



HAL
open science

An algorithm for state and parameter estimation through elimination

Krishnan Srinivasarengan, José Ragot, Didier Maquin, Christophe Aubrun

► **To cite this version:**

Krishnan Srinivasarengan, José Ragot, Didier Maquin, Christophe Aubrun. An algorithm for state and parameter estimation through elimination. 14th International Workshop on Advanced Control and Diagnosis, ACD 2017, Nov 2017, Bucharest, Romania. hal-01637827

HAL Id: hal-01637827

<https://hal.science/hal-01637827v1>

Submitted on 22 Nov 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

An Algorithm for State and Parameter Estimation through Elimination

Krishnan Srinivasarengan^{*†}, José Ragot^{*†}, Didier Maquin^{*†} and Christophe Aubrun^{*†}

^{*}CRAN UMR 7039, Université de Lorraine,

Vandoeuvre-lès-Nancy, Cedex, France.

[†]CRAN UMR 7039, CNRS, France.

Email: {krishnan.srinivasarengan, jose.ragot, didier.maquin, christophe.aubrun}@univ-lorraine.fr

Abstract—We are interested in the problem of the estimation of states and some unknown parameters for a class of discrete-time nonlinear state-space model. The class contains those nonlinear models written using a Linear Parameter Varying (LPV) or a quasi-LPV form. The unknown parameters appear affinely in the LPV/quasi-LPV form. An elimination approach forms the core of the proposed algorithm that leads to the estimation of the parameter and then the states in a sliding measurement window. Several examples illustrate the mechanics and the working of the algorithm. Some insights and implications of this approach are discussed.

I. INTRODUCTION

State-space modeling of dynamical systems offers a natural approach to capture the characteristics of many systems. The conditions that can guarantee the computation of the model parameters from input-output data have been investigated using many tools. For a large class of discrete-time nonlinear models, difference algebra based conditions for identifiability were formulated in [1]. Other methods for discrete-time case include the local-state isomorphism approach [2] and the realization theory of Affine LPV models [3].

A related situation is when a part of the parameter set is unknown and a joint estimation of those unknown parameters and the states of the system is required. Such scenarios arise when the parameter values are affected by model uncertainties, equipment faults, gradual degradation of performance etc. The structural conditions for joint estimation of states and parameters include the model observability, and the identifiability a form of identifiability condition that depends on the approach used for the estimation. For example, one approach is to design an extended Kalman filter (EKF) for an extended model where the unknown, but constant parameters are added to the state vector. These approaches have been shown practically useful in many applications. However, they may suffer from some structural issues, as discussed in [4].

One algebraic method of interest for the estimation of states and/or parameters of a state-space model is that of elimination. An elimination method takes a model in state-space form and obtains a set of equations in inputs, outputs and their future values (higher-order derivatives, in continuous-time) along with a limited set of unknown/unmeasured variables of interests while eliminating the others. In [5], an elimination theory is proposed to achieve this through the use of differential algebra. The author illustrates how to convert a

nonlinear state-space model to a set of differential equations in the inputs and outputs and some inequations. In [6], the author gives a procedure that results in two sets of equations, one that has only the inputs, outputs and their higher order derivatives and the other set contains all those in the first set along with state variables. The latter set allows for solving the states as a function of inputs, outputs and their higher order derivatives, giving a nonlinear extension to the interpretation of observability.

In this paper, we develop an elimination approach for models in Linear Parameter Varying (LPV) or quasi-LPV forms with the parameters appearing affinely on the model matrices. Over a sliding window of measurements, the algorithm sequentially estimates the parameters and then the states through elimination. Our contribution is in exploiting the recent tools in linear algebra and consolidate them into a full-fledged algorithm for state and parameter estimation. In this context, the paper is organized as follows. A brief on the recent advances in the applied mathematical tools used in the paper is in Sec. II. The proposed elimination approach is illustrated for a linear case in Sec. III. This is then extended to the LPV/quasi-LPV case in Sec. IV where the algorithm is summarized. A discussion on various aspects of the algorithm is given in Sec. V which is followed by several examples illustrating the algorithms in Sec. VI. Finally, the paper concludes with some remarks in Sec. VII.

