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# Boundary Control of a Wave Equation With an Anti-Damped Boundary Dynamics in Presence of an In-Domain Velocity Source Term

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**Abstract:** This extended abstract presents some backstepping control designs for a one-dimensional wave equation with an anti-damped boundary dynamics in presence of in-domain velocity source term. It discusses the merits and limitations of already carried out solutions and sketches directions of ongoing works. This paper is with the open invited track for the GdR MACS Young PhD researchers.

*Keywords:* Wave equation, Observer, Partial differential equation (PDE), Boundary control, In-domain parameter, Exponential stabilization.

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## 1. INTRODUCTION

This paper focuses on the boundary control of a wave equation, i.e., the stabilization of an infinite dimensional equation and its uncontrolled boundary, achieved with a control action located at the opposite boundary. In the literature, some papers consider Dirichlet boundary conditions and focus on the in-domain source terms difficulties [Smyshlyaev et al. (2010)]. Alternatively, sometimes the boundaries under consideration are more complicated, e.g., [Saldivar et al. (2011)]. In this case, a proposed approach is to introduce redundancy by decomposing the boundary dynamics into a usual boundary condition and an ordinary differential equation (ODE). This leads to the consideration of a PDE-ODE cascade, a more standard framework which has been frequently studied [d'Andréa-Novel and Coron (2000)]. Another interesting problem concerns the case of the unknown boundary parameters paradigm. Recently, adaptive approaches have been designed to tackle this difficulty [Krstic (2009), Bresch-Pietri and Krstic (2014a)]. Nevertheless, in both cases, and, more generally in all aforementioned cases where the uncontrolled boundary is not standard, a pure wave equation without source term is considered.

An interesting feature of a pure wave equation, it is that it can be decomposed into two transport phenomena. Thus, one is able to have the knowledge of the (delayed) state-space from the measurement of one boundary [Bresch-Pietri and Krstic (2014b)]. This property reveals to be particularly convenient as it significantly simplifies the design and also corresponds to practical constraints (limited point-wise measurements). Nevertheless, neglecting an in-domain source term, even a viscous damping term, is not anodyne. Intuitively, when one considers an additional damping term into a (finite or infinite dimensional) dynamics, it sounds reasonable to conclude that

this dissipativity term improves the stability margins of the system. However, from a finite-dimensional point of view, this conclusion is refuted by the existence of an overdamping phenomenon, i.e., a decrease of the settling time is induced by a damping increase. A similar counter-intuitive outcome can be observed for infinite-dimensional systems. In particular, for a wave PDE, the addition of a distributed damping term introduces coupling between the two aforementioned transport PDEs [Roman et al. (2016b)], and thus eliminates potential finite-time regulation (if not taken into account).

In this paper, we therefore focus on a wave equation with in-domain velocity source term (see further (1)-(3)). Up to our knowledge, there exist only two designs accounting for distributed viscous damping for this anti-damped dynamical boundary which achieve exponential stabilization. First, the solution in [Sagert et al. (2013)], which grounds on the backstepping methodology [Krstic and Smyshlyaev (2008)], and is a full state feedback which requires a previous change of variables (not straightforwardly invertible). Consequently, the obtained stability result is expressed in terms of space derivatives of the state. We have proposed a second solution in [Roman et al. (2016a)] which also grounds on backstepping methodology, but associated to an observer. This observer will be detailed further. It is worth noticing that even if the design of both controllers seems close there does not exist a choice of tuning parameters for which both controller are equal.

This extended abstract is organized as follows. In Section 2 the considered wave equation is presented and appearing difficulties are explained. Section 3 is devoted to the already developed solutions and ongoing pursuits to tackle the presented difficulties. The benefits of the presented design are illustrated through some simulations in Section 4. Finally Section 5 is devoted to the conclusion with some outlook.

