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# OPTIMAL EXTINCTION RATES FOR THE FAST DIFFUSION EQUATION WITH STRONG ABSORPTION

RAZVAN GABRIEL IAGAR AND PHILIPPE LAURENÇOT

ABSTRACT. Optimal extinction rates near the extinction time are derived for non-negative solutions to a fast diffusion equation with strong absorption, the power of the absorption exceeding that of the diffusion.

## 1. INTRODUCTION

Given  $m \in (0, \infty)$ ,  $q \in (0, 1)$ , and a non-negative initial condition  $u_0$  in  $BC(\mathbb{R}^N)$ ,  $u_0 \not\equiv 0$ , it is well-known that the initial value problem

$$\partial_t u - \Delta u^m + u^q = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^N, \quad (1.1a)$$

$$u(0) = u_0, \quad x \in \mathbb{R}^N, \quad (1.1b)$$

has a unique non-negative (weak) solution  $u$  which vanishes identically after a finite time, a phenomenon usually referred to as *finite time extinction* [19–21]. More precisely, introducing the extinction time

$$T_e := \sup\{t > 0 : u(t) \not\equiv 0\} > 0, \quad (1.2)$$

then  $T_e$  is finite and satisfies  $T_e \leq \|u_0\|_\infty^{(1-q)}/(1-q)$ , the latter upper bound being a straightforward consequence of (1.1) and the comparison principle. Moreover, there holds

$$u(t) \not\equiv 0 \text{ for } t \in [0, T_e) \text{ and } u(t) \equiv 0 \text{ for } t \geq T_e. \quad (1.3)$$

When  $q < m$  and  $u_0(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ , finite time extinction is accompanied by an even more striking phenomenon, the *instantaneous shrinking of the support*, that is, the positivity set  $\mathcal{P}(t) := \{x \in \mathbb{R}^N : u(t, x) > 0\}$  of  $u$  at time  $t$  is a relatively compact subset of  $\mathbb{R}^N$  for all  $t \in (0, T_e)$ , even if  $\mathcal{P}(0) = \mathbb{R}^N$  initially [1, 5, 7, 19]. Observe that the inequality  $q < m$  is always satisfied when the diffusion is linear ( $m = 1$ ) or slow ( $m > 1$ ). Additional information on the behaviour of  $\mathcal{P}(t)$  as  $t \rightarrow T_e$  is also available when  $m \geq 1$  and  $N = 1$  [6, 10–14].

Once finite time extinction is known to take place, gaining further insight into the underlying mechanism requires to identify the behaviour of  $u(t)$  as  $t \rightarrow T_e$ , a preliminary step being to determine

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the relevant space and time scales. Simple scaling arguments predict that, for  $r \in [1, \infty]$  and  $u_0 \in L^r(\mathbb{R}^N)$ , there is a constant  $\gamma_r > 0$  (depending on  $N, m, q, u_0$ , and  $r$ ) such that

$$\|u(t)\|_r \sim \gamma_r (T_e - t)^{\alpha - (N\beta/r)}, \quad (1.4)$$

where

$$\alpha := \frac{1}{1-q} > 0, \quad \beta := \frac{q-m}{2(1-q)} \in \mathbb{R}. \quad (1.5)$$

As already observed by several authors [9, 12, 15], a rather simple comparison argument provides a lower bound for the  $L^\infty$ -norm of the form (1.4). Indeed, consider  $t \in (0, T_e)$  and let  $x(t) \in \mathbb{R}^N$  be a point where  $u(t)$  reaches its maximum value, that is,  $u(t, x(t)) = \|u(t)\|_\infty$ . Then  $u(t)^m$  also attains its maximum value at this point, so that  $\Delta u^m(t, x(t)) \leq 0$  and we infer from (1.1a) that (at least formally)

$$\frac{d}{dt} \|u(t)\|_\infty = \partial_t u(t, x(t)) \leq -u(t, x(t))^q = -\|u(t)\|_\infty^q.$$

Integrating the above differential inequality over  $(t, T_e)$  gives the expected lower bound

$$\|u(t)\|_\infty \geq [(1-q)(T_e - t)]^{1/(1-q)}, \quad t \in [0, T_e). \quad (1.6)$$

The derivation of an upper bound of the form (1.4) turns out to be more involved and the results obtained so far are rather sparse: in one space dimension, the upper bound

$$\|u(t)\|_\infty \leq C_\infty (T_e - t)^{1/(1-q)}, \quad t \in [0, T_e), \quad (1.7)$$

is shown in [17, Proposition 2.2] for  $m = 1$  and in [9, Lemma 5.2, Lemma 7.2 & Lemma 9.2] for  $m \in (0, 1)$ , the latter being valid only for compactly supported initial data. The proofs are however of a completely different nature: in [17], properties of the linear heat equation are used while the approach in [9] relies on the intersection-comparison technique, which requires in particular the compactness of the support of the initial condition. Still for  $m = 1$  but in any space dimension, the upper bound (1.7) is derived in [12, Lemma 2.1] for radially symmetric initial data  $u_0$  having a non-increasing profile and satisfying  $\Delta u_0 + \mu u_0^q \geq 0$  in  $\mathbb{R}^N$  for some  $\mu > 0$ . The last case for which (1.7) is proved corresponds to the choice  $m = 2 - q > 1$  and the proof relies on the derivation of an Aronson-Bénilan estimate, which seems to be only available for this specific choice of the parameters  $m$  and  $q$  [15].

The purpose of this note is to contribute to the validity of (1.4) and derive optimal upper and lower bounds near the extinction time when the parameters  $m$  and  $q$  range in

$$\frac{(N-2)_+}{N} < m < q < 1. \quad (1.8)$$

Recalling that a lower bound in  $L^\infty$  is already available, see (1.6), we begin with upper bounds.