A. Notation

Some notation used in the paper were chosen for simpler representation. $X(\theta)$ and X^θ are used to represent the dependence of the matrix X on the parameter θ . If a matrix X depends on a scheduling variable ρ_k in an LPV/quasi-LPV model, it would be represented either as $X(\rho_k)$ or X_k . And if the same matrix depends on both the unknown parameter θ and the scheduling variable ρ_k , the notations $X(\rho_k, \theta)$ and X_k^θ are used interchangeably. The dimensions of the variables are given by $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$, $y_k \in \mathbb{R}^{n_y}$, $\rho_k \in \mathbb{R}^{n_\rho}$ and $\theta \in \mathbb{R}^{n_\theta}$ represent the states, the inputs, the outputs, the scheduling variables and the unknown parameters, respectively. For individual elements within each of these vectors, a subscript is used. For example, $x_{1,k}$ is the first state in the state vector. The system matrices used are of appropriate dimensions. I represents an identity matrix of appropriate

dimensions. The 0 entries in a matrix could correspond to a scalar or a zero matrix of appropriate dimension.

II. PRELIMINARIES

In this section, we outline some tools and perspectives that form the building blocks of the elimination procedure as proposed in this paper. Nonlinear embedding techniques such as in [7] can transform nonlinear models, under certain conditions, to LPV and quasi-LPV forms. However, in the rest of the paper, we assume that an LPV or a quasi-LPV model is available.

A. Parity-space approach

Parity-space approach (or Chow-Wilsky approach) was used for fault detection in [8], from the context of robustness to model uncertainties. Analytical redundancy relations of the underlying system are used to design residual generation methods which in-turn could aid in minimizing the effects of the uncertainties. In this paper, we propose to use some known and derived analytical redundancy relations based on the LPV/quasi-LPV form so as to eliminate the states of the system.

B. Polynomial null-space

Computation of null-space of polynomial matrices has achieved considerable attention in the last few years. A matrix pencil based approach was initiated in [9] and has been applied in many contexts including fault detection [10]. Non-pencil approaches exploiting the Toeplitz structure [11] have also been developed. Moreover, the null-space computation tools are now integrated into the symbolic computation toolboxes in software packages like Maple, MATLAB etc. In this paper, we suggest two ways to map the null-space computation problem to that of solving a set of linear equations.

III. STATE AND PARAMETER ESTIMATION THROUGH ELIMINATION: LINEAR CASE

In this section, we discuss the proposed algorithm. The unknown parameters are assumed to appear affine on the matrices A and B such that for a linear case,

$$A^\theta = A_0 + \sum_{i=1}^{n_\theta} \theta_i \bar{A}_i$$

Similarly for B . To illustrate the steps of the algorithm, this section elaborates it for a case where the underlying model is linear with a scalar unknown parameter such that,

$$\begin{aligned} x_{k+1} &= A^\theta x_k + B^\theta u_k \\ y_k &= C x_k \end{aligned} \quad (1)$$

with

$$A^\theta = A_0 + \theta \bar{A}, \quad B^\theta = B_0 + \theta \bar{B} \quad (2)$$

where the entries of the matrices $A_0, \bar{A}, B_0, \bar{B}$ are constant.

A. Observability forms

The proposed algorithm estimates the unknown parameters and the states within a given measurement window of size N . For a given measurement window, the dynamical model could be represented to reflect the relationship between the inputs and the measurements available during the measurement window with the states and unknown parameters. Two such relationships, termed as *Observability forms* are described here.

1) *Observability form 1*: Consider the formulation,

$$Y_1 = \mathbb{O}_1^\theta x_k + \mathbb{G}^\theta U \quad (3)$$

with

$$Y_1 = \begin{bmatrix} y_k \\ y_{k+1} \\ \vdots \\ y_{k+N-1} \end{bmatrix}, \quad \mathbb{O}_1^\theta = \begin{bmatrix} C \\ CA^\theta \\ \vdots \\ C(A^\theta)^{N-1} \end{bmatrix}, \quad U = \begin{bmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+N-2} \end{bmatrix}$$

$$\mathbb{G}^\theta = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ CB^\theta & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C(A^\theta)^{N-2}B^\theta & C(A^\theta)^{N-3}B^\theta & \cdots & CB^\theta \end{bmatrix}$$

This is the standard observability formulation and we will term the equation (3) in this paper as *observability form 1*.

2) *Observability form 2*: Also consider an alternative representation,

$$Y_2^\theta = \mathbb{O}_2^\theta X \quad (4)$$

with

$$Y_2^\theta = [y_k \quad B^\theta u_k \quad y_{k+1} \quad \cdots \quad B^\theta u_{k+N-2} \quad y_{k+N-1}]^T$$

$$X = [x_k \quad x_{k+1} \quad \cdots \quad x_{k+N-1}]^T$$

$$\mathbb{O}_2^\theta = \begin{bmatrix} C & 0 & 0 & \cdots & 0 & 0 \\ -A^\theta & I & 0 & \cdots & 0 & 0 \\ 0 & C & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -A^\theta & I \\ 0 & 0 & 0 & \cdots & 0 & C \end{bmatrix}$$

This equation is called the *observability form 2*.

