



HAL
open science

On a stochastic Hardy-Littlewood-Sobolev inequality with application to Strichartz estimates for a noisy dispersion

Romain Duboscq, Anthony Réveillac

► **To cite this version:**

Romain Duboscq, Anthony Réveillac. On a stochastic Hardy-Littlewood-Sobolev inequality with application to Strichartz estimates for a noisy dispersion. *Annales Henri Lebesgue*, 2022, 5, pp.263-274. 10.5802/ahl.122 . hal-01636293v2

HAL Id: hal-01636293

<https://hal.science/hal-01636293v2>

Submitted on 9 Jan 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On a stochastic Hardy-Littlewood-Sobolev inequality with application to Strichartz estimates for a noisy dispersion

Romain Duboscq ^{*} Anthony Réveillac [†]

INSA de Toulouse [‡]
IMT UMR CNRS 5219
Université de Toulouse

Abstract

In this paper, we investigate a stochastic Hardy-Littlewood-Sobolev inequality. Due to the non-homogenous nature of the potential in the inequality, a constant proportional to the length of the interval appears on the right-hand-side. As a direct application, we derive local Strichartz estimates for randomly modulated dispersions and solve the Cauchy problem of the critical nonlinear Schrödinger equation.

Key words: Stochastic regularization; Stochastic Partial Differential Equations; Non-linear Schrödinger equation, Hardy-Littlewood-Sobolev inequality.

AMS 2010 subject classification: Primary: 35Q55; Secondary: 60H15.

1 Introduction

Let (Ω, \mathbb{P}) be the standard probability space endowed with the Wiener filtration $(\mathcal{F}_t)_{t \geq 0}$. We consider the stochastic process W^H , a fractional brownian motion with Hurst parameter $H \in (0, 1)$, given by, $\forall t \in \mathbb{R}^+$,

$$W_t^H = \int_{-\infty}^t ((t-s)_+^H - (-s)_+^H) dW_s,$$

where W is a standard wiener process. The main objective of this paper is to derive a stochastic counter-part to the classical Hardy-Littlewood-Sobolev inequality [21, 22, 28]. To be more specific, we obtain the following result.

Theorem 1.1. *Let $(W_t^H)_{t \geq 0}$ be a fractional brownian motion of Hurst index $H \in (0, 1)$, $\beta \in (0, 1 - H)$, $p, q \in (1, \infty)$ and $\alpha \in (0, 1)$ such that*

$$2 - \alpha = \frac{1}{p} + \frac{1}{q}.$$

Then, there exist $T > 0$ and $C_{1.1} > 0$ such that, \mathbb{P} -a.s., $\forall f \in L^p([0, T])$, $\forall g \in L^q([0, T])$ the following inequality holds

$$\left| \int_0^T \int_0^T f(t) |W_t^H - W_s^H|^{-\alpha} g(s) ds dt \right| \leq C_{1.1} T^{\alpha\beta} \|f\|_{L^p([0, T])} \|g\|_{L^q([0, T])}. \quad (1.1)$$

^{*}romain.duboscq@insa-toulouse.fr

[†]anthony.reveillac@insa-toulouse.fr

[‡]135 avenue de Rangueil 31077 Toulouse Cedex 4 France

Our motivation to prove such result stems from the Cauchy problem of nonlinear evolution equations with a randomly modulated dispersion. Such equations are for instance: the nonlinear Schrödinger equation [13, 14, 8, 3], the Korteweg-de Vries equation [8] and the Benjamin-Bona-Mahony [7]. They have recently raised an interest due to the effects of the stochastic modulation. Here, we address the local Cauchy problem for the following nonlinear Schrödinger equation with noisy dispersion in its mild formulation

$$\psi(t, x) = P_{0,t}\psi_0(x) + \lambda \int_0^t P_{s,t}|\psi|^{2\sigma}\psi(s, x)ds, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^d, \quad (1.2)$$

where $\lambda \in \mathbb{C}$ and, $\forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$,

$$P_{s,t}\varphi := \mathcal{F}^{-1} \left(e^{-i|\xi|^2(W_t^H - W_s^H)} \hat{\varphi} \right).$$

For $d = 1$, $H = 1/2$ and $\sigma = 1$, this equation arises in the field of nonlinear optics as a limit model for the propagation of light pulse in an optical fiber where the dispersion varies along the fiber [2, 1]. These variations in the dispersion accounts for the so-called *dispersion management* which aims to improve the transmission of a light signal by constructing a zero-mean dispersion fiber in order to avoid the problem of the chromatic dispersion of the light signal. When the variations are assumed to be random, a noisy dispersion can be derived (see [26, 13]) which leads, in the white noise case, to Equation (1.2).

