq \eta$  for  $x \in \overline{\Omega}$ . In particular,  $v(t, x) \geq \eta$  for all  $t \geq 0$  because  $\eta \in (\theta, 1)$  is a subsolution to (1.2) by (1.6). This local uniform convergence implies that, for every  $T > 0$ , we can find  $n$  large enough such that

$$\forall t \in [0, T], \forall x \in \Omega \cap \overline{B_{\sqrt{T}}}, \quad u^n(t, x) > \eta'.$$

Hence,  $T_n \geq T$ , for  $n$  large enough. This means that  $T_n \rightarrow +\infty$  as  $n \rightarrow +\infty$ .

*Step 3. Reducing to a front-like entire solution.*

Consider the sequence  $(z_n)_{n \in \mathbb{N}}$  in  $\prod_{i=1}^N L_i \mathbb{Z}$  for which  $y_n := x_n - z_n \in \mathcal{C}$ . Then define

$$w_n(t, x) := u^n(t + T_n, x + z_n).$$

By the periodicity of the problem, the functions  $w_n$  are solutions to (1.2) for  $t > -T_n$  and satisfy

$$w_n(0, y_n) = \eta',$$

$$\forall t \in [-T_n, 0], \forall x \in \Omega \cap B_{\sqrt{t+T_n}}(-z_n), \quad w_n(t, x) \geq \eta'. \quad (3.24)$$

Because  $T_n \rightarrow +\infty$  as  $n \rightarrow +\infty$  by the previous step, the sequence  $(w_n)_{n \in \mathbb{N}}$  converges (up to subsequences) locally uniformly to an entire solution  $w_\infty$  of (1.2), i.e., a solution for all times  $t \in \mathbb{R}$ . Observe that  $w_\infty$  satisfies  $w_\infty(0, y) = \eta'$ , where  $y$  is the limit of (a subsequence of)  $(y_n)_{n \in \mathbb{N}}$ . Furthermore, defining for  $t \in \mathbb{R}$

$$H_t := \Omega \cap \bigcup_{M \geq 1} \bigcap_{n \geq M} B_{\sqrt{t+T_n}}(-z_n)$$

(by convention, we set  $B_{\sqrt{\tau}} = \emptyset$  if  $\tau \leq 0$ ) we see that

$$\forall t \leq 0, \forall x \in H_t, \quad w_\infty(t, x) \geq \eta'.$$

Now, assume that  $(x_n)_{n \in \mathbb{N}}$  is bounded independently of  $n$ . Then, so is  $(z_n)_{n \in \mathbb{N}}$  and thus  $B_{\sqrt{t+T_n}}(-z_n)$  invades  $\mathbb{R}^N$  as  $n$  goes to  $+\infty$ , which implies that  $H_t = \Omega$  for all  $t$ . As a consequence,

$$\forall t \leq 0, \forall x \in \Omega, \quad w_\infty(t, x) \geq \eta' \quad \text{and} \quad w_\infty(0, y) = \eta'.$$

Arguing as in the step 1, the fact that  $\eta' > \theta$  implies that  $w_\infty \geq 1$ , whence a contradiction.

We consider the remaining case:

$$|x_n| \xrightarrow{n \rightarrow +\infty} +\infty.$$

Let us show that there is  $e \in \mathbb{S}^{N-1}$  such that

$$\forall t \leq 0, \quad P := \{x \in \Omega : x \cdot e < y \cdot e\} \subset H_t.$$

Let  $e \in \mathbb{S}^{N-1}$  be such that, up to extraction,  $\hat{x}_n := \frac{x_n}{|x_n|} \rightarrow e$  as  $n$  goes to  $+\infty$ . Take  $t \leq 0$  and  $x \in P$ . Then, there is  $\varepsilon > 0$  such that  $x \cdot e < y \cdot e - \varepsilon$ . For  $n \in \mathbb{N}$ , we have

$$\begin{aligned} |x + z_n|^2 &= |x - y_n + x_n|^2 \\ &= |x - y_n|^2 + |x_n|^2 + 2(x - y) \cdot x_n + 2(y - y_n) \cdot x_n \\ &< |x - y_n|^2 + |x_n|^2 - 2\varepsilon|x_n| + 2(x - y) \cdot (x_n - |x_n|e) + 2(y - y_n) \cdot x_n \\ &\leq \left( \frac{|x - y_n|^2}{|x_n|} - 2\varepsilon + 2(x - y) \cdot (\hat{x}_n - e) + 2(y - y_n) \cdot \hat{x}_n \right) |x_n| + |x_n|^2. \end{aligned}$$

Because  $|x_n| \rightarrow +\infty$  and  $\hat{x}_n \rightarrow e$  as  $n$  goes to  $+\infty$ , we have that

$$\lim_{n \rightarrow \infty} \left( \frac{|x - y_n|^2}{|x_n|} - 2\varepsilon + 2(x - y) \cdot (\hat{x}_n - e) + 2(y - y_n) \cdot \hat{x}_n \right) = -2\varepsilon.$$

Therefore, recalling that  $|x_n| \leq \sqrt{T_n}$ , we derive for  $n$  large enough

$$|x + z_n|^2 \leq T_n - \varepsilon\sqrt{T_n} < T_n + t.$$

This means that  $x \in H_t$ . Hence

$$\forall t \leq 0, \forall x \in P, \quad w_\infty(t, x) \geq \eta' \quad \text{and} \quad w_\infty(0, y) = \eta'.$$

Now, because  $\eta' \in (\theta, 1)$ , [10, Lemma 1] ensures that the functions  $w_\infty(t, \cdot)$  are actually front-like as  $x \cdot e \rightarrow -\infty$  *uniformly* with respect to  $t < 0$ , in the sense that

$$\lim_{x \cdot e \rightarrow -\infty} \left( \inf_{t < 0} w_\infty(t, x) \right) = 1. \quad (3.25)$$

*Step 4. Proof of the lower bound (3.22).*

Here we use property (i). Owing to the previous step, the function  $\underline{w}_0$  defined by

$$\underline{w}_0(x) := \chi(x \cdot e) \left( \inf_{t < 0} w_\infty(t, x) \right),$$

with  $\chi$  smooth, decreasing and satisfying  $\chi(-\infty) = 1$ ,  $\chi(+\infty) = 0$ , fulfils the front-like condition (1.3). Therefore, by property (i), invasion occurs for the solution  $\underline{w}$  of (1.2) with initial datum  $\underline{w}_0$ . Now, for  $m > 0$ , there holds

$$\forall x \in \Omega, \quad \underline{w}_0(x) \leq w_\infty(-m, x),$$

and then, by comparison,  $\underline{w}(m, y) \leq w_\infty(0, y) = \eta'$ . This contradicts the invasion property of  $\underline{w}$ . We have proved that (3.22) holds for  $n$  large enough.  $\square$

We now prove our second main result about invasion.

*Proof of Theorem 1.5.* We prove the result by showing that property (ii) of Theorem 1.3 can still be derived by if one replaces property (i) with Hypothesis 1. Recall that the only step of the previous proof which makes use of property (i) is step 4. Let us check that it holds true under Hypothesis 1. Let  $v$  be a pulsating traveling front in the direction  $e$  with a positive speed  $c$ . Using the fact that  $c > 0$  and that

$w_\infty$  satisfies (3.25), together with the nonincreasing monotonicity of  $s \mapsto f(x, s)$  in some neighbourhoods of 0 and 1, one can prove that

$$\forall t \in \mathbb{R}, \forall x \in \Omega, \quad v(t, x) \leq w_\infty(t, x).$$

This is classical in the homogeneous case, in our heterogeneous framework we can invoke [10, Lemmas 1, 2]. Of course the function  $v(t, x)$  can be replaced by any temporal-translation  $v(t + \tau, x)$ , and thus, because  $v$  fulfils (1.8) with  $c \neq 0$ , we can assume without loss of generality that  $v(0, y) > \eta'$ . This is impossible because  $w_\infty(0, y) = \eta'$ . This proves (3.22), concluding the step 4.  $\square$

**Remark 1.** The validity of the invasion property for a compactly supported non-negative initial datum  $u_0$  satisfying  $\sup u_0 < 1$  readily implies that the invasion takes actually place with at least a linear speed. Indeed the associated solution  $u$  satisfies, for some  $T > 0$ ,

$$\forall x \in \Omega, \forall k \in \prod_{i=1}^N \{-L_i, 0, L_i\}, \quad u(T, x) \geq u_0(x + k),$$

and thus, since the spatial translations of  $u$  by  $L_1\mathbb{Z} \times \cdots \times L_N\mathbb{Z}$  are still solutions of (1.2), using the comparison principle we get, by iteration,

$$\forall n \in \mathbb{N}, \forall x \in \Omega, \forall k \in \prod_{i=1}^N \{-nL_i, \dots, 0, \dots, nL_i\}, \quad u(nT, x) \geq u_0(x + k).$$

Now, owing to the invasion property, for any  $\varepsilon > 0$  we can find  $\tau > 0$  such that  $u(t, x) > 1 - \varepsilon$  for  $t \geq \tau$  and  $x$  in the periodicity cell  $\mathcal{C}$ . We eventually infer that  $u(t, x) > 1 - \varepsilon$  for  $t \geq nT + \tau$  and  $x \in \overline{\Omega} \cap ([-nL_1, nL_1] \times \cdots \times [-nL_N, nL_N])$ . This means that the upper level set  $\{u > 1 - \varepsilon\}$  propagates with at least a linear speed.

## 3.2 Applications of the invasion result

In this section, we make use of Theorem 1.5 to derive Corollary 1.6 and two other results that apply to the bistable case. This essentially reduces to check that Hypothesis 1 holds.

Corollary 1.6 applies to non-negative nonlinearities, i.e., satisfying (1.10), such as the Fisher-KPP nonlinearity  $f(u) = u(1 - u)$  or the Arrhenius nonlinearity  $f(u) = e^{-\frac{1}{u}}u(1 - u)$ . To derive it, we shall need the following result from Berestycki and Hamel.

**Proposition 3.1** ([3, Theorem 1.13]). *Assume that  $f$  satisfies (1.5), (1.10) together with*

$$\begin{cases} \exists \vartheta \in (0, 1), f(x, s) = 0 \text{ if } s \leq \vartheta, \\ \forall s > \vartheta, \exists x \in \overline{\Omega}, \text{ such that } f(x, s) > 0, \end{cases} \quad (3.26)$$

*and that  $q$  satisfies (1.9). Then, Hypothesis 1 holds for the problem (1.2).*

*Proof of Corollary 1.6.* Assume that  $q$  and  $f$  satisfy the hypotheses of Corollary 1.6.  $f$  be a nonlinearity satisfying (1.5) and (1.10). We cannot apply Theorem 1.5 directly to  $f$ , because it may not be non-increasing in a neighbourhood of  $s = 0$ . To overcome this, we take  $\varepsilon \in (0, 1 - \theta)$ , with  $\theta$  given by (1.6), and we define a nonlinearity  $\tilde{f}$  independent of  $x$  satisfying (1.5), together with

$$\begin{cases} \tilde{f}(s) = 0, & \text{for } s \in [0, \theta + \varepsilon], \\ 0 < \tilde{f}(s) \leq \min_{x \in \bar{\Omega}} f(x, s), & \text{for } s \in (\theta + \varepsilon, 1). \end{cases}$$

Applying Proposition 3.1 to  $\tilde{f}$  (with  $\vartheta = \theta + \varepsilon$ ), we deduce that Hypothesis 1 is verified for the problem (1.2) with the nonlinearity  $\tilde{f}$ . Now, we can apply Theorem 1.5 to get that, for  $\eta \in (\theta + \varepsilon, 1)$ , there is  $r > 0$  such that, if

$$u_0 \geq \eta \quad \text{in } x \in B_r,$$

then the solution  $\tilde{u}(t, x)$  of (1.2) with nonlinearity  $\tilde{f}$  and initial datum  $u_0$  satisfies the invasion property. By comparison, the same holds true for the solution of (1.2) with initial datum  $u_0$  but with nonlinearity  $f$ , because  $\tilde{f} \leq f$ . As  $\varepsilon$  can be chosen arbitrarily close to 0, this yields the result.  $\square$

In the bistable case, some sufficient conditions for the existence of pulsating fronts are provided in the whole space with 1-periodic terms (i.e., satisfying our usual definition of periodicity with  $L_1, \dots, L_N = 1$ ) by Ducrot [11, Corollary 1.12] and Xin [28, Theorem 2.2]. These results, combined with Theorem 1.5, directly yield the following.