**Theorem 1.1** (Upper bounds). *Assume that  $m$  and  $q$  satisfy (1.8) and consider a non-negative initial condition  $u_0 \in BC(\mathbb{R}^N)$ ,  $u_0 \not\equiv 0$ , for which there is  $\kappa_0 > 0$  such that*

$$u_0(x) \leq \kappa_0 |x|^{-2/(q-m)}, \quad x \in \mathbb{R}^N. \quad (1.9)$$

Given  $r \in [1, \infty]$ , there is  $C_r > 0$  depending only on  $N, m, q, u_0$ , and  $r$  such that the solution  $u$  to (1.1a)-(1.1b) satisfies

$$\|u(t)\|_r \leq C_r (T_e - t)^{\alpha - (N\beta/r)}, \quad t \in (0, T_e), \quad (1.10)$$

the extinction time  $T_e$  being defined in (1.2).

Theorem 1.1 thus extends the validity of the upper bound (1.7) established in [9] for  $N = 1$  and  $r = \infty$  to any space dimension  $N \geq 1$  and  $r \in [1, \infty]$ , while relaxing the assumption of compact support required in [9]. It is worth mentioning that the validity of (1.10) for  $r \in [1, \infty)$  does not seem to be a simple consequence of (1.10) for  $r = \infty$  since  $u(t)$  is positive everywhere in  $\mathbb{R}^N$  for all  $t \in (0, T_e)$  even if  $u_0$  is compactly supported, see [9, Lemma 2.5] and Proposition 1.4 below.

To be able to cope with higher space dimensions and non-compactly supported initial data, the proof of Theorem 1.1 takes a different route from that in [9] and is carried out in two steps: we first show that the algebraic decay at infinity (1.9) enjoyed by  $u_0$  remains true throughout time evolution and combine it with (1.1a) to prove (1.10) for  $r = 1$ . We next use self-similar variables and Moser's iteration technique to derive (1.10) for all  $r \in (1, \infty]$ .

As a consequence of (1.6) and Theorem 1.1 for  $r = \infty$ , the correct time scale for the extinction phenomenon is identified. We now supplement the lower bound (1.6) in  $L^\infty$  with another one in  $L^{m+1}$ . On the one hand, it allows us to identify the right space scale. On the other hand, its derivation does not rely on the comparison principle but on energy estimates, a technique which is more likely to extend to other problems for which the former might not be available.

**Theorem 1.2** (Lower bound in  $L^{m+1}$ ). *Assume that  $m$  and  $q$  satisfy (1.8) and consider a non-negative initial condition  $u_0 \in BC(\mathbb{R}^N)$ ,  $u_0 \not\equiv 0$ , such that  $u_0 \in L^{m+1}(\mathbb{R}^N)$ . There is  $c_{m+1} > 0$  depending only on  $N, m, q$ , and  $u_0$  such that the solution  $u$  to (1.1a)-(1.1b) satisfies*

$$\|u(t)\|_{m+1} \geq c_{m+1} (T_e - t)^{\alpha - (N\beta/(m+1))}, \quad t \in (0, T_e), \quad (1.11)$$

the extinction time  $T_e$  being defined in (1.2).

Observing that Theorems 1.1 and 1.2 are shown without using the  $L^\infty$ -lower bound (1.6), the latter may be recovered from these two results by Hölder's inequality, with a less explicit constant though.

**Corollary 1.3.** *Assume that  $m$  and  $q$  satisfy (1.8) and consider a non-negative initial condition  $u_0 \in BC(\mathbb{R}^N)$ ,  $u_0 \not\equiv 0$ , enjoying the decay property (1.9). For  $r \in (m+1, \infty]$ , there is  $c_r > 0$  depending only on  $N, m, q, u_0$ , and  $r$  such that the solution  $u$  to (1.1a)-(1.1b) satisfies*

$$\|u(t)\|_r \geq c_r (T_e - t)^{\alpha - (N\beta/r)}, \quad t \in (0, T_e). \quad (1.12)$$

Summarizing the outcome of Theorem 1.1, Theorem 1.2, and Corollary 1.3, we have shown that, for all non-negative initial data  $u_0 \in BC(\mathbb{R}^N)$ ,  $u_0 \not\equiv 0$ , enjoying the decay property (1.9), the corresponding solution  $u$  to (1.1a)-(1.1b) is bounded in  $L^r(\mathbb{R}^N)$ ,  $r \in [m+1, \infty]$ , from above and from below at time  $t \in (0, T_e)$  by the same power of  $T_e - t$ . Such estimates pave the way towards a more precise description of the behaviour of  $u(t)$  as  $t \rightarrow T_e$ , which is expected to be self-similar. That this

is indeed the case is shown in [8, 9] in one space dimension, another building block of the proof being the uniqueness of self-similar solutions [8].

We end up this note with the already mentioned everywhere positivity of solutions to (1.1a)-(1.1b) for positive times prior to the extinction time. As we shall see below, this property holds true for a wider range of the parameters  $m$  and  $q$ , namely  $0 < m \leq q < 1$ . It is already observed in [9, Lemma 2.5] in one space dimension and we extend it herein to any space dimension. It is worth emphasizing that it includes the case  $q = m$  and contrasts markedly with the instantaneous shrinking of the support occurring when  $q < m$ .

