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# A Mathematical Proof for the Four-Color Problem

In honor to the Mathematician August Ferdinand Möbius (1790-1868)

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## **Abstract**

In this manuscript we present a mathematical proof in the traditional manner for the Four-Color Problem, without using any software program. We do so in two parts. In Part One we consider the four-color problem with the following constraint. Each country is a connected region. A proof using mathematical induction on the number of countries of a map is given. In Part Two we prove the Four-Color Problem without this constraint. No attempt is made to generalize the original four-color problem.

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## I.1. PART ONE: PRELIMINARY REMARKS.

The four-color problem says the following: on any two-dimensional map it is possible to label all countries using just four colors so that any two adjacent countries have different colors. In this context, a connected region is in fact simply connected. Two countries are adjacent if they share at least one common border. If they touch just at one point, then they aren't adjacent.

There have been numerous counter examples and “proofs” since the conjecture dawned in the mind of the mathematician August Ferdinand Möbius in 1840. It was formulated in a letter by another mathematician in 1852.

However, all these “proofs” and counter examples turned out to contain mistakes. With the help of digital computer software programs some authors came up with huge number of maps with all kinds of possible country constellations.

Even mathematical proofs using software programs are accepted as correct, we provide a mathematical proof in the traditional manner.

## I.2. PART ONE: THE INITIAL CASE.

A planar graph only partially describes the four-color problem. Each country is substituted by a vertex, and any two adjacent countries by an edge.

The initial case is a map with four countries using precisely four colors. As a planar graph it has four vertices and six edges. The Euler's formula  $V - E + R = 2$  holds, where  $V$ ,  $E$ , and  $R$  denote the number of vertices, edges, and connected regions respectively. Four colors are indeed needed in order to ensure that any two adjacent countries have distinct colors. We denote the colors of the four countries by 1, 2, 3, 4.



## I.3. PART ONE: THE INDUCTION HYPOTHESIS.

Let  $k$  be an integer. We assume that we are able to assign four given colors to  $k$  countries so that any two adjacent countries have distinct colors using the given four colors 1, 2, 3, 4.

Let  $l$  be an integer such that  $l < (k + 1)$  and  $l > (k/2 + 1)$ . Then  $(k + 1) - l < k + 1 - (k/2 + 1) \leq k/2 < k$ .

We add one more country to the  $k$  existing countries adjacent to the region made up of these  $k$  countries. We want to show that we can assign colors to  $k + 1$  countries satisfying the coloring conditions.

## I.4. PART ONE: THE INDUCTION CONCLUSION.

By induction hypothesis we are able to assign colors to the  $l$  countries satisfying coloring conditions, since  $l \leq k$ .

Again we apply the induction hypothesis. It is possible to assign colors to the  $(k + 1) - l$  countries satisfying the coloring conditions, since  $(k + 1) - l < k$ . Here we have to start assigning colors to countries adjacent to those countries that already have specific colors. Then we proceed assigning one color to each other remaining country until all  $(k + 1) - l$  countries are assigned colors.

Therefore, we are able to assign the given four colors to  $k + 1$  countries, and any two adjacent countries have distinct colors. The Part One is done. **q.e.d.**

## II.1. GENERAL CASE OF THE FOUR-COLOR PROBLEM.

II.1.1. Definition of a Big Country: A big country consists of more than one non-connected regions located on a continent. Non-connected regions can be just islands. They can have adjacent countries located on a different continent. The two continents are separated by oceans.

II.1.2. Example: Kaliningrad Oblast is a region of Russia, which is not connected to Russia. Between Kaliningrad Oblast and Russia there are Lithuania, then either Latvia or Belarus.

II.1.3. Remark: We take one region as the starting point of a big country. First, we link those non-connected regions to the big country, using paths that are constructed by linking one country after another like a chain. On each chain we choose countries so that they reach the big country. Note that those countries being used to build a chain don't belong to the big country.

In case a big country has just one non-connected region, we assign the region with the same color as the big country. Then we assign colors to the countries of the chain, to neighboring countries of the chain in order to obtain a simply connected superregion.

When a big country has a region located on a different continent, and has adjacent countries, then we assign colors to the latter in order to ensure that coloring conditions are met.