**Corollary 3.2.** *Consider the equation*

$$\partial_t u = d\Delta u + r(x)u(u - a(x))(1 - u), \quad t > 0, \quad x \in \mathbb{R}^2, \quad (3.27)$$

where  $a, r \in C^\alpha(\mathbb{R}^2)$ , for some  $\alpha \in (0, 1)$ , are 1-periodic and satisfy

$$\forall x \in \mathbb{R}^2, \quad 0 < a(x) < 1 \quad \text{and} \quad r(x) > 0,$$

and

$$\bar{\theta} := \frac{\int_{[0,1]^2} r(x)a(x)dx}{\int_{[0,1]^2} r(x)dx} < \frac{1}{2}.$$

Then, there is  $d_0 \geq 1$  large enough such that properties (i)-(ii) of Theorem 1.3 hold provided  $d \geq d_0$ .

**Corollary 3.3.** *Consider the equation*

$$\partial_t u = \nabla(A(x)\nabla u) + q(x) \cdot \nabla u + u(1 - u)(u - \theta), \quad t > 0, \quad x \in \mathbb{R}^N, \quad (3.28)$$

where  $A$  is a uniformly elliptic smooth matrix field and  $q$  is a smooth vector field which are 1-periodic and  $\theta \in (0, \frac{1}{2})$ . Then, there is  $\delta > 0$  and  $s > N + 1$ , such that properties (i)-(ii) of Theorem 1.3 hold provided

$$\left\| A - \int_{[0,1]^N} A \right\|_{H^s([0,1]^N)} < \delta \quad \text{and} \quad \|q\|_{H^s([0,1]^N)} < \delta. \quad (3.29)$$

## 4 Estimates on the spreading speed

This section is devoted to the proof of Theorem 1.7. As already mentioned, this result applies to equations set in the whole space  $\mathbb{R}^N$  with a nonlinearity that satisfies (1.10). It provides a lower estimate on the speed of invasion.

The philosophy of this section differs from the previous one in that we shall build “explicitly” a subsolution that invades the space with some given speed. We start with a technical lemma.

**Lemma 4.1.** *Assume that  $f$  is independent of  $x$  and satisfies (1.5), (1.10). Then, for any  $0 < \lambda \leq \Lambda$ , there exist  $H \in (\theta, 1)$ ,  $L > 0$  and a non-increasing function  $h \in W^{2,\infty}(\mathbb{R})$  such that*

$$\begin{cases} h(z) = H & \text{in } (-\infty, 0], \\ h(z) = 0 & \text{in } [L, +\infty), \end{cases}$$

and

$$Ah'' + Bh' + f(h) \geq 0 \quad \text{in } [0, L], \quad \text{for } A \in [\lambda, \Lambda], \quad B \leq \frac{\lambda}{\sqrt{\Lambda}} \sqrt{R(f)}. \quad (4.30)$$

Before proving Lemma 4.1, we show how it yields Theorem 1.7.

*Proof of Theorem 1.7.* First of all, up to replacing  $f(x, s)$  with  $\min_{x \in \bar{\Omega}} f(x, s)$ , it is not restrictive to assume that  $f$  is independent of  $x$  (and still satisfies (1.5), (1.10)). We have that  $f(s) > 0$  for  $s \in (\theta, 1)$ .

By hypothesis, there holds

$$w^* := \frac{\lambda}{\sqrt{\Lambda}} \sqrt{R(f)} - \limsup_{|x| \rightarrow +\infty} q(x) \cdot \frac{x}{|x|} > 0.$$

Fix  $\bar{c} \in (0, w^*)$ . Then take  $\rho > 0$  in such a way that

$$\forall x \in B_\rho^c, \quad \frac{N\Lambda}{\rho} + \bar{c} + q(x) \cdot \frac{x}{|x|} < \frac{\lambda}{\sqrt{\Lambda}} \sqrt{R(f)}.$$

Now, let  $H, L$  and  $h$  be the constants and the function provided by Lemma 4.1, associated with  $\lambda, \Lambda$  given by (1.4). Consider the function

$$v(t, x) := h(|x| - \bar{c}t - \rho).$$

We claim that  $v$  is a subsolution of (1.11). Equation (1.11) trivially holds outside the region  $0 < |x| - \bar{c}t - \rho < L$ , where  $v$  is constant, hence it suffices to show that

$$\forall t > 0, \quad \forall x \in B_{\rho+L+\bar{c}t} \setminus B_{\rho+\bar{c}t}, \quad \partial_t v - \text{Tr}(AD^2v) - q \cdot \nabla v - f(v) \leq 0.$$

Direct computation shows that this is equivalent to have, for such  $t$  and  $x$ ,

$$Ah''(|x| - \bar{c}t - \rho) + Bh'(|x| - \bar{c}t - \rho) + f(h(|x| - \bar{c}t - \rho)) \geq 0,$$

where

$$A = \left( A(x) \frac{x}{|x|} \cdot \frac{x}{|x|} \right), \quad B = \left( \frac{\text{Tr}(A(x))}{|x|} - \frac{1}{|x|} \left( A(x) \frac{x}{|x|} \cdot \frac{x}{|x|} \right) + q(x) \cdot \frac{x}{|x|} + \bar{c} \right).$$

Observing that  $\lambda \leq A \leq \Lambda$  and that

$$B \leq \frac{N\Lambda}{\rho} + q(x) \cdot \frac{x}{|x|} + k < \frac{\lambda}{\sqrt{\Lambda}} \sqrt{R(f)}$$

because  $|x| \geq \rho$ , the above inequality holds thanks to (4.30). This shows that  $v$  is a subsolution of (1.11).

Now, arguing as in the proof of Theorem 1.2, for any  $\eta > \theta$  we can find  $r, T > 0$  such that every solution  $u$  of (1.11) with a non-negative initial datum  $u_0 \geq \eta$  in  $B_r$  satisfies  $u(T, x) > v(0, x)$  for  $x \in \mathbb{R}^N$ . Hence, by comparison,  $u(t + T, x) > v(t, x)$  for  $t > 0, x \in \mathbb{R}^N$ . Consider such a solution  $u$  and take  $c \in (0, \bar{c})$ . We claim that

$$\inf_{|x| \leq ct} u(t, x) \xrightarrow{t \rightarrow +\infty} 1.$$

We proceed by contradiction: assume that there is  $\varepsilon \in (0, 1)$ , a diverging sequence  $(t_n)_{n \in \mathbb{N}}$  and a sequence  $(x_n)_{n \in \mathbb{N}}$  in  $\mathbb{R}^N$  such that  $|x_n| \leq ct_n$  and  $u(t_n, x_n) \leq 1 - \varepsilon$ . We define the sequence of translations  $u_n(t, x) := u(t + t_n, x + z_n)$ , with  $(z_n)_{n \in \mathbb{N}}$  in  $\prod_{i=1}^N L_i \mathbb{Z}$  satisfying  $x_n - z_n \in \mathcal{C} = \prod_{i=1}^N [0, L_i)$ . This sequence converges (up to extraction) to an entire solution  $u_\infty$  of (1.11). It satisfies  $\min_{x \in \bar{\mathcal{C}}} u_\infty(0, x) \leq 1 - \varepsilon$  and, for  $t \in \mathbb{R}$  and  $x \in \mathbb{R}^N$ ,

$$\begin{aligned} u_\infty(t + T, x) &= \lim_{n \rightarrow +\infty} u(t + T + t_n, x + z_n) \geq \lim_{n \rightarrow +\infty} v(t + t_n, x + z_n) \\ &= \lim_{n \rightarrow +\infty} h(|x + z_n| - \bar{c}(t + t_n) - \rho). \end{aligned}$$

The latter term is equal to  $H$  because  $|z_n| \leq |x_n| + \sqrt{L_1 \cdots L_N} \leq ct_n + \sqrt{L_1 \cdots L_N}$  and  $c < \bar{c}$ . Hence  $u_\infty$  is everywhere larger than  $H > \theta$ . Therefore, again by the same argument as in the proof of Theorem 1.2, we get  $u_\infty \geq 1$ , which contradicts  $\min_{x \in \bar{\mathcal{C}}} u_\infty(0, x) \leq 1 - \varepsilon$ . This yields the result.  $\square$

We now turn to the proof of Lemma 4.1.

*Proof of Lemma 4.1.* By definition of  $R(f)$  and  $\theta$ , there exist  $\theta < K < H < 1$  such that  $R(f) = (H - K) \min_{[K, H]} f$ . For  $0 < z_1 < z_2$  and  $\beta, \gamma, \mu > 0$  that will be chosen later, we define the function  $h$  on each interval  $(-\infty, 0]$ ,  $[0, z_1]$ ,  $[z_1, z_2]$  as follows:

$$\begin{cases} h(z) = H & \text{for } z \in (-\infty, 0], \\ h(z) = H - \frac{\gamma}{2} z^2 & \text{for } z \in [0, z_1], \\ h(z) = \mu(z_2 - z)^\beta & \text{for } z \in [z_1, z_2]. \end{cases} \quad (4.31)$$

We want to find  $z_1, z_2, \beta, \gamma, \mu$  so that  $h \in W^{2, \infty}(\mathbb{R})$  and (4.30) holds. We further impose  $h(z_1) = K$ . We shall take  $\beta \geq 2$ . Then, to have  $h \in W^{2, \infty}$  we need

$$K = H - \frac{\gamma}{2} z_1^2, \quad K = \mu(z_2 - z_1)^\beta,$$



$$\gamma z_1 = \mu \beta (z_2 - z_1)^{\beta-1} = K \beta (z_2 - z_1)^{-1}.$$

Now, let us see what conditions are needed to have (4.30). Call for short  $\bar{B} := \frac{\lambda}{\sqrt{\Lambda}} \sqrt{R(f)}$ . On one hand, if  $z \in [0, z_1]$ , property (4.30) holds as soon as

$$-\gamma \Lambda - \gamma \bar{B} z_1 + f(h(z)) \geq 0.$$

Then, using the fact that  $h(z) \in [K, H]$  for  $z \in [0, z_1]$ , it is sufficient to have

$$-\gamma(\Lambda + \bar{B} z_1) + \min_{[K, H]} f \geq 0.$$

On the other hand, for  $z \in [z_1, z_2]$ , property (4.30) holds as soon as (recall that  $f \geq 0$ )

$$\lambda(\beta - 1) - \bar{B}(z_2 - z_1) \geq 0.$$

For notational simplicity, we shall write  $\Delta = z_2 - z_1$ . Summing up,  $h$  satisfies (4.30) on  $\mathbb{R} \setminus \{z_1, z_2\}$  provided we can find  $z_1, \Delta, \beta, \mu, \gamma$  such that

$$\begin{cases} \lambda(\beta - 1) - \bar{B}\Delta \geq 0, \\ \gamma(\Lambda + \bar{B}z_1) \leq \min_{[K, H]} f, \\ \frac{\gamma}{2} z_1^2 = H - K, \\ \mu \Delta^\beta = K, \\ \gamma z_1 = K \beta \Delta^{-1}. \end{cases} \quad (4.32)$$

Let us show that this is solvable. We leave  $\beta$  as a free parameter and take

$$\gamma = \left( \frac{\bar{B} \beta K}{\lambda \sqrt{2(H - K)}(\beta - 1)} \right)^2,$$

and then

$$z_1 = \frac{\sqrt{2(H - K)}}{\sqrt{\gamma}}, \quad \Delta = \frac{\beta K}{\sqrt{2\gamma(H - K)}}, \quad \mu = \frac{(2\gamma(H - K))^{\frac{\beta}{2}}}{\beta^\beta K^{\beta-1}}.$$

Direct computation shows that all the equations of (4.32) are satisfied, with the possible exception of the second one. Let us show that the second equation holds as well, provided  $\beta$  is sufficiently large. To do so, we observe that

$$\gamma \xrightarrow{\beta \rightarrow +\infty} \left( \frac{\bar{B} K}{\lambda \sqrt{2(H - K)}} \right)^2 \quad \text{and} \quad z_1 \xrightarrow{\beta \rightarrow +\infty} \frac{2\lambda(H - K)}{\bar{B} K},$$

whence, recalling the expression of  $\bar{B}$ ,

$$\gamma(\Lambda + \bar{B}z_1) \xrightarrow{\beta \rightarrow +\infty} \frac{K^2}{2\Lambda} \left( \Lambda + \frac{2\lambda(H - K)}{K} \right) \min_{[K, H]} f.$$

Now, because  $K, H \in [0, 1]$  and  $\lambda \leq \Lambda$ , we find that

$$\frac{K^2}{2\Lambda} \left( \Lambda + \frac{2\lambda(H - K)}{K} \right) \leq \frac{1}{2} K (2H - K) < 1.$$

It follows that, choosing  $\beta$  large, the second equation of (4.32) is verified too.  $\square$

Now that we have given some sufficient conditions that ensure that invasion occurs, we focus to the case where  $f$  is a bistable nonlinearity. In this case, Hypothesis 1 is not known to hold in general, thus we cannot apply Theorem 1.5. In Section 5, we show how the geometry of the domain can either totally block or allow the invasion. In Section 6, we present a phenomenon that we call *oriented invasion*.