**Proposition 1.4** (Everywhere positivity). *Consider  $0 < m \leq q < 1$ . Let  $u_0 \in BC(\mathbb{R}^N)$  be a non-negative initial condition,  $u_0 \not\equiv 0$ , and denote the corresponding solution to (1.1a)-(1.1b) by  $u$  with extinction time  $T_e$ . For  $t \in (0, T_e)$ , there holds*

$$\mathcal{P}(t) := \{x \in \mathbb{R}^N : u(t, x) > 0\} = \mathbb{R}^N . \quad (1.13)$$

Before proving the results stated above, we point out once more that the energy techniques developed herein seem to be rather flexible and are expected to have a wider range of applicability. For instance, a related approach is used in the companion paper [18], where optimal (lower and upper) bounds near the extinction time are established for a different fast diffusion equation (featuring the  $p$ -Laplacian operator,  $p \in (1, 2)$ ) with a gradient absorption term.

## 2. UPPER BOUNDS NEAR THE EXTINCTION TIME

Throughout this section, we assume that  $m$  and  $q$  satisfy (1.8) and consider a non-negative initial condition  $u_0 \in BC(\mathbb{R}^N)$ ,  $u_0 \not\equiv 0$ , enjoying the decay property (1.9). Let  $u$  be the corresponding solution to (1.1a)-(1.1b).

**2.1.  $L^1$ -estimate.** We begin with the propagation throughout time evolution of the algebraic decay (1.9) and set

$$\kappa_* := \left( \frac{2m(m+q)}{(q-m)^2} \right)^{1/(q-m)} . \quad (2.1)$$

**Lemma 2.1.** *For  $t \in [0, \infty)$  and  $x \in \mathbb{R}^N \setminus \{0\}$ , there holds*

$$u(t, x) \leq \max\{\kappa_0, \kappa_*\} |x|^{-2/(q-m)} .$$

*Proof.* Set  $\Sigma_\kappa(x) := \kappa |x|^{-2/(q-m)}$  for  $x \in \mathbb{R}^N \setminus \{0\}$ , where  $\kappa$  is a positive constant yet to be determined. We note that

$$\begin{aligned} -\Delta \Sigma^m(x) + \Sigma(x)^q &= -\kappa^m \left[ \frac{2m(m+q)}{(q-m)^2} |x|^{-2q/(q-m)} - \frac{2m(N-1)}{(q-m)} |x|^{-2q/(q-m)} \right] \\ &\quad + \kappa^q |x|^{-2q/(q-m)} \\ &\geq \kappa^m (\kappa^{q-m} - \kappa_*^{q-m}) |x|^{-2q/(q-m)} \end{aligned}$$

for  $x \in \mathbb{R}^N \setminus \{0\}$ , so that  $\Sigma_\kappa$  is a supersolution to (1.1a) in  $\mathbb{R}^N \setminus \{0\}$  for all  $\kappa \geq \kappa_*$ . We then choose  $\kappa = \max\{\kappa_0, \kappa_*\}$  and use the comparison principle to complete the proof of Lemma 2.1.  $\square$

We are now in a position to derive the claimed upper bound near the extinction time for  $r = 1$ .

*Proof of Theorem 1.1:  $r = 1$ .* Let  $t \in [0, T_e)$ . Integrating (1.1a) over  $(t, T_e) \times \mathbb{R}^N$  gives

$$\|u(t)\|_1 = \int_t^{T_e} \int_{\mathbb{R}^N} u(s, x)^q dx ds . \quad (2.2)$$

Owing to (1.8), there holds  $2q/(q - m) > N$  and we infer from Lemma 2.1 and Hölder's inequality that, for  $s \in (t, T_e)$  and  $R > 0$ ,

$$\begin{aligned} \int_{\mathbb{R}^N} u(s, x)^q dx &= \int_{B(0, R)} u(s, x)^q dx + \int_{\mathbb{R}^N \setminus B(0, R)} u(s, x)^q dx \\ &\leq CR^{N(1-q)} \|u(s)\|_1^q + (\max\{\kappa_0, \kappa_*\})^q |\mathbb{S}^{N-1}| \int_R^\infty r^{N-1-(2q/(q-m))} dr \\ &\leq C \left( R^{N(1-q)} \|u(s)\|_1^q + R^{(N(q-m)-2q)/(q-m)} \right) . \end{aligned}$$

We next optimize in  $R$  in the previous inequality by setting  $R(s) := \|u(s)\|_1^{-(q-m)/(N(m-q)+2)}$ , which satisfies

$$R(s)^{N(1-q)} \|u(s)\|_1^q = R(s)^{(N(q-m)-2q)/(q-m)} = \|u(s)\|_1^{(N(m-q)+2q)/(N(m-q)+2)} .$$

Consequently, taking  $R = R(s)$  in the previous inequality, we obtain

$$\int_{\mathbb{R}^N} u(s, x)^q dx \leq C \|u(s)\|_1^{(N(m-q)+2q)/(N(m-q)+2)} ,$$

which gives, together with (2.2), the positivity of  $N(m - q) + 2q$ , and the time monotonicity of  $s \mapsto \|u(s)\|_1$ ,

$$\begin{aligned} \|u(t)\|_1 &\leq C \int_t^{T_e} \|u(s)\|_1^{(N(m-q)+2q)/(N(m-q)+2)} ds \\ &\leq C(T_e - t) \|u(t)\|_1^{(N(m-q)+2q)/(N(m-q)+2)} , \end{aligned}$$

from which (1.10) for  $r = 1$  readily follows.  $\square$

**2.2. Scaling variables and  $L^r$ -estimates,  $r \in (1, \infty]$ .** The next step is to take advantage of the just derived  $L^1$ -upper bound to derive the corresponding ones in  $L^r$  for  $r \in (1, \infty]$ . To this end, we introduce the scaling variables

$$s := \ln \left( \frac{T_e}{T_e - t} \right) , \quad y := x(T_e - t)^\beta , \quad (t, x) \in [0, T_e) \times \mathbb{R}^N , \quad (2.3)$$

and the new unknown function  $v$  defined by

$$u(t, x) = (T_e - t)^\alpha v \left( \ln(T_e) - \ln(T_e - t), x(T_e - t)^\beta \right) , \quad (t, x) \in [0, T_e) \times \mathbb{R}^N , \quad (2.4)$$

or, equivalently,

$$v(s, y) = T_e^{-\alpha} e^{\alpha s} u \left( T_e(1 - e^{-s}), y T_e^{-\beta} e^{\beta s} \right) , \quad (s, y) \in [0, \infty) \times \mathbb{R}^N . \quad (2.5)$$