Remark 1: The terms Y_1 , U , and Y_2^θ consist of known/measured variables and of a given measurement window. Similarly, the state vector X . Their contents depend on the sampling instant k , but their symbols do not signify it for notational convenience.

B. Null-space computation

We consider the left null-spaces of the matrices \mathbb{O}_1^θ and \mathbb{O}_2^θ defined as,

$$(\Omega_1^\theta)^T \mathbb{O}_1^\theta = 0 \quad \text{and} \quad (\Omega_2^\theta)^T \mathbb{O}_2^\theta = 0 \quad (5)$$

As outlined in the previous section, there are many approaches to compute the null-space of polynomial matrices (\mathbb{O}_1^θ or \mathbb{O}_2^θ). Two intuitive processes one for each observability form is given below.

1) *Null-space for observability form 1:* Given that the parametrization is affine, the entries direct that \mathbb{O}_1 will be a polynomial in θ with a degree $N - 1$ for a window of size N . Let us consider the case for $N = 3$, which allows to write \mathbb{O}_1 as

$$\mathbb{O}_1 = \mathbb{O}_{1,0} + \theta\mathbb{O}_{1,1} + \theta^2\mathbb{O}_{1,2}$$

with

$$\mathbb{O}_{1,0} = \begin{bmatrix} C \\ CA_0 \\ CA_0^2 \end{bmatrix} \mathbb{O}_{1,1} = \begin{bmatrix} 0 \\ C\bar{A} \\ C(\bar{A}A_0 + A_0\bar{A}) \end{bmatrix} \mathbb{O}_{1,2} = \begin{bmatrix} 0 \\ 0 \\ C\bar{A}^2 \end{bmatrix}$$

The null-space can also be parametrized similarly. This is given by

$$\Omega_1^\theta = \Omega_{1,0} + \theta\Omega_{1,1} + \dots + \theta^m\Omega_{1,m} \quad (6)$$

This type of parametrization allows a coefficient-wise comparison of the left and right hand side of the null-space equation in (5). This leads to,

$$[\Omega_{1,0}^T \quad \Omega_{1,1}^T \quad \dots \quad \Omega_{1,m}^T] \mathbb{P}_1 = 0 \quad (7)$$

where,

$$\mathbb{P}_1 = \begin{bmatrix} \mathbb{O}_{1,0} & \mathbb{O}_{1,1} & \mathbb{O}_{1,2} & 0 & 0 & \dots & 0 \\ 0 & \mathbb{O}_{1,0} & \mathbb{O}_{1,1} & \mathbb{O}_{1,2} & 0 & \dots & 0 \\ 0 & 0 & \mathbb{O}_{1,0} & \mathbb{O}_{1,1} & \mathbb{O}_{1,2} & \dots & 0 \\ \dots & \ddots & & \dots & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & \mathbb{O}_{1,0} & \mathbb{O}_{1,1} & \mathbb{O}_{1,2} \end{bmatrix}$$

The problem has now been transformed to a solution of a linear system of equations.

Remark 2: The value of m is to be chosen. The choice can be made through an iteration starting with 0 and then incrementing by 1 until the null-space exists. The case $m = 0$ is when the parameter appears only on the matrix B .

2) *Null-space for observability form 2:* For the observability form 2, the affine characterization and the presence of A^θ without a power would make the parametrization $\mathbb{O}_2(\theta)$ independent of the window size such that,

$$\mathbb{O}_2^\theta = \mathbb{O}_{2,0} + \theta\mathbb{O}_{2,1} \quad (8)$$

where (for $N = 3$),

$$\mathbb{O}_{2,0} = \begin{bmatrix} C & 0 & 0 \\ -A_0 & I & 0 \\ 0 & C & 0 \\ 0 & -A_0 & I \end{bmatrix} \mathbb{O}_{2,1} = \begin{bmatrix} 0 & 0 & 0 \\ -\bar{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\bar{A} & 0 \end{bmatrix}$$

If the null-space is parametrized as in the previous case,

$$\Omega_2^\theta = \Omega_{2,0} + \theta\Omega_{2,1} + \dots + \theta^m\Omega_{2,m} \quad (9)$$