As part of the problems concerning the propagation of waves in random media, there is a vast literature around random Schrödinger equations. Let us mention in particular the cases of random potentials [15, 16] and noisy potentials [10, 11, 12]. In these works, the effects of the stochastic potential greatly affect the dynamic of the Schrödinger equation and are, in a broader context, a motivation to introduce randomness in PDEs. Specifically, there is a well known effect which attracted a lot of attention: the so-called *regularization by noise* phenomenon (see [18] for a survey). This phenomenon can be summarized as an improvement, due to the presence of noise, of the well-posedness of differential equations and has been studied in the context of SDEs [31, 29, 24, 27, 5], transport equation [19, 17, 4], SPDEs [9] and scalar conservation laws [20]. We remark that obtaining a regularization by noise in the context of nonlinear random PDE is a challenging task and most of the results are obtained in a linear setting. For instance, an open problem is to obtain a regularization by noise for the Euler or Navier-Stokes equations.

As mentioned previously, we are not the first one to investigate the Cauchy problem of Equation (1.2). It was first studied in [13] where the global Cauchy problem was solved for $H = 1/2$ and $\sigma < 2/d$ (which corresponds to a classical L^2 -subcritical nonlinearity). In [14], the authors proved that, in the L^2 -critical case, when $d = 1$, $H = 1/2$ and $\sigma = 5$, the solutions are globally well-posed, which is not the case for the deterministic nonlinear Schrödinger equation and, thus, hints for a regularization by noise effect. In [8], the authors study the case for $d = 1$, $\sigma = 2$, H small enough and $d = 2$, $\sigma = 1$, $H \in (0, 1)$. By a simple scaling argument on the space and time variables of (1.2) and thanks to the scaling invariance of the Wiener process, it was conjectured in [3] that, in fact, the critical nonlinearity should be $\sigma = 4/d$ for $H = 1/2$, a L^2 -supercritical nonlinearity, which is twice as large as the deterministic L^2 -critical nonlinearity. Furthermore, this fact was supported by numerical simulations in 1D and leads to believe that the white noise dispersion has a strong stabilizing property.

In this paper, we prove the global Cauchy problem (1.2) for $d \in \mathbb{N}$, $\sigma \leq 2/d$ and $H \in (0, 1)$. To be more specific, we obtain the following result.

Theorem 1.2. *Let $\sigma \leq \frac{2}{d}$, $\psi_0 \in L^2(\mathbb{R}^d)$ and $a \in (2, \infty)$ such that $2/a = d(1/2 - 1/(2\sigma + 2))$. Then, \mathbb{P} -a.s., there exists a unique solution $\psi \in L^a([0, +\infty[; L^{2\sigma+2}(\mathbb{R}^d))$ to (1.2).*

Remark 1.1. *Thus, the modulation by a random noise of the dispersion operator leads to a regularizing effect since we are able to prove the global existence and uniqueness of solutions in the critical case $\sigma = 2/d$. The other interesting fact of our result is that, no matter how close to 1 the Hurst parameter is, we still reach the critical case. This problem was left open in [8] where H needs to be large enough.*

The classical approach to investigate the Cauchy problem for nonlinear Schrödinger equations is to derive local Strichartz estimates [6]. These estimates are a direct consequence of the dispersive property of the linear operator $i\Delta$. However, as pointed out in [14], it is much harder to obtain such estimates in the case of a white noise dispersion because of the presence of the Wiener process. We recall from [26, 13] that the propagator associated to the linear part of (1.2) is explicitly given by, $\forall t, s \in (0, \infty)$ and $\forall \varphi \in C_0^\infty(\mathbb{R}^d)$,

$$P_{s,t}\varphi(x) = \frac{1}{(4\pi(W_t^H - W_s^H))^{d/2}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4(W_t^H - W_s^H)}} \varphi(y) dy. \quad (1.3)$$

Following the classical proof of Strichartz estimates (see for instance [23]), a fundamental tool is the Hardy-Littlewood-Sobolev inequality. This is where Theorem 1.1 comes at hand since the classical potential $|t - s|^{-\alpha}$ is replaced by $|W_t - W_s|^{-\alpha}$. As a direct consequence, we obtain the following stochastic Strichartz estimates.