## 5 Invasion and blocking in domains with periodic holes

### 5.1 Invasion

This section is dedicated to the proof of Theorem 1.8. In the whole section,  $f$  will denote an unbalanced bistable nonlinearity, i.e, satisfying (1.13). Before turning to the proof itself, let us make some remarks, that shall prove useful here and also in Section 6. In order to derive the invasion property in a given domain  $\Omega$ , we shall use “sliding-type” arguments. Such arguments rely on the existence of compactly supported generalized subsolutions of the stationary problem associated with (1.12). The latter are given by solutions of the following Dirichlet problem:

$$\begin{cases} -\Delta u = f(u), & x \in B_R, \\ u = 0, & x \in \partial B_R. \end{cases} \quad (5.33)$$

If  $R > 0$  is large enough, the existence of a positive solution of (5.33) is classical. One could also reclaim the construction done in Section 2 for the more general problem (2.14). In the following, such solutions are denoted by  $u_R$ . Thanks to the celebrated result of Gidas, Ni and Nirenberg [16],  $u_R$  is radially symmetric and decreasing. In the sequel, we shall use functions of the form  $u_R(\cdot - z)$  as stationary subsolutions of (1.12). In this sense,  $u_R$  will be understood to be extended by 0 outside  $\overline{B}_R$ . We also mention, for future use, that, applying Proposition 2.5 we have

$$\forall \eta \in (\theta, 1), \forall r > 0, \exists R > 0, \min_{B_r} u_R > \eta. \quad (5.34)$$

*Proof of Theorem 1.8.* By the previous considerations, we can take  $R > 0$  large enough so that the function  $u_R$  is non-negative, radially symmetric and decreasing and satisfies

$$M := \max u_R = u_R(0) > \theta.$$

Moreover,  $M < 1$  by Lemma 2.3. Let  $K \subset \mathbb{R}^N$  be a star-shaped compact set. Up to a coordinate change, we can assume that it is star-shaped with respect to the origin. Set

$$L := 2 \operatorname{diam}(K) + 2R + 1,$$

where  $\operatorname{diam}(K)$  stands for the diameter of  $K$ . We then define

$$\Omega_1 := (K + LZ^N)^c.$$

In order to fall into our standing assumptions, we require that  $K^c$  has a  $C^3$  boundary. Call  $z := (\frac{L}{2}, 0, \dots, 0)$ . Let  $u$  be the solution of (1.12) on  $\Omega_1$  with initial datum

$u_R(\cdot - z)$ . Notice that  $u_R(\cdot - z)$  is compactly supported in  $\Omega_1$ , whence it is a generalized stationary subsolution of (1.12). Classical arguments show that  $u(t, x)$  is increasing with respect to  $t$  and converges locally uniformly in  $x \in \overline{\Omega}_1$  to a stationary solution  $u_\infty > u_R(\cdot - z)$  of (1.12) as  $t$  goes to  $+\infty$ . In particular,  $u_\infty > 0$  in  $\overline{\Omega}_1$  by the elliptic strong maximum principle and the Hopf lemma. We claim that  $u_\infty \equiv 1$ . The proof of this fact is divided into three steps.

*Step 1. Lower bound “far from the boundary”.*

Let us show that

$$\forall y \in \left( \bigcup_{k \in LZ^N} B_{\frac{L}{2}}(k) \right)^c, \quad u_\infty > u_R(\cdot - y). \quad (5.35)$$

Fix an arbitrary point  $y$  in the set  $\left( \bigcup_{k \in LZ^N} B_{\frac{L}{2}}(k) \right)^c$ . This set is closed, path-connected, contains the point  $z$  and its distance from  $\Omega_1^c$  is larger than or equal to  $L/2 - \text{diam}(K) > R$ . We can then find a continuous path  $\gamma : [0, 1] \mapsto \Omega_1$  such that

$$\gamma(0) = z, \quad \gamma(1) = y, \quad \text{and} \quad \overline{B}_R(\gamma(s)) \subset \Omega_1, \quad \text{for all } s \in [0, 1].$$

Then, for  $s \in [0, 1]$ , define

$$s^* := \sup \{ s \in [0, 1] : \forall s' \in [0, s], u_\infty > u_R(\cdot - \gamma(s')) \}.$$

Observe that the above set is non-empty because  $u_R(\cdot - z) < u_\infty$ . If we show that  $s^* = 1$  then (5.35) is proved. Assume by contradiction that  $s^* < 1$ . Then, the fact that  $u_R(x - \gamma(s^*))$  is continuous with respect to  $s$  and  $x$  and it is compactly supported implies that  $u_R(\cdot - \gamma(s^*))$  touches  $u_\infty$  from below, in the sense that  $\min(u_\infty - u_R(\cdot - \gamma(s^*))) = 0$ . The point(s) at which the minimum is attained necessarily belongs to  $B_R(\gamma(s^*)) \subset \Omega_1$  because  $u_\infty > 0$  in  $\overline{\Omega}_1$ . The elliptic strong maximum principle eventually yields  $u_\infty \equiv u_R(\cdot - \gamma(s^*))$ , which is impossible again because  $u_\infty > 0$ .

*Step 2. Lower bound “near the boundary”.*

We show now that

$$\forall x \in \overline{\Omega}_1 \cap \bigcup_{k \in LZ^N} B_{\frac{L}{2}}(k), \quad u_\infty(x) \geq M. \quad (5.36)$$

To do this we use the same sliding method as before, but we need a refinement. Namely, we need the support of the subsolution to cross the boundary of  $\Omega_1$ . This is where the star-shaped condition comes into play.

For  $l > 0$ , define the function

$$v_l := \max_{e \in \mathbb{S}^{N-1}} u_R(\cdot - le).$$

Recalling that  $u_R$  is radially symmetric and decreasing, we see that  $v_l$  is a radial function with respect to the origin which is increasing on  $\overline{B}_l \setminus \overline{B}_{l-R}$  (with the convention  $\overline{B}_{l-R} = \{0\}$  if  $l \leq R$ ) and decreasing on  $\overline{B}_{l+R} \setminus \overline{B}_l$ . We know from the previous step that  $u_\infty > v_{L/2}$ . We define

$$l^* := \inf \left\{ l \geq 0 : \forall l' \in \left[ l, \frac{L}{2} \right], u_\infty \geq v_{l'} \right\}.$$

Assume by contradiction that  $l^* > 0$ . By continuity,  $v_{l^*}$  touches  $u_\infty$  from below at some point  $\bar{x} \in \bar{\Omega}_1 \cap (B_{l+R} \setminus \bar{B}_{l-R})$  (the contact cannot happen on  $\partial B_{l+R} \cup \partial B_{l-R}$  because  $v_{l^*} = 0$  there if  $l \geq R$  and  $v_{l^*}$  cannot be touched from above at 0 by a smooth function if  $l < R$ ). We necessarily have that  $|\bar{x}| \leq l^*$ , because otherwise  $v_{l^*}(\bar{x})$  would be larger than  $v_{l^*}(\bar{x}) = u_\infty(\bar{x})$  for  $l < l^*$  close to  $l^*$ , contradicting the definition of  $l^*$ . It is easy to check that  $v_{l^*}(\bar{x}) = u_R(\bar{x} - l^*e)$ , with  $e = \bar{x}/|\bar{x}|$ , whence  $u_R(\cdot - l^*e)$  touches  $u_\infty$  from below at the point  $\bar{x}$ . If  $\bar{x} \in \Omega_1$  then the elliptic strong maximum principle implies that  $u_R(\cdot - l^*e) \equiv u_\infty$  on  $\bar{\Omega}_1$ , which is not the case because  $u_R(\cdot - l^*e)$  is not everywhere positive on  $\bar{\Omega}_1$ . Hence,  $\bar{x} \in \partial K$ . Recalling that  $|\bar{x}| \leq l^*$ , we find that  $\nabla u_R(\bar{x} - l^*e)$  is positively collinear to  $e$ , whence

$$(l^*e - \bar{x}) \cdot \nu(\bar{x}) = \left( \frac{l^*}{|\bar{x}|} - \bar{x} \right) \bar{x} \cdot \nu(\bar{x}) \leq 0.$$

$$\nu(\bar{x}) \cdot \nabla u_R(\bar{x} - l^*e) = (\nu(\bar{x}) \cdot e) |\nabla u_R(\bar{x} - l^*e)| \leq 0,$$

the latter term is non-positive because  $K$  is star-shaped with respect to 0. But then the Hopf lemma leads to the same contradiction  $u_R(\cdot - l^*e) \equiv u_\infty$  as before. We have reached a contradiction. This shows that  $l^* = 0$ . In particular, we find that  $u_\infty \geq M$  on  $\bar{\Omega}_1 \cap \bar{B}_{L/2}$ .

The above argument can be repeated with  $v_l$  replaced by  $v_l(\cdot + k)$  for any  $k \in LZ^N$ , leading to the property (5.36).

*Step 3. Conclusion.*

On one hand, (5.35)-(5.36) yield  $u_\infty \geq M$  in the whole  $\Omega_1$ . As seen in the first step of the proof of Theorem 1.3, the fact that  $u_\infty$  is an entire (stationary) solution of (1.12) which is larger than  $M > \theta$  implies  $u_\infty \geq 1$ . On the other hand, because  $\max u_R(\cdot - z) = M < 1$ , the comparison principle yields  $u_\infty \leq 1$ . This means that the invasion property holds for the initial datum  $u_R(\cdot - z)$ . We eventually derive property (ii) of Theorem 1.3 by arguing as in the proof of Theorem 1.2 in Section 2.  $\square$

## 5.2 Blocking

This section is dedicated to the proof of Theorem 1.9. Namely, we exhibit a smooth periodic domain  $\Omega_2$  where the blocking property holds for any compactly supported data  $\leq 1$ . We use a result by Bereszycki, Hamel and Matano, who construct a compact set  $K \subset B_{1/2}$  such that there is a classical solution to the problem

$$\begin{cases} -\Delta w = f(w) & \text{in } B_1 \setminus K \\ \partial_\nu w = 0 & \text{on } \partial K \\ w = 1 & \text{on } \partial B_1, \\ 0 < w < 1 & \text{in } B_1 \setminus K, \end{cases} \quad (5.37)$$

see [5, Theorem 6.5 and (6.8)]. The function  $w$  will act as a *barrier* which prevents invasion. Our domain  $\Omega_2$  is depicted in Figure 1.

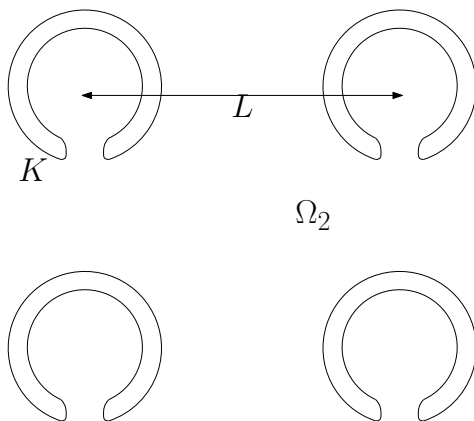


Figure 1: The domain  $\Omega_2$

*Proof of Theorem 1.9.* As at the beginning of the proof of Theorem 1.8 in the previous section, we start with taking  $R > 0$  so that the solution  $u_R$  to (5.33) is nonnegative and satisfies  $\theta < \max u_R < 1$ . Call  $L := 2R + 2$  and define

$$\Omega_2 := (K + LZ^N)^c.$$

We prove the three statements of the theorem separately.