II.1.4. Example: The French oversea territory La Guyane is located in South America. It is assigned the same color as France in Europe. It is a good idea to assign colors to Brasil and Surinam. This precaution ensures that only the given four colors are used and any two adjacent countries have distinct colors.

French oversea territories, which are islands, are easy to assign colors. We simply assign them the same color as France in Europe.

II.1.5. Definition of Superregion: If a big country has more than one non-connected regions, we proceed in the same manner as above until all non-connected regions are linked to the big country and assigned the same color as the original big country.

We assign colors to participating countries in order to eventually construct a simply connected superregion on the same continent.

II.1.6. Remark: We note again that participating countries to form a superregion don't belong to the big country, and are assigned colors so that only the given four colors are used and any two adjacent countries have distinct colors.

If there are more than one non-connected regions located on a different continent with the two continents separated by oceans, then the construction of a superregion on that continent is required to ensure that the coloring conditions are met.

If this situation occurs on several continents and the latter are all separated by oceans, then superregions are constructed on all pertinent continents.

Informally, a simply connected region has "no hole." When there is a "hole" of countries within a superregion, then we cannot be sure that the coloring conditions are met.

## II.2. CONSTRUCTION OF A HIERARCHY OF BIG COUNTRIES.

II.2.1. Remark: For any big country there are more than one possible superregions. We can add a few not yet colored countries, or take away a few of them. Precisely this flexibility enables us to construct superregions in such a manner to suit and meet coloring conditions.

Given a map. We take one region of any big country, and attempt to construct a superregion as described in Part II.1. We make a note of this region. This big country corresponds to a vertex of a graph in graph theory.

If in this process we encounter any participating countries that belong to other big countries, we make note of them and their big countries correspond to vertices right underneath the vertex in the graph of big countries. Then we draw edges connecting the vertex at the beginning and new vertices right underneath it.

When a big country whose superregion no longer contains participating countries of any other big countries, this big country corresponds to a terminal vertex of the graph being constructed.

We continue constructing the graph until all big countries related to each other, by having either participating countries belonging to other big countries, or a vertex is a terminal vertex, form a simply connected region comprising all related big countries.

Now we work on the colorings. We assign colors to big countries corresponding to vertices at the lowest level of the graph, one by one. In this manner the corresponding superregions are constructed.

Then we go one level higher in the hierarchy of vertices. And we construct superregions of big countries corresponding to these vertices. We construct superregions until we are done with all vertices of the graph.

II.2.2. Remark: It is possible that vertices at a lower level are linked to vertices at higher levels in the graph, and these vertices form the lowest level in a branch of the graph. Their corresponding big countries are also first processed like in the case of terminal vertices.

In other words, we work on vertices at the lowest level within the graph to construct corresponding superregions. Then we go to one level higher in the vertex hierarchy to construct superregions corresponding to these vertices.

Again we continue doing identical works to big countries until all vertices and their corresponding superregions are entirely constructed.

As a consequence, we assign colors to all related big countries satisfying the coloring conditions. The result is a simply connected region consisting of all related big countries.

### II.3. PROCESSING OTHER CLUSTERS OF BIG COUNTRIES.

If there are other clusters of related big countries like in Part II.2, we work in the same manner until all simply connected regions are colored satisfying the coloring conditions.

Islands surrounded by oceans and belonging to any big country are always assigned the same color as the remaining regions of that specific big country.

### II.4. THE FINAL STEP OF THE GENERAL PROOF.

After all individual clusters of big countries are processed, there remain only connected countries that are NOT located inside any simply connected regions.

By Part I we first assign colors to those countries adjacent to countries that already have a fixed color, keeping the coloring conditions in mind. One by one, all these remaining countries are assigned one of the four given colors and any two adjacent countries have distinct colors.

Therefore, the four-color conjecture, first initiated by the mathematician August Ferdinand Möbius in 1840, is proved. **Q. E. D.**

## III. PART THREE: CONCLUDING REMARKS AND COMMENTS.

1. When studying and scrutinizing this mathematical proof for the Four-Color Problem, I suggest that readers draw their own maps in any way they wish, in order to convince themselves of each logical step in the reasoning of my proof.

2. Since I didn't study any research articles published by any other authors, there is no bibliography at the end in this research manuscript.