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Ho Siao Hong

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# A Mathematical Proof for the Four-Color Problem

In honor to the Mathematician August Ferdinand Möbius (1790-1868)

Mr. Ho Siao Hong, Ph.D. and Diplom-Mathematiker

1709 South Valencia Street, Alhambra California 91801, United States of America

## **Abstract**

In this manuscript we present a mathematical proof in the traditional manner for the Four-Color Problem, without using any software program. We do so in two parts. In Part One we consider the four-color problem with the following constraint. Each country is a connected region. A proof using mathematical induction on the number of countries of a map is given. In Part Two we prove the Four-Color Problem without this constraint. No attempt is made to generalize the original four-color problem.

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## I.1. PART ONE: PRELIMINARY REMARKS.

The four-color problem says the following: on any two-dimensional map it is possible to label all countries using just four colors so that any two adjacent countries have different colors. In this context, a connected region is in fact simply connected. Two countries are adjacent if they share at least one common border. If they touch just at one point, then they aren't adjacent.

There have been numerous counter examples and “proofs” since the conjecture dawned in the mind of the mathematician August Ferdinand Möbius in 1840. It was formulated in a letter by another mathematician in 1852.

However, all these “proofs” and counter examples turned out to contain mistakes. With the help of digital computer software programs some authors came up with huge number of maps with all kinds of possible country constellations.

Even mathematical proofs using software programs are accepted as correct, we provide a mathematical proof in the traditional manner.

## I.2. PART ONE: THE INITIAL CASE.

A planar graph only partially describes the four-color problem. Each country is substituted by a vertex, and any two adjacent countries by an edge.

The initial case is a map with four countries using precisely four colors. As a planar graph it has four vertices and six edges. The Euler's formula  $V - E + R = 2$  holds, where  $V$ ,  $E$ , and  $R$  denote the number of vertices, edges, and connected regions respectively. Four colors are indeed needed in order to ensure that any two adjacent countries have distinct colors. We denote the colors of the four countries by 1, 2, 3, 4.



## I.3. PART ONE: THE INDUCTION HYPOTHESIS.

Let  $k$  be an integer. We assume that we are able to assign four given colors to  $k$  countries so that any two adjacent countries have distinct colors using the given four colors 1, 2, 3, 4.

Let  $l$  be an integer such that  $l < (k + 1)$  and  $l > (k/2 + 1)$ . Then  $(k + 1) - l < k + 1 - (k/2 + 1) \leq k/2 < k$ .

We add one more country to the  $k$  existing countries adjacent to the region made up of these  $k$  countries. We want to show that we can assign colors to  $k + 1$  countries satisfying the coloring conditions.

## I.4. PART ONE: THE INDUCTION CONCLUSION.

By induction hypothesis we are able to assign colors to the  $l$  countries satisfying coloring conditions, since  $l \leq k$ . These  $l$  countries form a simply connected colored surface area.

Again we apply the induction hypothesis. It is possible to assign colors to the  $(k + 1) - l$  countries satisfying the coloring conditions, since  $(k + 1) - l < k$ . Here we have to start assigning colors to countries adjacent to those countries that already have specific colors. Then we proceed assigning one color to each other remaining country until all  $(k + 1) - l$  countries are assigned colors.

Therefore, we are able to assign the given four colors to  $k + 1$  countries, and any two adjacent countries have distinct colors. The Part One is done. **q.e.d.**

## II.1. GENERAL CASE OF THE FOUR-COLOR PROBLEM.

II.1.1. Definition of a Big Country: A big country consists of more than one non-connected regions located on a continent. Non-connected regions can be just islands. They can have adjacent countries located on a different continent. The two continents are separated by oceans.

II.1.2. Example: Kaliningrad Oblast is a region of Russia, which is not connected to Russia. Between Kaliningrad Oblast and Russia there are Lithuania, then either Latvia or Belarus.

II.1.3. Remark: We take one region as the starting point of a big country. First, we link those non-connected regions to the big country, using paths that are constructed by linking one country after another like a chain. On each chain we choose countries so that they reach the big country. Note that those countries being used to build a chain don't belong to the big country.