*Statement (i).*

Let  $u$  be the solution of (1.12) arising from a compactly supported initial datum  $u_0 \leq 1$ . Let  $M > 0$  be large enough so that  $\text{supp}(u_0) \subset B_M$ . Consider the solution  $w$  of (5.37) given by [5]. Define the function  $\tilde{w}$  as follows:

$$\forall k \in LZ^N \text{ such that } |k| > M + 1, \quad \forall x \in \overline{\Omega_2} \cap B_1, \quad \tilde{w}(k + x) := w(x),$$

then extended to 1 in the rest of  $\overline{\Omega_2}$ . This is a generalized stationary supersolution of (1.12). Hence, because  $u_0 \leq \tilde{w}$ , the parabolic comparison principle yields that

$$\forall t \geq 0, \quad \forall x \in \Omega_2, \quad u(t, x) \leq \tilde{w}(x). \quad (5.38)$$

This implies that  $u$  does not satisfy the invasion property. Let us see that  $u$  is actually blocked in the sense of Definition 1.1. We argue by contradiction. If this were not the case, we would be able to find a diverging sequence  $(t_n)_{n \in \mathbb{N}}$  and a sequence  $(x_n)_{n \in \mathbb{N}}$  in  $\Omega_2$  such that  $u(t_n, x_n) \rightarrow 1$  as  $n$  goes to  $+\infty$ . Then, defining

$$u_n := u(\cdot + t_n, \cdot + k_n),$$

where  $k_n \in LZ^N$  is such that  $x_n - k_n \in \mathcal{C}$ , the parabolic estimates would allow us to extract a subsequence of  $(u_n)_{n \in \mathbb{N}}$  converging locally uniformly in  $\overline{\Omega_2}$  to some entire solution  $u_\infty$  of (1.12) such that  $\max u_\infty = 1$ . The parabolic comparison principle and the Hopf lemma would yield  $u_\infty \equiv 1$ . Hence, from (5.38) we would get that  $\tilde{w}(\cdot + k_n) \rightarrow 1$  as  $n$  goes to  $+\infty$  (up to subsequences), locally uniformly on  $\overline{\Omega_2}$ , which is impossible because  $\inf_{B_{M+L+2}} \tilde{w}(\cdot + k) \leq \inf w < 1$  for any  $k \in LZ^N$ .

*Statement (ii).*

Now, let  $\tilde{u}$  be the solution of (1.2) emerging from the initial datum  $u_R(\cdot - z)$ , where

$z := (\frac{L}{2}, 0, \dots, 0)$ . Observe that  $\text{supp } u_R(\cdot - z) = \overline{B}_R(z) \subset \Omega_2$ , whence  $u_R(\cdot - z)$  (extended by 0 outside  $\overline{B}_R(z)$ ) is a stationary generalized subsolution of (1.12). It follows that, as  $t$  goes to  $+\infty$ ,  $\tilde{u}(t, x)$  converges increasingly to a stationary solution  $p(x)$  of (1.12) satisfying  $u_R(\cdot - z) \leq p \leq 1$ . We have proved above that  $p \not\equiv 1$ . Let us show that  $p$  is periodic.

Take  $k \in LZ^N$ . We can find a continuous path  $\gamma : [0, 1] \mapsto \Omega_2$  such that

$$\gamma(0) = z, \quad \gamma(1) = z + k, \quad \text{and} \quad \overline{B}_R(\gamma(s)) \subset \Omega_2, \quad \text{for } s \in [0, 1].$$

Then, we can argue as in the proof of Theorem 1.8 to get that

$$p \geq u_R(\cdot - z - k).$$

Using  $u_R(\cdot - z - k)$  as an initial datum for (1.12), the parabolic comparison principle yields

$$p \geq p(\cdot - k).$$

This being true for all  $k \in LZ^N$ , we have that  $p$  is indeed periodic, with the same periodicity as  $\Omega_2$ . Finally, the fact that  $1 > \inf p > u_R(0) > \theta$  readily implies that  $p$  cannot be constant.

*Statement (iii).*

Let us show that invasion fails for front-like initial data. Let  $v_0$  be a front-like initial datum in a direction  $e \in \mathbb{S}^{N-1}$ , in the sense of (1.3). Consider again the solution  $w$  of (5.37), extended by 1 outside  $\overline{B}_1$ . Then, there is  $M > 0$  such that

$$\sup_{\substack{x \cdot e > M \\ x \in \Omega_2}} v_0(x) < \inf w.$$

We take  $k \in LZ^N$  such that  $k \cdot e > M + 1$ . It holds that  $w(\cdot - k) \geq v_0$ . Thus the parabolic comparison principle yields that the solution of (1.12) arising from  $v_0$  lies below  $w(\cdot - k)$ , whence cannot converge locally uniformly to 1 as  $t$  goes to  $+\infty$ .

Suppose now that (1.12) admits a pulsating front solution  $v$  (recall Definition 1.4). On one hand, we have just shown that the invasion property fails for  $v$ , which implies that its speed  $c$  cannot be positive. On the other hand, statement (ii) provides us with a solution with a compactly supported initial datum  $u_0 < 1$  which converges to a positive periodic steady state as  $t$  goes to  $+\infty$ . Up to translation,  $u_0$  can be fit below  $v(0, \cdot)$ , and thus we deduce by comparison that the speed  $c$  cannot be  $\leq 0$  either. Hence, pulsating fronts do not exist for problem (1.12) in  $\Omega_2$   $\square$

## 6 Oriented invasion

In this section, we construct some domains which exhibit a new phenomenon, that we call oriented invasion, which is between blocking and invasion. Namely, invasion occurs in a direction but is blocked in the opposite one.

Throughout this section, the nonlinearity  $f$  is of the unbalanced bistable type (1.13). As in Section 5, we shall use some ‘‘sliding-type’’ arguments to prove that invasion occurs in some directions, and some ‘‘barriers’’ to get the blocking in other

directions. Recall that these methods worked under suitable geometric conditions on the domain. The whole issue here is to construct a periodic domain which roughly satisfies one type of condition in some directions and the other type in other directions.

Let us make the geometric condition required in the sliding method explicit. We will slide the same functions  $u_R(\cdot - z)$  as in Section 5, extended by 0 outside  $\overline{B_R}(z)$  and restricted to  $\Omega$ , which are generalized subsolutions of the first equation in (1.12). We recall that for  $R$  large enough,  $u_R$  is positive in  $B_R$ , with  $\theta < \max u_R < 1$ . By [16], we further know that  $u_R$  is radially symmetric and decreasing on  $\overline{B_R}$ , whence

$$\forall x \in (\partial\Omega) \cap B_R(z), \quad \nu(x) \cdot \nabla u_R(x - z) = \nu(x) \cdot \frac{z - x}{|z - x|} |\nabla u_R(x - z)|.$$

It follows that  $u_R(\cdot - z)$  is a stationary subsolution of (1.12) if and only if  $\Omega$  fulfils the following geometric condition:

$$\forall x \in (\partial\Omega) \cap B_R(z), \quad (z - x) \cdot \nu(x) \leq 0. \quad (6.39)$$

## 6.1 Oriented invasion in a periodic cylinder

We start with showing that the oriented invasion occurs in cylindrical domains. We consider problem (1.12) set in a *periodic cylindrical* domain, that is, of the form

$$\Omega = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{N-1} : |x_2| \leq \omega(x_1)\},$$

with  $\omega : \mathbb{R} \rightarrow (0, +\infty)$  periodic of class  $C^3$ . Throughout this section, we will denote the points in  $\Omega$  by  $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{N-1}$ , and  $e_1 := (1, 0, \dots, 0)$  a unit vector in the direction of the axis of the cylinder.

**Theorem 6.1.** *There exists a periodic cylindrical domain  $\Omega$  and a positive constant  $w$  such that, for every  $\eta > \theta$ , there is  $r > 0$  for which the following properties hold for every solution to (1.12) arising from a compactly supported initial datum satisfying*

$$0 \leq u_0 \leq 1, \quad u_0 > \eta \quad \text{in } \Omega \cap B_r.$$

(i) Invasion to the right:

$$\forall 0 < c_1 < c_2 < w, \quad \min_{\substack{c_1 t \leq x_1 \leq c_2 t \\ |x_2| \leq \omega(x_1)}} |u(t, x_1, x_2) - 1| \xrightarrow{t \rightarrow +\infty} 1.$$

(ii) Blocking to the left:

$$u(t, x_1, x_2) \xrightarrow{x_1 \rightarrow -\infty} 0 \quad \text{uniformly in } t > 0, \quad |x_2| \leq \omega(x_1).$$

The blocking result makes use of the results by Berestycki, Bouhours and Chapuisat [2], which are in turn inspired by [5, Theorem 6.5] by Berestycki, Hamel and Matano, already used here in Section 5.2. In [2], the authors build an asymptotically straight cylinder for which all solutions initially confined in the half space  $\{x_1 < 0\}$  do not invade. The mechanism they exploit is that the propagation is hampered by the presence of a “narrow passage” which suddenly widens. More precisely, we shall need the following.

**Proposition 6.2.** *Let  $\Omega$  be a periodic cylinder and let  $a < b$  be given. There is  $\varepsilon > 0$  depending on*

$$\{(x_1, x_2) \in \Omega : x_1 \in (a-1, a)\}$$

*such that, if*

$$|\{(x_1, x_2) \in \Omega : x_1 \in (a, b)\}| < \varepsilon,$$

*then the following problem admits a positive solution:*

$$\begin{cases} -\Delta w = f(w), & (x_1, x_2) \in \Omega, x_1 < b \\ \partial_\nu w = 0, & (x_1, x_2) \in \partial\Omega, x_1 < b \\ w(b, x_2) = 1, & |x_2| \leq \omega(b) \\ w(x_1, x_2) \xrightarrow{x_1 \rightarrow -\infty} 0 & \text{uniformly in } |x_2| \leq \omega(x_1). \end{cases} \quad (6.40)$$

This proposition can be extracted from the proof of [2, Theorem 1.8]. We shall not redo the proof here, but we mention the ideas for reader's ease. They rely on the same energy method used in Section 2, with the energy functional defined on the truncated cylinders  $\Omega \cap \{-R < x_1 < b\}$  with boundary conditions  $w(-R, \cdot) = 0$ ,  $w(b, \cdot) = 1$ . The idea is then to take the limit  $R \rightarrow +\infty$ , and to hope that the Dirichlet condition will translate at the limit into the last condition of (6.40). One needs to be cautious there: it is crucial to have that the selected minimizers do not converge to 1 as  $R$  goes to  $+\infty$ , which could happen if one takes global minimizers. The authors take instead local minimizers in a suitable energy well which converge to a solution of (6.40).

Proposition 6.2 will allow us to derive the “blocking” property (ii) of Theorem 6.1 by considering a periodic cylindrical domain  $\Omega$  containing narrow passages which widen very suddenly in the leftward direction. Conversely, for the “invasion” property (i), we need such passages to open slowly in the rightward direction; this will allow us to construct a front-like subsolution by “bending” the level sets of a planar front. The domain  $\Omega$  is depicted in Figure 2.

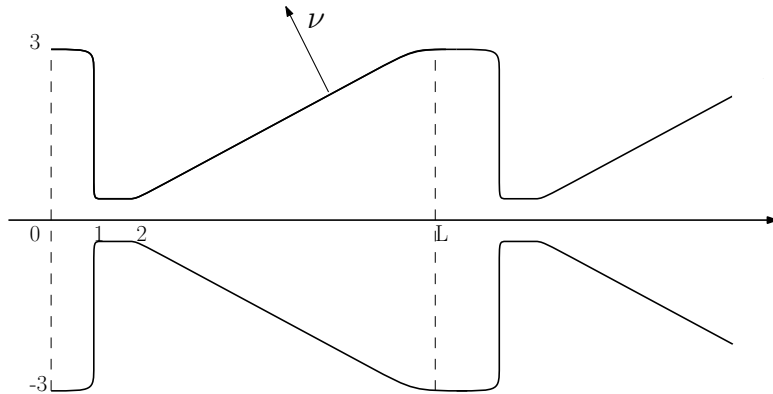


Figure 2: The periodic cylinder

*Proof.* The proof is divided into four steps.



