Localization due to Damage in Two-Direction Fiber-Reinforced Composites

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Fiber pull-out is one of the fracture features of fiber-reinforced ceramic matrix composites. The onset of this mechanism is predicted by using continuum damage mechanics, and corresponds to a localization of deformation. After deriving two damage models from a uniaxial bundle approach, different configurations are analyzed through numerical methods. For one model some very simple criteria can be derived, whereas for the second one none of these criteria can be derived and the general criterion of localization must be used.

1 Introduction

Ceramic matrix composites (CMCs) can either be reinforced by fibers in one direction or by fibers in two directions. The aim of this paper is to study composites reinforced with fibers in two perpendicular directions by extending a previous study on CMCs with fibers in one direction (Hild et al., 1992).

The rupture of most of the CMCs involves two separate failure mechanisms. The first mechanism is matrix cracking. The matrix cracks develop and their density saturates as the load level increases. The second mechanism is fiber breakage accompanied by fiber pull-out. Eventually, the final rupture will take place around one of the matrix cracks: it corresponds to *localized* fiber pull-out due to fiber breakage. The occurrence of this mechanism corresponds to the appearance of a macrocrack and will be described by a localization of deformation. The initiation of macrocracks in a structure during service often constitutes the early stage of the final failure of the structure. Starting from a material that is assumed free from any initial defect, the initiation of macrocracks can be predicted using continuum damage mechanics. The driving force is fiber breakage, which is accompanied by *distributed* pull-out. The approach using localization has successfully been used for ductile materials (Billardon and Doghri, 1989a, b; Doghri, 1989). The initiation stage is considered as the onset of a surface across which the velocity gradient is discontinuous. Under small deformation assumptions, this phenomenon is mainly driven by the damage mechanism that causes strain softening. For CMCs, the damage mechanism is related to fiber breakage, and the damage variable describes the percentage of broken fibers (Hild et al., 1992).

Although localization can be studied at the scale of fibers bonded to a matrix through an interface (Benallal et al., 1991a), i.e., at a micro level, localization also can be analyzed at a mesolevel, when the material is assumed to be homogeneous. Continuum damage mechanics, which represents a local approach to fracture (Benallal et al., 1991b), constitutes an efficient tool for this purpose. The progressive deterioration of the material is modeled by internal variables defined at the meso level. These variables are called *damage* variables. The damage state and the evolution of these variables is obtained through a uniaxial study based on fiber breakage (Coleman, 1958; Curtin, 1991). A two-dimensional plane-stress analysis is performed based on an extended model. The loss of uniqueness and the localization are studied for shear free states. A criterion referring to a critical value of the damage or to a maximum normal stress can describe the localization, which constitutes an objective criterion, from a design point of view.

2 Localization and Loss of Uniqueness

The failure at a meso level, with the initiation of a macrocrack, is defined as the bifurcation of the rate problem in certain modes, viz. the appearance of a surface across which the velocity gradient is discontinuous (Billardon and Doghri, 1989a). This phenomenon is referred to as *localization*, and corresponds to the failure of the ellipticity condition. The condition of localization can also be compared to the loss of uniqueness of the rate problem.

Stationary waves were studied by Hadamard (1903) in elasticity and by Hill (1962) and Mandel (1962) in elastoplasticity. Rice (1976) related the localization of plastic shear bands to jumps of the velocity gradient. Borré and Maier (1989) have given necessary and sufficient conditions for the onset of modes inside the body, and extended the results given by Rice (1976) and Rice and Rudnicki (1975, 1980).

Under small strain assumption and in elasticity coupled with damage, the behavior of a material is assumed to be described by the following piece-wise linear rate constitutive law:

$$\dot{\boldsymbol{\sigma}} = \begin{cases} \mathbf{E} : \dot{\boldsymbol{\epsilon}} & \text{if} \quad \dot{D} = 0 \\ \mathbf{H} : \dot{\boldsymbol{\epsilon}} & \text{if} \quad \dot{D} \neq 0 \end{cases} \tag{1}$$

where $\dot{\boldsymbol{\sigma}}$ and $\dot{\boldsymbol{\epsilon}}$, respectively, denote the stress and strain rates, $\mathbf{E}$ and $\mathbf{H}$ are fourth rank tensors, $\mathbf{E}$ is assumed to be positive definite, and D is either a single damage variable or a set of damage variables.