## Notations

The partial derivatives are denoted as follows

$$u_t(x, t) = \frac{\partial u(x, t)}{\partial t}, \quad u_x(x, t) = \frac{\partial u(x, t)}{\partial x}$$

$$u_{tt}(x, t) = \frac{\partial^2 u(x, t)}{\partial t^2}, \quad u_{xx}(x, t) = \frac{\partial^2 u(x, t)}{\partial x^2}$$

the same notations hold for  $w$ ,  $\hat{u}$ ,  $\tilde{u}$ ,  $S$ , and  $K$ .

The Lebesgue set of square-integrable function on  $[0, 1]$  is denoted  $L_2(0, 1)$ . The usual Sobolev space of function in  $L_2(0, 1)$  with first order derivative also in  $L_2(0, 1)$  is denoted  $H_1(0, 1)$ .

## 2. PROBLEM STATEMENT

Let us consider the following wave equation with distributed in-domain viscous damping, subject to an anti-damping boundary, with actuation on the opposite boundary

$$u_{tt}(x, t) = u_{xx}(x, t) - \lambda(x)u_t(x, t) \quad (1)$$

$$u_x(1, t) = U(t) \quad (2)$$

$$u_{tt}(0, t) = a q u_t(0, t) + a u_x(0, t) \quad (3)$$

in which  $U(t)$  is the scalar control input,  $(u, u_t)$  is the system state, with  $(u(\cdot, 0), u_t(\cdot, 0)) \in H_1(0, 1) \times L_2(0, 1)$ ,  $a > 0$  is a scalar constant. The in-domain velocity source term coefficient is  $\lambda(x)$ , in the case where it is non negative, is also called in-domain viscous damping. The anti-damping coefficient is  $q > 0$ .

*The control objective is the velocity regulation, i.e., to compute  $U(t)$  such that  $u_t(x, t) = u_t^{ref}$ ,  $x \in [0, 1]$ .*

The problem is challenging because, first, the uncontrolled boundary is not standard; it can be seen as a dynamical condition on  $u_t(0, t)$  driven by  $u_x(0, t)$ , and second, the presence of in-domain source term prevents to apply d'Alembert's transformation or Riemann invariants.

## 3. CONTROL AND OBSERVER DESIGNS

This section establishes the already developed solutions and the ongoing works, the advancement of which will be detailed during presentation.

### 3.1 Control design: backstepping approach

The purpose of backstepping is to find a control law such that the closed-loop system has the same behavior as the target one, i.e.,

$$w_{tt}(x, t) = w_{xx}(x, t) - \lambda_w(x)w_t(x, t) \quad (4)$$

$$w_x(1, t) = -c_t w_t(1, t) \quad (5)$$

$$w_{tt}(0, t) = -a_w q_w w_t(0, t) + a_w w_x(0, t) \quad (6)$$

using a change of variables. By using the following Lyapunov functional ( $|\delta| < 1/2$ )

$$V(t) = \frac{1}{2} \int_0^1 (w_x^2 + w_t^2) dx + \frac{1}{2a_w} w_t(0, t)^2$$

$$+ \delta \int_0^1 (1+x) w_x w_t dx \quad (7)$$

and adapting the proof in [Roman et al. (2016a)] the following result can be shown.

*Theorem 1.* If  $c_t$ ,  $a_w$ ,  $q_w$ , and  $\lambda_w(x)$  are non negative then  $w_x(x, t)$ ,  $w_t(x, t)$  and  $w_t(0, t)$  are exponentially stable.

Furthermore we list some cases with respect to some assumptions on  $\lambda(x)$  and  $\lambda_w(x)$ .