This would lead to the null-space equation (5) as,

$$[\Omega_{2,0}^T \quad \Omega_{2,1}^T \quad \dots \quad \Omega_{2,m}^T] \mathbb{P}_2 = 0 \quad (10)$$

where

$$\mathbb{P}_2 = \begin{bmatrix} \mathbb{O}_{2,0} & \mathbb{O}_{2,1} & 0 & 0 & 0 & \dots & 0 \\ 0 & \mathbb{O}_{2,0} & \mathbb{O}_{2,1} & 0 & 0 & \dots & 0 \\ 0 & 0 & \mathbb{O}_{2,0} & \mathbb{O}_{2,1} & 0 & \dots & 0 \\ \dots & \ddots & & \dots & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & 0 & \mathbb{O}_{2,0} & \mathbb{O}_{2,1} \end{bmatrix}$$

Remark 3: The structure of the matrices \mathbb{P}_1 and \mathbb{P}_2 is exploited in the works such as [11] where a non-pencil approach is developed to obtain polynomial null-space of a matrix.

Remark 4: The two observability forms have interesting properties. Given the degree 1 matrix polynomial in (8), the form 2 simplifies the null-space computation step. The form 1 is indispensable for the state estimation as it only has x_k instead of the state vector of each instance of the measurement window in form 2.

C. Solving nonlinear algebraic equations

Once the null-space equation in (5) is solved numerically (or symbolically, as the case may be), we obtain one of the two sets of parity equations, based on the chosen observability form:

$$(\text{form 1}) \Rightarrow (\Omega_1^\theta)^T (Y_1 - \mathbb{G}^\theta U) = 0 \quad (11)$$

$$(\text{form 2}) \Rightarrow (\Omega_2^\theta)^T Y_2^\theta = 0 \quad (12)$$

The parity equations (11)-(12) contain only the known values of inputs and outputs and the unknown parameters. The parity equations are a polynomial in the unknown parameter with the degree of the parameter depending on, among others, the size of the window N , represented as,

$$\psi(u_k, u_{k+1} \dots u_{k+N-2}, y_k, y_{k+1}, \dots, y_{k+N-1}, \theta) = 0 \quad (13)$$

The parity equations can be solved using one of the many methods to solve a system of nonlinear (polynomial) equations (see [12]). In case of a vector of unknown parameters, this would lead to a set of multivariate polynomial equations. Since there can be more than one solution for a given parity equations, extra conditions or constraints shall be added to obtain appropriate solutions.

Remark 5: The two observability forms result in the same parity equations at the end.

D. State estimation

Once an estimate of the parameter $\hat{\theta}$ is obtained, the next step is to employ the observability forms to obtain an estimate for the states of the system. For this purpose, the observability form 1 is more suitable and could be represented as follows.

$$\mathbb{O}_1^{\hat{\theta}} \hat{x}_k = Y_1 - \mathbb{G}^{\hat{\theta}} U \quad (14)$$

Depending upon the dimension and the rank of $\mathbb{O}_1^{\hat{\theta}}$, the solutions to the linear system of equations could be obtained. A least squares solution for this would be of the form

$$\hat{x}_k = \left((\mathbb{O}_1^{\hat{\theta}})^T (\mathbb{O}_1^{\hat{\theta}}) \right)^{-1} (\mathbb{O}_1^{\hat{\theta}})^T (Y_1 - \mathbb{G}^{\hat{\theta}} U) \quad (15)$$

Stable numerical methods could be used to compute the inverse.

IV. ELIMINATION APPROACH: LPV/QUASI-LPV CASE

We will now briefly summarize the algorithm in case of LPV or quasi-LPV models. The type of models considered are of the form

$$\begin{aligned} x_{k+1} &= A(\rho_k, \theta)x_k + B(\rho_k, \theta)u_k \\ y_k &= Cx_k \end{aligned} \quad (16)$$

where the scheduling variable ρ_k could be an external input or one of the system variables such as, inputs, outputs or its higher order derivatives if the nonlinear embedding method results in one. Further θ is a vector of unknown parameters. The parametrization is such that, we have

$$\begin{aligned} A(\rho_k, \theta) &= A_0(\rho_k) + \sum_{i=1}^{n_\theta} \bar{A}(\rho_k)\theta_i \\ B(\rho_k, \theta) &= B_0(\rho_k) + \sum_{i=1}^{n_\theta} \bar{B}(\rho_k)\theta_i \end{aligned} \quad (17)$$