Localization occurs *inside* the body, *if and only if* (Rudnicki and Rice, 1975; Borré and Maier, 1989; Benallal et al., 1991a)

$$\text{Det}\,(\boldsymbol{n} \cdot \mathbf{H} \cdot \boldsymbol{n}) = 0 \quad \text{for a vector } \boldsymbol{n} \neq 0 \quad \text{and at a point inside a structure } \Omega. \tag{2}$$

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This criterion corresponds to the failure of the ellipticity condition of the rate equilibrium equation; it also can be used as an indicator of the local failure of the material, at a meso scale (Billardon and Doghri, 1989a).

Furthermore, any loss of uniqueness, considered as bifurcation of the rate boundary value problem, is excluded provided

$$\dot{\boldsymbol{\sigma}} : \dot{\boldsymbol{\epsilon}} > 0. \tag{3}$$

In this study, the quantity that defines loss of uniqueness and localization is the linear tangent modulus $\mathbf{H}$. In the following, we analyze loss of uniqueness and loss of ellipticity (i.e., localization) for states when

$$\begin{cases} \epsilon_{11} = \alpha \epsilon_{22} \\ \epsilon_{12} = 0. \end{cases} \tag{4}$$

The parameter α is referred to as the strain ratio and its inverse is denoted by β. These particular states only are considered. When the hypothesis of Eq. (4) is satisfied, the nonvanishing components of the vector $\boldsymbol{n}$ are n_1 and n_2, and the matrix $\mathbf{A} = \boldsymbol{n} \cdot \mathbf{H} \cdot \boldsymbol{n}$ reduces to (Ortiz et al., 1987)

$$\mathbf{A} = \begin{bmatrix} n_1^2 H_{1111} + n_2^2 H_{1212} & n_1 n_2 (H_{1212} + H_{1122}) \\ n_1 n_2 (H_{1212} + H_{2211}) & n_1^2 H_{1212} + n_2^2 H_{2222} \end{bmatrix}. \tag{5}$$

If we rewrite $(n_1, n_2) = (\cos\theta, \sin\theta)$, $X = \tan^2\theta$, then the localization condition is equivalent to finding real positive roots of the following equation:

$$aX^2 + bX + c = 0 \tag{6}$$

with

$$\begin{aligned} a &= H_{1212} H_{2222} \\ b &= H_{1111} H_{2222} - H_{1122} H_{2211} - H_{1122} H_{1212} - H_{2211} H_{1212} \\ c &= H_{1212} H_{1111}. \end{aligned} \tag{7}$$

If real positive roots are found, then the localization direction is perpendicular to the vector $(n_1, n_2, 0) = (\cos\theta, \sin\theta, 0)$, characterized by the angle θ (Fig. 1). The values of H_{1111}, H_{2222}, H_{1122}, H_{2211} and H_{1212} are model dependent and specific models are now developed.

3 Constitutive Laws

This section is concerned with the development of two constitutive laws in the case of CMCs reinforced in two perpendicular directions. At constant temperature, the behavior of a CMC reinforced by unidirectional fibers in the x_2-direction (see Fig. 1) can be characterized by the Helmholtz free energy density ψ_2, which is a function of the state variables ϵ_{11}, ϵ_{22}, ϵ_{12}, and the damage variable D_2 in the x_2-direction

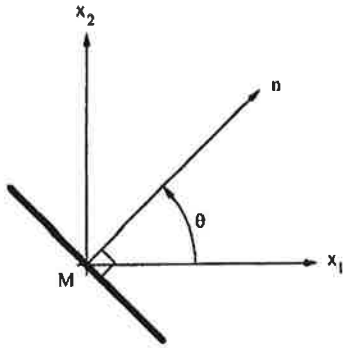

Fig. 1 Localization mode

$$\rho\psi_2 = \rho\psi(\epsilon_{11}, \epsilon_{22}, \epsilon_{12}, D_2, f_2, k_2), \tag{8}$$

where D_2 represents the fiber degradation in the x_2-direction, E_2 the Young's modulus in the x_2-direction, ν_{12} the Poisson's ratio, k_2 the ratio of the Young's modulus in the fiber direction (E_2) to the Young's modulus in the transverse direction (E_1), and G_{12} the shear modulus. It is worth noting that the elastic quantities depend on the volume fraction of fibers. The expression for the general Helmholtz free energy density ψ is given by