It readily follows from (1.1a)-(1.1b) that  $v$  solves

$$\partial_s v(s, y) = \alpha v(s, y) + \beta y \cdot \nabla v(s, y) + \Delta v^m(s, y) - v(s, y)^q, \quad (s, y) \in (0, \infty) \times \mathbb{R}^N, \quad (2.6)$$

$$v(0, y) = v_0(y) := T_e^{-\alpha} u_0(y T_e^{-\beta}), \quad y \in \mathbb{R}^N. \quad (2.7)$$

Since

$$\|u(t)\|_r = (T_e - t)^{\alpha - (N\beta/r)} \|v(s)\|_r, \quad t \in (0, T_e), \quad (2.8)$$

for all  $r \in [1, \infty]$ , we realize that an upper bound such as (1.10) on  $\|u(t)\|_r$  for  $t \in (0, T_e)$  obviously follows from a uniform upper bound on  $\|v(s)\|_r$  for  $s \geq 0$ , the converse being true as well. In particular, it follows from (2.8) and Theorem 1.1 for  $r = 1$  that

$$\|v(s)\|_1 \leq C_1, \quad s \geq 0, \quad (2.9)$$

and we may assume without loss of generality that  $C_1 \geq 1$ .

We now aim at using a bootstrap argument to deduce from (2.6) and (2.9) that  $v$  belongs to  $L^\infty(0, \infty; L^r(\mathbb{R}^N))$  for all  $r \in (1, \infty]$ . To this end, Moser's iteration technique is a suitable tool and the way we apply it is inspired from [2, Theorem 3.1]. But since [2, Theorem 3.1] is devoted to the slow diffusion case  $m > 1$ , some technical aspects of its proof do not seem to apply directly here and we borrow additional arguments from the proof of [3, Proposition 2].

**Lemma 2.2.** *Let  $r \in (0, \infty]$ . There is  $C_{r+1} > 0$  depending only on  $N, m, q, u_0$ , and  $r$  such that*

$$\|v(s)\|_{r+1} \leq C_{r+1}, \quad s \geq 0.$$

*Proof.* Let  $r \in [2 - m, \infty)$ . Multiplying (2.6) by  $v^r$ , integrating over  $\mathbb{R}^N$ , and using integration by parts, we obtain

$$\begin{aligned} \frac{1}{r+1} \frac{d}{ds} \|v\|_{r+1}^{r+1} + rm \int_{\mathbb{R}^N} v^{r+m-2} |\nabla v|^2 dy + \int_{\mathbb{R}^N} v^{r+q} dy &= \left( \alpha - \frac{N\beta}{r+1} \right) \|v\|_{r+1}^{r+1}, \\ \frac{d}{ds} \|v\|_{r+1}^{r+1} + \frac{4mr(r+1)}{(m+r)^2} \|\nabla v^{(m+r)/2}\|_2^2 &\leq \alpha(r+1) \|v\|_{r+1}^{r+1}. \end{aligned}$$

Since  $4mr(r+1) \geq 2m(m+r)^2$ , we end up with

$$\frac{d}{ds} \|v\|_{r+1}^{r+1} + 2m \|\nabla v^{(m+r)/2}\|_2^2 \leq \alpha(r+1) \|v\|_{r+1}^{r+1}. \quad (2.10)$$

We next fix  $\zeta \in (2/m, 2^*)$  where  $2^* := 2N/(N-2)_+$  (with  $2^* = \infty$  for  $N = 1, 2$ ). On the one hand, it follows from the Gagliardo-Nirenberg inequality that

$$\|v^{(m+r)/2}\|_\zeta \leq C \|\nabla v^{(m+r)/2}\|_2^\theta \|v^{(m+r)/2}\|_1^{1-\theta}, \quad (2.11)$$

with

$$\theta := \frac{2N(\zeta - 1)}{(N + 2)\zeta}.$$

On the other hand, since  $(m+r)/2 \in [1, \zeta(m+r)/2]$  for  $r \geq 2 - m$ , we infer from Hölder's inequality that

$$\|v\|_{(m+r)/2}^{(m+r)/2} \leq \|v\|_{\zeta(m+r)/2}^{\zeta(m+r)(m+r-2)/2[\zeta(m+r)-2]} \|v\|_1^{(\zeta-1)(m+r)/[\zeta(m+r)-2]}. \quad (2.12)$$