Definition 1.1. *For any $(q, p) \in (2, \infty)^2$, we say that (q, p) is an admissible pair if*

$$\frac{2}{q} = d \left(\frac{1}{2} - \frac{1}{p} \right).$$

Proposition 1.1. *Let $(W_t^H)_{t \geq 0}$ be a fractional brownian motion of Hurst index $H \in (0, 1)$, $\beta \in (0, 1 - H)$ and (q, p) an admissible pair. Then, \mathbb{P} -a.s., there exist $T > 0$ and $C_{1,1.1}, C_{2,1.1} > 0$ such that, $\forall f \in L^2(\mathbb{R}^d)$ and $\forall g \in L^{r'}([0, T]; L^l(\mathbb{R}^d))$, the following inequalities holds*

$$\|P_{0,\cdot} f\|_{L^q([0, T]; L^p(\mathbb{R}^d))} \leq C_{1,1.1} T^{\alpha\beta} \|f\|_{L^2}, \quad (1.4)$$

$$\left\| \int_0^T P_{s,\cdot} g(s) ds \right\|_{L^q([0, T]; L^p(\mathbb{R}^d))} \leq C_{2,1.1} T^{\alpha\beta} \|g\|_{L^{r'}([0, T]; L^l(\mathbb{R}^d))}, \quad (1.5)$$

for any (r, l) admissible pair.

These Strichartz estimates are more powerful due to the presence of the term $T^{\alpha\beta}$. Indeed, in the fixed-point argument, this term will be necessary to obtain the contraction of the mapping in the critical case $\sigma = 2/d$. This will be the main argument to prove Theorem 1.2. The rest of the paper is devoted to the proof of Theorem 1.1 in section 2 and the proofs of Proposition 1.1 and Theorem 1.2 in section 3.

2 Proof of Theorem 1.1

Before proceeding any further, let us remark that we can, without loss of generality, assume that $f \in L^p([0, T])$ and $g \in L^q([0, T])$ are non-negative functions and, furthermore, by a scaling argument, we can assume that $\|f\|_{L^p([0, T])} = \|g\|_{L^q([0, T])} = 1$. Our

strategy follows the proof of Lieb-Loss [25, Section 4.3]. It is based on the following layer cake representation

$$f(t) = \int_0^{+\infty} \mathbf{1}_{f(t) > a} da, \quad g(s) = \int_0^{+\infty} \mathbf{1}_{g(s) > b} db$$

$$\text{and } |W_t^H - W_s^H|^{-\alpha} = \alpha \int_0^{+\infty} c^{-1-\alpha} \mathbf{1}_{|W_t^H - W_s^H| < c} dc.$$

Then, by Fubini's theorem, the left-hand-side of (1.1) can be rewritten as

$$\int_0^T \int_0^T f(t) |W_t^H - W_s^H|^{-\alpha} g(s) ds dt =$$

$$\alpha \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} c^{-1-\alpha} \left(\int_0^T \int_0^T \mathbf{1}_{f(t) > a} \mathbf{1}_{g(s) > b} \mathbf{1}_{|W_t^H - W_s^H| < c} ds dt \right) dc db da. \quad (2.1)$$

By denoting

$$\check{f}(a) := \int_0^T \mathbf{1}_{f(t) > a} dt, \quad \check{g}(b) := \int_0^T \mathbf{1}_{g(s) > b} ds \quad \text{and} \quad \check{W}(c, T) := \sup_{t \in [0, T]} \int_0^T \mathbf{1}_{|W_t^H - W_s^H| < c} ds,$$

we have the following result whose proof is postponed.

Lemma 2.1. *Let $H \in (0, 1)$ and $\beta \in (0, 1 - H)$. There exists $T > 0$ and $C_{2.1} > 0$ such that, \mathbb{P} -a.s., $\forall c \in \mathbb{R}^{+*}$,*

$$\check{W}(c, T) \leq C_{2.1} T^\beta c.$$

We now set $p, q \in (2, +\infty)$ such that

$$\frac{1}{p} + \frac{1}{q} + \alpha = 2.$$

We see that we can bound each characteristic function by 1 in (2.1) and, thus, we deduce that

$$\int_0^T \int_0^T f(t) |W_t^H - W_s^H|^{-\alpha} g(s) ds dt \leq \alpha \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} c^{-\alpha-1} I(a, b, c) da db dc,$$

with

$$I(a, b, c) := \begin{cases} \check{f}(a) \check{g}(b), & \text{if } \mathbf{m}(a, b, c) = \check{h}(c), \\ \check{f}(a) \check{W}(c, T), & \text{if } \mathbf{m}(a, b, c) = \check{g}(b), \\ \check{W}(c, T) \check{g}(b), & \text{if } \mathbf{m}(a, b, c) = \check{f}(a), \end{cases}$$

where

$$\mathbf{m}(a, b, c) = \lim_{\iota \rightarrow 0} \max\{\check{f}(a), \check{g}(b), \check{h}(c) + \iota\} \quad \text{and} \quad \check{h}(c) := C_{2.1} T^\beta c.$$