$$\rho\psi(x, y, z, d, f, k) = \frac{E_2(f)}{2}\left[\frac{x^2 + 2\nu_{12}k(1-d)xy + ky^2}{k\{1 - \nu_{12}^2 k(1-d)\}}\right] + 2G_{12}z^2 \tag{9}$$

where ρ is the material density, x, y, z are dummy variables representing strains, d damage, f volume fraction, and k Young's moduli ratio. The stresses and the thermodynamic force Y_2 associated to the damage variable D_2 are derived from the Helmholtz free energy density ψ_2 as follows:

$$\sigma_{11} = \rho\frac{\partial\psi_2}{\partial\epsilon_{11}} \quad \sigma_{22} = \rho\frac{\partial\psi_2}{\partial\epsilon_{22}} \quad 2\sigma_{12} = \rho\frac{\partial\psi_2}{\partial\epsilon_{12}}$$

$$Y_2 = \rho\frac{\partial\psi_2}{\partial D_2}. \tag{10}$$

The explicit expressions for the stresses related to the strains and the damage variable modeling the fiber degradation in the x_2-direction are given by

$$\sigma_{11} = \frac{E_2}{k_2[1 - \nu_{12}^2(1-D_2)k_2]}[\epsilon_{11} + \nu_{12}(1-D_2)k_2\epsilon_{22}]$$

$$\sigma_{22} = \frac{E_2(1-D_2)}{1 - \nu_{12}^2(1-D_2)k_2}(\epsilon_{22} + \nu_{12}\epsilon_{11})$$

$$\sigma_{12} = 2G_{12}\epsilon_{12}. \tag{11}$$

The damage state of fibers in the x_2-direction, D_2 can be related to the stress (and is denoted by $D_2^{(1)}$) or strain state (and is denoted by $D_2^{(2)}$). The relationship is either implicit in terms of the normal stress in the x_2-direction (model #1)

$$D_2^{(1)} = 1 - \exp\left[-\left\{\frac{\sigma_{22}}{(1 - D_2^{(1)})f_2\sigma_c}\right\}^{m+1}\right]$$

$$\text{if} \quad \epsilon_{22} > 0 \quad \text{and} \quad \dot{\epsilon}_{22} > 0 \tag{12}$$

where m is the shape parameter of a Weibull law (Weibull, 1939), σ_c the characteristic strength (Henstenburg and Phoenix, 1989), and f_2 is the volume fraction of fibers in the x_2-direction; or explicit in terms of the normal strain in the x_2-direction (model #2)

$$D_2^{(2)} = 1 - \exp\left[-\left(\frac{\epsilon_{22}}{\epsilon_c}\right)^{m+1}\right]$$

$$\text{if} \quad \epsilon_{22} > 0 \quad \text{and} \quad \dot{\epsilon}_{22} > 0 \tag{13}$$

where ϵ_c is related to the characteristic strength σ_c by $\sigma_c = E_F\epsilon_c$ (E_F is the Young's modulus of the fibers). Both models describe the same material behavior when subjected to uniaxial tension. However, the models give different predictions for multiaxial loading states (Hild et al., 1992). It is worth noting that the damage evolution laws are a priori independent of the volume, since we assume that the local behavior of the fiber degradation is not dependent on the total length of the fiber (Curtin, 1991). This type of behavior is observed when *distributed* pull-out happens in conjunction with fiber breakage, and it can be shown that in most practical cases, the statistics driving the fiber breakage is independent of the total length of the composite. On the

other hand, if the composite length becomes very small, a length dependence is found again (Hild et al., 1994), and in this case the evolution of the damage variable is mainly given by a fiber-bundle-type of behavior, which leads to replacing $m + 1$ by m, the characteristic strength σ_c by $\sigma_0(L/L_0)^{-1/m}$, where σ_0 is the scale parameter of a Weibull law, and the scale strain ϵ_c by $\epsilon_0(L/L_0)^{-1/m}$, where L_0 is the gauge length at which the scale parameter has been identified, and $\sigma_0 = E_F\epsilon_0$. Since the results are the same for both damage evolution laws when the previous permutation is used, we will just express them in the case when the model is length independent, which is the most relevant in practice.