*Case 1*  $\lambda = \lambda_w \in \mathbb{R}^+$

The case where both viscous source terms are constant is solved in [Roman et al. (2016a)]. Indeed, considering the following backstepping transformation

$$w(x, t) = u(x, t) - \int_0^x s(x, y) u_t(y, t) dy$$

$$- \int_0^x m(x, y) u_x(y, t) dy - \int_0^x g(x, y) u_{tx}(y, t) dy \quad (8)$$

one shows that the kernels are solutions of

$$S_{xx}(x, y) - S_{yy}(x, y) = 0 \quad (9)$$

$$S(x, x) = F \quad (10)$$

$$S_y(x, 0) = HS(x, 0) \quad (11)$$

where  $S$  is the vector of the kernels,  $F$  is a constant vector and  $H$  a constant matrix. The idea of the proof is to decompose the wave equation, of which  $S$  is a solution (9), into two transport phenomena, see [Roman et al. (2016a)] for details.

*Case 2*  $\mathbb{R} \ni \lambda \neq \lambda_w \in \mathbb{R}$

The viscous source terms are now assumed constant but not equal. Considering a slightly different backstepping transformation one proves that the kernels are solutions of

$$K_{xx}(x, y) - K_{yy}(x, y) = BK_y(x, y) + CK(x, y) \quad (12)$$

$$K(x, x) = F_k(x) \quad (13)$$

$$K_y(x, 0) = H_k K(x, 0) \quad (14)$$

where  $K$  is the vector of the kernels,  $F_k(x)$  is a space variable vector and  $H_k$ ,  $B$ , and  $C$  are constant matrices. Here, in (12) the kernels do not satisfy a pure wave equation anymore. Nevertheless by using the method of successive approximations [Krstic and Smyshlyaev (2008)], we can prove that there exists a unique  $K(x, y)$ . Moreover one finds an implicit formulation and an iterative method to compute the kernel. The idea of the proof is to simplify the wave equation with a change of variable before using the method of successive approximations.

*Case 3*  $\lambda(x) \neq \lambda_w(x)$ .

The viscous source terms are now unspecified. Using the previous backstepping transformation one obtains that the kernels are solutions of

$$K_{xx}(x, y) - K_{yy}(x, y) = B(x, y)K_y(x, y)$$

$$+ C(x, y)K(x, y) \quad (15)$$

$$K(x, x) = F_k(x) \quad (16)$$

$$K_y(x, 0) = H_k(x)K(x, 0) \quad (17)$$

where  $K$  is the vector of the kernels,  $F_k(x)$  is a space-varying vector and  $H_k(x)$ ,  $B(x, y)$ , and  $C$  are matrices. We are currently trying to prove the existence of a solution,

with the method of successive approximations. The proof for (12)-(14) cannot be extended, because the first change of variable does not hold anymore as  $B(x, y)$  depends on  $x$  and  $y$ .

### 3.2 Design of a state-space observer

Previously, we were using a backstepping approach, giving rise to a full-state feedback. However, in practice, such a state is not available for measurement and the design of an observer is needed.

In [Roman et al. (2016a)] we have addressed the case where the output is the velocity at both boundaries. Consider the following observer

$$\hat{u}_{tt}(x, t) = \hat{u}_{xx}(x, t) - \lambda(x)\hat{u}_t(x, t) \quad (18)$$

$$\hat{u}_x(1, t) = U(t) + l_1[u_t(1, t) - \hat{u}_t(1, t)] \quad (19)$$

$$\begin{aligned} \hat{u}_{tt}(0, t) = & aq\hat{u}_t(0, t) + a\hat{u}_x(0, t) \\ & + l_2[u_t(0, t) - \hat{u}_t(0, t)] \end{aligned} \quad (20)$$

the estimate-error system  $\tilde{u} = u - \hat{u}$  is

$$\tilde{u}_{tt}(x, t) = \tilde{u}_{xx}(x, t) - \lambda(x)\tilde{u}_t(x, t) \quad (21)$$

$$\tilde{u}_x(1, t) = -l_1\tilde{u}_t(1, t) \quad (22)$$

$$\tilde{u}_{tt}(0, t) = (aq - l_2)\tilde{u}_t(0, t) + a\tilde{u}_x(0, t) \quad (23)$$

Using directly *Theorem 1* the following statement holds. If  $l_1 > 0$  and  $l_2 > aq$ , then the error system, in terms of  $\tilde{u}_t(x, t)$  and  $\tilde{u}_x(x, t)$ , is exponentially stable.