Remark 2.1. *In the previous definition of \mathbf{m} , we choose to have, in the case where $\max\{\check{f}(a), \check{g}(b), \check{h}(c)\} = \check{f}(a) = \check{h}(c)$, $\mathbf{m}(a, b, c) = \check{h}(c)$ (and similarly for \check{g}).*

From here, we essentially follow the arguments from [25]. We first assume that $\check{f}(a) \geq \check{g}(b)$. We deduce that

$$\int_0^{+\infty} c^{-\alpha-1} I(a, b, c) dc \leq$$

$$\check{g}(b) \int_0^{+\infty} c^{-\alpha-1} \check{h}(c) \mathbf{1}_{\check{h}(c) \leq \check{f}(a)} dc + \check{f}(a) \check{g}(b) \int_0^{+\infty} c^{-\alpha-1} \mathbf{1}_{\check{f}(a) \leq \check{h}(c)} dc.$$

We denote $\kappa_T = C_{2,1}T^\beta$. Since $\check{h}(c) \leq \check{f}(a)$ is equivalent to

$$c \leq \check{f}(a)/\kappa_T,$$

the first integral on the right-hand-side is estimated as

$$\begin{aligned} \int_0^{+\infty} c^{-\alpha-1} \check{h}(c) \mathbf{1}_{\check{h}(c) \leq \check{f}(a)} dc &\leq \kappa_T \int_0^{\check{f}(a)/\kappa_T} c^{-\alpha} dc \\ &\leq (1-\alpha)^{-1} \kappa_T (\check{f}(a)/\kappa_T)^{1-\alpha} = (1-\alpha)^{-1} \kappa_T^\alpha \check{f}(a)^{1-\alpha}. \end{aligned}$$

The second integral is bounded as

$$\int_0^{+\infty} c^{-\alpha-1} \mathbf{1}_{\check{f}(a) \leq \check{h}(c)} dc \leq \int_{\check{f}(a)/\kappa_T}^{+\infty} c^{-\alpha-1} dc \leq \alpha^{-1} \kappa_T^\alpha \check{f}(a)^{-\alpha}.$$

Hence, since by assumption $\check{g}(b)^{-\alpha} \leq \check{f}(a)^{-\alpha}$, it follows that

$$\int_0^{+\infty} c^{-\alpha-1} I(a, b, c) dc \lesssim_\alpha \kappa_T^\alpha \check{g}(b) \check{f}(a)^{1-\alpha} = \kappa_T^\alpha \min\{\check{g}(b) \check{f}(a)^{1-\alpha}, \check{f}(a) \check{g}(b)^{1-\alpha}\}.$$

By assuming that $\check{f}(a) \leq \check{g}(b)$ and following the same arguments, we obtain

$$\int_0^{+\infty} c^{-\alpha-1} I(a, b, c) dc \lesssim_\alpha \kappa_T^\alpha \min\{\check{g}(b) \check{f}(a)^{1-\alpha}, \check{f}(a) \check{g}(b)^{1-\alpha}\}.$$

We then proceed by integrating with respect to b and a . We have

$$\begin{aligned} &\int_0^{+\infty} \int_0^{+\infty} \min\{\check{g}(b) \check{f}(a)^{1-\alpha}, \check{f}(a) \check{g}(b)^{1-\alpha}\} db da = \\ &\int_0^{+\infty} \int_0^{a^{p/q}} \check{f}(a) \check{g}(b)^{1-\alpha} db da + \int_0^{+\infty} \int_{a^{p/q}}^{+\infty} \check{g}(b) \check{f}(a)^{1-\alpha} db da. \end{aligned}$$

Thanks to Hölder' inequality, we have, for $r = (q-1)(1-\alpha)$,

$$\begin{aligned} \int_0^{a^{p/q}} \check{g}(b)^{1-\alpha} db &= \int_0^{a^{p/q}} \check{g}(b)^{1-\alpha} b^{-r} b^r db \\ &\leq \left(\int_0^{a^{p/q}} \check{g}(b) b^{q-1} db \right)^{1-\alpha} \left(\int_0^{a^{p/q}} b^{-r/\alpha} db \right)^{\alpha/\beta_2}. \end{aligned}$$