*Step 1. Building the domain.*

Let  $\mu \in (0, 1 - \theta)$  to be chosen after. We define  $f_\mu$  to be a Lipschitz-continuous function on  $\mathbb{R}$  such that  $f_\mu(\mu) = f_\mu(1 - \mu) = 0$ ,  $f_\mu < 0$  in  $(-\mu, \theta)$ ,  $f_\mu > 0$  in  $(\theta, 1 - \mu)$  and  $\int_{-\mu}^{1-\mu} f_\mu(s) ds > 0$ . Then,  $f_\mu$  is an unbalanced bistable nonlinearity. Moreover, provided  $\mu$  is small enough, we take it so that

$$\begin{cases} f_\mu \leq f, \\ (1 + \mu)f_\mu(s) \leq f(s) \text{ when } f(s) > 0. \end{cases} \quad (6.41)$$

Let  $\phi$  be the 1-dimensional bistable travelling front connecting  $1 - \mu$  to  $-\mu$  with speed  $c_\mu > 0$  solution of

$$\phi'' + c_\mu \phi' + f_\mu(\phi) = 0.$$

Now, we take  $R > 0$  large enough such that there is a non-negative non-null solution  $u_R$  of (5.33). Owing to (5.34), we increase  $R$  so that there holds

$$u_R(x_1, x_2) \geq 1 - \mu \text{ for } (x_1, x_2) \in [0, 1] \times [-3, 3]. \quad (6.42)$$

Now, for  $L > 0$  to be chosen after, we define

$$\Omega := \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{N-1} : |x_2| \leq \omega(x_1)\},$$

where  $\omega$  is defined as follows. First, we define  $\omega$  on the interval  $[0, L]$  to be in  $C^\infty([0, L])$  and positive, such that:

$$\begin{cases} \omega = 3, & \text{on } [0, 1], \\ \omega' \leq 0, & \text{on } [1, 2], \\ 0 \leq \omega' \leq \frac{4}{L-4}, & \text{on } [2, L], \\ \omega^{(n)}(L) = 0, & \text{for } n \geq 1, \\ \omega(L) = 3, \\ \omega(\frac{L}{2}) = 1. \end{cases} \quad (6.43)$$

Moreover, we take  $\omega$  such that

$$|\{(x_1, x_2) : x_1 \in [1, 2], |x_2| \leq \omega(x_1)\}| \leq \varepsilon,$$

where  $\varepsilon$  is small enough so that we can apply Proposition 6.2 to the cylinder  $\Omega$  with  $a = 1$  and  $b = 2$ . Observe that  $\varepsilon$  does not depend on  $L$ . Then, we can extend  $\omega$  to be  $L$ -periodic. Observe that  $\min \omega = \omega(2)$ . Up to increasing  $L$ , we assume that the following technical condition holds:

$$\frac{4}{L-4} < \omega(2) \min\left(\frac{c_\mu}{N-1}, \frac{\sqrt{\mu}}{3}\right). \quad (6.44)$$

*Step 2. Building a subsolution.*

This step is dedicated to the construction of a subsolution to (1.12) moving rightward, that we shall use after to prove that invasion occurs in this direction. To do so, take  $c \in (0, c_\mu)$  and define

$$\psi(t, x_1, x_2) := \phi(x_1 + \eta|x_2|^2 - ct),$$

with  $\eta := \min(\frac{c_\mu - c}{2(N-1)}, \frac{\sqrt{\mu}}{6})$ . Thanks to (6.44), we can choose  $c$  small enough so that

$$\frac{4}{L-4} \leq 2\eta\omega(2). \quad (6.45)$$

Let us show that  $\psi$  is a subsolution of (1.12). An easy computation gives us

$$\begin{aligned} \partial_t \psi - \Delta \psi - f(\psi) &= (c_\mu(1 + 4\eta^2|x_2|^2) - c - 2\eta(N-1))\phi' + (1 + 4\eta^2|x_2|^2)f_\mu(\psi) - f(\psi) \\ &\leq (c_\mu - c - 2\eta(N-1))\phi' + (1 + 4\eta^2|x_2|^2)f_\mu(\psi) - f(\psi). \end{aligned} \quad (6.46)$$

If  $\psi \leq \theta$  we have  $(1 + 4\eta^2|x_2|^2)f_\mu(\psi) \leq f(\psi)$ . If  $\psi \geq \theta$ , thanks to our definition of  $f_\mu$  and  $\eta$ , and because  $|x_2| \leq \max \omega = 3$ , there holds

$$(1 + 4\eta^2|x_2|^2)f_\mu(\psi) \leq (1 + \mu)f_\mu(\psi) \leq f(\psi).$$

Either case, and because  $\phi' \leq 0$ , there holds

$$\forall t \in \mathbb{R}, \forall (x_1, x_2) \in \Omega, \quad \partial_t \psi - \Delta \psi - f(\psi) \leq 0.$$

To conclude that  $\psi$  is indeed a subsolution, it remains to show that  $\partial_\nu \psi \leq 0$  on  $\partial\Omega$ . Observe that the unit exterior normal to  $\Omega$  at  $(x_1, x_2) \in \partial\Omega$  is positively collinear to  $(-\omega'(x_1), \frac{x_2}{|x_2|})$ . Recalling that  $|x_2| \geq \omega(2)$ , it follows that, for  $t \in \mathbb{R}$  and  $(x_1, x_2) \in \partial\Omega$ , we have

$$\begin{aligned} |(-\omega'(x_1), \frac{x_2}{|x_2|})| \partial_\nu \psi(t, x_1, x_2) &= (-\omega'(x_1) + 2\eta|x_2|)\phi' \\ &\leq (-\frac{4}{L-4} + 2\eta\omega(2))\phi' \\ &\leq 0, \end{aligned}$$

the last inequality coming from (6.45). Hence,  $\psi$  is indeed a subsolution of (1.12).

*Step 3. Invasion to the right.*

We denote by  $u$  the solution of (1.12) with initial datum  $u_R(\cdot - \frac{L}{2}e_1)$ , restricted to  $\Omega$ , extended by zero outside of its support. First, let us observe that  $u_R(\cdot - \frac{L}{2}e_1)$  is a generalized stationary subsolution of (1.12): indeed, for  $y := (y_1, y_2) \in \partial\Omega \cap B_R(\frac{L}{2}e_1)$ , we find that

$$\begin{aligned} -\nu(y) \cdot (y - \frac{L}{2}e_1) &= \omega'(y_1)(y_1 - \frac{L}{2}) - \omega(y_1) \\ &\leq \frac{4}{L-4}R - \omega(\frac{L}{2} - R) \\ &\leq \frac{8}{L-4}R - 1, \end{aligned}$$

the last inequality coming from the mean value theorem together with  $\omega(\frac{L}{2}) = 1$ . Then, if  $L$  is large enough, (6.39) is verified and  $u_R(\cdot - \frac{L}{2}e_1)$  is a generalized stationary subsolution of (1.12) in the cylinder  $\Omega$ . Hence,  $u(t, \cdot, \cdot)$  is increasing with respect to  $t$  and, because of (6.42), there holds

$$\forall t \geq 0, \forall (x_1, x_2) \in \Omega, \quad x_1 \in \left[ \frac{L}{2}, \frac{L}{2} + 1 \right], \quad u(t, x_1, x_2) \geq 1 - \mu.$$

Define

$$\Omega^+ := \left\{ (x_1, x_2) \in \Omega : x_1 \geq \frac{L}{2} \right\}.$$

We have  $\psi(t, \frac{L}{2}, x_2) \leq u(t, \frac{L}{2}, x_2)$ , for all  $t \geq 0$ ,  $|x_2| \leq \omega(\frac{L}{2})$ . Moreover, up to some translation in time, we can assume that

$$\forall (x_1, x_2) \in \Omega^+, \quad \psi(0, x_1, x_2) \leq u(0, x_1, x_2).$$

Hence, the parabolic comparison principle yields that  $\psi(t, x_1, x_2) \leq u(t, x_1, x_2)$  for  $t \geq 0$ ,  $(x_1, x_2) \in \Omega^+$ . This implies that, for  $\gamma \in (0, c)$  ( $c$  is from second step):

$$\lim_{t \rightarrow +\infty} \left( \min_{\substack{0 \leq x_1 \leq \gamma t \\ |x_2| \leq \omega(x_1)}} u(t, x_1, x_2) \right) \geq 1 - \mu.$$

Because  $1 - \mu > \theta$ , we easily infer that, for  $\gamma \in (0, c)$  and for  $h \in (0, \gamma)$ , we have

$$\min_{\substack{ht \leq x_1 \leq \gamma t \\ x_2 \leq \omega(x_1)}} u(t, x_1, x_2) \xrightarrow{t \rightarrow +\infty} 1.$$

This proves the ‘‘invasion’’ part of the theorem for initial datum  $u_R(\cdot - \frac{L}{2})$ . For more general initial data, we can argue as in the proof of Theorem 1.2.

*Step 4. Blocking.*

We use the result of [2], Proposition 6.2 above, to prove that we have ‘‘blocking’’. Let  $w$  be the function given by Proposition 6.2 applied to  $\Omega$  with  $a = 1$  and  $b = 2$  (we recall that we chose  $\varepsilon$  so that this were possible). Then, we can extend  $w$  by setting  $w(x_1, \cdot) = 1$  for  $x_1 \geq 2$ . Then,  $w$  is a generalized supersolution of (1.12), and so is  $w(\cdot + kL, \cdot)$ , where  $k \in \mathbb{Z}$  and  $L$  is the period of our cylinder. Then, if  $u_0 \leq 1$  is compactly supported, we can find  $k \in \mathbb{Z}$  such that  $u_0 \leq w(\cdot + kL, \cdot)$ . The parabolic comparison principle yields that  $u(t, x_1, x_2) \leq w(x_1 + kL, x_2)$ , for all  $t > 0$ ,  $(x_1, x_2) \in \Omega$ . Because  $\sup_{\substack{x_1 \leq M \\ x_2 \leq \omega(x_1)}} w(x_1 + kL, x_2) \rightarrow 0$  as  $M$  goes to  $-\infty$ , the result follows.  $\square$

## 6.2 Oriented invasion in a periodic domain

This section is dedicated to the proof of Theorem 1.10. In Section 6.2.1, we design the periodic domain  $\Omega_3 \subset \mathbb{R}^2$ . In Section 6.2.2, we state some preliminary lemmas, used in Section 6.2.3 to prove the oriented invasion property. In the whole section, we denote  $e_1 := (1, 0)$  and  $e_2 := (0, 1)$  the unit vectors of the canonical basis.

### 6.2.1 Designing the domain $\Omega_3$

In this section, we build a periodic domain  $\Omega_3$ . We construct it as the plane  $\mathbb{R}^2$  with triangular-shaped holes. More specifically, we shall take  $L_1, L_2 > 0$  large enough and a compact set  $K \subset \mathbb{R}^2$ , and we define  $\Omega_3$  to be the following domain:

$$\Omega_3 := \left( K + L_1 \mathbb{Z} \times L_2 \mathbb{Z} \right)^c. \quad (6.47)$$

The domain  $\Omega_3$  we have in mind is depicted in Figure 3.

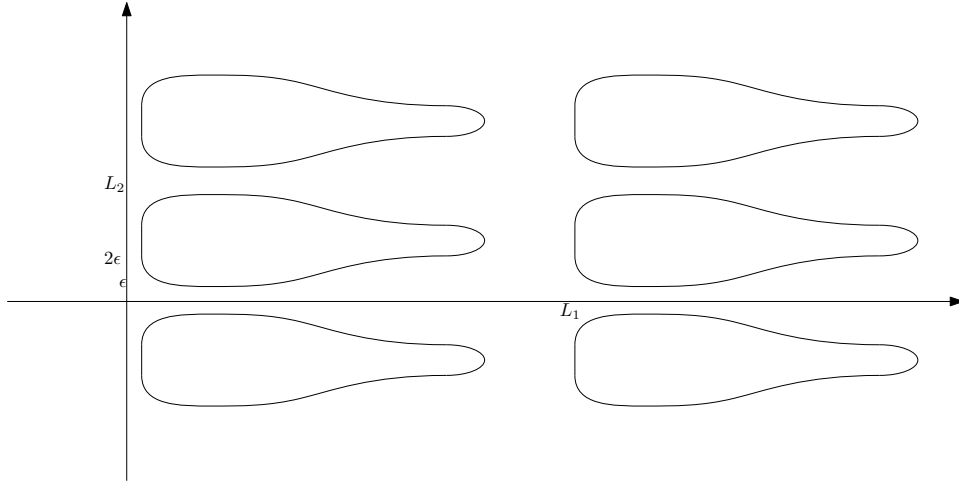


Figure 3: The periodic domain  $\Omega_3$

The idea is to take this domain so that the narrow passages open “widely” to the left, so that propagation will be blocked in this direction, but “slowly” to the right, so that the solution will be able to pass, as in the case of the periodic cylinder presented above.