The structural similarities of (1) and (16), the observability forms and the null-space computation are also related. The matrices in observability form 1 in (3), for the LPV/quasi-LPV form for the case of $N = 4$ could be given as

$$\begin{aligned} Y_1 &= \begin{bmatrix} y_k \\ y_{k+1} \\ y_{k+2} \\ y_{k+3} \end{bmatrix}, \quad \mathbb{O}_1^\theta = \begin{bmatrix} C \\ CA_k^\theta \\ CA_{k+1}^\theta A_k^\theta \\ CA_{k+2}^\theta A_{k+1}^\theta A_k^\theta \end{bmatrix}, \quad U = \begin{bmatrix} u_k \\ u_{k+1} \\ u_{k+2} \end{bmatrix} \\ \mathbb{G}^\theta &= \begin{bmatrix} 0 & 0 & 0 \\ CB_k^\theta & 0 & 0 \\ CA_{k+1}^\theta B_k^\theta & CB_{k+1}^\theta & 0 \\ CA_{k+2}^\theta A_{k+1}^\theta B_k^\theta & CA_{k+2}^\theta B_{k+1}^\theta & CB_{k+2}^\theta \end{bmatrix} \end{aligned} \quad (18)$$

Similarly, for the observability form 2 in (4), we have,

$$\begin{aligned} Y_2^\theta &= \begin{bmatrix} y_k \\ B_k^\theta u_k \\ y_{k+1} \\ B_{k+1}^\theta u_{k+1} \\ y_{k+2} \\ B_{k+2}^\theta u_{k+2} \\ y_{k+3} \end{bmatrix}, \quad X = \begin{bmatrix} x_k \\ x_{k+1} \\ x_{k+2} \\ x_{k+3} \end{bmatrix} \\ \mathbb{O}_2^\theta &= \begin{bmatrix} C & 0 & 0 & 0 \\ -A_k^\theta & I & 0 & 0 \\ 0 & C & 0 & 0 \\ 0 & -A_{k+1}^\theta & I & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & -A_{k+2}^\theta & I \\ 0 & 0 & 0 & C \end{bmatrix} \end{aligned} \quad (19)$$

The null-space computation, obtaining parity equation and the estimation of states all follow the steps in the linear case, given that the scheduling variables are made of known and/or measured variables. The overall procedure is summarized in Algorithm 1.

Algorithm 1 Steps for state and parameter estimation

- 1: Evaluate the window size (N) required and construct observability form(s)
 - 2: **for** every measurement window **do**
 - 3: Compute null-space and obtain parity equations
 - 4: Solve the set of nonlinear algebraic equations
 - 5: **if** Solution is not unique **then**
 - 6: Extra conditions to choose parameter estimates
 - 7: **end if**
 - 8: Estimate the state values
 - 9: **end for**
-

Remark 6: It is not necessary to compute the null-space and the parity equations during every measurement window. Both could be symbolically computed and evaluated inside for the corresponding values of inputs and outputs during the measurement window. This significantly reduces the computational burden.

Remark 7: The algorithm suggests computation of the parameter in every measurement window, which may not be required in an ideal case where the parameters are constant. However, they are useful in the presence of measurement noise, slowly time-varying parameters etc.

V. DISCUSSION

A. Time-varying parameter

In some scenarios such as process degradation, the unknown parameter varies slowly. In some types of faults, the parameter could change from one value to another over a period of time and then remain constant. In these circumstances, the algorithm works under the assumption that the parameter remains constant for a given measurement window. It then recalculates the parameters in the next window which indirectly compensates for the possible error if the parameter is varying.

B. Solvability issues

The set of polynomial equations that constitute the parity equation can have unique, multiple, infinitely many or no solution. Additional conditions could be used to find an appropriate solution among many, if one exists. However, the question of solvability of the parity equation for the given set of unknown parameters is not dealt with in the algorithm. An additional pre-processing could be performed to mitigate this. The parity equations could first be obtained through symbolic methods and then subjected to Buchberger algorithm to generate a Groebner basis [13]. This would provide a way to test the solvability.

Further, given the affine parametrization assumption, it is possible to derive the characteristics of the parity equations with respect to the degree of the polynomial terms, cross terms etc. This in turn could provide a test for solvability and hence the identifiability of a class of models.

C. Choosing the window size and multiple-output

A closely connected issue to solvability is choosing the measurement window size N . At the outset, this would depend on the relative degree of the output(s) with respect to the unknown parameters. A systematic way to find the right window size could be as follows:

- Choose $N = 1$.
- Obtain parity equations.
- Check the solvability of the parity equations. If not solvable go back to step 1 and increment N and repeat the procedure.

For multiple output case, this problem is further complicated. This is because the order in which measurements should be added for solvability is not known *a priori*. This could be solved as a combinatorial optimization problem for sensor placement.