In case a big country has just one non-connected region, we assign the region with the same color as the big country. Then we assign colors to the countries of the chain, to neighboring countries of the chain in order to obtain a simply connected superregion.

When a big country has a region located on a different continent, and has adjacent countries, then we assign colors to the latter in order to ensure that coloring conditions are met.

II.1.4. Example: The French oversea territory La Guyane is located in South America. It is assigned the same color as France in Europe. It is a good idea to assign colors to Brasil and Surinam. This precaution ensures that only the given four colors are used and any two adjacent countries have distinct colors.

French oversea territories, which are islands, are easy to assign colors. We simply assign them the same color as France in Europe.

II.1.5. Definition of Superregion: If a big country has more than one non-connected regions, we proceed in the same manner as above until all non-connected regions are linked to the big country and assigned the same color as the original big country.

We assign colors to participating countries in order to eventually construct a simply connected superregion on the same continent.

II.1.6. Remark: We note again that participating countries to form a superregion don't belong to the big country, and are assigned colors so that only the given four colors are used and any two adjacent countries have distinct colors.

If there are more than one non-connected regions located on a different continent with the two continents separated by oceans, then the construction of a superregion on that continent is required to ensure that the coloring conditions are met.

If this situation occurs on several continents and the latter are all separated by oceans, then superregions are constructed on all pertinent continents.

Informally, a simply connected region has “no hole.” When there is a “hole” of countries within a superregion, then we cannot be sure that the coloring conditions are met.

## II.2. CONSTRUCTION OF A HIERARCHY OF BIG COUNTRIES.

II.2.1. Remark: For any big country there are more than one possible superregions. We can add a few not yet colored countries, or take away a few of them. Precisely this flexibility enables us to construct superregions in such a manner to suit and meet coloring conditions. In addition, we ensure that any superregion becomes simply connected.

Given a map. We take one region of any big country, and attempt to construct a superregion as described in Part II.1.5. We make a note of this region. This big country corresponds to a vertex of a graph in graph theory.

Using a top-down approach we construct a graph representing a hierarchy of big countries, without yet assigning any colors to them.

If a big country consists of two non-connected regions, then it is always possible to link them using a chain of neighboring countries. None of which belongs to any big country.

But when there are three or more non-connected regions, then some of the participating countries may actually be regions of other big countries. Therefore, we need to establish a hierarchy of related big countries.

We exhaustively search for all participating countries which are actually regions of other big countries. If they exist, then we make a note of them and their corresponding big countries. Next we draw an edge between the vertex at the beginning and all new vertices underneath it.

We continue constructing the graph until all big countries related to each other, by having participating countries belonging to other big countries, are taken into consideration.

Such a graph has a finite number of vertices, because the number of big countries is finite.

What remains are other big countries which have nothing to do with this graph of vertices and their corresponding big countries. They will be processed later on.

Now we consolidate the hierarchy of vertices relative to each other, before we work on assigning colors.

Let us take one simple example to illustrate it and also show how we assign colors using a bottom-up approach.

### II.2.2. Example:

Let us consider a first big country to begin with. Suppose there are two participating countries  $A$  and  $C$  which are actually regions of two other big countries. Call them the second and third countries.

For the purpose of coloring we take a region  $B$  of the second country. Similarly we take a region  $D$  in the third country.

Assume that we have no more big countries as participating countries. We only have two levels of vertices.

If the second country contains a participating country  $E$  which is a region of the third big country, then the vertex corresponding to the third country has to be one level lower than the vertex corresponding to the second country. We also need to find a region  $F$  belonging to the third country to work with subsequently.

Now instead of two levels of vertices, the graph of big countries has three levels of vertices. This is how we consolidate the hierarchy of vertices.

Next we work on the coloring of regions and countries in this example.

Since the third country is at the lowest level, we start our coloring works with it.

$F$  and  $E$  have the same color. We assign colors to participating countries in a chain linking  $F$  to  $E$ . Likewise,  $D$  and  $C$  have the same color. We assign colors to a chain of participating countries linking  $D$  to  $C$ .