After computation, one gets that the decay rate of the Lyapunov functional (7) is limited, i.e., the  $\eta$  such that  $\dot{V} \leq -\eta V$  is bounded. This means that one cannot arbitrarily choose the behavior of the target/observer system, e.g., in the case of regulation, the settling time may have a lower bound for any tuning parameters used. Moreover the decay rate,  $\eta$  goes towards 0 when the in-domain damping,  $\lambda_w$ , goes to infinity. This seems like the overdamping phenomenon. Nevertheless the Lyapunov function only gives a conservative bound on the decay rate, so none can conclude that it is the overdamping effect.

We are currently trying to improve the behavior of this observer design, using the backstepping approach as done in [Smyshlyaev and Krstic (2005)] to change the in-domain damping for the observer. In the case where the wave PDE has no in-domain source term, one can design an observer using only the controlled boundary measurement. We would like to do the same with in-domain source term. Moreover in [Smyshlyaev and Krstic (2005)] there are lines of research which may be used to design a state observer using the controlled boundary measurement, i.e.,  $u_t(1, t)$ .

### 3.3 Unknown anti-damped boundary dynamics

We aspire to design an adaptive control law to control the system where the uncontrolled parameters are unknown. Nevertheless it is a hard task to realize from the very beginning. Our final purpose is to generalize the output adaptive prediction based control of [Bresch-Pietri and Krstic (2014a)] for in-domain viscous damping. In our development, we are still in the stage where the parameter are assumed to be known, i.e.,  $q$  and a feedforward bias denoted  $d$ . Moreover as  $d$  is a feedforward bias the solution

for  $d = 0$  stands for  $d \in \mathbb{R}$  through a change of variable, which is why we have not considered this bias previously, contrary to [Bresch-Pietri and Krstic (2014a) and Roman et al. (2016b)].

## 4. NUMERICAL SIMULATIONS

The wave equation (1)-(3) can represent a normalized and linearized model of the mechanical angular vibrations occurring in the drilling process, [e.g. Bresch-Pietri and Krstic (2014a), Bresch-Pietri and Krstic (2014b), and Sagert et al. (2013)]. Following simulations are performed with the control design in [Roman et al. (2016a)]. Following the same approach as of [Roman et al. (2016a)], the wave equation (1)-(3) is semi-discretized in space, i.e.,

$$\dot{X}(t) = AX(t) + BU(t) \quad (24)$$

in which  $X(t) = [u[1:n] \ u_t[1:n]]^T$ , with  $A \in \mathbb{R}^{2n \times 2n}$  where  $n$  is the number of spatial points considered,  $B \in \mathbb{R}^{2n \times 1}$ . For the presented simulation we considered  $n = 30$  spatial points. The parameters value are expressed in Table 1. Figure 1 depicts the evaluation of the boundary velocities (using  $X(0) = [u^0[1:n] \ u_t^0[1:n]]^T$ ) of the open-loop system, i.e., (1)-(3) with  $U(t) = 0$ . The closed-loop system evaluation of the boundary velocities in the care of the full state backstepping control law of [Roman et al. (2016a)] is displayed in Figure 2. The evaluation of the boundary velocities in the care of the observer based control is shown in Figure 3 and the evaluation of the associated control law is pictured in Figure 4.