We deduce from (2.11) and (2.12) that

$$\begin{aligned} \|v\|_{\zeta(m+r)/2}^{(m+r)/2} &= \|v^{(m+r)/2}\|_{\zeta} \leq C \|\nabla v^{(m+r)/2}\|_2^\theta \left( \|v\|_{\zeta(m+r)/2}^{(m+r)/2} \right)^{1-\theta} \\ &\leq C \|\nabla v^{(m+r)/2}\|_2^\theta \left[ \|v\|_{\zeta(m+r)/2}^{\zeta(m+r)(m+r-2)/2[\zeta(m+r)-2]} \|v\|_1^{(\zeta-1)(m+r)/[\zeta(m+r)-2]} \right]^{1-\theta}, \end{aligned}$$

hence

$$\|v\|_{\zeta(m+r)/2}^{\zeta(m+r)[N(m+r)+2-N]/N[\zeta(m+r)-2]} \leq C \|\nabla v^{(m+r)/2}\|_2^2 \|v\|_1^{[2N-(N-2)\zeta](m+r)/N[\zeta(m+r)-2]}. \quad (2.13)$$

Moreover, since  $\zeta > 2/m$  and  $m < 1$ , we have  $2r \leq m[\zeta(m+r) - 2]$ , hence

$$\begin{aligned} \frac{[2N - (N - 2)\zeta](m + r)}{N[\zeta(m + r) - 2]} &\leq \frac{[2N - (N - 2)\zeta] m(m + r)}{N \cdot 2r} \\ &\leq \frac{m[2N - (N - 2)\zeta]}{N}, \end{aligned}$$

so that

$$\begin{aligned} \|v\|_1^{[2N-(N-2)\zeta](m+r)/N[\zeta(m+r)-2]} &\leq C_1^{[2N-(N-2)\zeta](m+r)/N[\zeta(m+r)-2]} \\ &\leq C_1^{m[2N-(N-2)\zeta]/N}. \end{aligned} \quad (2.14)$$

Also,

$$1 - \frac{N[\zeta(m+r) - 2]}{\zeta[N(m+r) + 2 - N]} = \frac{2N - \zeta(N - 2)}{\zeta[N(m+r) + 2 - N]} > 0,$$

and we infer from (2.13), (2.14), and Young's inequality that

$$\begin{aligned} \|v\|_{\zeta(m+r)/2}^{m+r} &\leq \frac{N[\zeta(m+r) - 2]}{\zeta[N(m+r) + 2 - N]} \|v\|_{\zeta(m+r)/2}^{\zeta(m+r)[N(m+r)+2-N]/N[\zeta(m+r)-2]} \\ &\quad + \frac{2N - \zeta(N - 2)}{\zeta[N(m+r) + 2 - N]} \\ &\leq C \|\nabla v^{(m+r)/2}\|_2^2 + 1. \end{aligned}$$

Therefore, there is  $\nu \in (0, 1)$  depending only on  $N$ ,  $m$ ,  $q$ , and  $u_0$  such that

$$\nu \left( \|v\|_{\zeta(m+r)/2}^{m+r} - 1 \right) \leq \|\nabla v^{(m+r)/2}\|_2^2. \quad (2.15)$$

Moreover, since  $r + 1 \in [1, \zeta(m + r)/2]$ , it follows from (2.9) and Hölder's and Young's inequalities that

$$\begin{aligned}
\|v\|_{r+1}^{r+1} &\leq \|v\|_{\zeta(m+r)/2}^{\zeta r(r+m)/[\zeta(r+m)-2]} \|v\|_1^{[(\zeta-2)r+\zeta m-2]/(\zeta r+\zeta m-2)} \\
&\leq C_1^{[(\zeta-2)r+\zeta m-2]/(\zeta r+\zeta m-2)} \|v\|_{\zeta(m+r)/2}^{\zeta r(r+m)/[\zeta(r+m)-2]} \\
&\leq C_1 \|v\|_{\zeta(m+r)/2}^{\zeta r(r+m)/[\zeta(r+m)-2]} \\
&\leq \frac{\zeta m - 2}{\zeta(m+r) - 2} C_1^{[\zeta(m+r)-2]/(\zeta m-2)} + \frac{\zeta r}{\zeta(m+r) - 2} \|v\|_{\zeta(m+r)/2}^{m+r} \\
&\leq C_1^{[\zeta(m+r)-2]/(\zeta m-2)} + \|v\|_{\zeta(m+r)/2}^{m+r}
\end{aligned} \tag{2.16}$$

Next, let  $\sigma > 1$  to be chosen appropriately later on and set

$$\mathcal{I}_r(s) := \int_{\mathbb{R}^N} v(s, y)^{\zeta[r+m+\sigma(1-m)]/[\sigma(\zeta-2)+2]} dy, \quad s \geq 0.$$

Since  $\sigma(\zeta - 2) + 2 \in [\zeta, \sigma\zeta]$  and

$$r + 1 = \frac{\sigma - 1}{\sigma}(m + r) + \frac{m + r + \sigma(1 - m)}{\sigma},$$

we deduce from (2.9) and Hölder's and Young's inequalities that, for  $\delta > 0$ ,

$$\begin{aligned}
\|v\|_{r+1}^{r+1} &\leq \|v\|_{\zeta(m+r)/2}^{(\sigma-1)(m+r)/\sigma} \mathcal{I}_r^{[\sigma(\zeta-2)+2]/\sigma\zeta} \\
&\leq \frac{(\sigma - 1)\delta}{\sigma} \|v\|_{\zeta(m+r)/2}^{m+r} + \frac{1}{\sigma\delta^{\sigma-1}} \mathcal{I}_r^{[\sigma(\zeta-2)+2]/\zeta} \\
&\leq \delta \|v\|_{\zeta(m+r)/2}^{m+r} + \frac{1}{\delta^{\sigma-1}} \mathcal{I}_r^{[\sigma(\zeta-2)+2]/\zeta}.
\end{aligned} \tag{2.17}$$