D. Over-parametrization

The parity equations would typically be solved numerically and would find one of the possible solutions. Additional conditions to the optimization problem that solves it could be added that mirrors certain application constraints (e.g., $\theta > 0$, $0 \leq \theta \leq 1$). One part of reducing the multiple solutions and terms that lead to multiple solutions (e.g., θ^2) is to use the constraint that $\theta_{k+1} = \theta_k$. The given nonlinear equation could be shifted once and the original and the new equation could be solved to possibly obtain equations that doesn't have terms that lead to multiple solutions. This approach is termed prolongation in [6] and the regression terminology of over-parametrization in [1].

E. Identifiability

Identifiability for special classes of discrete-time models have been investigated in works such as [2], [1], [3] as discussed in Sec. I. In the algorithm proposed in this paper, identifiability is a condition that is required for the joint state and parameter estimation. However, it is possible to conceive the steps followed by the algorithm and its result to provide an indication for kind of identifiability satisfied (local/global, structural/algebraic) or lack of thereof by the given model. This would require further study.

VI. ILLUSTRATIVE EXAMPLES

In this section, we provide several examples to give an illustration of the algorithm under different circumstances.

Example 1: Consider the following model with the scheduling variables as $\rho_{1,k} \triangleq y_k$ and $\rho_{2,k} \triangleq u_k$,

$$x_{k+1} = \begin{bmatrix} \rho_{1,k} & -0.5 \\ 0.5\theta & \rho_{2,k} \end{bmatrix} x_k + \begin{bmatrix} \theta \\ 0 \end{bmatrix} u_k, \quad y_k = [1 \quad 0] x_k$$

The null-space of the matrix in observability form 1 (with $N = 3$) and the parity equation are given by,

$$\Omega_1^\theta = \begin{bmatrix} 0.25\theta + u_k y_k \\ -u_k - y_{k+1} \\ 1 \end{bmatrix}$$

$$y_{k+2} - y_{k+1}(u_k + y_{k+1}) - \theta u_{k+1} + y_k(0.25\theta + u_k y_k) + \theta u_k^2 = 0$$

In this example, given that there is a single parameter to be estimated, the parity equation could be rewritten to obtain $\hat{\theta}$ as a function of known and measured variables. For some multi-variable cases the solution could be found symbolically to represent all the parameters as a function of known and measured variables (e.g., *solve* in MATLAB). In a more complicated multi-variable case, a numerical approach shall be used. This estimate could be put into the observability form 1 which has the matrices

$$\mathbb{O}_1^{\hat{\theta}} = \begin{bmatrix} 1 & 0 \\ y_k & -0.5 \\ y_k y_{k+1} - 0.25\hat{\theta} & -0.5(u_k + y_{k+1}) \end{bmatrix}$$

$$\mathbb{G}^{\hat{\theta}} = \begin{bmatrix} 0 & 0 \\ \hat{\theta} & 0 \\ \hat{\theta} y_{k+1} & \theta \end{bmatrix}$$

and Y_1 and U are as described earlier to obtain an estimate for the state,

$$\hat{x}_k = \left((\mathbb{O}_1^{\hat{\theta}})^T (\mathbb{O}_1^{\hat{\theta}}) \right)^{-1} (\mathbb{O}_1^{\hat{\theta}})^T (Y_1 - \mathbb{G}^{\hat{\theta}} U) \quad (20)$$

Example 2: In this example, we illustrate the idea of over-parametrization through a linear example. Consider,

$$x_{k+1} = \begin{bmatrix} \theta & 1 \\ -0.2 & \theta \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k, \quad y_k = [1 \quad 0] x_k$$

For $N = 3$, the null-space of observability form 1 is given by,

$$\Omega_1^\theta = \begin{bmatrix} \theta^2 + 0.2 \\ -2\theta \\ 1 \end{bmatrix}$$

which leads to a parity equation of the form,

$$y_{k+2} - u_{k+1} + y_k(\theta^2 + 0.2) + \theta u_k - 2\theta y_{k+1} = 0 \quad (21)$$

Given that this equation would have two solution due to the presence of θ^2 , we can consider the condition, $\theta_{k+1} = \theta_k$ and shift (21) by 1 to obtain,

$$y_{k+3} - u_{k+2} + y_{k+1}(\theta^2 + 0.2) + \theta u_{k+1} - 2\theta y_{k+2} = 0$$

The term θ^2 can be eliminated using the above two equations and gives a unique estimate for θ with an extended window,

$$\hat{\theta} = \frac{u_{k+1}y_{k+1} - y_{k+1}y_{k+2} + y_k y_{k+3} - y_k u_{k+2}}{y_{k+1}(u_k - 2y_{k+1}) - y_k(u_{k+1} - 2y_{k+2})}$$