The norms of f and g are such that

$$\|f\|_{L^p([0,T])}^p = p \int_0^{+\infty} a^{-p-1} \check{f}(a) da = 1 \quad \text{and} \quad \|g\|_{L^q([0,T])}^q = q \int_0^{+\infty} b^{-q-1} \check{g}(b) db = 1.$$

Thus, since

$$\begin{aligned} \frac{p}{q} \left(1 - \frac{r}{\alpha}\right) \alpha &= \frac{p}{q} (\alpha - (q-1)(1-\alpha)) \\ &= \frac{p}{q} (1 - q(1-\alpha)) = p \left(\frac{1}{q} - 1 + \alpha\right) = p - 1, \end{aligned}$$

we obtain

$$\int_0^{+\infty} \check{f}(a) \int_0^{a^{p/q}} \check{g}(b)^{1-\alpha} db da \lesssim \|g\|_{L^q([0,T])}^{q(1-\alpha)} \int_0^{+\infty} \check{f}(a) a^{p-1} da = \|g\|_{L^q([0,T])}^{q(1-\alpha)} \|f\|_{L^p([0,T])}^p \leq 1.$$

By similar arguments, we deduce that

$$\int_0^{+\infty} \int_{a^{p/q}}^{+\infty} \check{g}(b) \check{f}(a)^{1-\alpha} db da \lesssim 1,$$

which concludes the proof of Theorem 1.1.

It remains to prove Lemma 2.1. We have that, $\forall c > 0$ and $\forall t \in [0, T]$,

$$\int_0^T \mathbf{1}_{|W_t^H - W_s^H| < c} ds = \int_{\mathbb{R}} \mathbf{1}_{|y| < c} \ell_{[0, T]}^{W_t^H - y} dy = \int_{-c}^c \ell_{[0, T]}^{W_t^H - y} dy,$$

where ℓ is the local time of W^H given as

$$\ell_{[s, t]}^z := \lim_{\varepsilon \rightarrow 0} \int_{[s, t]} P_\varepsilon \delta_z(W_u^H) du,$$

where $(P_t)_{t \geq 0}$ is the heat semigroup. We need the following result from [30].

Theorem 2.1. *Let $(W_t^H)_{t \geq 0}$ be a fractional brownian motion with Hurst parameter $H \in (0, 1)$. Then, for any interval $I \subset \mathbb{R}$, there exists a positive finite constant K such that, \mathbb{P} -a.s.,*

$$\limsup_{r \rightarrow 0} \sup_{t \in I} \frac{\sup_{x \in \mathbb{R}} \ell_{[t-r, t+r]}^x}{r^{1-H} \log(1/r)^H} \leq K.$$

We deduce from the previous result that for any $\beta \in (0, 1 - H)$ and $T > 0$ small enough, there exists a constant $C_{2.1} > 0$ such that, \mathbb{P} -a.s., we have

$$\sup_{x \in \mathbb{R}} \ell_{[0, T]}^x \leq C_{2.1} T^\beta.$$

It follows that, for T small enough,

$$\int_0^T \mathbf{1}_{|W_t^H - W_s^H| < c} ds \leq 2C_{2.1} c T^\beta,$$

which is exactly the desired result.

3 Proof of Proposition 1.1 and Theorem 1.2

3.1 Stochastic Strichartz estimates

Since $(P_{s,t})_{0 \leq s \leq t}$ is an isometry from L^2 to itself, we deduce by the Hausdorff-Young inequality and an interpolation argument, that, $\forall p \in [2, \infty]$,

$$\|P_{s,t} \varphi\|_{L^p(\mathbb{R}^d)} \lesssim \frac{1}{|W_t - W_s|^{d(1/2 - 1/p)}} \|\varphi\|_{L^{p'}(\mathbb{R}^d)}, \quad (3.1)$$

where p' is the Hölder conjugate of p . We denote $(P_{s,t}^*)_{0 \leq s \leq t}$ the adjoint of $(P_{s,t})_{0 \leq s \leq t}$, that is

$$P_{s,t}^* \varphi(x) := \mathcal{F}^{-1} \left(e^{-\frac{i}{2} |\xi|^2 (W_t^H - W_s^H)} \hat{\varphi}(\xi) \right) = P_{t,s}.$$

This leads, in particular, to the fact that

$$P_{s,t}^* = P_{t,s}, \quad P_{0,s}^* P_{0,t} = P_{s,t} \quad \text{and} \quad P_{s,t} P_{r,t}^* = P_{s,r}, \quad \forall r \in [s, t].$$