If the fibers are in the x_1-direction then the breakage can be modeled by a damage variable denoted by D_1. Using Eq. (9), the Helmholtz free energy density $\rho\psi_1$ is given by

$$\rho\psi_1 = \rho\psi(\epsilon_{22}, \epsilon_{11}, \epsilon_{12}, D_1, f_1, k_1). \tag{14}$$

If the fibers are in both x_1 and x_2-directions, then we assume as a first approximation that the total specific Helmholtz free energy $\rho\psi_{12}$ is given by a law of mixture of the Helmholtz free energy densities in the x_1 and in the x_2-directions

$$\rho\psi_{12} = (1 - f)\rho\psi_1 + f\rho\psi_2 \tag{15}$$

where f is the fraction of fibers in the x_2-direction ($f = f_2/(f_1 + f_2)$, and where f_1 and f_2 are the volume fraction of fibers in the x_1 and x_2-direction, respectively). This assumption also corresponds to a Lin-Taylor Hypothesis. The evolution of the stresses is given by

$$\sigma_{11} = \rho\frac{\partial\psi_{12}}{\partial\epsilon_{11}} = (1 - f)S_{11} + fS_{12}$$

$$\sigma_{22} = \rho\frac{\partial\psi_{12}}{\partial\epsilon_{22}} = (1 - f)S_{21} + fS_{22}$$

$$\sigma_{12} = \frac{1}{2}\rho\frac{\partial\psi_{12}}{\partial\epsilon_{12}} = 2G_{12}\epsilon_{12} \tag{16}$$

where the explicit expression for S_{ij} is given in Appendix A, and the corresponding thermodynamic forces associated to the two *independent* damage variables D_1 and D_2 are

$$Y_1 = \rho\frac{\partial\psi_{12}}{\partial D_1} = (1 - f)\rho\frac{\partial\psi_1}{\partial D_1}$$

$$Y_2 = \rho\frac{\partial\psi_{12}}{\partial D_2} = f\rho\frac{\partial\psi_2}{\partial D_2}. \tag{17}$$

Again, the evolution of the damage variables can either be implicit in terms of the respective normal stresses (model #1)

$$D_1^{(1)} = 1 - \exp\left[-\left\{\frac{\sigma_{11}}{(1 - D_1^{(1)})f_1\sigma_c}\right\}^{m+1}\right] \quad \text{if} \quad \epsilon_{11} > 0 \quad \text{and} \quad \dot{\epsilon}_{11} > 0$$

$$D_2^{(1)} = 1 - \exp\left[-\left\{\frac{\sigma_{22}}{(1 - D_2^{(1)})f_2\sigma_c}\right\}^{m+1}\right] \quad \text{if} \quad \epsilon_{22} > 0 \quad \text{and} \quad \dot{\epsilon}_{22} > 0 \tag{18}$$

or explicit in terms of the respective normal strains (model #2)

$$D_1^{(2)} = 1 - \exp\left[-\left(\frac{\epsilon_{11}}{\epsilon_c}\right)^{m+1}\right] \quad \text{if} \quad \epsilon_{11} > 0 \quad \text{and} \quad \dot{\epsilon}_{11} > 0$$

$$D_2^{(2)} = 1 - \exp\left[-\left(\frac{\epsilon_{22}}{\epsilon_c}\right)^{m+1}\right] \quad \text{if} \quad \epsilon_{22} > 0 \quad \text{and} \quad \dot{\epsilon}_{22} > 0. \tag{19}$$

It is worth noting that we assume that the statistical properties of the fibers are supposed to be identical in both directions. This hypothesis will be maintained throughout the paper since generalization would be straightforward. Both models are studied for shear free states when the strain ratio α (see Eq. (4)), and thus its inverse β are given.

3.1 Failure Criteria for Model #1. For model #1, the evolution of the damage variables is implicit in the sense that $D_1^{(1)}$ (respectively $D_2^{(1)}$) is a function of the normal stress σ_{11} (respectively σ_{22}) *and* the damage variable $D_2^{(1)}$ (respectively $D_2^{(1)}$) itself. The evolution is therefore computed by a numerical scheme based upon a Newton method. To study localization and loss of uniqueness, we need to compute the tangent operator, which takes the following form:

$$H_{1111} = \frac{[(1 - f)F_{11} + fF_{12}](1 + fF_{52}F_{72}) - fF_{22}F_{72}[(1 - f)F_{41} + fF_{42}]}{[1 + fF_{21}F_{72}](1 + fF_{52}F_{72}) - f(1 - f)F_{22}F_{72}F_{51}F_{71}}$$