Combining (2.10), (2.15), and (2.17) leads us to

$$\begin{aligned}
\frac{d}{ds} \|v\|_{r+1}^{r+1} + 2m\nu \left( \|v\|_{\zeta(m+r)/2}^{m+r} - 1 \right) &\leq \frac{d}{ds} \|v\|_{r+1}^{r+1} + 2m \|\nabla v^{(m+r)/2}\|_2^2 \\
&\leq \alpha(r + 1) \|v\|_{r+1}^{r+1} \\
&\leq \alpha\delta(r + 1) \|v\|_{\zeta(m+r)/2}^{m+r} + \frac{\alpha(r + 1)}{\delta^{\sigma-1}} \mathcal{I}_r^{[\sigma(\zeta-2)+2]/\zeta}.
\end{aligned}$$

We then choose  $\delta = m\nu/\alpha(r + 1)$  in the above inequality to obtain

$$\frac{d}{ds} \|v\|_{r+1}^{r+1} + m\nu \|v\|_{\zeta(m+r)/2}^{m+r} \leq 2m\nu + \frac{\alpha^\sigma(r + 1)^\sigma}{(m\nu)^{\sigma-1}} \mathcal{I}_r^{[\sigma(\zeta-2)+2]/\zeta}.$$

We finally use (2.16) to estimate from below the second term of the left-hand side of the previous inequality and end up with

$$\frac{d}{ds} \|v\|_{r+1}^{r+1} + m\nu \|v\|_{r+1}^{r+1} \leq 2m\nu + m\nu C_1^{[\zeta(m+r)-2]/(\zeta m-2)} + \frac{\alpha^\sigma(r + 1)^\sigma}{(m\nu)^{\sigma-1}} \mathcal{I}_r^{[\sigma(\zeta-2)+2]/\zeta}. \tag{2.18}$$

We first choose

$$\sigma = \frac{\zeta(m+r) - 2}{\zeta m - 2} > 1$$

in (2.18) and observe that this choice guarantees that

$$\zeta[m+r+\sigma(1-m)] = \sigma(\zeta - 2) + 2 .$$

Consequently,  $\mathcal{I}_r = \|v\|_1$  and we deduce from (2.9) and (2.18) that there is  $C(r) > 0$  depending on  $N, m, q, u_0$ , and  $r$  such that

$$\frac{d}{ds} \|v\|_{r+1}^{r+1} + m\nu \|v\|_{r+1}^{r+1} \leq C(r) .$$

Integrating the previous differential inequality entails that

$$\sup_{s \geq 0} \|v(s)\|_{r+1} < \infty . \quad (2.19)$$

The validity of (2.19) extends to all  $r \in (0, 2-m)$  by (2.9) and Hölder's inequality.

To complete the proof of Lemma 2.2, we are left to check the boundedness of  $v$  in  $L^\infty(\mathbb{R}^N)$ . To this end, we take  $\sigma = \sigma_0 := 2(\zeta - 1)/(\zeta - 2) > 1$  in (2.18) and obtain, after integration with respect to time,

$$\begin{aligned} \|v(s)\|_{r+1}^{r+1} &\leq \|v_0\|_{r+1}^{r+1} e^{-m\nu s} + 2 + C_1^{[\zeta(m+r)-2]/(\zeta m-2)} \\ &\quad + \left( \frac{\alpha(r+1)}{m\nu} \right)^{\sigma_0} \left[ \sup_{s_* \in [0, s]} \mathcal{I}_r(s_*) \right]^2 \\ &\leq \|v_0\|_1 \|v_0\|_\infty^r + 2 + C_1^{[\zeta(m+r)-2]/(\zeta m-2)} \\ &\quad + \left( \frac{\alpha(r+1)}{m\nu} \right)^{\sigma_0} \left[ \sup_{s_* \in [0, s]} \mathcal{I}_r(s_*) \right]^2 , \end{aligned}$$

and

$$\mathcal{I}_r = \|v\|_{[(r+m)+\sigma_0(1-m)]/2}^{[(r+m)+\sigma_0(1-m)]/2} .$$

Therefore, there are  $K_0 > 0$  and  $K_1 > 0$  depending only on  $N, m, q$ , and  $u_0$  such that

$$\sup_{s \geq 0} \{ \|v(s)\|_{r+1}^{r+1} \} \leq K_0 \left( K_1^{r+1} + (1+r)^{\sigma_0} \sup_{s \geq 0} \left\{ \|v(s)\|_{[(r+m)+\sigma_0(1-m)]/2}^{[(r+m)+\sigma_0(1-m)]} \right\} \right) . \quad (2.20)$$

We now define the sequence  $(r_j)_{j \geq 0}$  by

$$1 + r_{j+1} = 2(1 + r_j) - (1 - m)(\sigma_0 - 1) , \quad j \geq 0 , \quad r_0 := 2 - m ,$$

and set

$$V_j := \sup_{s \geq 0} \left\{ \|v(s)\|_{r_j+1}^{r_j+1} \right\} , \quad j \geq 0 .$$

For  $j \geq 0$ , we take  $r = r_{j+1}$  in (2.20) and realize that

$$\begin{aligned} V_{j+1} &\leq K_0 \left( K_1^{1+r_{j+1}} + (1+r_{j+1})^{\sigma_0} V_j^2 \right) \\ &\leq K_0 (1+r_{j+1})^{\sigma_0} \max \left\{ K_1^{1+r_{j+1}}, V_j^2 \right\}, \quad j \geq 0. \end{aligned}$$

Since  $\sigma_0 - 1 < 1/(1-m)$  thanks to the constraint  $\zeta > 2/m$ , one has  $1+r_0 - (1-m)(\sigma_0 - 1) > 0$  and we are in a position to apply [22, Lemma A.1], which we recall in Lemma 2.3 below for completeness, to conclude that there is  $K_2 > 0$  depending only on  $m$ ,  $\zeta$ ,  $K_0$ , and  $K_1$  such that

$$V_j^{1/(1+r_j)} \leq K_2, \quad j \geq 0.$$

Equivalently,

$$\sup_{s \geq 0} \{ \|v(s)\|_{1+r_j} \} \leq K_2, \quad j \geq 0,$$

and letting  $j \rightarrow \infty$  entails that  $\|v(s)\|_\infty \leq K_2$  for all  $s \geq 0$ , thereby completing the proof of Lemma 2.2.  $\square$

The proof of Theorem 1.1 for  $r \in (1, \infty]$  is now a straightforward consequence of (2.8) and Lemma 2.2.