The proof of Proposition 1.1 is based on the TT^* argument. We set $\alpha = d(1/2 - 1/p)$ and consider the integral, $\forall f, g \in \mathcal{C}([0, T], C_0^\infty(\mathbb{R}^d))$,

$$\begin{aligned} I(f, g) &:= \left| \int_0^T \int_0^T \langle P_{0,t}f(s), P_{0,s}g(t) \rangle_{L^2} ds dt \right| = \left| \int_0^T \int_0^T \langle P_{0,s}^*f(s), P_{0,t}^*g(t) \rangle_{L^2} ds dt \right| \\ &= \left| \int_0^T \int_0^T \langle P_{s,t}f(s), g(t) \rangle_{L^2} ds dt \right| \end{aligned}$$

By Hölder's inequality, (3.1) and Theorem 1.1, we deduce that

$$\begin{aligned} I(f, g) &\leq \int_0^T \int_0^T \|P_{s,t}f(s)\|_{L^p(\mathbb{R}^d)} \|g(t)\|_{L^{p'}(\mathbb{R}^d)} ds dt \\ &\lesssim \int_0^T \int_0^T |W_t - W_s|^{-\alpha} \|f(t)\|_{L^{p'}(\mathbb{R}^d)} \|g(s)\|_{L^{p'}(\mathbb{R}^d)} ds dt \\ &\lesssim T^{\alpha\beta} \|f\|_{L^{q'}([0, T], L^{p'}(\mathbb{R}^d))} \|g\|_{L^{q'}([0, T], L^{p'}(\mathbb{R}^d))}, \end{aligned}$$

since

$$2 - d \left(\frac{1}{2} - \frac{1}{p} \right) = \frac{1}{q'} + \frac{1}{q'} = 2 - \frac{2}{q}.$$

This yields, on one hand, that

$$\left\| \int_0^T P_{0,s}^*f(s) ds \right\|_{L^2(\mathbb{R}^d)}^2 = I(f, f) \lesssim T^{\alpha\beta} \|f\|_{L^{q'}([0, T], L^{p'}(\mathbb{R}^d))}^2, \quad (3.2)$$

and, on another hand, by a duality argument,

$$\left\| \int_0^T P_{s,\cdot}f(s) ds \right\|_{L^q([0, T], L^p(\mathbb{R}^d))} \lesssim T^{\alpha\beta} \|f\|_{L^{q'}([0, T], L^{p'}(\mathbb{R}^d))} \quad (3.3)$$

We are now in position to prove (1.4) and (1.5). It follows from (3.2) that, $\forall f \in L^2(\mathbb{R}^d)$ and $\forall g \in L^{q'}([0, T]; L^{p'}(\mathbb{R}^d))$,

$$\begin{aligned} \int_0^T \langle P_{0,t}f, g(t) \rangle_{L^2} dt &= \left\langle f, \int_0^T P_{0,t}^*g(t) \right\rangle_{L^2} \leq \|f\|_{L^2(\mathbb{R}^d)} \left\| \int_0^T P_{0,t}^*g(t) ds \right\|_{L^2(\mathbb{R}^d)} \\ &\lesssim T^{\alpha\beta} \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^{q'}([0, T], L^{p'}(\mathbb{R}^d))}, \end{aligned}$$

we obtain (1.4) by a duality argument. We now turn to (1.5). We have, by (3.2),

$$\begin{aligned} \left\| \int_0^T P_{s,\cdot}f(s) ds \right\|_{L^q([0, T]; L^p(\mathbb{R}^d))} &\leq \int_0^T \|P_{s,\cdot}f(s)\|_{L^q([0, T]; L^p(\mathbb{R}^d))} ds \\ &\lesssim T^{\alpha\beta} \int_0^T \|f(s)\|_{L^2(\mathbb{R}^d)} ds = T^{\alpha\beta} \|f\|_{L^1([0, T]; L^2(\mathbb{R}^d))}. \end{aligned}$$

Thanks to this estimate, by an interpolation argument with (3.3), we deduce (1.5).

3.2 Well-posedness of equation (1.2)

We can now apply the previous result to solve the local Cauchy problem of (1.2) The strategy is based on a fixed-point argument of the mapping Γ from $L^q([0, T]; L^p(\mathbb{R}^d))$ to itself given by

$$\Gamma(\psi)(t, x) = P_{0,t}\psi_0(x) - i\lambda \int_0^T P_{s,t}|\psi|^{2\sigma}\psi(s, x) ds. \quad (3.4)$$

We denote a closed ball of $L^q([0, T]; L^p(\mathbb{R}^d))$

$$B_{R, L^q([0, T]; L^p(\mathbb{R}^d))} := \left\{ \psi \in L^q([0, T]; L^p(\mathbb{R}^d)); \|\psi\|_{L^q([0, T]; L^p(\mathbb{R}^d))} \leq R \right\}.$$