Symbol	Description	Value
$\lambda(x)$	In-domain damping coefficient	0.1
$a$	Uncontrolled boundary coefficient	1
$q$	Anti-damping coefficient	1
$c_t$	Control boundary target coefficient	1
$\lambda_w(x)$	In-domain damping target coefficient	$\lambda(x)$
$a_w$	Uncontrolled boundary target coefficient	$a$
$q_w$	Boundary damping target coefficient	$q$
$l_1$	Observer controlled boundary coefficient	10
$l_2$	Observer uncontrolled boundary coefficient	120% $aq$
$u^0[1:n]$	Initial data for the state $u$	0, $\forall n$
$u_t^0[1:n]$	Initial data for the state $u_t$	1, $\forall n$
$\hat{u}^0[1:n]$	Initial data for the estimated state $u$	0, $\forall n$
$\hat{u}_t^0[1:n]$	Initial data for the state estimated state $u_t$	0, $\forall n$

Table 1. Values of the parameters for the simulation

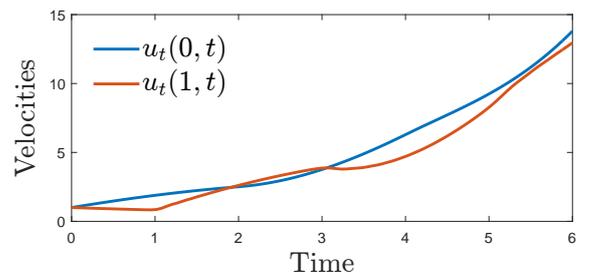


Fig. 1. Open-loop boundary velocities time-domain response for non-zero initial conditions. The open-loop system is unstable.

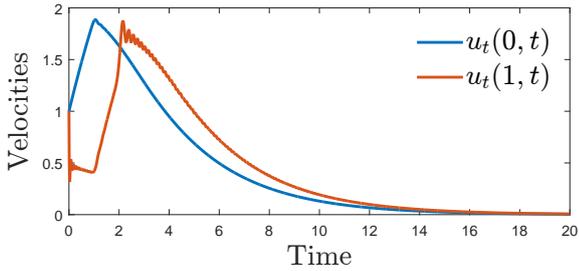


Fig. 2. Closed-loop boundary velocities time-domain response for non-zero initial conditions. The full-state backstepping controller stabilizes the system.

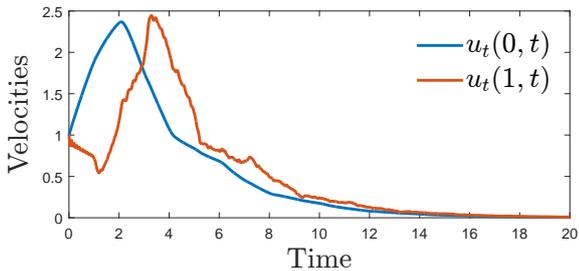


Fig. 3. Closed-loop boundary velocities time-domain response for non-zero initial conditions. The observer based backstepping controller stabilizes the system. The boundary velocities amplitude is more important than in Figure 2; due to the fact that the observer has not converged yet.

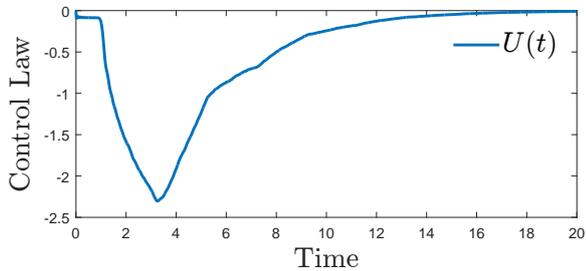


Fig. 4. Control law time-domain response for non-zero initial conditions for the observer based backstepping controller. The time-domain response of this control is somehow similar to the prediction based control for the system without in-domain damping.

## 5. CONCLUSION

We have developed an observer using the velocity of both boundaries to estimate the full-state space, i.e., in terms of  $u_t$  and  $u_x$ . Moreover using backstepping design we have developed a controller which can modify the in-domain damping (or velocity source term) between the target and the original system. In the case we did not modify the in-domain damping between them, the kernels can be explicitly given.

We devote ourselves to design of an adaptive control with an in-domain viscous source term. But first we would like to improve our work on the control and observer design, before adding adaptive approach.

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