**Lemma 2.3.** *Let  $a > 1$ ,  $b \geq 0$ ,  $c \in \mathbb{R}$ ,  $C_0 \geq 1$ ,  $C_1 \geq 1$ , and  $p_0 > 0$  be given such that  $p_0(a-1)+c > 0$ . We define the sequence  $(p_k)_{k \geq 0}$  of positive real numbers by  $p_{k+1} = ap_k + c$  for  $k \geq 0$  and assume that  $(Q_k)_{k \geq 0}$  is a sequence of positive real numbers satisfying*

$$Q_0 \leq C_1^{p_0}, \quad Q_{k+1} \leq C_0 p_{k+1}^b \max \{ C_1^{p_{k+1}}, Q_k^a \}, \quad k \geq 0.$$

*Then the sequence  $(Q_k^{1/p_k})_{k \geq 0}$  is bounded.*

### 3. LOWER BOUND NEAR THE EXTINCTION TIME

We now turn to the lower bound near the extinction time in  $L^{m+1}(\mathbb{R}^N)$ .

*Proof of Theorem 1.2.* For  $t \in [0, T_e]$ , we define

$$X(t) := \|u(t)\|_{m+1}^{m+1} \quad \text{and} \quad Y(t) := \int_{\mathbb{R}^N} u(t, x)^{m+q} dx.$$

Let  $t \in (0, T_e)$ . It follows from (1.1a) that

$$\frac{1}{m+1} \frac{dX}{dt}(t) + \|\nabla u^m(t)\|_2^2 + Y(t) = 0. \quad (3.1)$$

Since

$$1 < \frac{m+q}{m} < \frac{m+1}{m} < 2^* := \frac{2N}{(N-2)_+}$$

by (1.8) we infer from the Gagliardo-Nirenberg inequality that

$$\begin{aligned} X(t)^{m/(m+1)} &= \|u(t)^m\|_{(m+1)/m} \leq C \|\nabla u^m(t)\|_2^\theta \|u(t)^m\|_{(m+q)/m}^{1-\theta} \\ &\leq CY(t)^{m(1-\theta)/(m+q)} \|\nabla u^m(t)\|_2^\theta, \end{aligned}$$

where

$$\theta := \frac{2Nm(1-q)}{(m+1)[m(N+2) - q(N-2)]}.$$

Consequently, since  $u(t) \not\equiv 0$  as  $t \in (0, T_e)$ ,

$$\|\nabla u^m(t)\|_2^2 \geq CX(t)^{2m/\theta(m+1)} Y(t)^{-2m(1-\theta)/\theta(m+q)},$$

which gives, together with (3.1),

$$\frac{dX}{dt}(t) + CX(t)^{2m/\theta(m+1)} Y(t)^{-2m(1-\theta)/\theta(m+q)} + (m+1)Y(t) \leq 0. \quad (3.2)$$

Setting

$$\xi := 1 + \frac{2m(1-\theta)}{\theta(m+q)} > 1 \quad \text{and} \quad \gamma := \frac{2m}{\theta\xi(m+1)},$$

it follows from Young's inequality that

$$\begin{aligned} X(t)^\gamma &= X(t)^\gamma Y(t)^{-(\xi-1)/\xi} Y(t)^{(\xi-1)/\xi} \leq \frac{1}{\xi} X(t)^{\xi\gamma} Y(t)^{1-\xi} + \frac{\xi-1}{\xi} Y(t) \\ &\leq X(t)^{2m/\theta(m+1)} Y(t)^{-2m(1-\theta)/\theta(m+q)} + Y(t) \end{aligned}$$

Combining this inequality with (3.2) leads us to the differential inequality

$$\frac{dX}{dt}(t) + CX(t)^\gamma \leq 0, \quad t \in (0, T_e). \quad (3.3)$$

Now,

$$\begin{aligned} \gamma &= \frac{2m}{\theta\xi(m+1)} = \frac{2m(m+q)}{(2m + \theta(q-m))(m+1)} \\ &= \frac{2m(m+q)[m(N+2) - q(N-2)]}{2m\{(m+1)[m(N+2) - q(N-2)] + N(q-m)(1-q)\}} \\ &= \frac{m(N+2) - q(N-2)}{m(N+2) - qN + 2} \in (0, 1), \end{aligned}$$

and we integrate (3.3) over  $(t, T_e)$  to obtain

$$-X(t)^{1-\gamma} + (1-\gamma)C(T_e - t) \leq 0, \quad t \in (0, T_e).$$

Noticing that

$$(m+1)\alpha - N\beta = \frac{m+1}{1-q} - \frac{N(q-m)}{2(1-q)} = \frac{m(N+2) - qN + 2}{2(1-q)} = \frac{1}{1-\gamma},$$

the lower bound (1.11) readily follows from the previous inequality.  $\square$

We end up this section with the derivation of the lower bound for  $r \in (m+1, \infty]$  from Theorem 1.1 for  $r = 1$  and Theorem 1.2.