Now, let us define  $K, L_1$  and  $L_2$ . First, we take  $R > 0$  large enough so that there is a positive solution  $u_R$  of (5.33). thanks to (5.34) we can choose  $R > 4$  large enough so that

$$M := \min_{x \in B_2} u_R(x) > \theta. \quad (6.48)$$

We denote

$$M' := \max u_R = u_R(0) > M > \theta.$$

For  $\varepsilon \in (0, 1)$ , to be chosen later, we define

$$\kappa := \frac{2R}{\varepsilon}, \quad L_2 := 2(R^2 + 1) \quad \text{and} \quad L_1 := 2(\kappa + 8R + \frac{L_2}{2} + 1).$$

We shall build a function  $h \in C([0, \frac{L_1}{2}]) \cap C^2((0, \frac{L_1}{2}))$  to parametrize the boundary of  $K$ . We define  $K \subset [0, \frac{L_1}{2}] \times (0, L_2)$  in such a way that

$$K \cap (\{0\} \times \mathbb{R}) = \{0\} \times [2\varepsilon, L_2 - 2\varepsilon],$$

and

$$\partial K \cap \left( \left(0, \frac{L_1}{2}\right] \times \left[0, \frac{L_2}{2}\right] \right) = \left\{ (s, h(s)) : s \in \left[0, \frac{L_1}{2}\right] \right\},$$

and, in a symmetric way,

$$\partial K \cap \left( \left(0, \frac{L_1}{2}\right] \times \left[\frac{L_2}{2}, L_2\right] \right) = \left\{ (s, L_2 - h(s)) : s \in \left[0, \frac{L_1}{2}\right] \right\}.$$

Observe that, because we want  $\Omega_3$  to be smooth, we need to have  $h'(0^+) = -\infty$  and  $h'(\frac{L_1}{2}^-) = +\infty$ . Let us define  $h$ . First, we take  $\varphi \in C^\infty(\mathbb{R})$  to be a non-decreasing

function such that

$$\begin{cases} \varphi = 0 & \text{on } (-\infty, 0], \\ \varphi = 1 & \text{on } [1, +\infty), \\ \varphi' \leq 2 & \text{on } \mathbb{R}. \end{cases} \quad (6.49)$$

Now, we take  $h$  such that:

$$\begin{cases} h(0) = 2\varepsilon, \\ h'(s) \leq 0, & \text{for } s \in (0, 1), \\ h(s) = \varepsilon, & \text{for } s \in (1, 4R), \\ h(s) = (1 - \varepsilon)\varphi\left(\frac{s-4R}{\kappa}\right) + \varepsilon, & \text{for } s \in [4R, \kappa + 6R], \\ h'(s) \leq \frac{\varepsilon}{R}, & \text{for } s \in [\kappa + 6R, \kappa + 8R], \\ h\left(\frac{L_1}{2}\right) = \frac{L_2}{2}. \end{cases} \quad (6.50)$$

Moreover, we take  $h$  such that, for every  $x \in (\partial K) \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in [\kappa + 6R, \frac{L_1}{2}]\}$ , there is  $y \in K$  such that

$$B_{R^2}(y) \subset K \text{ and } x \in \partial B_{R^2}(y). \quad (6.51)$$

The reason why we do not simply take  $h$  so that  $K \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in [\kappa + 6R, \frac{L_1}{2}]\}$  is a half-ball is because  $h$  would not be  $C^\infty$ . For later use, we mention that (6.50) together with the fact that  $\varepsilon < 1$  yields

$$h(z) \leq 2 \quad \text{for } z \in [2R, \kappa + 7R]. \quad (6.52)$$

The function  $h$  is depicted in Figure 6.2.1.

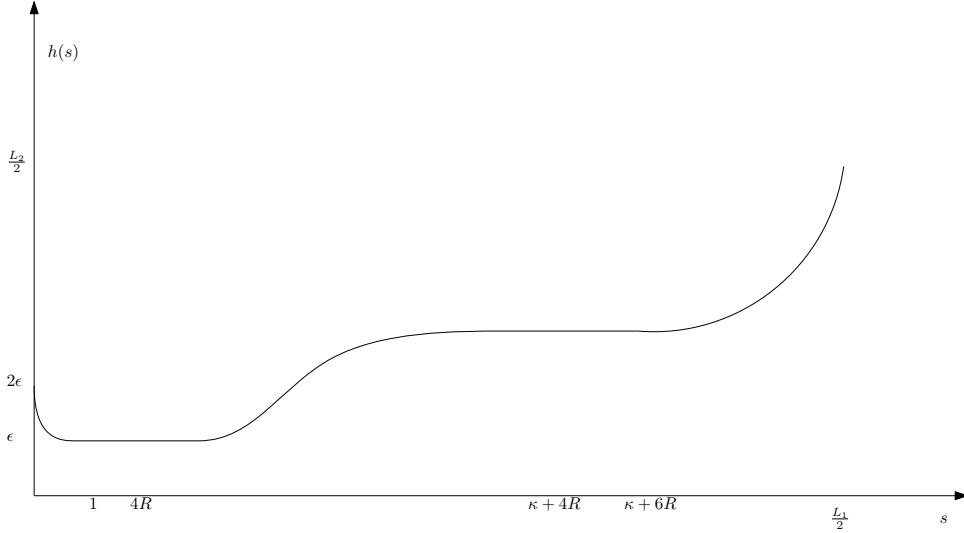


Figure 4: Graph of the function  $h$

The construction we made depends on  $\varepsilon$ , we choose it now. Define the periodic cylinder

$$\tilde{\Omega}_3 := \{(x_1, x_2) \in \Omega_3 : x_2 \in [0, L_2]\}.$$

By construction, the portion of cylinder

$$\left\{ (x_1, x_2) \in \tilde{\Omega}_3 : x_1 \in (-1, 0) \right\} = [-1, 0] \times [2, L_2]$$

does not depend on  $\varepsilon$ . We can then apply Proposition 6.2 to  $\Omega_3$  with  $a = 0$  and  $b = 1$  to see that we can take  $\varepsilon$  small enough so that

$$\left| \left\{ (x_1, x_2) \in \tilde{\Omega}_3 : x_1 \in (0, 1) \right\} \right|$$

is small enough to have a solution of (6.40) in the cylinder  $\tilde{\Omega}_3$ . Observe that this  $\varepsilon$  depends only on  $L_2$ , i.e., only on  $R$ . From now on, all the parameters are fixed. We denote the periodicity cell

$$\mathcal{C} := [0, L_1] \times \left[ -\frac{L_2}{2}, \frac{L_2}{2} \right].$$

The key result to prove Theorem 1.10 is the following:

**Proposition 6.3.** *Let  $u$  be the solution of (1.12), with domain  $\Omega_3$  given by (6.47), emerging from the initial datum  $u_R$ . Then, there is  $T > 0$  such that, for any  $n \in \mathbb{N}^*$ , we have*

$$u(t, x) > M \text{ for } x \in \Omega_3 \cap \bigcup_{\substack{a \in \{0, \dots, n\} \\ b \in \{-a, \dots, a\}}} (\mathcal{C} + aL_1e_1 + bL_2e_2), \quad t \geq nT.$$

### 6.2.2 Proof of Proposition 6.3

We recall that  $R, M'$  and  $M$  are chosen in the previous section. We state here four preliminary lemmas and prove Proposition 6.3.

For  $R' > R$ , we denote by  $v_{R'}$  the solution of the following parabolic problem

$$\begin{cases} \partial_t v - \Delta v = f(v), & t > 0, x \in B_{R'} \\ v = 0, & t > 0, x \in \partial B_{R'} \\ v(0, x) = u_R(x), & x \in B_{R'}. \end{cases} \quad (6.53)$$

A standard sliding argument yields the following (we recall that  $M' = \max u_R = u_R(0)$ ):

**Lemma 6.4.** *Let  $R' > R$ . Then, there is  $T > 0$  such that*

$$v_{R'}(t, x) > M' \text{ for } x \in B_{R'-R}, \quad t \geq T.$$

*Proof.* Because the initial datum  $u_R$  is a generalized stationary subsolution for the parabolic problem (6.53),  $v_{R'}(t, x)$  is increasing with respect to  $t$  and converges locally uniformly in  $x \in B_{R'}$  to a stationary solution of (6.53). We denote this stationary solution by  $v_\infty$ . Let  $z \in B_{R'-R}$ . For  $\lambda \in [0, 1]$ , we define

$$\underline{u}_\lambda := u_R(\cdot - \lambda z)$$

and

$$\lambda^* := \max \{ \lambda \in [0, 1] : \underline{u}_\lambda \leq v_\infty \}.$$

Let us show that  $\lambda^* = 1$ . We argue by contradiction. Assume that  $\lambda^* < 1$ , then there is a contact point between  $v_\infty$  and  $\underline{u}_{\lambda^*}$ , i.e.,

$$\exists x^* \in B_R(\lambda^* z) \subset B_{R'} \quad \text{such that} \quad v_\infty(x^*) = \underline{u}_{\lambda^*}(x^* - \lambda^* z).$$

The definition of  $\lambda^*$  yields that  $v_\infty \geq \underline{u}_{\lambda^*}$ . Because  $v_\infty$  is a stationary solution and  $\underline{u}_{\lambda^*}$  is a stationary subsolution of (6.53), the strong elliptic maximum principle yields  $v_\infty \equiv \underline{u}_{\lambda^*}$ , which is not possible, because  $\underline{u}_{\lambda^*}$  is compactly supported in  $B_{R'}$ . Hence,  $\lambda^* = 1$ . This implies that

$$v_\infty > u_R(0) = M' \quad \text{in} \quad B_{R'-R}.$$

Moreover, this yields that  $v_\infty \geq \underline{u}_1$ . The strong elliptic maximum implies that the only possible contact points between  $v_\infty$  and  $\underline{u}_1$  are on the boundary  $\partial B_{R'}$ , hence

$$v_\infty > M' \quad \text{in} \quad \overline{B}_{R'-R}.$$

Now, because  $v_{R'}(t, x)$  converges locally uniformly to  $v_\infty$  as  $t$  goes to  $+\infty$ , there is  $T > 0$  such that  $v_{R'}(t, x) > M'$  for  $x \in B_{R'-R}$  and  $t \geq T$ .  $\square$

As we mentioned before,  $u_R$  is a radial decreasing function by [16]. The following lemma states that this is still true for  $v_{R'}(t, \cdot)$ , for all  $t > 0$ .

**Lemma 6.5.** *Let  $R' > R$ . Then,  $v_{R'}(t, x)$  is radial decreasing for all  $t > 0$ .*

The proof relies on the use of a moving plane technique.

*Proof.* First, because  $u_R$  is radial decreasing, the uniqueness of the solution of the parabolic problem (6.53), yields that  $v_{R'}(t, \cdot)$  is also radial, for all  $t > 0$ . Let us see that it is also radial decreasing.

To do so, take  $H$  to be an affine hyperplane in  $\mathbb{R}^N$  intersecting  $B_{R'}(0)$ . Now, let  $\Sigma$  denote the orthogonal symmetry with respect to the hyperplane  $H$ . We define

$$w_{R'}(t, x) := v_{R'}(t, \Sigma x), \quad \text{for } t > 0, x \in \Sigma(B_{R'}).$$

Then,  $w_{R'}$  is solution of (6.53) set on  $\Sigma(B_{R'})$  arising from the initial datum  $u_{R'}(\Sigma \cdot)$ . Now, consider the domain

$$U := \left\{ x \in \Sigma(B_{R'}) : x \cdot \Sigma(0) < \frac{1}{2} \|\Sigma(0)\|^2 \right\}.$$

Then, because  $v_{R'}(t, \cdot)$  is radial for all  $t > 0$ , it is readily seen that

$$\forall x \in \partial U, \forall t > 0, \quad v_{R'}(t, x) \geq w_{R'}(t, x).$$

Moreover, if  $x \in U$ , then  $d(x, 0) < d(x, \Sigma(0))$ . Because  $u_R$  is radial decreasing, we have that  $u_{R'} \geq u_{R'}(\Sigma \cdot)$  in  $U$ . The parabolic comparison principle then yields

$$\forall x \in U, \forall t > 0, \quad v_{R'}(t, x) \geq w_{R'}(t, x).$$

This being true for every hyperplane  $H$ , it follows that the radial function  $v_{R'}(t, \cdot)$  is radial decreasing, hence the result.  $\square$

We now state two geometrical lemmas on the shape of  $\Omega_3$  that shall be employed later in “sliding-type” arguments.