$$H_{2222} = \frac{[(1 - f)F_{61} + fF_{62}][1 + (1 - f)F_{21}F_{71}] - (1 - f)F_{51}F_{71}[(1 - f)F_{41} + fF_{42}]}{[1 + fF_{21}F_{72}](1 + fF_{52}F_{72}) - f(1 - f)F_{22}F_{72}F_{51}F_{71}}$$

$$H_{1122} = \frac{[(1 - f)F_{41} + fF_{42}](1 + fF_{52}F_{72}) - fF_{22}F_{72}[(1 - f)F_{61} + fF_{62}]}{[1 + fF_{21}F_{72}](1 + fF_{52}F_{72}) - f(1 - f)F_{22}F_{72}F_{51}F_{71}}$$

$$H_{2211} = \frac{[(1 - f)F_{41} + fF_{42}][1 + (1 - f)F_{21}F_{71}] - (1 - f)F_{51}F_{71}[(1 - f)F_{11} + fF_{12}]}{[1 + fF_{21}F_{72}](1 + fF_{52}F_{72}) - f(1 - f)F_{22}F_{72}F_{51}F_{71}}$$

$$H_{1212} = 2G_{12} \tag{20}$$

where the explicit expressions for F_{ij} are given in Appendix B.

The loss of uniqueness and localization are investigated when the fiber fraction f and the strain ratio α vary. Although analytical results cannot be derived from criterion (2) in the general case, some simple results can be found when f is equal to 0 or 1. In these cases, the criteria derived by Hild et al. (1992) apply. If f is equal to 0 (fibers only in the x_1-direction), then localization and loss of uniqueness occur at the same load level when

$$D_1^{(1)} = D_c = 1 - \exp\left(\frac{-1}{m + 1}\right)$$

$$\sigma_{11} = \sigma_{u1} = f_1\sigma_c\left(\frac{1}{(m + 1)e}\right)^{1/(m+1)}$$

$$Y_1 = Y_c = \frac{\sigma_{u1}^2}{2E_1(1 - D_c)^2} \tag{21}$$

where the stress σ_{u1} corresponds to the ultimate tensile strength in the x_1-direction. It is worth noting that the three previous criteria are easier to compute than the general criterion (2). The direction of localization is $\theta = 0$ deg, i.e., a localization surface perpendicular to the fiber direction.

If f is equal to 1, the same kind of result apply and the direction of localization is $\theta = 90$ deg, i.e., a localization surface again perpendicular to the fiber direction. When $f \neq 0$ and $f \neq 1$, these results cannot be proved. However the computations show that loss of uniqueness and localization can be described very accurately by the two following criteria:

$$\mathrm{Max}(D_1^{(1)}, D_2^{(1)}) = D_c = 1 - \exp\left(\frac{-1}{m+1}\right) \qquad (22)$$

$$\sigma_{11} = \sigma_{u1} \quad \text{or} \quad \sigma_{22} = \sigma_{u2} \qquad (23)$$

when the fiber properties are the same in the two directions. The maximum error is .5 percent in terms of criteria (22) and (23).

Criterion (22) shows that for model #1, maximum damage at localization depends only on the Weibull exponent of the fibers. Furthermore, criterion (23) shows that the maximum normal stress σ_{11} (respectively σ_{22}) depends only on the volume fraction of fibers in the x_1 (respectively x_2-) direction and on the fiber characteristics. This result is consistent with some experimental observations on woven carbon matrix composites reinforced with SiC (Nicalon) fibers (Heredia et al., 1992). On the other hand, the localization angle is dependent on the fiber percentage f (see Fig. 2). When the fiber percentage f and the sign of the strains ϵ_{11} and ϵ_{22} are constant, the variation of the localization angle is due to the fact that the maximum tensile stress is either reached in the x_1 or in the x_2-direction.

Moreover, if the strain ratio α is different from 0 and 1 then there is a complete symmetry of the results. If the strain ratio α is positive, the strains ϵ_{11} and ϵ_{22} are positive, changing α into β, f_2 into f_1, changes f into $1 - f$, and alters the absolute value of the localization angle $|\theta|$ into $\pi/2 - |\theta|$ and keeps the maximum stresses and damage levels constant. These two properties are referred to as *symmetry properties,* and are mainly due to the features of Eqs. (4), (8), (14), and (15).