*Proof of Corollary 1.3.* We first note that, owing to (1.8), there holds  $2/(q-m) > N$  and (1.9) entails that  $u_0 \in L^1(\mathbb{R}^N)$ . Since  $u_0$  also belongs to  $L^\infty(\mathbb{R}^N)$ , we conclude that  $u_0 \in L^{m+1}(\mathbb{R}^N)$ .

Let  $r \in (m+1, \infty]$  and  $t \in (0, T_e)$ . We infer from Theorem 1.1, Theorem 1.2, and Hölder's inequality that

$$\begin{aligned} c_{m+1}^{m+1}(T_e - t)^{(m+1)\alpha - N\beta} &\leq \|u(t)\|_{m+1}^{m+1} \leq \|u(t)\|_r^{rm/(r-1)} \|u(t)\|_1^{(r-1-m)/(r-1)} \\ &\leq C_1(T_e - t)^{(\alpha - N\beta)(r-1-m)/(r-1)} \|u(t)\|_r^{rm/(r-1)}, \end{aligned}$$

from which (1.12) readily follows.  $\square$

#### 4. EVERYWHERE POSITIVITY

In this section, we assume that  $0 < m \leq q < 1$  and consider a non-negative initial condition  $u_0 \in BC(\mathbb{R}^N)$ ,  $u_0 \not\equiv 0$ . We denote the corresponding solution to (1.1a)-(1.1b) by  $u$  and define its extinction time by (1.2). As in [9], the proof relies on an upper bound for  $\partial_t u$  which we establish now.

**Lemma 4.1.** *For  $t > 0$  there holds*

$$\partial_t u(t) \leq \frac{u(t)}{(1-m)t} \quad \text{in } \mathbb{R}^N.$$

When  $q = m$ , Lemma 4.1 is a consequence of [4, Theorem 2], the proof relying on an homogeneity argument. Though the operator  $-\Delta u^m + u^q$  is not homogeneous, we may still adapt the proof of [4, Theorem 2] when  $q \geq m$ .

*Proof.* Given a non-negative initial condition  $u_0 \in BC(\mathbb{R}^N)$ , we denote the corresponding solution to (1.1a)-(1.1b) at time  $t \geq 0$  by  $S(t)u_0$ . Recall that, if  $u_0$  and  $v_0$  are two non-negative functions in  $BC(\mathbb{R}^N)$  satisfying  $u_0 \geq v_0$ , then the comparison principle entails  $S(t)u_0 \geq S(t)v_0$  for all  $t \geq 0$ .

**Step 1.** We first claim that, for  $\lambda \geq 1$ ,

$$S(\lambda t)u_0 \leq \lambda^{1/(1-m)} S(t) (\lambda^{1/(m-1)} u_0), \quad t \geq 0. \quad (4.1)$$

Indeed, setting  $u(t) := S(t)u_0$  for  $t \geq 0$ , the function  $v$  defined by  $v(t) := \lambda^{1/(m-1)} S(\lambda t)u_0$  satisfies

$$\begin{aligned} \partial_t v(t, x) - \Delta v^m(t, x) + v(t, x)^q &= \lambda^{m/(m-1)} \partial_t u(\lambda t, x) - \lambda^{m/(m-1)} \Delta u^m(\lambda t, x) \\ &\quad + \lambda^{q/(m-1)} u(\lambda t, x)^q \\ &= (\lambda^{q/(m-1)} - \lambda^{m/(m-1)}) u(\lambda t, x)^q \leq 0. \end{aligned}$$

Since  $v(0) = \lambda^{1/(m-1)} u_0 \leq u_0$ , we infer from the comparison principle that (4.1) holds true.

**Step 2.** Now, fix  $t > 0$  and consider  $h > 0$ . Since  $\lambda = (1 + h/t) > 1$  and  $m \in (0, 1)$ , we infer from (4.1) and the comparison principle that

$$\begin{aligned} S(t+h)u_0 - S(t)u_0 &= S(\lambda t)u_0 - S(t)u_0 \\ &\leq \lambda^{1/(1-m)} S(t) (\lambda^{1/(m-1)} u_0) - S(t)u_0 \\ &\leq \left[ \left(1 + \frac{h}{t}\right)^{1/(1-m)} - 1 \right] S(t)u_0 . \end{aligned}$$

Dividing the above inequality by  $h$  and passing to the limit as  $h \rightarrow 0$  complete the proof.  $\square$

We now argue as in the proof of [9, Lemma 2.5] to complete the proof of Proposition 1.4.

*Proof of Proposition 1.4.* Fix  $t \in (0, T_e)$  and assume for contradiction that  $u(t, x_0) = 0$  for some  $x_0 \in \mathbb{R}^N$ . By (1.1a) and Lemma 4.1, there holds

$$-\Delta u^m(t) + u(t)^q + \frac{u(t)}{(1-m)t} \geq 0 \quad \text{in } \mathbb{R}^N ,$$

so that  $u(t)^m$  is a supersolution to

$$-\Delta w + dw = 0 \quad \text{in } \mathbb{R}^N ,$$

with  $d(x) := u(t, x)^{q-m} + u(t, x)^{1-m}/((1-m)t)$  for  $x \in \mathbb{R}^N$ . Since  $t > 0$  and  $m \leq q < 1$ , the function  $d$  is non-negative and bounded and we infer from the strong maximum principle [16, Theorem 8.19] that  $u(t)^m \equiv 0$  in  $\mathbb{R}^N$ , contradicting  $t < T_e$ . Consequently,  $u(t)^m$  is positive everywhere in  $\mathbb{R}^N$  and the proof of Proposition 1.4 is complete.  $\square$

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