The components of (20) for state estimation are given by,

$$\mathbb{O}_1^{\hat{\theta}} = \begin{bmatrix} 1 & 0 \\ \hat{\theta} & 1 \\ \hat{\theta}^2 - 0.2 & 2\hat{\theta} \end{bmatrix} \quad \mathbb{G}^{\hat{\theta}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ \hat{\theta} & 1 \end{bmatrix}$$

Example 3: This example considers multiple unknown parameters with multiple outputs. A quasi-LPV model with the scheduling variable $\rho_k \triangleq u_k$ is,

$$x_{k+1} = \begin{bmatrix} 0.2 & 0 & 0 & \theta_2 \\ 0 & \theta_1 & -0.5 & 0 \\ 0 & \rho_k & 0 & 0.2 \\ 0 & 0 & 0.5 & 0.8 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_k$$

$$y_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x_k$$

For $N = 3$ the observability form 1 null-space obtained symbolically is given by,

$$\Omega_1^\theta = \begin{bmatrix} 0.16 & -0.02/\theta_2 \\ -\theta_1\theta_2 & 0.5u_0 \\ -1 & 0.1/\theta_2 \\ \theta_2 & \theta_1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The parity equations are given by,

$$0.8u_0 - u_1 + 0.16y_{1,0} - y_{1,1} + y_{1,2} + \theta_2 y_{2,1} - \theta_1 \theta_2 y_{2,0} = 0$$

$$y_{2,2} - \theta_1 y_{2,1} + 0.5u_0 y_{2,0} - 50 \frac{y_{1,0}}{\theta_2} - 0.1 \frac{u_0 - y_{1,1}}{\theta_2} = 0$$

With the obtained $\hat{\theta}$, we could compute the state using (20). The relevant matrices are not shown due to space constraints

Example 4: In this example, a case where the parameter only appears in the input matrix in the LPV form is given. If the parameter is considered time-varying in this case, this models a multiplicative actuator fault scenario. Considering the same scheduling variables as $\rho_{1,k} \triangleq y_k$ and $\rho_{2,k} \triangleq u_k$ we have,

$$x_{k+1} = \begin{bmatrix} 0.2 & -0.5 & 0 \\ 0 & \rho_{1,k} & 0.5 \\ 0 & 0.5\rho_{2,k} & -0.5 \end{bmatrix} x_k + \begin{bmatrix} 1 + \theta \\ 0 \end{bmatrix} u_k$$

$$y_k = [1 \ 0 \ 0] x_k$$

Given that the state transmission matrix A doesn't have θ appearing, the null-spaces of both the observability forms would not be in θ . Symbolically, the form 1 null-space and the parity equations are given by,

$$\Omega_1^\theta = \begin{bmatrix} 0.05u_k + 0.1y_k \\ 0.2y_{k+1} - 0.5y_k - 0.25u_k - 0.1 \\ 0.3 - y_{k+1} \\ 1 \end{bmatrix}$$

$$y_{k+3} + \frac{1+\theta}{4}(u_k + 2u_{k+1}) - y_{k+1}\left(\frac{u_k}{4} + \frac{y_k}{2} - \frac{y_{k+1}}{5} + 0.1\right) - y_{k+2}(y_{k+1} - 0.3) + y_k\left(\frac{u_k}{20} + 0.1y_k\right) = 0$$

For this example, we show the results of a time-varying parameter and state estimation. MATLAB computing environment was used to implement with the aid of the symbolic computation toolbox. A Gaussian, zero-mean measurement noise with a standard deviation of 0.01 was added to the output. In Fig. 1, the parameter estimation is illustrated for