**Lemma 6.6.** *Let  $\lambda \in [2R, \kappa + 7R]$ . Then property (6.39) holds with  $z = \lambda e_1$ , and therefore  $u_R(\cdot - \lambda e_1)$  is a generalized subsolution to (1.12).*

*Proof.* We have already observed at the beginning of Section 6 that property (6.39) is equivalent to have that  $u_R(\cdot - z)$  is a generalized subsolution of (1.12).

Take  $\lambda \in [2R, \kappa + 7R]$  and  $x \in (\partial\Omega_3) \cap B_R(\lambda e_1)$ . Without loss of generality, thanks to the symmetry of the domain, we assume that  $x \cdot e_2 > 0$ . For such  $x$ , we have  $R \leq x \cdot e_1 \leq \kappa + 8R$ , and then, by definition of  $h$ ,  $h'(x \cdot e_1) \leq \frac{\varepsilon}{R}$ . Because the unit exterior normal at point  $x \in \partial\Omega_3$  is positively collinear to  $(-h'(x \cdot e_1), 1)$ , there holds

$$\begin{aligned} |\nu(x)|(\lambda e_1 - x) \cdot \nu(x) &= -h'(x \cdot e_1)(\lambda - x \cdot e_1) - h(x \cdot e_1) \\ &\leq \frac{\varepsilon}{R}|x \cdot e_1 - \lambda| - \varepsilon \\ &\leq \frac{\varepsilon}{R}R - \varepsilon \\ &\leq 0, \end{aligned}$$

hence the result.  $\square$

**Lemma 6.7.** *Let  $z \in \Omega_3$  be such that  $z \cdot e_1 \in [\kappa + 7R, L_1 - R]$  and  $d(z, \partial\Omega_3) \geq 1$ , where  $d(z, \partial\Omega_3)$  denotes the distance between  $z$  and the boundary  $\partial\Omega_3$ . Then property (6.39) holds and therefore  $u_R(\cdot - z)$  is a generalized subsolution to (1.12).*

*Proof.* Let  $z \in \Omega_3$  be such that  $z \cdot e_1 \in [\kappa + 7R, L_1 - R]$  and  $d(z, \partial\Omega_3) \geq 1$ . Take  $x \in (\partial K) \cap B_R(z)$ . Thanks to the hypotheses on  $h$  and  $K$ , we can find  $y \in \Omega_3^c$  such that

$$B_{R^2}(y) \subset \Omega_3^c \quad \text{and} \quad x \in \partial B_{R^2}(y).$$

Then,  $|x - y| = R^2$ ,  $|x - z| \leq R$  and  $|z - y| \geq R^2 + 1$ . From that, we get

$$\begin{aligned} (z - x) \cdot \nu(x) &= \frac{(z-x) \cdot (y-x)}{|y-x|} \\ &= \frac{|z-x|^2 + |x-y|^2 - |z-y|^2}{2|y-x|} \\ &\leq \frac{R^2 + R^4 - (R^2 + 1)^2}{2|y-x|} \\ &\leq 0, \end{aligned}$$

hence the result.  $\square$

These lemmas at hand, we can turn to the proof of Proposition 6.3.

*Proof of Proposition 6.3.* The proof is divided into four steps. We denote by  $u$  the solution of (1.12) set on the domain  $\Omega_3$  given by (6.47) emerging from the initial datum  $u_0 := u_R$ . Its support  $B_R$  intersects  $\Omega_3$ . However, because  $h(x)$  is decreasing for  $x \in [0, R]$ , there holds:

$$-x \cdot \nu(x) \leq 0 \quad \text{for } x \in (\partial\Omega_3) \cap B_R,$$

and then  $u_R$  is a generalized stationary subsolution for (1.12). Hence,  $u(t, x)$  is increasing with respect to  $t$  and converges uniformly in  $x \in \overline{\Omega}_3$  to a stationary solution of (1.12), that we denote by  $u_\infty$ . We devote the steps 1,2 and 3 to prove that

$$u_\infty > M \quad \text{in } \Omega_3 \cap \mathcal{C}, \tag{6.54}$$



and

$$u_\infty > u_R(\cdot - L_1 e_1) + u_R(\cdot - L_1 e_1 - L_2 e_2) + u_R(\cdot - L_1 e_1 + L_2 e_2). \quad (6.55)$$

*Step 1. Estimate near the narrow passage.*

In this step, we show that

$$u_\infty > M' \quad \text{in } \Omega_3 \cap B_{3R}. \quad (6.56)$$

Let  $v_{4R}$  denote the function given by Lemma 6.4 with  $R' = 4R$ . Up to extending it by 0 out of its support, we consider its restriction to  $\Omega_3$ . Because  $h$  is non-increasing on  $[0, 4R]$ , we have that  $-x \cdot \nu(x) \leq 0$ , for all  $x \in (\partial\Omega_3) \cap B_{4R}$ . Hence, as seen at the beginning of Section 6, the fact that  $v_{4R}(t, \cdot)$  is radially symmetric and decreasing for every  $t > 0$  implies that  $v_{4R}$  is a generalized subsolution to (1.12).

Now, because  $u_\infty > u_R$ , the parabolic comparison principle yields that  $u_\infty > v_{4R}(t, \cdot)$  for all  $t > 0$ . Lemma 6.4 yields that there is  $\tilde{T} > 0$  such that  $v_{4R}(t, x) > M'$  for  $x \in \Omega_3 \cap B_{3R}$ ,  $t \geq \tilde{T}$ . Hence, (6.56) holds.

*Step 2. Estimate after the narrow passage.*

This step is dedicated to show that (recall that  $M = \min_{B_2} u_R$ , see (6.48)):

$$u_\infty > M \quad \text{in } \{z \in \Omega_3 : z \cdot e_1 \in [2R, L_1 - R]\} \cap \mathcal{C}. \quad (6.57)$$

First, we define, for  $\lambda \in [2R, \kappa + 7R]$ :

$$\underline{u}_\lambda := u_R(\cdot - \lambda e_1).$$

Lemma 6.6 implies that  $\underline{u}_\lambda$  is a generalized subsolution of (1.12) for all  $\lambda \in [2R, \kappa + 7R]$ . Moreover, the result of the first step, (6.56), yields (recall that  $M' = \max u_R$ ):

$$u_\infty > u_R(\cdot - 2R e_1) = \underline{u}_{2R}.$$

By the same token we used to prove Lemma 6.4, we have

$$u_\infty > \underline{u}_\lambda \quad \text{for } \lambda \in [2R, \kappa + 7R]. \quad (6.58)$$

Hence, if  $z \in \Omega_3 \cap \mathcal{C}$  is such that  $z \cdot e_1 \in [2R, \kappa + 7R]$ , we have

$$u_\infty(z) > u_R(z - (z \cdot e_1)e_1) \geq M,$$

the last inequality coming from the fact that  $|z - (z \cdot e_1)e_1| \leq h(\kappa + 7R) \leq 2$ , thanks to (6.52), together with the definition of  $M$ .

Now, take  $z \in \Omega_3 \cap \mathcal{C}$  such that  $z \cdot e_1 \in [\kappa + 7R, L_1 - R]$ . Because of (6.51), we can find  $y \in \Omega_3$  such that  $y \cdot e_1 \in [\kappa + 7R, L_1 - R]$ ,  $|z - y| \leq 1$  and  $d(y, \Omega_3) \geq 1$  (we possibly have  $y = z$ ). Owing to Lemma 6.6 and Lemma 6.7, we can find a continuous path  $\gamma : s \in [0, 1] \mapsto \Omega_3$  such that  $\gamma(0) = 2R e_1$ ,  $\gamma(1) = y$  and such that  $u_R(\cdot - \gamma(s))$  is a generalized subsolution of (1.12) for all  $s \in [0, 1]$ . Using again the same token we used to prove Lemma 6.4, we have that  $u_\infty > u_R(\cdot - y)$ , and then

$$u_\infty(z) > u_R(z - y) \geq M,$$

this inequality coming from the fact that  $|z - y| \leq 1$  together with the definition of  $M$ . We can then infer (6.57).

*Step 3. Estimate up to the next cell.*

Let us show that

$$u_\infty > M \quad \text{in} \quad \{z \in \Omega_3 : z \cdot e_1 \in [L_1 - R, L_1]\} \cap \mathcal{C}, \quad (6.59)$$

and

$$u_\infty > u_R(\cdot - L_1 e_1) + u_R(\cdot - L_1 e_1 - L_2 e_2) + u_R(\cdot - L_1 e_1 + L_2 e_2). \quad (6.60)$$

First, we define, for  $\lambda \in [2R, L_1]$ ,

$$\underline{u}_\lambda := u_R(\cdot - \lambda e_1).$$

Arguing as in the previous steps, Lemma 6.6, Lemma 6.7 and the fact that  $h$  is non-increasing on  $[0, 4R]$  imply that  $\underline{u}_\lambda$  is a generalized subsolution of (1.12) for all  $\lambda \in [2R, L_1]$ , and then we have that

$$u_\infty > u_R(\cdot - L_1 e_1).$$

Similar sliding arguments show (6.60).

Now, thanks to (6.60), and arguing as in the first step, we have that  $v_{4R}(t, \cdot - L_1 e_1)$ ,  $v_{4R}(t, \cdot - L_1 e_1 - L_2 e_2)$  and  $v_{4R}(t, \cdot - L_1 e_1 + L_2 e_2)$  are generalized subsolutions of (1.12) and hence

$$u_\infty > M' \quad \text{in} \quad B_{3R}(L_1 e_1) \cup B_{3R}(L_1 e_1 + L_2 e_2) \cup B_{3R}(L_1 e_1 - L_2 e_2).$$

Take now  $z \in \Omega_3 \cap \mathcal{C}$  such that  $z \cdot e_1 \in [L_1 - R, L_1]$  and let us show that  $u_\infty(z) > M$ . If  $z \in B_{3R}(L_1 e_1) \cup B_{3R}(L_1 e_1 + L_2 e_2) \cup B_{3R}(L_1 e_1 - L_2 e_2)$ , we are done. Otherwise, it is readily seen that (6.39) holds, and we know that this implies that  $u_R(\cdot - z)$  is a stationary subsolution of (1.12). Then, using Lemmas 6.6 and 6.7, we can find a continuous path  $\gamma : [0, 1] \mapsto \Omega_3$ ,  $\gamma(0) = 2R e_1$ ,  $\gamma(1) = z$  such that  $u_R(\cdot - \gamma(s))$  is a stationary subsolution of (1.12). Arguing as before, this yields that  $u_\infty(z) > M' > M$ . Gathering steps 1, 2 and 3 we get (6.54) and (6.55).

*Step 4. Conclusion.*

Recalling that the convergence of  $u(t, x)$  to  $u_\infty$  is locally uniform in  $x$  as  $t$  goes to  $+\infty$  and that  $u(t, x)$  is increasing with respect to  $t$ , (6.54) and (6.55) yields that there is  $T_1 > 0$  large enough such that:

$$\forall t \geq T_1, \quad \forall x \in \Omega_3 \cap \mathcal{C}, \quad u(t, x) > M$$

and

$$\forall t \geq T_1, \quad u(t, \cdot) > u_R(\cdot - L_1 e_1) + u_R(\cdot - L_1 e_1 - L_2 e_2) + u_R(\cdot - L_1 e_1 + L_2 e_2).$$

Now, define  $T := 2T_1$ . We can iterate the reasoning above, replacing  $u_R$  by  $u_R(\cdot - L e_1)$ ,  $u_R(\cdot - L e_1 - L e_2)$  and  $u_R(\cdot - L e_1 + L e_2)$  to get that

$$u(t, x) > M \quad \text{for} \quad x \in \Omega_3 \cap \bigcup_{\substack{0 \leq a \leq n \\ |b| \leq a}} (\mathcal{C} + a L_1 e_1 + b L_2 e_2), \quad t \geq nT,$$

hence the result.  $\square$

### 6.2.3 Proof of Theorem 1.10

We first show the invasion property in the direction  $e_1$ , and then the blocking in the direction  $-e_1$ .