When the strain ratio α is equal to 1 and the fiber percentage f is equal to .5, the localization angle is undetermined. This is due to the vanishing of the three constants a, b, and c in Eq. (6), for $H_{1111} = H_{1122} = H_{2211} = H_{2222} = 0$. Any value of the angle θ satisfies Eq. (6). This phenomenon can be observed when the fiber percentage f is different from 1: if $\sigma_{11} = \sigma_{u1}$ and $\sigma_{22} = \sigma_{u2}$ simultaneously, then $D_1^{(1)} = D_2^{(1)} = D_c$, and $H_{1111} = H_{1122} = H_{2211} = H_{2222} = 0$. This particular result shows that in terms of this model, for a given strain ratio α, it is possible to optimize locally a CMC reinforced by fibers in two perpendicular directions. Indeed, in terms of fiber breakage, a condition $\sigma_{11} = \sigma_{u1}$ and $\sigma_{22} = \sigma_{u2}$ leads to an optimum of the fiber behavior in both directions.

Model #1 constitutes a straightforward generalization of the fiber bundle models studied by Krajcinovic and Silva (1982), and Hult and Travnicek (1983). Finally, a shear stress has no influence on all the previous results since we assumed no coupling between the damage variables and the shear strain or stress for both model #1 and #2.

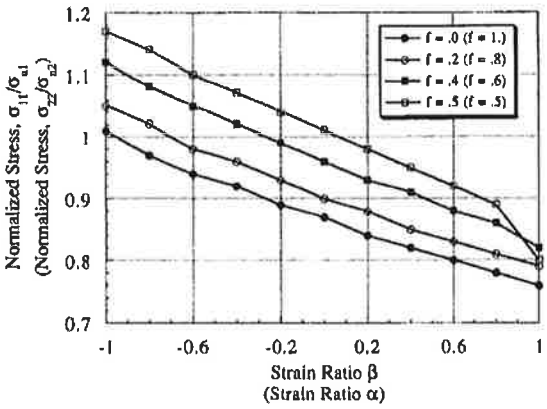

Fig. 3 Normalized maximum stress at localization for model #2, the main caption of the axes corresponds to the case where $f_2 = .5$, $f_1 = .0$, .125, .333, .5, and the captions in brackets correspond to the cases where $f_1 = .5$, $f_2 = .0$, .125, .333, .5

3.2 Study of Localization With Model #2. For model #2, the evolution of the damage variables is explicit and therefore is easier to compute. The tangent operator takes the form

$$\begin{aligned} H_{1111} &= (1-f)(F_{11} - F_{21}F_{31}) + fF_{12} \\ H_{2222} &= (1-f)F_{62} + f(F_{42} - F_{52}F_{32}) \\ H_{1122} &= (1-f)F_{41} + f(F_{42} - F_{22}F_{32}) \\ H_{2211} &= (1-f)(F_{41} - F_{21}F_{31}) + fF_{42} \\ H_{1212} &= 2G_{12} \end{aligned} \qquad (24)$$

where the explicit expressions for F_j are given in Appendix B. As shown in the case of fibers in only one direction (Hild et al., 1992), the localization criterion cannot be described by some simple criteria as those given by model #1. When fibers are in both directions the latter results are confirmed. A first consequence is that an optimization procedure can be performed since the maximum stress at localization, and the maximum damage at localization are dependent on both the strain ratio α and on the fiber percentage f.

Since the elastic law given in Eqs. (16) is identical for both models, the symmetry properties apply also for model #2 (see Figs. 3, 4, and 5). It can also be noticed that the maximum stress at localization varies with the fiber fraction f and with the strain ratio α.

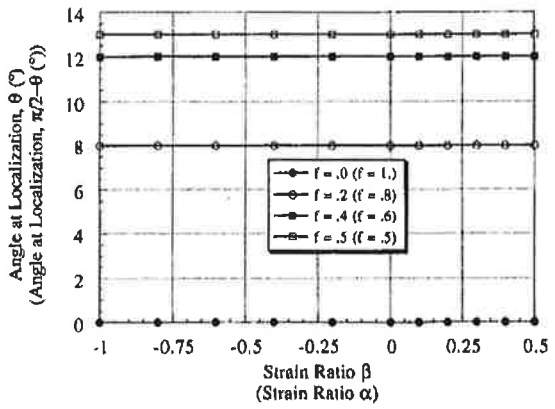

Fig. 2 Absolute value of the localization angle in degrees at localization for model #1, the main caption of the axes corresponds to the case where $f_2 = .5$, $f_1 = .0$, .125, .333, .5, and the captions in brackets correspond to the cases where $f_1 = .5$, $f_2 = .0$, .125, .333, .5