Fix $R > 0$ that will be set later. For any $\psi \in B_{R, L^q([0, T]; L^p(\mathbb{R}^d))}$, we apply the $L^q([0, T]; L^p(\mathbb{R}^d))$ norm to (3.4) and deduce, thanks to (1.4) and (1.5),

$$\|\Gamma(\psi)\|_{L^q([0, T]; L^p(\mathbb{R}^d))} \leq C_1 \|\psi_0\|_{L^2(\mathbb{R}^d)} + C_2 |\lambda| T^{\alpha\beta} \|\psi\|_{L^{r'(2\sigma+1)}([0, T]; L^{l'(2\sigma+1)}(\mathbb{R}^d))}^{2\sigma+1}$$

for any (r, l) admissible. By choosing $(q, p) = (r, l) = (a, 2\sigma + 2)$, we have

$$l' = \frac{l}{l-1} = \frac{2\sigma + 2}{2\sigma + 1}.$$

Hence, we obtain, by Hölder's inequality,

$$\|\psi\|_{L^{r'(2\sigma+1)}([0, T]; L^{l'(2\sigma+1)}(\mathbb{R}^d))}^{2\sigma+1} = \|\psi\|_{L^{r'(2\sigma+1)}([0, T]; L^{2\sigma+2}(\mathbb{R}^d))}^{2\sigma+1} \leq T^{1-\frac{2}{d\sigma}} \|\psi\|_{L^a([0, T]; L^{2\sigma+2}(\mathbb{R}^d))}^{2\sigma+1},$$

which gives us

$$\|\Gamma(\psi)\|_{L^q([0, T]; L^p(\mathbb{R}^d))} \leq C_1 \|\psi_0\|_{L^2(\mathbb{R}^d)} + C_2 |\lambda| T^{1+\alpha\beta-\frac{2}{d\sigma}} \|\psi\|_{L^a([0, T]; L^{2\sigma+2}(\mathbb{R}^d))}^{2\sigma+1}. \quad (3.5)$$

By similar computations, we obtain that, $\forall \psi_1, \psi_2 \in B_{R, L^q([0, T]; L^p(\mathbb{R}^d))}$,

$$\|\Gamma(\psi_1) - \Gamma(\psi_2)\|_{L^q([0, T]; L^p(\mathbb{R}^d))} \leq C_2 |\lambda| T^{1+\alpha\beta-\frac{2}{d\sigma}} R^{2\sigma} \|\psi_1 - \psi_2\|_{L^a([0, T]; L^{2\sigma+2}(\mathbb{R}^d))}. \quad (3.6)$$

We remark that, since $2/d\sigma \leq 1$ and $\alpha\beta > 0$, we have

$$1 + \alpha\beta - \frac{2}{d\sigma} > 0.$$

Hence, by setting

$$R = 2C_1 \|\psi_0\|_{L^2(\mathbb{R}^d)},$$

and taking $T > 0$ such that

$$C_2 |\lambda| T^{1+\alpha\beta-\frac{2}{d\sigma}} R^{2\sigma} < 1,$$

we can see that Γ is a contraction from $B_{R, L^a([0, T]; L^{2\sigma+2}(\mathbb{R}^d))}$ to itself. It follows from a Banach fixed point theorem that there exists a unique solution to (1.2). The proof of Theorem 1.2 then follows by iterating this argument on time intervals of length T since we have $\|\psi(T)\|_{L^2(\mathbb{R}^d)} = \|\psi_0\|_{L^2(\mathbb{R}^d)}$.

References

- [1] G. P. Agrawal. *Applications of nonlinear fiber optics*. Academic press, 2001.
- [2] G. P. Agrawal. *Nonlinear fiber optics*. Academic press, 2007.
- [3] R. Belaouar, A. de Bouard, and A. Debussche. Numerical analysis of the nonlinear schrödinger equation with white noise dispersion. *Stochastic Partial Differential Equations: Analysis and Computations*, 3(1):103–132, 2015.
- [4] R. Catellier. Rough linear transport equation with an irregular drift. *Stochastics and Partial Differential Equations: Analysis and Computations*, 4(3):477–534, 2016.