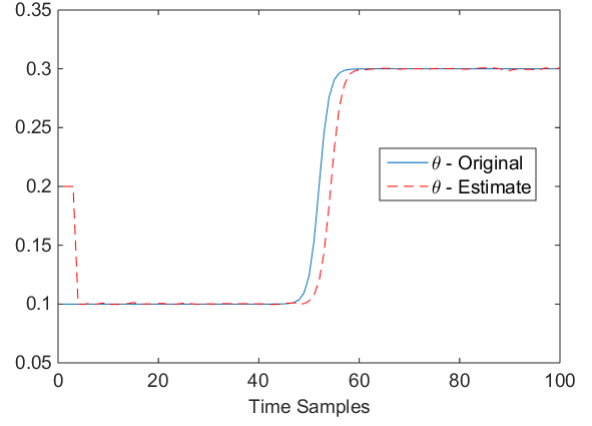


Fig. 1. True and the estimate of the parameter

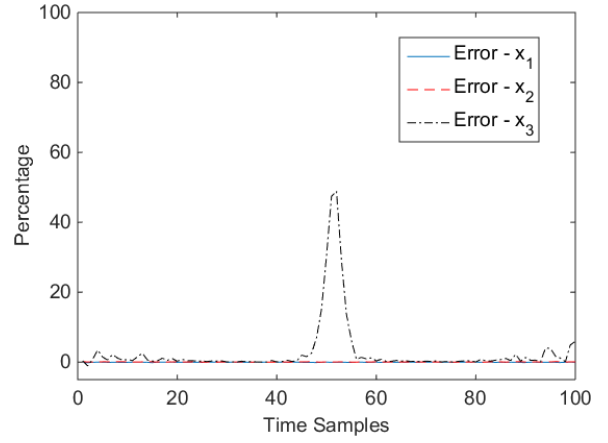


Fig. 2. State estimation errors

a time-varying case typical in fault scenarios with an arbitrary input. There is a delay because it takes $N = 4$ measurements to be available before an accurate estimation can be achieved. In Fig. 2, the errors of state estimation are illustrated as percentage of their values. We accounted for the delay to calculate these errors since the states have a dynamic nature. The errors on the states x_1 and x_2 are less than 1% and hence appear to be almost zero. The spike in the state estimation error for x_3 coincides with the transition on the parameter θ .

VII. CONCLUDING REMARKS

In this paper, we have outlined a method based on elimination which can estimate unknown parameters and then the states, but in a joint sense within a given measurement window. The models of interest are those discrete-time nonlinear models that could be represented as an LPV or a quasi-LPV model with affine parametrization. Several examples illustrated the approach and a detailed discussion suggested various open ends, extensions and possible applications. It would be interesting to consider them as future work.

ACKNOWLEDGMENT

The authors acknowledge the support from the FP7 project, Energy in Time (EiT), under the grant no. 608981.

REFERENCES

- [1] C. Lyzell, T. Glad, M. Enqvist, and L. Ljung, "Identification aspects of ritt's algorithm for discrete-time systems," *IFAC Proceedings Volumes*, vol. 42, no. 10, pp. 681–686, 2009.
- [2] F. Anstett, G. Bloch, G. Millérioux, and L. Denis-Vidal, "Identifiability of discrete-time nonlinear systems: The local state isomorphism approach," *Automatica*, vol. 44, no. 11, pp. 2884–2889, Nov. 2008.
- [3] Z. Alkhoury, M. Petreczky, and G. Mercère, "Identifiability of affine linear parameter-varying models," *Automatica*, vol. 80, pp. 62–74, 2017.
- [4] M. J. Yu and D. S. Bernstein, "Combined state and parameter estimation and identifiability of state space realizations," in *55th IEEE Conference on Decision and Control CDC'16*, 2016.
- [5] S. Diop, "Elimination in control theory," *Mathematics of Control, Signals and Systems*, vol. 4, no. 1, pp. 17–32, Mar. 1991.
- [6] A. J. van der Schaft, "Representing a nonlinear state space system as a set of higher-order differential equations in the inputs and outputs," *Systems & Control Letters*, vol. 12, no. 2, pp. 151–160, 1989.
- [7] A. Kwiatkowski, M. T. Boll, and H. Werner, "Automated Generation and Assessment of Affine LPV Models," in *45th IEEE Conference on Decision and Control CDC'06*, 2006.
- [8] E. Chow and A. Willsky, "Analytical redundancy and the design of robust failure detection systems," *IEEE Transactions on Automatic Control*, vol. 29, no. 7, pp. 603–614, 1984.
- [9] T. Beelen and P. Van Dooren, "A pencil approach for embedding a polynomial matrix into a unimodular matrix," *SIAM Journal on matrix analysis and applications*, vol. 9, no. 1, pp. 77–89, 1988.
- [10] A. Varga, "On computing nullspace bases A fault detection perspective," *IFAC Proceedings Volumes*, vol. 41, no. 2, pp. 6295–6300, 2008.
- [11] J. Z. Anaya and D. Henrion, "An improved toeplitz algorithm for polynomial matrix null-space computation," *Applied Mathematics and Computation*, vol. 207, no. 1, pp. 256–272, 2009.
- [12] W. C. Rheinboldt, *Methods for solving systems of nonlinear equations*. SIAM, 1998.
- [13] B. Buchberger, "An Algorithmic Criterion for the Solvability of a System of Algebraic Equations," in *Gröbner Bases and Applications*. Cambridge University Press, 1998, pp. 535–545.