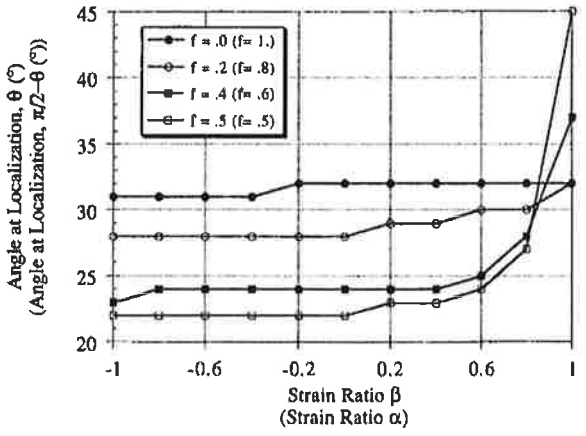

Fig. 4 Absolute value of the localization angle in degrees for model #2, the main caption of the axes corresponds to the case where $f_2 = .5$, $f_1 = .0$, .125, .333, .5, and the captions in brackets correspond to the cases where $f_1 = .5$, $f_2 = .0$, .125, .333, .5

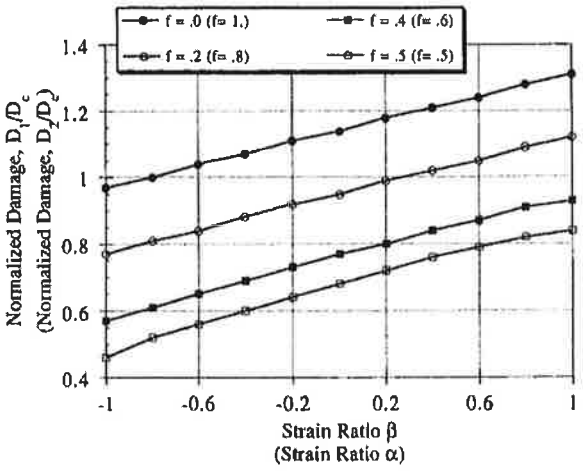

Fig. 5 Maximum normalized damage value at localization (m = 4.) for model #2, the main caption of the axes corresponds to the case where f_2 = .5, f_1 = .0, .125, .333, .5, and the captions in brackets correspond to the cases where f_1 = .5, f_2 = .0, .125, .333, .5

In the experiments reported by Heredia et al. (1992) the stress at localization was given by the ultimate tensile strength corresponding to the volume fraction of fibers in the same direction. This is not found by using model #2. Indeed, in a tensile test, when $f_1 = f_2 = .5$ the maximum stress σ_{22} normalized by the ultimate tensile strength σ_{u2} is given by .63, whereas the same tensile test when $f_1 = .0$ and $f_2 = .5$ would give a normalized tensile strength σ_{22}/σ_{u2} equal to 1. On the other hand, the damage at localization D_2 normalized by the critical damage D_c is equal to 1.04 when $f_1 = .5$ and $f_2 = .5$ and is equal to 1. when $f_1 = .0$ and $f_2 = .5$.

It is too early to draw a final conclusion, but it seems that the predictions of model #1 correspond more to reality than those of model #2. On the other hand, model #2 turned out to give results very close to model #1 when applied to structures with fibers in one direction (Hild et al., 1992). This will be addressed in the case of structures with fibers in two perpendicular directions such as spinning disks.

4 Conclusions

Using a one-dimensional study of fiber breakage modeled by a single damage variable, two models are derived. Both of them are then generalized to a two-dimensional plane stress analysis, with fibers in two perpendicular directions. Whereas model #1 constitutes a straightforward generalization of the elementary study, model #2 exhibits different features. Indeed, loss of uniqueness and localization can be described by some very simple criteria referring to Continuum Damage Mechanics for model #1. Conversely, these simple criteria do not apply for model #2. Physically, model #1 gives a better description of some experimental trends observed in the case of a carbon matrix reinforced with silicon carbide (Nicalon) fibers in two perpendicular directions. On the other hand, model #2 is easier to compute, and when applied to the study of spinning disc with fibers in one direction, it leads to load levels at localization of the same order of magnitude as model #1 (Hild et al., 1992).