*Proof of Theorem 1.10.* Let  $f$  be a bistable nonlinearity independent of  $x$  and take  $\Omega_3$  the domain built in (6.47).

*Step 1. Invasion.*

Let  $u$  be the solution of (1.12) arising from the initial datum  $u_R$ , where  $R$  is given in the construction of  $\Omega_3$ . Let us show that there is  $w > 0$  such that

$$\forall 0 < c < c' < w, \forall a > 0, \quad \min_{\substack{x \in \Omega \\ ct \leq x \cdot e_1 \leq c't \\ |x \cdot e_2| \leq a}} |u(t, x) - 1| \xrightarrow{t \rightarrow +\infty} 0. \quad (6.61)$$

First, because  $u_R \leq M < 1$ , the parabolic comparison principle yields that  $u \leq 1$ . Now, call  $w := L_1/T$ , where  $L_1$  is given in the construction of  $\Omega_3$  and  $T$  is given by Proposition 6.3. Take  $c, c'$  such that  $0 < c < c' < w$ . Consider a positive sequence  $(t_n)_{n \in \mathbb{N}}$  and a sequence  $(x_n)_{n \in \mathbb{N}}$  in  $\Omega$  such that

$$t_n \rightarrow +\infty, \quad \forall n \in \mathbb{N}, \quad ct_n \leq x_n \cdot e_1 \leq c't_n, \quad (x_n \cdot e_2)_{n \in \mathbb{N}} \text{ is bounded.}$$

Consider then the sequence  $(m_n)_{n \in \mathbb{N}}$  in  $\mathbb{N}$  for which  $0 \leq x_n \cdot e_1 - m_n L_1 < L_1$ . This sequence diverges to  $+\infty$  because  $(x_n \cdot e_1)_{n \in \mathbb{N}}$  does. We define the sequence of translated functions

$$u_n := u(\cdot + t_n, \cdot + m_n e_1 L_1).$$

The sequence of functions  $(u_n)_{n \in \mathbb{N}}$  converges as  $n$  goes to  $+\infty$  (up to extraction) locally uniformly in  $t \in \mathbb{R}$ ,  $x \in \Omega_3$ , to a function  $u_\infty$ , entire solution of (1.12). Now, let us show that

$$\forall t \in \mathbb{R}, x \in \Omega_3, \quad u_\infty(t, x) \geq M, \quad (6.62)$$

where  $M$  is given by (6.48). Take  $t \in \mathbb{R}$ ,  $x \in \Omega_3$ , and define  $k_n := \lfloor \frac{t+t_n}{T} \rfloor$  (the notation  $\lfloor \cdot \rfloor$  stands for the integer part). Then, for  $n$  large enough,

$$x \cdot e_2 \leq x \cdot e_1 + m_n L_1$$

because  $m_n \rightarrow +\infty$ . Moreover, because

$$\begin{aligned} k_n - m_n &\geq \frac{t+t_n}{T} - 1 - \frac{x_n \cdot e_1}{L_1} \\ &\geq \frac{wt_n}{L_1} - \frac{c't_n}{L_1} + \frac{t}{T} - 1, \end{aligned}$$

and because  $c' < w$ , we have that  $k_n - m_n \rightarrow +\infty$  as  $n$  goes to  $+\infty$ . Hence, for  $n$  large enough we have

$$x \cdot e_1 + m_n L_1 \leq k_n L_1.$$

So, for  $n$  large enough, we have

$$t + t_n \geq k_n T \quad \text{and} \quad x + m_n L_1 e_1 \in \bigcup_{\substack{0 \leq a \leq k_n \\ |b| \leq a}} (C + aL_1 e_1 + bL_1 e_2).$$

Hence, (6.54) yields that  $u_n(t, x) \geq M$  for  $n$  large enough, and passing to the limit  $n \rightarrow +\infty$ , we get (6.62).

Then, because  $u \leq 1$ , arguing as in the first step of the proof of Theorem 1.3, (6.62) yields that (6.61) holds.

Now, arguing as in the proof of Theorem 1.2, we get that, for every  $\eta > \theta$ , there is  $r > 0$  such that, if

$$u_0 \geq \eta \quad \text{in } \Omega_3 \cap B_r,$$

then (6.61) holds true for the solution of (1.12) arising from the initial datum  $u_0$ . Hence, the property (i) of Theorem 1.10 is proved.

*Step 2. Blocking.*

Let us prove the “blocking” part of Theorem 1.10. We define the cylinder

$$\tilde{\Omega}_3 := \{x \in \Omega_3 : x \cdot e_2 \in [0, L_2]\}.$$

Because of the choice of  $\varepsilon$  in the construction of  $\Omega_3$ , we can apply Proposition 6.2 to this cylinder with  $a = 0$  and  $b = 1$  to get a function  $\tilde{p}(x)$  solution of

$$\begin{cases} -\Delta \tilde{p} & = f(\tilde{p}), & x \in \tilde{\Omega}_3, \quad x \cdot e_1 < 1, \\ \partial_\nu \tilde{p} & = 0, & x \in \partial \tilde{\Omega}_3, \quad x \cdot e_1 < 1, \\ \tilde{p}(x) & = 1, & x \cdot e_1 = 1, \\ \sup_{x \cdot e_1 \leq L} \tilde{p}(x) & \xrightarrow{L \rightarrow -\infty} 0. \end{cases}$$

Now, we can extend  $\tilde{p}$  to the whole of  $\Omega_3$  as follows : first, we extend it to the whole of  $\tilde{\Omega}_3$  by setting  $\tilde{p}(x) = 1$  if  $x \cdot e_1 \geq 1$ . Now, define

$$p(x) := \tilde{p}(x) \quad \text{if } x \cdot e_1 \in [0, L_2], \quad p(x) := \tilde{p}(2L_2 - x) \quad \text{if } x \cdot e_2 \in [L_2, 2L_2] \\ \text{and } p(\cdot + 2L_2 k e_2) = p(\cdot), \quad \text{for } k \in \mathbb{Z}. \quad (6.63)$$

Then  $p$  is a generalized supersolution of (1.12),  $2L_2$ -periodic in the direction  $e_2$ , and so is  $p(\cdot + L_1 k e_1)$ , for all  $k \in \mathbb{Z}$ . Then, if  $u_0 \leq 1$  is compactly supported, we can find  $k \in \mathbb{Z}$  such that  $u_0 \leq p(\cdot + L_1 k e_1)$ . The parabolic comparison principle yields that  $u(t, x) \leq p(x + 2L_1 k e_1)$ , for all  $t > 0$ ,  $x \in \Omega_3$ . Because  $\sup_{x \cdot e_1 \leq M} p(x + 2L_1 k e_1) \rightarrow 0$  as  $M$  goes to  $-\infty$ , the result follows.  $\square$

## References

- [1] D. G. Aronson and H. F. Weinberger. Multidimensional nonlinear diffusion arising in population genetics. *Adv. in Math.*, 30(1):33–76, 1978.
- [2] H. Berestycki, J. Bouhours, and G. Chapuisat. Front blocking and propagation in cylinders with varying cross section. *Calc. Var. Partial Differential Equations*, 55(3):Paper No. 44, 32, 2016.
- [3] H. Berestycki and F. Hamel. Front propagation in periodic excitable media. *Comm. Pure Appl. Math.*, 55(8):949–1032, 2002.

- [4] H. Berestycki and F. Hamel. Non-existence of travelling front solutions of some bistable reaction-diffusion equations. *Adv. Differential Equations*, 5(4-6):723–746, 2000.
- [5] H. Berestycki, F. Hamel, and H. Matano. Bistable traveling waves around an obstacle. *Comm. Pure Appl. Math.*, 62(6):729–788, 2009.
- [6] H. Berestycki, F. Hamel, and N. Nadirashvili. The speed of propagation for KPP type problems. I. Periodic framework. *J. Eur. Math. Soc. (JEMS)*, 7(2):173–213, 2005.
- [7] H. Berestycki, F. Hamel, and L. Rossi. Liouville-type results for semilinear elliptic equations in unbounded domains. *Ann. Mat. Pura Appl. (4)*, 186(3):469–507, 2007.
- [8] H. Berestycki and P.-L. Lions. Une méthode locale pour l’existence de solutions positives de problèmes semi-linéaires elliptiques dans  $\mathbf{R}^N$ . *J. Analyse Math.*, 38:144–187, 1980.
- [9] A.-P. Calderón. Lebesgue spaces of differentiable functions and distributions. In *Proc. Sympos. Pure Math., Vol. IV*, pages 33–49. American Mathematical Society, Providence, R.I., 1961.
- [10] R. Ducasse. Propagation properties of reaction-diffusion equations in periodic domains. *Preprint*, 2017.
- [11] A. Ducrot. A multi-dimensional bistable nonlinear diffusion equation in a periodic medium. *Math. Ann.*, 366(1-2):783–818, 2016.
- [12] M. I. El Smaily. Min-max formulae for the speeds of pulsating travelling fronts in periodic excitable media. *Ann. Mat. Pura Appl. (4)*, 189(1):47–66, 2010.
- [13] R. A. Fisher. The wave of advantage of advantageous genes. *Ann. Eugenics*, 7:355–369, 1937.
- [14] Freidlin, M. I. On wavefront propagation in periodic media. in *Stochastic analysis and applications, Adv. Probab. Related Topics*, vol. 7, pp. 147–166, Dekker, New York, 1984.
- [15] J. Gärtner and M. I. Freidlin. The propagation of concentration waves in periodic and random media. *Dokl. Akad. Nauk SSSR*, 249(3):521–525, 1979.
- [16] B. Gidas, W. M. Ni, and L. Nirenberg. Symmetry and related properties via the maximum principle. *Comm. Math. Phys.*, 68(3):209–243, 1979.
- [17] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*, volume 224 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1983.

- [18] T. Giletti and L. Rossi. Pulsating fronts for multidimensional bistable and multistable equations. *Preprint*, 2017.
- [19] A. N. Kolmogorov, I. G. Petrovskii, and N. S. Piskunov. Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique. *Bull. Univ. Etat. Moscow Ser. Internat. Math. Mec. Sect. A*, 1:1–26, 1937.
- [20] O. A. Ladyzenskaja, V. A. Solonnikov, and N. N. Ural'ceva. *Linear and quasi-linear equations of parabolic type*. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1967.
- [21] G. M. Lieberman. *Second order parabolic differential equations*. World Scientific Publishing Co. Inc., River Edge, NJ, 1996.
- [22] A. P. Morse. The behavior of a function on its critical set. *Ann. of Math. (2)*, 40(1):62–70, 1939.
- [23] L. Rossi. The Freidlin-Gärtner formula for general reaction terms. *Adv. Math.*, 317:267–298, 2017.
- [24] N. Shigesada, K. Kawasaki, and E. Teramoto. Traveling periodic waves in heterogeneous environments. *Theoret. Population Biol.*, 30(1):143–160, 1986.
- [25] G. Stampacchia. Problemi al contorno ellittici, con dati discontinui, dotati di soluzioni hölderiane. *Ann. Mat. Pura Appl. (4)*, 51:1–37, 1960.
- [26] E. M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.
- [27] H. F. Weinberger. On spreading speeds and traveling waves for growth and migration models in a periodic habitat. *J. Math. Biol.*, 45(6):511–548, 2002.
- [28] J. X. Xin. Existence and nonexistence of traveling waves and reaction-diffusion front propagation in periodic media. *J. Statist. Phys.*, 73(5-6):893–926, 1993.
- [29] X. Xin. Existence and stability of traveling waves in periodic media governed by a bistable nonlinearity. *J. Dynam. Differential Equations*, 3(4):541–573, 1991.
- [30] X. Xin. Existence and uniqueness of travelling waves in a reaction-diffusion equation with combustion nonlinearity. *Indiana Univ. Math. J.*, 40(3):985–1008, 1991.
- [31] A. Zlatos. Existence and non-existence of transition fronts for bistable and ignition reactions. *Preprint*, 2015.