In this particular example we don't link  $A$  to  $C$  because they are just two participating regions of two distinct big countries and therefore have different colors. We work on them later.

$E$  is just a participating country in the second country, but is a region of the third country.  $E$  and  $B$  have different colors.

Since  $F$  and  $D$  belong to the third big country and have the same color, we link them through a chain of participating countries.

Next we assign the same color to one non-connected region of the third country either from the region  $F$  or  $D$ . Then we also assign colors of adjacent and neighboring countries to form a simply connected area before we connect to the next non-connected region.

We repeat the same task to all other remaining non-connected regions of the third country and ensure that the growing colored superregion remains simply connected.

The regions  $F$ ,  $D$ , and all other non-connected regions of the third big country have an identical color.

After this superregion is done, we work on coloring the second country which corresponds to a vertex at one higher level.

We link  $B$  to  $A$  through a chain of participating countries. Starting with the region  $B$  we construct a superregion to the second country, ensuring that the two superregions of the third and second big countries together remain simply connected.

Since  $A$  and  $C$  are just participating regions of the second and third big countries It is fairly simple to construct a superregion to the first big country, again ensuring that the new colored surface area remains simply connected.

### II.3. COLORING A HIERARCHY OF BIG COUNTRIES.

From the II.2.2. Example we get an idea how to assign colors in case of related big countries.

After the vertices of the graph are consolidated, we start with vertices at the lowest level in any branch of the graph.

From bottom-up we establish which vertices at higher levels are related to a vertex at a lowest level. Then we link one non-connected region of the big country corresponding to this vertex being considered, through a path of neighboring countries, to a participating region of a big country corresponding to a vertex at a higher level.

We repeat this task until all pertinent vertices at higher levels are taken care of.



Furthermore, when we know that any two non-connected regions belong to this same big country corresponding to the vertex at this lowest level, we connect them through a chain of neighboring countries.

In regard to assigning colors in the construction of a superregion:

1. Starting from one colored non-connected region we connect it to another one, with both belonging to the same big country.

2. Then we must assign colors to adjacent and neighboring countries to form a simply connected surface area. This activity also ensures that the coloring conditions will be met at the end of our works.

We repeat these two steps just mentioned until we are done with all non-connected regions. Then we continue assigning colors in order to construct a superregion of this big country.

Once a superregion is finished, we perform the same tasks to construct a superregion of the big country corresponding to a vertex at the same level.

If there is no more vertex at the same level, we go one level higher in the vertex hierarchy. Then we repeat the identical tasks keeping in mind that any two new superregions form a simply connected colored surface area.

We continue this identical process until all vertices of the given graph are processed. As a result we obtain a cluster of related big countries, whose superregions together remain simply connected.

#### II.4. PROCESSING OTHER CLUSTERS OF BIG COUNTRIES.

If there are other clusters of related big countries like in Part II.2 and Part II.3, we work in the same manner until all simply connected superregions are colored satisfying the coloring conditions.

When two simply connected colored clusters located on the same continent are going to merge, we have to assign colors so that the new growing merged surface area still remains simply connected.

Islands surrounded by oceans and belonging to any big country are always assigned the same color as the remaining regions of that specific big country.

## II.5. THE FINAL STEP OF THE GENERAL PROOF.

After all individual clusters of big countries are processed, there remain only connected countries that are NOT located inside any simply connected regions.

By Part I we first assign colors to those countries adjacent to countries that already have a fixed color, keeping the coloring conditions in mind. One by one, all these remaining countries are assigned one of the four given colors and any two adjacent countries have distinct colors.

Therefore, the four-color conjecture, first initiated by the mathematician August Ferdinand Möbius in 1840, is proved. **Q. E. D.**

## III. PART THREE: CONCLUDING REMARKS AND COMMENTS.

1. At each step of assigning colors to countries and/or regions of big countries, the growing colored surface must remain simply connected. This is the reason why I construct a hierarchy of related big countries in a top-down manner, then work in a bottom-up approach when it comes to assigning colors.

The notion of simple connectivity is indeed central to this proof of the Four-Color Problem.

2. Since I didn't study any research articles published by any other authors, there is no bibliography at the end in this research manuscript.