Lastly, this study shows that the localization for model #1 can be described by using criterion (23) derived from the general criterion of localization (2). This criterion can also be used for a computation in elasticity and may turn out to be sufficient in first approximation to predict load levels at which a macrocrack initiates, instead of using a computation in elasticity coupled with damage.

5 Acknowledgments

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Appendix A

$$k_1 = \frac{E_2(f_1)}{E_1}$$

$$k_2 = \frac{E_2(f_2)}{E_1}$$

$$S_{11} = \frac{E_2(f_1)(1 - D_1)}{1 - \nu_{12}^2(1 - D_1)k_1}(\epsilon_{11} + \nu_{12}\epsilon_{22})$$

$$S_{12} = \frac{E_2(f_2)}{k_2[1 - \nu_{12}^2(1 - D_2)k_2]}[\epsilon_{11} + \nu_{12}(1 - D_2)k_2\epsilon_{22}]$$

$$S_{21} = \frac{E_2(f_1)}{k_1[1 - \nu_{12}^2(1 - D_1)k_1]}[\epsilon_{22} + \nu_{12}(1 - D_1)k_1\epsilon_{11}]$$

$$S_{22} = \frac{E_2(f_2)(1 - D_2)}{1 - \nu_{12}^2(1 - D_2)k_2}(\epsilon_{22} + \nu_{12}\epsilon_{11})$$

Appendix B

$$F_{11} = \frac{E_2(f_1)(1 - D_1)}{1 - \nu_{12}^2(1 - D_1)k_1}$$

$$F_{12} = \frac{E_2(f_2)}{k_2(1 - \nu_{12}^2(1 - D_2)k_2)}$$

$$F_{21} = \frac{E_2(f_1)(\nu_{12}\epsilon_{11} + \epsilon_{22})}{(1 - \nu_{12}^2(1 - D_1)k_1)^2}$$

$$F_{22} = \frac{E_2(f_2)\nu_{12}(\nu_{12}\epsilon_{11} + \epsilon_{22})}{(1 - \nu_{12}^2(1 - D_2)k_2)^2}$$

$$F_{41} = \frac{E_2(f_1)\nu_{12}(1 - D_1)}{1 - \nu_{12}^2(1 - D_1)k_1}$$

$$F_{42} = \frac{E_2(f_2)\nu_{12}(1 - D_2)}{1 - \nu_{12}^2(1 - D_2)k_2}$$

$$F_{51} = \frac{E_2(f_1)\nu_{12}(\nu_{12}\epsilon_{11} + \epsilon_{22})}{(1 - \nu_{12}^2(1 - D_1)k_1)^2}$$

$$F_{52} = \frac{E_2(f_2)(\nu_{12}\epsilon_{11} + \epsilon_{22})}{(1 - \nu_{12}^2(1 - D_2)k_2)^2}$$

$$F_{61} = \frac{E_2(f_1)}{k_1(1 - \nu_{12}^2(1 - D_1)k_1)}$$

$$F_{62} = \frac{E_2(f_2)(1 - D_2)}{1 - \nu_{12}^2(1 - D_2)k_2}$$

$$F_{31} = \frac{m + 1}{\epsilon_c}\left(\frac{\epsilon_{11}}{\epsilon_c}\right)^m \exp\left[-\frac{\epsilon_{11}}{\epsilon_c}\right)^{m+1}\Bigg]$$

$$F_{32} = \frac{m + 1}{\epsilon_c}\left(\frac{\epsilon_{22}}{\epsilon_c}\right)^m \exp\left[-\left(\frac{\epsilon_{22}}{\epsilon_c}\right)^{m+1}\right]$$

$$F_{71} = \frac{\dfrac{m + 1}{f_1\sigma_c}\left[\dfrac{\sigma_{11}}{(1 - D_1)f_1\sigma_c}\right]^m}{1 - (m + 1)\left[\dfrac{\sigma_{11}}{(1 - D_1)f_1\sigma_c}\right]^{m+1}}$$

$$F_{72} = \frac{\dfrac{m + 1}{f_2\sigma_c}\left[\dfrac{\sigma_{22}}{(1 - D_2)f_2\sigma_c}\right]^m}{1 - (m + 1)\left[\dfrac{\sigma_{22}}{(1 - D_2)f_2\sigma_c}\right]^{m+1}}$$