- [5] R. Catellier and M. Gubinelli. Averaging along irregular curves and regularisation of odes. *Stochastic Processes and their Applications*, 126(8):2323 – 2366, 2016.
- [6] T. Cazenave. *Semilinear schrödinger equations*, volume 10. American Mathematical Soc., 2003.
- [7] M. Chen, O. Goubet, and Y. Mhammeri. Generalized regularized long wave equation with white noise dispersion. *Stochastics and Partial Differential Equations: Analysis and Computations*, 5(3):319–342, 2017.
- [8] K. Chouk and M. Gubinelli. Nonlinear pdes with modulated dispersion i: Nonlinear schrödinger equations. *Communications in Partial Differential Equations*, 40(11):2047–2081, 2015.
- [9] G. Da Prato, F. Flandoli, E. Priola, and M. Röckner. Strong uniqueness for stochastic evolution equations in hilbert spaces perturbed by a bounded measurable drift. *The Annals of Probability*, 41(5):3306–3344, 2013.
- [10] A. de Bouard and A. Debussche. A stochastic nonlinear schrödinger equation with multiplicative noise. *Communications in mathematical physics*, 205(1):161–181, 1999.
- [11] A. de Bouard and A. Debussche. On the effect of a noise on the solutions of the focusing supercritical nonlinear schrödinger equation. *Probability theory and related fields*, 123(1):76–96, 2002.
- [12] A. De Bouard and A. Debussche. Blow-up for the stochastic nonlinear schrödinger equation with multiplicative noise. *The Annals of Probability*, 33(3):1078–1110, 2005.
- [13] A. de Bouard and A. Debussche. The nonlinear schrödinger equation with white noise dispersion. *Journal of Functional Analysis*, 259(5):1300–1321, 2010.
- [14] A. Debussche and Y. Tsutsumi. 1d quintic nonlinear schrödinger equation with white noise dispersion. *Journal de mathématiques pures et appliquées*, 96(4):363–376, 2011.
- [15] L. Erdős, M. Salmhofer, and H.-T. Yau. Quantum diffusion of the random schrödinger evolution in the scaling limit. *Acta mathematica*, 200(2):211, 2008.
- [16] L. Erdős and H.-T. Yau. Linear boltzmann equation as the weak coupling limit of a random schrödinger equation. *Communications on Pure and Applied Mathematics*, 53(6):667–735, 2000.
- [17] E. Fedrizzi and F. Flandoli. Noise prevents singularities in linear transport equations. *Journal of Functional Analysis*, 264(6):1329–1354, 2013.
- [18] F. Flandoli. *Random Perturbation of PDEs and Fluid Dynamic Models: École d’été de Probabilités de Saint-Flour XL–2010*, volume 2015. Springer Science & Business Media, 2011.
- [19] F. Flandoli, M. Gubinelli, and E. Priola. Well-posedness of the transport equation by stochastic perturbation. *Inventiones mathematicae*, 180(1):1–53, 2010.
- [20] B. Gess and P. E. Souganidis. Long-time behavior, invariant measures, and regularizing effects for stochastic scalar conservation laws. *Communications on Pure and Applied Mathematics*, 70(8):1562–1597, 2017.
- [21] G.H. Hardy and J.E. Littlewood. Some properties of fractional integrals. i. *Mathematische Zeitschrift*, 27(1):565–606, 1928.

- [22] G.H. Hardy and J.E. Littlewood. Some properties of fractional integrals. ii. *Mathematische Zeitschrift*, 34(1):403–439, 1932.
- [23] M. Keel and T. Tao. Endpoint strichartz estimates. *American Journal of Mathematics*, 120(5):955–980, 1998.
- [24] N. V. Krylov and M. Roekner. Strong solutions of stochastic equations with singular time dependent drift. *Probability theory and related fields*, 131(2):154–196, 2005.
- [25] Elliott H Lieb and Michael Loss. Analysis. *Graduate studies in Mathematics*, 14(2), 2001.
- [26] R. Marty. On a splitting scheme for the nonlinear schrödinger equation in a random medium. *Communications in Mathematical Sciences*, 4(4):679–705, 2006.
- [27] E. Priola. Pathwise uniqueness for singular sdes driven by stable processes. *Osaka Journal of Mathematics*, 49(2):421–447, 2012.
- [28] S. L. Sobolev. On a theorem of functional analysis. *Mat. Sbornik*, 4:471–497, 1938.
- [29] A. J. Veretennikov. On strong solutions and explicit formulas for solutions of stochastic integral equations. *Sbornik: Mathematics*, 39(3):387–403, 1981.
- [30] Yimin Xiao. Hölder conditions for the local times and the hausdorff measure of the level sets of gaussian random fields. *Probability theory and related fields*, 109(1):129–157, 1997.
- [31] A. K. Zvonkin. A transformation of the phase space of a diffusion process that removes the drift. *Mathematics of the USSR-Sbornik*, 22(1):129, 1974.